



HAL
open science

Hitchin fibration under ramified coverings

Thiago Fassarella, Frank Loray

► **To cite this version:**

Thiago Fassarella, Frank Loray. Hitchin fibration under ramified coverings. *Journal of Pure and Applied Algebra*, 2023, 227 (12), pp.article n°107399. 10.1016/j.jpaa.2023.107399 . hal-03754844v2

HAL Id: hal-03754844

<https://hal.science/hal-03754844v2>

Submitted on 21 Mar 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

HITCHIN FIBRATION UNDER RAMIFIED COVERINGS

THIAGO FASSARELLA AND FRANK LORAY

ABSTRACT. We are interested in studying the variation of the Hitchin fibration in moduli spaces of parabolic Higgs bundles, under the action of a ramified covering. Given a degree two map $\pi : Y \rightarrow X$ between compact Riemann surfaces, we may pull back a Higgs bundle from X to Y , the lifted Higgs bundle tends to have many apparent singularities, then we perform a suitable birational transformation in order to eliminate them. This correspondence preserves the Hitchin fibrations and then its restriction to a general fiber gives a map between Abelian varieties. The aim of this paper is to describe this map.

1. INTRODUCTION

Let X be a compact Riemann surface of genus g_X and let $D = t_1 + \dots + t_n$ be a divisor on it determined by n distinct points. These points are called parabolic points. We are interested in moduli spaces of Higgs bundles on X with simple poles on D . Higgs bundles were originally introduced by Hitchin [7, 8] (see also [16]), without parabolic structure, and then extended to parabolic context in [19, 22, 23, 11, 15]. The Hitchin fibration is defined on these moduli spaces and it is a source of remarkable applications [6, 9, 17, 18].

Before going into technical definitions let us offer some motivation for the techniques developed in this paper. On the one hand, it is well known that the general fiber of the Hitchin fibration is isomorphic to an Abelian variety. On the other hand, singular fibers are difficult to deal with and this constitutes an interesting problem in the area. We believe that the symmetries given by elementary transformations play an important role in this study. Therefore, the first objective of this paper is to establish a dictionary that allows us to expressing elementary transformations in terms of the Beauville-Narasimhan-Ramanan (BNR) correspondence. As a consequence, we present a study of the variation of the Hitchin fibration with respect to a degree two ramified covering. Our main motivation is the application of these techniques to explicit examples, which are very rare in the current literature. Hence, the cases $(g_X, n) \in \{(0, 5), (0, 6), (1, 2), (2, 0)\}$ are considered at the end of the paper. It is also important to note that even in the case of five parabolic points on the Riemann sphere, a complete description of singular fibers is not known, see for example [21, Discussion page 14]. In ongoing work, we shall use the tools developed in this paper to describe all the singular fibers of the Hitchin fibration when $(g_X, n) \in \{(0, 5), (1, 2)\}$. Here, the role of elementary transformations and the variation of the Hitchin fibration with respect to the two

Date: March 21, 2023.

2020 Mathematics Subject Classification. Primary 14D20; Secondary 14H40, 14H70.

Key words and phrases. Higgs bundles, parabolic structure, Hitchin fibration, spectral curve, ramified covering.

The second author is supported by CNRS and Henri Lebesgue Center, program ANR-11-LABX-0020-0. The authors also thank Brazilian-French Network in Mathematics and CAPES-COFECUB project 932/19.

ramified elliptic cover are crucial. As a last piece of motivation, the description of singular fibers includes the study of the nilpotent cone and the locus of fixed points of the \mathbb{C}^* -action given by multiplication on Higgs field. In particular, these techniques might also be useful to investigate the foliation conjecture [20, Question 7.4]. Indeed, it follows from the work of C. T. Simpson that there is a decomposition in the moduli space of logarithmic connections obtained by looking at the limit in the moduli space of Higgs bundles, the limit point is a fixed point with respect to the \mathbb{C}^* -action and the foliation conjecture predicts that this decomposition forms a regular foliation.

Let ω_X denote the canonical sheaf of X . A rank two Higgs bundle on (X, D) consists of a rank two holomorphic vector bundle E on X , endowed with a homomorphism $\theta : E \rightarrow E \otimes \omega_X(D)$, which has nilpotent residual matrix $Res(\theta; t_i)$ at each parabolic point. By $Res(\theta; t_i)$ we mean the linear endomorphism of the fiber E_{t_i} defined by taking the residues at t_i of local 1-forms defining θ . For any integer d , there is a moduli space $\mathcal{H}(X, D, d)$ parametrizing Higgs bundles over (X, D) , with fixed degree $\deg E = d$. Actually, we consider triples (E, l, θ) where l intends to be the parabolic direction, described just below. See Section 3.2 for the detailed definition. The definition of these moduli spaces also depends on a choice of a weight vector (a sequence of reals numbers $0 \leq \mu_i \leq 1$, $i = 1, \dots, n$) which gives a notion of slope-stability. The moduli space $\mathcal{H}(X, D, d)$ is a normal variety of dimension $2(n - 3 + 4g_X)$, if it is nonempty. For instance see [23, Theorem 5.2] and our Remark 3.4.

Closely related, is the moduli space of parabolic vector bundles $\text{Bun}(X, D, d)$, which is a normal variety of dimension $n - 3 + 4g_X$, if the stable locus is nonempty. An element of it consists of a pair (E, l) , where the additional data l refers to a one dimensional subspace on the fiber of E over each parabolic point. There is a rational map $\mathcal{H}(X, D, d) \dashrightarrow \text{Bun}(X, D, d)$ which associates to a Higgs field its underlying parabolic vector bundle, where the parabolic direction over t_i coincides with the kernel of $Res(\theta; t_i)$. It turns out that the fiber of this map, at a generic point, identifies with the cotangent space $T_{(E, l, \theta)}^* \text{Bun}(X, D, d)$. In addition, $\mathcal{H}(X, D, d)$ admits a symplectic structure which coincides with that of $T^* \text{Bun}(X, D, d)$, given by the Liouville form.

An important player on this subject is the Hitchin map H_X , it associates to a Higgs bundle its characteristic polynomial. This map is known to be an algebraically integrable system, which turns out to say that the fiber over a generic point is Lagrangian and isomorphic to an Abelian variety. This last consists of the Picard variety

$$H_X^{-1}(s) \simeq \text{Pic}^{\mathbf{n}}(X_s)$$

(of the corresponding spectral curve X_s) which parametrizes line bundles of degree $\mathbf{n} = d + n + 2(g_X - 1)$ on X_s , by the BNR correspondence. This correspondence is reviewed in Section 2, as well as the notion of spectral curves.

The main goal of this paper is to study the behavior of the Hitchin fibration under a degree two ramified covering $\pi : Y \rightarrow X$, between compact Riemann surfaces. We can pull back parabolic Higgs bundles from X to Y . The lifted Higgs field, after pull back, tends to have apparent singularities, as well as the corresponding spectral curve has many singular points. In order to eliminate them, we perform a suitable elementary transformation, and this process gives a correspondence Φ between moduli spaces, which we now describe.

Assume that π is branched over the points x_1, \dots, x_n , $n \geq 1$, and let $B = \sum_{i=1}^n x_i$ be the reduced divisor on X defined by them. Let $T = \sum_{i=1}^k t_i$, $k \geq 0$, be a reduced divisor on X ($T = 0$ when $k = 0$) formed by points outside the support of B . We consider a map between moduli spaces

$$\Phi : \mathcal{H}(X, B + T, 0) \dashrightarrow \mathcal{H}(Y, \pi^*T, -n)$$

which is defined by doing the pull back π^* followed by an elementary transformation elem_R over R , where $R = \sum_{i=1}^n y_i$ denotes the divisor on Y defined by the set of ramification points, with $\pi(y_i) = x_i$. (see Section 2.4 for the detailed definition). This map preserves the Hitchin fibration and we denote by

$$\Phi_s : H_X^{-1}(s) \rightarrow H_Y^{-1}(r)$$

its restriction to a given fiber. Using the BNR correspondence, we obtain a map between the Abelian varieties that parametrize line bundles on the spectral curves. Hence, in order to describe the map Φ_s we are led to investigate the variation of the BNR correspondence with respect to elementary transformations, see Proposition 2.3.

Assuming $2g_X + n - 4 \geq 0$, with strict inequality if $k = 0$, we assure that the generic spectral curve in both moduli spaces is smooth and irreducible (see Proposition 4.1). We show that there is an étale morphism $\xi_s : Y_r \rightarrow X_s$ of degree two between spectral curves (Proposition 4.2), and the modular map Φ_s is determined by ξ_s^* , up to a translation. This is the main result of this paper, Theorem 5.1 in the main text.

Theorem 1.1. *Assume that $2g_X + n - 4 \geq 0$, with strict inequality if $k = 0$. Then Φ is a rational map of degree two onto its image, and it preserves the Hitchin fibrations. Moreover, its restriction to a general Hitchin fiber is the map*

$$\Phi_s : \text{Pic}^n(X_s) \longrightarrow \text{Pic}^{\bar{n}}(Y_r)$$

which sends a line bundle M in $\text{Pic}^n(X_s)$ to the line bundle $\xi_s^*(M)(q_r^*(-R))$ in $\text{Pic}^{\bar{n}}(Y_r)$.

We finish the paper by applying Theorem 1.1 to a degree two morphism $\pi : Y \rightarrow \mathbb{P}^1$ where $g_Y \in \{1, 2\}$. Let us briefly describe each case.

In the former case $g_Y = 1$, we consider the moduli space $\mathcal{H}(\mathbb{P}^1, \Lambda)$ of SL_2 -Higgs bundles on (\mathbb{P}^1, Λ) , where Λ is given by branch points $\{0, 1, \lambda, \infty\}$ of π with an extra point t . By SL_2 we mean that Higgs fields have vanishing trace and vector bundles have trivial determinant line bundle. After multiplication by a suitable line bundle to redress the determinant, we get a modular map $\Phi_0 : \mathcal{H}(\mathbb{P}^1, \Lambda) \rightarrow \mathcal{H}(Y, D)$ to the moduli space $\mathcal{H}(Y, D)$ of SL_2 -Higgs bundles over the elliptic curve Y , with simple poles on $D = t_1 + t_2$, with $\pi(t_i) = t$. The spectral curve X_s , on the \mathbb{P}^1 side, has genus 2; on the elliptic side, the spectral curve Y_r has genus 3.

In the second case $g_Y = 2$, we consider the moduli space $\mathcal{H}(\mathbb{P}^1, B)$ of SL_2 -Higgs bundles over (\mathbb{P}^1, B) , where B is formed by branch points of π . After renormalization of the determinant line bundle, we get a modular $\Phi_0 : \mathcal{H}(\mathbb{P}^1, B) \rightarrow \mathcal{H}(Y)$ map with values on the moduli space $\mathcal{H}(Y)$ of holomorphic Higgs bundles (without parabolic points) over Y . The spectral curve X_s over \mathbb{P}^1 has genus 3, and over Y , the spectral curve Y_r has genus 5. In this context, Theorem 1.1 yields the following result, which corresponds to Corollaries 5.2 and 5.3 in the main text.

Corollary 1.2. *Let the notation be as above. In both cases $g_Y = 1, 2$, the modular map Φ_0 is dominant and its restriction to a smooth Hitchin fiber is the degree two map*

$$\begin{aligned} \Phi_{0,s} : \text{Pic}^n(X_s) &\rightarrow \text{Prym}(Y_r/Y) \\ M &\mapsto \xi_s^*(M) \otimes q_r^*(L_0(-R)) \end{aligned}$$

where L_0 is a square root of $\mathcal{O}_Y(R)$ and $\text{Prym}(Y_r/Y)$ is the Prym variety of the covering $q_r : Y_r \rightarrow Y$.

2. BASIC TOOLS

2.1. Notations and conventions. Throughout this paper, we assume that X is a compact Riemann surface of genus $g_X \geq 0$. As usual, we denote by \mathcal{O}_X its structural sheaf and by ω_X its canonical sheaf. When L is a line bundle on X we let L^{-1} be its dual. If E is a vector bundle on X we often write $\Gamma(E)$ for $H^0(X, E)$.

2.2. Spectral curve. Let L be a line bundle on X . We denote by \mathbf{P} the ruled surface $\mathbb{P}(\mathcal{O}_X \oplus L^{-1})$ which is the projectivization of the rank two vector bundle $\mathcal{O}_X \oplus L^{-1}$. Let $q : \mathbf{P} \rightarrow X$ be the natural projection and let $\mathcal{O}_{\mathbf{P}}(1)$ be the hyperplane bundle along the fibers. We note that $q_*(\mathcal{O}_{\mathbf{P}}) = \mathcal{O}_X$ and $q_*(\mathcal{O}_{\mathbf{P}}(1)) = \mathcal{O}_X \oplus L^{-1}$, see for example [5, II - Proposition 7.11].

The ruled surface \mathbf{P} contains two disjoint sections $\mathbb{P}(\mathcal{O}_X)$ and $\mathbb{P}(L^{-1})$ corresponding to the embedding of \mathcal{O}_X and L^{-1} , respectively. The embedding of $\mathcal{O}_X \hookrightarrow q_*(\mathcal{O}_{\mathbf{P}}(1)) = \mathcal{O}_X \oplus L^{-1}$ gives a section \mathbf{w} of $\mathcal{O}_{\mathbf{P}}(1)$, via the adjoint formula (cf. [5, II - Section 5]) which has $\mathbb{P}(L^{-1})$ as its zero set. Similarly, since

$$q_*(q^*L \otimes \mathcal{O}_{\mathbf{P}}(1)) \simeq L \otimes (\mathcal{O}_X \oplus L^{-1}) = L \oplus \mathcal{O}_X$$

by the projection formula, we obtain a section \mathbf{z} of $q^*L \otimes \mathcal{O}_{\mathbf{P}}(1)$ which has $\mathbb{P}(\mathcal{O}_X)$ as its zero set.

Given $s = (s_1, s_2) \in \Gamma(L) \oplus \Gamma(L^2)$, we define the *spectral curve* $X_s \subset \mathbf{P}$ as the zero locus of the section

$$\mathbf{z}^2 + q^*(s_1) \cdot \mathbf{z} \cdot \mathbf{w} + q^*(s_2) \cdot \mathbf{w}^2 \in H^0(\mathbf{P}, q^*(L^2) \otimes \mathcal{O}_{\mathbf{P}}(2)).$$

It comes with a degree two map

$$q_s : X_s \rightarrow X$$

which is the restriction of q to X_s . Notice that in a local open subset U of X where $\mathbf{P}|_U \simeq U \times \mathbb{P}_{(w,z)}^1$, the spectral curve has equation

$$z^2 + s_1 z w + s_2 w^2 = 0.$$

Equivalently, we can define a structure of commutative ring on $\mathcal{O}_X \oplus L^{-1}$ induced by s :

$$(a_0, a_1) \cdot (b_0, b_1) := (a_0 b_0 - s_2 a_1 b_1, a_0 b_1 + a_1 b_0 - s_1 a_1 b_1). \quad (2.1)$$

This makes $\mathcal{O}_X \oplus L^{-1}$ an \mathcal{O}_X -algebra, which will be denoted by \mathcal{A}_s , and is locally given by

$$\mathcal{A}_s(U) = \frac{\mathcal{O}_X(U)[z]}{(z^2 + s_1 z + s_2)}. \quad (2.2)$$

The spectral curve may be seen as

$$X_s = \text{Spec}(\mathcal{A}_s).$$

We often write simply

$$X_s = \text{Spec}(\mathcal{O}_X \oplus L^{-1}) \quad (2.3)$$

where $\mathcal{O}_X \oplus L^{-1}$ is considered as \mathcal{O}_X -algebra. Note that

$$(q_s)_* \mathcal{O}_{X_s} = \mathcal{O}_X \oplus L^{-1}. \quad (2.4)$$

The (arithmetic) genus of X_s is given by

$$g_{X_s} := \dim H^1(X_s, \mathcal{O}_{X_s}) = \deg L + 2(g_X - 1) + 1 \quad (2.5)$$

see [2, Remark 3.2].

2.3. BNR correspondence. Let us briefly recall the correspondence between \mathcal{O}_X -homomorphisms $\theta : E \rightarrow E \otimes L$ having the same characteristic polynomial P_s and torsion free sheaves M_θ of rank one on X_s (cf. [2, Proposition 3.6]).

Let E be a rank two holomorphic vector bundle on X and let $\theta \in \Gamma(\mathcal{E}nd(E) \otimes L)$ be a homomorphism with $s_1 = -\text{tr}(\theta) \in \Gamma(L)$ and $s_2 = \det(\theta) \in \Gamma(L^2)$. By Cayley-Hamilton theorem it satisfies the equation $P_s(\theta) = 0$, where $P_s(z) = z^2 + s_1 z + s_2$ is the characteristic polynomial and

$$P_s(\theta) = \theta^2 + s_1 \cdot \theta + s_2 \cdot I$$

must be seen as a homomorphism $E \rightarrow E \otimes L^2$.

Let us assume that X_s is integral, i.e. irreducible and reduced, which turns out to say that \mathcal{A}_s is a sheaf of integral domains. We have a canonical isomorphism

$$\mathcal{E}nd(E) \otimes L \simeq \mathcal{H}om_{\mathcal{O}_X}(L^{-1}, \mathcal{E}nd(E))$$

which associates $\theta \in \mathcal{E}nd(E) \otimes L$ to the \mathcal{O}_X -homomorphism

$$\begin{aligned} L^{-1} &\rightarrow \mathcal{E}nd(E) \\ z &\rightarrow \theta_z \end{aligned}$$

where $\theta_z(e) = \theta(e) \cdot z$. Thus the homomorphism θ induces a structure of \mathcal{A}_s -module on E

$$\Theta : \mathcal{A}_s \times E \rightarrow E \quad (2.6)$$

which is locally given by (see (2.2))

$$\begin{aligned} \Theta_U : \mathcal{A}_s(U) \times E(U) &\rightarrow E(U) \\ (a + bz, e) &\rightarrow (aI + b\theta_z) \cdot e \end{aligned}$$

It defines a sheaf M_θ of \mathcal{A}_s -modules on X_s , which comes with a multiplication

$$(a + bz) \odot e := (aI + b\theta_z) \cdot e \quad (2.7)$$

for any $a + bz \in \mathcal{A}_s$, $e \in E$. Since X_s is integral then M_θ is also torsion free.

Conversely, if M is a torsion free sheaf of rank one on X_s then $E = (q_s)_* M$ is a locally free \mathcal{O}_X -module of rank two with an \mathcal{O}_X -linear map

$$\Theta : \mathcal{A}_s \times E \rightarrow E.$$

The section \mathbf{z} of q_s^*L gives a family of endomorphisms

$$\begin{aligned} \theta_U : E(U) &\rightarrow E(U) \\ s &\mapsto \Theta(z, s) \end{aligned}$$

for each open set U of X such that $\mathbf{P}|_U \simeq U \times \mathbb{P}_{(w:z)}^1$, which glue together to give a global homomorphism $\theta : E \rightarrow E \otimes L$ satisfying $\theta^2 + s_1\theta + s_2I = 0$. Since X_s is integral then $P_s(z) = z^2 + s_1z + s_2$ is irreducible over the function field of X . Thus P_s is the characteristic polynomial of θ .

Summarizing the discussion above, here is the version of BNR correspondence [2, Proposition 3.6] needed for our purposes:

Proposition 2.1. *Assume that the spectral curve X_s is integral. Then there is a bijective correspondence between isomorphism classes of torsion free rank one sheaves on X_s and isomorphism classes of pairs (E, θ) where E has rank two and $\theta : E \rightarrow E \otimes L$ is an \mathcal{O}_X -homomorphism having characteristic polynomial P_s .*

We conclude this section with a lemma which will be useful in the sequel.

Lemma 2.2. *Let M be the torsion free sheaf of rank one on X_s associated to $\theta : E \rightarrow E \otimes L$. Then M is a line bundle on X_s if and only if for every point $x \in X$ the \mathbb{C} -linear map $\theta_x : E_x \rightarrow E_x \otimes L_x \simeq E_x$ is not of the form λI , where $\lambda \in \mathbb{C}$ and $I : E_x \rightarrow E_x$ is the identity.*

Proof. First assume that M is a line bundle on X_s , i.e. it has rank one as \mathcal{A}_s -module. Then for any $x \in X$ there exist an open subset $U \subset X$ containing x and $v \in M(U)$ such that every $e \in M(U)$ writes as

$$e = (a + bz) \odot v, \quad a + bz \in \mathcal{A}_s(U).$$

This means that any section $e \in E(U)$ can be written as

$$e = (a + bz) \odot v = av + b\theta(v), \quad a, b \in \mathcal{O}_X(U).$$

In particular, v and $\theta(v)$ are \mathbb{C} -linearly independent when considered as elements of E_x , which implies that $\theta_x \neq \lambda I$, for all $\lambda \in \mathbb{C}$. Reciprocally, assuming that $\theta_x \neq \lambda I$ for all $\lambda \in \mathbb{C}$, there exist an open neighbourhood U of x and $v \in E(U)$ such that v and $\theta(v)$ are $\mathcal{O}_X(U)$ -linearly independent. This implies that $v \in M(U)$ generates $M(U)$ as $\mathcal{A}_s(U)$ -module. \square

2.4. Elementary transformations. We next introduce strongly parabolic endomorphisms and elementary transformations. The goal of this section is the behavior of M_θ (cf. Section 2.3) with respect to elementary transformations.

Fix $t_1, \dots, t_n \in X$ distinct points and denote by $D = t_1 + \dots + t_n$ the divisor determined by them.

A *quasiparabolic vector bundle* (E, l) , $l = \{l_i\}$, of rank two on (X, D) consists of a holomorphic vector bundle E of rank two on X and for each $i = 1, \dots, n$, a 1-dimensional linear subspace $l_i \subset E_{t_i}$. We refer to the points t_i 's as parabolic points, and to the subspaces $l_i \subset E_{t_i}$ as the parabolic direction of E at t_i .

There is a correspondence between quasiparabolic vector bundles, called elementary transformation, which we now describe. We consider the following exact sequence of sheaves

$$0 \rightarrow E' \xrightarrow{\alpha} E \xrightarrow{\beta} \bigoplus_{i=1}^n E/l_i \rightarrow 0$$

where E/l_i intends to be a skyscraper sheaf determined by E_{t_i}/l_i , i.e., for an open subset U of X we have $(E/l_i)(U) = E_{t_i}/l_i$ if $t_i \in U$ and $\{0\}$ otherwise. The map β sends s to $\bigoplus_{i=1}^n s(t_i)$. If E is locally generated by e_1, e_2 as \mathcal{O}_X -module near t_i with $e_1(t_i) \in l_i$, then E' is locally generated by e_1, e'_2 , with $e'_2 = xe_2$, where x is a local coordinate. In particular E' is locally free of rank two. We view E' as a quasiparabolic vector bundle (E', l') of rank two over (X, D) putting $l'_i := \ker \alpha_{t_i}$. We call it the *elementary transformation* of (E, l) over D :

$$\text{elem}_D(E, l) := (E', l').$$

Notice that we have the following equality

$$\det E' = \det E \otimes \mathcal{O}_X(-D).$$

An endomorphism $f : E \rightarrow E$ is called *parabolic* if $f(l_i) \subseteq l_i$ for every $i = 1, \dots, n$. We denote by $\mathcal{E}nd(E, l)$ the sheaf of parabolic endomorphisms. A parabolic endomorphism is called *strongly parabolic* if $f(E_{t_i}) \subseteq l_i$ and $f(l_i) = 0$ for every $i = 1, \dots, n$. The sheaf of strongly parabolic endomorphisms of E will be denoted by $\mathcal{S}\mathcal{E}nd(E, l)$.

Given a parabolic homomorphism $\theta : E \rightarrow E \otimes L$, i.e.

$$\theta \in \Gamma(\mathcal{E}nd(E, l) \otimes L)$$

we can perform an elementary transformation elem_D on the pair (E, θ) , centered in l . For instance, since θ is parabolic with respect to l , then $\theta(E') \subset E' \otimes L$ and its restriction induces a homomorphism

$$\theta' : E' \rightarrow E' \otimes L$$

which is parabolic with respect to the direction l' of E' . If e_1, e_2 are local sections which generate E near a parabolic point t_i with $e_1(t_i) \in l_i$ then θ is given by

$$\theta = \begin{pmatrix} a & b \\ xc & d \end{pmatrix}.$$

Since E' is locally generated by e_1, e'_2 , with $e'_2 = xe_2$, then the restriction of θ to E' corresponds to the matrix

$$\theta' = \begin{pmatrix} a & xb \\ c & d \end{pmatrix}.$$

Note that trace and determinant do not change after an elementary transformation.

The condition of being strongly parabolic, i.e. $\theta \in \Gamma(\mathcal{S}\mathcal{E}nd(E, l) \otimes L)$ yields

$$\begin{cases} \text{tr}(\theta) \in \Gamma(L(-D)) \\ \det(\theta) \in \Gamma(L^2(-D)) \end{cases}$$

and in this case the spectral curve X_s , determined by

$$s = (-\text{tr } \theta, \det \theta) \in \Gamma(L) \oplus \Gamma(L^2)$$

is ramified over D .

Before stating the next result, we introduce some notation. For this, let $\theta : E \rightarrow E \otimes L$ be a parabolic homomorphism with respect to l and assume that the corresponding spectral curve X_s is integral, in particular $\det \theta$ is nonzero. We shall consider the curve $X_\theta \subset \mathbb{P}E$ of eigendirections of θ , defined as follows. If we take a local trivialization $\mathbb{P}E|_U \simeq U \times \mathbb{P}^1$ of the ruled surface $\mathbb{P}E$ at a point $x \in X$ where $\det \theta_x \neq 0$, and if θ has local matrix

$$\theta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then we consider the curve in $\mathbb{P}E|_U$ having equation

$$bz^2 + (a - d)z - c = 0$$

where $(1 : z)$ is a coordinate for \mathbb{P}^1 . These local equations patch together to build a quasi-projective subvariety of $\mathbb{P}E$ and X_θ is defined as the closure of this subvariety. Notice that X_θ intersects the fiber $\mathbb{P}E_x$, at a general point $x \in X$, exactly at two eigendirections of the linear homomorphism $\theta_x : E_x \rightarrow E_x \otimes L_x \simeq E_x$.

Let us denote by

$$W_l = l_1 + \cdots + l_n$$

the *divisor in X_θ defined by parabolic directions*, see Figure 1. There is a birational morphism

$$\eta : X_\theta \rightarrow X_s$$

which associates to each eigendirection the corresponding eigenvalue, and then it defines a divisor

$$W_{l,s} = \eta(l_1) + \cdots + \eta(l_n)$$

on X_s . If in addition X_s is smooth, then η is an isomorphism. Note that when θ is strongly parabolic, and then X_s is ramified over D , one has $W_{l,s} = (q_s^* D)_{red}$. The subscript *red* denotes the effective reduced divisor defined by the support of $q_s^* D$.

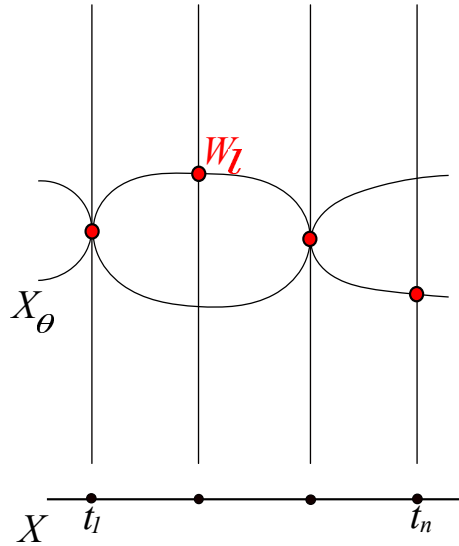


FIGURE 1. Divisor W_l defined by parabolic directions.

The next result describes the behavior of M_θ (cf. Section 2.3) under elementary transformations.

Proposition 2.3. *Assume that the spectral curve X_s is integral. Let M_θ be a line bundle on X_s corresponding to a parabolic homomorphism $\theta \in \Gamma(\mathcal{E}nd(E, l) \otimes L)$. Let $M_{\theta'}$ be the torsion free sheaf corresponding to $\theta' \in \Gamma(\mathcal{E}nd(E', l') \otimes L)$, obtained by performing an elementary transformation over D . Then*

$$M_{\theta'} = M_\theta \otimes \mathcal{I}$$

where \mathcal{I} is the ideal sheaf of the subscheme consisting of the support of $q_s^*D - W_{l,s}$.

Proof. The homomorphism $\theta : E \rightarrow E \otimes L$ induces a structure of \mathcal{O}_{X_s} -module

$$\Theta : \mathcal{O}_{X_s} \times E \rightarrow E$$

on E and M_θ is just E with this structure. See Section 2.3. Let $M_{\theta'}$ be the torsion free sheaf of rank one associated to θ' . Since θ' is the restriction of θ to E' , then $M_{\theta'}$ is the \mathcal{O}_{X_s} -submodule of M_θ determined by the restriction Θ' of Θ to E'

$$\Theta' : \mathcal{O}_{X_s} \times E' \rightarrow E'.$$

It is enough to analyse the behavior of M_θ at a point t of the support of D , because outside it does not change. We may assume, without loss of generality, that $t = 0$, $l_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and

$$\theta = \begin{pmatrix} a & b \\ xc & d \end{pmatrix}$$

and the spectral curve X_s has local equation

$$X_s = \{(x, z) : z^2 - (a + d)z + (ad - xbc) = 0\}.$$

Since M_θ is a line bundle, then there is a generator v for the \mathcal{O}_{X_s} -module M_θ , i.e., any local section e of M_θ , can be written as

$$e = (\alpha + \beta z) \odot v \in M_\theta$$

with $\alpha + \beta z \in \mathcal{O}_{X_s}$ ($\alpha, \beta \in \mathcal{O}_X$, cf. (2.7)), meaning that any local section e of E can be written as

$$e = (\alpha I + \beta \theta)(v) \in E.$$

Sections of E' are those of the form $e' = \alpha v + \beta \theta(v) \in E$, such that

$$e'_0 = (\alpha_0 I + \beta_0 \theta_0)(v_0)$$

is proportional to $l_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and the subscript means evaluation at the point $x = 0$. Since

$$\alpha_0 I + \beta_0 \theta_0 = \begin{pmatrix} \alpha_0 + \beta_0 a_0 & \beta_0 b_0 \\ 0 & \alpha_0 + \beta_0 d_0 \end{pmatrix}$$

one obtains that $\alpha_0 + \beta_0 d_0 = 0$, because v_0 is not proportional to l_0 . This last implies that the regular function $\alpha + \beta z \in \mathcal{O}_{X_s}$ vanishes at the point $(0, d_0) \in X_s$, which corresponds to the second eigendirection of θ when $a_0 \neq d_0$. When both eigendirections coincide, for

instance when X_s is singular or has a smooth ramification point, then $\alpha + \beta z$ vanishes at this point.

We conclude that the sections of $M_{\theta'}$ are those sections of M_θ which vanish at the support of the divisor $q_s^*D - W_{l,s}$. This gives $M_{\theta'} = M_\theta \otimes \mathcal{I}$ and this concludes the proof of the proposition. \square

Corollary 2.4. *With the same notation of Proposition 2.3, if X_s is smooth and irreducible, and θ is strongly parabolic then*

$$M_{\theta'} = M_\theta(-(q_s^*D)_{red}).$$

Proof. When θ is strongly parabolic, X_s is ramified over every point in the support of D , then $q_s^*D - W_{l,s} = (q_s^*D)_{red}$. In addition, since X_s is smooth then $\mathcal{I} = \mathcal{O}_{X_s}(-(q_s^*D)_{red})$. \square

2.5. Pullback and twist. Let Y be another compact Riemann surface and assume that $\pi : Y \rightarrow X$ is a finite morphism. Given $s = (s_1, s_2) \in \Gamma(L) \oplus \Gamma(L^2)$, let Y_r^* be the spectral curve associated to $r = \pi^*s \in \Gamma(\pi^*L) \oplus \Gamma(\pi^*L^2)$. The superscript $*$ in Y_r^* will be justified in Section 4, where a normalization Y_r of Y_r^* is constructed for our purposes.

We note that

$$Y_r^* = Y \times_X X_s.$$

Indeed, since π is finite and thus affine, we can assume $\pi : \text{Spec } B \rightarrow \text{Spec } A$, $X_s = \text{Spec } A_s$ and $Y_r^* = \text{Spec } B_{\pi^*s}$ with

$$A_s = \frac{A[z]}{(z^2 + s_1z + s_2)} \quad \text{and} \quad B_{\pi^*s} = \frac{B[z]}{(z^2 + \pi^*s_1z + \pi^*s_2)};$$

and then we see that $B_{\pi^*s} = B \otimes_A A_s$. In terms of sheaves of algebras, we have

$$\mathcal{B}_{\pi^*s} = \mathcal{O}_Y \otimes_{\pi^{-1}(\mathcal{O}_X)} \pi^{-1}(\mathcal{A}_s)$$

where \mathcal{B}_{π^*s} denotes the structural sheaf of Y_r^* .

In particular, there is a lifting $\pi_s : Y_r^* \rightarrow X_s$ making the following diagram commute

$$\begin{array}{ccc} Y_r^* & \xrightarrow{\pi_s} & X_s \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\pi} & X \end{array} \quad (2.8)$$

and which is locally given by the natural homomorphism $A_s \rightarrow A_s \otimes_A B = B_{\pi^*s}$. The next result describes the behavior of M_θ (cf. Section 2.3) under pullback.

Proposition 2.5. *Assume that both spectral curves X_s and Y_r^* are integral. Let $\theta : E \rightarrow E \otimes L$ be a homomorphism with characteristic coefficient s and let M be the torsion free sheaf of rank one on X_s associated to θ . Then π_s^*M is the torsion free sheaf of rank one associated to $\pi^*\theta$.*

Proof. The pullback of M by $\pi_s : Y_r^* \rightarrow X_s$ is given by

$$\pi_s^*M = \pi_s^{-1}(M) \otimes_{\pi_s^{-1}(\mathcal{O}_{X_s})} \mathcal{O}_{Y_r^*}.$$

If $U \subset Y$ and $V \subset X$ are local affine open sets where

$$\pi : U = \text{Spec } B \rightarrow V = \text{Spec } A$$

let us denote by $\tilde{U} \subset Y_r^*$ and $\tilde{V} \subset X_s$ their respective inverse images. We can write

$$(\pi_s^* M)(\tilde{U}) = M(\tilde{V}) \otimes_{A_s} B_{\pi^* s}.$$

Since the A_s -module structure on $M(\tilde{V})$ is induced by θ , then the $B_{\pi^* s}$ -module structure on $(\pi_s^* M)(\tilde{U})$ is induced by $\pi^* \theta$. Which is equivalent to give a $B_{\pi^* s}$ -module structure on

$$\pi^*(E)(U) = E(V) \otimes_A B$$

induced by $\pi^* \theta$. It turns out to say that $\pi_s^* M$ coincides with $\pi^*(E)$ with $B_{\pi^* s}$ -module structure induced by $\pi^* \theta$. This shows that $\pi_s^* M$ is the torsion free sheaf of rank one associated to $\pi^* \theta$. \square

We finish this section by analyzing the behavior of M under twist.

Proposition 2.6. *Assume that X_s is integral. Let $\theta : E \rightarrow E \otimes L$ be a homomorphism with characteristic coefficient s and let M be the torsion free sheaf of rank one on X_s associated to θ . Let L_0 be a line bundle on X and let $\theta_{L_0} : E \otimes L_0 \rightarrow (E \otimes L_0) \otimes L$ be the homomorphism induced by θ . Then the torsion free sheaf of rank one on X_s associated to θ_{L_0} is $M \otimes q_s^* L_0$.*

Proof. Let M_{L_0} be the torsion free sheaf of rank one on X_s associated to θ_{L_0} . It follows from projection formula that

$$(q_s)_*(M \otimes q_s^* L_0) = ((q_s)_*(M)) \otimes L_0 = E \otimes L_0$$

and then $M \otimes q_s^* L_0$ and M_{L_0} have the same pushforward via q_s . The reader can check that the structure of \mathcal{A}_s -module on $E \otimes L_0$ induced by θ_{L_0} coincides with that of $M \otimes q_s^* L_0$. \square

3. PARABOLIC HIGGS BUNDLES AND HITCHIN FIBRATION

3.1. Parabolic vector bundles. Let (E, l) be a quasiparabolic vector bundle. Fix a weight vector $\mu = (\mu_1, \dots, \mu_n)$ of real numbers $0 \leq \mu_i \leq 1$. The *parabolic slope* of (E, l) with respect to μ is

$$\text{Slope}^\mu(E) = \frac{\deg E + \sum_{i=1}^n \mu_i}{2}$$

where $\deg E = \deg(\det E)$. Let $N \subset E$ be a line subbundle. For each $i = 1, \dots, n$, set

$$\text{Slope}^\mu_i(N, E) = \begin{cases} \mu_i & \text{if } N_{t_i} = L_i, \\ 0 & \text{if } N_{t_i} \neq L_i. \end{cases}$$

The *parabolic slope* of $N \subset E$ with respect to μ is

$$\text{Slope}^\mu(N, E) = \deg N + \sum_{i=1}^n \text{Slope}^\mu_i(N, E).$$

A quasiparabolic vector bundle (E, l) is μ -*semistable* (respectively μ -*stable*) if for every line subbundle $N \subset E$ we have

$$\text{Slope}^\mu(N, E) \leq \text{Slope}^\mu(E)$$

(respectively $\text{Slope}^\mu(N, E) < \text{Slope}^\mu(E)$). A *parabolic vector bundle* is a quasiparabolic vector bundle together with a weight vector μ . We say that a parabolic vector bundle is *semistable* if the corresponding quasiparabolic vector bundle is μ -*semistable*.

For each $d \in \mathbb{Z}$ and a weight vector μ , there is a *moduli space* $\text{Bun}_\mu(X, D, d)$, parametrizing rank two parabolic vector bundles (E, l) on (X, D) , with $\deg E = d$, which are semistable. If N is a line subbundle with $\deg N = d$, we denote by $\text{Bun}_\mu(X, D, N)$ the subvariety of $\text{Bun}_\mu(X, D, d)$ given by those parabolic vector bundles with $\det E = N$.

The moduli space $\text{Bun}_\mu(X, D, d)$ has a structure of a normal projective variety of dimension $n - 3 + 4g_X$, if the stable locus is not empty, see [13, 3]. By twisting vector bundles with a fixed line bundle L_0 , we see that $\text{Bun}_\mu(X, D, N) \cong \text{Bun}_\mu(X, D, N \otimes L_0^2)$.

Elementary transformations give correspondences between moduli spaces. The stability condition is preserved after an appropriate modification of weights, if (E, l) is μ -semistable then $\text{elem}_D(E, l)$ is μ' -semistable with

$$\mu' = (1 - \mu_1, \dots, 1 - \mu_n).$$

In particular, this gives a map

$$\text{elem}_D: \text{Bun}_\mu(X, D, N) \rightarrow \text{Bun}_{\mu'}(X, D, N \otimes \mathcal{O}_X(-D)).$$

3.2. Parabolic Higgs bundles. A *parabolic Higgs bundle* is a triple (E, l, θ) where (E, l) is a quasiparabolic vector bundle over (X, D) and

$$\theta \in \Gamma(\mathcal{S}\mathcal{E}nd(E, l) \otimes \omega_X(D))$$

is a strongly parabolic homomorphism in the sense of Section 2.4. We say that θ is a *parabolic Higgs field*. We note that being strongly parabolic means that for each point $t \in X$ lying in the support of D , the endomorphism

$$\text{Res}(\theta; t) \in \text{End}(E_t)$$

is nilpotent with respect to the parabolic direction of E_t . The nilpotency condition means that if $p \subset E_t$ is the parabolic direction, then

$$\text{Res}(\theta; t)(p) = 0 \quad \text{and} \quad \text{Res}(\theta; t)(E_t) \subset p.$$

A line subbundle $N \subset E$ is called *invariant* under $\theta : E \rightarrow E \otimes \omega_X(D)$ if

$$\theta(N) \subset N \otimes \omega_X(D).$$

We say that θ is *irreducible* if it does not admit invariant line subbundle.

Fix a weight vector $\mu \in [0, 1]^n$. A parabolic Higgs bundle (E, l, θ) on (X, D) is called μ -*semistable* (respectively μ -*stable*) if for every line subbundle $N \subset E$ invariant under θ , we have

$$\text{Slope}^\mu(N, E) \leq \text{Slope}^\mu(E)$$

(respectively $\text{Slope}^\mu(N, E) < \text{Slope}^\mu(E)$). We say that (E, l, θ) is μ -*unstable* if it is not μ -*semistable*. Note that an irreducible Higgs field is stable for any choice of weight vector.

Given $d \in \mathbb{Z}$, let $\mathcal{H}_\mu(X, D, d)$ be the moduli space of μ -semistable parabolic Higgs bundles (E, l, θ) on (X, D) with $\deg E = d$. If N is a line bundle of degree d , we denote by $\mathcal{H}_\mu(X, D, N)$ the subvariety of $\mathcal{H}_\mu(X, D, d)$ given by those parabolic Higgs bundles with fixed determinant $\det E = N$.

The moduli space $\mathcal{H}_\mu(X, D, d)$ is a quasiprojective variety of dimension $2(n - 3 + 4g_X)$, see [22, 23, 4].

Similar to parabolic vector bundles, given (E, l, θ) we can perform an elementary transformation elem_D on it, see Section 2.4, and this provides a correspondence between moduli spaces of Higgs bundles

$$\text{elem}_D : \mathcal{H}_\mu(X, D, N) \rightarrow \mathcal{H}_\mu(X, D, N \otimes \mathcal{O}_X(-D)).$$

3.3. Hitchin map and its fibers. Let $(E, l, \theta) \in \mathcal{H}_\mu(X, D, d)$ be a parabolic Higgs bundle. Since $\text{Res}(\theta; t_i)$ is nilpotent for every parabolic point $t_i \in X$, one obtains

$$\begin{cases} \text{tr}(\theta) \in \Gamma(\omega_X) \\ \det(\theta) \in \Gamma(\omega_X^{\otimes 2}(D)) \subset \Gamma(\omega_X^{\otimes 2}(2D)). \end{cases}$$

In order to simplify notation we shall write Σ_D for the vector space

$$\Sigma_D = \Gamma(\omega_X) \oplus \Gamma(\omega_X^{\otimes 2}(D)). \quad (3.1)$$

The *Hitchin map* is defined as

$$\begin{aligned} H_X : \mathcal{H}_\mu(X, D, d) &\longrightarrow \Sigma_D \\ (E, l, \theta) &\longmapsto (-\text{tr}(\theta), \det(\theta)). \end{aligned}$$

Let $s \in \Sigma_D$. Using the correspondence of Section 2.3 we get an identification between the Jacobian variety of X_s and the fiber of H_X over s , when X_s is integral. Then, we are led to investigate the existence of integral spectral curves. To do this, we need a couple of results.

Proposition 3.1. *Let (E, l, θ) be a parabolic Higgs bundle on (X, D) and let $s = (-\text{tr} \theta, \det \theta)$. If the spectral curve X_s is non-integral then $\det \theta \in \Gamma(\omega_X^{\otimes 2})$.*

Proof. If X_s is non-integral then the characteristic polynomial $P_s(z) = z^2 - \text{tr} \theta z + \det \theta$ is reducible over the function field of X . Hence, near a parabolic point t_i in the support of D we can write

$$P_s(z) = (z - a)(z - b)$$

with $a, b \in \mathcal{O}_X(U)$. The nilpotence condition on θ implies that $a + b$ and ab vanish at t_i . In particular, a and b vanish at t_i and then $\det \theta \in \Gamma(\omega_X^{\otimes 2}(2D))$ has a zero of order at least two at t_i , $i = 1, \dots, n$, which means that $\det \theta \in \Gamma(\omega_X^{\otimes 2})$. \square

Proposition 3.2. *Let (E, l, θ) be a parabolic Higgs bundle on (X, D) . If θ has an invariant line bundle $N \subset E$, then X_s is non-integral.*

Proof. We will show that the characteristic polynomial $P_s(z) = z^2 - \text{tr} \theta z + \det \theta$ is reducible over the function field of X . Taking a Zariski open subset U of X and $e \in N(U)$ there exists $\lambda \in \omega_X(D)(U)$ such that

$$\theta(e) = \lambda \cdot e.$$

In particular, the rational function $z - \lambda$ on X divides $P_s(z)$. \square

Let us assume that X_s is integral and let M be a line bundle on X_s . We can compute the degree of $E = (q_s)_* M$ using (2.4) and the following identity

$$\det(E) \simeq \det((q_s)_* \mathcal{O}_{X_s}) \otimes \text{Nm}(M) \quad (3.2)$$

where $\text{Nm}(M)$ is the norm map (see [5, Cap. IV Ex. 2.6]). The result is

$$\deg M = \deg E + \deg L$$

where $L = \omega_X(D)$, and this yields

$$\deg M = \deg E + \deg D + 2g_X - 2.$$

This computation can be extended to torsion free rank one sheaves on X_s . Therefore, given $d \in \mathbb{Z}$ we are led to define

$$\mathfrak{n} = \mathfrak{n}(d) := d + \deg D + 2g_X - 2.$$

Let us denote by $\overline{\text{Pic}}^{\mathfrak{n}}(X_s)$ the variety parametrizing isomorphism classes of torsion free sheaves of rank one on X_s of degree \mathfrak{n} and let $\text{Pic}^{\mathfrak{n}}(X_s)$ denote its corresponding subset formed by line bundles. For an irreducible curve having only planar singularities, $\overline{\text{Pic}}^{\mathfrak{n}}(X_s)$ is an irreducible variety that contains $\text{Pic}^{\mathfrak{n}}(X_s)$ as a dense open subset (c.f. [1]). If X_s has a non-planar singularity then $\overline{\text{Pic}}^{\mathfrak{n}}(X_s)$ has at least two irreducible components (c.f. [10]).

As we have seen in Section 2.3, there is a bijective correspondence between $\overline{\text{Pic}}^{\mathfrak{n}}(X_s)$ and isomorphism classes of pairs (E, θ) where E is a vector bundle of rank two and degree d , and $\theta : E \rightarrow E \otimes \omega_X(D)$ is a homomorphism with $\text{tr}(\theta) = -s_1$ and $\det(\theta) = s_2$. In particular we have the following result (see also [12]).

Proposition 3.3. *Assuming that $s = (s_1, s_2) \in \Sigma_D$, with*

$$s_2 \in \Gamma(\omega_X^2(D)) \setminus \Gamma(\omega_X^2(D - t_i)), \quad \forall i = 1, \dots, n$$

then X_s is integral and there is a bijective correspondence

$$\overline{\text{Pic}}^{\mathfrak{n}}(X_s) \longleftrightarrow H_X^{-1}(s).$$

Moreover, any Higgs bundle in $H_X^{-1}(s)$ has irreducible Higgs field.

Proof. Our hypothesis on s_2 ensures, in particular, that $s_2 \notin \Gamma(\omega_X^{\otimes 2})$, then X_s is integral by Proposition 3.1.

Let $L = \omega_X(D)$. For each $M \in \overline{\text{Pic}}^{\mathfrak{n}}(X_s)$ we can associate a pair (E, θ) , where $\theta : E \rightarrow E \otimes L$ is a homomorphism with characteristic polynomial s , see Section 2.3. Since s_2 lies in the subspace $\Gamma(L^2(-D))$ of $\Gamma(L^2)$ then any residual matrix $\text{Res}(\theta; t_i)$ has vanishing determinant. This and $\text{tr}(\theta) \in \Gamma(L(-D))$ yield θ strongly parabolic. Note that $\text{Res}(\theta; t_i)$ is non-null because $s_2 \notin \Gamma(L^2(-D - t_i))$. Thus the parabolic direction $l_i \subset E_{t_i}$ is defined as the kernel of $\text{Res}(\theta; t_i)$. Now we show that (E, l, θ) is μ -semistable. Actually, we can show that θ is irreducible and this also completes the last assertion of the statement of the proposition. Indeed, by Proposition 3.2, if there is a line subbundle $N \subset E$ invariant by θ , then $\det(\theta) = s_2 \in \Gamma(\omega_X^{\otimes 2})$ and this gives a contradiction.

Conversely, for each pair (E, θ) with characteristic coefficient s , BNR correspondence gives an element $M \in \overline{\text{Pic}}^{\mathfrak{n}}(X_s)$. This finishes the proof of the proposition. \square

Remark 3.4. Let $\mathcal{U} = \Gamma(\omega_X^2(D)) \setminus \cup_{i=1}^n \Gamma(\omega_X^2(D - t_i))$ be the open set given by Proposition 3.3. If $2g_X + \deg D - 4 \geq 0$ then \mathcal{U} is nonempty. To see this, note that this inequality yields $\deg \omega_X^2(D) \geq 2g_X$ and then $\omega_X^2(D)$ has no base points. In particular, by Proposition 3.3, if $2g_X + \deg D - 4 \geq 0$ then there exist irreducible parabolic Higgs fields $\theta : E \rightarrow E \otimes \omega_X(D)$, and therefore $\mathcal{H}_\mu(X, D, d)$ contains stable Higgs bundles for any choice of weight vector μ . \lrcorner

We finish this section by considering traceless Higgs fields. In view of Lemma 2.2 and the next result, Higgs fields having non-vanishing residual part at every parabolic point play a key role. We say that $\theta : E \rightarrow E \otimes \omega_X(D)$ is *nowhere-holomorphic in D* if

$$\text{Res}(\theta; t_i) \neq 0$$

for every t_i at the support of D .

Proposition 3.5. *Let $s = (0, s_2) \in \Sigma_D$. Assume that X_s is integral and smooth at every point over the complement of the support of D . Then there is a bijective correspondence*

$$\text{Pic}^n(X_s) \longleftrightarrow \{(E, l, \theta) \in H_X^{-1}(s) : \theta \text{ is nowhere-holomorphic in } D\}$$

Proof. By Lemma 2.2 there is a bijective correspondence between $\text{Pic}^n(X_s)$ and the isomorphism classes of pairs (E, θ) where E has degree d , $\theta : E \rightarrow E \otimes \omega_X(D)$ is a homomorphism with $\text{tr}(\theta) = 0$ and $\det(\theta) = s_2$, and θ_x is not of the form λI for every $x \in X$.

Since $\text{tr} \theta = 0$, then $\theta_x = \lambda I$ if and only if $\theta_x = 0$. But, θ_x is nonvanishing at every point over the complement of the support of D , because X_s is smooth at that point. For a point t_i of the support of D , we have

$$\theta_{t_i} = \text{Res}(\theta; t_i)$$

and then $\theta_{t_i} \neq 0, \forall i = 1, \dots, n$, if and only if θ is nowhere-holomorphic in D . To conclude the proof of the proposition we note that, in this case, the parabolic direction over t_i is determined by the kernel of $\text{Res}(\theta; t_i)$. \square

4. SPECTRAL CURVES UNDER DEGREE TWO RAMIFIED COVERINGS

Let $\pi : Y \rightarrow X$ be a finite morphism of degree two between irreducible smooth complex curves. Assume that $\{x_1, \dots, x_n\}$, $n \geq 1$, is the set of branch points and let $B = \sum_{i=1}^n x_i$ be the reduced divisor on X defined by them. Let $R = \sum_{i=1}^n y_i$ be the divisor on Y formed by ramification points: $\pi(y_i) = x_i$.

Let us consider a reduced divisor $T = \sum_{i=1}^k t_i$, $k \geq 0$, on X formed by points outside the support of B . We put $T = 0$ when $k = 0$. Then we can write

$$\pi^*(T + B) = \pi^*(T) + 2R.$$

Note that π induces a linear map (see Section 3.3 for the definition of Σ_D)

$$\pi^* : \Sigma_{T+B} \rightarrow \Sigma_{\pi^*(T)},$$

because poles over branch points became regular after pullback. Indeed, if y and x denote local coordinates of Y and X , respectively, over a branch point, then π is locally given by $\pi(y) = y^2$ and

$$\pi^* \left(\frac{dx^{\otimes 2}}{x} \right) = 4dy^{\otimes 2}.$$

We want to pull back Higgs fields over $(X, T + B)$ to Higgs fields over $(Y, \pi^*(T))$. Let us first investigate the relation between their spectral curves.

4.1. General spectral curve. In this section we start with an element $s \in \Sigma_{T+B}$, which gives a spectral curve X_s . Since $\pi^*(s)$ becomes regular over ramification points, we can see it as an element $r = \pi^*(s) \in \Sigma_{\pi^*(T)}$. Let Y_r be the corresponding spectral curve given by r .

We set $L = \omega_X(T + B)$ and consider the respective spectral curve (cf. (2.3))

$$X_s = \text{Spec}(\mathcal{O}_X \oplus L^{-1})$$

with its degree two map $q_s : X_s \rightarrow X$. The divisor R_{q_s} of ramification points of q_s is given by zeros of $s_1^2 - 4s_2$ and since s lies in the subspace Σ_{T+B} , then q_s is ramified over every point in the support of $T + B$.

Note that

$$\pi^*L \simeq \omega_Y(R + \pi^*(T)).$$

Following Section 2.5, the element

$$r = \pi^*s \in \Gamma(\pi^*L) \otimes \Gamma(\pi^*L^2)$$

gives the spectral curve

$$Y_r^* = \text{Spec}(\mathcal{O}_Y \oplus (\pi^*L)^{-1}) = Y \times_X X_s$$

which is singular over a point lying in the support of R . To avoid singular points of Y_r^* , it is convenient to switch π^*L . Then we consider

$$N = \pi^*L(-R) = \omega_Y(\pi^*T)$$

and the element

$$r = \pi^*s \in \Sigma_{\pi^*(T)} \subset \Gamma(N) \oplus \Gamma(N^2)$$

gives the spectral curve

$$Y_r = \text{Spec}(\mathcal{O}_Y \oplus N^{-1}).$$

We summarize the discussion above in Figure 2.

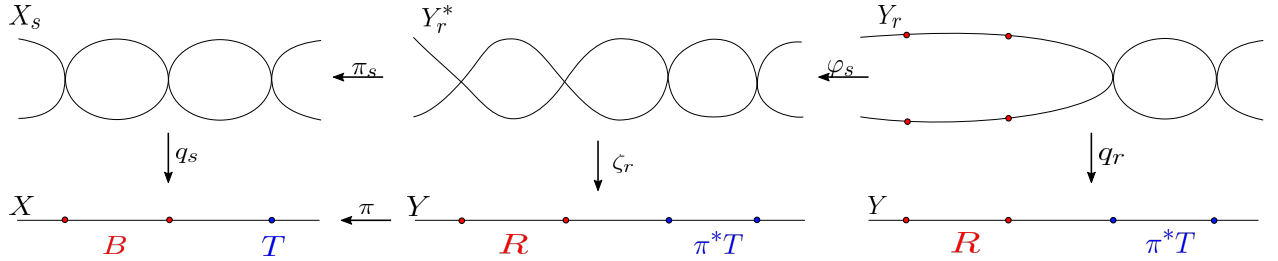


FIGURE 2. Three spectral curves.

Proposition 4.1. *Assume that $2g_X + n - 4 \geq 0$, with strict inequality if $k = 0$ ($n = \deg B$ and $k = \deg T$). If s is a general element of Σ_{B+T} , then X_s and Y_r are irreducible and smooth.*

Proof. The hypothesis $2g_X + n - 4 \geq 0$ ensures that X_s is integral for general $s \in \Sigma_{B+T}$, see Remark 3.4. The locus of sections $s \in \Sigma_{B+T}$, $s = (s_1, s_2)$, such that X_s is smooth is open and nonempty. Besides that if s_2 has only simple zeros then the same holds for $\pi^*(s_2) \in \Sigma_{\pi^*(T)}$. In particular, the curve Y_r , $r = \pi^*(s)$, is smooth for general s . Now we show that Y_r is integral.

First let us assume $k > 0$ and let y be a point with $\pi(y) = t_i$, where t_i lies in the support of T . If $s_2 \notin \Gamma(\omega_X^{\otimes 2}(B + T - t_i))$ then $\pi^*s_2 \notin \Gamma(\omega_Y^{\otimes 2}(\pi^*T - y))$ because π is étale at y . Therefore we can use Proposition 3.3 again to conclude that Y_r is integral.

Now suppose $k = 0$. We claim that the image of $\pi^* : \Sigma_{B+T} \rightarrow \Sigma_{\pi^*(T)}$ intersects the open subset of $\Sigma_{\pi^*(T)}$ formed by all sections for which the spectral curve is integral. To do this, we can assume $s_1 = 0$. A section $s_2 \in \Gamma(\omega_Y^{\otimes 2})$ gives a reducible spectral curve if and only if it lies in the image of the map $\psi : \Gamma(\omega_Y) \rightarrow \Gamma(\omega_Y^{\otimes 2})$, $a \mapsto a^2$. By Riemann-Roch theorem, the injective linear map $\pi^* : \Gamma(\omega_X^{\otimes 2}(B)) \rightarrow \Gamma(\omega_Y^{\otimes 2})$ has image of dimension $3(g_X - 1) + n$, whereas the image of ψ has dimension at most g_Y . Thus if $3(g_X - 1) + n > g_Y$, the claim above will follow. By Riemann-Hurwitz formula, the previous inequality is equivalent to $2g_X + n - 4 > 0$. This concludes the proof of the proposition. \square

4.2. A map between spectral curves.

Proposition 4.2. *Let the notation be as above. The following statements hold:*

(1) *there is a morphism $\xi_s : Y_r \rightarrow X_s$ of degree two making the following diagram commute*

$$\begin{array}{ccc} Y_r & \xrightarrow{\xi_s} & X_s \\ q_r \downarrow & & \downarrow q_s \\ Y & \xrightarrow{\pi} & X \end{array} .$$

- (2) *if X_s is smooth then Y_r is the normalization of Y_r^* ;*
 (3) *ξ_s is étale when X_s and Y_r are smooth and irreducible.*

Proof. Let us prove (1). Following Section 4.1, setting $L = \omega_X(T + B)$ and $N = \omega_Y(\pi^*T)$ we have

$$\begin{cases} X_s = \text{Spec}(\mathcal{O}_X \oplus L^{-1}) \\ Y_r^* = \text{Spec}(\mathcal{O}_Y \oplus (\pi^*L)^{-1}) = Y \times_X X_s \\ Y_r = \text{Spec}(\mathcal{O}_Y \oplus N^{-1}). \end{cases}$$

Since $(\pi^*L)^{-1} \subset N^{-1}$, this gives an inclusion $i : \mathcal{O}_Y \oplus (\pi^*L)^{-1} \rightarrow \mathcal{O}_Y \oplus N^{-1}$ of \mathcal{O}_Y -modules. Note that their structure of commutative ring induced by π^*s (c.f. (2.1)) are compatible and thus $\mathcal{O}_Y \oplus (\pi^*L)^{-1}$ is a subring of $\mathcal{O}_Y \oplus N^{-1}$. In particular i yields a map $\varphi_s : Y_r \rightarrow Y_r^*$. Then we set $\xi_s = \pi_s \circ \varphi_s$, where $\pi_s : Y_r^* \rightarrow X_s$ is given by (2.8). Note that $(\pi^*L)^{-1}$ and N^{-1} coincide when restricted to an open set U away from the support of R , then $\varphi_s|_U$ is an isomorphism. Since $\pi_s : Y_r^* \rightarrow X_s$ has degree two and $\varphi_s : Y_r \rightarrow Y_r^*$ is birational, this concludes the proof of (1).

In order to prove (2), we note first that if X_s is smooth then s_2 has only simple zeros. Consequently, the same is true for π^*s_2 , seen as a section of $\Gamma(N^2)$. This shows that Y_r is smooth and then $\varphi_s : Y_r \rightarrow Y_r^*$ is the normalization of Y_r^* . This proves (2).

Finally, when X_s and Y_r are smooth and irreducible, we can use (2.5) and Riemann-Hurwitz formula to conclude (3). \square

In view of Proposition 4.2 we conclude this section with the following well-known result, see also [14, Lemma p. 332]. Here, we include a proof for the convenience to the reader.

Proposition 4.3. *Let $\xi : Z \rightarrow W$ be an étale morphism of degree 2 between compact Riemann surfaces. Then the induced morphism $\xi^* : \text{Pic}^0(W) \rightarrow \text{Pic}^0(Z)$ has degree 2 onto its image.*

Proof. There is a correspondence between étale morphisms of degree 2 and 2-torsion elements in $\text{Pic}^0(W)$. We recall briefly this correspondence. Let $L \in \text{Pic}^0(W)$ be a 2-torsion point, $\mathbf{P} = \mathbb{P}(\mathcal{O}_W \oplus L)$ and $q : \mathbf{P} \rightarrow W$ the natural projection. Roughly speaking, the curve Z can be seen as the inverse image of the unit section of $L^2 \simeq \mathcal{O}_W$ under the map $L \rightarrow L^2$ which sends a to a^2 . The subvariety $\mathbb{P}(\mathcal{O}_W) \subset \mathbf{P}$ is the zero locus of a section

$$\mathbf{z} : \mathcal{O}_{\mathbf{P}} \rightarrow q^*(L) \otimes \mathcal{O}_{\mathbf{P}}(1)$$

and $\mathbb{P}(L)$ is the zero locus of a section

$$\mathbf{w} : \mathcal{O}_{\mathbf{P}} \rightarrow \mathcal{O}_{\mathbf{P}}(1)$$

and $\xi = q|_Z$. If we denote by s a nonzero section of L^2 , the curve $Z \subset \mathbf{P}$ corresponds to the zero locus of the section

$$\mathbf{z}^2 - q^*(s) \cdot \mathbf{w}^2.$$

Note that s is not a power a^2 of a section a of L , otherwise L would be trivial.

Now we can show that $\xi^*(L) = \mathcal{O}_Z$. To see this, observe that \mathbf{z} and \mathbf{w} do not vanish in Z , then the restriction of $\mathcal{O}_{\mathbf{P}}(1)$ and $q^*(L) \otimes \mathcal{O}_{\mathbf{P}}(1)$ to Z are trivial. It follows that

$$\xi^*(L) = (q^*(L) \otimes \mathcal{O}_{\mathbf{P}}(1))|_Z$$

is trivial.

Assume that $M \in \text{Pic}^0(W)$ is a line bundle with $\xi^*(M) = \mathcal{O}_Z$. Then projection formula yields

$$\mathcal{O}_W \oplus L \simeq \xi_*(\mathcal{O}_Z) = \xi_*((\pi^*M) \otimes \mathcal{O}_Z) \simeq M \oplus (M \otimes L).$$

In particular, $M \oplus (M \otimes L)$ has a nonzero section. Since M and L have degree zero, one obtains that either M is \mathcal{O}_W or $M \otimes L$ is \mathcal{O}_W . This concludes the proof of the proposition. \square

5. A MAP BETWEEN MODULI SPACES

5.1. Main result. We keep the notation of Section 4, where B and R denote the divisors of branch points and ramification points of $\pi : Y \rightarrow X$, respectively. Also, let T be an effective reduced divisor on X whose support does not intersect the support of B .

In this section we will consider a rational map between moduli spaces of parabolic Higgs bundles

$$\Phi : \mathcal{H}_\mu(X, B + T, 0) \dashrightarrow \mathcal{H}_{\mu'}(Y, \pi^*(T), -n). \quad (5.1)$$

This map is defined by doing π^* followed by an elementary transformation over R

$$(\mathcal{E}, \theta) \xrightarrow[\pi^*]{\Phi} (\pi^*\mathcal{E}, \pi^*\theta) \xrightarrow[\text{elem}_R]{} ((\pi^*\mathcal{E})', (\pi^*\theta)')$$

where $\mathcal{E} = (E, l)$. In the last step, parabolic directions over ramification points are forgotten, this justifies the lack of ramification points in the divisor of parabolic points on the target in (5.1). We describe below each step of this correspondence.

At first, Φ might send each semistable Higgs bundle to an unstable Higgs bundle. So, in view of Proposition 4.1, we assume that $2g_X + n - 4 \geq 0$, with strict inequality if $k = 0$ ($n = \deg B$ and $k = \deg T$). Let us fix $s \in \Sigma_{B+T}$ such that X_s and Y_r are integral.

We can define a correspondence

$$(E, l, \theta) \mapsto \Phi(E, l, \theta)$$

which associates to a Higgs bundle on $(X, T + B)$, with characteristic coefficient s , a Higgs bundle on $(Y, \pi^*(T))$ with characteristic coefficient $r = \pi^*s$, by doing the following series of transformations:

- (1) Let (E, l, θ) be a parabolic Higgs bundle, with $\deg E = 0$, and with characteristic coefficient s .
- (2) Pulling back (E, l, θ) to Y via π , we obtain a parabolic Higgs bundle $(\pi^*(E, l), \pi^*\theta)$ on $(Y, \pi^*(B + T))$ with characteristic coefficient $r \in \Sigma_{\pi^*(T)}$.
- (3) After performing an elementary transformation over R (cf. Section 2.4), the resulting parabolic Higgs bundle

$$\Phi(E, l, \theta) := (((\pi^*E)', l'), (\pi^*\theta)')$$

is regular over each ramification point, and $\deg(\pi^*E)' = -n$. Then we get a parabolic Higgs bundle over $(Y, \pi^*(T))$. Since X_s and Y_r are integral, θ and $(\pi^*\theta)'$ are irreducible (cf. Proposition 3.2), then they are both stable.

This correspondence gives a rational map between moduli spaces of parabolic Higgs bundles

$$\Phi : \mathcal{H}_\mu(X, B + T, 0) \dashrightarrow \mathcal{H}_{\mu'}(Y, \pi^*(T), -n) \quad (5.2)$$

which makes the following diagram commute

$$\begin{array}{ccc} \mathcal{H}_\mu(X, B + T, 0) & \xrightarrow{\Phi} & \mathcal{H}_{\mu'}(Y, \pi^*(T), -n) \\ H_X \downarrow & & \downarrow H_Y \\ \Sigma_{B+T} & \xrightarrow{\pi^*} & \Sigma_{\pi^*(T)} \end{array} \quad (5.3)$$

We want to investigate the behavior of Φ with respect to the Hitchin fibration. Let us consider its restriction to a (general) Hitchin fiber

$$\Phi_s : H_X^{-1}(s) \rightarrow H_Y^{-1}(r)$$

with $r = \pi^*s$. When X_s and Y_r are irreducible and smooth, then

$$H_X^{-1}(s) \simeq \text{Pic}^n(X_s) \quad \text{and} \quad H_Y^{-1}(r) \simeq \text{Pic}^{\tilde{n}}(Y_r).$$

where $\mathbf{n} = k + n + 2g_X - 2$ and $\tilde{\mathbf{n}} = 2k - n + 2g_Y - 2$. This gives a map

$$\Phi_s : \text{Pic}^{\mathbf{n}}(X_s) \rightarrow \text{Pic}^{\tilde{\mathbf{n}}}(Y_r).$$

In the next result we show that this map is determined by ξ_s^* (see Proposition 4.2), up to a translation on the target.

Theorem 5.1. *Assume that $2g_X + n - 4 \geq 0$, with strict inequality if $k = 0$. Then Φ is a rational map of degree two onto its image which preserves the Hitchin fibrations. Moreover, its restriction to a general Hitchin fiber is the map*

$$\Phi_s : \text{Pic}^{\mathbf{n}}(X_s) \longrightarrow \text{Pic}^{\tilde{\mathbf{n}}}(Y_r)$$

which sends a line bundle M in $\text{Pic}^{\mathbf{n}}(X_s)$ to the line bundle $\xi_s^*(M)(q_r^*(-R))$ in $\text{Pic}^{\tilde{\mathbf{n}}}(Y_r)$, where ξ_s is given by Proposition 4.2.

Proof. By Proposition 4.1, X_s and Y_r are irreducible and smooth for general s . We want to show that $\Phi_s(M) = \xi_s^*(M)(-S)$ for any line bundle M on X_s , of degree \mathbf{n} . Recall that ξ_s is a composition $\xi_s = \pi_s \circ \varphi_s$

$$Y_r \begin{array}{c} \xrightarrow{\xi_s} \\ \xrightarrow{\varphi_s} Y_r^* \xrightarrow{\pi_s} X_s \end{array}$$

where π_s is the natural projection of the fiber product $Y_r^* = Y \times_X X_s$ and φ_s is the normalization of Y_r^* , see Proposition 4.2 and Figure 2.

The map Φ is defined using two correspondences, the former is a pullback and the second is an elementary transformation followed by a suitable reduction to the curve Y_r (c.f. Section 4.1). Thus we need to follow the behavior of M under these correspondences:

$$\text{Pic}^{\mathbf{n}}(X_s) \begin{array}{c} \xrightarrow{\Phi_s} \\ \xrightarrow{\pi_s^*} \text{Pic}^{2\mathbf{n}}(Y_r^*) \xrightarrow{\text{elem}_R} \text{Pic}^{2\mathbf{n}}(Y_r^*) \xrightarrow{\varphi_s^*} \text{Pic}^{\tilde{\mathbf{n}}}(Y_r) \end{array}$$

Let (E, l, θ) be the Higgs bundle which corresponds to $M \in \text{Pic}^{\mathbf{n}}(X_s)$. By Proposition 2.5, π_s^*M is the line bundle in $\text{Pic}^{2\mathbf{n}}(Y_r^*)$ corresponding to $(\pi^*(E, l), \pi^*\theta)$, and it follows from Proposition 2.3 that the elementary transformation elem_R transforms π_s^*M into $\pi_s^*M \otimes \mathcal{I}$, where \mathcal{I} is the ideal sheaf of $(\zeta_r^*R)_{\text{red}}$ given by the 2:1 cover $\zeta_r : Y_r^* \rightarrow Y$.

The last operation is given by the normalization of Y_r^* . Since $\xi_s = \pi_s \circ \varphi_s$, then

$$\Phi_s(M) = \varphi_s^*(\pi_s^*M \otimes \mathcal{I}) = \xi_s^*(M) \otimes \varphi_s^*(\mathcal{I}).$$

The conclusion follows from the fact that Y_r is smooth, and then $\varphi_s^*(\mathcal{I}) = \mathcal{O}_{Y_r}(-S)$ is the ideal sheaf of $S = q_r^*R$, where $q_r : Y_r \rightarrow Y$ is the 2:1 cover.

To see that Φ has degree two onto its image, first remark that π^* of (5.3) is a one-to-one map between Hitchin basis and Proposition 4.3. □

We point out that it would be interesting to compare the techniques used in this paper with the methods of [24].

We close the paper by presenting two situations where the modular map Φ is dominant, which deal with pullback of SL_2 -Higgs bundles over \mathbb{P}^1 via $\pi : Y \rightarrow \mathbb{P}^1$, where $g_Y \in \{1, 2\}$.

By SL_2 we mean that $\theta : E \rightarrow E \otimes \omega_X(D)$ has vanishing trace and E has trivial determinant line bundle.

5.2. **The case** $(g_X, g_Y) = (0, 1)$. Let us consider the degree two elliptic cover $\pi : Y \rightarrow \mathbb{P}^1$, branched over four distinct points $\{0, 1, \lambda, \infty\}$. Let us fix $t_1, t_2 \in Y$ with $\pi(t_1) = \pi(t_2) = t$, $t \notin \{0, 1, \lambda, \infty\}$.

On the \mathbb{P}^1 side, we consider the moduli space $\mathcal{H}(\mathbb{P}^1, \Lambda)$ of irreducible SL_2 parabolic Higgs bundles over (\mathbb{P}^1, Λ) , where $\Lambda = 0 + 1 + \lambda + \infty + t$. On the elliptic side, we consider the moduli space $\mathcal{H}(Y, D)$ of irreducible SL_2 parabolic Higgs bundles over (Y, D) , where $D = t_1 + t_2$.

It turns out that $\mathcal{H}(\mathbb{P}^1, \Lambda)$ and $\mathcal{H}(Y, D)$ have the same dimension: four. We can associate to a Higgs bundle in $\mathcal{H}(\mathbb{P}^1, \Lambda)$ a Higgs bundle in $\mathcal{H}(Y, D)$ by doing the pullback π^* followed by a composition of an elementary transformation $elem_R$ over the divisor

$$R = w_0 + w_1 + w_\lambda + w_\infty$$

formed by ramification points of π . This is our map Φ . Note that, starting with a vector bundle E of degree 0 on \mathbb{P}^1 , the transformed vector bundle on Y has determinant $\mathcal{O}_Y(-R)$, then degree -4 . Since $R \sim 4w_\infty$, we then perform a last transformation, twisting by $\mathcal{O}_Y(2w_\infty)$, which results in trivial determinant.

This correspondence gives a map

$$\Phi_0 : \mathcal{H}(\mathbb{P}^1, \Lambda) \dashrightarrow \mathcal{H}(Y, D)$$

which consists in $[\otimes \mathcal{O}_Y(2w_\infty)] \circ \Phi$, and fits in the following commutative diagram

$$\begin{array}{ccc} \mathcal{H}(\mathbb{P}^1, \Lambda) & \xrightarrow{\Phi_0} & \mathcal{H}(Y, D) \\ \det \downarrow & & \downarrow \det \\ \Gamma(\omega_{\mathbb{P}^1}^{\otimes 2}(\Lambda)) & \xrightarrow{\pi^*} & \Gamma(\omega_Y^{\otimes 2}(t_1 + t_2)) \end{array}$$

(both spaces of quadratic differentials are two dimensional).

Over \mathbb{P}^1 , the smooth spectral curve X_s has genus 2 and is branched over 6 distinct points $0, 1, \lambda, \infty, t, \rho$. The corresponding Hitchin fiber $\det^{-1}(s)$ is isomorphic $\text{Pic}^3(X_s)$. Over Y , the generic spectral curve Y_r is a hyperelliptic curve of genus 3 branched over 4 distinct points t_1, t_2, u_1, u_2 , with $\rho = \pi(u_1) = \pi(u_2)$. The fiber $\det^{-1}(r)$ over Y_r is an Abelian variety isomorphic to the Prym variety

$$\text{Prym}(Y_r/Y) = \{M \in \text{Pic}^2(Y_r) : \det((q_r)_*M) = \mathcal{O}_Y\}.$$

Since q_r is a ramified covering, then $\text{Prym}(Y_r/Y)$ is an irreducible variety, this follows from [14, (iv) in p. 329].

We can choose local coordinates where

$$\begin{cases} X_s = \{(x, w) & : w^2 = x(x-1)(x-\lambda)(x-t)(x-\rho)\} \\ Y_r = \{(x, y, z) & : y^2 = x(x-1)(x-\lambda) \text{ and } z^2 = (x-t)(x-\rho)\} \end{cases}$$

and both curves fit in the commutative diagram below

$$\begin{array}{ccccc}
 & & Y_r & & \\
 & \swarrow & \downarrow & \searrow & \\
 & & \mathbb{P}^1_\rho & & \\
 & \swarrow & \downarrow & \searrow & \\
 Y & & \mathbb{P}^1 & & X_s
 \end{array}
 \tag{5.4}$$

$\begin{array}{l} \text{2:1} \\ \text{2:1} \\ \text{2:1} \end{array}$

where

$$\mathbb{P}^1_\rho = \{(x, z) \quad : \quad z^2 = (x - t)(x - \rho)\}.$$

The étale map $\xi_s : Y_r \rightarrow X_s$ is the same of Proposition 4.2, explicitly given here by $(x, y, z) \mapsto (x, yz)$.

In this context, Theorem 5.1 can be written as

Corollary 5.2. *The rational map $\Phi_0 : \mathcal{H}(\mathbb{P}^1, \Lambda) \dashrightarrow \mathcal{H}(Y, D)$ has degree two. Moreover, its restriction to a general fiber of the Hitchin map is a morphism of degree two given by*

$$\begin{aligned}
 \Phi_{0,s} : \text{Pic}^3(X_s) &\rightarrow \text{Prym}(Y_r/Y) \\
 M &\mapsto \xi_s^*(M)(S_0)
 \end{aligned}$$

where $S_0 = q_r^*(2w_\infty - R)$.

Proof. As mentioned above, the map Φ_0 consists of $[\otimes \mathcal{O}_Y(2w_\infty)] \circ \Phi$ where Φ is the map of Theorem 5.1. Let us fix a point (E, l, θ) in the image of Φ_0 with characteristic coefficient r . By construction, we have $\det E = \mathcal{O}_Y$, then it follows by BNR correspondence that (E, l, θ) corresponds to a line bundle $N \in \text{Pic}^2(Y_r)$ satisfying $\det((q_r)_*N) = \det E = \mathcal{O}_Y$. Hence, $N \in \text{Prym}(Y_r/Y)$ and the proof of the corollary follows from Theorem 5.1. \square

In a forthcoming paper, we will give the full description (at singular Hitchin fibers) of the map Φ_0 .

5.3. The case $(g_X, g_Y) = (0, 2)$. We now take a degree two morphism $\pi : Y \rightarrow \mathbb{P}^1$ from a curve Y of genus 2. It is branched over 6 distinct points t_1, \dots, t_6 of \mathbb{P}^1 . Let $B = t_1 + \dots + t_6$ be the reduced divisor on \mathbb{P}^1 defined by those points, and let $R = w_1 + \dots + w_6$ be the divisor on Y formed by ramification points of π .

Let us consider the moduli space $\mathcal{H}(\mathbb{P}^1, B)$ of irreducible SL_2 -Higgs bundles over (\mathbb{P}^1, B) . Over Y , we consider the moduli space $\mathcal{H}(Y)$ of irreducible holomorphic SL_2 -Higgs bundles over Y , without parabolic points. These moduli spaces have both dimension 6.

The spectral curve on the \mathbb{P}^1 side has genus 3 and the corresponding smooth Hitchin fiber is isomorphic to $\text{Pic}^4(X_s)$. Over Y , the spectral curve Y_r has genus 5 and the smooth Hitchin fiber $\mathcal{H}(Y)$ is isomorphic to

$$\text{Prym}(Y_r/Y) = \{M \in \text{Pic}^2(Y_r) \quad : \quad \det((q_r)_*M) = \mathcal{O}_Y\}$$

which is three dimensional, and irreducible ([14, (iv) in p. 329]).

We consider the correspondence $\Phi_0 : \mathcal{H}(\mathbb{P}^1, B) \dashrightarrow \mathcal{H}(Y)$ given by $[\otimes L_0] \circ elem_R \circ \pi^*$, where L_0 is a square root of $\mathcal{O}_Y(R)$. Note that the following diagram is commutative

$$\begin{array}{ccc} \mathcal{H}(\mathbb{P}^1, B) & \xrightarrow{\Phi_0} & \mathcal{H}(Y) \\ \det \downarrow & & \downarrow \det \\ \Gamma(\omega_{\mathbb{P}^1}^{\otimes 2}(B)) & \xrightarrow{\pi^*} & \Gamma(\omega_Y^{\otimes 2}) \end{array}$$

where the Hitchin basis are three dimensional. From Theorem 5.1, we obtain a result similar to Corollary 5.2:

Corollary 5.3. *The rational map $\Phi_0 : \mathcal{H}(\mathbb{P}^1, B) \dashrightarrow \mathcal{H}(Y)$ has degree two and its restriction to a general fiber of the Hitchin map consists of*

$$\begin{aligned} \Phi_{0,s} : \text{Pic}^4(X_s) &\rightarrow \text{Prym}(Y_r/Y) \\ M &\mapsto \xi_s^*(M) \otimes q_r^*(L_0(-R)). \end{aligned}$$

Proof. Similar to the proof of Corollary 5.2. □

REFERENCES

- [1] A. B. Altman, A. Iarrobino, and S. L. Kleiman, *Irreducibility of the compactified Jacobian*, pp. 1-12, Sijthoff and Noordhoff, Alphen aan den Rijn, (1977).
- [2] A. Beauville, M.S. Narasimhan and S. Ramanan, *Spectral curves and the generalised theta divisor*. J. reine angew. Math. (Crelles Journal), **398** (1989) 169-179.
- [3] U. N. Bhosle, *Generalized parabolic bundles and applications - II*, Proceedings of the Indian Academy of Sciences - Mathematical Sciences, **106** 4 (1996) 403-420.
- [4] H. U. Boden and K. Yokogawa, *Moduli spaces of parabolic Higgs bundles and parabolic $K(D)$ pairs over smooth curves: I*, International Journal of Mathematics, **07**, (1996), 573-598.
- [5] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics, Springer-Verlag (1997).
- [6] T. Hausel and M. Thaddeus, *Mirror symmetry, Langlands duality, and the Hitchin system*, Invent. Math., **153**, (2003), 197-229.
- [7] N. Hitchin, *Stable bundles and integrable systems*, Duke Math. J. **54**, (1987), 91-114.
- [8] N. Hitchin, *The self-duality equations and on a Riemann surface*, Proc. London Math. Soc, **55**, (1987), 59-126.
- [9] A. Kapustin and E. Witten, *Electric-magnetic duality and the geometric Langlands program*, Commun. Number Theory Phys. **1**, (2007), 1-236.
- [10] H. Kleppe and S. L. Kleiman, *Reducibility of the compactified Jacobian*, Compositio Math. **43** (1981), no. 2, 277-280.
- [11] H. Konno, *Construction of the moduli space of stable parabolic Higgs bundles on a Riemann surface*. J. Math. Soc. Japan, **45** no. 2 (1993), 253-276.
- [12] M. Logares and J. Martens, *Moduli of parabolic Higgs bundles and Atiyah algebroids*, J. reine angew. Math. (Crelles Journal), **649** (2010) 89-116.
- [13] V. B. Mehta and C. S. Seshadri, *Moduli of vector bundles on curves with parabolic structures*. Math. Ann. **248** (1980) 205-239.
- [14] D. Mumford, *Prym varieties I in: Contributions to analysis* (L. V. Ahlfors et al. eds.), Academic Press, New York (1974), 325-350.
- [15] H. Nakajima, *Hyper-Kähler structures on moduli spaces of parabolic Higgs bundles on Riemann surfaces*, Lecture Notes in Pure and Appl. Math. **179**, 199-208.
- [16] N. Nitsure, *Moduli space of semistable pairs on a curve*, Proc. London Math. Soc. **62**, (1991), 275-300.
- [17] B. C. Ngô, *Fibration de Hitchin et endoscopie*, Invent. Math. **164**, (2006), 399-453.

- [18] B. C. Ngô, *Le lemme fondamental pour les algèbres de Lie*, Publ. Math., Inst. Hautes Étud. Sci. **111**, (2010), 11-69.
- [19] C. T. Simpson, *Harmonic bundles on noncompact curves*, Journal of the A.M.S. **3**, (1990), 713-770.
- [20] C. T. Simpson. *Iterated destabilizing modifications for vector bundles with connection*. Vector Bundles and Complex Geometry (Conference in Honor of Ramanan, Miraflores 2008) Contemporary Math. 522 (2010) 183-206.
- [21] C. T. Simpson, *An explicit view of the Hitchin fibration on the Betti side for \mathbb{P}^1 minus five points*. Geometry and Physics: Volume 2: A Festschrift in honour of Nigel Hitchin (2018): 705.
- [22] K. Yokogawa *Compactification of moduli of parabolic sheaves and moduli of parabolic Higgs sheaves*, J. Math. Kyoto Univ. **33**, (1993), 451-504.
- [23] K. Yokogawa, *Infinitesimal deformation of parabolic Higgs sheaves*, International Journal of Mathematics, **6** (1995), 125–148.
- [24] H. Zelaci, *Hitchin systems for invariant and anti-invariant vector bundles*, Trans. Amer. Math. Soc. **375** (2022), 3665–3711.

UNIVERSIDADE FEDERAL FLUMINENSE, INSTITUTO DE MATEMÁTICA E ESTATÍSTICA, RUA ALEXANDRE MOURA 8, SÃO DOMINGOS, 24210-200 NITERÓI RJ, BRAZIL.

Email address: `tfassarella@id.uff.br`

UNIV RENNES, CNRS, IRMAR - UMR 6625, F-35000 RENNES, FRANCE

Email address: `frank.loray@univ-rennes1.fr`