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AN UNIVERSAL MINORATION OF THE MAHLER MEASURE OF REAL RECIPROCAL ALGEBRAIC INTEGERS

JEAN-LOUIS VERGER-GAUGRY

ABSTRACT. This paper proves the existence of an universal nontrivial minorant of the set of the Mahler measures $M(\beta)$ where β runs over the set of reciprocal algebraic integers > 1 . It represents a part of the solution of the Conjecture of Lehmer, the one restricted to the set of real reciprocal algebraic integers > 1 . The proof relies upon, when β tends to one: (i) the asymptotic geometry of the lenticular zeroes of the Parry Upper functions $f_\beta(z)$ associated with the dynamical zeta functions $\zeta_\beta(z)$ of the Rényi–Parry arithmetical dynamical systems (β -shift), (ii) the identification of these lenticular zeroes with conjugates of β using Kala-Vavra’s Theorem, (iii) a Dobrowolski-type inequality deduced from the Poincaré asymptotic expansions of these lenticular zeroes as a function of the dynamical degree of β .

The dynamical zeta function is used to investigate the domain of very small Mahler measures of algebraic integers in the range $(1, 1.176280\dots]$, where 1.176280 is Lehmer’s number, if any.

Keywords: Mahler measure, Dynamical zeta function, Rényi–Parry β -shift, beta transformation, Dobrowolski-type inequality, lenticular measure, minoration.

2020 Mathematics Subject Classification: 37C30, 37B05, 30C15, 37F05, 30B10, 11R06

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1. INTRODUCTION

Let β be an algebraic integer of degree d , with minimal polynomial $P_\beta(X) = X^d + a_{d-1}X^{d-1} + \dots + a_1X + a_0 \in \mathbb{Z}[X]$, with $a_0 \neq 0$. Let $\beta_1 = \beta, \beta_2, \beta_3, \dots, \beta_d$ its conjugates. Recall that the Mahler measure $M(\beta)$ of β is by definition (Smyth [41] [42]) the Mahler measure of its minimal polynomial $M(P_\beta) := \prod_{i=1}^d \max\{1, |\beta_i|\}$. If $\beta = 0$ or $\beta = \pm 1$, then $M(\beta) = 1$. We say that $\beta (\neq 0, \neq \pm 1)$ is reciprocal if β^{-1} is a conjugate of β ; if β is reciprocal, then $d = \deg \beta$ is even and the minimal polynomial of β^{-1} is the reciprocal polynomial $P_\beta^*(X) := X^d P_\beta(1/X)$ of $P_\beta(X)$, which satisfies: $P_\beta^*(X) = P_\beta(X)$. If $\beta \neq 0$, we have: $M(\beta) = M(-\beta) = M(\pm\beta^{-1})$.

In this note we consider the problem of the existence of a non-trivial minorant of the set

$$(1.1) \quad \{M(\beta) : \beta \text{ is a real algebraic integer, } \beta \neq 0, \beta \neq \pm 1\}.$$

It is sufficient to restrict β to the half-line $(1, +\infty)$. For the minorant, by “non-trivial” we mean “ > 1 ”. By a Theorem of Smyth in [41], if $\beta > 1$ is a real non-reciprocal algebraic integer, i.e. for which β^{-1} is not a conjugate of β , then $M(\beta) \geq \Theta$, where Θ is the smallest Pisot number. Recall that Siegel [40] proved that the smallest Pisot number $\Theta = 1.3247\dots$ is the root > 1 of the trinomial $X^3 - X - 1$. What is the smallest element of (1.1) known? *Lehmer’s number*, say $\tau = 1.176280\dots$, is the dominant root of *Lehmer’s polynomial*, irreducible and reciprocal, of degree 10,

$$(1.2) \quad P_\tau(X) = X^{10} + X^9 - X^7 - X^6 - X^5 - X^4 - X^3 + X + 1.$$

It is the smallest Salem number known [22] [23]. As such $M(\tau) = \tau$, and τ belongs to the set (1.1). Lehmer’s number τ is the smallest element of (1.1) known [23].

Then the problem we consider is the problem of the existence of a non-trivial minorant of the set

$$(1.3) \quad \{M(\beta) : \beta \text{ is a real reciprocal algebraic integer, } \beta > 1, M(\beta) < \tau\}.$$

In this paper we solve this problem. It is the Problem of Lehmer, but restricted to the real nonzero algebraic numbers $\neq 1$. The existence of a non-trivial minorant is proved in Theorem 1.1. It is effective, but this paper does not say that the set (1.3) contains elements between this universal minorant and τ . It may be empty; this is an open problem.

Let us sketch the organization of the paper. For solving this problem we suppose that the set (1.3) is not empty; we assume the existence of a real reciprocal algebraic integer β in $(1, \tau)$ such that $M(\beta) < \tau$. Recall that, if $|\overline{\beta}| := \max\{|\beta_i| : i = 1, 2, \dots, d\}$ denotes the house of β , we have: $\beta \leq |\overline{\beta}| \leq M(\beta)$. Solving the problem amounts to investigating minorants of $M(\beta)$ when $\beta > 1$ tends to 1 and $|\overline{\beta}| > 1$ lies in a very small neighbourhood of 1. The originality of the present note is the following: to the condition “ $\beta > 1$ very close to 1” are associated an integer, called the dynamical degree of β , denoted by $\text{dyg}(\beta)$, and two analytic functions which are complementary: the first one is $z \mapsto P_\beta(z)$, the second one is the Parry Upper function $z \mapsto f_\beta(z)$ deduced from the dynamical zeta function $\zeta_\beta(z)$ of the β -shift (Ito and Takahashi [15] [44] [45] [46] [20], cf Section 2). The present introduction of the dynamical zeta function $\zeta_\beta(z)$ of the β -shift in the search for small Mahler measures has an impact on the Problem of Lehmer.

The (non-trivial) universal minorant of $M(\beta)$ established in Theorem 1.1 is deduced from the existence of the lenticular poles of $\zeta_\beta(z)$, i.e. the lenticular zeroes of $f_\beta(z)$ (recalled in Section 2.1); to this set of lenticular poles a lenticular measure $\mathfrak{S}_{lent}(\beta)$ can be defined (in (2.16)) and admits a lower bound depending upon $\text{dyg}(\beta)$ (Section 2.2, Theorem 2.8). This lower bound, given explicitly by Theorem 2.8, is a Dobrowolski-type inequality, comparable to the one obtained by Dobrowolski in [7], but where the usual degree $\text{deg}(\beta)$ of β is replaced by the dynamical degree $\text{dyg}(\beta)$. This is based on a formulation of the lenticular poles as *Poincaré asymptotic expansions* of the variable $\text{dyg}(\beta)$ (divergent series) [48] [49]. In Section 3, we recall how this Dobrowolski-type inequality with $\text{dyg}(\beta)$ is a minorant function of the Mahler measure $M(\beta)$, as function of $\text{dyg}(\beta)$, using the method of rewriting trails between $P_\beta(z)$ and $f_\beta(z)$ (introduced in [51] and its earlier versions, explicated in [9]), Kala-Vavra's Theorem [16], and the identification of the lenticular zeroes of $f_\beta(z)$ with some conjugates of β .

With this identification, the (Galois) conjugates of β which are lenticular poles of $\zeta_\beta(z)$ are called lenticular. The identification process between the lenticular poles of $\zeta_\beta(z)$ and some Galois conjugates of β obey some sufficient conditions recalled in Section 3.2. In Section 3.3 two examples of sequences $(\beta_j)_{j \geq 1}$ of reciprocal algebraic integers > 1 tending to 1 in which these sufficient conditions are not satisfied are given: the first one for which $M(\beta_j)$ remains constant with j , the second one, outside the Problem of Lehmer, for which the sequence of houses $(\lceil \beta_j \rceil)_{j \geq 1}$ tends to 2 as j tends to infinity. Finally, in Section 4, we apply the Dobrowolski-type inequality numerically to prove the non-triviality of the universal minorant of $M(\beta)$.

In this note, θ_n denotes the unique zero of $-1 + z + z^n$ in $(0, 1)$, $n \geq 2$ (cf [49] [52] for the properties of the Perron numbers θ_n^{-1}).

Theorem 1.1. *There exists an integer $\eta \geq 260$ such that, for any nonzero real reciprocal algebraic integer β , which is $\neq \pm 1$,*

$$M(\beta) \geq \theta_\eta^{-1}.$$

2. THE DYNAMICAL ZETA FUNCTION $\zeta_\beta(z)$ OF THE β -SHIFT AS β TENDS TO 1

In this Section, β denotes any real number in the interval $(1, (1 + \sqrt{5})/2)$.

2.1. The lenticular poles of $\zeta_\beta(z)$. The β -transformation is the map $[0, 1] \rightarrow [0, 1], x \mapsto \{\beta x\}$, where $\{\cdot\}$ denotes the fractional part, and $\lfloor y \rfloor = y - \{y\}$ the integer part of y . The β -transformation is denoted by T_β , and its iterates by: $T_\beta^{(0)} = \text{Id}$, $T_\beta^{(1)} = T_\beta$, $T_\beta^{(j)} = T_\beta(T_\beta^{(j-1)})$, for $j \geq 1$.

The dynamical system $([0, 1], \mathcal{A}, T_\beta, \mu_\beta)$, where \mathcal{A} is the σ -algebra of borelians and μ_β the invariant measure [13], introduced by Rényi [30], was initially studied in ergodic theory by Parry [24] [14]. We call *dynamical zeta function of the β -shift*, denoted by $\zeta_\beta(z)$, the dynamical zeta function of this dynamical system [1] [25] [28] [29]. After Takahashi [44]

[45] [46] and Ito and Takahashi [15] [20], this function takes the form

$$(2.1) \quad \zeta_\beta(z) = \frac{1 - z^N}{(1 - \beta z) \left(\sum_{n=0}^{\infty} T_\beta^n(1) z^n \right)} = - \frac{1 - z^N}{-1 + t_1 z + t_2 z^2 + t_3 z^3 + \dots}$$

where N , which depends upon β , is the minimal positive integer such that $T_\beta^N(1) = 0$; in the case where $T_\beta^j(1) \neq 0$ for all $j \geq 1$, “ z^N ” has to be replaced by “0”. In (2.1) the digits t_i are defined from β by: $t_i := \lfloor \beta T_\beta^{(i-1)}(1) \rfloor$, $i \geq 1$, and depend upon β . Writing $t_i = t_i(\beta)$, the integers $t_i = t_i(\beta)$ all belong to the alphabet $\mathcal{A} = \{0, 1\}$. The sequence $(t_i = t_i(\beta))_{i \geq 1}$ is such that $0.t_1 t_2 t_3 \dots$ is the Rényi β -expansion of unity $d_\beta(1)$ ([21], Chapt 7), i.e. $1 = \sum_{i \geq 1} t_i \beta^{-i}$. The sequence (t_i) uniquely characterizes β . Multiplying $d_\beta(1) = 0.t_1 t_2 t_3 \dots$ by β provides β as a Laurent series of β^{-1} from (t_i) , as:

$$(2.2) \quad \beta = t_1(\beta) + \frac{t_2(\beta)}{\beta} + \frac{t_3(\beta)}{\beta^2} + \dots$$

Up to the sign, by definition, the expansion of the power series of the denominator (2.1) is the Parry Upper function $f_\beta(z)$ at β . It satisfies

$$(2.3) \quad (i) \quad f_\beta(z) = - \frac{1 - z^N}{\zeta_\beta(z)} \quad \text{in the first case,}$$

$$(2.4) \quad (ii) \quad f_\beta(z) = - \frac{1}{\zeta_\beta(z)} \quad \text{in the second case.}$$

We have:

$$(2.5) \quad -1 + t_1 z + t_2 z^2 + t_3 z^3 + \dots = f_\beta(z) = -(1 - \beta z) \left(\sum_{n=0}^{\infty} T_\beta^n(1) z^n \right),$$

with $d_\beta(1) = 0.t_1 t_2 t_3 \dots$. The Parry Upper function $f_\beta(z)$ has no zero in $\{z \in \mathbb{C} : |z| \leq 1/\beta\}$ except $z = 1/\beta$ which is a simple zero.

The lexicographical ordering $<_{lex}$ on the Rényi expansions of 1 is uniquely in correspondence with the total ordering $<$ on $(1, +\infty)$ by the following Proposition, which is Lemma 3 in Parry [24].

Proposition 2.1. *Let $\alpha > 1$ and $\beta > 1$. If the Rényi α -expansion of 1 is*

$$d_\alpha(1) = 0.t'_1 t'_2 t'_3 \dots, \quad \text{i.e.} \quad 1 = \frac{t'_1}{\alpha} + \frac{t'_2}{\alpha^2} + \frac{t'_3}{\alpha^3} + \dots$$

and the Rényi β -expansion of 1 is

$$d_\beta(1) = 0.t_1 t_2 t_3 \dots, \quad \text{i.e.} \quad 1 = \frac{t_1}{\beta} + \frac{t_2}{\beta^2} + \frac{t_3}{\beta^3} + \dots,$$

then $\alpha < \beta$ if and only if $(t'_1, t'_2, t'_3, \dots) <_{lex} (t_1, t_2, t_3, \dots)$.

Theorem 2.2. *Let $n \geq 2$. A real number $\beta \in (1, \frac{1+\sqrt{5}}{2}]$ belongs to $[\theta_{n+1}^{-1}, \theta_n^{-1})$ if and only if the Rényi β -expansion of unity $d_\beta(1)$ is of the form (finite or infinite)*

$$(2.6) \quad d_\beta(1) = 0.10^{n-1} 10^{n_1} 10^{n_2} 10^{n_3} \dots,$$

with $n_k \geq n - 1$ for all $k \geq 1$.

Proof. Since $d_{\theta_{n+1}^{-1}}(1) = 0.10^{n-1}1$ and $d_{\theta_n^{-1}}(1) = 0.10^{n-2}1$, Proposition 2.1 implies that the condition is sufficient. It is also necessary: $d_\beta(1)$ begins as $0.10^{n-1}1$ for all β such that $\theta_{n+1}^{-1} \leq \beta < \theta_n^{-1}$. For such β s we write $d_\beta(1) = 0.10^{n-1}1u$ with digits in the alphabet $\mathcal{A}_\beta = \{0, 1\}$ common to all β s, that is

$$u = 1^{h_0}0^{n_1}1^{h_1}0^{n_2}1^{h_2} \dots$$

and $h_0, n_1, h_1, n_2, h_2, \dots$ integers ≥ 0 . The *Conditions of Parry* ([21] Chap. 7) applied to the sequence $(1, 0^{n-1}, 1^{1+h_0}, 0^{n_1}, 1^{h_1}, 0^{n_2}, 1^{h_2}, \dots)$, which characterizes uniquely the base of numeration β , readily implies $h_0 = 0$ and $h_k = 1$ and $n_k \geq n - 1$ for all $k \geq 1$. \square

Definition 2.3. Let $n \geq 2$ (ou ? 3). The *class* \mathcal{B} is the collection of all the polynomial sections of the power series $f_\beta(z)$ for all β s in the interval $(1, (1 + \sqrt{5})/2)$. From (2.5) and (2.6), the power series $f_\beta(z)$ in (2.5) take the form:

$$(2.7) \quad -1 + x + x^n + x^{m_1} + x^{m_2} + \dots + x^{m_s} + \dots$$

with the distanciation conditions: $m_1 - n \geq n - 1$, $m_{q+1} - m_q \geq n - 1$ for all $1 \leq q$. The polynomials of the class \mathcal{B} are integer lacunary polynomials having coefficients in the alphabet $\{0, 1\}$ except the constant term equal to -1 , of the form

$$-1 + x + x^n + x^{m_1} + x^{m_2} + \dots + x^{m_s}$$

for some integer $s \geq 0$ (the case $s = 0$ corresponds to the trinomials $-1 + x + x^n$), with the same distanciation conditions: $m_1 - n \geq n - 1$, $m_{q+1} - m_q \geq n - 1$ for all $1 \leq q < s$.

The domain of definition of $\zeta_\beta(z)$, as a meromorphic function, contains the open unit disk $D(0, 1) = \{z \in \mathbb{C} : |z| < 1\}$. Indeed, in all the cases where $\beta > 1$ is such that the denominator $-1 + t_1z + t_2z^2 + t_3z^3 + \dots$ of (2.1) has infinitely many non-zero terms, Hadamard's formula gives $R = 1$ as radius of convergence. For the other cases, the domain of definition of $\zeta_\beta(z)$ as a meromorphic function is \mathbb{C} . From (2.2), the real number β^{-1} is always a simple pole of $\zeta_\beta(z)$ in $D(0, 1)$. The function $\zeta_\beta(z)$ is holomorphic on $\{z \in \mathbb{C} : |z| < 1/\beta\}$ and is extended meromorphically to $D(0, 1)$.

In $D(0, 1)$, the poles of $\zeta_\beta(z)$ are separated into two subcollections [52]:

- the *lenticular poles*,
- the non-lenticular poles (the ‘‘other poles’’).

This classification is not canonical; it is due to the method of Rouché, applied to the zeroes of modulus < 1 of the trinomials $-1 + x + x^n$, $n \geq 3$ for the identification of the lenticular poles. However, it is powerful enough to deduce the Dobrowolski-type inequality in Theorem 2.8, and this inequality is used successfully in Section 4. A finer method would lead to better constants in (2.18). The subset of the *lenticular poles* is denoted by \mathcal{L}_β , all simple poles, and recalled below. Lenticuli of poles are symmetrical with respect to the real axis. Examples of lenticuli of lenticular poles are given in [8]. A lenticular pole of $\zeta_\beta(z)$ is a zero of (2.7), the Parry Upper function $f_\beta(z)$ at β . Below, in Definition 2.7), we recall the definition of a lenticular pole of $\zeta_\beta(z)$, from Theorem 2.6.

The terminology ‘‘lenticular pole’’ comes from the particular case where β belongs to $\cup_{n \geq 2} \{\theta_n^{-1}\}$ (cf [49]). Recall that the sequence $(\theta_n^{-1})_{n \geq 2}$ is strictly decreasing: $1 < \dots < \theta_{n+1}^{-1} < \theta_n^{-1} < \dots < \theta_2^{-1} = (1 + \sqrt{5})/2$, with $\lim_{n \rightarrow \infty} \theta_n^{-1} = 1$. Indeed, in this particular case, if $\beta = \theta_n^{-1}$ for some $n \geq 2$, the set of poles of $\zeta_\beta(z) = -(1 - z^n)/(-1 + z + z^n)$ inside $D(0, 1)$ forms a lenticulus of simple poles in $\{z \in D(0, 1) : |\arg z| < \pi/3\}$; all the poles of modulus < 1 are lenticular and there is no ‘‘other pole’’ [49]. In the general case, ‘‘other poles’’ exist, in a thin annulus of the unit circle in $D(0, 1)$, and, if $\theta_n^{-1} < \beta < \theta_{n-1}^{-1}$ for some $n \geq 3$, the lenticulus \mathcal{L}_β of lenticular poles can be defined ‘‘optimally’’ with respect to $\mathcal{L}_{\theta_n^{-1}}$.

Let us fix the notations; we follow Section 2 in [49], and Appendix in [52]. They come from the factorization of $G_n(X) := -1 + X + X^n$ in Selmer [39]. Summing in pairs over complex conjugated imaginary roots, the indexation of the roots and the factorization of $G_n(X)$ are taken as follows:

$$(2.8) \quad G_n(X) = (X - \theta_n) \left(\prod_{j=1}^{\lfloor \frac{n}{6} \rfloor} (X - z_{j,n})(X - \bar{z}_{j,n}) \right) \times q_n(X),$$

where θ_n is the only (real) root of $G_n(X)$ in the interval $(0, 1)$, where

$$q_n(X) = \begin{cases} \left(\prod_{j=1+\lfloor \frac{n}{6} \rfloor}^{\frac{n-2}{2}} (X - z_{j,n})(X - \bar{z}_{j,n}) \right) \times (X - z_{\frac{n}{2},n}) & \text{if } n \text{ is even, with} \\ & z_{\frac{n}{2},n} \text{ real} < -1, \\ \prod_{j=1+\lfloor \frac{n}{6} \rfloor}^{\frac{n-1}{2}} (X - z_{j,n})(X - \bar{z}_{j,n}) & \text{if } n \text{ is odd,} \end{cases}$$

where the index $j = 1, 2, \dots$ is such that $z_{j,n}$ is a (nonreal) complex zero of $G_n(X)$, except if n is even and $j = n/2$, such that the argument $\arg(z_{j,n})$ of $z_{j,n}$ is roughly equal to $2\pi j/n$ and that the family of arguments $(\arg(z_{j,n}))_{1 \leq j < \lfloor n/2 \rfloor}$ forms a strictly increasing sequence with j :

$$0 < \arg(z_{1,n}) < \arg(z_{2,n}) < \dots < \arg(z_{\lfloor \frac{n}{2} \rfloor, n}) \leq \pi.$$

For $n \geq 2$ all the roots of $G_n(X)$ are simple.

Denote by $a_{\max} = 5.87433\dots$ the abscissa of the maximum of the function

$$a \mapsto (1 - \exp(\frac{-\pi}{a})) / (2 \exp(\frac{\pi}{a}) - 1)$$

on $(0, \infty)$. Let $\kappa := 0.171573\dots$ be the value of its maximum, at $a = a_{\max}$ (Figure 1 in [52]). From a numerical viewpoint we have: $2 \arcsin(\kappa/2) = 0.171784\dots$. The sequel follows from Section 2 in [52].

Definition 2.4. Let $n \geq 195$. Denote by J_n the largest integer $j \geq 1$ such that the root $z_{j,n}$ of G_n satisfies

$$(2.9) \quad \frac{|-1 + z_{j,n}|}{|z_{j,n}|} \leq \kappa = \frac{1 - \exp(\frac{-\pi}{a_{\max}})}{2 \exp(\frac{\pi}{a_{\max}}) - 1} = 0.171573\dots,$$

and by c_n the real number > 0 such that $|z_{J_n, n}| = 1 - \frac{c_n}{n}$.

Proposition 2.5. *Let $n \geq 195$. Then*

$$(2.10) \quad \arg(z_{J_n, n}) = 2 \arcsin\left(\frac{\kappa}{2}\right) + \frac{\kappa \operatorname{Log} \kappa}{n\sqrt{4 - \kappa^2}} + \frac{1}{n} O\left(\left(\frac{\operatorname{Log} \operatorname{Log} n}{\operatorname{Log} n}\right)^2\right),$$

$$(2.11) \quad J_n = \frac{n}{\pi} \left(\arcsin\left(\frac{\kappa}{2}\right)\right) + \frac{\kappa \operatorname{Log} \kappa}{\pi\sqrt{4 - \kappa^2}} + O\left(\left(\frac{\operatorname{Log} \operatorname{Log} n}{\operatorname{Log} n}\right)^2\right)$$

with, at the limit,

$$(2.12) \quad \lim_{n \rightarrow +\infty} \arg(z_{J_n, n}) = \lim_{n \rightarrow +\infty} 2\pi \frac{J_n}{n} = 2 \arcsin\left(\frac{\kappa}{2}\right) = 0.171784\dots,$$

$$(2.13) \quad c_n = -(\operatorname{Log} \kappa) \left(1 + \frac{1}{n}\right) + \frac{1}{n} O\left(\left(\frac{\operatorname{Log} \operatorname{Log} n}{\operatorname{Log} n}\right)^2\right),$$

with $c = \lim_{n \rightarrow +\infty} c_n = -\operatorname{Log} \kappa = 1.76274\dots$

Proof. Proposition 2.7 and Lemma 2.14 in [52]. □

Let $n \geq 195$. Denote

$$D_{0, n} := \{z \in \mathbb{C} : |z - \theta_n| < \theta_n - \theta_{n-1}\}$$

the open disk of radius $\theta_n - \theta_{n-1} \approx \frac{t_{0, n}}{n}$ with $t_{0, n} := \left(\frac{\operatorname{Log} \operatorname{Log} n}{\operatorname{Log} n}\right)^2$. Denote

$$(2.14) \quad D_{j, n} := \{z \mid |z - z_{j, n}| < \frac{\pi |z_{j, n}|}{n a_{\max}}\}, \quad j = 1, 2, \dots, J_n,$$

with additional constraints of the radii for the small values of j , from Theorem 2.11 in [52].

Theorem 2.6. *Let $c_{\text{lent}} := \min_{n \geq 260} (c_n - \frac{\pi}{a_{\max}})$. Let $n \geq 260$ and $\beta > 1$ be a real number such that $\operatorname{dyg}(\beta) = n$, Denote by*

$$f_\beta(z) = -1 + z + z^n + z^{m_1} + z^{m_2} + \dots + z^{m_j} + z^{m_{j+1}} + \dots,$$

where $m_1 - n \geq n - 1$, $m_{j+1} - m_j \geq n - 1$ for $j \geq 1$, the Parry Upper function at β . Then the zeroes of $f_\beta(z)$ of modulus < 1 which lie in $-\arg(z_{J_n, n}) - \frac{\pi}{n a_{\max}} < \arg z < +\arg(z_{J_n, n}) + \frac{\pi}{n a_{\max}}$ either belong to

$$\left\{z \mid \left||z| - 1\right| < \frac{1}{3} \frac{c_{\text{lent}}}{n}\right\} \quad \text{or to} \quad \left\{z \mid \left||z| - 1\right| > \frac{c_{\text{lent}}}{n}\right\}.$$

In the second class of zeroes, all the zeroes, say $\omega_{j, n}$, are simple, and lie in the union

$$D_{0, n} \cup \bigcup_{j=1}^{J_n} (\overline{D_{j, n}} \cup D_{j, n});$$

there is one zero per disk $\omega_{j, n} \in D_{j, n}$, $\overline{\omega_{j, n}} \in \overline{D_{j, n}}$, the disk $D_{0, n}$ containing the element β^{-1} .

Proof. Theorem 2.19 in [52]. □

Definition 2.7. Let $n \geq 260$. Let $\beta > 1$ be a real number such that $\text{dyg}(\beta) = n$. The poles of $\zeta_\beta(z)$ which belong to the angular sector

$$(2.15) \quad \left\{ z \in \mathbb{C} : |z| < 1 - \frac{c_{lent}}{n}, |\arg z| \leq \arg(z_{J_n, n}) + \frac{\pi}{na_{max}} \right\}$$

are called the *lenticular poles* of $\zeta_\beta(z)$.

The lenticuli of lenticular poles, relative to θ_n^{-1} and β respectively, are:

$$\mathcal{L}_{\theta_n^{-1}} := \{\overline{z_{\lfloor n/6 \rfloor, n}}, \dots, \overline{z_{2, n}}, \overline{z_{1, n}}, \theta_n, z_{1, n}, z_{2, n}, \dots, z_{\lfloor n/6 \rfloor, n}\},$$

$$\mathcal{L}_\beta = \{\overline{\omega_{J_n, n}}, \dots, \overline{\omega_{1, n}}, \beta^{-1}, \omega_{1, n}, \dots, \omega_{J_n, n}\}.$$

The lenticular zeroes of f_β are peculiar zeroes, off the unit circle. By (2.12), a good approximation of the angular dependence in (2.15) is

$$“|\arg z| \leq \pi/18” \quad \text{instead of} \quad “|\arg z| \leq \arg(z_{J_n, n}) + \frac{\pi}{na_{max}}.”$$

This approximation is used in Theorem 4 in [8] (and Theorem 2.2 in [10]).

2.2. A Dobrowolski-type inequality for the lenticular measure. If $\theta_n^{-1} < \beta < \theta_{n-1}^{-1}$ for some $n \geq 3$, and

$$\mathcal{L}_\beta = \{\overline{\omega_{J_n, n}}, \dots, \overline{\omega_{1, n}}, \beta^{-1}, \omega_{1, n}, \dots, \omega_{J_n, n}\}$$

denotes the set of the lenticular poles of $\zeta_\beta(z)$, all depending upon β , then we attribute to \mathcal{L}_β a measure, that we call *lenticular measure* of β , by the expression

$$(2.16) \quad \mathfrak{S}_{lent}(\beta) := \prod_{\omega \in \mathcal{L}_\beta} |\omega|^{-1} = \beta \prod_{j=1}^{J_n} |\omega_{j, n}|^{-2}.$$

By construction, $\mathfrak{S}_{lent}(\beta) \geq 1$. If $\beta = \theta_n^{-1}$, then the identification with the Mahler measure of θ_n^{-1} holds: $\mathfrak{S}_{lent}(\theta_n^{-1}) = M(\theta_n^{-1})$. The definition of $\mathfrak{S}_{lent}(\beta)$ is mimicked on the definition of the Mahler measure of an algebraic integer. It is an analogue of the Mahler measure. The properties of $\mathfrak{S}_{lent}(\beta)$, $\theta_n^{-1} < \beta < \theta_{n-1}^{-1}$, are closely related to those of the Mahler measure of β if β is an algebraic integer.

Theorem 1.1 in [52] proves the continuity of $\beta \mapsto \mathfrak{S}_{lent}(\beta)$ for $\theta_n^{-1} < \beta < \theta_{n-1}^{-1}$, as soon as n is large enough. In [52] lower bounds of the infimum $\liminf_{\beta \rightarrow 1} \mathfrak{S}_{lent}(\beta)$ by comparison with $\lim_{n \rightarrow \infty} M(\theta_n^{-1})$ are studied. The method used is an extension of the one developed in [49] for the zeroes of the trinomials $-1 + x + x^n$: in there the lenticular poles are developed as asymptotic expansions “à la Poincaré” as a function of n [48], which leads to a Dobrowolski-type inequality of the Mahler measure, better than the classical Dobrowolski’s inequality in [7]. Let us recall more precisely these inequalities from [7], [49] and [52].

Denote

$$(2.17) \quad C := \exp\left(\frac{-1}{\pi} \int_0^{2 \arcsin(\frac{\sqrt{3}}{2})} \text{Log} \left[\frac{1 + 2 \sin(\frac{x}{2}) - \sqrt{1 - 12 \sin(\frac{x}{2}) + 4(\sin(\frac{x}{2}))^2}}{4} \right] dx\right) = 1.15411 \dots$$

The main theorem is:

Theorem 2.8. (Verger-Gaugry [52]) *There exists an integer $\eta \geq 260$ such that the following inequality holds:*

$$(2.18) \quad \mathfrak{S}_{lent}(\beta) \geq C - C \frac{\arcsin(\kappa/2)}{\pi} \frac{1}{\text{Log}(n)}, \quad \text{for all } n \geq \eta$$

and any $\beta \in (\theta_n^{-1}, \theta_{n-1}^{-1})$.

This theorem extends the following result, proved in [49]:

$$(2.19) \quad M(\theta_n^{-1}) > \Lambda - \frac{\Lambda}{6} \left(\frac{1}{\text{Log } n} \right), \quad n \geq 2,$$

where Λ is the following constant

$$(2.20) \quad \Lambda := \exp\left(\frac{3\sqrt{3}}{4\pi} L(2, \chi_3)\right) = \exp\left(\frac{-1}{\pi} \int_0^{\pi/3} \text{Log}\left(2 \sin\left(\frac{x}{2}\right)\right) dx\right) = 1.38135\dots,$$

higher than $C = 1.1541\dots$, and $L(s, \chi_3) := \sum_{m \geq 1} \frac{\chi_3(m)}{m^s}$ the Dirichlet L-series for the character χ_3 , with χ_3 the uniquely specified odd character of conductor 3 ($\chi_3(m) = 0, 1$ or -1 according to whether $m \equiv 0, 1$ or $2 \pmod{3}$), equivalently $\chi_3(m) = \left(\frac{m}{3}\right)$ the Jacobi symbol). Recall that, from Selmer [39], in (2.19), we have: $n = \deg(\theta_n^{-1})$ if $n \not\equiv 5 \pmod{6}$, and $n = \deg(\theta_n^{-1}) + 2$ if $n \equiv 5 \pmod{6}$.

In 1979, Dobrowolski [7], using an auxiliary function, obtained the general asymptotic minoration, with $n = \deg(\alpha)$ the degree of the nonzero algebraic integer α , which is not a root of unity,

$$(2.21) \quad M(\alpha) > 1 + (1 - \varepsilon) \left(\frac{\text{Log Log } n}{\text{Log } n} \right)^3, \quad n > n_0(\varepsilon),$$

with $1 - \varepsilon$ replaced by $1/1200$ for $n \geq 2$, for an effective version of the minoration.

In the right hand sides of (2.19) and (2.18) the constant term is not any more 1 as in (2.21) but $\Lambda = 1.38135\dots$ and $C = 1.15411\dots$ respectively, and the sign of the n -dependent term is “−”, instead of “+”, with an appreciable gain of $(\text{Log } n)^2$ in the denominator. In the case where β is a reciprocal algebraic integer in Theorem 2.8, (2.18) represents an improvement of Dobrowolski’s inequality, in some sense, but in which the degree $\deg(\beta)$ is replaced by the integer n such that $\beta \in (\theta_n^{-1}, \theta_{n-1}^{-1})$. To emphasize, the integer variable “ n ” is the true algebraic degree of β in (2.21), is almost all the time the algebraic degree of β in (2.20), but is different in (2.18). In [52] it was suggested to name this peculiar integer n , appearing in (2.18), the *dynamical degree of β* , and to denote it by $\text{dyg}(\beta)$. This name is used in the sequel. It has the obvious following property:

β tends to 1 if and only if $\text{dyg}(\beta)$ tends to infinity.

When $\beta > 1$ is a reciprocal algebraic integer and $\text{dyg}(\beta)$ is large enough, a simple relation between $\text{dyg}(\beta)$ and $\deg(\beta)$ can be deduced (cf Section 3.4). From Theorem 2.8, we have:

$$C \leq \liminf_{\beta \rightarrow 1^+} \mathfrak{S}_{lent}(\beta) \leq \Lambda.$$

3. LENTICULAR MINORATION OF THE MAHLER MEASURE

In this section we assume the existence of a reciprocal algebraic integer β in the interval $(\theta_n^{-1}, \theta_{n-1}^{-1})$ for some integer $n \geq 3$ (n is fixed), having $M(\beta) < 1.176280\dots$, i.e. less than Lehmer's number. The minimal polynomial P_β of β is monic and reciprocal in the sense: $P_\beta^*(X) := X^{\deg P_\beta} P_\beta(1/X) = P_\beta(X)$. It can be written

$$(3.1) \quad P_\beta(X) = \widetilde{P}_\beta(X^r)$$

for some integer $r \geq 1$ and some \mathbb{Z} -minimal integer polynomial $\widetilde{P}_\beta(X)$. The integer $r = r(\beta)$ is the largest one such that (3.1) holds; it depends upon β . It suffices to restrict ourselves to the β s for which $r(\beta) = 1$. The β s such that $r(\beta) \geq 2$ are excluded in the following, as explained in Section 3.3.1, since they do not bring any new contribution to the minoration problem of the Mahler measure; they constitute the "Second case". In the following we assume $r(\beta) = 1$.

Under these assumptions and the notations of Theorem 2.6, we have

Proposition 3.1.

$$(3.2) \quad f_\beta(\omega_{j,n}) = 0 \quad \implies \quad P_\beta(\omega_{j,n}) = 0.$$

Proof. Proposition 3 in [9]. □

The objective of this Section is to recall briefly the main ingredients of its proof. In Section 3.2 we detail the rewriting trails between the minimal polynomial " P_β " and the polynomial sections of " f_β ", and the way convergence occurs towards " f_β " using the theorem of Kala-Vávra, with respect to Galois conjugation. The domain of applicability of Kala-Vávra's Theorem is due to the type of factorization of the polynomial sections of " f_β ", and is recalled in Section 3.1.

The identification (3.2) gives the following minoration to the Mahler measure of β :

$$(3.3) \quad M(\beta) = M(\beta^{-1}) = \prod_{\omega \notin \mathcal{L}_\beta, |\omega| < 1} |\omega|^{-1} \times \prod_{\omega \in \mathcal{L}_\beta} |\omega|^{-1} \geq \prod_{\omega \in \mathcal{L}_\beta} |\omega|^{-1} = \mathfrak{S}_{lent}(\beta).$$

3.1. Factorization of the lacunary almost-Newman polynomials of the class \mathcal{B} and Kala-Vávra's Theorem. Newman polynomials are polynomials with coefficients in $\{0, 1\}$. In [8] *almost-Newman polynomials* have been introduced: by definition, an almost-Newman polynomial is an integer polynomial which has coefficients in $\{0, 1\}$ except the constant term equal to -1 . Recall that Newman polynomials are irreducible with probability one, under the Generalized Riemann Hypothesis [4].

By Definition 2.3 the polynomials of the type:

$$f(x) := -1 + x + x^n + x^{m_1} + x^{m_2} + \dots + x^{m_s}$$

where $n \geq 2$, $s \geq 0$, $m_1 - n \geq n - 1$, $m_{q+1} - m_q \geq n - 1$ for $1 \leq q < s$, constituting the class \mathcal{B} , are all almost-Newman polynomials, for which the lacunarity is controlled a minima by the (unique) integer $n - 1$.

The case “ $s = 0$ ” corresponds to the trinomials $G_n(z) := -1 + z + z^n$. The subclass $\mathcal{B}_n \subset \mathcal{B}$ is the set of polynomials $f(x) \in \mathcal{B}$ whose third monomial is exactly x^n , so that the union $\mathcal{B} = \cup_{n \geq 2} \mathcal{B}_n$ is disjoint.

The factorization of the polynomials of the class \mathcal{B} has been investigated in [8] and the results are summarized in Theorem 3.2 and Theorem 3.3. Theorem 3.3 is more precise than the general theorems of factorization of lacunary polynomials of Schinzel [33] [34] [35] [36] [37] [38]. Two conjectures have appeared in this study:

- (i) the “Asymptotic Reducibility Conjecture” says that 75% of the polynomials $f(x) \in \mathcal{B}$ are irreducible,
- (ii) the Conjecture B, which says that the factorization of any $f \in \mathcal{B}$ never contains factors which are reciprocal non-cyclotomic.

Conjecture B has been established in [10] for some infinite families of polynomials f of \mathcal{B} , using criteria of Sawin, Shusterman and Stoll [32]: cf Proposition 4.7 and Remark 4.8 in [10].

Theorem 3.2 (Selmer [39]). *Let $n \geq 2$. The trinomials $G_n(x)$ are irreducible if $n \not\equiv 5 \pmod{6}$, and, for $n \equiv 5 \pmod{6}$, are reducible as product of two irreducible factors whose one is the cyclotomic factor $x^2 - x + 1$, the other factor $(-1 + x + x^n)/(x^2 - x + 1)$ being nonreciprocal of degree $n - 2$.*

Theorem 3.3 (Dutykh - Verger-Gaugry [8]). *For any $f \in \mathcal{B}_n$, $n \geq 3$, denote by*

$$(3.4) \quad f(x) = A(x)B(x)C(x) = -1 + x + x^n + x^{m_1} + x^{m_2} + \dots + x^{m_s},$$

where $s \geq 1$, $m_1 - n \geq n - 1$, $m_{j+1} - m_j \geq n - 1$ for $1 \leq j < s$, the factorization of f where A is the cyclotomic part, B the reciprocal noncyclotomic part, C the nonreciprocal part. Then

- (i) the nonreciprocal part C is nontrivial, irreducible and never vanishes on the unit circle,
- (ii) if $\gamma > 1$ denotes the real algebraic number uniquely determined by the sequence $(n, m_1, m_2, \dots, m_s)$ such that $1/\gamma$ is the unique real root of f in (θ_{n-1}, θ_n) , $-C^*(X)$ is the minimal polynomial $P_\gamma(X)$ of γ , and γ is a nonreciprocal algebraic integer.

In (3.4), in 75% of the cases, $A(x)B(x) = 1$. In the 25 % remaining cases, the polynomial $A(x)$ is not trivial and is a product of cyclotomic polynomials Φ_j of small indices j [8].

Remark 3.4. For applying Kala-Vávra’s Theorem 3.5 to rewriting trails and representations in Section 3.2, the important point in Theorem 3.3 is

$$|\gamma'| \neq 1 \text{ for any conjugate } \gamma' \text{ of } \gamma \in (\theta_n^{-1}, \theta_{n-1}^{-1}).$$

As a consequence, any $x \in \mathbb{Q}(\gamma)$ will be an eventually periodic (γ, \mathcal{A}) -representation for some alphabet \mathcal{A} depending only upon γ , and, for any conjugation \mathbb{Q} -automorphism σ , such that $|\gamma' = \sigma(\gamma)| > 1$, $\sigma(x)$ will be the eventually periodic $(\sigma(\gamma), \mathcal{A})$ -representation, with the same coefficient vector.

Let us recall the definitions. For a general complex number $\beta \in \mathbb{C}$, $|\beta| > 1$, and a finite alphabet $\mathcal{A} \subset \mathbb{C}$, we define the (β, \mathcal{A}) -representations as expressions of the form

$\sum_{k \geq -L} a_k \beta^{-k}$, $a_k \in \mathcal{A}$, for some integer $L \in \mathbb{Z}$. They are Laurent series of $1/\beta$. We define

$$\text{Per}_{\mathcal{A}}(\beta) := \{x \in \mathbb{C} \mid x \text{ has an eventually periodic } (\beta, \mathcal{A})\text{-representation}\}.$$

Attention will be focused on the complex numbers β which are real algebraic integers > 1 , close to 1, assuming that β has no conjugate on the unit circle, and on the alphabets $\mathcal{A} \subset \mathbb{Z}$, depending upon β , involved in the identity:

$$\mathbb{Q}(\beta) = \text{Per}_{\mathcal{A}}(\beta).$$

Those β s which are the real roots > 1 of the polynomials of the class \mathcal{B} will be of special interest in the next section.

We write \mathbb{Q} for the set of rational numbers, $\mathbb{Q}(\beta)$ for the smallest sub-field of \mathbb{C} containing β . Indeed, such an identity always holds by the following theorem.

Theorem 3.5 (Kala -Vávra [16]). *Let $\beta \in \mathbb{C}$ be an algebraic number of degree d , $|\beta| > 1$, and let $a_d x^d - a_{d-1} x^{d-1} - \dots - a_1 x - a_0 \in \mathbb{Z}[x]$ be its minimal polynomial. Suppose that $|\beta'| \neq 1$ for any conjugate β' of β . Then there exists a finite alphabet $\mathcal{A} \subset \mathbb{Z}$ such that*

$$(3.5) \quad \mathbb{Q}(\beta) = \text{Per}_{\mathcal{A}}(\beta).$$

Daróczy and Kátai [5], and later Thurston [47], have proved that for any non-real $\beta \in \mathbb{C}$, $|\beta| > 1$, there exists a finite alphabet $\mathcal{A} \subset \mathbb{C}$ such that every $x \in \mathbb{C}$ has a (β, \mathcal{A}) -representation. The search for periodic representations in radix systems goes back to Kovács [18] and to Kovács and Környei [19] (see also Pethő [26]). For the Rényi-Parry numeration system in base $\beta > 1$, the idea of the enlargement of the alphabet to obtain the eventual periodicity for the representations of the elements of the number field $\mathbb{Q}(\beta)$ has been recurrent.

In the context of automorphisms of complex numbers (Kestelman [17], Yales [53]); the absence of continuity of the \mathbb{Q} -automorphisms of conjugation is compensated in some sense by the eventual periodicity of the representations in Theorem 3.5.

3.2. Identification, rewriting trails, Galois conjugation and convergence. Since $\beta \in (\theta_n^{-1}, \theta_{n-1}^{-1})$ is a reciprocal algebraic integer for some integer $n \geq 3$ (n is fixed), the Parry Upper function $f_{\beta}(z) = -1 + z + z^n + z^{m_1} + z^{m_2} + \dots + z^{m_j} + z^{m_{j+1}} + \dots$, where $m_1 - n \geq n - 1$, $m_{j+1} - m_j \geq n - 1$ for $j \geq 1$, written $= -1 + \sum_{i \geq 1} t_i z^i$, is a power series which is never a polynomial. Indeed, the number of positive real roots of $f_{\beta}(z)$ is 2, and if we assume it is a polynomial, we obtain a contradiction, with Descartes's rule. Then the sequence $(t_i = t_i(\beta))_{i \geq 1}$, satisfying $t_1 = 1, t_2 = \dots = t_{n-1} = 0, t_n = 1, t_{n+1} = 0, \dots$, in $\{0, +1\}$, contains infinitely many nonzero terms.

For every $s \geq 1$, we denote by $S_{\gamma_s}^*(X) = X^s S_{\gamma_s}(1/X) = 1 - t_1 X - t_2 X^2 - \dots - t_{s-1} X^{s-1} - t_s X^s$ the reciprocal polynomial of $S_{\gamma_s}(X) := X^s - \sum_{i=0}^{s-1} t_{s-i} X^i$. The polynomial $-S_{\gamma_s}^*(z)$ is the s th polynomial section of $f_{\beta}(z)$. We denote by $\gamma_s > 1$ the nonreciprocal algebraic integer which is the unique real root of $S_{\gamma_s}(z)$. We have: $S_{\gamma_s}^*(\gamma_s^{-1}) = 0$ and $\lim_{s \rightarrow +\infty} \gamma_s^{-1} = \beta^{-1}$; $\deg \gamma_s^{-1} = \deg S_{\gamma_s}^*$ if and only if $S_{\gamma_s}^*(x)$ is irreducible. . The number of distinct polynomial sections of $f_{\beta}(z)$ is infinite.

The minimal polynomial P_{β} of β is monic and reciprocal. Denote it

$$P_{\beta}(x) = 1 + a_1 x + a_2 x^2 + \dots + a_{d-1} x^{d-1} + a_d x^d \quad (a_{d-i} = a_i).$$

Let $H := \max\{|a_i| : 1 \leq i \leq d-1\} \geq 1$ be the naïve height of P_β .

We proceed in two steps:

- (i) first we want to express $P_\beta(\gamma_s)$ as a (γ_s, \mathcal{A}) -eventually periodic representation with the symmetrical alphabet $\mathcal{A} = \{-m, \dots, 0, \dots, +m\} \subset \mathbb{Z}$ which does not depend upon s , with $m = \lceil 2((2^d - 1)H + 2^d)/3 \rceil$. This expression of $P_\beta(\gamma_s)$, as a Laurent series of $1/\gamma_s$, will be obtained using Theorem 3 in [9], and Theorem 3.5,
- (ii) we allow s to tend to infinity (which has a sense by the above) to obtain the convergence of $P_\beta(\gamma_s)$ to 0, what will prove Proposition 3.1.

(i) For simplifying the notations, we write P instead of P_β and γ instead of γ_s . Let us recall the construction called “rewriting trail from “ S_γ^* ” to “ P ”, at γ^{-1} ” from [8].

The starting point is the identity $1 = 1$, to which we add $0 = -S_\gamma^*(\gamma^{-1})$ in the right hand side. Then we define a rewriting trail from

$$(3.6) \quad 1 = 1 - S_\gamma^*(\gamma^{-1}) = t_1\gamma^{-1} + t_2\gamma^{-2} + \dots + t_{s-1}\gamma^{-(s-1)} + t_s\gamma^{-s}$$

to

$$-a_1\gamma^{-1} - a_2\gamma^{-2} + \dots - a_{d-1}\gamma^{-(d-1)} - a_d\gamma^{-d} = 1 - P(\gamma^{-1}).$$

A rewriting trail will be a sequence of integer polynomials, whose role will consist in “restoring” the coefficients of $1 - P(\gamma^{-1})$ one after the other, from the left, by adding “0” conveniently at each step to both sides of (3.6). At the first step we add $0 = (-a_1 - t_1)\gamma^{-1}S_\gamma^*(\gamma^{-1})$; and we obtain

$$1 = -a_1\gamma^{-1} + (-(a_1 - t_1)t_1 + t_2)\gamma^{-2} + (-(a_1 - t_1)t_2 + t_3)\gamma^{-3} + \dots$$

so that the height of the polynomial

$$(-(a_1 - t_1)t_1 + t_2)X^2 + (-(a_1 - t_1)t_2 + t_3)X^3 + \dots$$

is $\leq H + 2$. At the second step we add $0 = (-a_2 - (-(a_1 - t_1)t_1 + t_2))\gamma^{-2}S_\gamma^*(\gamma^{-1})$. Then we obtain

$$1 = -a_1\gamma^{-1} - a_2\gamma^{-2} + [(-a_2 - (-(a_1 - t_1)t_1 + t_2))t_1 + (-(a_1 - t_1)t_2 + t_3)]\gamma^{-3} + \dots$$

where the height of the polynomial

$$[(-a_2 - (-(a_1 - t_1)t_1 + t_2))t_1 + (-(a_1 - t_1)t_2 + t_3)]X^3 + \dots$$

is $\leq H + (H + 2) + (H + 2) = 3H + 4$. Iterating this process d times we obtain

$$1 = -a_1\gamma^{-1} - a_2\gamma^{-2} - \dots - a_d\gamma^{-d} + \text{polynomial remainder in } \gamma^{-1}.$$

Denote by $V(\gamma^{-1})$ this polynomial remainder in γ^{-1} , for some $V(X) \in \mathbb{Z}[X]$, and X specializing in γ^{-1} . If we denote the upper bound of the height of the polynomial remainder $V(X)$, at step q , by $\lambda_q H + v_q$, we readily deduce: $v_q = 2^q$, and $\lambda_{q+1} = 2\lambda_q + 1$, $q \geq 1$, with $\lambda_1 = 1$; then $\lambda_q = 2^q - 1$.

To summarize, we obtain a sequence $(A'_q(X))_{q \geq 1}$ of rewriting polynomials involved in this rewriting trail; for $q \geq 1$, $A'_q \in \mathbb{Z}[X]$, $\deg(A'_q) \leq q$ and $A'_q(0) = -1$. The first polynomial

$A'_1(X)$ is $-1 + (-a_1 - t_1)X$. The second polynomial $A'_2(X)$ is $-1 + (-a_1 - t_1)X + (-a_2 - (-(-a_1 - t_1)t_1 + t_2))X^2$, etc.

For $q \geq \deg(P)$, all the coefficients of P are restored; denote by $(h'_{q,j})_{j=0,1,\dots,s-1}$ the s -tuple of integers produced by this rewriting trail, at step q . It is such that

$$(3.7) \quad A'_q(\gamma^{-1})S_\gamma^*(\gamma^{-1}) = -P(\gamma^{-1}) + \gamma^{-q-1} \left(\sum_{j=0}^{s-1} h_{q,j} \gamma^{-j} \right).$$

Then take $q = d$. The lhs of (3.7) is equal to 0. Thus

$$P(\gamma^{-1}) = \gamma^{-d-1} \left(\sum_{j=0}^{s-1} h_{d,j} \gamma^{-j} \right) \quad \implies \quad P(\gamma) = \sum_{j=0}^{s-1} h_{d,j} \gamma^{-j-1}.$$

The height of the polynomial $W(X) := \sum_{j=0}^{s-1} h_{d,j} X^{j+1}$ is $\leq (2^d - 1)H + 2^d$. We now assume $|P(\gamma)| < \eta$. By Kala-Vavra's Theorem 3.5 and Theorem 3 in [9] there exist a preperiod $R(X) \in \mathcal{A}[X]$, a period $T(X) \in \mathcal{A}[X]$ such that

$$(3.8) \quad W(\gamma^{-1}) = P(\gamma) = R(\gamma^{-1}) + \gamma^{-\deg R - 1} \sum_{j=0}^{\infty} \frac{1}{\gamma^{j(\deg T + 1)}} T(\gamma^{-1}),$$

$\deg R \leq s - 1$, $\deg T \leq s - 1$, with $\mathcal{A} \subset \mathbb{Z}$ the symmetrical alphabet $= \{-m, \dots, 0, \dots, +m\}$ where $m = \lceil 2((2^d - 1)H + 2^d)/3 \rceil$.

Since the algebraic norm $N(\gamma)$ is equal to ± 1 the uniqueness of the representation (3.8) is not expected [19]. However, for any (γ, \mathcal{A}) -eventually periodic representation of $P(\gamma)$

$$P(\gamma) = W(\gamma^{-1}) = \frac{a'_w}{\gamma^w} + \frac{a'_{w+1}}{\gamma^{w+1}} + \frac{a'_{w+2}}{\gamma^{w+2}} + \dots, \quad \text{with } |a'_j| \leq m, j = w, w+1, \dots$$

with $a'_w \neq 0$, the exponent w appearing in the first term tends to infinity if η tends to 0. Indeed, from Theorem 4, Remarks 5 to 7, in [2], there exists a positive real number $\kappa_{\gamma, \mathcal{A}} > 0$ such that w is the minimal integer such that

$$\gamma^{w-1} \geq \frac{\kappa_{\gamma, \mathcal{A}}}{|P(\gamma)|} \geq \frac{\kappa_{\gamma, \mathcal{A}}}{\eta}.$$

(ii) The integer $n = \text{dyg}(\beta) \geq 3$ is fixed. By Theorem 3.3, for any $s \geq 0$, γ_s^{-1} has no conjugate on the unit circle. For all s such that $\deg S_{\gamma_s}^* \geq \deg P_\beta$, the identity

$$\mathbb{Q}(\gamma_s) = \text{Per}_{\mathcal{A}}(\gamma_s),$$

holds with $\mathcal{A} = \{-m, \dots, +m\} \subset \mathbb{Z}$, $m = \lceil 2((2^d - 1)H + 2^d)/3 \rceil$.

The polynomial value $P_\beta(\gamma_s) \in \mathbb{Q}(\gamma_s)$ is eventually periodic

$$(3.9) \quad P_\beta(\gamma_s) = R(\gamma_s^{-1}) + \frac{1}{\gamma_s^L} \sum_{j=0}^{\infty} \frac{1}{\gamma_s^{j\rho}} T(\gamma_s^{-1}) \in \text{Per}_{\mathcal{A}}(\gamma_s)$$

with L, ρ and $R(X), T(X)$, depending upon s . This representation of $P_\beta(\gamma_s)$ starts as

$$= \frac{a_{w,(s)}}{\gamma_s^w} + \frac{a_{w+1,(s)}}{\gamma_s^{w+1}} + \dots + \frac{a_{w+m_s-1,(s)}}{\gamma_s^{w+m_s-1}} + \dots, \quad a_{j,(s)} \in \mathcal{A}, j = w, \dots, w + m_s - 1,$$

with $a_{w,(s)} \neq 0$ and $w = w_s \geq 1$, depending upon s , satisfies $\frac{\kappa_{\gamma_s, \mathcal{A}}}{\eta} \leq \gamma_s^{w_s-1}$ for some positive constant $\kappa_{\gamma_s, \mathcal{A}}$ depending upon γ and \mathcal{A} . Since \mathcal{A} is independent of s , and that the sequence (γ_s) is convergent with limit $\beta > 1$, there exists a (true) constant $\kappa > 0$ such that $\frac{\kappa}{\eta} \leq \gamma_s^{w_s-1}$ from Theorem 4, Remarks 5 to 7, in [2]. Since $\lim_{s \rightarrow \infty} P_\beta(\gamma_s) = 0 = P_\beta(\beta)$, we take $\eta = \eta_s := |P_\beta(\gamma_s)|$. The sequence (η_s) tends to 0. We deduce $\lim_{s \rightarrow \infty} w_s = +\infty$.

Let $\omega_{j,n} \neq \beta^{-1}$ be a lenticular zero of $f_\beta(z)$. We assume the existence of a small disk $D(\omega_{j,n}, r)$ centered at $\omega_{j,n}$ of radius $r > 0$, included in the open unit disk, which has the property that the only zero of $f_\beta(z)$ in $D(\omega_{j,n}, r)$ is $\omega_{j,n}$.

The lenticular zero $\omega_{j,n}$ is limit point of a sequence of zeroes of the polynomial sections of $f_\beta(z)$. As soon as $s \geq s_0$ for some s_0 , we assume that the disk $D(\omega_{j,n}, r)$ contains only one zero of $S_{\gamma_s}^*(z)$. Denote by r_s this zero. Let us assume that r_s is a Galois conjugate of γ_s^{-1} , and denote by $\sigma_s : \gamma_s^{-1} \rightarrow r_s$ the \mathbb{Q} -automorphism of conjugation. This assumption is reasonable by the Asymptotic Reducibility Conjecture.

To summarize, for $s \geq s_0$:

$$f_\beta(\omega_{j,n}) = 0, \quad S_{\gamma_s}^*(r_s) = 0, \quad r_s = \sigma_s(\gamma_s^{-1}), \quad |\omega_{j,n} - \sigma_s(\gamma_s^{-1})| < r, \quad \lim_{s \rightarrow \infty} r_s = \omega_{j,n}.$$

Let us show that $P_\beta(\omega_{j,n}) = 0$. Let us conjugate (3.8) for $X = \gamma_s^{-1}$. The power series (3.9) specialized at γ_s^{-1} is eventually periodic, therefore can be conjugated term by term, once the image of γ_s^{-1} by the conjugation σ_s is such that $|\sigma_s(\gamma_s^{-1})| < 1$, to ensure convergence. Then

$$(3.10) \quad \sigma_s(P_\beta(\gamma_s)) = W(r_s) = R(r_s) + r_s^L \frac{T(r_s)}{1 - r_s^\rho} = a_{w_s, (s)} r_s^{w_s} + a_{w_s+1, (s)} r_s^{w_s+1} + \dots$$

with

$$\frac{\kappa}{|P_\beta(\gamma_s)|} \leq \gamma_s^{w_s-1} \quad \text{and} \quad |r_s| < |\omega_{j,n}| + r < 1, \quad s \geq s_0.$$

We have, with $m = \lceil 2((2^d - 1)H + 2^d)/3 \rceil$,

$$\begin{aligned} |W(r_s)| &\leq |a_{w_s, (s)}| |r_s^{w_s}| + |a_{w_s+1, (s)}| |r_s^{w_s+1}| + \dots \\ &\leq m(|r_s^{w_s}| + |r_s^{w_s+1}| + \dots) = m|r_s^{w_s}|(1 + |r_s| + |r_s|^2 + \dots). \end{aligned}$$

Then

$$(3.11) \quad |W(r_s)| \leq |r_s|^{w_s} \frac{m}{1 - |r_s|}.$$

Let us observe that within a period of period length ρ in the power series (3.10) a certain number of coefficients are equal to zero, and therefore that the upper bound (3.11) can be improved using the period length ρ and the degree of T . However it is sufficient for below. We deduce

$$P_\beta(\omega_{j,n}) = \lim_{s \rightarrow \infty} W(r_s) = 0,$$

that is Proposition 3.1.

3.3. Examples without identification of the lenticular poles. Let $(\beta_q)_{q \geq 1}$ be an infinite sequence of real reciprocal algebraic integers > 1 tending to 1. We assume that it is strictly decreasing, with $\beta_1 \leq 1.176280\dots$, $\overline{|\beta_1|} \leq 1.176280\dots$, $M(\beta_1) \leq 1.176280\dots$, and that there exists only one β_q in an interval $(\theta_n^{-1}, \theta_{n-1}^{-1})$ for some n . We denote it by β_{q_n} and have:

$$\theta_n^{-1} < \beta_{q_n} < \theta_{n-1}^{-1}, \quad \beta_{q_n} \leq \overline{|\beta_{q_n}|} \leq M(\beta_{q_n}) \quad (n \geq n_0).$$

We allow n to run over a strictly increasing sequence \mathcal{J} of integers n_0, n_1, n_2, \dots . We assume that the sequence of Mahler measures $(M(\beta_{q_n}))_{n \in \mathcal{J}}$ and the sequence of houses $(\overline{|\beta_{q_n}|})_{n \in \mathcal{J}}$ are decreasing:

$$\begin{aligned} 1 &< \dots < \beta_{q_{n+1}} < \beta_{q_n} < \beta_{q_{n-1}} < \dots < \beta_{q_{n_0}} \leq \dots \leq 1.176280\dots, \\ 1 &< \dots \leq \overline{|\beta_{q_{n+1}}|} \leq \overline{|\beta_{q_n}|} \leq \overline{|\beta_{q_{n-1}}|} \leq \dots \leq \overline{|\beta_{q_{n_0}}|} \leq \dots \leq 1.176280\dots, \\ 1 &\leq \dots \leq M(\beta_{q_{n+1}}) \leq M(\beta_{q_n}) \leq M(\beta_{q_{n-1}}) \leq \dots \leq 1.176280\dots \end{aligned}$$

If they exist, let us call "Main Case" when the sequence $(M(\beta_{q_n}))_{n \in \mathcal{J}}$ is not stationary after a certain rank; and "Second Case" when the sequence $(M(\beta_{q_n}))_{n \in \mathcal{J}}$ is stationary after a certain rank. Whatever \mathcal{J} , from Theorem 2.8, Proposition 3.1 and (3.3), we have the universal lower bound

$$\lim_{n \in \mathcal{J}, n \rightarrow +\infty} M(\beta_{q_n}) \geq C,$$

and, for $n \in \mathcal{J}$ large enough, the Dobrowolsky-type inequality

$$M(\beta_{q_n}) \geq \mathfrak{S}_{lent}(\beta_{q_n}) \geq C - C \frac{\arcsin(\kappa/2)}{\pi} \frac{1}{\text{Log}(\text{dyg}(\beta_{q_n}))}.$$

Let us give two examples of sequences of reciprocal algebraic integers $\beta_{q_n} > 1$ which converge to 1 which do not possess any lenticular conjugate (except $\beta_{q_n}^{-1}$), to better understand the assumptions involved. Recall that the definition of a lenticular zero is given in Definition 2.7.

3.3.1. "Second Case": Minimal polynomial $P_\beta(X) = \widetilde{P}_\beta(X^r)$ for $r \geq 2$ - Roots of unity.

Let $\beta_1 = 2 + \sqrt{3} \in (1, +\infty)$. The minimal polynomial of β_1 is $P_{\beta_1}(X) = X^2 - 4X + 1$. It is reciprocal.

Lemma 3.6. *Let $r \geq 1$ be an integer and denote by β_r the real number in $(1, \infty)$ with $\beta_r^r = \beta_1$. Then β_r is a reciprocal algebraic integer, $\lim_{r \rightarrow \infty} \beta_r = 1$, with minimal polynomial $X^{2r} - 4X^r + 1$. The Mahler measure is constant on the family $(\beta_r)_{r \geq 1}$, i.e., for all $r \geq 1$, $M(\beta_r) = \beta_1$.*

Proof. Say $r \geq 2$ and let $\gamma \in \mathbb{C}$ satisfy $\gamma^r = \beta_1$. Then γ is an algebraic integer. We claim $\gamma \notin K = \mathbb{Q}(\sqrt{3})$. Indeed, if $\gamma \in K$, then $|\gamma| \leq \sqrt{2 + \sqrt{3}} < 2$ and $|\gamma'| \leq \sqrt{2 - \sqrt{3}} < 0.6$ with γ' the conjugate of γ . We write $\gamma = a + b\sqrt{3}$ with $a, b \in \mathbb{Z}$, so $\gamma' = a - b\sqrt{3}$. Then $b = (\gamma - \gamma')/(2\sqrt{3})$ and so $|b| \leq (2 + 0.6)/(2\sqrt{3}) < 1$ which implies $b = 0$. So $|a| = |\gamma| < 2$ implies $\gamma \in \{0, \pm 1\}$, a contradiction.

So β_r is not the r -th power of any element of K if $r \geq 2$. Moreover $-\beta_1/4 < 0$ is not a fourth power in the real quadratic field K . From this and from a classical result from the

theory of fields, Theorem VI.9.1 in S. Lang, *Algebra*, Graduate Texts in Mathematics, **211**, (2002), we conclude that $X^r - \beta_1$ is irreducible in $K[X]$ for all $r \geq 1$.

So

$$X^{2r} - 4X^r + 1 = P_{\beta_1}(X^r)$$

is the \mathbb{Z} -minimal polynomial of β_r . In particular β_r is reciprocal.

Now, if ε runs over the set of r -th roots of unity, $\varepsilon^r = 1$, then the set of the conjugates of β_r is $\{\beta_r \varepsilon \mid \text{all } \varepsilon, \varepsilon^r = 1\}$, since $(\beta_r \varepsilon)^r = \beta_1$ which implies $M(\beta_r) = M(\beta_1)$ for all $r \geq 2$.

The action of the r -th roots of unity, for r tending to infinity, does not produce reciprocal algebraic integers of smaller Mahler measure than $M(\beta_1)$. We have

$$1 \dots < \beta_{r+1} < \beta_r < \dots < \beta_2 < \beta_1 = 2 + \sqrt{3}$$

with

$$\dots = M(\beta_r + 1) = M(\beta_r) = \dots = M(\beta_2) = M(\beta_1) = 2 + \sqrt{3}.$$

□

In this note we are concerned with the attack of the Conjecture of Lehmer, not with sequences of algebraic integers on which the Mahler measure remains constant. Therefore, for obtaining a minorant of $M(\beta_r)$, because of the Mahler measure remains constant on the sequence $(\beta_r)_{r \geq 1}$, we replace $\text{dyg}(\beta_r)$ by $\text{dyg}(\beta_r^r)$ and use the lenticular minorant relative to $f_{\beta_r}(z) = f_{\beta_1}(z)$.

The fact is that $\beta_1 = 2 + \sqrt{3} \geq (1 + \sqrt{5})/2$ and that “dyg” has not yet be defined on $[(1 + \sqrt{5})/2, \infty)$. By convention, say $\text{dyg}(\gamma) = 1$ when $\gamma > (1 + \sqrt{5})/2$.

So $\text{dyg}(\beta_r)$ is replaced by $\text{dyg}(\beta_r^r) = 1$. The minimal polynomial of β_1 is relative to the Main Case. Now the set of lenticular roots of $f_{\beta_1}(z)$ is only $= \{1/\beta_1\}$, and $M(1/\beta_1) = \beta_1$. We have proved $M(\beta_r) \geq \beta_1$. The equality holds; in this case the lenticular minorant is exactly the Mahler measure $M(\beta_1) = \beta_1$.

The general strategy is the same. Let $\beta > 1$ be a reciprocal algebraic integer of dynamical degree $\text{dyg}(\beta) \geq 260$, for which the minimal polynomial

$$P_\beta(X) = 1 + \sum_{j=1}^{d-1} a_j X^j + X^d = \prod_{k=1}^d (X - \beta^{(k)}), \quad \beta^{(1)} = \beta,$$

is relative to the Main Case. Then, for any $q \geq 2$, we define the reciprocal algebraic integer $\beta_q > 1$, root of the reciprocal polynomial

$$P_{\beta_q}(X) = 1 + \sum_{j=1}^{d-1} a_j X^{qj} + X^{qd} = P_\beta(X^q).$$

For all $q \geq 2$, we have: $M(\beta_q) = M(\beta)$, considering the equations $X^q - \beta^{(k)}$, $|\beta^{(k)}| \geq 1$, and if $\gamma^q = \beta^{(k)}$, $(\gamma\varepsilon)^q = \beta^{(k)}$ for any ε , $\varepsilon^q = 1$. For any $q \geq 2$, there exists $n \geq 260$ such that

$$\theta_n^{-1} < \beta_q < \theta_{n-1}^{-1}.$$

Then to find a minorant of $M(\beta_q)$ we

$$\text{replace } n = \text{dyg}(\beta_q) \quad \text{by} \quad \text{dyg}(\beta)$$

and consider the set of the lenticular zeroes of $f_{\beta^q}(z) = f_{\beta}(z)$. Then we obtain the lenticular minorant

$$\mathfrak{S}_{lent}(\beta) \leq M(\beta_q).$$

This amounts to the case: the ‘‘Main Case relative to β and $P_{\beta}(X)$ ’’.

3.3.2. *An example outside the Problem of Lehmer.* Let $n \geq 2$ be an integer. Let β_n the unique root > 1 of the reciprocal integer polynomial

$$P_n(X) = X^{2n} - 2X^{2n-1} - 8X^{2n-1} - 2X + 1 \in \mathbb{Z}[X].$$

We show that the Conjecture of Lehmer is true for the family $\{\beta_n : n \geq 2\}$ (in the sense that there is a common lower bound > 1 for all the Mahler measures $M(\beta_n)$). This is due to Lemma 3.8 and to the fact that the house $|\overline{\beta_n}| = \beta_n$ tends to 2 as $n \rightarrow \infty$ ((iv) in Proposition 3.7). Since 2 does not belong to the interval $(1, 1.32\dots)$, a dynamical degree dyg of $|\overline{\beta_n}|$ cannot be defined. Therefore a lenticular minorant of the Mahler measure $M(\beta_n)$ has no sense. We cannot expect any help from any lenticular root of $f_{|\overline{\beta_n}|}(z)$ in this case, since lenticular roots do not exist.

This example does not constitute an attack of the problem of Lehmer, but, interestingly, when we consider the integer polynomial

$$P_n(-X) = X^{2n} + 2X^{2n-1} - 8X^{2n-1} + 2X + 1 \in \mathbb{Z}[X],$$

we find the sequence of reciprocal algebraic integers $(-\gamma_n)_{n \geq 2}$ in $(1, +\infty)$ which are roots of $P_n(-z)$ and which converges to 1, by (iv) in Proposition 3.7. For this family it is tempting to try to use the lenticular roots of $f_{-\gamma_n}(z)$ for n large enough, which exist, as n tends to infinity, to establish a minorant of $M(-\gamma_n) = M(\beta_n)$. It is hopeless. The conditions of identification of the lenticular roots of $f_{-\gamma_n}(z)$ with some zeroes of $P_n(-z)$ in $\{z \in \mathbb{C} : -\pi/18 \leq \arg z \leq +\pi/18, |z| < 1\}$, are not satisfied (cf [9]) since the Mahler measure $M(\beta_n) = M(\gamma_n)$ is far too large (cf Lemma 3.8 and Remark 3.9); in this angular sector, the only zero which is common to $f_{-\gamma_n}(z)$ and $P_n(-z)$ is $(-\gamma_n)^{-1}$ (cf (v) in Proposition 3.7).

Proposition 3.7. *Let $n \geq 2$. Then*

- (i) *the polynomial $P_n(X)$ is not of the form $Q(X^r)$ for some integer $r \geq 2$ and some integer polynomial Q ,*
- (ii) *P_n is irreducible in $\mathbb{Q}[X]$,*
- (iii) *P_n has no root on the unit circle,*
- (iv) *for n large enough, P_n admits four real roots $\gamma_n, \gamma_n^{-1}, \beta_n, \beta_n^{-1}$ which have the following properties:*

$$\gamma_n < -1 < \gamma_n^{-1} < 0 < \beta_n^{-1} < 1 < \beta_n,$$

$$\beta_n = |\overline{\gamma}| > 2 \text{ for any conjugate } \gamma \text{ of } \beta_n, \text{ in particular } \beta_n = |\overline{\gamma_n}| > 2,$$

$$\lim_{n \rightarrow +\infty} \gamma_n = -1, \quad \lim_{n \rightarrow +\infty} \beta_n = 2,$$

- (v) *for n large enough, P_n has no root in the annulus $\{z \in \mathbb{C} : |\gamma_n^{-1}| < |z| < 1\}$.*

Proof. (i) This is readily due to the fact that 2^n and $2^n - 1$ are coprime.

(ii) The shifted polynomial $P_n(X + 1)$ modulo 2 satisfies:

$$P_n(X + 1) \equiv (X + 1)^{2^n} + 1 \equiv X^{2^n} \pmod{2\mathbb{Z}[X]}.$$

Moreover $P_n(-1) = -2$ is not divisible by 2^2 . By Eisenstein's criterium, $P_n(X + 1)$ is irreducible in $\mathbb{Z}[X]$.

(iii) Assume z , $|z| = 1$, is a zero of P_n . We have:

$$|z^{2^n} + 2z^{2^n-1} + 2z + 1| \leq 1 + 2 + 2 + 1 = 6.$$

Then

$$|P_n(z)| \geq \left| \left| -8z^{2^{n-1}} \right| - |z^{2^n} + 2z^{2^n-1} + 2z + 1| \right| \geq 2.$$

Contradiction.

(iv) For n large enough, let us prove

$$-1 - \frac{1}{2^{n-1}} < \gamma_n < -1.$$

Let us abbreviate $m = 2^{n-1}$. Then

$$\begin{aligned} P_n\left(-1 - \frac{1}{m}\right) &= \left(-1 - \frac{1}{m}\right)^{2m} - 2\left(-1 - \frac{1}{m}\right)^{2m-1} - 8\left(-1 - \frac{1}{m}\right)^m - 2\left(-1 - \frac{1}{m}\right) + 1 \\ &= \left(1 + \frac{1}{m}\right)^{2m} + 2\left(1 + \frac{1}{m}\right)^{2m-1} - 8\left(1 + \frac{1}{m}\right)^m + 2\left(1 + \frac{1}{m}\right) + 1, \end{aligned}$$

which converges to $e^2 + 2e^2 - 8e + 3 > 0$ as $m \rightarrow \infty$. Since $P_n(-1) < 0$, we obtain the existence of a root in the interval $(-1 - \frac{1}{2^{n-1}}, -1)$. By Descartes's rule, the number of positive real roots of $P_n(-X)$ is equal to the number of sign changes of the polynomial, which is equal to 2. Since $P_n(-X)$ is reciprocal, there is only one root in $(-1 - \frac{1}{m}, -1)$; γ_n is the only root of $P_n(X)$ in this interval.

By Descartes's rule applied to $P_n(X)$, since P_n is reciprocal, there is only one root of P_n in $(1, \infty)$. For $n \geq 2$, observe that $P_n(-2) < 0$, $P_n(-3) > 0$, and then $-3 < \beta_n < -2$. For n large enough, let us prove that

$$\beta_n - 2 = 2^{-2^{n-1}+4}(1 + o(1)), \quad \text{as } n \rightarrow \infty.$$

Indeed, if we set $\beta_n = 2 + u$,

$$P_n(2 + u) = 0 = (2 + u)^{2m} - 2(2 + u)^{2m-1} - 8(2 + u)^m - 2(2 + u) + 1$$

gives, expanding at the first order,

$$u [2^{2m-1} - 8m2^{m-1} - 2 + \dots] = 3 + 8 \times 2^m,$$

hence the result.

Now, let ξ a primitive root of unity, $\xi^m = 1$. Let us prove that the other roots γ' of P_n , $\gamma' \neq \beta_n$, $|\gamma'| > 1$, are

$$(3.12) \quad \xi^j \gamma_n + v_j \quad \text{with } |v_j| \leq 2|\gamma_n|^{-2m+3}, \quad j = 1, 2, \dots, m-1.$$

Indeed, if we set $\gamma' = \xi^j \gamma_n + v_j$, we have

$$P_n(\gamma') = 0 = (\xi^j \gamma_n + v_j)^{2m} - 2(\xi^j \gamma_n + v_j)^{2m-1} - 8(\xi^j \gamma_n + v_j)^m - 2(\xi^j \gamma_n + v_j) + 1.$$

Consequently, for $j \neq 0$, expanding at the first order,

$$v_j [2 \times (\xi^j \gamma_n)^{2m-2} - 8m(\xi^j \gamma_n)^{m-1} + \dots] = 2(\xi^j - 1)\gamma_n,$$

hence the result. Since $|\gamma_n| < 1 + 1/2^{n-1}$ we deduce from (3.12), for any $\varepsilon > 0$, that

$$\lim_{n \rightarrow \infty} |\gamma_n|^{-2 \times 2^{n-1} + 3} \leq e^{-2} + \varepsilon.$$

Thus, since $e^{-2} < 0.5$, for any root γ' of P_n , for n large enough, we have: $|\gamma'| \leq |\gamma_n| + |v_j| \leq 1 + 1/m + e^{-2} + \varepsilon$, hence $|\gamma'| = \beta_n$.

(v) We use Rouché's Theorem and assume that m is large enough. The root $1/\gamma_n$ is the unique root in $[-1, 0)$ of $P_n(x)$, from (iii). For all $r \in [-1, 1/\gamma_n)$, we find $0 > P_n(r) = r^{2^n} - 2r^{2^{n-1}} - 8r^{2^{n-1}} - 2r + 1$, hence

$$|r|^{2m} + 2|r|^{2m-1} + 2|r| + 1 < 8|r|^m.$$

Therefore, for all $z \in \mathbb{C}$ with $|z| = |r|$, we find

$$|P_n(z) + 8z^m| = |z^{2m} - 2z^{2m-1} - 2z + 1| \leq |z|^{2m} + 2|z|^{2m-1} + 2|z| + 1 < |8z^m|.$$

By Rouché's Theorem $z \mapsto P_n(z)$ and $z \mapsto -8z^m$ have the same number of roots, counted with multiplicities, in $\{z \in \mathbb{C} : |z| < |r|\}$. This number equals m for all $r \in [-1, 1/\gamma_n)$. Therefore $P_n(z)$ has no roots with absolute value in $(|1/\gamma_n|, 1)$. \square

Lemma 3.8. *For n large enough, $M(\beta_n) = M(\gamma_n) \geq 4.6$.*

Proof. From Proposition 3.7, the number of conjugates of β_n of modulus $> -\gamma_n$ is exactly 2^{n-1} . Therefore, the Mahler measure of β_n satisfies

$$M(\beta_n) > \beta_n(-\gamma_n)^{2^{n-1}-1}.$$

A lower bound of $-\gamma_n$, as a function of n , is obtained as follows. We have: $P_n(-1) = -2$, $P_n(-1 - \frac{1}{m})$ which tends to $e^2 + 2e^2 - 8e + 3 > 0$ as $m \rightarrow \infty$. Observe that $P'_n(-1) = -2^n$ which tends to $-\infty$ as $n \rightarrow \infty$, and

$$P'_n(-1 - \frac{1}{m}) = 2m(-1 - \frac{1}{m})^{2m-1} - 2(2m-1)(-1 - \frac{1}{m})^{2m-2} - 8m(-1 - \frac{1}{m})^{m-1} - 2$$

which tends to $-\infty$ as $m \rightarrow \infty$, with $\frac{1}{m}P'_n(-1 - \frac{1}{m})$ converging to $-6e^2 + 8e < 0$. Since $P_n(-1 - 1/m) - P_n(\gamma_n) = P_n(-1 - 1/m) = (-1 - \frac{1}{m} - \gamma_n)P'_n(-1 - \frac{1}{m})$ at the first order, we deduce that $m(1 + \frac{1}{m} + \gamma_n)$ converges to $\frac{3e^2 - 8e + 3}{6e^2 - 8e}$ as m tends to $+\infty$. Therefore, when m is large, we have approximately

$$-\gamma_n = +1 + \frac{1}{m} \frac{3(e^2 - 1)}{6e^2 - 8e}.$$

We deduce, for any $\varepsilon > 0$,

$$M(P_n) = M(\beta_n) > 2 \times e^{\left(\frac{3(e^2-1)}{6e^2-8e}\right)} - \varepsilon, \approx 4.66\dots - \varepsilon.$$

as soon as n is large enough. \square

Remark 3.9. Lemma 3.8 shows that the family (β_n) is not concerned with the Problem of Lehmer. It is concerned with the problem of the topology and the search for limit points of the set of Mahler measures of algebraic numbers in the half-line $(1, \infty)$. Indeed, the Mahler measure $M(\beta_n)$ (calculated by Graeffe's method) tends to a limit, as $n \rightarrow \infty$.

n	$M(\beta_n) = M(-\gamma_n)$
2	7.095126...
3	7.273581...
4	7.275408...
5	7.275409...
6	7.275409...
...	...

Whether this limit 7.275409... is algebraic or transcendental is not known. The topology of the set of Mahler measures of algebraic integers, and its limit points, is a general problem [50]. The characterization of the small limit points has been tackled by Boyd and Mossinghoff [3] and Deninger [6] for instance.

3.4. Algebraic degree and dynamical degree. By Northcott Theorem, the (algebraic) degree $\deg(\beta)$ of the reciprocal algebraic integer $\beta > 1$ tends to infinity, when β tends to one. In this paragraph we still assume the proviso on β , stated at the beginning of section 3, which is that the integer $r = r(\beta) = 1$ is the largest one such that

$$P_\beta(X) = \widetilde{P}_\beta(X^r)$$

holds for some \mathbb{Z} -minimal integer polynomial $\widetilde{P}_\beta(X)$, discarding the other cases which are not relevant. The following Theorem 3.10 makes precise the relations between the dynamical degree $n = \text{dyg}(\beta)$ of β , coming from the inequalities $\theta_n^{-1} \leq \beta < \theta_{n-1}^{-1}$, and $\deg(\beta)$.

First, by Selmer's Theorem 3.2, we have, for $\beta = \theta_n^{-1}$ and $n \geq 2$:

$$(3.13) \quad \text{dyg}(\theta_n^{-1}) = n = \begin{cases} \deg(\theta_n^{-1}) & \text{if } n \not\equiv 5 \pmod{6}, \\ \deg(\theta_n^{-1}) + 2 & \text{if } n \equiv 5 \pmod{6}, \end{cases}$$

Secondly, from the Appendix in [52], if β is a any real number in $(\theta_n^{-1}, \theta_{n-1}^{-1})$, $n \geq 2$, then the asymptotic expansion of the dynamical degree $\text{dyg}(\beta) = \text{dyg}(\theta_n^{-1}) = n$ of β is:

$$(3.14) \quad \text{dyg}(\beta) = -\frac{\text{Log}(\beta - 1)}{\beta - 1} \left[1 + O\left(\left(\frac{\text{Log}(-\text{Log}(\beta - 1))}{\text{Log}(\beta - 1)} \right)^2 \right) \right],$$

with the constant 1 in the Big O. It is in particular the case when β is a reciprocal algebraic integer; the asymptotic expansion of $\text{dyg}(\beta)$ is a locally constant function of β .

Then we have the asymptotic minorant of the degree of β , as:

Theorem 3.10. *If $\beta \in (\theta_n^{-1}, \theta_{n-1}^{-1})$, $n \geq 260$, is a reciprocal algebraic integer of degree $\deg(\beta)$, with $M(\beta) < 1.176280\dots$, then*

$$(3.15) \quad C_1 \text{dyg}(\beta) + C_2 \leq \deg(\beta).$$

with

$$C_1 := \frac{2 \arcsin\left(\frac{\kappa}{2}\right)}{\pi}, \quad C_2 := \frac{2\kappa \operatorname{Log} \kappa}{\pi \sqrt{4 - \kappa^2}}.$$

Proof. The lenticulus of lenticular conjugates relative to β is:

$$\mathcal{L}_\beta = \{\overline{\omega_{J_n, n}}, \dots, \overline{\omega_{1, n}}, \beta^{-1}, \omega_{1, n}, \dots, \omega_{J_n, n}\}.$$

The number of elements in the lenticulus \mathcal{L}_β is $1 + 2J_n$, with $n = \operatorname{dyg}(\beta)$. The total number of conjugates of β is the degree $\operatorname{deg}(\beta)$ of the minimal polynomial P_β . By Proposition 2.5,

$$1 + 2J_n = \frac{2n}{\pi} \left(\arcsin\left(\frac{\kappa}{2}\right) \right) + \left(\frac{2\kappa \operatorname{Log} \kappa}{\pi \sqrt{4 - \kappa^2}} \right) + \left(1 + \frac{1}{n} O\left(\left(\frac{\operatorname{Log} \operatorname{Log} n}{\operatorname{Log} n}\right)^2\right) \right).$$

The inequality (3.15) follows. □

4. PROOF OF THEOREM 1.1

Let $\beta \neq 0$ be a reciprocal algebraic integer which is not a root of unity, such that $\operatorname{dyg}(\beta) \geq \eta$ with $\eta \geq 260$. Since $M(\beta) = M(\beta^{-1})$ there are three cases to be considered:

- (i) the house of β satisfies $|\overline{\beta}| \geq \theta_5^{-1}$,
- (ii) the dynamical degree of β satisfies: $6 \leq \operatorname{dyg}(\beta) < \eta$, with $M(\beta) < 1.176280\dots$,
- (iii) the dynamical degree of β satisfies: $\operatorname{dyg}(\beta) \geq \eta$, with $M(\beta) < 1.176280\dots$

In the first case, $M(\beta) \geq \theta_5^{-1} > \theta_{260}^{-1} \geq \theta_\eta^{-1} > 1$. In the second case, $M(\beta) \geq \theta_\eta^{-1}$. In the third case, the Dobrowolski-type inequality in Theorem (2.8), Proposition 3.1, and (3.3), give the following lower bound of the Mahler measure

$$\begin{aligned} M(\beta) &\geq C - C \frac{\arcsin(\kappa/2)}{\pi \operatorname{Log}(\operatorname{dyg}(\beta))} \\ &\geq C - C \frac{\arcsin(\kappa/2)}{\pi \operatorname{Log}(\eta)} \geq C - C \frac{\arcsin(\kappa/2)}{\pi \operatorname{Log}(260)}, \approx 1.14\dots \end{aligned}$$

This lower bound is numerically greater than $\theta_{260}^{-1} = 1.01\dots$, itself greater than θ_η^{-1} . Therefore, in any case, the universal lower bound θ_η^{-1} of $M(\beta)$ holds true. We deduce the theorem.

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