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Conceptual understanding of solutions to differential equations: How do assessment tasks promote it?

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Differential equations (DEs) are used for modelling real-life phenomena in economics, engineering, natural and social sciences. Courses in DEs constitute an important part of engineering curricula at many universities but educational research on teaching DEs is rather scarce. Recent empirical studies point towards students' difficulties with the conceptual understanding of DEs and solutions to DEs. We use Conceptual Change Theory to analyse five tasks suggested in the literature for assessing students conceptual understanding of solutions to DEs arguing that only one problem contributes to the goal. The task on the Existence and Uniqueness Theorem for first-order linear DEs designed by the authors for engineering students revealed gaps in their conceptual understanding of the general solution to a DE. Our analysis emphases the role of rigour and non-standard tasks for students' conceptual thinking.

Keywords: Differential equations, general solution, particular solutions, scientific concept, conceptual understanding.

Abstract mathematical reasoning in engineering education.

The prominent role of rigour and abstraction in mathematics courses for engineering students is highly debated in the literature. Speaking at the thematic Conference The Teaching of Mathematics for Engineers, Bickley emphasized that "abstraction and generalization are the life and the soul of mathematics, and these should be made clear, along with the ubiquity of mathematical concepts and techniques" (1964, p. 382). Nearly sixty years later, Devlin confirmed that "the main benefit they [software engineers] got from the mathematics they learned in academia was the experience of rigorous reasoning with purely abstract objects and structures. Moreover, mathematics was the only subject that gave them that experience" (2001, p. 22). Flegg et al. reported that most engineering students either agreed or strongly agreed that mathematics is relevant to their future career and study with the highest ratings for "the ways of thinking 82%, ideas 79%, mathematical skills 76%, communicating using mathematical arguments 94%" (2012, p. 723). Furthermore, "68% of students thought that the rigorous aspects of mathematics would be important to them in the future" (Flegg et al., 2012, p. 729).

Logical reasoning, abstraction, rigour are grounding components of advanced mathematical thinking understood as "thinking that requires deductive and rigorous reasoning about mathematical notions that are not entirely accessible to us through our five senses" (Edwards et al., 2005, pp. 17–18). Development of advanced mathematical thinking is a complex process which "produces a wide variety of cognitive conflict which can act as an obstacle to learning" (Tall, 1992, p. 495). Attempts to reduce the cognitive demand can become adverse because "oversimplified environments designed to protect students from confusion may only serve to provide implicit regularities that students

abstract, causing serious conflict at a later stage" (Tall, 1992, p. 508). The importance of abstract mathematical reasoning is acknowledged by students but "very few 'becoming engineers' realized how mathematically demanding their courses would prove to be, and evidently some would not have chosen the course if they had known!" (Harris et al., 2015, p. 333). Many engineering students choose an instrumental approach to learning "marked by a motivation to pass exams in order to obtain a degree (and hence a job), rather than being driven by an interest in learning" (Ditcher 2001, p. 25).

Possible ways for improving the situation have been discussed in the literature. Booth (2008, p. 383) suggested a qualitative shift "from teaching as an act performed by the teacher to teaching as creating conditions for students to learn, with prime focus on learning rather than teaching." To this end, instructors should develop "stimulating and interesting tasks, activities, and materials, including some novelty and variety in tasks and activities" (Pintrich, 2003, p. 672). Good mathematics tasks can be created by employing the inquiry by design technique (Richards, 1991) where teachers "design tasks and projects that stimulate students to ask questions, pose problems, and set goals" and students "must learn to inquire systematically" and "must actively construct their own knowledge" (p. 38).

Educational research on differential equations.

Teaching and learning DEs at the university level is a relatively new area of educational research, and "we need to explore the variety of ways in which content, instruction, and technology can be profitably coordinated to promote student learning" (Rasmussen, 2001, p. 84). A recent survey reported "fewer than two dozen empirical studies published in top journals" which is surprising "given the centrality of differential equations (DEs) in the undergraduate curriculum, as well as the move away from a "cookbook" course to one that emphasizes modelling, qualitative, graphical and numerical methods of analysis" (Rasmussen & Wawro, 2017, p. 555).

Modern approaches promoting inquiry and active learning challenge traditional views on DEs employing non-standard problems and geometric ideas supported by active use of graphical and numerical methods (Marrongelle & Rasmussen, 2008; Rasmussen, 2001; Rogovchenko et al., 2020; Treffert et al., 2018). Understanding DEs is important since they model different real-life phenomena, but even relatively simple DEs "seem to be a stumbling block for many students" (Sazhin, 1998, p. 147) who tend to concentrate attention on specific solution techniques and "often fail to relate them to other concepts or ideas" (Camacho-Machín et al., 2012, p. 76). In fact, "algebraic solutions of DEs can be found even without a deep understanding and conceptualization of DEs, which is why students do not feel any need to understand DEs and related concepts" (Arslan, 2010, p. 887). The three most striking difficulties experienced by students in DEs courses are related to the concepts of a DE, the general and particular solutions (Arslan, 2010). For instance, "the general solution was considered as if it consisted of a unique solution" (Arslan, 2010, p. 880), students "failed to report general solutions" (Camacho-Machín et al., 2012, p. 74), or "made little or no attempt to place the solution in context, be it a solution to an equation or a DE" (Raychaudhuri, 2014, p. 48).

The purpose of this exploratory research study is to analyse several tasks used in mathematics education literature for testing students' conceptual understanding of solutions to DEs. Advanced search in the ISI Web of Science® database with the search string "differential equations AND solution AND conceptual understanding" returned thirty results. Restriction to items in categories

"Education Educational Research OR Education Scientific Disciplines OR Social Sciences Interdisciplinary" reduced the list to eight items. The full text analysis of accessible items and analysis of abstracts for those non-accessible reduced the list to only one paper by Keene et al. (2011) which is consistent with the scarcity of research on teaching DEs (Rasmussen & Wawro, 2017). In addition, we also analyse several tasks from the papers by Camacho-Machín et al. (2012) and Arslan (2010) not included in the ISI Web of Science® database. The research question we address in this paper is: *How well do the tasks designed for assessing students' conceptual understanding of the general and particular solutions to differential equations (DEs) serve the intended goal?*

Tasks assessing conceptual understanding of solutions to differential equations.

In mathematics, a new knowledge may be in conflict with that already acquired. Mathematical concepts with the same name may be defined differently in different areas of mathematics; students need to know which definition they have to use. This ambiguity applies to the concept of solution during its evolution from a number as a solution to an algebraic equation to a function as a solution to a DE. It is known that the "switch from conceptualizing solutions as numbers to conceptualizing solutions as functions is nontrivial for students" (Rasmussen, 2001, p. 67). Students have to form a good conceptual understanding of a DE and different types of solutions before they start applying DEs for modelling. This goal is better achieved by "stretching the students' minds to the utmost limits of cultural breadth of which they are capable, and by pitching the material at a level that is just a little higher than they can reach" (Rota, 1997, p. 9). The reasons for students' confusion with the concept of solution to a DE are well understood:

> Consider the numerous words that are used as prefixes to the word solution – the general solution, particular solution, unique solution, superposed solution, fundamental solution etc. According to the definition – solution to a DE is a differentiable function that satisfies the DE in some domain, any of the aforementioned would be a solution to the DE. But with the defining process pushed to the background, each of them may stand alone in students' minds. (Raychaudhuri, 2008, p. 175)

Assessment is an important pedagogic instrument influencing both students' motivation and learning outcomes. In fact, "one way to encourage the use of deep learning approaches is to set assessment tasks which test a student's understanding, rather than tasks which require memorizing and reproducing knowledge or processes" (Ditcher, 2001, p. 27). To address students' difficulties with the conceptual understanding of DEs and their solutions, Arslan (2010), Camacho-Machín et al. (2012), and Keene et al. (2011) designed the tasks presented in Figure 1. We analyse how these tasks contribute to the goal drawing on the Conceptual Change Theory (CCT) which proved to be useful for teaching concepts that are difficult for students (Vosniadou, 2008). CCT is based on Piaget's notions of assimilation and accommodation and Thomas Kuhn's vision of scientific revolution, in particular, on a concept of a "shift of paradigm".

Solutions to a DE in Task A are referred to in several ways: (a) all; (b) all except equilibrium; (c) equilibrium; (d) extraneous. The authors claim that the "question invokes conceptual thinking about DEs that differs from mere mastery of the separation of variables technique" (Keene et al., 2011, p. 4). They expected students to recognise that equilibria for autonomous DEs are defined by $y' = 0$ and notice that the process of solving separable DEs may lead to the loss of solutions (as it happens

in this case with the equilibrium solution $y = -3$). Conceptual thinking about DEs in this task requires understanding of several important notions: (a) a *solution* as a continuously differentiable function satisfying a DE (this is usually done by a direct verification or integration of a DE), (b) *the general solution* including *all possible solutions*; it is usually obtained by the integration of a DE which yields a family of solutions that depend on one or more parameters; and (c) an *equilibrium solution* obtained for autonomous equations by finding the points where the slope is zero; it may not be included in the general solution as in Task A. The term 'general solution' is not used in the problem and its relationship with the notion '*all solutions*' is unclear. We tend to believe that (a) means the general solution found by Jensen and the equilibrium solution $y = -3$; (b) is the general solution; (c) an equilibrium solution; and (d) is not relevant for DEs since it is used for solving equations with radicals where squaring of both sides leads to extraneous solutions. CCT suggests that avoidance of the notion 'general solution' in (b) and reference to previous experience with algebraic equations in (d) might confuse students complicating their understanding of DEs. In fact, Keene et al. (2011) reported that this task was the hardest for students, but students' conceptual thinking about DEs was not analysed in detail.

Task A. Jensen solves the separable differential equation $\mathrm{d}y/\mathrm{d}t = (y+3)^2$ and correctly gets $y = -1/(t+C) - 3$. Jensen's use of separation of variables gives you:

(a) All solutions.
(b) All solutions except equilibrium solutions.
(c) Equilibrium solutions only.
(d) Extraneous solutions.

Task B. Say whether the following statements are true or false and give reasons for your answer:

(a) The function $y = e^{\int e^{t^2}\mathrm{d}t}$ is a solution for the differential equation $\mathrm{d}y/\mathrm{d}t = 4e^{t^2}y$.

(b) The functions $y = f(x)$ which satisfy $-x^3 + 3y - y^3 = C$ are solutions for the differential equation $\mathrm{d}y/\mathrm{d}x = x^2/(1-y^2)$.

Task C. (P1): Is the function $xy^2 + x^3/3 = 1$ a particular solution to the differential equation $(x^2+y^2)\mathrm{d}x + 2xy\mathrm{d}y = 0$?

(C2): Could a given function $f(x,y)$ be a particular solution of different DEs? Justify your answer.

Figure 1: Tasks for the assessment of conceptual understanding of solutions to DEs: A (Keene et al., 2011, p. 3); B (Camacho-Machín et al., 2012, p. 78); C (Arslan, 2010, pp. 877, 883)

Task B, part (a), requires a straightforward differentiation of a given function or a direct integration of a DE; both procedures are important for learning DEs but contribute very little to their conceptual understanding. In part (b), the general solution is provided but not named (as in Task A); instead, students have to verify that a function $y = f(x)$, defined implicitly by an algebraic equation, satisfies the given DE. As in part (a), one uses the standard Calculus procedure of implicit differentiation or a direct integration of the exact DE employing relevant techniques from Calculus or DEs. Camacho-Machín et al. (2012) reported that students "encountered difficulties in identifying and accessing proper knowledge and strategies to operate that knowledge" (p. 74). Clearly, the difficulties remain

at the procedural level. There is no need for students to invoke higher level thinking; one can simply interpret Task B as a differentiation task. According to the CCT, no conceptual change occurs.

Problem (P1) in Task C is similar to Task B (b), but one deals with one solution to an exact DE defined implicitly by an algebraic equation rather than with a one-parameter family of solutions. Thus, there is no conceptual change here as well. Problem (C2) requires a deeper reflection about the notions of a DE, the general and particular solutions and may lead to the conceptual change. This task revealed several students' misconceptions which signal the lack of conceptual understanding. For instance, students believed that DEs have infinitely many solutions and thus two different DEs may and even should have common solutions; they also thought that the choice of the solution method or the transformation of the equation may affect particular solutions (Arslan, 2010, pp. 884-885).

An assessment task for mechatronics students.

Now we analyse students' work on one of the problems designed by the first author to check students conceptual understanding of the Existence and Uniqueness Theorems (EUTs) for first order DEs. Students first worked on the problems individually in class and at home, discussed solutions in small groups, audio-recorded the discussion and presented solutions to their peers. Final individual reports were marked and contributed to the final grade in the course because the "assessment needs to be seen as part of the teaching of a course, rather than an add-on" (Ditcher, 2001, p. 27). We analysed transcripts of audio recordings of small group work and three written scripts; more details can be found in Rogovchenko et al. (2020) and Treffert et al. (2018).

Problem P. (a) Verify that $y(x) = 2/x - C_1/x^2$ is the general solution of a differential equation $x^2y' + 2xy = 2$.

(b) Show that both initial conditions $y(1) = 1$ and $y(-1) = -3$ result in an identical particular solution. Does this violate the Existence and Uniqueness Theorem? Explain your answer.

Figure 2: Problem in the assessment on Existence and Uniqueness Theorems for DEs

Part (a) was intended to facilitate the work on part (b). Students employed two methods: (i) solving a linear DE using an integrating factor and (ii) substituting a function and its derivative into a DE. For first order linear DEs, (i) provides the general solution; students had to explain this after solving the DE. Computation in (ii) is easier but students had to explain that the given DE has no solutions other than those defined by the function given in (a). The analysis of written scripts and recorded small group discussions revealed that, irrelevant to the solution method used, most students did not verify that the given function is the general solution and skipped relevant explanations. Engineering students in our study were introduced to first order DEs in a Calculus course and were familiar with the basic concepts. However, the focus was always on the verification that a function is a solution to the given DE; these problems were just differentiation tasks disguised as DE problems. The latter could be one of possible explanations for students' predominant choice of the method (ii).

Reflecting further about the reasons for this negligence, we turned our attention to the course textbook (Boyce & DiPrima, 2012). The general solution was first introduced in Example 1 (p. 11) but was not termed *general* yet; it was defined later as an expression that "contains all possible solutions" followed by its geometric interpretation as "an infinite family of curves called integral curves. Each

integral curve is associated with a particular value of c and is the graph of the solution corresponding to that value of c" (p. 12). Example 1 and the following comments loosely defined a particular solution, but a better explanation was given in relation to the existence and uniqueness of solutions.

> If we assume that a given differential equation has at least one solution, then we may need to consider how many solutions it has, and what additional conditions must be specified to single out a particular solution. This is the question of uniqueness. In general, solutions of differential equations contain one or more arbitrary constants of integration. (p. 23)

We tend to believe that an inconsistent use of the terminology is one of the reasons for why students focused attention on the term "solution" and pay much less attention, if at all, to its combination with the important terms "particular" and "general" – already on page 13, right after defining the general solution, the authors refer to it simply omitting the word "*general*." Apparently, all information students needed for solving Problem P correctly is collected in a short but informative summary.

> Assuming that the coefficients are continuous, there is a general solution, containing an arbitrary constant, that includes all solutions of the differential equation. … The possible points of discontinuity, or singularities, of the solution can be identified … merely by finding the points of discontinuity of the coefficients. Thus, if the coefficients are continuous for all t, then the solution also exists and is differentiable for all t. (p. 75–76)

Unfortunately, our students did not distinguish between the general and particular solutions regardless of their previous knowledge from the mathematics courses and the information in the textbook. We believe that the pedagogical approach of the textbook where the general solution to the linear DEs is defined through a sequence of examples gradually becoming more complicated did not achieve the goal; the use of this notion only for linear DEs does not emphasise its importance and adds even more confusion. Students could not use properly the information scattered through the course textbook and need explicit and more precise definitions of solutions to a DE. This might be a problem not only for our students since the textbook by Boyce & DiPrima (2012) is used in many universities worldwide.

Conclusions.

The analysis of the tasks designed for assessing students' conceptual understanding of DEs, the general and particular solutions reveals many deficiencies; only one out of five problems contributes to the goal and may eventually lead to the conceptual change. We argue that the understanding of the multifaceted concept of solution to a DE, in all its forms, can be complete only when it is firmly set in accordance with the views of the mathematics community, that is, when it forms as a *scientific concept*. Attention should be paid to the rigorous explanation of all relevant notions and results, including existence and uniqueness theorems which introduce students to situations where multiple solutions are possible. A new knowledge and a new way of thinking should settle with those previously acquired by students through the use of purposedly designed challenging tasks like Problem (C2) that support conceptual change and are not only focusing on routine computation.

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