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A DOBROWOLSKI-TYPE INEQUALITY FOR THE POLES OF THE DYNAMICAL ZETA FUNCTION OF THE BETA-SHIFT

JEAN-LOUIS VERGER-GAUGRY

ABSTRACT. The poles of the dynamical zeta function of the β -shift in the open unit disk in \mathbb{C} are investigated, when $\beta > 1$ is a real number close to 1 and tends to 1. We show that to β is associated an integer, called dynamical degree of β , and that two types of poles can be characterized as a function of $\text{dyg}(\beta)$: the lenticular poles and the other poles in a narrow annular neighbourhood of the unit circle. From the lenticulus of lenticular poles we show that we can construct a lenticular measure associated to it. Its definition is mimicked on that of the Mahler measure, and both are related. We prove that this lenticular measure admits a lower bound, when $\text{dyg}(\beta)$ is large enough, as a function of $\text{dyg}(\beta)$, which is an analogue of the inequality of Dobrowolski relative to the Mahler measure of algebraic integers. This Dobrowolski-type inequality generalizes a previous one obtained by the author on the Mahler measure of the trinomials $-1 + x + x^n$ for n large enough.

Keywords: Dynamical zeta function, Rényi-Parry β -shift, beta transformation, Mahler measure, Dobrowolski-type inequality, lenticular measure, poles, minoration.

2020 Mathematics Subject Classification: 37C30, 37B05, 30C15, 37F05, 30B10, 11R06

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1. INTRODUCTION

Let $\beta > 1$ be a real number. The β -transformation is the map $[0, 1] \rightarrow [0, 1], x \mapsto \{\beta x\}$, where $\{\cdot\}$ denotes the fractional part, and $\lfloor y \rfloor = y - \{y\}$ the integer part of y . The β -transformation is denoted by T_β , and its iterates by: $T_\beta^{(0)} = \text{Id}, T_\beta^{(1)} = T_\beta, T_\beta^{(j)} = T_\beta(T_\beta^{(j-1)})$, for $j \geq 1$. The dynamical system $([0, 1], \mathcal{A}, T_\beta, \mu_\beta)$, where \mathcal{A} is the σ -algebra of borelians and μ_β the invariant measure [7], was introduced by Rényi [17] and initially studied in ergodic theory by Parry [12] [8]. We call *dynamical zeta function of the β -shift*, denoted by $\zeta_\beta(z)$, the dynamical zeta function of this dynamical system [1] [13] [15] [16]. After Takahashi [24] [25] [26] and Ito and Takahashi [9] [10], this function takes the form

$$(1.1) \quad \zeta_\beta(z) = \frac{1 - z^N}{(1 - \beta z) \left(\sum_{n=0}^{\infty} T_\beta^n(1) z^n \right)} = - \frac{1 - z^N}{-1 + t_1 z + t_2 z^2 + t_3 z^3 + \dots}$$

where N , which depends upon β , is the minimal positive integer such that $T_\beta^N(1) = 0$; in the case where $T_\beta^j(1) \neq 0$ for all $j \geq 1$, “ z^N ” has to be replaced by “0”. In (1.1) the digits t_i are defined from β by: $t_i := \lfloor \beta T_\beta^{(i-1)}(1) \rfloor, i \geq 1$, and depend upon β . All the t_i s are integers in the alphabet $\mathcal{A}_\beta = \{0, 1, \dots, \lceil \beta - 1 \rceil\}$ where $\lceil \beta - 1 \rceil$ is the upper integer part of $\beta - 1$; if β is not an integer, then $\lceil \beta - 1 \rceil = \lfloor \beta \rfloor$. The sequence $(t_i)_{i \geq 1}$ is such that $0.t_1 t_2 t_3 \dots$ is the Rényi β -expansion of unity $d_\beta(1)$ ([11], Chapt 7), i.e. $1 = \sum_{i \geq 1} t_i \beta^{-i}$. Multiplying this identity by β uniquely characterizes β as a Laurent series of β^{-1} from (t_i) , as:

$$(1.2) \quad \beta = t_1 + \frac{t_2}{\beta} + \frac{t_3}{\beta^2} + \dots$$

The domain of definition of $\zeta_\beta(z)$, as a meromorphic function, contains the open unit disk $D(0, 1) = \{z : |z| < 1\}$. Indeed, in all the cases where $\beta > 1$ is such that the power series $-1 + t_1 z + t_2 z^2 + t_3 z^3 + \dots$ at the denominator of (1.1) has infinitely many non-zero terms, Hadamard’s formula gives $R = 1$ as radius of convergence. For the other cases, the domain of definition of $\zeta_\beta(z)$ as a meromorphic function is \mathbb{C} .

From (1.2) β^{-1} is always a simple pole of $\zeta_\beta(z)$ in $D(0, 1)$. The function $\zeta_\beta(z)$ is holomorphic on $\{z \in \mathbb{C} : |z| < 1/\beta\}$ and is extended meromorphically to $D(0, 1)$. What are the other poles of $\zeta_\beta(z)$ in $D(0, 1)$? Could it be possible to classify them into classes and, on each class, that a global measure can be defined and have interesting bounded properties associated with β itself? Let us ask these two questions only when β lies in the neighbourhood of 1, precisely when $\beta \in (1, (1 + \sqrt{5})/2)$; the alphabet \mathcal{A}_β being common, equal to $\{0, 1\}$, for all such β s.

In this paper, for $1 < \beta \leq (1 + \sqrt{5})/2$ any real number, we show two results; first that the set of the poles of $\zeta_\beta(z)$ in $D(0, 1)$ can be partitioned into two parts: the subset of the *lenticular poles*, denoted by \mathcal{L}_β , all simple poles, and the subset of the *other poles*. The definition of a lenticular pole is given in Definition 2.12. The terminology “lenticular pole” comes from the particular case where β belongs to $\cup_{n \geq 2} \{\theta_n^{-1}\}$, where θ_n is the unique zero of $-1 + z + z^n$ in $(0, 1)$ (cf Appendix and [27]). The sequence $(\theta_n^{-1})_{n \geq 2}$ is strictly decreasing and tends to 1 if n tends to infinity: $1 < \dots < \theta_{n+1}^{-1} < \theta_n^{-1} < \dots < \theta_2^{-1} =$

$(1 + \sqrt{5})/2$. Indeed, in this particular case, if $\beta = \theta_n^{-1}$ for some $n \geq 2$, the set of poles of $\zeta_\beta(z) = -(1 - z^n)/(-1 + z + z^n)$ inside $D(0, 1)$ forms a lenticulus of simple poles in $\{z \in D(0, 1) : |\arg z| < \pi/3\}$; all the poles of modulus < 1 are lenticular and there is no “other pole”. In the general case, in Theorem 2.18 we show that “other poles” exist, in a neighbourhood of the unit circle, and that, if $\theta_n^{-1} < \beta < \theta_{n-1}^{-1}$ for some $n \geq 3$, the lenticulus \mathcal{L}_β of lenticular poles can be defined “optimally” with respect to $\mathcal{L}_{\theta_n^{-1}}$, using the method of Rouché circles. The notion of “optimality” is made precise in Theorem 2.4. Lenticuli of poles are symmetrical with respect to the real axis. Examples of lenticuli of lenticular poles are given in [5].

Second, if $\theta_n^{-1} < \beta < \theta_{n-1}^{-1}$ for some $n \geq 3$, and $\mathcal{L}_\beta = \{\overline{\omega_{J_n, n}}, \dots, \overline{\omega_{1, n}}, \beta^{-1}, \omega_{1, n}, \dots, \omega_{J_n, n}\}$ denotes the set of lenticular poles of $\zeta_\beta(z)$, all depending upon β , then we attribute to \mathcal{L}_β a measure, that we call *lenticular measure* of β , by the expression

$$(1.3) \quad \mathfrak{S}_{lent}(\beta) := \prod_{\omega \in \mathcal{L}_\beta} |\omega|^{-1} = \beta \prod_{j=1}^{J_n} |\omega_{j, n}|^{-2}.$$

By construction, $\mathfrak{S}_{lent}(\beta) \geq 1$. If $\beta = \theta_n^{-1}$ for some $n \geq 2$, then we have the identification with the Mahler measure of θ_n^{-1} : $\mathfrak{S}_{lent}(\theta_n^{-1}) = M(\theta_n^{-1})$ (cf Appendix Section 5). Recall that the Mahler measure of a nonzero algebraic number β , of minimal polynomial $P_\beta(X) = a_0 X^m + a_1 X^{m-1} + \dots + a_m = a_0 \prod_i (X - \alpha^{(i)}) \in \mathbb{Z}[X]$, is $M(\beta) = |a_0| \prod_i \max\{1, |\alpha^{(i)}|\} =: M(P_\beta)$ [21] [27]. The definition of $\mathfrak{S}_{lent}(\beta)$ is mimicked on the definition of the Mahler measure of an algebraic integer. It is an analogue of the Mahler measure. Hence, the properties of $\mathfrak{S}_{lent}(\beta)$, $\theta_n^{-1} < \beta < \theta_{n-1}^{-1}$, are expected to be closely related to those of the Mahler measure of β if β is an algebraic integer. It will be the purpose of a future work to clarify this point. In this note we are just interested in (i) proving the continuity of $\mathfrak{S}_{lent}(\beta)$ (Theorem 1.1), (ii) obtaining a lower bound of $\liminf_{\beta \rightarrow 1} \mathfrak{S}_{lent}(\beta)$ by comparison with $\lim_{n \rightarrow \infty} M(\theta_n^{-1})$, and n -depending lower bounds of $\mathfrak{S}_{lent}(\beta)$ when $\beta \in (\theta_n^{-1}, \theta_{n-1}^{-1})$ and n large enough (Theorem 1.2). The method used is an extension of the one developed in [27]: in there the lenticular poles are developed as asymptotic expansions “à la Poincaré” as a function of n , which leads to a Dobrowolski-type inequality of the Mahler measure, better than the classical Dobrowolski’s inequality in [4].

Let us recall more precisely these inequalities from [4] and [27]. In 1979, Dobrowolski [4], using an auxiliary function, obtained the asymptotic minoration, with $n = \deg(\alpha)$ the degree of the nonzero algebraic integer α , which is not a root of unity,

$$(1.4) \quad M(\alpha) > 1 + (1 - \varepsilon) \left(\frac{\text{Log Log } n}{\text{Log } n} \right)^3, \quad n > n_0(\varepsilon),$$

with $1 - \varepsilon$ replaced by $1/1200$ for $n \geq 2$, for an effective version of the minoration. The minoration (1.4) is general and admits a better lower bound, in a similar formulation, when α only runs over the set $\{\theta_n^{-1} : n \geq 2\}$. Indeed, in this case, it is proved in [27] that

$$(1.5) \quad M(\theta_n^{-1}) > \Lambda - \frac{\Lambda}{6} \left(\frac{1}{\text{Log } n} \right), \quad n \geq 2,$$

holds with the following constant

$$(1.6) \quad \Lambda := \exp\left(\frac{3\sqrt{3}}{4\pi}L(2, \chi_3)\right) = \exp\left(\frac{-1}{\pi} \int_0^{\pi/3} \text{Log}\left(2 \sin\left(\frac{x}{2}\right)\right) dx\right) = 1.38135\dots,$$

higher than $1.1541\dots$, and $L(s, \chi_3) := \sum_{m \geq 1} \frac{\chi_3(m)}{m^s}$ the Dirichlet L-series for the character χ_3 , with χ_3 the uniquely specified odd character of conductor 3 ($\chi_3(m) = 0, 1$ or -1 according to whether $m \equiv 0, 1$ or $2 \pmod{3}$), equivalently $\chi_3(m) = \left(\frac{m}{3}\right)$ the Jacobi symbol). From Selmer [19], in (1.5), we have: $n = \deg(\theta_n^{-1})$ if $n \not\equiv 5 \pmod{6}$, and $n = \deg(\theta_n^{-1}) + 2$ if $n \equiv 5 \pmod{6}$.

Denote by $a_{\max} = 5.87433\dots$ the abscissa of the maximum of the function $a \rightarrow (1 - \exp(\frac{-\pi}{a})) / (2 \exp(\frac{\pi}{a}) - 1)$ on $(0, \infty)$ (Figure 1). Let $\kappa := 0.171573\dots$ be the value of its maximum, at $a = a_{\max}$. From a numerical viewpoint we have: $2 \arcsin(\kappa/2) = 0.171784\dots$

Denote

$$(1.7) \quad C := \exp\left(\frac{-1}{\pi} \int_0^{2 \arcsin(\frac{\kappa}{2})} \text{Log}\left[\frac{1 + 2 \sin(\frac{x}{2}) - \sqrt{1 - 12 \sin(\frac{x}{2}) + 4(\sin(\frac{x}{2}))^2}}{4}\right] dx\right) = 1.15411\dots$$

The objective of this paper is to prove the two following theorems.

Theorem 1.1. *For all $n \geq 260$, the map*

$$(1.8) \quad \mathfrak{S}_{lent} : \beta \rightarrow \mathfrak{S}_{lent}(\beta)$$

is continuous on the interval $(\theta_n^{-1}, \theta_{n-1}^{-1})$.

Theorem 1.2. *There exists an integer $\eta \geq 260$ such that the following inequality holds:*

$$(1.9) \quad \mathfrak{S}_{lent}(\beta) \geq C - C \frac{\arcsin(\kappa/2)}{\pi} \frac{1}{\text{Log}(n)}, \quad \text{for all } n \geq \eta$$

and any $\beta \in (\theta_n^{-1}, \theta_{n-1}^{-1})$.

In (1.5) or (1.9) the constant in the minorant is not any more 1 as in (1.4) but $\Lambda = 1.38135\dots$ and $C = 1.15411\dots$ respectively, and the sign of the n -dependent term is “−”, instead of “+”, with an appreciable gain of $(\text{Log } n)^2$ in the denominator. In the case where β is an algebraic integer in Theorem 1.2, (1.9) represents an improvement of Dobrowolski’s inequality, in some sense, but in which the degree $\deg(\beta)$ is replaced by the integer n such that $\beta \in (\theta_n^{-1}, \theta_{n-1}^{-1})$. We suggest to name this peculiar integer n the *dynamical degree* of β , and to denote it by $\text{dyg}(\beta)$. It has the obvious following property:

β tends to 1 if and only if $\text{dyg}(\beta)$ tends to infinity.

Then, from Theorem 1.2, we have: $C \leq \liminf_{\beta \rightarrow 1^+} \mathfrak{S}_{lent}(\beta) \leq \Lambda$.

The paper is organized as follows. In Section 2 we show the optimal existence of lenticuli of poles of $\zeta_\beta(z)$, by the method of Rouché, by comparison with the sets of lenticular zeroes of the trinomials $-1 + x + x^n$. To be self-contained the Appendix fixes notations, recalls useful results concerning the roots of the trinomials $-1 + x + x^n$, and establishes some new relations between them. We prove Theorem 1.1 in Section 3, and establish Theorem 1.2 in Section 4 from the asymptotic expansions bounding from below the lenticular measures, the notations for the asymptotic expansions of the poles of $\zeta_{\theta_n^{-1}}(z)$ being recalled in Section 5.

2. GEOMETRY OF THE POLES OF $\zeta_\beta(z)$ INSIDE THE OPEN UNIT DISK

2.1. Parry Upper function and lacunarity. Let $1 < \beta \leq (1 + \sqrt{5})/2$. The Rényi β -expansion of unity, denoted by $d_\beta(1) = 0.t_1t_2t_3\dots$, is such that the digits $t_i = \lfloor \beta T_\beta^{(i-1)}(1) \rfloor$, $i \geq 1$, are equal to 1 or 0. The sequence (t_i) entirely controls the set of admissible sequences of digits in $\{0, 1\}$, by the Conditions of Parry [12], [11] Chapt. 7: it defines the upper bound, in the lexicographical sense, of all admissible sequences. The adherence of the set of the admissible sequences in $\{0, 1\}^{\mathbb{N}}$ is called the β -shift. In honour of W. Parry, because of this upper bound, we introduce the following terminology.

Definition 2.1. The analytic function $-1 + t_1z + t_2z^2 + t_3z^3 + \dots$, whose coefficient vector is the sequence (t_i) of $d_\beta(1) = 0.t_1t_2t_3\dots$, is denoted by $f_\beta(z)$ and is called the *Parry Upper function at β* .

Because of (1.1), the identities

$$(2.1) \quad (i) \quad f_\beta(z) = -\frac{1-z^N}{\zeta_\beta(z)} \quad \text{if } T_\beta^i(1) \neq 0 \text{ for } 1 \leq i \leq N-1, T_\beta^N(1) = 0,$$

$$(2.2) \quad (ii) \quad f_\beta(z) = -\frac{1}{\zeta_\beta(z)} \quad \text{otherwise,}$$

hold. Therefore the zeroes of modulus < 1 of $f_\beta(z)$ are exactly the poles of $\zeta_\beta(z)$. For convenience in the sequel, because the Parry Upper function is easier to manipulate, we will mostly consider $f_\beta(z)$ and its zeroes inside $D(0, 1)$, instead of $\zeta_\beta(z)$.

The lacunarity of the Parry Upper function $f_\beta(z)$ is a consequence of the two following results.

Let us observe that, in general, the total ordering $<$ on $(1, +\infty)$ is uniquely in correspondence with the lexicographical ordering $<_{lex}$ on Rényi expansions of 1 by the following Proposition, which is Lemma 3 in Parry [12].

Proposition 2.2. *Let $\alpha > 1$ and $\beta > 1$. If the Rényi α -expansion of 1 is*

$$d_\alpha(1) = 0.t'_1t'_2t'_3\dots, \quad \text{i.e.} \quad 1 = \frac{t'_1}{\alpha} + \frac{t'_2}{\alpha^2} + \frac{t'_3}{\alpha^3} + \dots$$

and the Rényi β -expansion of 1 is

$$d_\beta(1) = 0.t_1t_2t_3\dots, \quad \text{i.e.} \quad 1 = \frac{t_1}{\beta} + \frac{t_2}{\beta^2} + \frac{t_3}{\beta^3} + \dots,$$

then $\alpha < \beta$ if and only if $(t'_1, t'_2, t'_3, \dots) <_{lex} (t_1, t_2, t_3, \dots)$.

Theorem 2.3. *A real number β , $1 < \beta < \frac{1+\sqrt{5}}{2}$, belongs to $[\theta_{n+1}^{-1}, \theta_n^{-1})$ for some integer $n \geq 2$ if and only if the Rényi β -expansion of unity $d_\beta(1)$ has the form*

$$(2.3) \quad d_\beta(1) = 0.10^{n-1}10^{n_1}10^{n_2}10^{n_3}\dots,$$

with $n_k \geq n-1$ for all $k \geq 1$.

Proof. The condition is necessary: $d_\beta(1)$ begins as $0.10^{n-1}1$ for all β s such that $\theta_{n+1}^{-1} \leq \beta < \theta_n^{-1}$. For such β s we write $d_\beta(1) = 0.10^{n-1}1u$ with u a string of digits in $\{0, 1\}$, that is

$$u = 1^{h_0}0^{n_1}1^{h_1}0^{n_2}1^{h_2} \dots$$

and $h_0, n_1, h_1, n_2, h_2, \dots$ integers ≥ 0 . The Conditions of Parry ([11] Chap. 7) applied to the sequence $(1, 0^{n-1}, 1^{1+h_0}, 0^{n_1}, 1^{h_1}, 0^{n_2}, 1^{h_2}, \dots)$, which characterizes uniquely the base of numeration β , readily implies $h_0 = 0$ and $h_k = 1$ and $n_k \geq n - 1$ for all $k \geq 1$.

The condition is sufficient: since $d_{\theta_{n+1}^{-1}}(1) = 0.10^{n-1}1$ and $d_{\theta_n^{-1}}(1) = 0.10^{n-2}1$, Proposition 2.2 implies the result. \square

From (2.3), for any real number β in the interval $(1, (1 + \sqrt{5})/2)$, we deduce that the Parry Upper function $f_\beta(z)$ takes the form, for some integer $n \geq 3$:

$$(2.4) \quad f_\beta(z) = -1 + x + x^n + x^{m_1} + x^{m_2} + \dots + x^{m_s} + \dots$$

with the distanciation conditions: $m_1 - n \geq n - 1$, $m_{q+1} - m_q \geq n - 1$ for $1 \leq q$. The real number β has dynamical degree $\text{dyg}(\beta) = n$ if the third monomial in (2.4) is exactly “ x^n ”.

2.2. Lenticular poles. In this subsection $\beta \in (1, \theta_6^{-1})$ is a real number such that $\beta \notin \{\theta_n^{-1} \mid n \geq 7\}$. The Rényi β -expansion $d_\beta(1)$ of 1 is infinite or not. If it is finite $f_\beta(z)$ is a polynomial of degree m_q for some integer $m_q, q \geq 2$. In Theorem 2.11 it will be proved that, to such a β , is associated a lenticulus \mathcal{L}_β of zeroes of $f_\beta(z)$, located in the angular sector

$$|\arg(z)| < \pi/18.2880, \quad |z| < 1,$$

each lenticular zero being uniquely associated with a lenticular zero of a trinomial $-1 + x + x^n$ for some n . Examples of lenticuli can be visualized in [5].

The Appendix (Section 5) gives a summary on the geometry of the lenticular zeroes of $f_\beta(z)$ for $\beta \in \{\theta_n^{-1} \mid n \geq 3\}$, together with their asymptotic expansions in regions of validity. In the following we will use the notation $G_n(z) := -1 + z + z^n$ instead of $f_{\theta_n^{-1}}(z), n \geq 3$.

The method which will be used to detect the lenticulus of lenticular zeroes of $f_\beta(z)$ is the method of Rouché, where the lenticular zeroes are expressed as asymptotic expansions of the integer $\text{dyg}(\beta)$. Let us give a few comments on the way it will be applied.

Let $n := \text{dyg}(\beta)$. From §5.1, let

$$\mathcal{L}_{\theta_n^{-1}} = \{\overline{z_{[n/6],n}}, \dots, \overline{z_{2,n}}, \overline{z_{1,n}}, \theta_n, z_{1,n}, z_{2,n}, \dots, z_{[n/6],n}\}$$

denote the lenticulus of lenticular roots of $G_n = f_{\theta_n^{-1}}$. Since it is symmetrical with respect to the real axis, we will only consider the lenticular zeroes in the (upper) Poincaré half-plane. $\mathcal{L}_{\theta_n^{-1}}$ lies in the angular sector

$$|\arg(z)| < \pi/3, \quad |z| < 1.$$

In Theorem 2.4 and Proposition 2.9, we prove the existence of real numbers $t_{j,n} \in (0, 1)$ such that the small Rouché circles

$$C_{j,n} := \{z \in \mathbb{C} : |z - z_{j,n}| = \frac{t_{j,n}}{n}\}$$

of respective centers $z_{j,n}, z_{j,n} \in \mathcal{L}_{\theta_n^{-1}}$, all satisfy the Rouché conditions:

$$(2.5) \quad |f_\beta(z) - G_n(z)| = \left| \sum_{q \geq 1} z^{mq} \right| < |G_n(z)| \quad \text{for } z \in C_{j,n}, \text{ for } j = 1, 2, \dots, J_n,$$

are pairwise disjoint, small enough to avoid a nonempty intersection with $\{z : |z| = 1\}$, and where the integer $J_n \leq \lfloor \frac{n}{6} \rfloor$ is the largest possible integer. The maximality of J_n means that the conditions of Rouché cannot be satisfied as soon as $J_n < j \leq \lfloor \frac{n}{6} \rfloor$ for the reason that the circles $C_{j,n}$ are too close to $|z| = 1$.

As a consequence, the number of zeroes of $f_\beta(z)$ and $G_n(z)$ in the open disk $D_{j,n} := \{z \mid |z - z_{j,n}| < \frac{t_{j,n}}{n}\}$ will be equal, implying the existence of a simple zero of the Parry Upper function $f_\beta(z)$ in each disk $D_{j,n}$.

For any real number $\beta \in (1, \theta_6^{-1})$ such that $\beta \notin \{\theta_n^{-1} \mid n \geq 7\}$, the quantities $t_{j,n}$ will be deduced from the following inequalities:

$$(2.6) \quad \frac{|z|^{2n-1}}{1 - |z|^{n-1}} < |G_n(z)| \quad \text{for } z \in C_{j,n}, \quad j = 1, 2, \dots, J_n,$$

since

$$|f_\beta(z) - G_n(z)| = \left| \sum_{q \geq 1} z^{mq} \right| \leq \sum_{q \geq 1} |z^{mq}| \leq \frac{|z|^{2n-1}}{1 - |z|^{n-1}}, \quad |z| < 1,$$

instead of (2.5).

The existence of the lenticular roots $\omega_{j,n}$ of \mathcal{L}_β in the main angular subsector is proved in Theorem 2.4, then in Theorem 2.11 in a refined version. Proposition 2.9 completes the proof of their existence in the bump angular sector (cf Section 5.2), where the asymptotic expansions of the lenticular zeroes $z_{j,n}$ take another form.

Theorem 2.4. *Let $n \geq n_1 = 195$, $a \geq 1$, and $j \in \{\lceil v_n \rceil, \lceil v_n \rceil + 1, \dots, \lfloor n/6 \rfloor\}$. Denote by $C_{j,n} := \{z \mid |z - z_{j,n}| = \frac{t_{j,n}}{n}\}$ the circle centered at the j -th root $z_{j,n}$ of $-1 + X + X^n$, with $t_{j,n} = \frac{\pi |z_{j,n}|}{a}$. Then the condition of Rouché*

$$(2.7) \quad \frac{|z|^{2n-1}}{1 - |z|^{n-1}} < |-1 + z + z^n|, \quad \text{for all } z \in C_{j,n},$$

holds true on the circle $C_{j,n}$ for which the center $z_{j,n}$ satisfies

$$(2.8) \quad \frac{|-1 + z_{j,n}|}{|z_{j,n}|} < \frac{1 - \exp\left(\frac{-\pi}{a}\right)}{2 \exp\left(\frac{\pi}{a}\right) - 1}.$$

The condition $n \geq 195$ ensures the existence of such roots $z_{j,n}$. Taking the value $a = a_{\max} = 5.87433\dots$ for which the upper bound of (2.8) is maximal, equal to $0.171573\dots$, the roots $z_{j,n}$ which satisfy (2.8) all belong to the angular sector, independent of n :

$$(2.9) \quad \arg(z) \in \left[0, +\frac{\pi}{18.2880}\right].$$

For any real number $\beta > 1$ having $\text{dyg}(\beta) = n$, $f_\beta(z)$ admits a simple zero $\omega_{j,n}$ in $D_{j,n}$ for which the center $z_{j,n}$ satisfies (2.8) with $a = a_{\max}$, and j in the range $\{\lceil v_n \rceil, \lceil v_n \rceil + 1, \dots, \lfloor n/6 \rfloor\}$.

Proof. Denote by $\varphi := \arg(z_{j,n})$ the argument of the j -th root $z_{j,n}$.

Since $-1 + z_{j,n} + z_{j,n}^n = 0$, we have $|z_{j,n}|^n = |-1 + z_{j,n}|$. Let us write $z = z_{j,n} + \frac{t_{j,n}}{n} e^{i\psi} = z_{j,n}(1 + \frac{\pi}{an} e^{i(\psi-\varphi)})$ the generic element belonging to $C_{j,n}$, with $\psi \in [0, 2\pi]$.

Let $X := \cos(\psi - \varphi)$. Let us show that if the inequality (2.7) of Rouché holds true for $X = +1$, for a certain circle $C_{j,n}$, then it holds true for all $X \in [-1, +1]$, that is for every argument $\psi \in [0, 2\pi]$, i.e. for every $z \in C_{j,n}$. Let us show

$$\left| 1 + \frac{\pi}{an} e^{i(\psi-\varphi)} \right|^n = \exp\left(\frac{\pi X}{a}\right) \times \left(1 - \frac{\pi^2}{2a^2 n} (2X^2 - 1) + O\left(\frac{1}{n^2}\right) \right)$$

and

$$\arg\left(\left(1 + \frac{\pi}{an} e^{i(\psi-\varphi)}\right)^n\right) = \operatorname{sgn}(\sin(\psi - \varphi)) \times \left(\frac{\pi\sqrt{1-X^2}}{a} \left[1 - \frac{\pi X}{an}\right] + O\left(\frac{1}{n^2}\right) \right).$$

Indeed, since $\sin(\psi - \varphi) = \pm\sqrt{1-X^2}$, then

$$\begin{aligned} \left(1 + \frac{\pi}{an} e^{i(\psi-\varphi)}\right)^n &= \exp\left(n \operatorname{Log}\left(1 + \frac{\pi}{an} e^{i(\psi-\varphi)}\right)\right) \\ &= \exp\left(\frac{\pi}{a}(X \pm i\sqrt{1-X^2}) + \left[-\frac{n}{2}\left(\frac{\pi}{an}(X \pm i\sqrt{1-X^2})\right)^2 + O\left(\frac{1}{n^2}\right)\right]\right) \\ &= \exp\left(\frac{\pi X}{a} - \frac{\pi^2}{2a^2 n}(2X^2 - 1) + O\left(\frac{1}{n^2}\right)\right) \times \exp\left(\pm i \frac{\pi\sqrt{1-X^2}}{a} \left[1 - \frac{\pi X}{an}\right] + O\left(\frac{1}{n^2}\right)\right). \end{aligned}$$

Moreover,

$$\left| 1 + \frac{\pi}{an} e^{i(\psi-\varphi)} \right| = \left| 1 + \frac{\pi}{an} (X \pm i\sqrt{1-X^2}) \right| = 1 + \frac{\pi X}{an} + O\left(\frac{1}{n^2}\right).$$

with

$$\arg\left(1 + \frac{\pi}{an} e^{i(\psi-\varphi)}\right) = \operatorname{sgn}(\sin(\psi - \varphi)) \times \frac{\pi\sqrt{1-X^2}}{an} + O\left(\frac{1}{n^2}\right).$$

For all $n \geq 18$ (Proposition 3.5 in [27]), let us recall that

$$(2.10) \quad |z_{j,n}| = 1 + \frac{1}{n} \operatorname{Log}\left(2 \sin \frac{\pi j}{n}\right) + \frac{1}{n} O\left(\frac{\operatorname{Log} \operatorname{Log} n}{\operatorname{Log} n}\right)^2.$$

Then the left-hand side term of (2.7) is

$$(2.11) \quad \frac{|z|^{2n-1}}{1 - |z|^{n-1}} = \frac{|-1 + z_{j,n}|^2 \left| 1 + \frac{\pi}{an} e^{i(\psi-\varphi)} \right|^{2n}}{|z_{j,n}| \left| 1 + \frac{\pi}{an} e^{i(\psi-\varphi)} \right| - |-1 + z_{j,n}| \left| 1 + \frac{\pi}{an} e^{i(\psi-\varphi)} \right|^n}$$

$$= \frac{|-1 + z_{j,n}|^2 \left(1 - \frac{\pi^2}{an} (2X^2 - 1) \right) \exp\left(\frac{2\pi X}{a}\right)}{\left(1 + \frac{1}{n} \operatorname{Log}\left(2 \sin \frac{\pi j}{n}\right) + \frac{\pi X}{an} \right) - |-1 + z_{j,n}| \left(1 - \frac{\pi^2}{2an} (2X^2 - 1) \right) \exp\left(\frac{\pi X}{a}\right)}$$

up to $\frac{1}{n} O\left(\frac{\operatorname{Log} \operatorname{Log} n}{\operatorname{Log} n}\right)^2$ -terms (in the terminant). The right-hand side term of (2.7) is

$$|-1 + z + z^n| = \left| -1 + z_{j,n} \left(1 + \frac{\pi}{na} e^{i(\psi-\varphi)} \right) + z_{j,n}^n \left(1 + \frac{\pi}{na} e^{i(\psi-\varphi)} \right)^n \right|$$

$$\begin{aligned}
 &= \left| -1 + z_{j,n} \left(1 \pm i \frac{\pi \sqrt{1-X^2}}{an} \right) \left(1 + \frac{\pi X}{an} \right) + \right. \\
 (2.12) \quad & \left. (1 - z_{j,n}) \left(1 - \frac{\pi^2}{2a^2 n} (2X^2 - 1) \right) \exp\left(\frac{\pi X}{a}\right) \exp\left(\pm i \left(\frac{\pi \sqrt{1-X^2}}{a} \left[1 - \frac{\pi X}{an} \right] \right)\right) + O\left(\frac{1}{n^2}\right) \right|
 \end{aligned}$$

Let us consider (2.11) and (2.12) at the first order for the asymptotic expansions, i.e. up to $O(1/n)$ - terms instead of up to $O\left(\frac{1}{n}(\text{Log Log } n / \text{Log } n)^2\right)$ - terms or $O(1/n^2)$ - terms. (2.11) becomes:

$$\frac{|-1 + z_{j,n}|^2 \exp\left(\frac{2\pi X}{a}\right)}{|z_{j,n}| - |-1 + z_{j,n}| \exp\left(\frac{\pi X}{a}\right)}$$

and (2.12) is equal to:

$$\left| -1 + z_{j,n} \left| 1 - \exp\left(\frac{\pi X}{a}\right) \exp\left(\pm i \frac{\pi \sqrt{1-X^2}}{a}\right) \right| \right|$$

and is independent of the sign of $\sin(\psi - \varphi)$. Then the inequality (2.7) is equivalent to

$$(2.13) \quad \frac{|-1 + z_{j,n}|^2 \exp\left(\frac{2\pi X}{a}\right)}{|z_{j,n}| - |-1 + z_{j,n}| \exp\left(\frac{\pi X}{a}\right)} < \left| -1 + z_{j,n} \left| 1 - \exp\left(\frac{\pi X}{a}\right) \exp\left(\pm i \frac{\pi \sqrt{1-X^2}}{a}\right) \right| \right|,$$

and (2.13) to

$$(2.14) \quad \frac{|-1 + z_{j,n}|}{|z_{j,n}|} < \frac{\left| 1 - \exp\left(\frac{\pi X}{a}\right) \exp\left(i \frac{\pi \sqrt{1-X^2}}{a}\right) \right| \exp\left(\frac{-\pi X}{a}\right)}{\exp\left(\frac{\pi X}{a}\right) + \left| 1 - \exp\left(\frac{\pi X}{a}\right) \exp\left(i \frac{\pi \sqrt{1-X^2}}{a}\right) \right|} =: \kappa(X, a).$$

Denote by $\kappa(X, a)$ the right-hand side term, as a function of (X, a) , on $[-1, +1] \times [1, +\infty)$. It is routine to show that, for any fixed a , the partial derivative $\partial \kappa_X$ of $\kappa(X, a)$ with respect to X is strictly negative on the interior of the domain. The function $x \rightarrow \kappa(x, a)$ takes its minimum at $X = 1$, and this minimum is always strictly positive. Hence the inequality of Rouché (2.7) will be satisfied on $C_{j,n}$ once it is satisfied at $X = 1$.

For which range of values of j/n ? Up to $O(1/n)$ -terms in (2.14), it is given by the set of integers j for which $z_{j,n}$ satisfies:

$$(2.15) \quad \frac{|-1 + z_{j,n}|}{|z_{j,n}|} < \kappa(1, a) = \frac{\left| 1 - \exp\left(\frac{\pi}{a}\right) \right| \exp\left(\frac{-\pi}{a}\right)}{\exp\left(\frac{\pi}{a}\right) + \left| 1 - \exp\left(\frac{\pi}{a}\right) \right|}.$$

In order to take into account a collection of roots of $z_{j,n}$ as large as possible, i.e. in order to have a minorant of the lenticular measure $\mathfrak{S}_{lent}(\beta)$ the largest possible, the value of $a \geq 1$ has to be chosen such that $a \rightarrow \kappa(1, a)$ is maximal in (2.15).

The function $a \rightarrow \kappa(1, a)$ reaches its maximum $\kappa(1, a_{\max}) := 0.171573\dots$ at $a_{\max} = 5.8743\dots$ (Figure 1). Denote by J_n the maximal integer j for which (2.15) is satisfied and in which a is taken equal to a_{\max} (Definition 2.6 and Proposition 2.7). From Proposition 2.7, in which are reported the asymptotic expansions of J_n and $\arg(z_{J_n, n})$, we deduce

$$(2.16) \quad \arg(z_{j,n}) < \frac{\pi}{18.2880\dots} = 0.171784\dots \quad \text{for } j = \lceil v_n \rceil, \lceil v_n \rceil + 1, \dots, J_n.$$

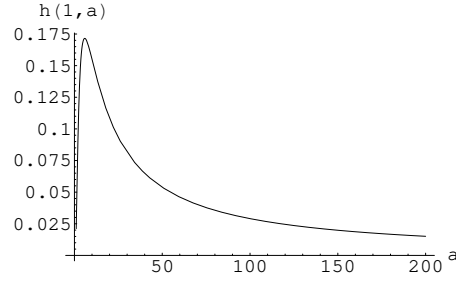


FIGURE 1. Curve of the Rouché condition $a \rightarrow \kappa(1, a)$ (upper bound in (2.8)), for the circles $C_{j,n} = \{z \mid |z - z_{j,n}| = \pi|z_{j,n}|/(an)\}$ centered at the zeroes $z_{j,n}$ of the trinomial $-1 + X + X^n$, as a function of the size of the circles $C_{j,n}$ parametrized by the adjustable real number $a \geq 1$.

Remark 2.5. The minimal value $n_1 = 195$ is calculated by the condition $2\pi \frac{v_n}{n} < \frac{\pi}{18.2880\dots} = 0.171784\dots$, for all $n \geq n_1$, for having a strict inclusion, of the “bump sector” inside the angular sector defined by the maximal opening angle $0.171784\dots$ (cf Appendix for the sequence (v_n))

This finishes the proof. \square

Let us calculate the argument of the last root $z_{j,n}$ for which (2.14) is an equality with $X = 1$.

Definition 2.6. Let $n \geq 195$. Denote by J_n the largest integer $j \geq 1$ such that the root $z_{j,n}$ of G_n satisfies

$$(2.17) \quad \frac{|-1 + z_{j,n}|}{|z_{j,n}|} \leq \kappa(1, a_{\max}) = \frac{1 - \exp\left(\frac{-\pi}{a_{\max}}\right)}{2 \exp\left(\frac{\pi}{a_{\max}}\right) - 1} = 0.171573\dots$$

Let us observe that the upper bound $\kappa(1, a_{\max})$ is independent of n . From this independence we deduce the following “limit” angular sector in which the Rouché conditions can be applied.

Proposition 2.7. Let $n \geq 195$. Let us put $\kappa := \kappa(1, a_{\max})$ for short. Then

$$(2.18) \quad \arg(z_{J_n, n}) = 2 \arcsin\left(\frac{\kappa}{2}\right) + \frac{\kappa \operatorname{Log} \kappa}{n\sqrt{4 - \kappa^2}} + \frac{1}{n} O\left(\left(\frac{\operatorname{Log} \operatorname{Log} n}{\operatorname{Log} n}\right)^2\right),$$

$$(2.19) \quad J_n = \frac{n}{\pi} \left(\arcsin\left(\frac{\kappa}{2}\right)\right) + \frac{\kappa \operatorname{Log} \kappa}{\pi \sqrt{4 - \kappa^2}} + O\left(\left(\frac{\operatorname{Log} \operatorname{Log} n}{\operatorname{Log} n}\right)^2\right)$$

with, at the limit,

$$(2.20) \quad \lim_{n \rightarrow +\infty} \arg(z_{J_n, n}) = \lim_{n \rightarrow +\infty} 2\pi \frac{J_n}{n} = 2 \arcsin\left(\frac{\kappa}{2}\right) = 0.171784\dots$$

Proof. Since $\lim_{n \rightarrow +\infty} |z_{J_n, n}| = 1$, we deduce from (2.17) that the limit argument φ_{lim} of $z_{J_n, n}$ satisfies $|-1 + \cos(\varphi_{lim}) + i \sin(\varphi_{lim})| = 2 \sin(\varphi_{lim}/2) = \kappa(1, a_{\max})$. We deduce (2.20), and the equality between the two limits from (2.21).

From (2.17), the inequality $|-1 + z_{j,n}| \leq |z_{j,n}| \kappa(1, a_{\max})$ already implies that $\arg(z_{J_n,n}) < \varphi_{lim}$. In the sequel, we will use the asymptotic expansions of the roots $z_{J_n,n}$. From Section 6 in [27] the argument of $z_{J_n,n}$ takes the following form

$$(2.21) \quad \arg(z_{J_n,n}) = 2\pi\left(\frac{J_n}{n} + \Re\right) \quad \text{with} \quad \Re = -\frac{1}{2\pi n} \left[\frac{1 - \cos(\frac{2\pi J_n}{n})}{\sin(\frac{2\pi J_n}{n})} \text{Log}(2 \sin(\frac{\pi J_n}{n})) \right]$$

with

$$\text{tl}(\arg(z_{J_n,n})) = +\frac{1}{n} O\left(\left(\frac{\text{Log Log } n}{\text{Log } n}\right)^2\right).$$

Its modulus is

$$(2.22) \quad |z_{J_n,n}| = 1 + \frac{1}{n} \text{Log}(2 \sin \frac{\pi J_n}{n}) + \frac{1}{n} O\left(\frac{\text{Log Log } n}{\text{Log } n}\right)^2.$$

Denote $\varphi := \arg(z_{J_n,n})$. Up to $\frac{1}{n} O\left(\left(\frac{\text{Log Log } n}{\text{Log } n}\right)^2\right)$ -terms, we have

$$\begin{aligned} | -1 + z_{J_n,n} |^2 &= \left| -1 + \left[1 + \frac{1}{n} \text{Log}(2 \sin \frac{\pi J_n}{n})\right] (\cos(\varphi) + i \sin(\varphi)) \right|^2 \\ &= \left[-1 + \left[1 + \frac{1}{n} \text{Log}(2 \sin \frac{\pi J_n}{n})\right] \cos(\varphi) \right]^2 + \left[1 + \frac{1}{n} \text{Log}(2 \sin \frac{\pi J_n}{n})\right]^2 (\sin(\varphi))^2 \\ &= 1 + \left[1 + \frac{1}{n} \text{Log}(2 \sin \frac{\pi J_n}{n})\right]^2 - 2 \left[1 + \frac{1}{n} \text{Log}(2 \sin \frac{\pi J_n}{n})\right] \cos(\varphi) \\ (2.23) \quad &= 4 \left(\sin\left(\frac{\varphi}{2}\right)\right)^2 + \frac{4}{n} \left(\sin\left(\frac{\varphi}{2}\right)\right)^2 \text{Log}(2 \sin \frac{\pi J_n}{n}) = 4 \left(\sin\left(\frac{\varphi}{2}\right)\right)^2 \left[1 + \frac{1}{n} \text{Log}(2 \sin \frac{\pi J_n}{n})\right]. \end{aligned}$$

Up to $\frac{1}{n} O\left(\left(\frac{\text{Log Log } n}{\text{Log } n}\right)^2\right)$ -terms, due to the definition of J_n , let us consider (2.17) as an equality; hence, from (2.23) and (2.22), the following identity should be satisfied

$$(2.24) \quad 2 \sin\left(\frac{\varphi}{2}\right) = \kappa \left[1 + \frac{1}{2n} \text{Log}(2 \sin \frac{\pi J_n}{n})\right]$$

We now use (2.24) to obtain an asymptotic expansion of $\psi_n := 2\pi \frac{J_n}{n} - \varphi_{lim}$ as a function of n and φ_{lim} up to $\frac{1}{n} O\left(\left(\frac{\text{Log Log } n}{\text{Log } n}\right)^2\right)$ -terms. First, at the first order in ψ_n ,

$$\begin{aligned} \sin\left(\frac{\pi J_n}{n}\right) &= \frac{\psi_n}{2} \cos\left(\frac{\varphi_{lim}}{2}\right) + \sin\left(\frac{\varphi_{lim}}{2}\right), \quad \cos\left(\frac{\pi J_n}{n}\right) = -\frac{\psi_n}{2} \sin\left(\frac{\varphi_{lim}}{2}\right) + \cos\left(\frac{\varphi_{lim}}{2}\right), \\ \text{Log}(2 \sin\left(\frac{\pi J_n}{n}\right)) &= \text{Log}(2 \sin\left(\frac{\varphi_{lim}}{2}\right)) + \psi_n \frac{\cos\left(\frac{\varphi_{lim}}{2}\right)}{2 \sin\left(\frac{\varphi_{lim}}{2}\right)} = \text{Log } \kappa + \psi_n \frac{\cos\left(\frac{\varphi_{lim}}{2}\right)}{h}. \end{aligned}$$

Moreover,

$$(2.25) \quad \begin{aligned} &\left[\frac{1 - \cos(\frac{2\pi J_n}{n})}{\sin(\frac{2\pi J_n}{n})} \text{Log}(2 \sin(\frac{\pi J_n}{n})) \right] \\ &= \tan\left(\frac{\varphi_{lim}}{2}\right) (\text{Log } \kappa) \left[1 + \psi_n \left(\frac{1}{\sin(\varphi_{lim})} + \frac{\cos(\frac{\varphi_{lim}}{2})}{\kappa \text{Log } \kappa} \right) \right]. \end{aligned}$$

Hence, with $2 \sin(\varphi/2) = 2 \sin(\pi J_n/n) \cos(\pi \mathfrak{R}) + 2 \cos(\pi J_n/n) \sin(\pi \mathfrak{R})$, and from (2.21), up to $\frac{1}{n} O\left(\left(\frac{\text{Log Log } n}{\text{Log } n}\right)^2\right)$ -terms, the identity (2.24) becomes

$$[\psi_n \cos\left(\frac{\varphi_{lim}}{2}\right) + 2 \sin\left(\frac{\varphi_{lim}}{2}\right)] + \left(\frac{-2 \cos\left(\frac{\varphi_{lim}}{2}\right) \tan\left(\frac{\varphi_{lim}}{2}\right) \text{Log } \kappa}{2n}\right) = \kappa \left[1 + \frac{\text{Log } \kappa}{2n}\right].$$

We deduce

$$(2.26) \quad \psi_n = \frac{\kappa \text{Log } \kappa}{n \cos\left(\frac{\varphi_{lim}}{2}\right)} + \frac{1}{n} O\left(\left(\frac{\text{Log Log } n}{\text{Log } n}\right)^2\right),$$

then $2\pi J_n/n = \psi_n + \varphi_{lim}$ and (2.19). With $2\pi J_n/n$, and from (2.21) and (2.25) we deduce (2.18). This finishes the proof. \square

Remark 2.8. (i) The maximal half-opening angle of the sector in which one can detect zeroes of $f_\beta(z)$, for any β such that $\theta_{n-1} < \beta^{-1} < \theta_n$, by the method of Rouché, is $0.17178\dots = 2 \arcsin\left(\frac{\kappa(1, a_{max})}{2}\right)$. Remarkably this upper bound $2 \arcsin\left(\frac{\kappa(1, a_{max})}{2}\right)$ is independent of n . By comparison it is fairly small with respect to $\pi/3$ for the Perron numbers θ_n^{-1} .

(ii) The curve $a \rightarrow \kappa(1, a)$, given by Figure 1, is such that any value in the interval $(0, \kappa(1, a_{max}))$ is reached by the function $\kappa(1, a)$ from two values say a_1 and a_2 , of a , satisfying $a_1 < a_{max} < a_2$. On the contrary, the correspondence $a_{max} \leftrightarrow \kappa(1, a_{max})$ is unique, corresponding to a double root. Denote $D := \exp(\pi/a_{max})$ and $\kappa := \kappa(1, a_{max})$. It means that the quadratic algebraic equation $2\kappa D^2 - (\kappa + 1)D + 1 = 0$ deduced from the upper bound in (2.17) has necessarily a discriminant equal to zero. The discriminant is $\kappa^2 - 6\kappa + 1$. Therefore $D = (\kappa + 1)/(4\kappa)$ and the limit value $x = 2 \arcsin(\kappa/2)$ in (2.20) satisfies the quadratic algebraic equation

$$4(\sin(x/2))^2 - 12 \sin(x/2) + 1 = 0.$$

Proposition 2.9. *Let $n \geq n_1 = 195$. The circles $C_{j,n} := \{z \mid |z - z_{j,n}| = \frac{\pi |z_{j,n}|}{n a_{max}}\}$ centered at the roots $z_{j,n}$ of the trinomial $-1 + z + z^n$ which belong to the “bump sector”, namely for $j \in \{1, 2, \dots, \lfloor v_n \rfloor\}$, are such that the conditions of Rouché*

$$(2.27) \quad \frac{|z|^{2n-1}}{1 - |z|^{n-1}} < |-1 + z + z^n|, \quad \text{for all } z \in C_{j,n}, \quad 1 \leq j \leq \lfloor v_n \rfloor,$$

hold true. For any real number $\beta > 1$ having $\text{dyg}(\beta) = n$, $f_\beta(z)$ admits a simple zero $\omega_{j,n}$ in $D_{j,n}$ (with $a = a_{max}$), for j in the range $\{1, 2, \dots, \lfloor v_n \rfloor\}$.

Proof. The development terms “ $D(|z_{j,n}|)$ ” of the asymptotic expansions of $|z_{j,n}|$ change from the main angular sector $\arg z \in (2\pi(\text{Log } n)/n, \pi/3)$ to the first transition region $\arg z \asymp 2\pi(\text{Log } n)/n$, the “bump sector”, further to the second transition region $\arg z \asymp 2\pi\sqrt{(\text{Log } n)(\text{Log Log } n)}/n$, and to a small neighbourhood of θ_n (cf Section 5.2).

Then the proof of (2.27) is the same as that of Theorem 2.4 once (2.10) is substituted by the suitable asymptotic expansions which correspond to the angular sector of the “bump”. The terminants $\text{tl}(|z_{j,n}|)$ of the respective asymptotic expansions of $|z_{j,n}|$ also change: this change obliges to reformulate (2.11) and (2.12) up to $\text{Log } n/n$ -terms, and not up to $1/n$ -terms, as in the proof of Theorem 2.4. It is remarkable that the inequality (2.14) remains the same, with the same upper bound function $\kappa(X, a)$. Then the equation of the curve of the

Rouché condition $a \rightarrow \kappa(1, a)$, on $[1, +\infty)$, is the same as in Theorem 2.4 for controlling the conditions of Rouché. The optimal value a_{\max} of a also remains the same, and (2.7) also holds true for those $z_{j,n}$ in the bump sector. \square

Lemma 2.10. *Let $n \geq 7$. Any real number β of dynamical degree n , $\beta \neq \theta_n^{-1}$, is such that $\beta^{-1} \in C_{0,n} := \{z \in \mathbb{C} : |z - \theta_n| = \frac{t_{0,n}}{n}\}$ with*

$$t_{0,n} := \left(\frac{\text{Log Log } n}{\text{Log } n} \right)^2.$$

Proof. Since β^{-1} runs over the open interval (θ_{n-1}, θ_n) , this interval (θ_{n-1}, θ_n) is necessarily completely included in $D_{0,n} := \{z \in \mathbb{C} : |z - \theta_n| < \frac{t_{0,n}}{n}\}$, and the radius of $C_{0,n}$ is $\theta_n - \theta_{n-1}$. We deduce the result from Lemma 5.7. From Proposition 5.10 the root $z_{1,n}$ admits $\Im(z_{1,n}) = \frac{2\pi}{n}(1 - \frac{1}{\text{Log } n} + \dots)$ as imaginary part. Then, for any $t_{1,n} \in (0, 1)$, the circle $C_{0,n}$, of radius $t_{0,n}/n$, and $C_{1,n}$ are disjoint and do not intersect $|z| = 1$. \square

From the inequalities (2.8) in Theorem 2.4, also used in the proof of Proposition 2.9, we now obtain a finer localization of a subcollection of the roots $\omega_{j,n}$ of the Parry Upper function $f_\beta(z)$, and a definition of the lenticulus \mathcal{L}_β of β , as follows.

Theorem 2.11. *Let $n \geq n_1 = 195$. Let $\beta > 1$ be any real number having $\text{dyg}(\beta) = n$. The Parry Upper function $f_\beta(z)$ has an unique simple zero $\omega_{j,n}$ in each disk $D_{j,n} := \{z \mid |z - z_{j,n}| < \frac{\pi|z_{j,n}|}{na_{\max}}\}$, $j = 1, 2, \dots, J_n$, which satisfies the additional inequality:*

$$(2.28) \quad |\omega_{j,n} - z_{j,n}| < \frac{\pi|z_{j,n}|}{na_{j,n}} \quad \text{for } j = \lceil v_n \rceil, \lceil v_n \rceil + 1, \dots, J_n,$$

where $a_{J_n,n} = a_{\max}$ and, for $j = \lceil v_n \rceil, \dots, J_n - 1$, the value $a_{j,n} > a_{\max}$, is defined by

$$(2.29) \quad D\left(\frac{\pi}{a_{j,n}}\right) = \text{Log} \left[\frac{1 + B_{j,n} - \sqrt{1 - 6B_{j,n} + B_{j,n}^2}}{4B_{j,n}} \right]$$

with $B_{j,n} := 2 \sin\left(\frac{\pi j}{n}\right) \left(1 - \frac{1}{n} \text{Log} \left(2 \sin\left(\frac{\pi j}{n}\right)\right)\right),$

and, putting $D := D\left(\frac{\pi}{a_{j,n}}\right)$ for short,

$$(2.30) \quad \text{tl}\left(\frac{\pi}{a_{j,n}}\right) = \frac{2}{n} \times B_{j,n}^{-1} \left(\frac{-3 + \exp(-D) + 2 \exp(D)}{4 - \exp(-D) - 2 \exp(D)} \right) \times \left(\frac{\text{Log Log } n}{\text{Log } n} \right)^2.$$

An upper bound of the tails, independent of j , is given by

$$(2.31) \quad O\left(\frac{(\text{Log Log } n)^2}{(\text{Log } n)^3}\right)$$

with the constant $\frac{1}{7\pi}$ in the Big O .

Definition 2.12. Under the notations of Theorem 2.11, the lenticulus \mathcal{L}_β associated with β is constituted by the following subset of lenticular roots of $f_\beta(z)$:

$$(2.32) \quad \mathcal{L}_\beta := \{1/\beta\} \cup \bigcup_{j=1}^{J_n} (\{\omega_{j,n}\} \cup \{\overline{\omega_{j,n}}\}).$$

Proof. The existence of the zeroes comes from Proposition 2.9 and Theorem 2.4, with the maximal value J_n of the index j given by Proposition 2.6. To refine the localization of $\omega_{j,n}$ in the neighbourhood of $z_{j,n}$, in the main angular sector, i.e. for $j \in \{\lceil v_n \rceil, \lceil v_n \rceil + 1, \dots, J_n\}$, the conditions of Rouché (2.7) are now used to define the new radii.

The value $a_{j,n}$ is defined by the development term $D(\frac{\pi}{a_{j,n}})$, itself defined as follows:

$$(2.33) \quad D\left(\frac{|-1+z_{j,n}|}{|z_{j,n}|}\right) =: \frac{1 - \exp(-D(\frac{\pi}{a_{j,n}}))}{2 \exp(D(\frac{\pi}{a_{j,n}})) - 1}$$

and the tail $tl(\frac{\pi}{a_{j,n}})$ calculated from $tl\left(\frac{|-1+z_{j,n}|}{|z_{j,n}|}\right)$ so that the Rouché condition

$$(2.34) \quad \frac{|-1+z_{j,n}|}{|z_{j,n}|} = D\left(\frac{|-1+z_{j,n}|}{|z_{j,n}|}\right) + tl\left(\frac{|-1+z_{j,n}|}{|z_{j,n}|}\right) < \frac{1 - \exp(-\frac{\pi}{a_{j,n}})}{2 \exp(\frac{\pi}{a_{j,n}}) - 1}$$

holds true. From Proposition 5.15, denote

$$B_{j,n} := D\left(\frac{|-1+z_{j,n}|}{|z_{j,n}|}\right) = 2 \sin\left(\frac{\pi j}{n}\right) \left(1 - \frac{1}{n} \text{Log}(2 \sin\left(\frac{\pi j}{n}\right))\right).$$

Let $W := \exp(D(\frac{\pi}{a_{j,n}}))$. The identity (2.33) transforms into the equation of degree 2:

$$(2.35) \quad 2B_{j,n}W^2 - (B_{j,n} + 1)W + 1 = 0$$

from which (2.36) is deduced. For the calculation of $tl(\frac{\pi}{a_{j,n}})$, denote $D := D(\frac{\pi}{a_{j,n}})$ and

$$tl_{j,n} := tl\left(\frac{\pi}{a_{j,n}}\right). \text{ Then, at the first order, } \frac{1 - \exp(-\frac{\pi}{a_{j,n}})}{2 \exp(\frac{\pi}{a_{j,n}}) - 1}$$

$$= \frac{1 - \exp(-D - tl_{j,n})}{2 \exp(D + tl_{j,n}) - 1} = B_{j,n} \left[1 + tl_{j,n} \times \left(\frac{4 - \exp(-D) - 2 \exp(D)}{-3 + \exp(-D) + 2 \exp(D)}\right)\right].$$

From (2.34) and (5.23) the following inequality should be satisfied, with the constant 2 in the Big O,

$$\frac{1}{n} O\left(\left(\frac{\text{Log Log } n}{\text{Log } n}\right)^2\right) = tl\left(\frac{|-1+z_{j,n}|}{|z_{j,n}|}\right) < tl_{j,n} \times B_{j,n} \left(\frac{4 - \exp(-D) - 2 \exp(D)}{-3 + \exp(-D) + 2 \exp(D)}\right).$$

The expression of $tl_{j,n}$ in (2.30) follows, to obtain a strict inequality in (2.34). By (2.29) the quantity $\exp(D)$ is a function of $B_{j,n}$, which tends to $\frac{3}{4}$ when $B_{j,n}$ tends to 0; hence, at the first order, a lower bound of the function $B_{j,n} \rightarrow |B_{j,n} \left(\frac{4 - \exp(-D) - 2 \exp(D)}{-3 + \exp(-D) + 2 \exp(D)}\right)|$ is obtained for $j = \lceil v_n \rceil$, and given by $2\pi \frac{\text{Log } n}{n} \times 7$. Then it suffices to take

$$tl_{j,n} = cste \left(\frac{(\text{Log Log } n)^2}{(\text{Log } n)^3}\right)$$

with $cste = 1/(7\pi)$, to obtain a tail independent of j , and therefore the conditions of Rouché (2.34) satisfied with these new smaller radii and tails in the main angular sector. \square

Remark 2.13. For n very large, up to second-order terms, (2.35) reduces to

$$4 \sin\left(\frac{\pi j}{n}\right) W^2 - \left(2 \sin\left(\frac{\pi j}{n}\right) + 1\right) W + 1 = 0$$

and (2.29) to

$$(2.36) \quad D\left(\frac{\pi}{a_{j,n}}\right) = \text{Log} \left[\frac{1 + 2 \sin\left(\frac{\pi j}{n}\right) - \sqrt{1 - 12 \sin\left(\frac{\pi j}{n}\right) + 4(\sin\left(\frac{\pi j}{n}\right))^2}}{8 \sin\left(\frac{\pi j}{n}\right)} \right].$$

Lemma 2.14. Let $n \geq 195$ and c_n defined by $|z_{J_n,n}| = 1 - \frac{c_n}{n}$. Let us put $\kappa := \kappa(1, a_{\max})$ for short. Then

$$(2.37) \quad c_n = -(\text{Log } \kappa) \left(1 + \frac{1}{n}\right) + \frac{1}{n} O\left(\left(\frac{\text{Log Log } n}{\text{Log } n}\right)^2\right),$$

with $c = \lim_{n \rightarrow +\infty} c_n = -\text{Log } \kappa = 1.76274\dots$, and, up to $O\left(\frac{1}{n} \left(\left(\frac{\text{Log Log } n}{\text{Log } n}\right)^2\right)\right)$ -terms,

$$(2.38) \quad \frac{(1 - \frac{c_n}{n})^{2n}}{(1 - \frac{c_n}{n}) - (1 - \frac{c_n}{n})^n} = \frac{e^{-2c}}{1 - e^{-c}} \left(1 + \frac{c}{2n(1 - e^{-c})} [2 - ce^{-c} - 2c]\right)$$

with $e^{-2c}/(1 - e^{-c}) = 0.0355344\dots$

Proof. The asymptotic expansion (2.37) of c_n is deduced from the asymptotic expansions of ψ_n and $z_{J_n,n}$ given by (2.26) and (2.22) (Proposition 3.5 in [27]). We deduce the limit $c := -\text{Log}(\kappa(1, a_{\max})) = 1.76274\dots$ and then (2.38) follows. \square

Definition 2.15. Let $n \geq n_2 := 260$. We denote by H_n the largest integer $j \geq \lceil v_n \rceil$ such that

$$(2.39) \quad \arg(z_{J_n,n}) - \arg(z_{j,n}) \geq \frac{(1 - \frac{c_n}{n})^{2n}}{(1 - \frac{c_n}{n}) - (1 - \frac{c_n}{n})^n}.$$

Proposition 2.16. Let $n \geq 260$. Let denote $\kappa := \kappa(1, a_{\max})$ for short. Then

$$(2.40) \quad \arg(z_{H_n,n}) = 2 \arcsin\left(\frac{\kappa}{2}\right) - \frac{\kappa^2}{1 - \kappa} + \frac{\text{Log } \kappa}{n} \left[\frac{\kappa}{\sqrt{4 - \kappa^2}} + \frac{2 + \kappa \text{Log}(\kappa) + 2 \text{Log}(\kappa)}{2(1 - \kappa)} \right] + \frac{1}{n} O\left(\left(\frac{\text{Log Log } n}{\text{Log } n}\right)^2\right),$$

with, at the limit,

$$\lim_{n \rightarrow +\infty} \arg(z_{H_n,n}) = 2 \arcsin\left(\frac{\kappa}{2}\right) - \frac{\kappa^2}{1 - \kappa} = 0.13625.$$

Proof. The asymptotic expansion of the right-hand side term of (2.39) is

$$(2.41) \quad \frac{(1 - \frac{c_n}{n})^{2n}}{(1 - \frac{c_n}{n}) - (1 - \frac{c_n}{n})^n} = \frac{e^{-2c}}{1 - e^{-c}} \left(1 + \frac{c(2 - ce^{-c} - 2c)}{2n(1 - e^{-c})}\right) + \dots$$

Then the asymptotic expansion of $\arg(z_{H_n,n})$ comes from (2.39) in which the inequality is replaced by an equality, and from the asymptotic expansion (2.18) of $\arg(z_{J_n,n})$ (Proposition 2.7). \square

For n large enough, $\arg(z_{H_n, n})$ is equal to $2\pi \frac{H_n}{n}$, up to higher order - terms, and a definition of H_n in terms of asymptotic expansions could be:

$$(2.42) \quad H_n = \lfloor \frac{n}{2\pi} \left(2 \arcsin\left(\frac{\kappa}{2}\right) - \frac{\kappa^2}{1-\kappa} \right) - \text{Log}(\kappa) \left[\frac{\kappa}{\sqrt{4-\kappa^2}} + \frac{2 + \kappa \text{Log}(\kappa) + 2 \text{Log}(\kappa)}{2(1-\kappa)} \right] \rfloor,$$

For simplicity's sake, we will take the following definition of H_n

$$(2.43) \quad H_n := \lfloor \frac{n}{2\pi} \left(2 \arcsin\left(\frac{\kappa}{2}\right) - \frac{\kappa^2}{1-\kappa} \right) - 1 \rfloor.$$

Remark 2.17. The value $n_2 = 260$ is calculated to be the smallest value such that the inequality $\frac{2\pi v_n}{n} < \arg(z_{H_n, n})$ holds, inequality which should be valid for all $n \geq n_2$; here H_n is given by (2.43), $\arg(z_{H_n, n})$ by (2.40), where (v_n) is the delimiting sequence (cf Appendix Section 5.2) of the transition region of the boundary of the bump sector. A first minimal value of n is first estimated by $2\pi \frac{\text{Log} n}{n} < D(\arg(z_{H_n, n}))$ using (2.40). Then it is corrected so that the numerical value of the tail of the asymptotic expansion in (2.40) be taken into account in this inequality.

2.3. Other poles. In the complement of the family of the adjustable Rouché disks Theorem 2.18 asserts the existence of a zerofree region depending upon the dynamical degree of β .

Theorem 2.18. *Let $n \geq n_2 := 260$. Denote by \mathcal{D}_n the subdomain of the open unit disk, symmetrical with respect to the real axis, defined by the conditions:*

$$(2.44) \quad |z| < 1 - \frac{c_n}{n}, \quad \frac{1}{n} \left(\frac{\text{Log} \text{Log} n}{\text{Log} n} \right)^2 < |z - \theta_n|,$$

$$(2.45) \quad \frac{\pi |z_{j,n}|}{n a_{\max}} < |z - z_{j,n}|, \quad \text{for } j = 1, 2, \dots, J_n,$$

and, for $j = J_n + 1, \dots, 2J_n - H_n + 1$,

$$(2.46) \quad \frac{\pi |z_{j,n}|}{n s_{j,n}} < |z - z_{j,n}|, \quad \text{with } s_{j,n} = a_{\max} \left[1 + \frac{a_{\max}^2 (j - J_n)^2}{\pi^2 J_n^2} \right]^{-1/2}.$$

Then, for any real number $\beta > 1$ having $\text{dyg}(\beta) = n$, the Parry Upper function $f_\beta(z)$ does not vanish at any point z in \mathcal{D}_n .

Proof. Assume $\beta > 1$ such that $\theta_{n-1} < \beta^{-1} < \theta_n$. We will apply the general form of the Theorem of Rouché to the compact \mathcal{K}_n which is the adherence of the domain \mathcal{D}_n , i.e. we will show that the inequality (and symmetrically with respect to the real axis)

$$(2.47) \quad |f_\beta(z) - G_n(z)| < |G_n(z)|, \quad z \in \partial \mathcal{K}_n^{\text{ext}} \cup C_{1,n} \cup C_{2,n} \cup \dots \cup C_{J_n,n}$$

holds, with $z \in \text{Im}(z) \geq 0$, where $\partial \mathcal{K}_n$ is the union of: (i) the arcs of the circles defined by the equalities in (2.45) and (2.46), arcs which lie in $|z| \leq 1 - c_n/n$, and circles for which the intersection with $|z| = 1 - c_n/n$ is not empty, (ii) the arcs of $C(0, 1 - c_n/n)$ which have empty intersections with the interiors of the disks defined by the inequalities “>”, instead of “<”, in (2.45) and (2.46), which join two successive circles. The two functions $f_\beta(z)$

and $G_n(z)$ are continuous on the compact \mathcal{K}_n , holomorphic in its interior \mathcal{D}_n , and G_n has no zero in \mathcal{K}_n . As a consequence the function $f_\beta(z)$ will have no zero in the interior \mathcal{D}_n of \mathcal{K}_n .

Instead of using $f_\beta(z)$ itself in (2.47), we will show that the following inequality holds true

$$(2.48) \quad \frac{|z|^{2n-1}}{1-|z|^{n-1}} < |-1+z+z^n|, \quad \text{for all } z \in \partial \mathcal{K}_n^{ext}$$

what will imply the claim.

The Rouché inequalities (2.47) (2.48) hold true on the (complete) circles $C_{j,n}$, $1 \leq j \leq J_n$ by Theorem 2.4 and Proposition 2.9; these conditions become out of reach for j taking higher values (i.e. in $\{J_n+1, \dots, \lfloor n/6 \rfloor\}$), but we will show that they remain true on the arcs defined by the equalities in (2.46).

The domain \mathcal{D}_n only depends upon the dynamical degree n of β , not of β itself.

Let us prove that the external Rouché circle $|z| = 1 - c_n/n$ intersects all the circles $C_{J_n-k,n}$, $k = 0, 1, \dots, k_{\max}$, with $k_{\max} := \lfloor J_n(\frac{\pi}{a_{\max}}) \rfloor$. Indeed, up to $\frac{1}{n}O((\frac{\text{Log Log } n}{\text{Log } n})^2)$ - terms, from Proposition 2.7,

$$(2.49) \quad \begin{aligned} \text{Log}(2 \sin(\pi \frac{J_n}{n})) &= \text{Log}(2 \sin(\pi \frac{(J_n-k)+k}{n})) = \text{Log}(2\pi \frac{J_n-k}{n} (1 + \frac{k}{J_n-k})) \\ &= \text{Log}(2 \sin(\pi \frac{J_n-k}{n})) + \frac{k}{J_n}. \end{aligned}$$

Since $|z_{J_n,n}| = 1 - c_n/n = 1 + \frac{1}{n} \text{Log}(2 \sin(\pi \frac{J_n}{n})) + \frac{1}{n} O((\frac{\text{Log Log } n}{\text{Log } n})^2)$, we deduce from (2.49), with $k \leq k_{\max}$, that the point $z \in C(0, 1 - c_n/n)$ for which $\arg(z) = \arg(z_{J_n-k,n})$ is such that

$$|z_{J_n-k,n} - z| = \frac{k}{nJ_n} \leq \frac{\lfloor J_n(\frac{\pi}{a_{\max}}) \rfloor}{nJ_n} \leq \frac{\pi}{na_{\max}}$$

up to $\frac{1}{n}O((\frac{\text{Log Log } n}{\text{Log } n})^2)$ - terms. As soon as n is large enough, we deduce that z lies in the interior of $D_{J_n-k,n}$. Since the function $x \rightarrow \text{Log}(2 \sin(\pi x))$ is negative and strictly increasing on $(0, 1/6)$, the sequence $(|z_{j,n}|)_{j=H_n, \dots, J_n}$ is strictly increasing, by (2.22). Hence we deduce that the circle $|z| = 1 - c_n/n$ intersects all the circles $C_{j,n}$ for $j = J_n - k_{\max}, \dots, J_n$.

The same arguments show that the external Rouché circle $|z| = 1 - c_n/n$ intersects all the circles $C(z_{j,n}, \frac{\pi|z_{j,n}|}{ns_{j,n}})$ for $j = J_n + 1, J_n + 2, \dots, 2J_n - H_n + 1$.

The quantities $s_{j,n}$, for $j = J_n + 1, \dots, 2J_n - H_n + 1$, are easily calculated (left to the reader) so that the distance (length of the j -th circle segment)

$$\left| \frac{z_{j,n}}{|z_{j,n}|} (1 - \frac{c_n}{n}) - y_j \right| = \left| \frac{z_{j,n}}{|z_{j,n}|} (1 - \frac{c_n}{n}) - y'_j \right|$$

for $y_j, y'_j \in C(z_{j,n}, \frac{\pi|z_{j,n}|}{ns_{j,n}}) \cap C(0, 1 - \frac{c_n}{n})$, $y_j \neq y'_j$, be independent of j in the interval $\{J_n + 1, \dots, 2J_n - H_n + 1\}$ and equal to

$$(2.50) \quad \frac{\pi|z_{J_n,n}|}{na_{\max}}.$$

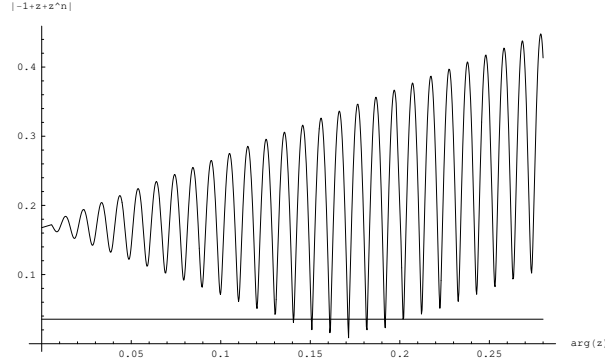


FIGURE 2. Oscillations of the upper bound $|-1 + z + z^n|$ of the Rouché inequality (2.7), for z running over the curve $|z| = 1 - c_n/n$ (here represented with $n = 615$) as a function of $\arg(z)$ in $[0, 0.28]$. The minima correspond to the angular positions of the zeroes $z_{j,n}$ of the trinomial $-1 + X + X^n$, for $j = 1, 2, \dots, H_n, \dots, J_n, \dots, 2J_n - H_n + 1, \dots$ ($J_{615} = 17, H_{615} = 12$). The angular separation between two successive minima is $\approx 2\pi/n$. The difference between two successive minima is $\approx 2\pi/n$. For $n = 615$, the arguments $2\pi(\text{Log } n)/n$ (limiting the bump sector), $\arg(z_{H_n,n})$ and $\arg(z_{J_n,n})$ are respectively equal to $0.0656\dots, 0.12189\dots, 0.17129\dots$. The horizontal line at the y -coordinate $0.0354\dots$ is the value of the left-hand side term of the Rouché inequality (2.7) (Proposition 2.47); it is always strictly smaller than the minimal value of the oscillating function $|-1 + z + z^n|$ on the external boundary $\partial \mathcal{K}_n^{\text{ext}}$, whose geometry surrounds the roots $z_{j,n}$ for j between $H_n + 1$ and $2J_n - H_n + 1$.

Then the two sequences of moduli of centers $(|z_{j,n}|)_{j=J_n+1, \dots, 2J_n-H_n+1}$ and of radii $(\frac{\pi|z_{j,n}|}{ns_{j,n}})_{j=J_n+1, \dots, 2J_n-H_n+1}$ are both increasing, with the fact that the corresponding disks $D(z_{j,n}, \frac{\pi|z_{j,n}|}{ns_{j,n}})$ keep constant the intersection chord $\arg(y_j) - \arg(y'_j) = \frac{\pi|z_{j,n}|}{na_{\max}}$ with the external Rouché circle $|z| = 1 - c_n/n$.

Let $z \in C(0, 1 - \frac{c_n}{n})$, $\varphi := \arg(z) \in [0, \pi]$. Denote by $Z(\varphi) := |G_n((1 - \frac{c_n}{n})e^{i\varphi})|^2 = |-1 + (1 - \frac{c_n}{n})e^{i\varphi} + (1 - \frac{c_n}{n})^n e^{in\varphi}|^2$. The expansion of the function $Z(\varphi)$ as a function of φ , up to $O(1/n)$ -terms, is the following: $Z(\varphi) =$

$$\begin{aligned}
& (-1 + (1 - \frac{c_n}{n})\cos(\varphi) + (1 - \frac{c_n}{n})^n \cos(n\varphi))^2 + ((1 - \frac{c_n}{n})\sin(\varphi) + (1 - \frac{c_n}{n})^n \sin(n\varphi))^2 \\
& = 2 + e^{-2c} - 2\cos(\varphi) - 2e^{-c}\cos(n\varphi) + 2e^{-c}\cos(\varphi)\cos(n\varphi) + 2e^{-c}\sin(\varphi)\sin(n\varphi) \\
& = 2 + e^{-2c} - 2\cos(\varphi) - 4e^{-c}\sin(\frac{\varphi}{2})\left(\cos(n\varphi)\sin(\frac{\varphi}{2}) - \sin(n\varphi)\cos(\frac{\varphi}{2})\right) \\
(2.51) \quad & = 2 + e^{-2c} - 2\cos(\varphi) + 4e^{-c}\sin(\frac{\varphi}{2})\sin(n\varphi - \frac{\varphi}{2}).
\end{aligned}$$

The function $Z(\varphi)$, defined on $[0, \pi/3]$, is almost-periodic (in the sense of Besicovitch and Bohr), takes the value 0 at $\varphi = \arg(z_{J_n,n})$, and therefore, up to $O(1/n)$ -terms, has its minima

at the successive arguments $\arg(z_{J_n,n}) + \frac{2k\pi}{n}$ for $|k| = 0, 1, 2, \dots, J_n - H_n + 1, \dots$ (Figure 2). For such integers k , from (2.51), we deduce the successive minima

$$(2.52) \quad |-1 + z_{J_n,n}e^{-2ik\pi/n} + (z_{J_n,n}e^{-2ik\pi/n})^n| = |G_n(z_{J_n,n})| + \frac{2|k|\pi}{n} = \frac{2|k|\pi}{n}$$

up to $\frac{1}{n}O\left(\left(\frac{\text{Log Log } n}{\text{Log } n}\right)^2\right)$ - terms, with $\arg(z_{J_n,n}e^{-2ik\pi/n}) = \arg(z_{J_n-k,n})$ up to $O(1/n)$ -terms.

With the above notations, denote by y_j, y'_j the two points of $C(0, 1 - \frac{c_n}{n})$ which belong to $C_{j,n}$ for $2H_n - J_n \leq j \leq J_n$, to $C(z_{j,n}, \frac{\pi|z_{j,n}|}{ns_{j,n}})$ for $J_n + 1 \leq j \leq 2J_n - H_n + 1$. Writing them by increasing argument, we have:

$$(2.53) \quad y_{2H_n-J_n}, y'_{2H_n-J_n}, \dots, y_{H_n}, y'_{H_n}, \dots, y_{J_n}, y'_{J_n}, y_{J_n+1}, y'_{J_n+1}, \dots, y_{2J_n-H_n+1}, y'_{2J_n-H_n+1}.$$

The Rouché inequality (2.47) is obviously satisfied at each y_j and y'_j for $j = 2H_n - J_n, \dots, J_n$. Let us show that this inequality holds at each point y_j and y'_j for $j = J_n + 1, \dots, 2J_n - H_n + 1$. Indeed, for such a point, say y_j , there exists

$$\xi_j = w_j z_{J_n,n} e^{2i(j-J_n)\pi/n} + (1 - w_j)y_j, \quad \text{for some } w_j \in [0, 1],$$

lying in the segment $\left[z_{J_n,n}e^{2i(j-J_n)\pi/n}, y_j\right]$ such that

$$G_n(y_j) = G_n(z_{J_n,n}e^{2i(j-J_n)\pi/n}) + (y_j - z_{J_n,n}e^{2i(j-J_n)\pi/n})G'_n(\xi_j)$$

with, using (2.50),

$$|G_n(y_j) - G_n(z_{J_n,n}e^{2i(j-J_n)\pi/n})| = |y_j - z_{J_n,n}e^{2i(j-J_n)\pi/n}| |G'_n(\xi_j)| = \frac{\pi|z_{J_n,n}|}{na_{\max}} |G'_n(\xi_j)|.$$

The derivative of $G_n(z)$ is $G'_n(z) = 1 + nz^{n-1}$. Up to $O(1/n)$ -terms, the line generated by the segment $\left[z_{J_n,n}e^{2i(j-J_n)\pi/n}, y_j\right]$ is tangent to the circle $C(0, 1 - c_n/n)$, and the modulus $\frac{1}{n}|G'_n(\xi_j)|$ satisfies

$$\frac{1}{n}|G'_n(\xi_j)| = \frac{1}{n}|G'_n(z_{J_n,n}e^{2i(j-J_n)\pi/n})| = \frac{1}{n}|G'_n(z_{J_n,n})| = \lim_{n \rightarrow +\infty} \frac{1}{n}|G'_n(z_{J_n,n})| = e^{-c}.$$

From $|G_n(y_j)| \geq |G_n(y_j) - G_n(z_{J_n,n}e^{2i(j-J_n)\pi/n})| - |G_n(z_{J_n,n}e^{2i(j-J_n)\pi/n})|$ and (2.50) we deduce

$$(2.54) \quad |G_n(y_j)| \geq \frac{\pi|z_{J_n,n}|}{a_{\max}} e^{-c} - \frac{2\pi|j - J_n|}{n}.$$

But, by definition of H_n , still up to $O(1/n)$ -terms, for $|j - J_n| \leq J_n - H_n - 1$,

$$(2.55) \quad \frac{2\pi|j - J_n|}{n} \leq \frac{2\pi(J_n - H_n - 1)}{n} = \arg(z_{J_n,n}) - \arg(z_{H_n+1,n}) \leq \frac{e^{-2c}}{1 - e^{-c}}.$$

This inequality is in particular satisfied for the last two values of $|j - J_n|$ which are $J_n - H_n$ and $J_n - H_n + 1$ up to $O(1/n)$ -terms.

Since the inequality

$$(2.56) \quad 0.0710\dots = 2 \frac{e^{-2c}}{1 - e^{-c}} < \frac{\pi|z_{J_n,n}|}{a_{\max}} e^{-c} = 0.0914\dots$$

holds, from (2.54), (2.55) and (2.56), as soon as n is large enough, we deduce the Rouché inequality

$$|G_n(y_j)| \geq \frac{\pi|z_{J_n,n}|}{a_{\max}} - \frac{e^{-2c}}{1-e^{-c}} \geq \frac{e^{-2c}}{1-e^{-c}}.$$

Therefore the conditions of Rouché (2.48) hold at all the points y_j and y'_j of (2.53).

Let us prove that the conditions of Rouché (2.48) hold on each arc $y'_j y_{j+1}$ of the circle $|z| = 1 - c_n/n$, for $j = 2H_n - J_n, 2H_n - J_n + 1, \dots, 2J_n - H_n$. Indeed, from (2.51), the derivative $Z'(\varphi)$ takes a positive value at the extremity y'_j while it takes a negative value at the other extremity y_{j+1} . $Z(\varphi)$ is almost-periodic of almost-period $2\pi/n$. The function $\sqrt{Z(\varphi)}$ is increasing on $(\arg(z_{j,n}), \arg(z_{j,n}) + \frac{\pi}{n})$ and decreasing on $(\arg(z_{j,n}) + \frac{\pi}{n}, \arg(z_{j,n}) + 2\frac{\pi}{n})$; on the arc $y'_j y_{j+1}$ it takes the value $|G_n(y'_j)| \geq \frac{e^{-2c}}{1-e^{-c}}$, admits a maximum, and decreases to $|G_n(y_{j+1})| \geq \frac{e^{-2c}}{1-e^{-c}}$. Hence, (2.7) holds true for all $z \in C(0, 1 - c_n/n)$ with $\arg(y'_j) \leq \arg(z) \leq \arg(y_{j+1})$.

Let us now prove that the condition of Rouché (2.7) is satisfied in the angular sector $0 \leq \arg(z) \leq \arg(z_{H_n,n})$. Indeed, in this angular sector, the successive minima of $\sqrt{Z(\varphi)}$ are all above $\frac{e^{-2c}}{1-e^{-c}}$ by the definition of H_n and (2.52). Hence the claim.

Let us prove that the condition of Rouché (2.7) is satisfied in the angular sector $\arg(z_{2J_n-H_n+1,n}) \leq \arg(z) \leq \frac{\pi}{2}$. In this angular sector, the oscillations of $\sqrt{Z(\varphi)}$ still occur by the form of (2.51) and the successive minima of $\sqrt{Z(\varphi)}$ are all above $\frac{e^{-2c}}{1-e^{-c}}$ for $\frac{2J_n-H_n+2}{J_n} \leq \arg(z) \leq \pi/2$, by (2.52) for $k \geq J_n - H_n + 1$. We deduce the claim.

The condition of Rouché (2.7) is also satisfied in the angular sector $\pi \leq \arg(z) \leq \pi/2$, since then $\cos(\varphi) \leq 0$ and therefore $\sqrt{Z(\varphi)} \geq \sqrt{2 + e^{-2c} - 4e^{-c}} = 1.15\dots$. Since this lower bound is greater than the value $\frac{e^{-2c}}{1-e^{-c}} = 0.0354\dots$ we deduce the claim.

Let us show that the conditions of Rouché (2.7) are also satisfied on the arcs $C(z_{j,n}, \frac{\pi|z_{j,n}|}{ns_{j,n}}) \cap \overline{D}(0, 1 - \frac{c_n}{n})$ for $j = J_n + 1, \dots, 2J_n - H_n + 1$. For such an integer j , let us denote such an arc by $y_j y'_j$. The two extremities y_j and y'_j of the arc $y_j y'_j$ of the circle $C(z_{j,n}, \frac{\pi|z_{j,n}|}{ns_{j,n}})$ define the same value of the difference cosine, say $X_j := \cos(\arg(y_j - z_{j,n}) - \arg(z_{j,n})) = \cos(\arg(y'_j - z_{j,n}) - \arg(z_{j,n}))$, by (2.50). The conditions of Rouché are already satisfied at the points y_j and y'_j by the above. Recall that, for any fixed $a \geq 1$, the function $\kappa(X, a)$, defined in (2.14), is such that the partial derivative $\partial \kappa_X$ of $\kappa(X, a)$ is strictly negative on the interior of $[-1, +1] \times [1, +\infty)$. In particular the function $\kappa(X, s_{j,n})$ is decreasing. For any point Ω of the arc $y_j y'_j$, we denote by $X = \cos(\arg(\Omega - z_{j,n}) - \arg(z_{j,n}))$. We deduce, up to $O(1/n)$ -terms,

$$\frac{e^{-2c}}{1-e^{-c}} \leq \kappa(X_j, s_{j,n}) \leq \kappa(X, s_{j,n}), \quad \text{for all } X \in [-1, X_j],$$

hence the result. \square

2.4. A dissociation theorem and an alternate definition.

Theorem 2.19. *Let $c_{lent} := \min_{n \geq 260} (c_n - \frac{\pi}{a_{max}})$. Let $n \geq 260$ and $\beta > 1$ be a real number such that $\text{dyg}(\beta) = n$, Denote by*

$$f_\beta(z) = -1 + z + z^n + z^{m_1} + z^{m_2} + \dots + z^{m_j} + z^{m_{j+1}} + \dots,$$

where $m_1 - n \geq n - 1$, $m_{j+1} - m_j \geq n - 1$ for $j \geq 1$, the Parry Upper function at β . Then the zeroes of $f_\beta(z)$ of modulus < 1 which lie in $-\arg(z_{J_n, n}) - \frac{\pi}{na_{max}} < \arg z < +\arg(z_{J_n, n}) + \frac{\pi}{na_{max}}$ either belong to

$$\left\{ z \mid ||z| - 1| < \frac{1}{3} \frac{c_{lent}}{n} \right\} \quad \text{or to} \quad \left\{ z \mid ||z| - 1| > \frac{c_{lent}}{n} \right\}.$$

In the second class of zeroes, all the zeroes are simple, and lie in the union

$$D_{0,n} \cup \bigcup_{j=1}^{J_n} (\overline{D_{j,n}} \cup D_{j,n});$$

there is one zero per disk $D_{j,n}$, $\overline{D_{j,n}}$, the disk $D_{0,n}$ containing the element β^{-1} .

Proof. Denote by

$$\mathcal{S}_n := \left\{ z \mid \theta_{n-1} \leq |z| < 1, -\arg(z_{J_n, n}) - \frac{\pi}{na_{max}} \leq \arg z \leq +\arg(z_{J_n, n}) + \frac{\pi}{na_{max}} \right\}$$

the truncated angular sector and let

$$\widehat{\mathcal{S}}_n := \overset{\circ}{\mathcal{S}}_n \setminus \left(\bigcup_{j=1}^{J_n} (D_{j,n} \cup \overline{D_{j,n}}) \cup D(\theta_n, \theta_n - \theta_{n-1}) \right)^{cl}$$

the open truncated angular sector obtained from \mathcal{S}_n by removing the closure of the Rouché disks $\overline{D_{j,n}}, D_{j,n}$ centered at the zeroes $z_{j,n}$ of $G_n(z)$ in \mathcal{S}_n of respective radius $\frac{\pi|z_{j,n}|}{na_{max}}$, and of $D(\theta_n, \theta_n - \theta_{n-1})$. The argument $\arg(z_{J_n, n})$ is defined in (2.18). The analytic function $G_n(z)$ has no zero in the adherence $\widehat{\mathcal{S}}_n$ of $\widehat{\mathcal{S}}_n$ and reaches its infimum $\inf_{z \in \widehat{\mathcal{S}}_n} |-1 + z + z^n| > 0$ on the boundary $\partial \widehat{\mathcal{S}}_n$ of $\widehat{\mathcal{S}}_n$. On the Rouché circles $C_{j,n}, \overline{C_{j,n}}$, $j = 1, \dots, J_n$, using (2.7) and (2.27), this infimum is bounded from below by

$$\frac{|Z|^{2n-1}}{1 - |Z|^{n-1}}$$

where Z is the point of $C_{1,n}$ of smallest modulus, which is such that $|Z| = |\theta_n - \frac{\pi}{na_{max}} \theta_n|$ at the first order. Putting aside the Rouché circles, using the inequality $|-1 + z + z^n| \geq ||-1 + z| - |z^n||$, the minimum of $|-1 + z + z^n|$ on the arcs $|z| = 1, |z| = \theta_{n-1}$, the segments $\arg x = \pm(\arg(z_{J_n, n}) + \frac{\pi}{na_{max}})$ and the circle $C(\theta_n, \theta_n - \theta_{n-1})$ on $\partial \widehat{\mathcal{S}}_n$ is bounded from below by

$$|-1 + \theta_{n-1}| - |\theta_{n-1}^n| = (1 - \theta_{n-1})^2.$$

Denote

$$\delta_n := \min \left\{ (1 - \theta_{n-1})^2, \frac{|Z|^{2n-1}}{1 - |Z|^{n-1}} \right\}.$$

We have: $0 < \delta_n \leq \inf_{z \in \widehat{\mathcal{F}}_n} |-1 + z + z^n|$ and $\lim_{n \rightarrow \infty} \delta_n = 0$. It is easy to show that

$$\lim_{n \rightarrow \infty} \frac{\text{Log } \delta_n}{n} = 0.$$

Using §5.3 in [5] this limit condition allows to calculate a first-order estimate of the thickness of the annular neighbourhood of the unit circle, in $\widehat{\mathcal{F}}_n$, which contains the roots of a polynomial section $-1 + z + z^n + z^{m_1} + z^{m_2} + \dots + z^{m_s}$ of $f_\beta(z)$; this estimate is

$$(2.57) \quad e(s) = 1 - \left(1 - 2 \frac{(n-1)(s - \delta_n)}{(n-1)(s^2 + s) + 2(m_s - n)} \right)^{1/(n-1)}.$$

In the expression (2.57) n is fixed, as well as the sequence $(m_j)_{j \geq 1}$ since β is fixed, therefore $f_\beta(z)$ also; the integer m_s tends to infinity, if s tends to infinity, since $m_s - n \geq (m_1 - n) + \sum_{j=2}^s (m_j - m_{j-1}) \geq s(n-1)$; the integer s is large enough (at least to have $s - \delta_n > 0$) and $\lim_{s \rightarrow \infty} e(s) = 0$.

Among all the Rouché disks $D_{j,n}$, $1 \leq j \leq J_n$, the J_n th Rouché disk $D_{J_n,n}$ is the closest to the unit circle (by (iii-2) in Proposition 5.2). By Lemma 2.14 its center is $z_{J_n,n}$, of modulus $|z_{J_n,n}| = 1 - \frac{c_n}{n}$, and its radius is $\frac{\pi|z_{J_n,n}|}{na_{\max}} < \frac{\pi}{na_{\max}}$. By Lemma 2.14 the limit $c = \lim_{n \rightarrow \infty} c_n$ exists, is positive, and, from a numerical viewpoint, $c - \frac{\pi}{a_{\max}} = 1.76274 - 0.53479 = 1.22794 \dots$. By the asymptotic expansion of c_n in Lemma 2.14, the constant $c_{\text{lent}} := \min_{n \geq 260} (c_n - \frac{\pi}{a_{\max}})$ is positive. The disk $\{z \mid |z| < 1 - \frac{c_{\text{lent}}}{n}\}$ contains all the Rouché disks $D_{j,n}, \overline{D_{j,n}}, 1 \leq j \leq J_n$, and $D_{0,n}$.

Let assume that $f_\beta(z)$ has a zero in

$$\widehat{\mathcal{F}}_n \cap \left\{ z \mid |z| < 1 - \frac{1}{3} \frac{c_{\text{lent}}}{n} \right\}.$$

Denote it by z , counted with multiplicity. There exists $r > 0$ small enough such that the open disk $D(z, r)$ be included in $\widehat{\mathcal{F}}_n \cap \{z \mid |z| < 1 - \frac{1}{3} \frac{c_{\text{lent}}}{n}\}$ and only contains the zero z of $f_\beta(z)$. By Hurwitz Theorem (for instance cf §11 in Chap. 2 in [18]) the number of zeroes of any polynomial section $-1 + z + z^n + z^{m_1} + z^{m_2} + \dots + z^{m_s}$ of $f_\beta(z)$ in $D(z, r)$ should be equal to the multiplicity ≥ 1 of z , as soon as s is large enough, say $s \geq s_0$ for some s_0 .

Since $\lim_{s \rightarrow 0} e(s) = 0$, we obtain a contradiction by taking s_0 such that $e(s) \leq \frac{c_{\text{lent}}}{10n}$ for all $s \geq s_0$. The constant 10, at the denominator, is arbitrary and may be taken eventually larger. This means that all the zeroes of all the polynomial sections of $f_\beta(z)$, in $\widehat{\mathcal{F}}_n$, for all $s \geq s_0$, are contained in

$$1 - \frac{c_{\text{lent}}}{3n} < |z| < 1.$$

But $\{z \mid 1 - \frac{c_{\text{lent}}}{3n} < |z| < 1\} \cap D(z, r) = \emptyset$. Contradiction.

Therefore the zeroes of $f_\beta(z)$ which lie in the open angular sector

$$\left\{ z \in \mathbb{C} : |z| < 1, -\arg(z_{J_n,n}) - \frac{\pi}{na_{\max}} < \arg z < \arg(z_{J_n,n}) + \frac{\pi}{na_{\max}} \right\}$$

are located either in the Rouché disks by Theorem 2.4 and Theorem 2.11, or in a small neighbourhood of the unit circle included in $\{z \mid 1 - \frac{c_{\text{lent}}}{3n} < |z| < 1\}$. This dichotomy naturally extends to the zeroes of any polynomial section of $f_\beta(z)$ (cf the proofs of Theorem 2.4, Theorem 2.11 and Theorem 2.19). \square

Definition 2.20. Let $n \geq 260$. Let $\beta > 1$ be a real number such that $\text{dyg}(\beta) = n$. With the notations of Theorem 2.19, the poles of $\zeta_\beta(z)$ which belong to the angular sector

$$(2.58) \quad \left\{ z \in \mathbb{C} : |z| < 1 - \frac{c_{lent}}{n}, |\arg z| \leq \arg(z_{J_n, n}) + \frac{\pi}{na_{max}} \right\}$$

are called the *lenticular poles* of $\zeta_\beta(z)$.

3. CONTINUITY OF THE LENTICULAR POLES - PROOF OF THEOREM 1.1

Let $n \geq 260$. To n is associated the set of the lenticular zeroes of the trinomial $-1 + x + x^n$ of imaginary part ≥ 0 , as:

$$\{\theta_n, z_{1,n}, z_{2,n}, \dots, z_{J_n, n}\},$$

and the set of Rouché disks (defined in Theorem 2.11)

$$\{D_{1,n}, D_{2,n}, \dots, D_{J_n, n}\},$$

the j th-disk $D_{j,n}$ being centered at $z_{j,n}$. For $j = 1, \dots, J_n$, the disks $D_{j,n}$ satisfy: $\overline{D_{j,n}} \subset D(0, 1)$, $\overline{D_{j,n}} \cap C(0, 1) = \emptyset$, $z_{j,n-1} \in D_{j,n}$, and, for any real number $\beta \in [\theta_n^{-1}, \theta_{n-1}^{-1}]$, $f_\beta(z)$ admits a unique zero $\omega_{j,n}$ in $D_{j,n}$, which is simple by Theorem 2.11. Since $\beta \rightarrow f_\beta(z)$ is injective on $[\theta_n^{-1}, \theta_{n-1}^{-1}]$, by Proposition 2.2, the map $\beta \rightarrow \omega_{j,n} = \omega_{j,n}(\beta)$ is well-defined. Let us denote this map by $\omega_{j,n}$ (with no ambiguity) and by $\omega_{j,n}(\beta)$ the image of β (instead of $\omega_{j,n}$). Let $\widetilde{\mathcal{S}}_{j,n} := \overline{\omega_{j,n}([\theta_n^{-1}, \theta_{n-1}^{-1}])}$ be the adherence of the image of the closed interval $[\theta_n^{-1}, \theta_{n-1}^{-1}]$; it is a compact subset of $D_{j,n}$ such that $\widetilde{\mathcal{S}}_{j,n} \cap \partial D_{j,n} = \emptyset$, for which the image of the left extremity of the interval is the center of the disk $D_{j,n}$: $\omega_{j,n}(\theta_n^{-1}) = z_{j,n}$.

Lemma 3.1. *Let $n \geq 3$. The two analytic functions $-1 + z + z^{n-1}$ and $-1 + z + z^n + \sum_{q=1}^{\infty} z^{q(n-1)+n}$ have the same zeroes (with the same multiplicities = 1) inside the open unit disk.*

Proof. Let $x_0, |x_0| < 1$, be a zero of the trinomial $-1 + z + z^{n-1}$. Then

$$0 = -1 + x_0 + x_0^{n-1} = -1 + x_0 + x_0^n \frac{1}{x_0} = -1 + x_0 + x_0^n (1 + x_0^{n-2}) = -1 + x_0 + x_0^n + x_0^{n+n-2+1} \frac{1}{x_0}.$$

Let us replace the last $\frac{1}{x_0}$ by $1 + x_0^{n-2}$. And so on, iteratively. Doing this operation infinitely many times provides the identity

$$0 = -1 + x_0 + x_0^{n-1} = -1 + x_0 + x_0^n + \sum_{q=1}^{\infty} x_0^{q(n-1)+n}.$$

The converse comes from

$$-1 + z + z^n + \sum_{q=1}^{\infty} z^{q(n-1)+n} = \frac{-1 + z + z^{n-1}}{1 - z^{n-1}}, \quad |z| < 1.$$

Multiplicities are equal to 1 by [19]. □

Denote $\delta := \min\{1 - |z| \mid z \in \cup_{j=1}^{J_n} \widetilde{\mathcal{S}}_{j,n}\} > 0$.

Proposition 3.2. *Let $n \geq 260$. For all $1 \leq j \leq J_n$, the map*

$$\omega_{j,n} : [\theta_n^{-1}, \theta_{n-1}^{-1}] \rightarrow D_{j,n}, \quad \beta \rightarrow \omega_{j,n}(\beta)$$

is continuous.

Proof. Let β_1, β_2 be two real numbers in the open interval $(\theta_n^{-1}, \theta_{n-1}^{-1})$, and assume $\beta_1 < \beta_2$. To β_1 , resp. β_2 , is associated uniquely the sequence $(t_i)_{i \geq 1} \in \{0, 1\}$, resp. $(t'_i)_{i \geq 1} \in \{0, 1\}$, of the coefficients of the Rényi β_1 -expansion of unity $d_{\beta_1}(1) = 0.t_1 t_2 t_3 \dots$, resp. of $d_{\beta_2}(1) = 0.t'_1 t'_2 t'_3 \dots$; the two Parry Upper functions $f_{\beta_1}(z)$ and $f_{\beta_2}(z)$ are deduced from them:

$$f_{\beta_1}(z) = -1 + \sum_{j \geq 1} t_j z^j, \quad f_{\beta_2}(z) = -1 + \sum_{j \geq 1} t'_j z^j.$$

The inequality ' $<$ ' is translated by the lexicographical inequality ' $<_{lex}$ ' on the sequences $(t_i)_{i \geq 1}$, resp. (t'_i) , by Proposition 2.2. When β_2 is close to β_1 , the first digits are the same. We define the lexicographical metric \bar{d} by:

$$\bar{d}(\beta_1, \beta_2) := e^{-r} \quad \text{iff} \quad t_1 = t'_1, t_2 = t'_2, \dots, t_r = t'_r, t_{r+1} \neq t'_{r+1}.$$

To prove the continuity of $\omega_{j,n}$ it suffices to show that, for all $\varepsilon > 0$, there exists $\eta > 0$ such that

$$\bar{d}(\beta_1, \beta_2) < \eta \quad \implies \quad |\omega_{j,n}(\beta_1) - \omega_{j,n}(\beta_2)| < \varepsilon,$$

and to establish the continuity at the extremity θ_{n-1}^{-1} of the closed interval $[\theta_n^{-1}, \theta_{n-1}^{-1}]$.

We have

$$f_{\beta_1}(\beta_1^{-1}) = f_{\beta_1}(\omega_{j,n}(\beta_1)) = f_{\beta_2}(\beta_2^{-1}) = f_{\beta_2}(\omega_{j,n}(\beta_2)) = 0.$$

Since $\beta_1 \neq \beta_2$ and that the disks $D_{j,n}$ have the property that $f_{\beta_2}(z)$ contains an unique zero in it (cf Theorem 2.11),

$$\begin{aligned} 0 \neq f_{\beta_2}(\omega_{j,n}(\beta_1)) &= f_{\beta_2}(\omega_{j,n}(\beta_2) + (\omega_{j,n}(\beta_1) - \omega_{j,n}(\beta_2))) \\ &= (\omega_{j,n}(\beta_1) - \omega_{j,n}(\beta_2)) \left[\sum_{q=1}^{\infty} \frac{f_{\beta_2}^{(q)}(\omega_{j,n}(\beta_2))}{q!} (\omega_{j,n}(\beta_1) - \omega_{j,n}(\beta_2))^{q-1} \right]. \end{aligned}$$

Now by Theorem 2.11 the unique zero $\omega_{j,n}(\beta_2^{-1})$ in $D_{j,n}$ is simple. Thus $f'_{\beta_2}(\omega_{j,n}(\beta_2)) \neq 0$ and the function

$$z \rightarrow f_{\beta_2}(z)/(z - \omega_{j,n}(\beta_2)),$$

extended by continuity by $f'_{\beta_2}(\omega_{j,n}(\beta_2))$ at $\omega_{j,n}(\beta_2)$, does not take the value 0 on the compact $\widetilde{\mathcal{S}}_{j,n}$. Hence, the function

$$z \rightarrow \left| \sum_{q=1}^{\infty} \frac{f_{\beta_2}^{(q)}(\omega_{j,n}(\beta_2))}{q!} (z - \omega_{j,n}(\beta_2))^{q-1} \right|$$

admits an infimum, say $\widetilde{\mu}_j$ on $\widetilde{\mathcal{S}}_{j,n}$. Denote $\widetilde{\mu} := \min\{\widetilde{\mu}_j \mid j = 1, 2, \dots, J_n\} > 0$ the infimum, common to all the disks $D_{j,n}$.

Now, with $\bar{d}(\beta_1, \beta_2) := e^{-r}$,

$$f_{\beta_2}(\omega_{j,n}(\beta_1)) = (f_{\beta_2} - f_{\beta_1})(\omega_{j,n}(\beta_1)) = \sum_{k=r+1}^{\infty} (t_k - t'_k)(\omega_{j,n}(\beta_1))^k.$$

Therefore

$$|f_{\beta_2}(\omega_{j,n}(\beta_1))| \leq \sum_{k=r+1}^{\infty} |\omega_{j,n}(\beta_1)|^k \leq \sum_{k=r+1}^{\infty} (1 - \delta)^k \leq (1 - \delta)^{r+1} \frac{1}{\delta}.$$

Hence

$$|\omega_{j,n}(\beta_1) - \omega_{j,n}(\beta_2)| \leq \frac{1}{\mu} (1 - \delta)^{r+1} \frac{1}{\delta}.$$

It suffices to take r large enough to have the property of continuity.

Let us consider the case of the extremity $\beta_2 = \theta_{n-1}^{-1}$. By Lemma 3.1, since θ_{n-1} is root of $-1 + x + x^{n-1}$ it satisfies:

$$0 = -1 + \theta_{n-1} + \theta_{n-1}^n + \sum_{q=1}^{\infty} \theta_{n-1}^{q(n-1)+n}.$$

The metric \bar{d} can be extended to $(\theta_n^{-1}, \theta_{n-1}^{-1}] \times (\theta_n^{-1}, \theta_{n-1}^{-1}]$ since the sequence of coefficients of the power series $-1 + z + z^n + \sum_{q \geq 1} z^{q(n-1)+n}$ is obviously in the adherence of the set

$$\{(t_i) \mid \text{the digits } t_i \text{ being those of } d_{\beta}(1) \text{ for all } \beta \in (\theta_n^{-1}, \theta_{n-1}^{-1})\},$$

and not an isolated point. The case is simpler for the other extremity $\beta_1 = \theta_n^{-1}$. □

Remark 3.3. In [6] Flatto, Lagarias and Poonen study the continuity of the modulus of the first root (given in Proposition 5.13)

$$(3.1) \quad \beta \rightarrow |\omega_{1,n}(\beta)|, \quad 1 < \beta < 2$$

over the union of the intervals $[\theta_n^{-1}, \theta_{n-1}^{-1}]$. The curve given by Figure 1 in [6] suggests that the map $\omega_{1,n}$ is injective on $(\theta_n^{-1}, \theta_{n-1}^{-1})$ but that the union $\widetilde{\mathcal{S}}_{1,n-1} \cup \widetilde{\mathcal{S}}_{1,n} \cup \widetilde{\mathcal{S}}_{1,n+1}$, for all n large enough, is a self-intersecting curve.

4. THE LENTICULAR MEASURE

The method of Rouché allows to define optimally the lenticulus \mathcal{L}_{β} of the lenticular poles of $\zeta_{\beta}(z)$ (Theorem 2.4 and Proposition 2.9), for any real number $1 < \beta < \theta_{259}^{-1}$ and $\beta \notin \{\theta_n^{-1} : n \geq 260\}$. Then we define the lenticular measure of β by

$$(4.1) \quad \mathfrak{S}_{lentic}(\beta) := \prod_{\omega \in \mathcal{L}_{\beta}} |\omega|^{-1} = \beta \prod_{j=1}^{J_n} |\omega_{j,n}(\beta)|^{-2}.$$

In Section 4.1 Theorem 4.1 proves the existence of a lower bound > 1.15 to the infimum $\liminf_{\beta \rightarrow 1} \mathfrak{S}_{lentic}(\beta)$. In Section 4.2, using

$$(4.2) \quad \beta \times \prod_{j=1}^{J_n} |\omega_{j,n}(\beta)|^{-2} \geq \beta \times \prod_{j=1}^{J_n} (|z_{j,n}| + \frac{t_{j,n}}{n})^{-2},$$

and denoting

$$L_r(\beta) := \text{Log} \left(\beta \times \prod_{j=1}^{J_n} (|z_{j,n}| + \frac{t_{j,n}}{n})^{-2} \right),$$

we obtain the asymptotic expansion of a lower bound of $L_r(\beta)$ as a function of $n = \text{dyg}(\beta)$ in Theorem 4.2. From $\mathfrak{S}_{lent}(\beta) \geq \exp(L_r(\beta))$, the Dobrowolski-type inequality (1.9) is deduced from this asymptotic expansion, in Section 4.3.

This generalizes the case where $\beta \in \{\theta_n^{-1} : n \geq 3\}$ treated in [27]; let us recall it briefly. Let n_0 be an integer such that $\frac{\pi}{3} > 2\pi \frac{\text{Log } n_0}{n_0}$, and let $n \geq n_0$. In there the Mahler measure (cf the Appendix)

$$\mathbf{M}(\theta_n^{-1}) = \mathbf{M}(G_n) = \theta_n^{-1} \prod_{j=1}^{\lfloor n/6 \rfloor} |z_{j,n}|^{-2},$$

which is equal to

$$\mathfrak{S}_{lent}(\theta_n^{-1}) = \prod_{\mathcal{L}_{\theta_n^{-1}}} |z|^{-1},$$

is shown to admit the following asymptotic expansion

$$(4.3) \quad \mathbf{M}(G_n) = \Lambda \left(1 + r(n) \frac{1}{\text{Log } n} + O \left(\frac{\text{Log } \text{Log } n}{\text{Log } n} \right)^2 \right), \quad n \geq n_0,$$

with the constant $1/6$ involved in the Big O, and with $r(n)$ real, $|r(n)| \leq 1/6$. The Dobrowolski-type inequality (1.5) is then deduced from (4.3). Theorem 4.2 generalizes (4.3).

4.1. A lower bound of the liminf of the lenticular measure, β tending to 1. Recall $\kappa = \kappa(1, a_{\max})$ from Definition 2.6. The value $2 \arcsin(\kappa(1, a_{\max})/2) = 0.171784\dots$ is given by Proposition 2.7, and $a_{\max} = 5.8743\dots$ by Theorem 2.4.

Theorem 4.1. *Let $\beta > 1$ be a real number such that $\text{dyg}(\beta) \geq 260$. Let*

$$\Lambda_r := \exp \left(\frac{-1}{\pi} \int_0^{2 \arcsin(\frac{\kappa}{2})} \text{Log} \left(2 \sin \left(\frac{x}{2} \right) \right) dx \right) = 1.16302\dots,$$

and

$$\begin{aligned} \mu_r &:= \exp \left(\frac{-1}{\pi} \int_0^{2 \arcsin(\frac{\kappa}{2})} \text{Log} \left[\frac{1 + 2 \sin(\frac{x}{2}) - \sqrt{1 - 12 \sin(\frac{x}{2}) + 4(\sin(\frac{x}{2}))^2}}{8 \sin(\frac{x}{2})} \right] dx \right) \\ &= 0.992337\dots \end{aligned}$$

The lenticular measure $\mathfrak{S}_{lent}(\beta)$ of β admits a liminf when β tends to 1^+ , bounded from below as

$$(4.4) \quad \liminf_{\text{dyg}(\beta) \rightarrow +\infty} \mathfrak{S}_{lent}(\beta) \geq \Lambda_r \cdot \mu_r = 1.15411\dots$$

Proof. Assume $\beta \notin \{\theta_n^{-1} : n \geq 259\}$. From (4.1) we have

$$\text{Log } \mathfrak{S}_{lent}(\beta) = -\text{Log} \left(\frac{1}{\beta} \right) - 2 \sum_{j=1}^{J_n} \text{Log} |\omega_{j,n}| = \text{Log} \left(\frac{1}{\beta} \right) - 2 \sum_{j=1}^{J_n} \text{Log} |(\omega_{j,n} - z_{j,n}) + z_{j,n}|$$

$$(4.5) \quad = \text{Log}(\beta) - 2 \sum_{j=1}^{J_n} \text{Log} |z_{j,n}| - 2 \sum_{j=1}^{J_n} \text{Log} \left| 1 + \frac{\omega_{j,n} - z_{j,n}}{z_{j,n}} \right|.$$

The first term of (4.5) tends to 0 if $\text{dyg}(\beta)$ tends to $+\infty$ since $\lim_{n \rightarrow \infty} \theta_n = 1$ (Proposition 5.5). Let us turn to the third summation in (4.5). The j -th root $\omega_{j,n} \in \mathcal{L}_\beta$ of $f_\beta(z)$ is the unique root of $f_\beta(z)$ in the disk $D_{j,n} = \{z \mid |\omega_{j,n} - z_{j,n}| < \frac{\pi |z_{j,n}|}{n a_{\max}}\}$. From Theorem 2.11 we have the more precise localization in $D_{j,n}$: $|\omega_{j,n} - z_{j,n}| < \frac{\pi |z_{j,n}|}{n a_{j,n}}$ for $j = \lceil v_n \rceil, \dots, J_n$ (main angular sector), with

$$D\left(\frac{\pi}{a_{j,n}}\right) = \text{Log} \left[\frac{1 + B_{j,n} - \sqrt{1 - 6B_{j,n} + B_{j,n}^2}}{4B_{j,n}} \right]$$

and $B_{j,n} = 2 \sin\left(\frac{\pi j}{n}\right) \left(1 - \frac{1}{n} \text{Log} \left(2 \sin\left(\frac{\pi j}{n}\right)\right)\right)$ (from (2.29)).

For $j = \lceil v_n \rceil, \dots, J_n$ the following inequalities hold:

$$1 - \frac{1}{n} D\left(\frac{\pi}{a_{j,n}}\right) \leq \left| 1 + \frac{\omega_{j,n} - z_{j,n}}{z_{j,n}} \right| \leq 1 + \frac{1}{n} D\left(\frac{\pi}{a_{j,n}}\right),$$

up to second order terms. Let us apply the remainder Theorem of alternating series: for x real, $|x| < 1$, $|\text{Log}(1+x) - x| \leq \frac{x^2}{2}$. Then the third summation in (4.5) satisfies

$$(4.6) \quad -2 \lim_{n \rightarrow \infty} \sum_{j=1}^{J_n} \frac{1}{n} \text{Log} \left[\frac{1 + 2 \sin\left(\frac{\pi j}{n}\right) - \sqrt{1 - 12 \sin\left(\frac{\pi j}{n}\right) + 4 \left(\sin\left(\frac{\pi j}{n}\right)\right)^2}}{8 \sin\left(\frac{\pi j}{n}\right)} \right] \\ \leq \liminf_{n \rightarrow \infty} \left(-2 \sum_{j=1}^{J_n} \text{Log} \left| 1 + \frac{\omega_{j,n} - z_{j,n}}{z_{j,n}} \right| \right).$$

Let us convert the limit to an integral. The Riemann-Stieltjes sum

$$S(F, n) := -2 \sum_{j=1}^{J_n} \frac{1}{n} \text{Log} \left[\frac{1 + 2 \sin\left(\frac{\pi j}{n}\right) - \sqrt{1 - 12 \sin\left(\frac{\pi j}{n}\right) + 4 \left(\sin\left(\frac{\pi j}{n}\right)\right)^2}}{8 \sin\left(\frac{\pi j}{n}\right)} \right] \\ = \frac{-1}{\pi} \sum_{j=1}^{J_n} (x_j - x_{j-1}) F(x_j)$$

with $x_j = \frac{2\pi j}{n}$ and $F(x) := \text{Log} \left[\frac{1 + 2 \sin\left(\frac{x}{2}\right) - \sqrt{1 - 12 \sin\left(\frac{x}{2}\right) + 4 \left(\sin\left(\frac{x}{2}\right)\right)^2}}{8 \sin\left(\frac{x}{2}\right)} \right]$ converges to the limit

$$(4.7) \quad \lim_{n \rightarrow \infty} S(F, n) = \frac{-1}{\pi} \int_0^{0.171784\dots} F(x) dx = \text{Log} \mu_r \quad \text{with} \quad \mu_r = 0.992337\dots$$

In a similar way we transform the limit, as n tends to infinity, of the second summation in (4.5) into an integral in order to obtain the term Λ_r . By Proposition 2.7 let us observe that the Riemann-Stieltjes sum

$$S(f, n) := -2 \sum_{j=1}^{J_n} \frac{1}{n} \text{Log} \left(2 \sin\left(\frac{\pi j}{n}\right) \right) = \frac{-1}{\pi} \sum_{j=1}^{J_n} (x_j - x_{j-1}) f(x_j)$$

with $x_j = \frac{2\pi j}{n}$ and $f(x) := \text{Log} \left(2 \sin\left(\frac{x}{2}\right) \right)$ converges to the limit

$$(4.8) \quad \begin{aligned} \lim_{n \rightarrow \infty} S(f, n) &= \frac{-1}{\pi} \int_0^{\lim_{n \rightarrow \infty} 2\pi \frac{J_n}{n}} f(x) dx = \frac{-1}{\pi} \int_0^{0.171784\dots} f(x) dx \\ &= \text{Log } \Lambda_r = \text{Log}(1.16302\dots). \end{aligned}$$

This limit is a log-sine integral [2]. From [27] §4.2, pp 111–115, in a similar way, we have

$$(4.9) \quad \lim_{n \rightarrow \infty} S(f, n) = -2 \lim_{n \rightarrow \infty} \sum_{j=1}^{J_n} \text{Log} |z_{j,n}|$$

It is easy to check that the corrections due to the roots which lie inside the bump sector (cf Appendix Section 5.1), not counted above, are zero contributions. Since $\mathfrak{S}_{lent}(\beta) \geq \exp(L_r(\beta))$, we deduce the lower bound $\Lambda_r \mu_r$ in (4.4) from (4.6), (4.7), (4.8) and (4.9).

Assume $\beta \in \{\theta_n^{-1} : n \geq 259\}$. By construction we have: $\mathfrak{S}_{lent}(\theta_n^{-1}) = M(G_n)$ for all $n \geq 3$, and $\lim_{n \rightarrow +\infty} M(G_n) = \Lambda = 1.38\dots$ (recalled in (1.6)) which is greater than $1.15\dots$. This establishes completely the inequality (4.4). \square

Let us observe that the constant C given in Theorem 1.2 is

$$C = \Lambda_r \mu_r.$$

4.2. Poincaré asymptotic expansions. The aim of this Section is to prove Theorem 4.2. Assume $\beta > 1$ and $\text{dyg}(\beta) \geq 260$, with $\beta \notin \{\theta_n^{-1} : n \geq 259\}$. In the continuation of the last paragraph, we expand asymptotically a lower bound of $L_r(\beta)$ as a function of $n = \text{dyg}(\beta)$. As a starting point, we have:

$$(4.10) \quad L_r(\beta) = \text{Log}(\beta) - 2 \sum_{j=1}^{J_n} \text{Log} |z_{j,n}| - 2 \sum_{j=1}^{\lfloor v_n \rfloor} \text{Log} \left(1 + \frac{\pi}{n a_{j,n}} \right) - 2 \sum_{j=\lfloor v_n \rfloor}^{J_n} \text{Log} \left(1 + \frac{\pi}{n a_{j,n}} \right),$$

where the $a_{j,n}$ s are given by Theorem 2.11, the sequence (v_n) by the Appendix, and J_n by Definition 2.6 and Proposition 2.7. From Section 4.1 we have $\lim_{\text{dyg}(\beta) \rightarrow \infty} L_r(\beta) \geq \text{Log } \Lambda_r + \text{Log } \mu_r$. In Theorem 4.2, we will gather the asymptotic contributions of each term in (4.10) and obtain the asymptotic expansion of a minorant of $L_r(\beta)$ as a function of n .

(i) First term in (4.10): from Lemma 5.6 and Theorem 5.8,

$$(4.11) \quad \text{Log}(\beta) = \frac{\text{Log } n}{n} (1 - \lambda_n) + \frac{1}{n} O \left(\left(\frac{\text{Log } \text{Log } n}{\text{Log } n} \right)^2 \right) = O \left(\frac{\text{Log } n}{n} \right);$$

(ii) second term in (4.10): from Proposition 5.12, $\sum_{j=\lfloor v_n \rfloor}^{J_n} \text{Log} |z_{j,n}| =$

$$\sum_{j=\lfloor v_n \rfloor}^{J_n} \text{Log} \left(1 + \frac{1}{n} \text{Log} \left(2 \sin\left(\frac{\pi j}{n}\right) \right) \right) + \frac{1}{2n} \left(\frac{\text{Log } \text{Log } n}{\text{Log } n} \right)^2 + \frac{1}{n} O \left(\frac{(\text{Log } \text{Log } n)^2}{(\text{Log } n)^3} \right)$$

with the constant 1 involved in the Big O. Let us apply the remainder Theorem of alternating series: for x real, $|x| < 1$, $|\text{Log}(1+x) - x| \leq \frac{x^2}{2}$. Then

$$(4.12) \quad \left| \sum_{j=\lceil v_n \rceil}^{J_n} \text{Log} |z_{j,n}| - \sum_{j=\lceil v_n \rceil}^{J_n} \frac{1}{n} \text{Log} \left(2 \sin \left(\frac{\pi j}{n} \right) \right) - \sum_{j=\lceil v_n \rceil}^{J_n} \frac{1}{2n} \left(\frac{\text{Log Log } n}{\text{Log } n} \right)^2 \right|$$

$$\leq \sum_{j=\lceil v_n \rceil}^{J_n} \frac{1}{n} \left| O \left(\frac{(\text{Log Log } n)^2}{(\text{Log } n)^3} \right) \right|$$

$$+ \frac{1}{2} \sum_{j=\lceil v_n \rceil}^{J_n} \frac{1}{n^2} \left[\text{Log} \left(2 \sin \left(\frac{\pi j}{n} \right) \right) + \frac{1}{2} \left(\frac{\text{Log Log } n}{\text{Log } n} \right)^2 + O \left(\frac{(\text{Log Log } n)^2}{(\text{Log } n)^3} \right) \right]^2.$$

For $1 \leq j \leq J_n$, the inequalities $0 < 2 \sin(\pi j/n) \leq 1$ and $\text{Log}(2 \sin(\pi j/n)) < 0$ hold. Then $|\text{Log}(2 \sin(\pi j/n))| \leq |\text{Log}(2 \sin(\pi/n))| = O(\text{Log } n)$. On the other hand, the two $O(\cdot)$ s in the rhs of (4.12) involve a constant which does not depend upon j . Therefore, from Proposition 2.7, the rhs of (4.12) is

$$= O \left(\left(\frac{(\text{Log Log } n)^2}{(\text{Log } n)^3} \right) \right) + O \left(\frac{\text{Log}^2 n}{n} \right) = O \left(\left(\frac{(\text{Log Log } n)^2}{(\text{Log } n)^3} \right) \right).$$

On the other hand, the two regimes of asymptotic expansions in the Bump sector (cf Appendix) give

$$\sum_{j=\lceil u_n \rceil}^{\lfloor v_n \rfloor} \text{Log} |z_{j,n}| = O \left(\frac{(\text{Log } n)^{2+\varepsilon}}{n} \right), \quad \sum_{j=1}^{\lfloor u_n \rfloor} \text{Log} |z_{j,n}| = O \left(\frac{(\text{Log } n)^2}{n} \right)$$

and

$$\sum_{j=\lceil \text{Log } n \rceil}^{\lfloor v_n \rfloor} \frac{2}{n} \text{Log} \left(2 \sin \left(\frac{\pi j}{n} \right) \right) = O \left(\frac{(\text{Log } n)^{2+\varepsilon}}{n} \right).$$

Therefore

$$(4.13) \quad -2 \sum_{j=1}^{J_n} \text{Log} |z_{j,n}| = - \sum_{j=\lceil \text{Log } n \rceil}^{J_n} \frac{2}{n} \text{Log} \left(2 \sin \left(\frac{\pi j}{n} \right) \right) + O \left(\left(\frac{\text{Log Log } n}{\text{Log } n} \right)^2 \right)$$

with the constant $\frac{1}{2\pi} \arcsin(\frac{\kappa}{2})$ (from Proposition 2.7) involved in the Big O.

(iii) third term in (4.10): with the definition of ε and (v_n) (Appendix),

$$(4.14) \quad -2 \sum_{j=1}^{\lfloor v_n \rfloor} \text{Log} \left(1 + \frac{\pi}{n a_{j,n}} \right) = O \left(\frac{(\text{Log } n)^{1+\varepsilon}}{n} \right);$$

(iv) fourth term in (4.10): from the Theorem of alternating series,

$$(4.15) \quad \left| \sum_{j=\lceil v_n \rceil}^{J_n} \text{Log} \left(1 + \frac{\pi}{n a_{j,n}} \right) - \sum_{j=\lceil v_n \rceil}^{J_n} \frac{1}{n} D \left(\frac{\pi}{a_{j,n}} \right) - \sum_{j=\lceil v_n \rceil}^{J_n} \frac{1}{n} \text{tl} \left(\frac{\pi}{a_{j,n}} \right) \right| \leq \frac{1}{2} \sum_{j=\lceil v_n \rceil}^{J_n} \left(\frac{\pi}{n a_{j,n}} \right)^2.$$

The terminant $\text{tl}(\frac{\pi}{a_{j,n}}) = O\left(\frac{(\text{Log Log } n)^2}{(\text{Log } n)^3}\right)$ is given by (2.31). From Theorem 2.11, with $B_{j,n} = 2 \sin(\frac{\pi j}{n}) \left(1 - \frac{1}{n} \text{Log}(2 \sin(\frac{\pi j}{n}))\right)$, it is easy to show

$$\begin{aligned} D\left(\frac{\pi}{a_{j,n}}\right) &= \text{Log} \left[\frac{1 + B_{j,n} - \sqrt{1 - 6B_{j,n} + B_{j,n}^2}}{4B_{j,n}} \right] \\ &= \text{Log} \left[\frac{1 + 2 \sin(\frac{\pi j}{n}) - \sqrt{1 - 12 \sin(\frac{\pi j}{n}) + 4 \sin(\frac{\pi j}{n})^2}}{8 \sin(\frac{\pi j}{n})} \right] + O\left(\frac{\text{Log } n}{n}\right). \end{aligned}$$

The rhs of (4.15) is $= O\left(\frac{1}{n}\right)$. Then $-2 \sum_{j=\lceil v_n \rceil}^{J_n} \text{Log}\left(1 + \frac{\pi}{na_{j,n}}\right) =$
(4.16)

$$\sum_{j=\lceil v_n \rceil}^{J_n} \frac{-2}{n} \text{Log} \left[\frac{1 + 2 \sin(\frac{\pi j}{n}) - \sqrt{1 - 12 \sin(\frac{\pi j}{n}) + 4 \sin(\frac{\pi j}{n})^2}}{8 \sin(\frac{\pi j}{n})} \right] + O\left(\frac{(\text{Log Log } n)^2}{(\text{Log } n)^3}\right).$$

The summation $\sum_{j=\lceil v_n \rceil}^{J_n}$ can be replaced by $\sum_{j=\lceil \text{Log } n \rceil}^{J_n}$. Indeed, from the definition of the sequence (v_n) (cf Appendix),

$$\sum_{j=\lceil \text{Log } n \rceil}^{\lceil v_n \rceil} \frac{2}{n} \text{Log} \left[\frac{1 + 2 \sin(\frac{\pi j}{n}) - \sqrt{1 - 12 \sin(\frac{\pi j}{n}) + 4 \sin(\frac{\pi j}{n})^2}}{8 \sin(\frac{\pi j}{n})} \right] = O\left(\frac{(\text{Log } n)^{2+\varepsilon}}{n}\right).$$

Inserting the contributions (4.11) (4.13) (4.14) (4.16) in (4.10) leads to

$$\begin{aligned} L_r(\beta) &\geq \text{Log } \Lambda_r + \text{Log } \mu_r + \left(-\text{Log } \Lambda_r - \sum_{j=\lceil \text{Log } n \rceil}^{J_n} \frac{2}{n} \text{Log} \left(2 \sin\left(\frac{\pi j}{n}\right) \right) \right) \\ &+ \left(-\text{Log } \mu_r - \sum_{j=\lceil \text{Log } n \rceil}^{J_n} \frac{2}{n} \text{Log} \left(\frac{1 + 2 \sin(\frac{\pi j}{n}) - \sqrt{1 - 12 \sin(\frac{\pi j}{n}) + 4 \sin(\frac{\pi j}{n})^2}}{8 \sin(\frac{\pi j}{n})} \right) \right) \\ &+ O\left(\left(\frac{\text{Log Log } n}{\text{Log } n}\right)^2\right) \end{aligned} \tag{4.17}$$

with the constant $\frac{1}{2\pi} \arcsin(\frac{\kappa}{2})$ involved in the Big O. Let us denote by Δ_1 the first term within brackets, resp. Δ_2 the second term within brackets, in (4.17) so that

$$(4.18) \quad D(L_r(\beta)) = \text{Log}(\Lambda_r \mu_r) + \Delta_1 + \Delta_2.$$

Calculation of $|\Delta_1|$: let us estimate and give an upper bound of $|\Delta_1| =$

$$(4.19) \quad \left| \frac{-1}{\pi} \int_0^{2 \arcsin(\kappa/2)} \text{Log} \left(2 \sin(x/2) \right) dx - \sum_{j=\lceil \text{Log } n \rceil}^{J_n} \frac{-2}{n} \text{Log} \left(2 \sin\left(\frac{\pi j}{n}\right) \right) \right|.$$

In (4.19) the sums are truncated Riemann-Stieltjes sums of $\text{Log } \Lambda_r$, the integral being $\text{Log } \Lambda_r$. Referring to Stoer and Bulirsch ([23], pp 126–128) we now replace $\text{Log } \Lambda_r$ by

an approximate value obtained by integration of an interpolation polynomial by the methods of Newton-Cotes; we just need to know this approximate value up to $O\left(\left(\frac{\text{Log Log } n}{\text{Log } n}\right)^2\right)$.

Up to $O\left(\left(\frac{\text{Log Log } n}{\text{Log } n}\right)^2\right)$, we will show that:

(i–1) an upper bound of (4.19) is (κ stands for $\kappa(1, a_{\max})$ as in Proposition 2.7)

$$\frac{\arcsin(\kappa/2)}{\pi} \frac{1}{\text{Log } n},$$

(ii–1) the approximate value of $\text{Log } \Lambda_r$ is independent of the integer m (i.e. step length) used in the Newton-Cotes formulas, assuming the weights $(\alpha_q)_{q=0,1,\dots,m}$ associated with m all positive. Indeed, if m is arbitrarily large, the estimate of the integral should be very good by these methods, ideally exact at the limit ($m \rightarrow +\infty$).

Proof of (i–1): we consider the decomposition of the interval of integration as $(0, 2 \arcsin(\kappa/2)] =$

$$(4.20) \quad \left(0, \frac{2\pi \lceil \text{Log } n \rceil}{n}\right] \cup \left(\bigcup_{j=\lceil \text{Log } n \rceil}^{J_n-1} \left[\frac{2\pi j}{n}, \frac{2\pi(j+1)}{n}\right]\right) \cup \left[\frac{2\pi J_n}{n}, 2 \arcsin(\kappa/2)\right]$$

and proceed by calculating the estimations of

$$(4.21) \quad \left| \frac{-1}{\pi} \int_{\frac{2\pi j}{n}}^{\frac{2\pi(j+1)}{n}} \text{Log} \left(2 \sin(x/2)\right) dx - \frac{-2}{n} \text{Log} \left(2 \sin\left(\frac{\pi j}{n}\right)\right) \right|$$

on the intervals $\mathcal{I}_j := \left[\frac{2\pi j}{n}, \frac{2\pi(j+1)}{n}\right]$, $j = \lceil \text{Log } n \rceil, \lceil \text{Log } n \rceil + 1, \dots, J_n - 1$. On each such \mathcal{I}_j , the function $f(x)$ is approximated by its interpolation polynomial $P_m(x)$, where $m \geq 1$ is the number of subintervals forming an uniform partition of \mathcal{I}_j given by

$$(4.22) \quad y_q = \frac{2\pi j}{n} + q \frac{2\pi}{n m}, \quad q = 0, 1, \dots, m,$$

of step length $h_{NC} := \frac{2\pi}{nm}$, and P_m the interpolating polynomial of degree m or less with

$$P_m(y_q) = f(y_q), \quad \text{for } q = 0, 1, \dots, m.$$

The Newton-Cotes formulas

$$\int_{\frac{2\pi j}{n}}^{\frac{2\pi(j+1)}{n}} P_m(x) dx = h_{NC} \sum_{q=0}^m \alpha_q f(y_q)$$

provide approximate values of $\int_{\frac{2\pi j}{n}}^{\frac{2\pi(j+1)}{n}} f(x) dx$, where the α_q are the weights obtained by integrating the Lagrange's interpolation polynomials. Steffensen [?] ([23], p 127) showed that the approximation error may be expressed as follows:

$$\int_{\frac{2\pi j}{n}}^{\frac{2\pi(j+1)}{n}} P_m(x) dx - \int_{\frac{2\pi j}{n}}^{\frac{2\pi(j+1)}{n}} f(x) dx = h_{NC}^{p+1} \cdot K \cdot f^{(p+1)}(\xi), \quad \xi \in \overset{\circ}{\mathcal{I}}_j,$$

where $p \geq 2$ is an integer related to m , and K a constant.

Using [23], p. 128, and $m = 1$, the method being the ‘‘Trapezoidal rule’’, we have: ‘‘ $p = 2$, $K = 1/12$, $\alpha_0 = \alpha_1 = 1/2$ ’’. Then (4.21) is estimated by

$$(4.23) \quad \left| \frac{1}{2} \frac{2\pi}{n} \left[\frac{-1}{\pi} \text{Log} \left(2 \sin \left(\frac{\pi j}{n} \right) \right) + \frac{-1}{\pi} \text{Log} \left(2 \sin \left(\frac{\pi(j+1)}{n} \right) \right) \right] - \frac{-2}{n} \text{Log} \left(2 \sin \left(\frac{\pi j}{n} \right) \right) \right|$$

$$= \frac{1}{n} \left| \text{Log} \left(2 \sin \left(\frac{\pi j}{n} \right) \right) - \text{Log} \left(2 \sin \left(\frac{\pi(j+1)}{n} \right) \right) \right| = \frac{2\pi}{n^2} \left| \frac{\cos(\xi/2)}{2 \sin(\xi/2)} \right| \leq \frac{1}{n} \frac{1}{\text{Log} n}$$

for some $\xi \in \mathcal{I}_j$, for large n . The (Steffensen’s) approximation error ‘‘ $h_{NC}^3 \cdot (1/12) \cdot f^{(2)}(\xi)$ ’’ for the trapezoidal rule, relative to (4.21), is

$$(4.24) \quad \frac{1}{\pi} \left(\frac{2\pi}{n} \right)^3 \frac{1}{12} \left| \frac{-1}{4 \sin^2(\xi/2)} \right| \leq \frac{1}{6n} \frac{1}{(\text{Log} n)^2}.$$

By Proposition 2.7 the integral

$$\left| \frac{-1}{\pi} \int_{\frac{2\pi j}{n}}^{2 \arcsin \kappa/2} \text{Log} \left(2 \sin(x/2) \right) dx \right| \quad \text{is a} \quad O\left(\frac{1}{n}\right).$$

Then, summing up the contributions of all the intervals \mathcal{I}_j , we obtain the following upper bound of (4.19)

$$(4.25) \quad \left| \frac{-1}{\pi} \int_0^{(2\pi \text{Log} n)/n} \text{Log} \left(2 \sin(x/2) \right) dx \right| + \frac{\arcsin(\kappa/2)}{\pi} \frac{1}{\text{Log} n}.$$

with global (Steffensen’s) approximation error, from (4.24),

$$O\left(\frac{1}{(\text{Log} n)^2}\right)$$

By integrating by parts the integral in (4.25), for large n , it is easy to show that this integral is $O\left(\frac{(\text{Log} n)^2}{n}\right)$. We deduce the following asymptotic expansion

$$(4.26) \quad \Delta_1 = \frac{\mathcal{R}}{\text{Log} n} + O\left(\frac{1}{(\text{Log} n)^2}\right) \quad \text{with} \quad |\mathcal{R}| < \frac{\arcsin(\kappa/2)}{\pi}.$$

Proof of (ii-1): Let us show that the upper bound $\frac{\arcsin(\kappa/2)}{\pi} \frac{1}{\text{Log} n}$ is independent of the integer m used, once assumed the positivity of the weights $(\alpha_q)_{q=0,1,\dots,m}$. For $m \geq 1$ fixed, this is merely a consequence of the relation between the weights in the Newton-Cotes formulas. Indeed, we have $\sum_{q=0}^m \alpha_q = m$, and therefore

$$\left| \int_{\frac{2\pi j}{n}}^{\frac{2\pi(j+1)}{n}} P_m(x) dx - h_{NC} m f(y_0) \right| = h_{NC} \left| \sum_{q=0}^m \alpha_q (f(y_q) - f(y_0)) \right|$$

$$\leq h_{NC} \left(\sum_{q=0}^m |\alpha_q| \right) \sup_{\xi \in \mathcal{L}_j} |f'(\xi)|.$$

Since $h_{NC} m = \frac{2\pi}{n}$ and that the inequality $\sup_{\xi \in \mathcal{L}_j} |f'(\xi)| \leq |f'((2\pi \text{Log} n)/n)|$ holds uniformly for all j , we deduce the same upper bound as in (4.23) for the Trapezoidal rule. Summing up the contributions over all the intervals \mathcal{I}_j , we obtain the same upper bound (4.25) of (4.19) as before.

As for the (Steffensen's) approximation errors, they make use of the successive derivatives of the function $f(x) = \text{Log}(2 \sin(x/2))$. We have:

$$f'(x) = \frac{\cos(x/2)}{2 \sin(x/2)}, \quad f''(x) = -\frac{1}{4 \sin^2(x/2)}, \quad f'''(x) = \frac{\cos(x/2)}{4 \sin^3(x/2)} \dots$$

Recursively, it is easy to show that the q -th derivative of $f(x)$, $q \geq 1$, is a rational function of the two quantities $\cos(x/2)$ and $\sin(x/2)$ with bounded numerator on the interval $(0, \pi/3]$, and a denominator which is $\sin^q(x/2)$. For the needs of majoration in the Newton-Cotes formulas over each interval of the collection (\mathcal{I}_j) , this denominator takes its smallest value at $\xi = (2\pi \lceil \text{Log } n \rceil) / n$. Therefore, for large n , the (Steffensen's) approximation error " $h_{NC}^{p+1} \cdot K \cdot f^{(p)}(\xi)$ " on one interval \mathcal{I}_j is

$$O\left(\left(\frac{2\pi}{nm}\right)^{p+1} \cdot K \cdot \frac{n^p}{(\pi \text{Log } n)^p}\right) = O\left(\frac{1}{n(\text{Log } n)^p}\right).$$

By summing up over the intervals \mathcal{I}_j , we obtain the global (Steffensen's) approximation error ($p \geq 2$)

$$O\left(\frac{1}{(\text{Log } n)^p}\right) \quad \text{which is a} \quad O\left(\left(\frac{\text{Log } \text{Log } n}{\text{Log } n}\right)^2\right).$$

Calculation of $|\Delta_2|$: we proceed as above for establishing an upper bound of

$$(4.27) \quad |\Delta_2| = \left| \frac{-1}{\pi} \int_0^{2 \arcsin(\frac{\kappa(1,4\max)}{2})} \text{Log} \left[\frac{1 + 2 \sin(\frac{x}{2}) - \sqrt{1 - 12 \sin(\frac{x}{2}) + 4(\sin(\frac{x}{2}))^2}}{8 \sin(\frac{x}{2})} \right] dx \right. \\ \left. - \sum_{j=\lceil \text{Log } n \rceil}^{J_n} \frac{-2}{n} \text{Log} \left(\frac{1 + 2 \sin(\frac{\pi j}{n}) - \sqrt{1 - 12 \sin(\frac{\pi j}{n}) + 4 \sin(\frac{\pi j}{n})^2}}{8 \sin(\frac{\pi j}{n})} \right) \right|$$

In (4.27) the sums are truncated Riemann-Stieltjes sums of $\text{Log } \mu_r$, the integral being $\text{Log } \mu_r$. As above, the methods of Newton-Cotes (Stoer and Bulirsch ([23], pp 126–128) will be applied to compute an approximate value of the integral up to $O\left(\left(\frac{\text{Log } \text{Log } n}{\text{Log } n}\right)^2\right)$. Up to $O\left(\left(\frac{\text{Log } \text{Log } n}{\text{Log } n}\right)^2\right)$, we will show that:

(i-2) *an upper bound of (4.27) is*

$$(4.28) \quad \frac{4 \arcsin(\kappa/2)}{\kappa \sqrt{2\kappa(3-\kappa)} \text{Log}(1/\kappa)} \frac{1}{\sqrt{n}} \quad \text{which is a} \quad O\left(\left(\frac{\text{Log } \text{Log } n}{\text{Log } n}\right)^2\right),$$

in other terms that (4.27) is equal to zero up to $O\left(\left(\frac{\text{Log } \text{Log } n}{\text{Log } n}\right)^2\right)$,

(ii-2) *the approximate value of $\text{Log } \mu_r$ is independent of the step length m used in the Newton-Cotes formulas, assuming the weights $(\alpha_q)_{q=0,1,\dots,m}$ associated with m all positive.*

Proof of (i-2): The decomposition of the interval of integration $(0, 2 \arcsin(\kappa/2)]$ remains the same as above, given by (4.20). Let us treat the complete interval of integration

$(0, 2 \arcsin(\kappa/2)]$ by subintervals. We first proceed by estimating an upper bound of

$$(4.29) \quad \left| \frac{-1}{\pi} \int_{\frac{2\pi j}{n}}^{\frac{2\pi(j+1)}{n}} \text{Log} \left[\frac{1 + 2 \sin(\frac{x}{2}) - \sqrt{1 - 12 \sin(\frac{x}{2}) + 4(\sin(\frac{x}{2}))^2}}{8 \sin(\frac{x}{2})} \right] dx \right. \\ \left. - \frac{-2}{n} \text{Log} \left(\frac{1 + 2 \sin(\frac{\pi j}{n}) - \sqrt{1 - 12 \sin(\frac{\pi j}{n}) + 4 \sin(\frac{\pi j}{n})^2}}{8 \sin(\frac{\pi j}{n})} \right) \right|$$

on the intervals $\mathcal{I}_j := [\frac{2\pi j}{n}, \frac{2\pi(j+1)}{n}]$, $j = \lceil \text{Log } n \rceil, \lceil \text{Log } n \rceil + 1, \dots, J_n - 1$. Let

$$F(x) := \text{Log} \left[\frac{1 + 2 \sin(\frac{x}{2}) - \sqrt{1 - 12 \sin(\frac{x}{2}) + 4(\sin(\frac{x}{2}))^2}}{8 \sin(\frac{x}{2})} \right].$$

On each interval \mathcal{I}_j the function $F(x)$ is approximated by its interpolation polynomial (say) $P_{F,m}(x)$, where $m \geq 1$ is the number of subintervals of \mathcal{I}_j given by their extremities y_q by (4.22), of step length $h_{NC} := \frac{2\pi}{nm}$, and $P_{F,m}$ the interpolating polynomial of degree m or less with

$$P_{F,m}(y_q) = F(y_q), \quad \text{for } q = 0, 1, \dots, m.$$

The Newton-Cotes formulas

$$(4.30) \quad \int_{\frac{2\pi j}{n}}^{\frac{2\pi(j+1)}{n}} P_{F,m}(x) dx = h_{NC} \sum_{q=0}^m \alpha_q F(y_q)$$

provide the approximate values $\int_{\frac{2\pi j}{n}}^{\frac{2\pi(j+1)}{n}} F(x) dx$, where the α_q s are the weights obtained by integrating the Lagrange's interpolation polynomials. Using [23], p. 128, and $m = 1$, the method being the "Trapezoidal rule", we have: $p = 2$, $K = 1/12$, $\alpha_0 = \alpha_1 = 1/2$. Then (4.29) is estimated by

$$(4.31) \quad \left| \frac{1}{2} \frac{2\pi}{n} \left[\frac{-1}{\pi} F\left(\frac{2\pi j}{n}\right) + \frac{-1}{\pi} F\left(\frac{2\pi(j+1)}{n}\right) \right] - \frac{-2}{n} F\left(\frac{2\pi j}{n}\right) \right| \\ = \frac{1}{n} \left| F\left(\frac{2\pi j}{n}\right) - F\left(\frac{2\pi(j+1)}{n}\right) \right| = \frac{2\pi}{n^2} |F'(\xi)|$$

for some $\xi \in \overset{o}{\mathcal{I}_j}$, for large n . As in Remark 2.8, let $x = 2 \arcsin(\kappa/2)$. The derivative

$$(4.32) \quad F'(y) = \frac{\cos(y/2)(-2 \sin(y/2) + 1 - \sqrt{4 \sin^2(y/2) - 12 \sin(y/2) + 1})}{4 \sin(y/2) \sqrt{4 \sin^2(y/2) - 12 \sin(y/2) + 1}} > 0$$

is increasing on the interval $(0, x)$. When $y = \frac{2\pi J_n}{n} < x$ tends to x^- , by Proposition 2.7 and Remark 2.8, since $0 < \sqrt{4 \sin^2(y/2) - 12 \sin(y/2) + 1} \leq 1$ is close to zero for $y = 2\pi J_n/n$, the following inequality holds

$$(4.33) \quad F'\left(\frac{2\pi J_n}{n}\right) \leq \frac{2/\kappa}{\sqrt{4 \sin^2\left(\frac{\pi J_n}{n}\right) - 12 \sin\left(\frac{\pi J_n}{n}\right) + 1}}.$$

The upper bound is a function of n which comes from the asymptotic expansion of $\frac{\pi J_n}{n} - \frac{x}{2}$, as deduced from (2.19). Indeed, from (2.19) and using Remark 2.8 (ii),

$$(4.34) \quad \begin{aligned} 4 \sin^2\left(\frac{\pi J_n}{n}\right) - 12 \sin\left(\frac{\pi J_n}{n}\right) + 1 &= \left(\frac{\pi J_n}{n} - \frac{x}{2}\right) [8 \sin(x/2) \cos(x/2) - 12 \cos(x/2)] + O\left(\frac{1}{n^2}\right) \\ &= \frac{2\kappa(3-\kappa)\text{Log}(1/\kappa)}{n} + \frac{1}{n} O\left(\left(\frac{\text{Log Log } n}{\text{Log } n}\right)^2\right) \end{aligned}$$

From (4.33) and (4.34) we deduce $|F'(\frac{2\pi J_n}{n})| < \frac{(2/\kappa)}{\sqrt{2\kappa(3-\kappa)\text{Log}(1/\kappa)}} \sqrt{n}$. From (4.31), we deduce the following upper bound of (4.29) on each $\mathcal{I}_j := [\frac{2\pi j}{n}, \frac{2\pi(j+1)}{n}]$:

$$(4.35) \quad \frac{4\pi}{\kappa \sqrt{2\kappa(3-\kappa)\text{Log}(1/\kappa)}} \frac{1}{n^{3/2}}.$$

By summing up the contributions, for $j = \lceil \text{Log } n \rceil, \dots, J_n - 1$, from (4.35) and the asymptotics of J_n given by (2.19), we deduce the upper bound (4.28) of $|\Delta_2|$.

Let us prove that the method of numerical integration we use leads to a (Steffensen's) approximation error which is a $O\left(\left(\frac{\text{Log Log } n}{\text{Log } n}\right)^2\right)$. The (Steffensen's) approximation error " $h_{NC}^3 \cdot (1/12) \cdot F^{(2)}(\xi)$ " for the trapezoidal rule applied to (4.29) ([23], p. 127–128) is

$$(4.36) \quad \frac{1}{\pi} \left(\frac{2\pi}{n}\right)^3 \frac{1}{12} \left|F^{(2)}(\xi)\right| \quad \text{for some } \xi \in \overset{o}{\mathcal{I}}_j.$$

The second derivative $F''(y)$ is positive and increasing on $(0, \frac{2\pi J_n}{n})$. It is easy to show that there exists a constant $C_0 > 0$ such that

$$F''\left(\frac{2\pi J_n}{n}\right) \leq \frac{C_0}{(4 \sin^2\left(\frac{\pi J_n}{n}\right) - 12 \sin\left(\frac{\pi J_n}{n}\right) + 1)^{3/2}}.$$

Using the asymptotic expansion of J_n ((2.19); Remark 2.8 (ii); (4.34)), there exist $C_1 > 0$ such that

$$(4.37) \quad F''\left(\frac{2\pi J_n}{n}\right) \leq C_1 n^{3/2}.$$

From (4.36) and (4.37), summing up the contributions for $j = \lceil \text{Log } n \rceil, \dots, J_n - 1$, the global (Steffensen's) approximation error of (4.27) for $|\Delta_2|$ admits the following upper bound, for some constants $C'_2 > 0, C_2 > 0$,

$$C'_2 \frac{J_n}{n^3} n^{3/2} = C_2 \frac{1}{\sqrt{n}} \quad \text{which is a } O\left(\left(\frac{\text{Log Log } n}{\text{Log } n}\right)^2\right).$$

Now let us turn to the extremity intervals. Using the Appendix, and (2.19) in Proposition 2.7, it is easy to show that the two integrals

$$\frac{-1}{\pi} \int_0^{\frac{2\pi \lceil \text{Log } n \rceil}{n}} \quad \text{and} \quad \frac{-1}{\pi} \int_{\frac{2\pi J_n}{n}}^{2 \arcsin(\kappa/2)} \quad \text{are} \quad O\left(\left(\frac{\text{Log Log } n}{\text{Log } n}\right)^2\right).$$

Proof of (ii-2): On each interval $\mathcal{I}_j := [\frac{2\pi j}{n}, \frac{2\pi(j+1)}{n}]$, $j = \lceil \text{Log } n \rceil, \dots, J_n - 1$, let us assume that the number m of subintervals of \mathcal{I}_j given by their extremities y_q by (4.22), is ≥ 2 . The weights α_q in (4.30) are assumed to be positive.

The upper bound $\frac{4 \arcsin(\kappa/2)}{\kappa \sqrt{2\kappa(3-\kappa)} \text{Log}(1/\kappa)} \frac{1}{\sqrt{n}}$ of (4.27) is independent of $m \geq 2$, once assumed the positivity of the weights $(\alpha_q)_{q=0,1,\dots,m}$, since, due to the relation between the weights in the Newton-Cotes formulas $\sum_{q=0}^m \alpha_q = m$,

$$\begin{aligned} \left| \int_{\frac{2\pi j}{n}}^{\frac{2\pi(j+1)}{n}} P_m(x) dx - h_{NC} m F(y_0) \right| &= h_{NC} \left| \sum_{q=0}^m \alpha_q (F(y_q) - F(y_0)) \right| \\ &\leq h_{NC} \left(\sum_{q=0}^m |\alpha_q| \right) \sup_{\xi \in \mathcal{I}_j} |F'(\xi)|. \end{aligned}$$

Since $h_{NC} m = \frac{2\pi}{n}$ and that $\sup_{\xi \in \mathcal{I}_j} |F'(\xi)| \leq |F'((2\pi J_n)/n)|$ holds uniformly for all $j = \lceil \text{Log} n \rceil, \dots, J_n - 1$, we deduce the same upper bound (4.35) as for the Trapezoidal rule. Summing up the contributions over all the intervals \mathcal{I}_j , we obtain the same upper bound (4.28) of (4.27), as before.

As for the (Steffensen's) approximation errors involved in the numerical integration (4.30) there are " $h_{NC}^{p+1} \cdot K \cdot F^{(p)}(\xi)$ " on one interval \mathcal{I}_j , for some $p \geq 2$. They make use of the successive derivatives of the function $F(x)$. It can be shown that they contribute negligibly, after summing up over all the intervals \mathcal{I}_j , as $O\left(\left(\frac{\text{Log Log } n}{\text{Log } n}\right)^2\right)$.

Gathering the different terms from (i-1)(i-2), the Steffensen's error terms and the error terms due to the numerical integration by the Newton-Cotes method (ii-1)(ii-2), we have proved the following theorem.

Theorem 4.2. *Let $\beta > 1$ be a real number such that $n = \text{dyg}(\beta) \geq 260$, with $\beta \notin \{\theta_n^{-1} : n \geq 259\}$. The minorant $L_r(\beta)$ of $\text{Log } \mathfrak{S}_{\text{lent}}(\beta)$ admits the following asymptotic expansion as lower bound, as a function of $n = \text{dyg}(\beta)$,*

$$(4.38) \quad L_r(\beta) \geq \text{Log } \Lambda_r \mu_r + \frac{\mathcal{R}}{\text{Log } n} + O\left(\left(\frac{\text{Log Log } n}{\text{Log } n}\right)^2\right), \quad \text{with } 0 < \mathcal{R} < \frac{\arcsin(\kappa/2)}{\pi}$$

and \mathcal{R} depending upon n .

4.3. A Dobrowolski-type inequality - Proof of Theorem 1.2. Let \mathcal{R}_n denote the n -depending positive real number \mathcal{R} in (4.38). Let us show that it is substantially smaller than the bound $\frac{\arcsin(\kappa/2)}{\pi}$.

Lemma 4.3. *With the same notations as in Theorem 4.2, there exists an integer $\eta \geq 260$ such that*

$$(4.39) \quad \left| \mathcal{R}_n + O\left(\frac{(\text{Log Log } n)^2}{\text{Log } n}\right) \right| < \frac{\arcsin(\kappa/2)}{\pi}, \quad \text{for all } n \geq \eta.$$

Proof. Let $X_n := c \lceil \text{Log } n \rceil$ with c a positive constant such that $\lceil \text{Log } n \rceil < X_n < J_n$. The limit $\lim_{n \rightarrow \infty} X_n/n = 0$ holds. Recall that J_n is given by (2.19).

The quantity \mathcal{R}_n comes from the integration of the j th-subdivision step (4.21) by (4.23), in order to give an estimate of the development term of $|\Delta_1|$ given by (4.19). This j th-subdivision step of integration provides the estimated term (cf (4.23))

$$(4.40) \quad \frac{2\pi}{n^2} \left| \frac{\cos(\xi/2)}{2\sin(\xi/2)} \right| \quad \text{for some } \xi \in \left(\frac{2\pi j}{n}, \frac{2\pi(j+1)}{n} \right).$$

Since the cotangent function is positive and strictly decreasing on $(0, \pi/2)$, the upper bound of (4.40) is naturally the one given by the first interval of the subdivision $\left[\frac{2\pi \lceil \text{Log} n \rceil}{n}, \frac{2\pi(\lceil \text{Log} n \rceil + 1)}{n} \right]$, that is $\frac{1}{n} \frac{1}{\text{Log} n}$. We now separate the following summation into two parts.

$$(4.41) \quad \frac{-1}{\pi} \int_{\frac{2\pi \lceil \text{Log} n \rceil}{n}}^{\frac{2\pi J_n}{n}} \text{Log} \left(2 \sin(x/2) \right) dx - \sum_{j=\lceil \text{Log} n \rceil}^{J_n} \frac{-2}{n} \text{Log} \left(2 \sin \left(\frac{\pi j}{n} \right) \right)$$

$$(4.42) \quad = \sum_{j=\lceil \text{Log} n \rceil}^{X_n-1} \left(\frac{-1}{\pi} \int_{\frac{2\pi j}{n}}^{\frac{2\pi(j+1)}{n}} \text{Log} \left(2 \sin(x/2) \right) dx - \frac{-2}{n} \text{Log} \left(2 \sin \left(\frac{\pi j}{n} \right) \right) \right)$$

$$(4.43) \quad + \sum_{j=X_n}^{J_n-1} \left(\frac{-1}{\pi} \int_{\frac{2\pi j}{n}}^{\frac{2\pi(j+1)}{n}} \text{Log} \left(2 \sin(x/2) \right) dx - \frac{-2}{n} \text{Log} \left(2 \sin \left(\frac{\pi j}{n} \right) \right) \right).$$

Each term of (4.42) is bounded by $\frac{1}{n} \frac{1}{\text{Log} n}$ from above, as previously. On the contrary, each term of (4.43) is such that

$$\frac{2\pi}{n^2} \left| \frac{\cos(\xi/2)}{2\sin(\xi/2)} \right| \leq \frac{1}{n^2} \left| \frac{1}{X_n/n} \right| = \frac{1}{n} \frac{1}{c \text{Log} n}.$$

Summing up the two contributions, we obtain the following upper bound of (4.41):

$$\begin{aligned} & (X_n - 1 - \lceil \text{Log} n \rceil) \frac{1}{n} \frac{1}{\text{Log} n} + (J_n - X_n - 1) \frac{1}{n} \frac{1}{c \text{Log} n} \\ & \leq (c - 1) \frac{1}{n} + \frac{1}{c} \frac{\arcsin(\kappa/2)}{\pi \text{Log} n} \end{aligned}$$

The first term $(c - 1) \frac{1}{n}$ is a $O\left(\frac{1}{n}\right)$ and, multiplied by $\text{Log} n$, is inserted in the Big O of (4.39). The second term $\frac{1}{c} \frac{\arcsin(\kappa/2)}{\pi \text{Log} n}$ is an upper bound of $\mathcal{R}_n / \text{Log} n$. Let us fix the constant c . Take for instance $c = 3$. The function $(\text{Log} \text{Log} x)^2 / \text{Log} x$ tends to 0 when x goes to infinity. Therefore there exists an integer η such that all the functions, depending upon n , “grouped in the Big O” of (4.39) satisfy (in short form):

$$\left| O\left(\frac{(\text{Log} \text{Log} n)^2}{\text{Log} n} \right) \right| < \frac{2}{3} \frac{\arcsin(\kappa/2)}{\pi}, \quad n \geq \eta.$$

We deduce Lemma 4.3. □

Proof of Theorem 1.2.— For any real number $\beta > 1$, $\text{dyg}(\beta) \geq 260$ and $\beta \notin \{\theta_n^{-1} : n \geq 259\}$, we have

$$\mathfrak{S}_{lent}(\beta) \geq \exp(L_r(\beta)) \geq \Lambda_r \mu_r \left(1 + \frac{\mathcal{R}}{\text{Log } n} + O\left(\left(\frac{\text{Log Log } n}{\text{Log } n}\right)^2\right) \right)$$

from Theorem 4.2. Using Lemma 4.3, for all $n \geq \eta$ for some $\eta \geq 260$, we can improve the above inequality to

$$\mathfrak{S}_{lent}(\beta) \geq \exp(L_r(\beta)) \geq \Lambda_r \mu_r \left(1 - \frac{1}{\text{Log } n} \frac{\arcsin(\kappa/2)}{\pi} \right).$$

Recall that $C = \Lambda_r \mu_r$. We deduce Theorem 1.2.

5. APPENDIX: TRINOMIALS $-1 + z + z^n$, MAHLER MEASURES AND ANGULAR SECTORIZATION

5.1. Notations, factorization, lenticular roots. The following notations are used throughout this paper. They come from the factorization of $G_n(X) := -1 + X + X^n$ ([19], Section 2 in [27]). Summing in pairs over complex conjugated imaginary roots, the indexation of the roots and the factorization of $G_n(X)$ are taken as follows:

$$(5.1) \quad G_n(X) = (X - \theta_n) \left(\prod_{j=1}^{\lfloor \frac{n}{6} \rfloor} (X - z_{j,n})(X - \overline{z_{j,n}}) \right) \times q_n(X),$$

where θ_n is the only (real) root of $G_n(X)$ in the interval $(0, 1)$, where

$$q_n(X) = \begin{cases} \left(\prod_{j=1+\lfloor \frac{n}{6} \rfloor}^{\frac{n-2}{2}} (X - z_{j,n})(X - \overline{z_{j,n}}) \right) \times (X - z_{\frac{n}{2},n}) & \text{if } n \text{ is even, with} \\ & z_{\frac{n}{2},n} \text{ real} < -1, \\ \prod_{j=1+\lfloor \frac{n}{6} \rfloor}^{\frac{n-1}{2}} (X - z_{j,n})(X - \overline{z_{j,n}}) & \text{if } n \text{ is odd,} \end{cases}$$

where the index $j = 1, 2, \dots$ is such that $z_{j,n}$ is a (nonreal) complex zero of $G_n(X)$, except if n is even and $j = n/2$, such that the argument $\arg(z_{j,n})$ of $z_{j,n}$ is roughly equal to $2\pi j/n$ (Proposition 5.11) and that the family of arguments $(\arg(z_{j,n}))_{1 \leq j < \lfloor n/2 \rfloor}$ forms a strictly increasing sequence with j :

$$0 < \arg(z_{1,n}) < \arg(z_{2,n}) < \dots < \arg(z_{\lfloor \frac{n}{2} \rfloor, n}) \leq \pi.$$

For $n \geq 2$ all the roots of $G_n(X)$ are simple, and the roots of $G_n^*(X) = X^n G_n(1/X) = 1 + X^{n-1} - X^n$, as inverses of the roots of $G_n(X)$, are classified in the reversed order (Figure 3).

Proposition 5.1. *Let $n \geq 2$. If $n \not\equiv 5 \pmod{6}$, then $G_n(X)$ is irreducible over \mathbb{Q} . If $n \equiv 5 \pmod{6}$, then the polynomial $G_n(X)$ admits $X^2 - X + 1$ as irreducible factor in its factorization and $G_n(X)/(X^2 - X + 1)$ is irreducible.*

Proof. Selmer [19]. □

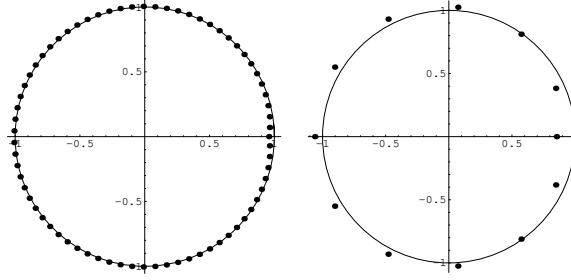


FIGURE 3. The roots (black bullets) of $G_n(z)$ (represented here with $n = 71$ and $n = 12$) are uniformly distributed near $|z| = 1$. A slight bump appears in the half-plane $\Re(z) > 1/2$ in the neighbourhood of 1, at the origin of the different regimes of asymptotic expansions. The dominant root of $G_n^*(z) = z^n G_n(1/z)$ is the Perron number $\theta_n^{-1} > 1$, with θ_n the unique root of G_n in the interval $(0, 1)$.

Proposition 5.2. *For all $n \geq 2$, all zeros $z_{j,n}$ and θ_n of the polynomials $G_n(X)$ have a modulus in the interval*

$$(5.2) \quad \left[1 - \frac{2\text{Log } n}{n}, 1 + \frac{2\text{Log } 2}{n} \right],$$

(ii) *the trinomial $G_n(X)$ admits a unique real root θ_n in the interval $(0, 1)$. The sequence $(\theta_n)_{n \geq 2}$ is strictly increasing, $\lim_{n \rightarrow +\infty} \theta_n = 1$, with $\theta_2 = \frac{2}{1+\sqrt{5}} = 0.618\dots$,*

(iii) *the root θ_n is the unique root of smallest modulus among all the roots of $G_n(X)$; if $n \geq 6$, the roots of modulus < 1 of $G_n(z)$ in the closed upper half-plane have the following properties:*

(iii-1) $\theta_n < |z_{1,n}|$,

(iii-2) *for any pair of successive indices $j, j + 1$ in $\{1, 2, \dots, \lfloor n/6 \rfloor\}$,*

$$|z_{j,n}| < |z_{j+1,n}|.$$

Proof. (i)(ii) Selmer [19], pp 291–292; (iii-1) Flatto, Lagarias and Poonen [6], (iii-2) Verger-Gaugry [27]. □

From [27], or by Smyth’s Theorem [20] since the trinomials $G_n(X)$ are not reciprocal, the Mahler measure of G_n satisfies

$$(5.3) \quad M(\theta_n) = M(G_n) \geq \Theta = 1.3247\dots, \quad n \geq 2,$$

where $\Theta = \theta_5^{-1}$ is the smallest Pisot number, dominant root of the Pisot polynomial $X^3 - X - 1 = -X^5 G_5(1/X)/(X^2 - X + 1)$.

Proposition 5.3. *Let $n \geq 2$. Then (i) the number p_n of roots of $G_n(X)$ which lie inside the open sector $\mathcal{S} = \{z \mid |\arg(z)| < \pi/3\}$ is equal to*

$$(5.4) \quad 1 + 2 \left\lfloor \frac{n}{6} \right\rfloor,$$

(ii) the correlation between the geometry of the roots of $G_n(X)$ which lie inside the unit disk and the upper half-plane and their indexation is given by:

$$(5.5) \quad j \in \{1, 2, \dots, \lfloor \frac{n}{6} \rfloor\} \iff \Re(z_{j,n}) > \frac{1}{2} \iff |z_{j,n}| < 1,$$

and the Mahler measure $M(G_n)$ of the trinomial $G_n(X)$ is

$$(5.6) \quad M(G_n) = M(G_n^*) = \theta_n^{-1} \prod_{j=1}^{\lfloor n/6 \rfloor} |z_{j,n}|^{-2}.$$

Proof. [27], Proposition 3.7. □

Definition 5.4. Let $n \geq 2$. The set of the lenticular poles of $\zeta_{\theta_n^{-1}}(z)$, equivalently the set of the lenticular roots of $G_n(z)$, is $\mathcal{L}_{\theta_n^{-1}} = \{\overline{z_{\lfloor n/6 \rfloor, n}}, \dots, \overline{z_{1, n}}, \theta_n, z_{1, n}, \dots, z_{\lfloor n/6 \rfloor, n}\}$.

5.2. Asymptotic expansions: poles of $\zeta_{\theta_n^{-1}}(z)$ and useful relations. The poles of $\zeta_{\theta_n^{-1}}(z)$, $n \geq 3$, inside the open unit disk are the zeroes of $G_n(z)$ of modulus < 1 , in \mathbb{C} . In the following, for being self-contained, we summarize the main results on the asymptotic expansions of the roots of the G_n s useful to the expansions of the lower bounds of the lenticular measures $\mathfrak{S}_{lent}(\beta)$ in Section 4.

The (Poincaré) asymptotic expansions of the roots $z_{j,n}$ of G_n are generically written: $\Re(z_{j,n}) = D(\Re(z_{j,n})) + \text{tl}(\Re(z_{j,n}))$, $\text{Im}(z_{j,n}) = D(\text{Im}(z_{j,n})) + \text{tl}(\text{Im}(z_{j,n}))$, $\theta_n = D(\theta_n) + \text{tl}(\theta_n)$, where "D" and "tl" stands for "development", or "limited expansion", or "lowest order terms", and "tl" for "tail" or "remainder", or "terminant" [3].

Poincaré [14] introduced this method of divergent series for the N -body problem in celestial mechanics; the analogue of the variable time t (of celestial mechanics) is the dynamical degree $\text{dyg}(\beta)$ when $\beta > 1$ is a real number; for the trinomial G_n it is n .

The Poincaré asymptotic expansions of the roots $z_{j,n}$ of $G_n(z) = -1 + z + z^n$, lying in the first quadrant of \mathbb{C} , are recalled in Proposition 5.10. They are divergent formal series of functions of the couple of *two variables* which is:

- $(n, \frac{j}{n})$, in the ("main" sector) angular sector: $\frac{\pi}{2} > \arg z > 2\pi \frac{\text{Log } n}{n}$,
- $(n, \frac{j}{\text{Log } n})$, in the angular sector ("bump" sector): $2\pi \frac{\text{Log } n}{n} > \arg z \geq 0$.

In the bump sector (cusp sector of Solomyak's fractal [22] in the open unit disk), the roots $z_{j,n}$ are dispatched into the two subsectors:

- $2\pi \frac{\sqrt{(\text{Log } n)(\text{Log } \text{Log } n)}}{n} > \arg z > 0$,
- $2\pi \frac{\text{Log } n}{n} > \arg z > 2\pi \frac{\sqrt{(\text{Log } n)(\text{Log } \text{Log } n)}}{n}$.

The relative angular size of the bump sector, as $(2\pi \frac{\text{Log } n}{n}) / (\frac{\pi}{2})$, tends to zero, as soon as n is large enough. By transition region, we mean a small neighbourhood of the argument :

$$\arg z = 2\pi \frac{\text{Log } n}{n} \quad \text{or of} \quad 2\pi \frac{\sqrt{(\text{Log } n)(\text{Log } \text{Log } n)}}{n}.$$

Outside these two transition regions, a dominant asymptotic expansion of $z_{j,n}$ exists. In a transition region an asymptotic expansion contains more n -th order terms of the same order of magnitude ($n = 2, 3, 4$). These two neighbourhoods are defined as follows. Let $\varepsilon \in (0, 1)$

small enough. Two strictly increasing sequences of real numbers $(u_n), (v_n)$ are introduced, which satisfy:

$$\lfloor n/6 \rfloor > v_n > \text{Log} n, \quad \text{Log} n > u_n > \sqrt{(\text{Log} n)(\text{Log} \text{Log} n)}, \quad \text{for } n \geq n_0 = 18,$$

such that

$$\lim_{n \rightarrow \infty} \frac{v_n}{n} = \lim_{n \rightarrow \infty} \frac{\sqrt{(\text{Log} n)(\text{Log} \text{Log} n)}}{u_n} = \lim_{n \rightarrow \infty} \frac{u_n}{\text{Log} n} = \lim_{n \rightarrow \infty} \frac{\text{Log} n}{v_n} = 0$$

and

$$(5.7) \quad v_n - u_n = O((\text{Log} n)^{1+\varepsilon})$$

with the constant 1 involved in the big O. The roots $z_{j,n}$ lying in the first transition region about $2\pi(\text{Log} n)/n$ are such that:

$$2\pi \frac{v_n}{n} > \arg z_{j,n} > 2\pi \frac{(2\text{Log} n - v_n)}{n},$$

and the roots $z_{j,n}$ lying in the second transition region about $\frac{2\pi\sqrt{(\text{Log} n)(\text{Log} \text{Log} n)}}{n}$ are such that:

$$2\pi \frac{u_n}{n} > \arg z_{j,n} > 2\pi \frac{2\sqrt{(\text{Log} n)(\text{Log} \text{Log} n)} - u_n}{n}.$$

In Proposition 5.10, for simplicity's sake, these two transition regions are schematically denoted by

$$\arg z \asymp 2\pi \frac{(\text{Log} n)}{n} \quad \text{resp.} \quad \arg z \asymp 2\pi \frac{\sqrt{(\text{Log} n)(\text{Log} \text{Log} n)}}{n}.$$

By complementarity, the other sectors are schematically written:

$$2\pi \frac{\sqrt{(\text{Log} n)(\text{Log} \text{Log} n)}}{n} > \arg z > 0$$

instead of

$$2\pi \frac{2\sqrt{(\text{Log} n)(\text{Log} \text{Log} n)} - u_n}{n} > \arg z > 0;$$

resp.

$$2\pi \frac{\text{Log} n}{n} > \arg z > 2\pi \frac{\sqrt{(\text{Log} n)(\text{Log} \text{Log} n)}}{n}$$

instead of

$$2\pi \frac{2\text{Log} n - v_n}{n} > \arg z > 2\pi \frac{u_n}{n};$$

resp.

$$\frac{\pi}{2} > \arg z > 2\pi \frac{\text{Log} n}{n} \quad \text{instead of} \quad \frac{\pi}{2} > \arg z > 2\pi \frac{v_n}{n}.$$

Proposition 5.5. *Let $n \geq 2$. The root θ_n can be expressed as: $\theta_n = D(\theta_n) + \text{tl}(\theta_n)$ with $D(\theta_n) = 1 -$*

$$(5.8) \quad \frac{\text{Log} n}{n} \left(1 - \left(\frac{n - \text{Log} n}{n \text{Log} n + n - \text{Log} n} \right) \left(\text{Log} \text{Log} n - n \text{Log} \left(1 - \frac{\text{Log} n}{n} \right) - \text{Log} n \right) \right)$$

and

$$(5.9) \quad \text{tl}(\theta_n) = \frac{1}{n} O \left(\left(\frac{\text{Log Log } n}{\text{Log } n} \right)^2 \right),$$

with the constant $1/2$ involved in $O(\cdot)$.

Proof. [27] Proposition 3.1. □

Lemma 5.6. *Given the limited expansion $D(\theta_n)$ of θ_n as in (5.8), denote*

$$\lambda_n := 1 - (1 - D(\theta_n)) \frac{n}{\text{Log } n}.$$

Then $\lambda_n = D(\lambda_n) + \text{tl}(\lambda_n)$, with

$$(5.10) \quad D(\lambda_n) = \frac{\text{Log Log } n}{\text{Log } n} \left(\frac{1}{1 + \frac{1}{\text{Log } n}} \right), \quad \text{tl}(\lambda_n) = O \left(\frac{\text{Log Log } n}{n} \right)$$

with the constant 1 in the Big O .

Proof. [27] Lemma 3.2. □

Lemma 5.7. *Let $n \geq 6$. The difference $\theta_n - \theta_{n-1} > 0$ admits the following asymptotic expansion, reduced to its terminant:*

$$(5.11) \quad \theta_n - \theta_{n-1} = \frac{1}{n} O \left(\left(\frac{\text{Log Log } n}{\text{Log } n} \right)^2 \right),$$

with the constant 1 involved in $O(\cdot)$.

Proof. From (5.8) and Lemma 5.6, we have

$$\theta_n = 1 - \frac{\text{Log } n}{n} (1 - \lambda_n) + \frac{1}{n} O \left(\left(\frac{\text{Log Log } n}{\text{Log } n} \right)^2 \right)$$

with the constant $1/2$ involved in $O(\cdot)$, and

$$\lambda_n = \frac{\text{Log Log } n}{\text{Log } n} \left(\frac{1}{1 + \frac{1}{\text{Log } n}} \right) + O \left(\frac{\text{Log Log } n}{n} \right)$$

with the constant 1 in the Big O . Then we deduce

$$D(\theta_n) - D(\theta_{n-1}) = \frac{\text{Log } n}{n^2} + O \left(\frac{\text{Log Log } n}{n^2} \right).$$

The real function $x^{-2} \text{Log } x$ on $(1, +\infty)$ is decreasing for $x \geq \sqrt{e}$. Hence the sequence $(D(\theta_n) - D(\theta_{n-1}))$ is decreasing for n large enough. By Proposition 5.2 $(\theta_n - \theta_{n-1})_n$ is already known to tend to 0 .

Since $\text{tl}(\theta_n) = \frac{1}{n} O \left(\left(\frac{\text{Log Log } n}{\text{Log } n} \right)^2 \right)$, we have

$$\begin{aligned} \theta_n - \theta_{n-1} &= (\theta_n - D(\theta_n)) + [D(\theta_n) - D(\theta_{n-1})] - (\theta_{n-1} - D(\theta_{n-1})) \\ &= \text{tl}(\theta_n) + \left(\frac{\text{Log } n}{n^2} + O \left(\frac{\text{Log Log } n}{n^2} \right) \right) - \text{tl}(\theta_{n-1}) \end{aligned}$$

$$(5.12) \quad = \frac{1}{n} O \left(\left(\frac{\text{Log Log } n}{\text{Log } n} \right)^2 \right)$$

where the constant involved in $O(\cdot)$ is now $1 = 1/2 + 1/2$. Hence the claim. \square

Theorem 5.8. *Let $n \geq 6$. Let $\beta > 1$ be a real number of dynamical degree $\text{dyg}(\beta) = n$. Then β^{-1} can be expressed as: $\beta^{-1} = D(\beta^{-1}) + \text{tl}(\beta^{-1})$ with $D(\beta^{-1}) = 1 -$*

$$(5.13) \quad \frac{\text{Log } n}{n} \left(1 - \left(\frac{n - \text{Log } n}{n \text{Log } n + n - \text{Log } n} \right) \left(\text{Log Log } n - n \text{Log} \left(1 - \frac{\text{Log } n}{n} \right) - \text{Log } n \right) \right)$$

and

$$(5.14) \quad \text{tl}(\beta^{-1}) = \frac{1}{n} O \left(\left(\frac{\text{Log Log } n}{\text{Log } n} \right)^2 \right),$$

with the constant 1 involved in $O(\cdot)$.

Proof. By definition $\theta_n \leq \beta^{-1} < \theta_{n-1}$. The development term of β^{-1} is $D(\beta^{-1}) = D(\beta^{-1} - \theta_n) + D(\theta_n)$, with $|\beta^{-1} - \theta_n| < \theta_n - \theta_{n-1}$. By Lemma 5.7, $D(\theta_n - \theta_{n-1}) = 0$. Therefore $\beta^{-1} = D(\beta^{-1}) + \text{tl}(\beta^{-1})$ is deduced from $D(\theta_n)$ in (5.8). \square

Theorem 5.9. *Let $\beta \in (1, \theta_6^{-1})$ be a real number. The asymptotic expansion of the locally constant function $n = \text{dyg}(\beta)$, as a function of the variable $\beta - 1$, is*

$$(5.15) \quad n = -\frac{\text{Log}(\beta - 1)}{\beta - 1} \left[1 + O \left(\left(\frac{\text{Log}(-\text{Log}(\beta - 1))}{\text{Log}(\beta - 1)} \right)^2 \right) \right]$$

with the constant 1 in $O(\cdot)$.

Proof. Inverting (5.13) gives the asymptotic expansion of n as a function of β : from (5.13) readily comes

$$(5.16) \quad n = \frac{\beta}{\beta - 1} \text{Log} \left(\frac{\beta}{\beta - 1} \right) \left[1 + O \left(\left(\frac{\text{Log Log} \left(\frac{\beta}{\beta - 1} \right)}{\text{Log} \left(\frac{\beta}{\beta - 1} \right)} \right)^2 \right) \right]$$

then (5.15) as $\beta \rightarrow 1$. \square

In the sequel, for short, we write λ_n instead of $D(\lambda_n)$.

Proposition 5.10. *Let $n \geq n_0 = 18$ and $1 \leq j \leq \lfloor \frac{n-1}{4} \rfloor$. The roots $z_{j,n}$ of $G_n(X)$ have the following asymptotic expansions: $z_{j,n} = D(z_{j,n}) + \text{tl}(z_{j,n})$ in the following angular sectors:*

(i) Sector $\frac{\pi}{2} > \arg z > 2\pi \frac{\text{Log } n}{n}$ (main sector):

$$D(\Re(z_{j,n})) = \cos(2\pi \frac{j}{n}) + \frac{\text{Log}(2 \sin(\pi \frac{j}{n}))}{n},$$

$$D(\Im(z_{j,n})) = \sin(2\pi \frac{j}{n}) + \tan(\pi \frac{j}{n}) \frac{\text{Log}(2 \sin(\pi \frac{j}{n}))}{n},$$

with

$$\text{tl}(\Re(z_{j,n})) = \text{tl}(\Im(z_{j,n})) = \frac{1}{n} O \left(\left(\frac{\text{Log Log } n}{\text{Log } n} \right)^2 \right)$$

and the constant 1 in the Big O ,

(ii) “Bump” sector $2\pi \frac{\text{Log } n}{n} > \arg z > 0$:

• Subsector $2\pi \frac{\sqrt{(\text{Log } n)(\text{Log } \text{Log } n)}}{n} > \arg z > 0$:

$$D(\Re(z_{j,n})) = \theta_n + \frac{2\pi^2}{n} \left(\frac{j}{\text{Log } n} \right)^2 (1 + 2\lambda_n),$$

$$D(\Im(z_{j,n})) = \frac{2\pi \text{Log } n}{n} \left(\frac{j}{\text{Log } n} \right) \left[1 - \frac{1}{\text{Log } n} (1 + \lambda_n) \right],$$

with

$$\text{tl}(\Re(z_{j,n})) = \frac{1}{n \text{Log } n} \left(\frac{j}{\text{Log } n} \right)^2 O \left(\left(\frac{\text{Log } \text{Log } n}{\text{Log } n} \right)^2 \right),$$

$$\text{tl}(\Im(z_{j,n})) = \frac{1}{n \text{Log } n} \left(\frac{j}{\text{Log } n} \right) O \left(\left(\frac{\text{Log } \text{Log } n}{\text{Log } n} \right)^2 \right),$$

• Subsector $2\pi \frac{\text{Log } n}{n} > \arg z > 2\pi \frac{\sqrt{(\text{Log } n)(\text{Log } \text{Log } n)}}{n}$:

$$D(\Re(z_{j,n})) = \theta_n + \frac{2\pi^2}{n} \left(\frac{j}{\text{Log } n} \right)^2 \left(1 + \frac{2\pi^2}{3} \left(\frac{j}{\text{Log } n} \right)^2 (1 + \lambda_n) \right)$$

$$D(\Im(z_{j,n})) =$$

$$\frac{2\pi \text{Log } n}{n} \left(\frac{j}{\text{Log } n} \right) \left[1 - \frac{1}{\text{Log } n} \left(1 - \frac{4\pi^2}{3} \left(\frac{j}{\text{Log } n} \right)^2 \left(1 - \frac{1}{\text{Log } n} (1 - \lambda_n) \right) \right) \right],$$

with

$$\text{tl}(\Re(z_{j,n})) = \frac{1}{n} O \left(\left(\frac{j}{\text{Log } n} \right)^6 \right), \text{tl}(\Im(z_{j,n})) = \frac{1}{n} O \left(\left(\frac{j}{\text{Log } n} \right)^5 \right).$$

Proof. [27] Proposition 3.4. □

Outside the “bump sector” the moduli of the roots $z_{j,n}$ are readily obtained as (Proposition 3.5 in [27]):

$$(5.17) \quad |z_{j,n}| = 1 + \frac{1}{n} \text{Log} \left(2 \sin \left(\frac{\pi j}{n} \right) \right) + \frac{1}{n} O \left(\frac{(\text{Log } \text{Log } n)^2}{(\text{Log } n)^2} \right),$$

with the constant 1 in the Big O (independent of j).

Proposition 5.11.

$$\arg(z_{j,n}) = 2\pi \left(\frac{j}{n} + A_{j,n} \right) \quad \text{with} \quad A_{j,n} = -\frac{1}{2\pi n} \left[\frac{1 - \cos \left(\frac{2\pi j}{n} \right)}{\sin \left(\frac{2\pi j}{n} \right)} \text{Log} \left(2 \sin \left(\frac{\pi j}{n} \right) \right) \right]$$

$$\text{and} \quad \text{tl}(\arg(z_{j,n})) = \frac{1}{n} O \left(\left(\frac{\text{Log } \text{Log } n}{\text{Log } n} \right)^2 \right).$$

Proof. §6 in [27]. □

Proposition 5.12. *For all j such that $\pi/3 \geq \arg z_{j,n} > 2\pi \frac{[v_n]}{n}$, the asymptotic expansions of the moduli of the roots $z_{j,n}$ are*

$$|z_{j,n}| = D(|z_{j,n}|) + \text{tl}(|z_{j,n}|)$$

with

$$(5.18) \quad D(|z_{j,n}|) = 1 + \frac{1}{n} \text{Log} \left(2 \sin \left(\frac{\pi j}{n} \right) \right) + \frac{1}{2n} \left(\frac{\text{Log Log } n}{\text{Log } n} \right)^2$$

and

$$(5.19) \quad \text{tl}(|z_{j,n}|) = \frac{1}{n} O \left(\frac{(\text{Log Log } n)^2}{(\text{Log } n)^3} \right)$$

where the constant involved in $O(\cdot)$ is 1 (does not depend upon j).

Proof. [27] Section 5.1. □

Proposition 5.13. *For $n \geq 18$, the modulus of the first root $z_{1,n}$ of $G_n(z) = -1 + z + z^n$ is*

$$(5.20) \quad |z_{1,n}| = 1 - \frac{\text{Log } n - \text{Log Log } n}{n} + \frac{1}{n} O \left(\frac{\text{Log Log } n}{\text{Log } n} \right)$$

and

$$(5.21) \quad |-1 + z_{1,n}| = \frac{\text{Log } n - \text{Log Log } n}{n} + \frac{1}{n} O \left(\frac{\text{Log Log } n}{\text{Log } n} \right)$$

with the constant 1 in the two Big Os.

Proof. The root $z_{1,n}$ belongs to the subsector $2\pi \frac{\sqrt{(\text{Log } n)(\text{Log Log } n)}}{n} > \arg z > 0$: first, from Lemma 5.6, the asymptotic expansion of λ_n is

$$\lambda_n = \frac{\text{Log Log } n}{\text{Log } n} + O \left(\frac{\text{Log Log } n}{(\text{Log } n)^2} \right)$$

with the constant 1 in the Big O . Since $D(|z_{1,n}|) = D(\Re(z_{1,n})) \left(1 + \left(\frac{D(\Im(z_{1,n}))}{D(\Re(z_{1,n}))} \right)^2 \right)^{1/2}$, that

$$D(\Re(z_{1,n})) = \theta_n + \frac{2\pi^2}{n} \left(\frac{1}{\text{Log } n} \right)^2 (1 + 2\lambda_n), \quad D(\Im(z_{1,n})) = \frac{2\pi}{n} \left[1 - \frac{1}{\text{Log } n} (1 + \lambda_n) \right]$$

(Proposition 5.10) and

$$\theta_n = 1 - \frac{\text{Log } n}{n} (1 - \lambda_n) + \frac{1}{n} O \left(\left(\frac{\text{Log Log } n}{\text{Log } n} \right)^2 \right)$$

(Proposition 5.5) we deduce (5.20) and the expansion (5.21) from the expansion of λ_n . □

Proposition 5.14. *For $n \geq 18$, the modulus of $-1 + z_{j,n}$, where $z_{j,n}$ is the j -th root of $G_n(z) = -1 + z + z^n$, $[v_n] \leq j \leq [n/6]$, is*

$$(5.22) \quad |-1 + z_{j,n}| = 2 \sin \left(\frac{\pi j}{n} \right) + \frac{1}{n} O \left(\left(\frac{\text{Log Log } n}{\text{Log } n} \right)^2 \right)$$

with the constant 1 in the Big O .

Proof. From (5.17), Proposition 5.10 and Proposition 5.12, the identity

$$|-1 + z_{j,n}|^2 = (-1 + \Re(z_{j,n}))^2 + (\Im(z_{j,n}))^2 = 1 + |z_{j,n}|^2 - 2\Re(z_{j,n})$$

implies: $|-1 + z_{j,n}|^2 =$

$$2 - 2\cos\left(2\pi\frac{j}{n}\right) + \frac{1}{n}O\left(\left(\frac{\text{Log Log } n}{\text{Log } n}\right)^2\right) = 4\sin^2\left(\frac{\pi j}{n}\right) + \frac{1}{n}O\left(\left(\frac{\text{Log Log } n}{\text{Log } n}\right)^2\right)$$

with the constant 4 in the Big O . We deduce (5.22). \square

Proposition 5.15. *For $n \geq 18$, the modulus of $(-1 + z_{j,n})/z_{j,n}$, where $z_{j,n}$ is the j -th root of $G_n(z) = -1 + z + z^n$, $\lceil v_n \rceil \leq j \leq \lfloor n/6 \rfloor$, is*

$$(5.23) \quad \frac{|-1 + z_{j,n}|}{|z_{j,n}|} = 2\sin\left(\frac{\pi j}{n}\right)\left(1 - \frac{1}{n}\text{Log}\left(2\sin\left(\frac{\pi j}{n}\right)\right)\right) + \frac{1}{n}O\left(\left(\frac{\text{Log Log } n}{\text{Log } n}\right)^2\right)$$

with the constant 2 in the Big O .

Proof. The expansion (5.23) readily comes (5.22) and $|z_{j,n}|$ given by Proposition 5.12. \square

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