

Subjective probability in use: reasoning about a chess game

Luis J. Rodríguez-Muñiz, Lorena Rodríguez-Suárez and Laura Muñiz-Rodríguez

Universidad de Oviedo, Spain; luisj@uniovi.es

Subjective probability is often left aside in elementary school curricula, despite it is often used in daily-life. In this paper, we expose a didactic experiment with 5th grade students (10-11 years old) in which they were asked to handle subjective probability. Findings show how students were able to make probability judgements even without a numerical support, they changed their initial judgements when more quality information was provided, and they refrained from state a founded judgement when they considered available information was not enough. Results endorse the importance of introducing this type of probabilistic reasoning in elementary school.

Keywords: Elementary school, Probabilistic reasoning, Subjective probability, Verbal communication.

Introduction

The notion of chance is polysemic and, often, controversially interpreted in common contexts, including scientific ones. This affects how probability is understood in our society, and how it is perceived in school contexts. Despite the axiomatic approach solved the formal mathematics obstacles, “there are still controversies over the interpretation of basic concepts and about their impact on the practice of statistics” (Batanero et al., 2005, pp. 15–16). Classical and frequentist (or experimental) approaches are the most common in elementary school when teaching probability, being the subjective (or Bayesian) approach often barely mentioned, or totally ignored (Carranza & Kuzniak, 2008; Gómez-Torres et al., 2014). However, subjective probability is largely present in daily-life, where events cannot be simplified into counting possible outcomes of the random experiment, or repeating it under the same conditions, as when predicting sport results or assessing the risk of being infected by a virus (Muñiz-Rodríguez et al., 2020).

At early school ages, the subjective meaning of probability is strongly related to the use of verbal reasoning and, particularly, chance language (Kazak & Leavy, 2018). This conceptualization is often described as the intuitive meaning of probability (Batanero, 2005), in which the use of linguistic quantifiers and terms about chance helps children making qualitative probabilistic judgements. In this paper, the results of a didactic experiment developed with 5th grade students (10-11 years old) in a Spanish school, consisting of probabilistic reasoning about a chess game, are described.

Theoretical framework

The intimate relationship between probabilistic reasoning and the notion of chance produces a greater space for intuition (and for counterintuition) when learning and dealing with probability. Handling this intuition and making it coherent with the mathematical development of probability is a challenge in probabilistic education at the elementary school, as Fischbein (1975) already pointed out. Even when subjective probability has not been a trending topic on probabilistic education research literature, the didactic experiment described here can be framed within the general paradigm of probability literacy (Gal, 2005), due to its link to the basic probabilistic instruction at elementary

school. Gal's framework includes five knowledge and three dispositional elements (see Figure 1). This research study focuses on figuring probabilities (estimates), language (of chance), and context (a chess game), and mainly dealing with beliefs and attitudes.

Knowledge elements
1. <i>Big ideas</i> : Variation, Randomness, Independence, Predictability/Uncertainty.
2. <i>Figuring probabilities</i> : Ways to find or estimate the probability of events.
3. <i>Language</i> : The terms and methods used to communicate about chance.
4. <i>Context</i> : Understanding the role and implications of probabilistic issues and messages in various contexts and in personal and public discourse.
5. <i>Critical questions</i> : Issues to reflect upon when dealing with probabilities.
Dispositional elements
1. <i>Critical stance</i> .
2. <i>Beliefs and attitudes</i> .
3. <i>Personal sentiments regarding uncertainty and risk (e.g., risk aversion)</i> .

Figure 1: Elements of probability literacy (Gal, 2005, p. 51)

De Finetti's claim (1974) "probability does not exist" illustrates the subjective meaning of probability, neglecting the objective existence of "one" probability, and assuming it as the degree of belief about the occurrence of a given event. Such a personal judgement about the outcome of a random experiment is necessarily based on the information about the experiment or similar experiments that the involved person has. Thus, estimating the probability is dependent not only on the quantity of information but also on its quality. Different people have different type of information, and process it into very different ways, depending on his/her knowledge of the problem/context, experiences, and beliefs. And when new information is provided, people revise their initial probability assignment (the formalization of this revision process should be made in terms of Bayes' formula, which led to alternatively use Bayesian for this subjective approach). The subjective theoretical approach differs from the classical one, based on Laplace's rule, which is only applicable under equiprobability of the elementary events in a finite sample space, and assumes the existence of a real/certain value of probability for a given event. Nevertheless, subjective does not mean ungrounded: the probability elicitation depends on the person, but in obvious situations, as simple random experiments with games (coins, cards, dice, etc.), the subjective probability assignment of a minimally informed person should converge with the classical one, unless evidence from the running experiment is opposed (for instance, an unfair dice). Hence, Pratt and Kazak (2018, p. 206) discussed "the dual notion of probability" as a degree of belief and as the stable frequency in a long run.

Going into the elementary school framework, probability is introduced in different countries by means of the language of probability (Vásquez Ortiz & Alsina, 2019). Some experiences can be found in the research literature to promote a proper probabilistic reasoning by linguistically labelling, interpreting, and ranking the probability of certain events (e.g., Alsina et al., 2020; Kazak & Leavy, 2018). But, after this linguistic approach, most national curricular guidelines go directly to the formal definition of probability based either on Laplace's rule or on the experimental probability (Batanero et al, 2005). These guidelines rarely move towards the subjective definition, despite many daily-life situations are not suitable to apply Laplace's rule, neither experimentation is possible (either similar conditions are not reached, or repetition is impossible).

This paper addresses Pratt and Kazak's claim (2018, p. 222): "there is still a scarcity of research [...] in the area of subjective probability at the school level". By means of a didactic experiment based on a chess game, we aim at analyzing 5th grade students' probabilistic reasoning in a context of

uncertainty without conditions for applying Laplace's rule nor for experimenting by repetition. Previous research (Helmerich, 2015; Huber & Huber, 1987; Kazak, 2015; Pratt & Kazak, 2018; Vásquez Ortiz & Alsina, 2019) showed that students can deal with the language of probability and produce qualitative probability estimates, and that, under certain conditions, they mobilize their personal knowledge and beliefs to confront with experimental information. However, these studies considered situations which could be analyzed in combinatorial terms, that is, counting cases, even when they were not undertaken by using that strategy. The task that we present here is not suitable for determining probabilities by counting cases and, therefore, the cognitive demand is greater. Thus, we raise the following research question: How do 5th-graders integrate contextual information to deal with probability estimates not based on counting cases?

Methodology

Context

The didactic experiment was carried out in April 2021, at an elementary school in Oviedo (Spain). Thirty-one students (20 boys and 11 girls) in 5th grade (10-11 years old) participated, 4 of which played chess (3 boys and 1 girl). Curricular guidelines in 5th include to identify random situations and to make estimates on some game results (coins, dice, cards). In previous years, students worked with probability language (impossible, likely, certain), whereas the definition of probability based on Laplace's rule is to be studied in 6th grade. The experiment was conducted by one of the researchers.

Experiment

The didactic experiment consisted of a chess-based situation about a game between two elite players, organized into three steps:

1st step: In Grenke's chess tournament Anish Giri (white) will face the current world champion, Magnus Carlsen, in black. Of all the games that Anish Giri has played (908) almost half (456) were drawn. By contrast, Magnus Carlsen lost just three games in the last two and a half years. What do you think the result will be? Will Anish Giri win the world champion with white? Justify your answer. Would Giri be less likely to win if he played black?

2nd step: Due to the COVID-19 pandemic, both players spent several months without playing slow games. The three games that Carlsen lost in the last two and a half years were after pandemic lockdown. Meanwhile, Anish Giri did not lose any games after the lockdown. Do you think confinement was more beneficial for one than the other? After knowing these data, would you change the answer in the previous section?

3rd step: If that game had happened before the pandemic, when Carlsen had 120 games without losing, what do you think would be Giri's probability of winning?

A sheet with the wording of the situation was given to each student. First, each student individually wrote down his/her answer to each question on the sheet. Then, students orally shared their answers in a group discussion (due to COVID-19 restrictions personal contact was limited).

Obviously, in this situation the probability cannot be determined by using Laplace's rule, as there are too many variables involved (pieces color, opening preparation, level of concentration, etc.).

Information from previous matches can help to make an estimate but, clearly, not the repetition of the same random experiment under similar conditions. Moreover, the data provided for both players refer to different periods. Students must mobilize their knowledge about the context and handle their beliefs about games. There was not a right or wrong answer, but a better or worse reasoning based on the way students integrated and aggregated information and beliefs, and how they handled possible biases in probabilistic reasoning. Depending on the amount and quality of information, students could critically analyze the situation and produce an argued answer. During the didactic experiment, some explanations about chess had to be provided regarding the supposed advantage that the player with white pieces has. Also, a brief profile of Giri and Carlsen was required by students, explaining why they are considered world top players. Even being a complex argumentation, this task can be framed within the aforementioned Gal's three knowledge elements in probabilistic literacy, combined with beliefs and attitudes as a dispositional element.

Data collection and analysis

Students' written answers were considered as the primary unit of analysis. One of the researchers acted as teacher during the didactic experiment. She gave some instructions at the beginning about how to tackle the task. During the group discussion, she asked few questions to prompt students to justify their answers and took observation notes. Data was analyzed using a mixed technique, with a frequentist analysis and also a qualitative content analysis (Krippendorff, 2018) to find evidence of the use of probability language, the estimation of probabilities from a subjective point of view, and changes in the initial estimates when new information is provided. Considering that students' answers were rather brief and not too elaborated, data was first examined using a grounded theory approach. Each of the emerging categories was later analyzed in relation to the aforementioned Gal's knowledge and dispositional elements. For instance, students' answers to the 1st step questions were classified into blank or unclear justification supporting that Giri wins, justifications based on Giri's trajectory, justifications based on Carlsen's trajectory, and justifications supporting a draw between both players. Later, each category was reviewed to understand how students build their justifications using probability language, estimates, and information from the context (knowledge elements) together with their beliefs and attitudes (dispositional element).

Results

Table 1 shows the answers to 1st step questions. Justifications about Giri's win with white were based on players' trajectories: "Anish Giri would win because he has not lost a game, while Carlsen lost three", "Carlsen is the champion", or "Carlsen only lost 3 games in one year and a half". There were unclear justifications ("Giri is going to be lucky"). Those who supported the draw argued that both players are excellent. When asked if Giri would be less likely to win if he played black, 6 students agreed, and the rest (25) said there would be no difference. One of the arguments supporting the less likelihood was expressed with probability language ("The piece color matters because there are higher probabilities of winning when playing white"), while the rest were mainly based on students' beliefs ("Black pieces give worse luck"). Most students supporting there was no less probability argued that the color did not matter. Between the first and the second step, the teacher conducted a discussion with the students about their answers. Many of the students who did not know too much

about chess waited to hear the argumentation provided by those who played chess, looking for endorsement or extra information. Key points of such discussion are stressed in the next section.

Table 1: Answers to “Will Giri win with white? Would Giri be less likely to win with black?”

Type of answers and justifications	With white				With black	
	Giri wins, blank or unclear	Giri wins, based on his trajectory	Carlsen wins, based on his trajectory	Draw	Less likely	No difference
Frequency	14	5	9	3	6	25

The 2nd step of the experiment was about reasoning if the confinement was more beneficial for Carlsen or Giri and checking how students integrated the related information. Table 2 shows the answers. Most who agreed that the confinement was more beneficial for Giri (“He did not lose any game after it”) kept their initial answer about Carlsen’s favoritism, based on his role as champion (“I trust in Carlsen, he is the champion”), but there were three answers providing an intuitive probabilistic reasoning (for instance, “Maybe Carlsen just had a bad run”). There was one student supporting the confinement was not beneficial for any of the players, who noted that the provided information was about the number of losses, but nothing was said about the number of draws (where Giri had much more than Carlsen), so it justified keeping Carlsen as the favorite player to win. We did not find solid argumentations among those changing their initial view.

Table 2: Answers to “Do you think confinement was more beneficial for one than the other? After knowing these data, would you change the answer in the previous section?”

Type of answers	For Giri, not changing	For Giri, changing	To any of them, not changing
Frequency	18	4	9

In the 3rd step of the experiment, students were asked about Giri’s probability of winning if the game had been held before the pandemic. Table 3 shows the types of arguments used by students. Some recognized the difficulty of estimating a probability (“I do not know how to measure probabilities, but I think it would be around 45%”), others simply provided a value (“From 1 to 10, an 8”). Some other reasonings were based on Carlsen’s apparent superiority, eliciting a low estimate for the probability (“Very low, 2 out of 10, 20%, because Carlsen was undefeated”, “About 33% or very low, because Carlsen was undefeated during 120 games”). Two answers explicitly based the argument on the previous results: “Giri would draw, because he usually draws”, and “Giri is going to win because he was never the champion and Carlsen started losing games”.

Table 3: Answers to “If that game had happened before the pandemic, [...] what do you think would be Giri’s probability of winning?”

Type of arguments	Blank	No reason	Verbal	Numerical	Verbal and numerical
Frequency	3	2	11	14	1

Discussion and conclusions

Results revealed different examples of probabilistic heuristics, as well as some biases in students' reasoning. In the first step, students used their previous knowledge about the context (a knowledge element in Gal, 2005) to guess the possible result of the game, but their beliefs (a dispositional element in Gal, 2005) were also involved. Since previous knowledge differed from one student to another, we found answers based on a proper integration of previous information and knowledge about chess ("Anish Giri would win because he did not lose a game, while Carlsen lost three") and also other answers built on maximum uncertainty ("There will be a draw because both players are excellent"). There were also some mistakes regarding information processing or reading comprehension, getting confused between Carlsen having lost three games and having won all but three (that is, not considering draw as a possible result). Thus, this illustrates how students differently integrate and process the contextual information in terms of deciding what is the most probable event.

In the second step, extra information was added, so the students had to decide whether they keep or change their initial decision, again within a meaningful context (Gal, 2005). It was striking how most of the students integrated the information about the confinement and the players but decided to keep their initial decision, because most answers were supported by the personal belief on Carlsen's quality as player, and also a poorly explained favoritism to him (Batanero, 2005; Vásquez Ortiz & Alsina, 2019). There was also an interesting answer trying to find a rationale between Carlsen's category as world champion and the three games he lost after confinement ("Maybe Carlsen just had a bad run"). Also remarkable was how students used personal experience regarding chess (for instance, the one arguing that whites have greater chances to win was one of the students playing chess and he used his records to support this idea). In line with Kazak and Leavy (2018), we confirmed how the meaning of subjectivity is used when dealing with previous and added information and personal experiences and beliefs to estimate a probability, stressing in this experience the importance of beliefs as an attitudinal element (Gal, 2005).

In the third step, the difficulties to estimate a probability without a counting or experimental situation were revealed. Nevertheless, students were able to find a rationale supporting their linguistic or numerical assignments (knowledge elements in Gal, 2005). Rather curious was how students expressed probability in many different scales (percentage, 1-10 scale, linguistic quantifiers, etc.), and how they were able to compare even without a numerical assignment. Besides numerical language, half of the students used verbal language to express such probability using different terms ("impossible", "very low", "none"). In some cases, a wrong understanding of the probabilistic language was reflected when considering an improbable or unlikely event as impossible. Students also used different approaches to integrate the information from previous games: some used it to find a trend in terms of experimental probability ("He usually draws"), others reasoned by using a version of gambler's fallacy ("Giri is going to win because he was never the champion"). The latter shows the outcome approach described by Konold (1989): when asked about a probability there are students answering about the outcome of the next trial.

Regarding research question, the development of the discussion between the first and second steps became particularly crucial. We witnessed how students who were not familiar with the context

(chess game) started from a total uncertainty (even ignorance) about the answer, but when they heard their colleagues who regularly play chess reasoning about the situation, they started either totally following them or modifying their opinions in terms of what was expressed by experts. That is, they acknowledged the quality of the information provided by others and modified their initial beliefs about the probability in an informal Bayesian reasoning. As stated by Kazak and Leavy (2018), such reformulations seem rather intuitive for children of this age.

The implications on this didactic experiment are rather promising for future research and practice. Despite the difficulties encountered on probabilistic reasoning, students are able to quite naturally handle probability in informal and even numerical terms. Evidence pointed to the use of Bayesian procedures to integrate added information about the experiment. Choosing a chess-based situation instead of a more popular sport (as football or basket) made students felt too much apart from the context and impeded arguing about their reasonings. Thus, to promote students' probability literacy, particularly probabilistic reasoning, it seems necessary to widely implement similar contextualized situations that are relevant to students and help them to understand the different meanings of probability. Similar experiences, with an increasing complexity in the situations and the cognitive demand, could be also designed for secondary education, where the use of the subjective meaning of probability is also usually omitted (Blanco-Fernández et al., 2016; Rodríguez-Muñiz et al., 2019). More precisely, the presence of didactic experiments about subjective probability should be increased by teachers making use of daily-life situations instead of archetypical examples, such as urns, coins, dice, cards, or lottery.

Acknowledgment

This research was partially funded by Grant TIN2017-87600P from the Spanish Ministry of Science and Innovation. Authors want to express their gratitude to the participating children.

References

- Alsina, Á., Vásquez Ortiz, C. A., Muñiz-Rodríguez, L., & Rodríguez Muñiz, L. J. (2020). ¿Cómo promover la alfabetización estadística y probabilística en contexto? Estrategias y recursos a partir de la COVID-19 para Educación Primaria. *Épsilon. Revista de la Sociedad Andaluza de Educación Matemática*, 104, 99–128.
- Batanero, C. (2005). Significados de la probabilidad en la educación secundaria. *Revista Latinoamericana de Investigación en Matemática Educativa*, 8(3), 247–264.
- Batanero, C., Henry, M., & Parzysz, B. (2005). The nature of chance and probability. In G. A. Jones (Ed.), *Exploring Probability in School: Challenges for Teaching and Learning* (pp. 15–37). Springer. https://doi.org/10.1007/0-387-24530-8_2
- Blanco-Fernández, Á., Díaz-Díaz, P., García-Honrado, I., Ramos-Guajardo, A. B., & Rodríguez-Muñiz, L. J. (2016). A proposal for assessing imprecise concepts in Spanish primary and secondary schools. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 24(suppl. 2), 71–91. <https://doi.org/10.1142/S0218488516400110>
- Carranza, P., & Kuzniak, A. (2008). Duality of probability and statistics teaching in French education. In C. Batanero, G. Burrill, C. Reading, & A. Rossman (Eds.), *Joint ICMI/IASE Study: Teaching Statistics in School Mathematics. Challenges for Teaching and Teacher Education*. ICMI and IASE.

- De Finetti, B. (1974). *Theory of probability*. John Wiley.
- Fischbein, E. (1975). *The intuitive sources of probabilistic thinking in children*. Reidel. <https://doi.org/10.1007/978-94-010-1858-6>
- Gal, I. (2005). Towards “probability literacy” for all citizens: Building blocks and instructional dilemmas. In G. A. Jones (Ed.), *Exploring probability in school* (pp. 39–63). Springer. https://doi.org/10.1007/0-387-24530-8_3
- Gómez-Torres, E., Ortiz, J. J., & Gea, M. M. (2014). Conceptos y propiedades de probabilidad en los libros de texto españoles de Educación Primaria. *Avances de Investigación en Educación Matemática*, 5, 49–71. <https://doi.org/10.35763/aiem.v1i5.63>
- Helmerich, M. (2015). Rolling the dice: Exploring different approaches to probability with primary school students. In K. Krainer & N. Vondrová (Eds.), *Proceedings of the 9th Congress of the European Society for Research in Mathematics Education* (pp. 678–684). Charles Univ. & ERME.
- Huber, B. L., & Huber, O. (1987). Development of the concept of comparative subjective probability. *Journal of Experimental Child Psychology*, 44(3), 304–316. [https://doi.org/10.1016/0022-0965\(87\)90036-1](https://doi.org/10.1016/0022-0965(87)90036-1)
- Kazak, S. (2015). A Bayesian inspired approach to reasoning about uncertainty: ‘How confident are you?’ In K. Krainer & N. Vondrová (Eds.), *Proceedings of the 9th Congress of the European Society for Research in Mathematics Education* (pp. 700–706). Charles Univ. & ERME.
- Kazak, S., & Leavy, A. M. (2018). Emergent reasoning about uncertainty in primary school children with a focus on subjective probability. In A. M. Leavy, M. Meletiou-Mavrotheris, & E. Paparistodemou (Eds.), *Statistics in Early Childhood and Primary Education. Supporting Early Statistical and Probabilistic Thinking* (pp. 37–54). Springer. https://doi.org/10.1007/978-981-13-1044-7_3
- Konold, C. (1989). Informal conceptions of probability. *Cognition and Instruction*, 6, 59–98. https://doi.org/10.1207/s1532690xci0601_3
- Krippendorff, K. (2018). *Content analysis: An introduction to its methodology*. Sage. <https://doi.org/10.4135/9781071878781>
- Muñiz-Rodríguez, L., Rodríguez-Muñiz, L. J., & Alsina, Á. (2020). Deficits in the statistical and probabilistic literacy of citizens: Effects in a world in crisis. *Mathematics*, 8(11), e1872. <https://doi.org/10.3390/math8111872>
- Pratt, D., & Kazak, S. (2018). Research on uncertainty. In D. Ben-Zvi, K. Makar, & J. Garfield (Eds.), *International handbook of research in statistics education* (pp. 193–227). Springer. https://doi.org/10.1007/978-3-319-66195-7_6
- Rodríguez-Muñiz, L. J., Díaz, P., & Muñiz-Rodríguez, L. (2019). Statistics and Probability in the Spanish Baccalaureate: Intended Curriculum and Implementation in Textbooks. In Y. Shimizu & R. Vithal (Eds.), *ICMI Study 24. Conference Proceedings. School Mathematics Curriculum Reforms: Challenges, Changes and Opportunities* (pp. 413–420). ICMI & University of Tsukuba.
- Vásquez Ortiz, C., & Alsina, Á. (2019). Intuitive ideas about chance and probability in children from 4 to 6 years old. *Acta Scientiae*, 21(3), 131–154. <https://doi.org/10.17648/acta.scientiae.v21iss3id5215>