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Manifestations of the collapse metaphor and their possible implications on personal meanings of integral and accumulation

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In this paper we report case study findings regarding two forms of the collapse metaphor and their implications on learners' personal meanings about accumulation and the integral. We found that students can interpret adding up lines as both adding up their lengths and adding up their areas. These conceptions appear to be related to personal meaning and context.

Keywords: Accumulation, integral, collapse metaphor, meaning.

Rationale

Learning and teaching calculus is considered to be an important part of mathematics education practice and research. There is quite broad agreement among mathematics educators that calculus concepts are highly challenging for learners. This difficulty may result from learning that leads to acquiring narrow, non-productive meanings of the concepts, or even meaningless learning.

In this paper we report on an empirical case study that was conducted in the framework of a larger project focusing on personal meanings of high school students for derivative and integral. We found that the collapse metaphor (Oehrtman, 2009) played an interesting and perhaps crucial role in students' meaning regarding accumulation. We report our findings concerning this issue.

Meaning

Students learn calculus concepts over many years, from middle school to university. Research shows that many acquire formal problem-solving techniques (e.g., Artigue, 1991; Tall, 1993) rather than comprehension. Researchers have emphasized the importance of learning with meaning in mathematics in general, and in calculus in particular. Thompson (2013) noted that although researchers talk about meaning, they do not address it, and that research in mathematics education rarely attends to the issue of meaning seriously. Thompson (2013) speaks of "a lack of systematic-cultural reference to mathematical meaning and coherence" (p.57). In this study we interpret "meaning" using the considerations suggested by Thompson (2013, 2016). Our aim is to identify what personal meaning students have.

For that purpose, we adopted the taxonomy that was suggested by Thompson and his colleagues (Thompson et al., 2014) as follows (p.13):

- Understanding (in the moment) is the cognitive state resulting from an assimilation.
- Meaning (in the moment) is the space of implications existing at the moment of understanding.
- Understanding (stable) is the cognitive state resulting from an assimilation to a scheme.
- Meaning (stable) is the space of implications that results from having assimilated to a scheme. The scheme is the meaning.
- Way of Thinking is the habitual anticipation of specific meanings or ways of thinking in reasoning.

This taxonomy is strongly rooted in Piaget's theory of meaning (e.g. Johnckheere, Mandelbrot, & Piaget, 1958). Thus, for example, assimilation is association with inference (ibid., p. 59). For us, the taxonomy has a strong operative methodological aspect (see data collection section for the details). The key notions of this taxonomy are the notion of scheme and the notion of the space of implications. The notion of scheme is described and analyzed by Thompson and colleagues (Thompson et al., 2014). Following this analysis, we consider schemes as "organizations of mental activity that express themselves in behavior, from which we, as observers, discern meanings and ways of thinking. Scheme is a theoretical construct that we impute to individuals to explain their behavior (ibid., p.10)". The individual aspect of this consideration is crucial for us. We search the space of implications through microanalysis of the individual's learning behavior.

Collapse metaphor

Oehrtman (2009) examined students' use of metaphors to reason about limits. Students were found to employ similar metaphors across multiple problem contexts. Moreover, the application of these metaphors was found to influence students' perception of problem, solution methods and results.

One of these metaphors is the collapse metaphor (Oehrtman, 2009). It involves a multidimensional object varying in size along one of its dimensions. The varying measurement is determined by the indexing variable in the limit. Students describe this measurement as decreasing in value until it reaches zero. When the measurement reaches zero the corresponding dimension "collapses", and what is left is a lower dimensional object.

The collapse metaphor involves the collapse of a physical dimension and is therefore relevant in a visual and geometric context. Students employing the metaphor to reason regarding accumulation functions and the FTC are thinking of an image instead of a quantity. Thus, their understanding of the FTC is disconnected from the concept of rate of change (Thompson & Silverman, 2008).

Methodology

Population

Nathan is an experienced high school teacher. He completed several analysis courses at university and is very skilled in the subject. Sigal is a pre-service teacher in her last year of studies at a teachers college. She completed several analysis courses with high grades. Sigal and Nathan volunteered to be interviewed.

Data collection and analysis

Research suggests that "students' understanding of integrals is largely procedural, lacking quantitative meanings of accumulated change" (Thompson & Harel, 2021, p. 509). One difficulty students face when thinking about accumulation is that they cannot conceptualize the "bits" that accumulate (Thompson & Silverman, 2008). In this paper we explore the effect of reasoning with the collapse metaphor on learners' conceptions of these bits in task-based interviews on:

- Evaluating the accumulated amount of cash given a cash flow graph
- Calculating the length of a curve represented by a smooth function
- Finding the mass of a thin wire, given the wire's density

- Finding the graph of the accumulation function for a given step-function
- Finding the volume of a solid of revolution.

The interviews were recorded and transcribed.

When analyzing the interviews, we needed to establish which utterances are indicative of the interviewee's personal meaning. To do this, we identified several relevant criteria involving distinct language, repetition, reasoning, unexpectedness, and statement of opinion.

Distinct language. We identify utterances as elucidating personal meaning if they contain concepts and language specific to the learner. By distinct we mean that it has not been used by the interviewer or in the task, and is not prompted in any way, but is the interviewee's own language.

Repetition. Utterances allude to personal meaning if they contain concepts that repeat themselves, whether within a certain task, or across different tasks and contexts. Repetition signifies that these concepts are readily available and relevant to the interviewee, and therefore are a salient part of their perception. Repetition across multiple contexts suggests stable meaning.

Reasoning. Utterances that are intended to explain or justify the mathematical concepts are indicative of the interviewee's thought process, and therefore are indicative of personal meaning.

Unexpectedness. If an utterance is surprising or unexpected to the researcher, then it was not solicited by the interviewer or the task, and is an expression of the interviewee's personal meaning.

Statement of opinion. Utterances explicitly qualified by the interviewee as their own belief, opinion or interpretation will be regarded as indicative of personal meaning. This includes, for example, utterances containing phrases such as "to me", "in my opinion", "the way I understand it".

Findings

Because of limitations of space, we present only excerpts of the relevant data.

Nathan exhibited use of the collapse metaphor across multiple tasks. The "accumulated cash" task presented the printed graph in Figure 1. as representing the cash flow (in 1000s of NIS per second) of a bank from 8:30 to 11:30 one morning, a polynomial expression for the graphed function, and the initial amount (5 million NIS) at 8:30. Students were asked at which hours there was less than the initial amount in the bank.

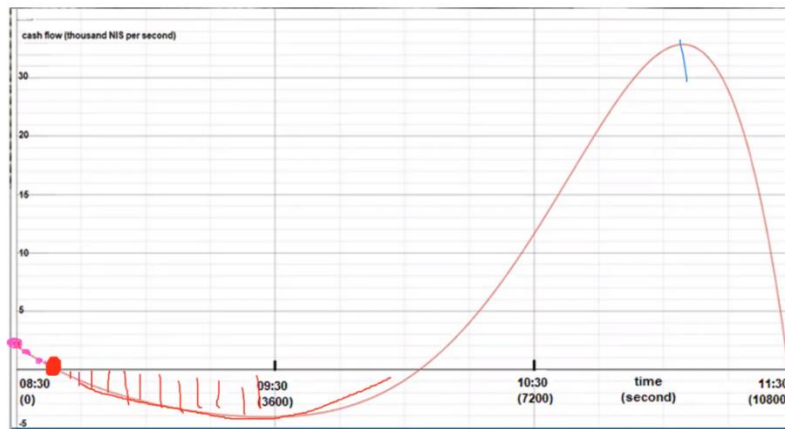


Figure 1: For Nathan, the amount of money exiting the bank is the sum of the lengths of the red lines

Nathan employed the collapse metaphor graphically, by drawing lines (Figure 1), and verbally:

Nathan: My logic is that when Δx approaches zero, or is even equal to zero, the size of the – I don't want to say rectangle, it's a line - it has no width. It's just a line, and since its width is zero, when we add up all of these lines, we will get the area under the curve.

When Nathan was asked to calculate the length of a curve of a given function, he once again exhibited use of the collapse metaphor by saying “I want to take all the points and add them up”. When asked to clarify his intention, Nathan replied: “I want to take two points, calculate the distance between them, and make Δx approach zero. Then I'll get the length of a single segment, and that's going to be for any pair of points on the graph of the function. If I integrate this, I will get the length of this segment along the graph of the function”.

Other utterances reveal a deeper influence of the collapse metaphor on Nathan's meanings. When asked to clarify what “adding up the lines” meant in the “accumulated cash” task, Nathan replied:

Nathan: I mean adding up the lengths of all the lines, meaning adding up the y-coordinates of all the points, the infinite number of points.
 Interviewer: You're adding up an infinite number of values, how are you doing that in practice?"
 Nathan: Using limits and using an integral on a graph of a function.

For Nathan the collapse metaphor connects to integrating as adding up line-lengths (or y-values):

Nathan: [The integral] sums the y-values when Δx approaches zero. I mean, that's the visualization I have in my head for an integral, that's how I perceive it, like rectangles with zero width – straight perpendiculars.
 Interviewer: Is there such a thing as a rectangle whose width is zero?
 Nathan: There is a straight line. A rectangle whose width approaches zero. It comes from Riemann and Darboux sums.

The wire AB is 8 meters long and made from cutting edge material.

A B

The density of matter of the wire is not constant but the density (in grams per meter) at any point X is numerically equal the length AX. Find the mass of the wire (in grams)

Figure 2: Mass from density task

Nathan was very consistent in his use of this conception. While the collapse metaphor is visual in nature, and involves the collapse of a physical dimension, the implications Nathan drew from this metaphor – integrating as adding-up-y-values, persisted even when the collapsing dimension was not explicitly present in the task, as in the “mass from density task” (Figure 2), where Nathan successfully identified the algebraic formula for the density but did not relate integrating the density over x with mass. Rather, he claimed that to find the total mass, he needs to integrate the mass function. He explained that if the mass at x is $g(x)$ then the total mass will be $\int_0^8 g(x)dx$.

The exception was the step-function task. Students were presented with the graph of a function $f(x)$ defined on $[0, 5]$ whose value is 4 on $[0,3)$ and 2 on $[3,5]$ and asked what they can say about $g(x) = \int_0^x f(t)dt, x \geq 0$. In this task, Nathan seems to gradually move from the concrete zero-width lines back to the abstract infinitesimally small rectangles they replaced, in an apparent process of re-abstractation. He refers to “strips, tiny rectangles, strips [drawing lines] strip, strip, strip, strip, strip.”

In this task, Nathan used the word “strip” instead of “line”. A “strip” is associated with a small but non-zero width. He also spoke of rectangles of decreasing width. He acknowledged that this was a forced deviation from his regular thinking pattern. About both ways of thinking, he stated that they were rooted in Riemann sums.

Finally, while Nathan explicitly agreed at times the area is the quantity being accumulated, at other times he seemed to struggle with this notion, strengthening the claim that students have a hard time conceptualizing “the ‘bits’ that accumulate”:

Nathan: It’s more accumulation to me than area.... It doesn’t change its accumulation. It doesn’t matter how much – I don’t want to say area, it doesn’t change the accumulation, it has no effect on the accumulation.

Sigal also exhibited use of the collapse metaphor across multiple problems, but the way in which she used the metaphor was not consistent and was context dependent. She interpreted adding-up-lines as both, adding up lengths (adding-up-y-values) and adding up areas (adding-up-zeros).

Sigal seemed not to be familiar with solids of revolution. She initially attempted to approximate the volume by dividing it into several parts. Next, she claimed that since the area bounded by the graph of the function and the x -axis is about half the area of the bounding rectangle, the volume would be about half the volume of the bounding cylinder. This suggests that Sigal thinks the area bounded by the function and the volume of the solid of revolution are proportional to one another:

Sigal: I would like to know the ratio between the integral, the specific area that is created, and the volume. Because there has to be some connection... If I got a connection of multiplication by 360 or something, then I would know that I just multiply the area by 360.

To explain this claim, Sigal adds: “if for example the area is 14, then its $14+14+14+14+14...$ ” This suggests that Sigal sees the revolution as an accumulation, and to her “the ‘bits’ that accumulate” are the bounded areas. This is consistent with a collapse of dimension, as like Nathan, she is also accumulating area to get volume (instead of an infinitesimal sector).

In the mass from density task, Sigal converts the problem into a geometrical one by representing each point on the wire as a segment in a Cartesian system of axes. Each segment represents a density to

Sigal, and adding up all the densities gives the mass of the wire. What Sigal struggles with is the infinite number of lines: “it’s an infinite number of lines, that I’m putting on top of one another. I don’t know what this height is because what it does is count how many lines I put there, it’s infinity.” In Sigal’s eyes, one of the sides of her triangle is infinite in length, and thus the area and mass are also infinite, which creates a conflict for her. While she isn’t using an integral, she is adding up line lengths to get area, thus providing further evidence of a conception of accumulation involving the collapse metaphor.

Sigal’s meaning for integral is intrinsically connected with area and antiderivative. In the length of a curve task, even though she perfectly describes the process of evaluating the length using line segments, and taking smaller segments to improve accuracy, she sees no connection to integrals in the task: “Integral to me is area, or antiderivative, or the connection between area and antiderivative”.

Sigal’s reasoning in the step-function task is consistent with her reasoning about integrals in the other tasks – it involves area and antiderivative. By her own account “all the time when I’m solving the question I’m thinking about areas”. However, Sigal also kept referring to accumulation: “I’m not starting to accumulate from 0 at $x=3$, I’m starting to accumulate, I’ve already accumulated a lot – 12, this is where I start”, “To me, integral is related to accumulation, so there’s a problem here”. Sigal also exhibited evidence of the collapse metaphor in this task, by reasoning with lines:

Sigal: I started from things I know about a continuous function, and relied on accumulation, and that’s basically what I was working by I kind of ignored 3, where there is a hole, because I figured that the line that goes through 3, doesn’t really affect the area... ;it has no width.

However, this manifestation of the collapse metaphor is different from Nathan’s, as it interprets adding-up-lines as adding up their areas (zeros), and not their lengths (y-values). Sigal’s inconsistent use of the collapse metaphor both as adding-up-y-values and as adding-up-zero-areas, depending on the task, is evidence of compartmentalization – her inconsistent interpretations in different contexts suggests she has conflicting elements in her concept image for accumulation that she is unaware of.

Discussion

Throughout Nathan’s interviews, he repeatedly made use of the collapse metaphor. The collapse metaphor was entirely brought up by him, with no prompting. Moreover, he invoked it to reason about integration, and qualified it as his personal conception: “My logic is”, “that’s the visualization I have in my head for an integral”. This suggests that the collapse metaphor is a part of Nathan’s personal meaning for integral.

Sigal also exhibited a collapse in dimension during her interviews: adding up areas to get volume in the solid of revolution task, adding up the lengths of lines to get area in the mass from density task, and considering the effect of subtracting a line from an area in the step-function task. This collapse of dimension was also repeated across contexts, unprompted, used in her reasoning, and qualified as opinion with phrases such as: “I think. It makes sense to me”, “according to my logic”.

However, there were several key differences between Nathan and Sigal. Firstly, Nathan explicitly used the collapse metaphor to make sense of integration and limit, collapsing the dimension that corresponds to the variable that approaches zero, in accordance with the Oehrtman’s (2009) findings.

Sigal, on the other hand, exhibited a collapse of dimension even where she did not necessarily perceive limit or integration to be relevant. In the solid of revolution task this was not the dimension she collapsed. Instead, she seemed to relate to accumulating the revolving area to create volume. In the mass from density task, even though Sigal represented the mass as area, which could be consistent with her meaning for integral, the lines she added were not the ones she was “integrating”. This suggests that for Sigal, the collapse metaphor is related to her meaning of accumulation, rather than to limit and integral.

Secondly, Nathan used the collapse metaphor consistently across contexts. Wherever he added up lines, he was always adding-up-y-values or line lengths. In the only instance where he did not add-up-y-values (step-function task), he admitted to forcing himself to think differently than usual. His repeated consistent way of reasoning with collapse metaphor suggests that Nathan’s meaning of integral is stable. Furthermore, Nathan’s reasoning is habitual: “My thoughts go to the same place all the time”. This indicates that adding-up-y-values is a way of thinking for him.

In contrast, Sigal’s use of the collapse metaphor was inconsistent. Although she exhibited signs of adding-up-y-values in both the solid of revolution task and the mass from density task, in the step-function task, when reasoning about the point of discontinuity, she claimed a single line is insignificant when accumulating area, as its width is zero. This suggests that in this instance Sigal is interpreting adding-up-lines as adding-up-zeros. The inconsistency in Sigal’s use of the collapse metaphor suggests her meanings for accumulation are meanings in the moment. Her contradictory and context dependent interpretation of “the ‘bits’ that accumulate” suggests compartmentalization.

Considering both Nathan and Sigal, we see two different manifestations of the collapse metaphor, distinct in their implications regarding “the ‘bits’ that accumulate”. Since adding-up-lines is not a mathematically well-defined concept, it leaves the learner with room for interpretation. Our findings suggest two such interpretations: adding-up-lines means adding up their lengths and adding-up-lines means adding up their areas. Both interpretations are problematic; however, depending on the interpretation and the context, they can lead to the correct conclusions. Considering that neither Sigal nor Nathan used adding-up-y-values in the step-function task, it is possible that the interpretation used depends at least in part on context. One possible explanation could be that use of adding-up-y-values is more likely when reasoning about an infinite number of lines, whereas adding-up-zeros is more likely when reasoning about a single line – as Sigal did in the step-function task.

According to Oehrtman (2009), to use the collapse metaphor is to apply an intuitively understood concept, so that it provides a concrete carrier for mathematics’ abstract structures, while systemizing the intuitive concepts being applied. The intuitive nature of the collapse metaphor is strengthened by our findings that Sigal exhibits a collapse in dimension even when no limit is explicitly present. Different studies have identified use of the collapse metaphor (Thompson, 1994; Oehrtman, 2009; Fisher et al., 2016). This implies that use of the collapse metaphor is rather prevalent. Thus, understanding its implications on meanings learners develop is necessary.

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