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To cite this version:
Igor Kontorovich, Sina Greenwood. Mathematics learning through a progressive transformation of a proof: A case from a topology classroom. Twelfth Congress of the European Society for Research in Mathematics Education (CERME12), Feb 2022, Bozen-Bolzano, Italy. hal-03754695

HAL Id: hal-03754695
https://hal.science/hal-03754695
Submitted on 19 Aug 2022

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Mathematics learning through a progressive transformation of a proof: A case from a topology classroom

Igor’ Kontorovich¹ and Sina Greenwood²

¹The University of Auckland, New Zealand; i.kontorovich@auckland.ac.nz
²The University of Auckland, New Zealand; s.greenwood@auckland.ac.nz

We report on an ongoing project in a cross-level topology course, where students have been provided with opportunities to prove the same mathematical statement in different social situations. This paper focuses on a pair of students who proved a statement collaboratively before one of them volunteered to re-prove it at the board for the whole class to observe. We offer a commognitive analysis of students’ discursive activity in each situation and trace the transformations of their proof throughout the process. This process is discussed with a focus on students’ mathematics learning.

Keywords: Commognition, graduate students, proof and proving, topology, university mathematics.

Introduction

In their comprehensive overview of the mathematics education literature, Stylianides et al. (2017) identify three broad perspectives in the area of proof: the cognitive – proving as problem solving, the constructivist – proving as convincing, and the social – proving as an activity that is embedded in communities. Within the latter, typical proof-related tasks (e.g., constructing or presenting a proof) are not viewed in isolation but as constituents of a broader mathematical activity. Stylianides et al. (2017) further explain that “If a student or teacher produces a proof, research in this perspective would frequently place emphasis on the meaning of this artifact and how that individual and members of his or her community could subsequently use it” (p. 247).

Stylianides et al. (2017) describe the social perspective as less developed, not yet coherent, and lacking “common, widely used concepts” (p. 248). These descriptors seem especially appropriate in the university mathematics education literature where the cognitive and the constructivist perspectives dominate the area of proof. This situation opens the space for socially-oriented studies on how proof is practiced in university classrooms. Stylianides et al. (2017) propose considering these practices in relation to the mathematics community.

Much has been written about proof in the mathematics community. For instance, Rav (1999) stresses that “the intricate role of proofs [is] in generating mathematical knowledge and understanding, [that goes] way beyond their purely logical-deductive function” (p. 6). He associates this role with “inventing methods, tools, strategies and concepts” (Rav, 1999, p. 6) for solving problems.

We suggest that proof often actualizes its intricate role when mathematicians prove collaboratively, participate in research seminars where proofs are communicated, engage with published proofs, et cetera. These are structured situations with particular rules of the game that shape the course of a mathematical activity and yield proof transformation. In the case of a new proof, the transformations often occur through interactions between the proof constructors and outsiders to the construction process. For instance, as a response to the feedback from reviewers and editors, the constructors can revise the proof. Alternatively, a familiar proof can be restructured in a way that makes it more readily
available “to build on it”. Morgan (1998) highlights the importance of the media through which mathematics is conveyed, which draws attention to instances where a proof transfuses from one communicational channel to another (e.g., from oral to written). Accordingly, we propose that proof transformation is a multifaceted process that can unfold in various social situations.

This paper comes from our ongoing project that unfolds in a proof-based course in topology (Kontorovich, 2021). This relatively advanced mathematical context has received limited attention in TWG14 (for an exception, see Stewart et al., 2017). Our project features a sequence of classroom situations where students develop proofs individually or in small groups, share them with the whole class at the board, and receive feedback from their peers and the course teacher. The project’s overarching aim is to explore opportunities for mathematics learning that emerge when students engage in progressive transformations of a proof. In this paper, we analyze an interaction between two students as they collaboratively constructed a proof, and the subsequent public re-proving of the same statement by one of them at the classroom board.

Theoretical framework

Our project is grounded in the commognitive framework (Sfard, 2008). This framework has been acknowledged for its capability to account for the complexity of university mathematics education and for offering tools to analyze learning and teaching in fine grain (e.g., Nardi et al., 2014). Commognition posits that mathematics as a whole and its particular disciplines (e.g., topology) can be construed as a discourse. Discourses are distinguishable through keywords (e.g., “Hausdorff space”) and their use, visual mediators (e.g., diagrams) and their use, endorsed narratives (e.g., a proof), and routines (e.g., proving). A person’s participation in a discourse is viewed as a patterned activity, when features of one’s public communication that remain relatively stable across interactions with different interlocutors constitutes a personal discourse.

Lavie et al. (2019) introduce the notion of a task situation to refer to “any setting in which a person considers herself bound to act—to do something” (p. 159). Then, they define “a routine performed in a given task situation by a given person is the task, as seen by the performer, together with the procedure she executed to perform the task” (p. 161, our italics). A procedure can be constructed through abstracting the commonalities of steps that a performer undertakes in similar task situations. Having no access to the performer’s interpretation of a task situation, one approach to deducing their task is attending to what the implemented procedure achieved.

The task situations at the heart of our project invite students to prove mathematical statements. We propose that in university mathematics classrooms, the keyword “proof” is often used to refer to a narrative that is targeted at endorsing a statement. This substantiating narrative is expected to unfold as a sequence of utterances (or sub-narratives), each either an “accepted fact”, or derived according to a well-defined set of rules (e.g., deduction, induction). Sfard (2008) comments that “routines of substantiation are probably the least uniform aspect of mathematical discourses. The very term endorsement may be interpreted differently by different people” (p. 231–232, italics in the original). These intrapersonal differences may emerge in students’ interpretation of which elements of their substantiating narratives constitutes a “classroom fact” and when a substantiation is required. We associate these interpretations with students’ tasks (cf. Lavie et al., 2019). We also note that the same
dilemmas feature in the mathematics community, when different resolutions were offered in different historical periods. Even today the approaches to proof vary greatly across mathematical communities. Whether the narrative “truly” endorses the statement or not, is a matter of social sanctioning. Manin (1977) writes that “a proof becomes a proof after the social act of ‘accepting it as a proof’” (p. 48).

In a similar vein, we use “proof” as a discursive label that is allocated by a particular community to a substantiating narrative (Kontorovich, 2021). This community can be as small as the person who constructed the narrative in the first place.

Lastly, Sfard (2008) defines learning as a lasting change in one’s discourse. This change can be triggered by learning opportunities – “circumstances that call for, and support, a change in the learner’s participation in a discourse, a transformation that would bring him or her closer to the discourse required by school curricula” (Chan & Sfard, 2020, p. 3, italics in the original). Chan and Sfard distinguish between opportunities for a change in the learner’s command of the discourse and for a change in the discourse itself. Within the former, the learner becomes more fluent in the target discourse by realizing the opportunity to mathematize according to its rules. In the latter, the learner enriches their discursive repertoire of endorsed narratives and routines. Students’ discourses are expected to get closer to the university version of a topological discourse in our project.

The case of Grace and Jonah

The project data comes from a semester-long course in a large New Zealand university. The course cohort consisted of six students: four were studying towards post-graduate degrees in mathematics, and two undergraduates were in their final year of a mathematics major. This was the only course in topology offered by the university’s mathematics department, and it covered standard topics in point-set and algebraic topologies (e.g., continuity, convergence, homology). For illustrative purposes, we selected a case where transformations in students’ proof were evident.

The data is extracted from the lesson on Hausdorff spaces, which took place towards the end of the first third of the semester. At the beginning of the lesson, the teacher defined Hausdorff spaces as those where every two elements can be separated by open sets (i.e., for each \( x \neq y \) in \( X \) there are open sets \( U, V \subset X \) such that \( x \in U, y \in V \) and \( U \cap V = \emptyset \)). After discussing this definition and specific examples, the students self-divided in pairs, and the teacher invited them “to have a go at proving” that if \( f:X \rightarrow Y \) is a continuous one-to-one function and \( Y \) is Hausdorff, then \( X \) is also Hausdorff. The protagonists of our case are a doctoral student, Grace, and an undergraduate, Jonah. They collaborated for nearly 4 minutes before the teacher asked “who is ready to present?”; then, Jonah volunteered to prove the statement at the classroom board.

The data corpus consisted of video-recordings of students’ activity and written work. We embarked on the analysis with two questions: “what routines did the students implement in each task situation?” and “how did their proof transform in the progression from one task situation to another?” After transcribing the data, we examined students’ activity to identify routines implemented in each task situation. We scrutinized the utterances to delineate substantiating narratives and routines before characterizing discursive similarities and differences between them. To present the findings, we begin with analyzing students’ collaboration, and then turn to Jonah’s mathematizing at the board.
Collaborative work

1. Jonah: Okay, so… [sketches two ovals for the sets $X$ and $Y$ in his notebook]
   
2. Jonah: So we want to show that in $X$, yeah
   [Jonah completes the diagram reproduced in Figure 1]
3. Jonah: That’s basically it.
4. Grace: [a] Yeah, that is kind of it, right? [b] Well, if they weren’t disjoint…
6. Grace: It’s almost too simple. [pause of 5 seconds]
7. Jonah: I feel like, I feel like something’s missing.
8. Grace: Yeah, I feel like something is missing as well. [pause of 15 seconds]
9. Jonah: Oh, doesn’t it imply that these two actually are in the same… You have an intersection.
10. Grace: Yeah, that seems wrong because then it [the statement] is true for all functions. Oh, but the fact that it is a continues function…
11. Grace: But why do we need one-to-one? I feel that we got to use that.
12. Grace: So, so… Let’s actually do this super logically. We start with two points.
14. Grace: We want to put an open set around each.
16. Gemma: We go to $f(x)$ and $f(y)$.
18. Grace: We put an open set here and here, which we can do because it is Hausdorff.
20. Gemma: We can pull these back and get two open sets here [pre-images in $X$].

**Figure 1: Jonah’s diagram**

22. Grace: If there was a point in this intersection but it can’t get mapped to two points.
24. Grace: One-to-one means that these two… Oh!!! These two can’t get mapped to the same point. Because if they got mapped to the same point, this argument wouldn’t work. It has to be two different open sets. That’s why [1:1].
25. Jonah: Oh, wait, what? I still don’t see where the one-to-one.
26. Grace: So our argument would fail. If $f$ wasn’t… because of if $f$ wasn’t one-to-one, then you could have $f(x)$ equals $f(y)$.
27. Jonah: Oh, oh, oh [as if realizing this]… Yeah.
28. Grace: [a] And then you definitely couldn’t do this picture. [b] So I think that’s where it happened.

The presented transcript features three rounds, differing in task situations and students’ routines. In the first round in [1-5], Jonah sketches a diagram that both students treat as a visual mediator of a proof of the assigned statement. Note that in [2] Grace appears to commence the construction of the substantiating narrative, but once Jonah completes the diagram the construction is relinquished. Accordingly, we suggest that generating a verbal version of the proof was not within the students’ task (in this round). This suggestion explains why the mathematically experienced students endorsed a diagram as “basically it” and “kind of it” in a proof-requiring task situation.

In [3-8], the pair implements what we term as a proof-monitoring routine: a procedure of “looking back” at the previous discursive activity with the task of assessing whether or not it can be sanctioned as a proof of the assigned statement. Herein, the students monitor the diagram visually and only the
routine outcomes are articulated: Grace and Jonah both do not identify issues with their never verbalized proof. Notwithstanding, both agree that “it” (the diagram, the proof, or their construction) was “too simple”. In tune with our approach to proof, we interpret the appearing tension between not identifying an issue with their work and being not satisfied with it as students monitoring not only their previous activity but also how it appears to them in the broader context. For instance, they could recall that the task situation was set up by a research mathematician in a cross-level course. Within this view, it may seem unlikely that the dyad could generate a proof just in seconds.

In [7-8], the wholistic “it” turns into a focused “something’s missing”, and the identification of a potentially problematic element turns into a task for the second round. Pursuing this task seems impossible without narrating the proof, and this is what happens in [9-11]. We refer to these students’ utterances and actions as the implementation of a proof-growing routine: a procedure through which a substantiating narrative is not constructed “from scratch” but becomes more extensive, elaborate and detailed based on previously conducted work. In this case, Grace uses Jonah’s diagram to name the sets and points, and in [10] Jonah appears to rephrase Grace’s utterance from [4b].

This round’s task is completed in [11-12] with Grace identifying that their (still partially narrated) proof does not capitalize on $f$ being one-to-one. This identification is not unlike the one in the previous round where the problematic spot was also not detected in proof-monitoring. The progress is in students delineating an element that they expect to feature in the substantiating narrative, and that it is currently not there. This recognition illustrates that a proof-growing routine can impact the substantiating narrative, not only by broadening its previously recognized constituents with new details, but also through its expansion to elements that were not addressed beforehand.

But why do Grace and Jonah expect the function’s injectivity to play a role in their emerging proof? Both appear to agree that their work substantiates the assigned statement “for all functions”, and injective functions are a subset of “all” – so why wouldn’t the pair see their proving mission as accomplished? As before, we propose that the students’ proof-monitoring went beyond their discursive activity to account for how this activity may appear in a broader context. For instance, drawing on their previous experiences, they could be driven by such considerations as “a teacher would not provide a redundant condition” or “we would be asked to prove a stronger version of the statement if it was possible”.

In tune with the above, delineating the role of the function’s injectivity becomes the task for the third round. The [13-30]-section features proof-growing and proof-monitoring, but the students’ interaction changes: Grace leads the implementation of both, narrating one proof element at a time, while Jonah endorses her statements. In [25], this interaction bears fruit: Grace realizes that their diagram had highlighted the function’s injectivity at the start, by depicting $f(x)$ and $f(y)$ as distinct points. Thus, in this round, their substantiating narrative grew by generating a verbal utterance about a visual element of the diagram. To appreciate what comes next, note that other parts of the diagram, especially those substantiating the Hausdorff-ness of $X$, did not feature in students’ discussion.

**Jonah’s mathematizing at the board**

Due to space limitations, we present an abbreviation of Jonah’s work at the board (for the full transcript see Kontorovich et al., in press). Jonah approached the board leaving the notebook with his
much-discussed diagram on the desk. He stood facing the board and with his back to the class throughout the process, often blocking the board with his body. Figure 2 shows a snapshot of Jonah’s board when he finished.

Jonah began by articulating every word in the sentences that he wrote on the board. The first three lines in Figure 2 were generated in this way. The fourth line emerged in silence. Then, Jonah appeared hesitant: he stopped writing and his gaze oscillated between the target statement and what he had written to that point. He took a step aside and sketched a diagram with two ovals, points \( x \) and \( y \), and dotted circles \( U \) and \( V \). After a few seconds, he giggled and smiled as if embarrassed, went back to his desk and returned with his notebook featuring the original diagram presented in Figure 1. After a quick glance at it, Jonah exclaimed “oh yeah!” and generated lines (5-6). Then he returned to the diagram on the board and completed it (see Figure 2). He wrote the lines (7-8) in silence, and instantly went back to his seat while cracking a smile to the video-camera.

In terms of routines, there is a visible change in how Jonah proceeded with his proof in this task situation: from utterance-duplicating in the first three lines, where he articulated what he put on the board as he wrote, to silent writing towards the end. We propose that while proving publicly for peers and the teacher to observe, Jonah mathematized for himself. This explains him investing almost no effort in elaboration on his text and eventually “turning off” the oral component – when one is communicating with themselves, the talk is loud even when all others hear is silence. Furthermore, his task appears to generate a written and self-contained proof, only parts of which were earlier discussed with Grace. Given that Grace did most of the “heavy lifting” in the earlier generation of the substantiating narrative, Jonah’s proving appears as a self-imposed challenge of constructing the board narrative on his own. This is in tune with him volunteering to prove at the board, initially leaving his notebook behind, and using it half-heartedly when getting stuck.

The transition from collaborative work to the board entailed a transformation of the proof in terms of restructuring, formalization, and growth. Indeed, recall the insight on the function’s injectivity in [25-29]. The dyad clarified it in the third interactive round and referred to it as a matter of contradiction (see [25,29a]). On the board, Jonah transformed this element into the symbolic “\( f(x) \neq f(y) \)”, eloquently substantiating it with \( f \) being “1-1” (see line (2)). The formalization aspect can be captured through what Sfard (2008) terms as reification: a discursive process in which humanized formulations about processes turn into a talk about mathematical objects (e.g., [19] vs line (2), [21] vs line (5)). Perhaps the most visible instance of proof growth is evident in the transition from “if
there was a point in this interaction […] but it can’t” in [23] to the deductive sequence in lines (6-7), where Jonah declares his intention to obtain a contradiction and explicates the steps that do so.

**Summary and Concluding Remarks**

Focusing on a single mathematical statement, we analyzed the progression of students’ proof between two task situations. Collaborative work on a statement “from scratch” and its consequent proving at the board are distinct social circumstances, and thus, it is barely surprising that different substantiating narratives emerged in each of them. What seems less obvious is what transformations a narrative can go through when “rolling over” such task situations. In the presented case, the proof was restructured, it grew previously not articulated elements and became more formal and elaborate. We acknowledge that these developments relate to the shift from the oral to written communicational medium, at least partially. Indeed, academic mathematical texts are renowned for being dense with terminology and symbols, modest in their use of “grammatical words”, and having impersonal and authoritative formulations (e.g., Morgan, 1998). Then, it may be expected that such an experienced mathematics student as Jonah would write in this way in the presence of a mathematically mature audience.

Can these transformations count as evidence of Jonah’s mathematics learning? The answer depends on whether Jonah’s discourse underwent a lasting change, which is out of the research scope in this paper. Our analysis captures the short-term developments in Jonah’s communication: from his limited contribution to the generation of a substantiating narrative in the first task situation to a fully-fledged narrative in the second. Some may argue with our claim about Jonah’s discursive development, noting that he was the one to generate the original diagram and declare “that’s basically it”. He delivered on this declaration at the board after a short glance at that diagram, something that may be explained by him as “holding the proof in his mind” all along. We remind the skeptics that commognition operates with communication that rests in a publicly accessible space and it recognizes the effort that is often needed to switch from inner dialogue with oneself (i.e., thinking) to conversing with others (e.g., recall how Jonah got stuck at his board proof). In the minimal case, by volunteering to mathematize at the board, Jonah realized a learning opportunity to change his command of an academic topological discourse. Indeed, he not only mathematized through using conventional keywords, symbols, narratives, and routines, but he did that at the board, which is characteristic to research mathematics.

A somewhat similar argument can be made regarding Grace’s discursive development. Throughout the interaction, Grace led the proof growth to broaden its components and expand it to new elements. Specifically, Grace’s discourse was enriched by a narrative about the role of the function’s injectivity in the assigned statement, a condition that initially appeared redundant.

Two interrelated aspects of the students’ interaction are noteworthy. First, the case illustrates how a substantiating narrative can develop through the alternation of proof-monitoring and proof-growing. Students’ monitoring is interesting since its first two implementations did not remove a blind spot in the proof, even though the dyad was convinced that this spot existed. We accounted for this conviction by proposing that the pair monitored not only their activity but also how it may appear in a broader context. The inaptness of their activity within this broader context explains why the students kept growing their proof further. This account illuminates how social considerations can permeate what could be expected to be a “purely logical-deductive function” (cf. Rav, 1999) of proof-monitoring.
Second, the students knew that in their topology course and beyond, a request for a proof is tantamount to generation of a written self-contained narrative. Yet, the dyad referred to a wordless diagram as “basically it” and “kind of it”. This illustrates that students’ familiarity with “a certain kind of action” (Lavie et al., 2019, p. 159) that the task setters expect does not always prevent students from pursuing a different task. Lavie et al. refer to “task” as “a person’s interpretation of a given task situation” (p. 161), but this case shows that it can be a deliberate choice. Janah’s proof at the board offers a colorful example that the choice of an alternative action in one task situation is not necessarily evidence of one’s incapability to undertake an expected action in different circumstances.

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