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Exploring a property of mixed product in the middle school inspired by Bézout's use of units

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In his arithmetic treatise, Bézout teaches arithmetic as grounded on quantities and units. It is from this perspective that we consider a property of a “mixed product”, involving a multiplication on sets of quantities. In this note, we present the main lines of some didactical research on the use of this property with students of years 7 and 8 (12–14 years old). We focus mainly on the conception of units and related notions, as the multiplicative relation between units of the same kind.

Keywords: Bézout, quantities, unities, multiplication, mixed product.

Historical introduction.

Étienne Bézout (1730–1783) is well known in algebraic geometry for his theorem on the number of intersection points of two plane algebraic curves. However, we are interested here in his « Cours de mathématiques à l'usage des gardes du pavillon et de la marine », first printed in Paris, in 1764¹. The French Royal Academy of Sciences praised his particular way of presenting the different notions in his « Éléments d'arithmétique »: “so M. Bezout has often treated these objects in a way which is absolutely his own & which makes them surprisingly simple.” (Histoire de l'Académie Royale des Sciences, 1764, p.97). His success did not diminish until the end of the 19th century (Alfonsi, 2011).

In his treatise on arithmetic, Bézout presents an application that we find of great didactic potential. It is a multiplication in which the unit of measurement is made to appear. The property could be stated as follows: multiplying a quantity (the multiplicand) by a multiplier is equivalent to multiplying the number of units of the multiplicand by the multiplier, and is also equivalent to multiplying the unit of the multiplicand by the multiplier. This particular use of units was remarked upon by the Académie des Sciences as an innovation.

These subjects, so often treated, become, by the manner in which he offers them, absolutely new & of the most luminous simplicity. We can put in the same rank what he says on the nature of the units in multiplicand, multiplier & product [...] (p.98)

Bézout gives the following example:

¹ The references in this text point to the edition printed by J.B.G. Musier fils in 1779, identical to the first one (Alfonsi, 2011) and the oldest that is available in a digitized text. Any quotation is our free translation from French.

112. Let us ask, for example, what is the value of $5/7$ of a pound? Since the $5/7$ of a pound is the same thing (96) as the seventh of 5 pounds, I reduce the 5 pounds to pennies (57) and [...]. If one were to ask for the $5/7$ of 24 pounds, it is obvious that one could first take, as we have just done, the $5/7$ of a pound, and then multiply by 24 what this operation would have given but it is more convenient to multiply $5/7$ by 24 pounds, which gives $120/7$ pounds (107), and then to evaluate this last fraction which will be found to be worth 17 pounds 2 pennies 10 denarii $2/7$. (Bézout, 1779, item 112. Our free trans.)

He also gives the following example involving fractions:

96. For example, in $4/5$, 4 can be considered as representing any four things, 4 pounds, for example, which must be divided into five parts; for it is obvious that it is the same thing to divide 4 pounds into five parts in order to take one of these parts, or to divide one pound into five parts in order to take 4 of these parts.

Bézout doesn't justify this property in his manuscript. Nevertheless, he gives a meaning thanks to the use of the different kind of units and stressing the multiplicative relations between units of the same system. It is also through the use of units that Bézout gives meaning to the different numbers, including decimal numbers and fractions, and also gives theoretical justifications to the algorithms of arithmetic operations and meaning to the rules of calculation and to the properties satisfied by the numbers under consideration.

The heritage, in the sense of Grattan-Guinness (2004), of Bézout's work, and in particular his use of units as a didactic tool, and even as a theoretical tool, is part of the inspiration for the research of which we present some of the results in this paper.

Mathematical perspective.

From a contemporary mathematical point of view, the property quoted above is a *mixed product*, which involves two multiplications in different sets. First, the *external law*, which acts on the set of quantities, and secondly, the *internal law of composition* on the set of numbers considered (*e.g.* integers or rationals).² Nowadays, and contrary to Bézout's usage, we would write the first step of the calculation of the example (112) (before dealing with sub-units), using the multiplication and equality signs and parentheses, which provides 3 equalities, as follows:

$$5/7 \times (24\text{£}) = 24 \times (5/7 \text{ £}) = (24 \times 5/7) \text{ £}.$$

In order to be demonstrated rigorously in some generality, it requires mathematical arguments that were out of use in Bézout's time (the modern axiomatics developed in the nineteenth century), and which are beyond the scope of the teaching curriculum of middle school today.

Theoretical framework and didactical motivation.

In this former example, the initial quantity is 24 £, in the term $(24 \times 5/7) \text{ £}$ the number of units is scaled, and in the term $24 \times (5/7 \text{ £})$ is the unit itself that is scaled. We consider both the possibility

² This is different from considering only the multiplication in the set of rational/integral numbers with the commutativity and associativity properties of it, disregarding the set quantities (and the units).

of varying the number of units considered and the one of scaling the unit itself. We have three different expressions for the same quantity, so in each concrete case we can choose to use the first or the second way of proceeding, depending on what would, for example, simplify the calculations. This is one of the (technical) interests of this property. However, what motivates us was to better understand what competencies are the ones reflected when using the unit-scaling strategy. We also stress that the reasoning involved can be used to conceive multiplication involving fractions, as, *e.g.*, in item (96)³. Indeed, Behr et al. (1997) consider the rational number as an operator acting, either on the number of units, or in the unit itself of the operand. In their experiment, the students (preservice teachers) were reluctant to scale the unit as if this strategy were “cognitively more demanding” (p.65). The unit-scaling perspective relies on quantities and measures, and differs from the more studied one about “unitizing” and “composite units”, as in Lamon 1996. Indeed, although she defines *unitizing* as the “cognitive assignment of a unit of measurement to a given quantity” (p. 170), it seems to us that her examples refer rather to a grouping notion. Nevertheless, she states that “The ability to form and operate with increasingly complex unit structures appears to be an important mechanism by which more sophisticated reasoning develops” and pointed out that this perspective has been shown to be successful in several mathematical teaching domains. Behr et al. (1997, p.50) also agree with the relevance of conceptual units in learning. See the references in therein.

We hypothesize that a perspective based on quantities with a broad approach to units, could support the development of the unit-scaling reasoning, as well as could perform the multiplicative relations between units of the same family, since they refer to the size of the quantity, instead of referring to the number of unities composing of the quantity. We focus on these concepts (quantities, units, multiplicative relations between units, quantity-scaling strategies) and the relations between them.

Chambris, Coulange, Rinaldi, & Train (2021) pointed out some other deeper potentialities of the mixed product property, related to the multiplication of fractions, the equivalence of fractions (in terms of “compensation theorem”), and the knowledge about metric units. They previously identified this property (in terms of “multiplicative version of the compensation theorem”). They show that the understanding and the teaching, of it could rely on the knowledge of the “related units”. Indeed, Chambris (2021) have introduced into didactic research the notion of *related units*⁴, based on quantities as a foundation for numbers. We can interpret these units as obtained by enlarging or reducing, let’s say, a *standard* unit (*e.g.* pound, gram)⁵. For instance, to justify his calculation in his item 96 quoted above, Bézout uses a unit, and in item 112, we can see 24 pounds as a unit.

³ Let us remark that a cognitive gap of abstraction is involved when considering $4/5$ as $4/5$ of the unit 1.

⁴ Previously named relative units, she actually calls them *related units*, whose meaning is shown in this example: both the sizes (the weights) of “1 gram” and of “250 grams” can be used as units, units that are related one to the other.

⁵ We prefer this point of view, to that of considering a *related unit* as the unit *composed* by many equal standard units, a notion that lead to a *grouping* approach, and makes it more difficult to consider a *related unit* smaller than the standard one, and then making obstacles for the multiplicative relations (in both directions) between units of the same family.

In the same article, Chambris, Coulange, Rinaldi, & Train (2021) stress that the *related units* are an implicit knowledge, mostly missing in the French curriculum. Furthermore, the lack of this knowledge seems to be in connection to some recurrent difficulties in the students, as for example, conceiving the multiplicative relations between different units of the same family, *e.g.* the ten is ten times smaller than the hundred, and the hundred is ten times bigger than the ten. Indeed, when working with metric units and “numeration units” (Chambris, 2021), Chambris, Coulange, & Train (2021) noticed difficulties in the students and the teachers in managing the possible links between

- *related units* (metric and *numeration*),
- multiplicative comparison relations (*e.g.* 1 cm is ten times smaller than 1 dm),
- the composition of these relationships (*e.g.* ten times smaller than a tenth).

Observe that these items appear in Bézout’s work (with a different point of view), even though he uses the partitive approach to define fractional units and fractions, instead of the “fraction as comparer approach”, as named by Freudenthal (1983), which leads to a better mastery of the multiplicative relations between units (Cortina et al., 2014).

On an ongoing project, we therefore sought to go further in identifying the lack in the competencies that is reflected in the missing of the mixed product. We asked ourselves what treatment involving units are needed, or at least useful, to support the quantity-scaling strategies in a meaningful way. In this note, we outline some explorations thought the interview of a middle school student.

A didactical reading of the use of units and multiplicative relations in Bézout’s treatise.

Through some citations, we show here that the units are at the base of Bézout’s arithmetic treatise, using them as a tool, as well as means of meaning. At the beginning of his treatise, Bézout defines units as “a quantity that is taken (usually arbitrarily) to serve as a term of comparison for all quantities of the same kind.” (item 3). He then adds “The number expresses how many units or parts of units a quantity is composed of.” (item 5). Bézout makes the distinction between numbers, which he calls *abstract numbers* (*e.g.* “three or three times”), and numbers expressed in units, which he calls *concrete numbers* (*e.g.* “four pounds”)⁶. His multiplicand of item 112 is as well, from which we took as our first example. Bézout defines the sub-units of the simple unit as parts of it. About tenths he writes the following:

21. In order to evaluate in decimals the parts smaller than the unit, one conceives that this unit [...] is composed of ten parts [...]

This is a partitioning approach. Nevertheless, he considers these sub-units as units in their own right, and hastens to mention one of the multiplicative relationships between them and the simple unit: “they [the tenths] are ten times smaller than this one [the simple unit] (item 21)”, which will make possible, later on, an explanation of the proposed calculation techniques and the exposed properties, including that of the mixed product. Indeed, to mention the multiplicative relations bet-

⁶ We observe that what he defines as a *concrete number* seems to be, nowadays, rather (the measure of) a quantity in a given unit.

ween units of the same system is recurrent is Bézout's treatise, in particular when introducing the tens, hundreds,... and the decimal place-value system: "a number followed by two others, [...] marks a number a hundred times greater than if it were alone" (item 11); "as one moves from right to left, the units of which each number is composed are ten times larger" (item 15); "these new units, ten times smaller than the tenths, will be one hundred times smaller than the main units" (item 22).

Furthermore, we find it interesting from a didactical point of view that before introducing decimal numbers, he first introduces another system of units, in which the ratio between one unit and the next is *not* division by ten. Indeed, the first example proposed is that of the pound (as a currency), "the pound is divided into 20 parts, which are called pennies, the penny into 12 parts, which are called denarii." (item 17). He then quotes the ounces, the toise, the day and the marc, which were common units of volume, length, time and currency at the time, before introducing the divisions and subdivisions by decimals, the convenience of which he praises. A multiplicative relation is also quoted to explain the meaning of multiplication of fractional numbers, "multiplying the denominator 3 by 5 changes the thirds into fifteenths, *i.e.* into parts five times smaller" (item 106).

To describe the process of multiplication, Bézout uses the notion of unit twice. Firstly, in the multiplication algorithm itself, to name the different digits of the multiplicand and the multiplier "[...] and retain the tens, which are hundreds, to add to the next product which will also be hundreds." (item 50). Secondly, he uses the units to give meaning to this technique ("[...] because the number by which I multiply is a number of hundreds." (item 51).

The notion of unit, and in particular that of *unit fractions*, remains fundamental in Bézout's treatment of fractions. He then considers a second definition, based on division, with a "unit" point of view, "Another way of looking at a fraction is to consider the numerator as representing a certain quantity that must be divided into as many parts as there are units in the denominator." (item 96), and uses the property of the mixed product to interpret it. We quoted the end of item 96 in our introduction.

Didactical exploration. The problem type. Interviews.

We want to explore whether, in specific problems where the didactic variables encourage it, the unit-scaling part of the mixed product property is used. We also look for the skills that are involved in it. The first author began her experimental approach with an interview between directive and semi-directive, with an interview guide with questions in a precise order, but left the possibility of asking other questions, depending on the interviewee's answers. On the one hand, we tested some knowledge on notions of units, conversion of units and measures, which are part of the curriculum. On the other hand, we checked the state of mastery of a certain point of view on units, less present in the teaching, such as the multiplicative relations between units of the same family. We finish with two problems, whose written productions we have collected. The two problems proposed aim to observe whether students are leveraging units to use the mixed product property by scaling the quantity and make calculations easier. Among the knowledge potentially involved in solving the problems this way, we sought to test which ones are mastered by the students, which ones are not, and then we wanted to know if the students use this property to solve the two problems.

The basic problem we propose to study is the following. Given a quantity of a certain magnitude whose measurement is expressed by the number \mathbf{a} relative to a quantity \mathbf{u} taken as a unit, multiply this magnitude by a number \mathbf{k} . The situations to which the learner is exposed highlight the various multiplications ($\mathbf{k} \times \mathbf{a}$, and $\mathbf{k} \times \mathbf{u}$), as well as the fact that a unit, like any quantity, can be enlarged (the scalar \mathbf{k} is greater than 1) or reduced (the scalar \mathbf{k} is smaller than 1).

In our starting example, “Rice pudding at the Indian festival”, the initial quantity is enlarged.

Problem (Rice pudding). For an Indian festival, it has been decided to make rice pudding. The rice is sold in packets of 250 grams and 1 kilogram. We calculated the quantity of rice needed, according to the number of guests. We need 7 packages of 250 grams. Then we say that there will be 4 times as many guests. How much rice is needed in total?

This is a special case, because the initial quantity (7 times 250 grams) is measured in relation to another quantity, \mathbf{u} , which is 250 grams, taken as a unit (here 7 stands for \mathbf{a}). To take 4 times this initial quantity, we could proceed by multiplying 7 by 4, then computing 250 grams 28 times. More simply, we can also choose to first enlarge our unit by 4 times, which gives 1000 grams, and then take 7 times 1000 grams.

We chose to interview students from the Collège International de l'Esplanade, Strasbourg, during the school year 2020-2021, one in year 8 (13 years old) and four in year 7 (12 years old). Here we briefly present part of the analysis of the interview with the 8 year student, which lasted 18 minutes.

Analysis of Persephone's interview

After analyzing the interview, we present here some evidences on the skills of the student. We have sought to highlight whether the mathematical knowledge existed for the student, whether the student had to find it by herself, with some adaptation or not. We also point out some knowledge that seems to be lacking in her studies, or some competencies that are not fully developed.

Indeed, she knows several families of units with their different units (*e.g.* “meters, kilograms” (A019), “hectograms decagrams decigrams” (A021)), knows what quantities they measure (*e.g.* “meters distances” (A025)), and knows how to convert between units of the same kind: to the interviewer's question in A106, “one meter equals how many centimeters”, she answers “one hundred” (A107), then adds that “one millimeter equals zero point one centimeters” (A115). She also masters the ratio between units of the same kind: the researcher : “how do you go from ten to a hundred?” (A078), the student : “by doing times ten” (A079).

The student knows the multiplicative relations between hours and minutes (“[the hour is] sixty times [larger than a minute]”, in A087) and “[a minute is] sixty times [smaller than the hour]”, in A089), although she does not state them spontaneously. We can ask ourselves whether this is due to a lack of practice in these manipulations because she has not been taught, for example, that a minute corresponds to a unit sixty times smaller than an hour, that it is one sixtieth of an hour.

The student knows that one hour is sixty minutes (A081). In contrast, the opposite conversion appears more difficult to the student. To the question “one minute is how many hours?” (A082), she starts to answer with an order of magnitude that is right “one minute is / zero decimal point” (A083), then she hesitates, then makes a mistake. The question seems to be out of reach at this point for the student.

The interview also shows that the student is more comfortable with families of units whose ratio between them is ten (so are every example of units given by the student). The dozen seems to be more remote to her (“by tens it is much easier” (A057)), and she may not consider the ten as a unit ten times larger than the single unit, or the hundred as a unit ten times larger than the ten. Sub-units, such as the tenth, are not named, and naming the multiplicative relations between units of the same species, when this relation is less than one, is also more difficult for her as shown in what follows.

A118 Interviewer: the centimeter is larger / and how many times larger than a millimeter?

A119 Student: since there is a decimeter / zero decimal point // by no means / is bigger / ten times uh a hundred times

With questions about the change in the measure when the unit have been changed, the student succeeds in giving the right answers. However, when the researcher asks “had you ever thought about that?” (152) , she replies “yes, but it's still paradoxical” (A153), then adds “I wasn't taught that, we weren't told about that no / on the other hand it's still visible” (A155).

The student doesn't succeed to correctly solve the « Rice pudding problem ». Despite an initial impulse to enlarge the unit in the first problem, she abandons this strategy, and does not use the unit-scaling part of the mixed product property yet leading to simpler calculations. The student begins by answering the question about the quantity of rice needed:

A181 Student: it will take um a kilogram of rice because we make two hundred and fifty times four so there are four times more and so twenty-one um not at all / twenty-eight packets of um rice of two hundred and fifty grams

Note that she starts by multiplying 250 grams by 4. Then she multiplies 7 times 4, which would be too much. Then she keeps this second product, giving as a result twenty-eight packets [...] of two-hundred-and-fifty rice, which is right, and does not take into account her first multiplication, that is 250 times 4, and which should have been multiplied by the number of initial packets, that is 7, to obtain a right final result. The researcher unsuccessfully tries to get her back to the first multiplication.

A194 Interviewer: at one point you had multiplied the four here / by four times as many guest

A195 Student: yes

A196 Interviewer: by the two hundred and fifty grams of rice / and you got a kilo / isn't that another way of calculating?

A197 Student: well, if you wait (silence) well, we don't know if it's the packets or if it's the grams // is that it?

Conclusion on Persephone's interview

The student interviewed has mastered all of the concepts related to the units taught in primary school and at the beginning of middle school. In contrast, the student does not manage to solve the problems correctly. In between, we identified some notions that are not completely out of reach, but that they don't appear as a solid background, and not taught at the school. We hypothesize that these are the notions that are lacking, or are at least useful, to give the mixed product a ground that can bring it meaning. This study shows that a broader view and knowledge about units and at least the following notions are lacking: families of units whose ratio between them is *not* ten, multiplicative relations -in the two directions- between units of the same family, fractional units, as well as considering the metric units and *numeration units* as units in their own right. It seems to us that a point of

view that Bézout gives in his treatise, where the conception of units is based on quantities and that the relations between their sizes are highlighted, could support the conceptualization of quantities into units and make closer the competency to apply it in a given task by using the quantity-scaling strategy.

Perspectives.

The aim of the interview was to test the knowledge of the students in the concepts involved in the application of the mixed product when *enlarging* or *reducing the unit*, as well as the ability to apply it. After a deeper analysis of the tasks, we sought to assess some of the involved concepts more precisely, before implementing a teaching experiment with students of year 7 (the results of which will appear in a forthcoming article). Indeed, the teaching experiment is based on the idea of better identifying the concepts at stake in the mixed product, by testing the students before and after exposing them to didactic situations with the aim of supporting them in the development of the missing competencies (including those highlighted in the interview), and in particular, the multiplicative relations between quantities of the same kind, as units of the same family and fractional units, through a “fraction as comparer approach” (Freudenthal, 1983), as well as handling tasks.

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