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Mathematic-historical example for university teaching of mathematics - how can methodical knowledge be taught as well as the culture of mathematics be made accessible?

Nicola Oswald¹

¹Universität Würzburg, AG Zahlentheorie, Würzburg, Germany; nicola.oswald@mathematik.uni-wuerzburg.de

In our presentation, we introduce a teaching unit for university mathematics teaching, which is based on historical documents. Specifically, these are two articles by David Hilbert (1862-1943) and Adolf Hurwitz (1859-1919) on proofs of the transcendence of the number e and an accompanying letter exchange between the two authors. We illustrate that these archival finds are suitable to deepening students' understanding of both mathematical methods and mathematical culture.

Keywords: Historical documents, Proofs of transcendence of e , David Hilbert and Adolf Hurwitz, Mathematical culture

Introductory reflections on history of mathematics in university teaching

Incorporating the historical development of mathematics can be used as a didactic element in teaching mathematics. In textbooks, monographs and lectures, it may help mathematicians to classify the mathematics dealt with and thus at the same time enables the reader a systematic approach to the mathematical content.

In a case study on the use of so-called knowledge maps in a history of mathematics course in university teaching (Khellaf et. al, 2018), we found that students develop very individual and confident ideas and cross-references about the development of mathematical thinking. Figure 1 illustrates different examples of student views on structuring mathematical development steps.

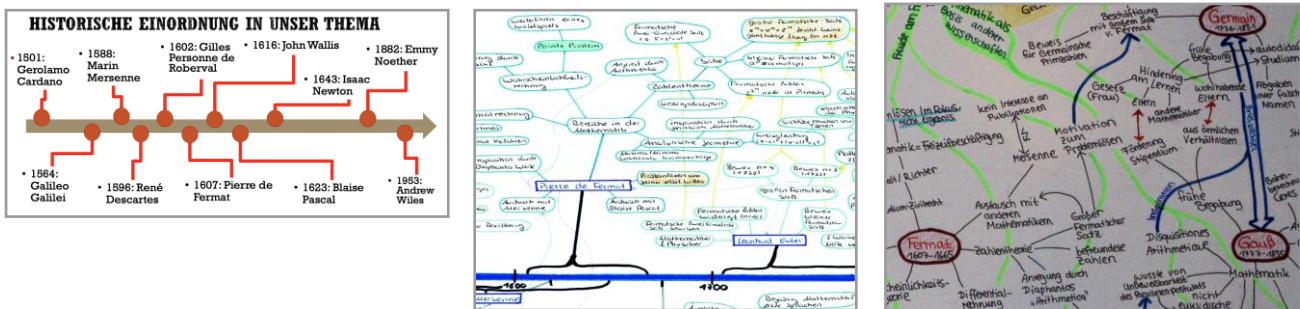


Figure 1: Excerpts from a lecture and two knowledge maps by students, created in the seminar "Development of mathematical ways of thinking" (University of Hanover).

The importance of mathematics history in university teaching has been discussed at various points in the past. The mathematician Max Dehn (1878 - 1952), for example, took a particularly firm standpoint. In a statement on the teaching of mathematics at the University of Frankfurt in 1932, he named the history of mathematics as fundamental subject matter in addition to "the structure of analysis", "axiomatics in geometry, the foundation of arithmetic". He justified his view by writing that the treatment of mathematics history ensures "[...] to consider again and again the multiple

connections between [mathematical topics], and this is more important than to bring no matter how [new] original results.“ (Dehn, 1932, p. 72) In addition, Dehn underscored the high value of a mathematics-history seminar, which was aimed particularly at faculty members, as a "kind of continuing educational class".

The mathematics didactician Gregor Nickel argues in his article “Zur Rolle von Philosophie und Geschichte der Mathematik für die universitäre Lehrerbildung“ (Nickel, 2015), translated "On the Role of Philosophy and History of Mathematics for University Teacher Education“, that mathematics history must be understood not as an accessory but as a "subject of teaching in its own right."

"The historically reasonably adequate presentation of a topic from the history of mathematics certainly cannot be done incidentally, but requires time and attention, on the part of the teacher a certain mathematical-historical professionalism, and on the part of the students a solid prior mathematical experience." ((Nickel, 2015, p. 216) translated from German by the author)

In the following we show a practically oriented teaching unit on transcendence proofs of Euler's number e , which on the one hand is thematically suitable for the current teaching material in early university semesters (after two basic lectures on analysis), and which on the other hand can be built on a limited corpus of mathematics-historical documents.

Historical material for the construction of a teaching unit

The transcendence proofs of e of Adolf Hurwitz (1859-1919) and David Hilbert (1862-1943) from 1893, which are the subject of this work, can be juxtaposed with an exchange of letters between the mathematicians, which shall make both the genesis of the proofs themselves and the contemporary context tangible and raise questions about the "culture of mathematics". In the following, we will first discuss excerpts of this correspondence, then give a sketch of the proofs' main ideas.

Correspondence between David Hilbert and Adolf Hurwitz

The correspondence in question essentially comprises five letters written by David Hilbert and Adolf Hurwitz between October 1892 and February 1893. The life and work of these two mathematicians was characterized by enormous and groundbreaking developments within the research and teaching of mathematics as well as the institutional conditions in the German-speaking world (see some aspects below in the introduction of the teaching unit).

Hurwitz and Hilbert (as junior scientists and professors in Königsberg, Zurich, and Göttingen) took an active part in developments of mathematics and exchanged views on them in a correspondence comprising a total of at least 198 postcards and letters, which is now kept in the Göttingen State and University Library (under the directories Cod_Ms_D_Hilbert_160 and Cod_Ms_Math_Arch_76). They had met when Hurwitz was appointed associate professor in Königsberg in 1884, where Hilbert attended his lectures as a committed and ambitious mathematics student. Since then they had an ongoing collegial friendship, which was maintained through regular written exchanges, especially after Hurwitz's call to the ETH Zurich (then still Polytechnikum). In their letters the topics ranged from general remarks on the situation in higher education and personal matters to very concrete research results and, where appropriate, mutual references and suggestions for improvement. In the latter sense, for example, the exchange of letters on new proofs of

transcendence of the number e quoted below was written. This begins with a letter in which Hilbert presented his simplification of a known proof (by Thomas Stieltjes). In the following we will quote some relevant passages of the letters, which are mostly three to five pages long. The focus at this point will be on the nature of the communication regarding the proofs.

A new development of the proof was communicated by Hilbert in his letter of December 31, 1892:

"Concerning my proof of the transcendence of e , I realized very soon after I wrote to you that one can still considerably shorten the same by omitting the whole Stieltje point. [...]"

- The proof follows, in particular the choice of the polynomial $f(z)$, see next section. -

"[...] You see that hereby the proof also contains a different conclusion at all, in that not from the integral sum Π but from the integer I the dissimilarity from 0 is shown. The use of this conclusion also gives the proof of the transcendence of π a simplification which seems to me not inconsiderable."

Hurwitz replied on January 10, 1893, again with a new proof and an idea:

"Your scientific communication concerning the number e has, as you can imagine, interested me very much. [...] The matter did not let me rest and I discovered a further simplification, [so] that one can now bring the proof in the first hours of a lecture on differential calculus. [...] Have you already edited your proof? If so, please write me whether you have already sent it to Klein. I would then have the above further simplification printed in a short note behind your work. I would prefer it if we chose the Göttinger Nachrichten. [...] So please answer me quickly, even if only by card. That your punch line can also be applied to π is clear; but I haven't quite thought it through yet."

Within three days Hilbert replied (on January 13, 1893):

"I have already worked out my proof for e and π during the Christmas vacations, and in the process - especially in the part dealing with π - a number of advantageous and simplifying things have emerged, so that the whole thing will now take up 4 - 5 printed pages, and my presentation is by no means brief. In your proof, of course, the integral is avoided; but whether the presentation of the proof becomes shorter and clearer is not yet quite clear to me. [...]. But it is my conviction that the proof with the help of the integral will always remain the clearest and most capable of development [...]"

Hurwitz wrote about a month later (on February 08 or 13, 1893):

"For a long time I wanted to answer your dear letter of 13/I, but - as it goes - the answer was postponed from day to day. Today, as a guiding impulse, your transcendence note arrives, which I immediately typed into a café. You have written the note with Gaussian classicism. I hope that you will agree with the short note (2 printed pages presumably) in which I will make the modification of your proof, [...]. Felix Klein submitted it to the Göttingen Society on 4/II. As an advantage of my modification I see that it becomes clear in the proof that only the addition theorem and the differential equation [...] The idea of modifying your proof by replacing the integrals by limit values had also occurred to me. However, there still seem to be difficulties [...]"

The letters already clarify the core of the proof differences: The preference of integral or differential calculus, which still lay the foundation of every analysis course today. It is interesting that Hilbert and Hurwitz emphasize the respective greater "simplicity" of their proofs (and remain in disagreement).

Two proofs of transcendence of the number e

In the following, we will give a sketch of the two proof lines, published in (Hilbert, 1893) and (Hurwitz, 1893). We mainly limit ourselves here to the mentioned crucial difference in the approach of the proofs (the „differential analogue“), which is illustrated by excerpts from a lecture by students in a seminar (see more details in the next section).

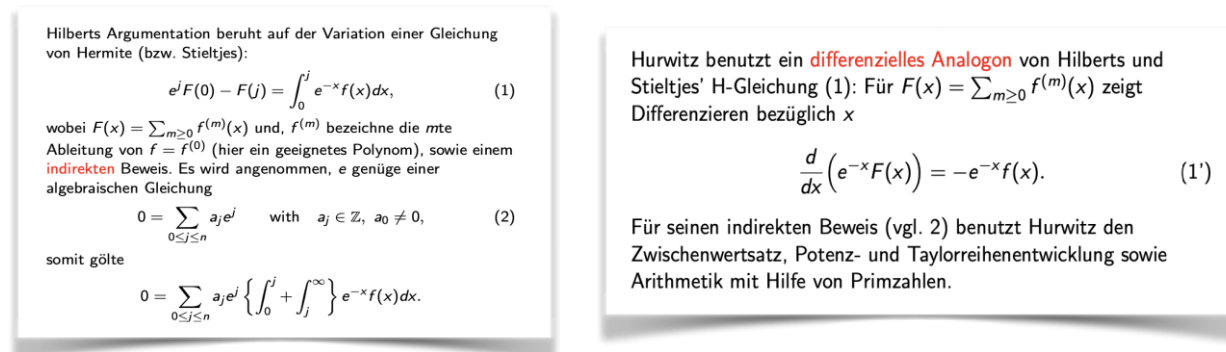


Figure 2: left: Excerpt from an introductory seminar lecture "Basic Idea Proof Hilbert, right: Excerpt from an introductory seminar lecture "Basic Idea Proof Hurwitz"

Figure 2 describes that Hilbert's argument is based on a variation of equation (1), where $F(x)$ is defined as given and $f^{(m)}$ describes the m th derivative of $f^{(0)}$. Hilbert then goes on to give an indirect proof by estimating the individual integrals, respectively their arithmetic properties, and leading equation (2) to contradiction. In contrast, Hurwitz argues with a "differential analogue," see equation (1'). For his indirect proof Hurwitz further uses the intermediate value theorem, power and Taylor series expansion, and arithmetic of primes. We cannot go into detail through all the mathematical steps here, however, we want to state that the proofs are non-trivial, but quite short, and all methods are usually discussed in the first two analysis lectures of a university course.

In favor of the two proofs as well as on the corresponding historical documents for practical teaching purposes is the fact that the basic idea of the proofs provides an application of the fundamental theorem of integral and differential calculus. An integral proof is contrasted with a differential proof. Students thus learn, using a concrete example from modern history, that the methods and foundations experienced in their first university semesters are sufficient to understand and discuss seminal mathematics. In addition, the parallel use of original proofs and accompanying correspondence allows a variety of cultural aspects of mathematics or the mathematical community to be reflected, see next section.

Implementation in a teaching unit

Based on the historical documents, we constructed a teaching unit, which was so far conducted in two different seminars: A seminar on selected topics in analysis (10 participants; University of Würzburg) and a mathematics-history seminar on developments in mathematical ways of thinking (13 participants, University of Hanover). In both seminars (each 90 minutes per week), the content

was divided into three parts: Mathematics-historical context and the letter exchange, the proof of David Hilbert, and the proof of Adolf Hurwitz before the background of the correspondence. The introduction was done by the lecturer, the proofs were presented by one student each. Time (about half an hour) was then allowed in each lesson so that the content presented could be discussed. In the following we will refer to the analysis seminar. The students were all in their third or fifth semester (Bachelor Mathematics or Computational Mathematics).

In the introduction, we first gave a brief historical contextualization of the seminal period of the late 19th century for mathematics in the German-speaking world: The first International Congress of Mathematicians was held in Zurich in 1897, journals and publishing houses were founded or changed their direction, lecture notes of mathematicians were published, polytechnics gained importance, interdisciplinary fields (for example, geometry of numbers, algebraic number theory, and algebraic geometry) were discussed and evolved, teacher training was professionalized - to name only a few contemporary aspects, some of which still influence our teaching and research practices in mathematics today. We then introduced the mathematicians Hurwitz and Hilbert, and their correspondence, which spanned a total of several decades. Finally, the letters referring to the proofs of transcendence of e were presented in detail. In the discussion that followed, the students were particularly interested in the influence of the mathematicians Charles Hermite (1822 - 1901) and Felix Klein (1849 - 1925). Furthermore, we focussed the way of discussing mathematics and developing proofs by letters - in contrast to today's mathematical culture and speed (e.g. by zoom meeting).

All students were given preparatory questions for the seminar sessions, and the presenting students in particular were asked to integrate these into their presentations. As an example, we show here the questions which especially refer to the difference integration-differentiation:

In your presentation, address the following questions:

(I) Historically, integral or differential calculus – what came first? Since when were these related to each other? Treat these questions discursively and be critical in your literature review.

(II) Fundamental theorem of differential and integral calculus: prove that Hurwitz's approach is indeed the „differential analogue“ of Hilbert's approach.

(III) Partial integration: prove the Hermite equation. What is the differential analogue to partial integration?

The presenters were encouraged to prepare together to some extent. The questions served as a guideline. They were deliberately kept both general and concrete. Question (I) was directly stimulated by the correspondence and the question whether differential calculus or integral calculus should be studied first and why. Especially in their search for sources, the students became active: The discussion led us to different protagonists of the history of mathematics (from Archimedes to Newton / Leibniz) and their influence on our mathematics today. Questions (II) and (III) were more specific to the respective lecture. By differentiating Hilbert's integral, question (II) could be addressed; the differential analog (question (III)) to partial integration is the product rule used for Hurwitz's equation (1).

With all students we summarized: Both proofs so far rely essentially on the differential equation $\exp' = \exp$ for the exponential function (or, equivalently, the integral equation for \exp). This kind of reproduction property appears in the proof of Hermite's identity (1) as well as in its differential counterpart (1'). The similarity of the proof ideas is due to the main theorem of integral and

differential calculus. The real differences of the two proofs lie mainly in the mathematical methods and tools used in the estimation of the arithmetic properties. Together with students, we have decomposed the proof into individual steps, see a summary of the results of the proof analysis in Table 1.

Table 1. Comparison of the steps of the proofs, highlighted with frames: integration-differentiation difference (blue), differences concerning mathematical tools (red)

Steps of the proof	Hilbert	Hurwitz	Background & Application
Basic starting point	Integral equation (1) Partial integration	Differential equation (1') Product rule	Duality in fundamental theorem of analysis Exponential function $\exp = \exp'$
Assumption	Algebraic equation = 0 (2)	Algebraic equation = 0 (2)	Concept of the proofs: Proof by contradiction
Defining ingredients	Definition of a polynomial f	Definition of the polynomial (7)	Comparing the polynomials: only slight differences due to prime numbers
Construction	Combining (2) and (1) (algebraic equation)	Combining (2) and (1') (algebraic equation)	Similar basic construction: Integration of equation (1) resp. (1') in algebraic equation (2)
Mathematical Tools	Investigation of Arithmetical Properties: Elementary formula (3) Arithmetical estimation (integers)	Investigation of Arithmetical Properties: Mean value theorem to receive equation (6) binomial theorem / Taylor series expansion Arithmetical estimation	Different proof lines: Tricky methods, Tools in principle elementary / understandable, Execution sophisticated
Conclusion	Assumption not possible	Assumption not possible	Conclusion: e is transcendental

In addition to the mathematical facts, further facets of mathematics were addressed. Especially with regard to students in their first semesters, a lack of understanding of the "culture of mathematics" is often thematized. This fuzzy term is used for very different aspects of the subject: For example, technical skills such as writing down a mathematical conclusion or the correct use of operators are addressed, or an understanding of the difference between examples and a rigorous proof is named. Especially the discussion of proofs is treated repeatedly and with different perspectives: The fact that mathematical facts can be proved in many ways (and by methods of different subdisciplines) and that it is ultimately a matter of perspective and context which proof lends itself or even "prevails" at which point is an integral (and challenging) part of understanding mathematics as an inquiry-based discipline. Looking at the related process of the creation of mathematics, its culture certainly includes the negotiation of the content itself: How is mathematics discussed and circumscribed, and by whom? Here again there is the possibility of taking different perspectives. In addition to the very concrete question of the use of terms and attributes themselves, it is also

possible to look much more indirectly at the way in which results are published and thereby made known. Even science, which seek to be as objective as possible, is determined by subjective people.

One aim of the teaching unit was to initiate a process of understanding, that the culture of mathematics carries this richness of facets within itself and thus, despite the immanent formal rigor, can be thought much more creatively than perhaps suspected. Respective aspects were directly or indirectly addressed: Proofs in general, the need for variations of proofs, formal language and notations, mathematical methods and tools, meta-level: when is mathematics "simple"? or the evolution of mathematics in general.

Some conclusions concerning the culture of mathematics

In some concluding remarks we want to take up two of the mentioned aspects that have met with a particular response from students with reference to our teaching unit.

Proofs and variations of proofs

The fact that proofs represent a fundamental difference between school and university didactics and thus form a decisive hurdle for students, especially first-year students, has been noted in numerous works. Students in particular have difficulties "acquiring an understanding of the culture of proof and argumentation in scientific mathematics." (Jahnke & Ufer 2015, p. 350) Yet proofs represent the heart of mathematical culture (cf. (Grieser, 2015)), they should themselves be able to be seen as a means of developing knowledge and understanding. Georg Pólya already stated that "beginners [must] be convinced first of all that learning proofs is worthwhile, that they have a purpose, that they are interesting." (Pólya (1967, p. 195)) The question here is how appropriate motivation can happen. In some cases it is underlined that just an "upstream exploratory phase for understanding an assertion" [helps] in the "eventual formation of a need for proof and is indispensable for making out a proof idea." Accordingly, the need is expressed to give students "ample opportunity to develop problem awareness." (Grieser, 2015, p. 662)

Our thesis is that by including the letter exchange of Hilbert and Hurwitz in the formation of the two proofs of transcendence of e (as „exploratory phase“, this process of proof development may be made more comprehensible and tangible. In addition, we assume that a reflection process on proof variations in general can be triggered. In a qualitative survey in the seminar on selected topics of analysis, students gave preliminary feedback on this. Here, we give some excerpts from the answers to the questions about the meaning of variations of proofs (connected with the correspondence) and about their own perception of the discussed proofs of transcendence of e against the historical background: Proof variations were described by students as "representing developments within mathematics" and as "very important, especially that they become easier and more understandable", "through the correspondence the [...] background" was recognized and it became apparent that "through discussion (competition) [of Hilbert and Hurwitz] development" occurs. All students found the historical or personal reference to mathematicians interesting, and some even motivating. Students highlighted that the "writing became more understandable" and "clearer why [...] some methods were used". One student, who found the letter exchange interesting and motivating, evaluated that the proofs, however, did not become easier to understand. In two additional oral discussions with the students (each at the end of the respective seminar presentations), it became clear that it was precisely the comparison of the two proofs by the authors

in the letters that made them reflect. One student, for example, brought up the general difference between "elementary" and „simple" mathematics, another remarked that Hilbert's integrals seem rather "scary", though mathematically more "universal". Together we then noted that in fact Hurwitz's more "elementary" proof was more often used in textbooks, while Hilbert's proof was more likely to be used or developed in more advanced research. Overall, students rated the difficulty of the mathematics as adequate for their prior knowledge (from lectures on analysis) and felt Hurwitz's proof was easier. Although we are of course referring to a small number of students and have no quantitative study to show, consequently, we see a tendency that our teaching unit has triggered an interesting thought process among students regarding the role of proofs and variations of proofs. Naturally, not all students have a strong interest in history of mathematics, e. g. only some indicated in the survey the wish to have a dedicated course on history of mathematics, however, all indicated they would like to see more mathematics-historical aspects in their studies (which is not provided for in their regular curriculum).

In summary, we would like to conclude that a teaching unit based on historical documents is able to broaden the students' horizons both from a mathematical as well as a cultural point of view. The documents presented above in excerpts are suitable for this purpose, since they allow not only a mathematical depth but also evolutionary access to the development of mathematics itself.

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