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## GENERALIZING LUSZTIG'S TOTAL POSITIVITY

OLIVIER GUICHARD AND ANNA WIENHARD

ABSTRACT. We introduce the notion of  $\Theta$ -positivity in real semisimple Lie groups. This notion at the same time generalizes Lusztig's total positivity in split real Lie groups and invariant orders in Lie groups of Hermitian type. We show that there are four families of simple Lie groups which admit  $\Theta$ -positive structures and investigate fundamental properties of  $\Theta$ -positivity. We define and describe the positive and nonnegative unipotent semigroups and show that they give rise to a notion of positive  $n$ -tuples in flag varieties.

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## 1. INTRODUCTION

The theory of totally positive matrices arose in the beginning of the 20th century through work of Schoenberg [Sch30] and Gantmacher and Krein [GK35]. Total positivity has since become an important concept in several mathematical fields. In the 1990's the theory has been generalized widely by Lusztig [Lus98] who introduced the total positive semigroup of a general split real semisimple Lie group. Lusztig's total positivity plays an important role in representation theory, cluster algebras, and has many relations to other areas in mathematics as well as in theoretical physics. In this article we introduce a generalization of Lusztig's total positivity in the context of real semisimple Lie groups  $G$  that are not necessarily split. We call this generalization  $\Theta$ -positivity because it depends on the choice of a flag variety associated with  $G$  which is determined by a subset  $\Theta$  of the set of simple roots. The notion of  $\Theta$ -positivity generalizes at the same time Lusztig's total positivity (when the group is split and the flag variety is the complete flag variety) as well as Lie semigroups of Lie groups of Hermitian type, which are related to bi-invariant orders and causal structures (and the flag variety is the Shilov boundary). We give a classification of semisimple Lie groups admitting a  $\Theta$ -positive structure. Besides split real Lie groups and Hermitian Lie groups of tube type, two further families of simple Lie groups admit a  $\Theta$ -positive structure: the groups locally isomorphic to indefinite orthogonal groups  $SO(p, q)$ , with  $2 \leq p < q$ , and an exceptional family, whose system of restricted roots is of type  $F_4$  (cf. Theorem 1.1 below). To our knowledge, for these two families no positive structure was known before.

Our interest in  $\Theta$ -positivity arose from higher rank Teichmüller theory, in particular from trying to find a unifying framework that explains the similarities between Hitchin representations and maximal representations. The notion of  $\Theta$ -positivity provides such a unifying systematic framework. It also leads to several conjectures regarding higher rank Teichmüller spaces, some of which have been formulated in [GW18] and [Wie18], and partly proven in [Col20, AABC<sup>+</sup>19, BCGP<sup>+</sup>21], [GLW21], [BP21], and [BGL<sup>+</sup>24].

However, the reach of  $\Theta$ -positivity goes far beyond higher rank Teichmüller theory. For example, as we will discuss in a bit more detail below,  $\Theta$ -positivity suggests that Hermitian Lie groups of tube type should be considered as groups of type  $A_1$  over noncommutative algebras. Something, which has been made precise in [ABR<sup>+</sup>22] for most classical groups.

Before we describe the notion of  $\Theta$ -positivity and our results in more detail, let us note that we described the notion of  $\Theta$ -positivity several years ago in a survey paper [GW18] with only few proofs and a hands on description of the notion for indefinite orthogonal groups  $SO(p, q)$ ,  $2 \leq p < q$ . Several of the properties we described there have been already used by other people. This article now finally gives the foundation of  $\Theta$ -positivity. We focus here on the structure of unipotent subgroups in  $G$  and on  $\Theta$ -positivity in generalized flag varieties. In particular, we provide all the background needed to introduce positive configurations in flag varieties, positive maps or positive representations of surface groups. We give all the background and prove the results used in [Col20, AABC<sup>+</sup>19, BCGP<sup>+</sup>21], in [PSW23, BP21], as well as in [GLW21] and [BGL<sup>+</sup>24]. Ideas from this paper and [GLW21] were also used in [BCL20]. In a forthcoming second foundational article we will focus on the finer structure of  $\Theta$ -positivity, including braid relations, Gauß decomposition theorems, and geometric properties of positive elements in  $G$ .

We now describe the results of the paper in more detail.

**1.1.  $\Theta$ -positive structures.** Totally positive matrices are defined by requiring all minors of the matrix to be positive. Anne Whitney showed in [Whi52] that the semigroup of totally positive, or more general totally nonnegative, matrices can be generated by an explicit set of simple matrices. This reduction theorem allowed the generalization of total positivity to all semisimple split real Lie groups by Lusztig [Lus98]. The key building block in Lusztig's approach is the construction of a closed nonnegative semigroup  $U^{\geq 0}$  in the unipotent radical  $U$  of the Borel subgroup  $P$  of a split real Lie group  $G$ . This semigroup is constructed as follows. The Lie algebra  $\mathfrak{u}$  can be described as the sum of the root spaces  $\mathfrak{g}_\alpha$  for all positive roots  $\alpha$ . Since  $G$  is split, any root space  $\mathfrak{g}_\alpha$  is of dimension 1. Choosing an element  $X_\alpha$  in  $\mathfrak{g}_\alpha$  it can hence be identified with  $\mathbf{R}$ . Given the set of simple positive roots  $\Delta$ , consider the map  $x_\alpha: \mathfrak{g}_\alpha \cong \mathbf{R} \rightarrow U$ ,  $s \mapsto \exp(sX_\alpha)$ . Lusztig's nonnegative semigroup  $U^{\geq 0}$  is then the semigroup generated by  $x_\alpha(\mathbf{R}_{\geq 0})$ ,  $\alpha \in \Delta$ .

Turning to our situation, when  $G$  is a semisimple real Lie group, which is not necessarily split, we consider a subset  $\Theta \subset \Delta$  of the set of simple roots. This defines a standard parabolic subgroup  $P_\Theta$  in  $G$  and a flag variety  $\mathbf{F}_\Theta = G/P_\Theta$ . (The convention is that  $P_\Delta$  is the minimal parabolic subgroup of  $G$ .) The standard opposite parabolic subgroup  $P_\Theta^{\text{opp}}$  is such that  $L_\Theta = P_\Theta \cap P_\Theta^{\text{opp}}$  is a Levi factor and the group  $P_\Theta$  is the semidirect product of its unipotent radical  $U_\Theta$  and the Lie group  $L_\Theta$ . The Lie algebra  $\mathfrak{u}_\Theta$  of  $U_\Theta$  carries the adjoint action by  $L_\Theta$  and can hence be decomposed into its  $L_\Theta$ -irreducible pieces. For every simple root  $\alpha \in \Theta$  there is an irreducible piece  $\mathfrak{u}_\alpha$ , which in general is not of

dimension 1. When  $G$  is split and  $\Delta = \Theta$  these are precisely the root spaces  $\mathfrak{g}_\alpha$ .

We say that  $G$  admits a  $\Theta$ -positive structure if it satisfies the following property: For every  $\alpha \in \Theta$  there exists an acute convex cone  $c_\alpha \subset \mathfrak{u}_\alpha$  that is invariant under the action of the connected component  $L_\Theta^\circ$  of  $L_\Theta$ .

In Section 3, we first show that the notion of  $\Theta$ -positivity defined for semisimple groups reduces to the case of simple groups (i.e. every simple factor must have a positive structure). We then prove

**Theorem 1.1** (see Theorem 3.4). *A simple Lie group  $G$  admits a  $\Theta$ -positive structure if and only if the pair  $(G, \Theta)$  belongs to the following list:*

- (1)  $G$  is a split real form,  $\Theta = \Delta$ , and  $F_\Theta$  is the complete flag variety of  $G$ ;
- (2)  $G$  is Hermitian of tube type and of real rank  $r$ ,  $\Theta = \{\alpha_r\}$ , where  $\alpha_r$  is the long simple restricted root, and  $F_\Theta$  is the Shilov boundary of the symmetric space associated with  $G$ ;
- (3)  $G$  is locally isomorphic to  $\mathrm{SO}(p+1, p+k)$ ,  $p > 1$ ,  $k > 1$ ,  $\Theta = \{\alpha_1, \dots, \alpha_p\}$ , where  $\alpha_1, \dots, \alpha_p$  are the long simple restricted roots, and  $F_\Theta$  is the variety of flags  $(F_1, \dots, F_p)$  with  $F_i \subset \mathbf{R}^{2p+k+1}$  isotropic subspace of dimension  $i$  and  $F_i \subset F_{i+1}$ ;
- (4)  $G$  is the real form of  $F_4$ ,  $E_6$ ,  $E_7$ , or of  $E_8$  whose system of restricted roots is of type  $F_4$ , and  $\Theta = \{\alpha_1, \alpha_2\}$ , where  $\alpha_1, \alpha_2$  are the long simple restricted roots.

The cones  $c_\alpha \subset \mathfrak{u}_\alpha$ ,  $\alpha \in \Theta$ , allow to define the nonnegative semigroup  $U_\Theta^{\geq 0}$  to be the subsemigroup of  $U_\Theta$  generated by the elements  $\exp(v)$ , with  $v \in c_\alpha$  for some  $\alpha \in \Theta$ .

When  $G$  is a split real Lie group which also belongs to one of the other three families in Theorem 1.1, then  $G$  carries two different positive structures, one with respect to  $\Delta$  and one with respect to a proper subset  $\Theta \subset \Delta$ .

Theorem 1.1 takes a more algebraic approach to  $\Theta$ -positivity, but we show further that the existence of a  $\Theta$ -positive structure can be characterized geometrically in terms of positive triples in the associated flag varieties  $F_\Theta$  (see Section 13). For this it is important to prove that essentially any  $L_\Theta^\circ$ -invariant semigroup in  $U_\Theta$  comes from a  $\Theta$ -positive structure.

To give a precise statement, let us make a few observations. To every root system there is a corresponding *opposition involution* from  $\Delta$  to  $\Delta$ , but which might not preserve  $\Theta$ . However in Theorem 1.1, whenever  $\Theta \neq \Delta$ , the opposition involution is just the identity on  $\Delta$  and thus preserves  $\Theta$ . This implies that in all cases  $P_\Theta$  is conjugate to its opposite parabolic subgroup  $P_\Theta^{\mathrm{opp}}$ , and the flag varieties defined by  $P_\Theta$  and by  $P_\Theta^{\mathrm{opp}}$  coincide. Any parabolic subgroup  $P$  of  $G$  defines a Bruhat decomposition (see Section 9.1), which is the orbit decomposition under the action of  $P \times P$ , and that admits a unique open orbit. We denote the open Bruhat orbit with respect to  $P_\Theta^{\mathrm{opp}} \times P_\Theta^{\mathrm{opp}}$  by  $\Omega_\Theta^{\mathrm{opp}}$ , it can be described as  $\Omega_\Theta^{\mathrm{opp}} = \{g \in G \mid gP_\Theta^{\mathrm{opp}}g^{-1} \cap U_\Theta^{\mathrm{opp}} \text{ is trivial}\}$  ( $U_\Theta^{\mathrm{opp}}$  is the unipotent radical of  $P_\Theta^{\mathrm{opp}}$ ).

**Theorem 1.2** (see Theorem 12.1 and Corollary 12.3). *Let  $G$  be a connected simple Lie group. Suppose that there is  $U^+ \subset U_\Theta$  such that  $U^+$  is a closed  $L_\Theta^\circ$ -invariant semigroup of nonempty interior, which contains no nontrivial invertible element. Then  $G$  admits a  $\Theta$ -positive structure and the semigroup  $U^+$  contains the semigroup  $U_\Theta^{\geq 0}$ .*

*If we further assume that the interior of  $U^+$  is contained in the open Bruhat cell  $\Omega_\Theta^{\text{opp}}$  with respect to  $P_\Theta^{\text{opp}}$ , then  $U^+ = U_\Theta^{\geq 0}$ .*

## 1.2. The unipotent positive semigroup and the $\Theta$ -Weyl group.

Whereas the definition of the nonnegative unipotent semigroup in Lusztig's and in our work is straight forward, the understanding of the positive unipotent semigroup is more delicate. In Lusztig's work it involves the Weyl group  $W$ . Recall that the Weyl group is generated by the elements  $s_\alpha$ ,  $\alpha \in \Delta$ . There is a unique longest length element  $w_\Delta \in W$ . To construct the positive unipotent semigroup  $U^{>0}$  one fixes a reduced expression  $s_{\alpha_1} \cdots s_{\alpha_k}$  of  $w_\Delta$ . Making this choice Lusztig defines a map  $F: \mathbf{R}^k \cong \mathfrak{g}_{\alpha_1} \times \cdots \times \mathfrak{g}_{\alpha_k} \rightarrow U$ ,  $(x_1, \dots, x_k) \mapsto \exp(x_1) \cdots \exp(x_k)$ . The image of  $\mathbf{R}_{>0}^k$  under this map  $F$  is the positive unipotent semigroup. A key point is of course to show that this image is independent of the choice of the reduced expression. In [Lus98] this is done by first a reduction to the simply-laced cases and then by giving explicit formulas for the change of coordinates given by a braid relation, i.e. exchanging  $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$  in the reduced expression. The change of coordinates is given by positive rational functions. (Berenstein and Zelevinsky [BZ97] give explicit formulas for the braid relations in the case of non simply-laced Dynkin diagrams.) Since any two reduced expressions of  $w_\Delta$  are related by a sequence of braid relations, this thus proves that the image  $F(\mathbf{R}_{>0}^k)$  is independent of the chosen reduced expression. This image is the positive unipotent semigroup  $U^{>0}$  and the map  $F$  thus gives a parametrization of  $U^{>0}$ . One can then conclude that the semigroup  $U^{>0}$  is in fact the interior of  $U^{\geq 0}$ .

Here we follow a slightly different approach. We define the nonnegative semigroup  $U_\Theta^{\geq 0}$  to be the semigroup, in the unipotent radical  $U_\Theta$  of  $P_\Theta$ , generated by  $\bigcup_{\alpha \in \Theta} \exp(c_\alpha)$ ; we then define the positive unipotent semigroup  $U_\Theta^{>0}$  to be the interior (relative to  $U_\Theta$ ) of the nonnegative semigroup  $U_\Theta^{\geq 0}$ .

We then give explicit parametrizations of the semigroup  $U_\Theta^{>0}$ . For this, the relevant object to control the combinatorial data underlying these parametrizations is the  $\Theta$ -Weyl group. The  $\Theta$ -Weyl group  $W(\Theta)$  is a subgroup of the Weyl group  $W$  of  $G$  defined by an explicit generating set  $R(\Theta) = \{\sigma_\alpha\}_{\alpha \in \Theta}$  (see Section 8.3) obtained by choosing shortest length representatives for some cosets of the subgroup  $W_{\Delta \setminus \Theta}$  corresponding to the Levi factor of  $P_\Theta$  in  $W$ . The  $\Theta$ -Weyl group is isomorphic to the Weyl group of a different root system. This is a priori be a bit surprising, but indeed there is even a natural embedding of a split real Lie group of type  $W(\Theta)$  into  $G$ . The  $\Theta$ -Weyl group  $W(\Theta)$  then plays the same role as the Weyl group in order to parametrize the positive unipotent semigroup and give an alternative description of it.

**Theorem 1.3** (see Section 10). *Given any reduced expression  $\sigma_{\gamma_1} \cdots \sigma_{\gamma_m}$  of the longest length element  $w_{\max}^\Theta \in W(\Theta)$  and setting  $c := c_{\gamma_1} \times \cdots \times$*



$c_{\gamma_n}$ , the map

$$F: c \rightarrow U_{\Theta},$$

defined as the product of the exponential map on each factor, has the following properties:

- (1)  $F$  is proper;
- (2)  $F|_{\mathring{c}}$  is open and injective;
- (3)  $F(\mathring{c})$  is a connected component of  $\Omega_{\Theta}^{\text{opp}} \cap U_{\Theta}$  (as above  $\Omega_{\Theta}^{\text{opp}}$  is the open Bruhat cell with respect to  $P_{\Theta}^{\text{opp}}$ );
- (4) The image  $F(\mathring{c})$  is independent of the reduced expression of  $w_{\max}^{\Theta}$  and is equal to the positive unipotent semigroup  $U_{\Theta}^{>0}$ .

Further properties of the unipotent semigroups follow from this:

**Theorem 1.4** (see Corollary 10.19). *The positive and nonnegative semigroups have the following properties:*

- (1) The semigroup  $U_{\Theta}^{>0}$  is invariant by conjugation by  $L_{\Theta}^{\circ}$ .
- (2) The closure of  $U_{\Theta}^{>0}$  is the nonnegative semigroup  $U_{\Theta}^{\geq 0}$ , in particular  $U_{\Theta}^{\geq 0}$  is closed.
- (3) For every reduced expression of  $w_{\max}^{\Theta}$ , one has  $U_{\Theta}^{\geq 0} = F(c)$  (with  $c$  and  $F$  as in Theorem 1.3).
- (4)  $U_{\Theta}^{\geq 0} U_{\Theta}^{>0} \subset U_{\Theta}^{>0}$  and  $U_{\Theta}^{>0} U_{\Theta}^{\geq 0} \subset U_{\Theta}^{>0}$ .

The proofs of Theorem 1.3 and Theorem 1.4 rely on a fine understanding of the image of the map  $F$  relative to the Bruhat decompositions of  $G$  with respect to  $P_{\Theta}^{\text{opp}}$  and with respect to  $P_{\Delta}^{\text{opp}}$ .

Analogously we construct the nonnegative semigroup  $U_{\Theta}^{\text{opp}, \geq 0}$  and the positive semigroup  $U_{\Theta}^{\text{opp}, > 0}$  in  $U_{\Theta}^{\text{opp}}$  and similar statements hold for them.

Using this, the  $\Theta$ -nonnegative semigroup  $G_{\Theta}^{\geq 0}$  is defined to be the semigroup of  $G$  generated by  $U_{\Theta}^{\text{opp}, \geq 0}$ ,  $U_{\Theta}^{\geq 0}$ , and  $L_{\Theta}^{\circ}$ , and the  $\Theta$ -positive semigroup  $G_{\Theta}^{> 0}$  is defined to be its interior. These semigroups, and the geometric properties of their elements will be analyzed in a forthcoming article.

*Remark 1.5.* Note that to prove these results, we do not need to establish the explicit change of coordinates with respect to modifying the reduced expression of  $w_{\max}^{\Theta}$  by a braid relation. However, to get a finer understanding, it is in fact interesting to write down explicit braid relations. In this paper we will only shortly discuss the braid relations in the case that  $G = \text{SO}(p, q)$  derived from explicit matrix equations. The more general treatment of braid relations involves  $\Theta$ -versions of universal enveloping algebras, adapting the strategy in [BZ97]. This will be appear in [GW24].

**1.3. Positivity in flag varieties.** We now turn our attention to the flag variety  $\mathbb{F}_{\Theta} \cong G/P_{\Theta}$ . There is a unique flag  $f_{\Theta} \in \mathbb{F}_{\Theta}$  fixed by  $P_{\Theta}$  and a unique flag  $f_{\Theta}^{\text{opp}} \in \mathbb{F}_{\Theta}$  fixed by the opposite parabolic subgroup  $P_{\Theta}^{\text{opp}}$  (recall that we are in a situation where  $G/P_{\Theta} \cong G/P_{\Theta}^{\text{opp}}$ ). It is well known that  $U_{\Theta}$  acts simply transitively on the set of flags that are transverse to  $f_{\Theta}$ . The positive unipotent semigroup  $U_{\Theta}^{>0}$  allows us to introduce the notion of positive  $n$ -tuples of flags in  $\mathbb{F}_{\Theta}$ . In order to have a more intrinsic notion, we will consider all the possible such subsemigroups, i.e. the subsemigroups  $U_{\Theta}^{\varepsilon > 0}$  where  $\varepsilon$  varies in  $\{\pm 1\}^{\Theta}$  and



$U_{\Theta}^{\varepsilon > 0}$  is the image of the map  $F$  of Theorem 1.3 where the factors  $\mathring{c}_{\alpha}$  are replaced by  $-\mathring{c}_{\alpha}$  when the  $\alpha$ -component of  $\varepsilon$  is equal to  $-1$ . (Alternatively  $U_{\Theta}^{\varepsilon \geq 0}$  is the semigroup generated by  $\exp(\bigcup_{\alpha \in \Theta} \varepsilon_{\alpha} c_{\alpha})$  and  $U_{\Theta}^{\varepsilon > 0}$  is its interior.)

An  $n$ -tuple of flags  $(f_0, f_1, \dots, f_{n-2}, f_{\infty})$  is said to be  $\Theta$ -positive if there exist an element  $g \in G$ ,  $\varepsilon$  in  $\{\pm 1\}^{\Theta}$ , and  $u_1, \dots, u_{n-2} \in U_{\Theta}^{\varepsilon > 0}$  such that  $(f_0, f_1, \dots, f_{n-2}, f_{\infty}) = g \cdot (f_{\Theta}^{\text{opp}}, u_1 \cdot f_{\Theta}^{\text{opp}}, \dots, u_i \cdots u_1 \cdot f_{\Theta}^{\text{opp}}, \dots, u_{n-2} \cdots u_1 \cdot f_{\Theta}^{\text{opp}}, f_{\Theta})$ .

A consequence of Theorem 1.3 is

**Theorem 1.6.** *The set of flags  $U_{\Theta}^{\varepsilon > 0} \cdot f_{\Theta}^{\text{opp}}$  is a connected component of the set of flags that are transverse to  $f_{\Theta}$  and to  $f_{\Theta}^{\text{opp}}$ . The stabilizer in  $G$  of a  $\Theta$ -positive triple is compact.*

Note that the set of  $\Theta$ -positive triples will in general not be connected, but can consist of several connected components of the set of triples of pairwise transverse flags.

We establish crucial properties of positive  $n$ -tuples in the flag variety  $F_{\Theta}$  in Section 13. For this we use the notion of diamonds, which turns out to be very helpful to keep track of positivity of triples and more general  $n$ -tuples of flags. Given two points  $f_0, f_1 \in F_{\Theta}$ , a diamond with extremities  $f_0, f_1$  is a (special) connected component of the set of flags that are transverse to  $f_0$  and  $f_1$ . Diamonds should be thought of as generalized intervals. In the case of total positivity they were introduced (without using this terminology) in [BCL20]. The term ‘‘diamond’’ was coined in [LT23] in the case of  $\text{SO}(2, n)$ , where their basic metric properties were established. For the case of  $\Theta$ -positivity, diamonds were introduced and studied in [GLW21]. Diamonds give a geometric characterization of  $\Theta$ -positivity. In Section 13.3 we show that the existence of a family of diamonds in the flag variety  $F_{\Theta}$  is equivalent to the existence of a  $\Theta$ -positive structure. This geometric characterization is used in [GLW21] to investigate positive representations.

We already noticed that the Lie groups  $\text{SO}(n, n+1)$ ,  $\text{Sp}(2n, \mathbf{R})$ , or the split real form of  $F_4$  admit two  $\Theta$ -positive structures, one for  $\Theta = \Delta$  and one for  $\Theta$  being a strict subset of  $\Delta$ . We prove that the natural projection  $F_{\Delta} \rightarrow F_{\Theta}$  maps positive  $n$ -tuples to positive  $n$ -tuples (Section 13.10).

**1.4. The  $\Theta$ -principal  $\mathfrak{sl}_2$  and positive circles.** It is well known that the Lie algebra of a split real Lie group contains a distinguished conjugacy class of 3-dimensional simple subalgebras, the principal  $\mathfrak{sl}_2$ . This principal  $\mathfrak{sl}_2$  plays an important role for the moduli space of Higgs bundles, and in particular for the Hitchin section and consequently the Hitchin component [Hit92]. When  $G$  admits a  $\Theta$ -positive structure, we introduce a  $\Theta$ -system, a family in the Lie algebra consisting of  $\mathfrak{sl}_2$ -triples associated with nilpotent elements in the cone  $\mathring{c}_{\alpha}$ , for each  $\alpha \in \Theta$ . This  $\Theta$ -system generates a split real subalgebra  $\mathfrak{h}_{\Theta}$  in the Lie algebra  $\mathfrak{g}$  of  $G$ , as well as a principal 3-dimensional simple subalgebra in  $\mathfrak{h}_{\Theta}$ . We call this special  $\mathfrak{sl}_2$  the  $\Theta$ -principal  $\mathfrak{sl}_2$ . This  $\Theta$ -principal  $\mathfrak{sl}_2$  gives rise to a positive circle in the flag variety  $F_{\Theta}$ . The  $\Theta$ -principal  $\mathfrak{sl}_2$  is related

to magical  $\mathfrak{sl}_2$ -triples introduced in [BCGP<sup>+</sup>21]. The explicit relation is stated in Section 7.

### 1.5. Positive representations and higher rank Teichmüller

**spaces.** Lusztig's total positivity on the one hand and positive semigroups in Hermitian Lie groups of tube type on the other play an important role in higher rank Teichmüller theory. Hitchin components and spaces of maximal representations have been characterized as positive representations [Lab06, Gui05, FG06, BILW05, BIW10]. At the same time, total positivity plays an important role in the construction of cluster coordinates by Fock and Goncharov [FG06].

As  $\Theta$ -positivity realizes total positivity in split real Lie groups and Lie semigroups in Hermitian Lie groups of tube type as two incarnations of the same concept, it provides an interesting systematic framework for higher rank Teichmüller spaces. This is based on the notion of positive  $n$ -tuples in the flag variety  $F_\Theta$ , which allows us to introduce the notion of positive representations.

Let  $\pi_1(S)$  be the fundamental group of an oriented surface of negative Euler characteristic. Then the boundary  $\partial_\infty \pi_1(S)$  carries a natural cyclic ordering. Given a simple Lie group  $G$  with a  $\Theta$ -positive structure, we call a representation  $\rho: \pi_1(S) \rightarrow G$   $\Theta$ -positive if there exists a  $\rho$ -equivariant map from  $\partial_\infty \pi_1(S)$  to the flag variety  $F_\Theta$ , which sends every cyclically ordered  $n$ -tuple of points in  $\partial_\infty \pi_1(S)$  to a positive  $n$ -tuple of flags in  $F_\Theta$ .

In [GW18] we conjectured that the set of positive representations forms higher rank Teichmüller spaces, i.e. it is open and closed and consists entirely of discrete and faithful representations. Using several of the results obtained in this paper, this conjecture has been proven to a large extent in [GLW21] and [BP21], relying partly on [Col20, BCGP<sup>+</sup>21], and is fully proven in [BGL<sup>+</sup>24].

However,  $\Theta$ -positivity suggests an even deeper connection between the different families of higher rank Teichmüller spaces, including extensions of Fock–Goncharov's construction of cluster coordinates to spaces of positive representations. The expectation is that there are cluster coordinates for positive representations into the Lie group  $G$  with respect to its  $\Theta$ -positive structure provide examples of noncommutative cluster algebras of type  $W(\Theta)$ . This has been proven for the symplectic group (as a Hermitian Lie group of tube type) in [AGRW19], where appropriate noncommutative coordinates give a geometric realization of the noncommutative cluster algebra of type  $A_1$  introduced in [BR18], see also [KR22]. Moreover in [ABR<sup>+</sup>22] classical Hermitian Lie groups of tube type are realized as groups of type  $A_1$  over a noncommutative ring.

For further conjectures and questions regarding  $\Theta$ -positivity, we refer the reader to [Wie18].

### 1.6. Structure of the paper.

The necessary background on semisimple Lie algebras, their roots systems and their parabolic subgroups is recalled in Section 2. There, we explain in particular how a semisimple group decomposes into its simple factors and the subsequent decompositions of the parabolic subgroups, the flag varieties, etc.

We introduce and classify  $\Theta$ -positive structures in Section 3. Their first immediate properties are then proved (the special root, the Hermitian tube type subgroup).

In Section 4 we introduce the unipotent nonnegative and positive semigroups and state a few simple properties of semigroups in topological groups.

In Section 5 we describe the invariant cones that enter in the construction of the semigroups, explicitly for the family of classical groups and with a table for the  $F_4$  cases. We construct  $\Theta$ -systems and prove some commutation relations used in the sequel.

We show that every  $\Theta$ -system generates a split real subalgebra in Section 6 and show that it gives rise to a  $\Theta$ -principal three-dimensional subalgebra in Section 7. Here we also show that the longest length element of the Weyl group of this split subgroup belongs to the image of the  $\Theta$ -principal  $\mathrm{SL}_2(\mathbf{R})$ .

In Section 8 we introduce the  $\Theta$ -Weyl group and show it is isomorphic to the Weyl group of the split real subgroup. The  $\Theta$ -length is introduced and we show that the  $\Theta$ -Weyl group normalizes the Weyl group of the Levi factor of  $L_\Theta$ .

In Section 9, relying on the elements in the cones  $c_\alpha$  constructed in Section 5, we establish important properties of the relations of the invariant cones and Bruhat decompositions, which are used in Section 10 to give explicit parametrizations of the unipotent positive semigroup. Theorem 1.3 is proved there. We also determine the tangent cone at the identity of the semigroup  $U_\Theta^{\geq 0}$ .

Section 11 takes a particular look at the positive structure for the family of indefinite orthogonal groups, including a parametrization of the nonnegative unipotent semigroup for  $\mathrm{SO}(3, q)$ .

In Section 12 we prove the relation of general invariant unipotent semigroups with those arising from a  $\Theta$ -positive structure, i.e. Theorem 1.2 above. Finally in Section 13 we consider the flag varieties, introduce the notion of diamonds and the positivity of  $n$ -tuples of flags.

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## 2. BACKGROUND AND PRELIMINARIES

In this section we recall classical material on the structure of semisimple Lie groups and their parabolic subgroups.

As, starting from the next section, we will mainly focus on simple Lie groups, we explain here how a semisimple Lie group decomposes into its simple factors and the induced decompositions of the parabolic subgroups, the flag varieties, the Weyl groups, etc.

Unless explicitly stated otherwise all vector spaces, Lie algebras, Lie groups, and their representations are defined over  $\mathbf{R}$ .

**2.1. Cartan involution.** Let  $G$  be a connected real semisimple Lie group with finite center and denote by  $\mathfrak{g}$  its Lie algebra. The simple factors of the Lie algebra  $\mathfrak{g}$  are denoted  $\{\mathfrak{g}_s\}_{s \in \mathcal{S}}$  so that  $\mathfrak{g} = \bigoplus_{s \in \mathcal{S}} \mathfrak{g}_s$ . For every  $s$  in  $\mathcal{S}$ , the connected Lie subgroup of  $G$  with Lie algebra  $\mathfrak{g}_s$  will be denoted by  $G_s$ ; it is a closed subgroup and a simple Lie group. The subgroups  $\{G_s\}_{s \in \mathcal{S}}$  pairwise commute and the product map  $\prod_{s \in \mathcal{S}} G_s \rightarrow G$

is onto with finite kernel (it is sometimes said that  $G$  is the *almost product* of the  $G_s$ ).

Let  $\mathfrak{k}$  be the Lie algebra of a maximal compact subgroup  $K \subset G$  ( $\mathfrak{k}$  is called a maximal compact subalgebra). Then  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{k}^\perp$  where  $\mathfrak{k}^\perp$  is the orthogonal of  $\mathfrak{k}$  with respect to the Killing form  $B$  on  $\mathfrak{g}$ . We denote by  $\tau$  the *Cartan involution* with respect to  $\mathfrak{k}$ , it is the Lie algebra automorphism  $\tau: \mathfrak{g} \rightarrow \mathfrak{g}$  whose fixed point set is  $\mathfrak{k}$  and which is an involution; namely  $\tau|_{\mathfrak{k}} = \text{Id}$  and  $\tau|_{\mathfrak{k}^\perp} = -\text{Id}$ . The bilinear form  $B(X, \tau(-Y))$  on  $\mathfrak{g}$  is symmetric and positive definite.

For all  $s$ ,  $\mathfrak{k}_s = \mathfrak{k} \cap \mathfrak{g}_s$  is the Lie algebra of a maximal compact subgroup  $K_s$  of  $G_s$ . For all  $s$  we denote by  $\mathfrak{k}_s^\perp$  the orthogonal complement in  $\mathfrak{g}_s$  of the Lie subalgebra  $\mathfrak{k}_s$  with respect to the Killing form  $B_s$  of  $\mathfrak{g}_s$ , one has the equality  $B = \sum_{s \in \mathcal{S}} B_s$  (under the decomposition  $\mathfrak{g} = \bigoplus_{s \in \mathcal{S}} \mathfrak{g}_s$ ) and the decompositions  $\mathfrak{k} = \bigoplus_{s \in \mathcal{S}} \mathfrak{k}_s$ ,  $\mathfrak{k}^\perp = \bigoplus_{s \in \mathcal{S}} \mathfrak{k}_s^\perp$ . Furthermore the Cartan involution  $\tau$  is the sum of the Cartan involutions  $\tau_s$  of  $\mathfrak{g}_s$  with respect to  $\mathfrak{k}_s$  (again using the decomposition  $\mathfrak{g} = \bigoplus_{s \in \mathcal{S}} \mathfrak{g}_s$ ). The Lie group  $K$  is the almost product of the  $K_s$ .

**2.2. Restricted roots.** We now choose a maximal Abelian subspace  $\mathfrak{a}$  contained in  $\mathfrak{k}^\perp$ ;  $\mathfrak{a}$  is called a *Cartan subspace* of  $G$ . Denoting  $\mathfrak{a}_s = \mathfrak{a} \cap \mathfrak{g}_s$ , one has that  $\mathfrak{a}_s$  is a Cartan subspace of  $G_s$ . The equality  $\mathfrak{a} = \bigoplus_{s \in \mathcal{S}} \mathfrak{a}_s$  holds and induces an identification of  $\mathfrak{a}^*$  with the direct sum  $\bigoplus_{s \in \mathcal{S}} \mathfrak{a}_s^*$ .

We denote by  $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$  the system of restricted roots, i.e.  $\Sigma \subset \mathfrak{a}^*$  is the set of nonzero weights for the adjoint action of  $\mathfrak{a}$  on  $\mathfrak{g}$ ; the corresponding weight spaces are denoted  $\mathfrak{g}_\alpha$  ( $\alpha \in \Sigma$ ). Namely, for every  $\alpha$  in  $\mathfrak{a}^*$ , we set

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g} \mid \text{ad}(H)(X) = \alpha(H)X \text{ for all } H \in \mathfrak{a}\}$$

and  $\Sigma$  is the family of elements  $\alpha$  in  $\mathfrak{a}^* \setminus \{0\}$  such that  $\mathfrak{g}_\alpha \neq 0$ ; the subspace  $\mathfrak{g}_\alpha$  is called a *root space* for  $\alpha \in \Sigma$ .

For every root  $\alpha \in \Sigma$ , the intersection of  $\mathfrak{g}_\alpha$  and  $\mathfrak{k}$  is reduced to  $\{0\}$  and one has  $\tau(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$ .

The root system  $\Sigma$  is the disjoint union of the root systems  $\Sigma_s = \Sigma(\mathfrak{g}_s, \mathfrak{a}_s)$  for  $s$  in  $\mathcal{S}$  and, when  $\alpha$  belongs to  $\Sigma_s$ , one has  $(\mathfrak{g}_s)_\alpha = \mathfrak{g}_\alpha$ . Furthermore  $\Sigma_s = \Sigma \cap \mathfrak{a}_s^*$ .

The weight space  $\mathfrak{g}_0$  is the centralizer  $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})$  of  $\mathfrak{a}$  in  $\mathfrak{g}$  and is the direct sum of the centralizers  $\mathfrak{z}_{\mathfrak{g}_s}(\mathfrak{a}_s)$ . Also  $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}) = \mathfrak{m} \oplus \mathfrak{a}$ , where  $\mathfrak{m} = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a}) = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}) \cap \mathfrak{k}$  is the centralizer of  $\mathfrak{a}$  in  $\mathfrak{k}$  (the decomposition  $\mathfrak{z}_{\mathfrak{k}}(\mathfrak{a}) = \bigoplus_{s \in \mathcal{S}} \mathfrak{z}_{\mathfrak{k}_s}(\mathfrak{a}_s)$  also holds) and the sum is orthogonal with respect to the Killing form  $B$ . Therefore we have a  $B$ -orthogonal decomposition  $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \left(\bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha\right)$ .

Let  $<_{\mathfrak{a}^*}$  be a total linear ordering on  $\mathfrak{a}^*$ : for every  $\alpha$  and  $\beta$  in  $\mathfrak{a}^*$ , one has either  $\alpha = \beta$ ,  $\alpha <_{\mathfrak{a}^*} \beta$ , or  $\beta <_{\mathfrak{a}^*} \alpha$  (which we also denote  $\alpha >_{\mathfrak{a}^*} \beta$ ) and if  $\alpha <_{\mathfrak{a}^*} \beta$  then, for every  $\gamma$  in  $\mathfrak{a}^*$ ,  $\alpha + \gamma <_{\mathfrak{a}^*} \beta + \gamma$ , and for every  $\lambda$  in  $\mathbf{R}_{>0}$ ,  $\lambda\alpha <_{\mathfrak{a}^*} \lambda\beta$ .

We then denote by  $\Sigma^+ = \{\alpha \in \Sigma \mid \alpha >_{\mathfrak{a}^*} 0\}$  the set of positive roots and by  $\Sigma^- = \{\alpha \in \Sigma \mid \alpha <_{\mathfrak{a}^*} 0\}$  the set of negative roots; one has  $\Sigma^- = -\Sigma^+$ . The positive roots that cannot be written nontrivially as a sum of positive roots are called *simple* roots; their set is denoted by  $\Delta \subset \Sigma^+$ . It is well known that  $\Delta$  is a basis of  $\mathfrak{a}^*$  and that, for every  $\beta$  in  $\Sigma$ , the coefficients of  $\beta$  as a linear combination with respect

to the basis  $\Delta$  are integers all nonnegative or all nonpositive according to whether  $\beta$  belongs to  $\Sigma^+$  or to  $\Sigma^-$ .

The set of positive roots  $\Sigma^+$  is the disjoint union of the sets  $\Sigma_s^+ = \Sigma_s \cap \Sigma^+$  of positive roots for the simple factors. This holds similarly for the set of negative roots. The set  $\Delta$  of simple roots is also the disjoint union of the  $\Delta_s = \Delta \cap \Sigma_s$ .

The Killing form induces, by restriction and duality, a Euclidean scalar product  $(\cdot|\cdot)$  on  $\mathfrak{a}^*$ . It is well known that, for every  $s$  in  $\mathcal{S}$ , either all the roots in  $\Sigma_s$  have the same norm or the set  $\Delta_s$  of simple roots is the disjoint union of two nonempty subsets  $\Delta_{s,p}$  and  $\Delta_{s,g}$  with the following properties: all the elements of  $\Delta_{s,p}$  have the same norm, all the elements of  $\Delta_{s,g}$  have the same norm, and the norm of elements in  $\Delta_{s,g}$  is greater than the norm of elements in  $\Delta_{s,p}$ . Elements of  $\Delta_{s,g}$  are called *long roots*, elements of  $\Delta_{s,p}$  are called *short roots*. For  $\alpha \in \Delta_{s,g}$  and  $\beta \in \Delta_{s,p}$ , one has in fact  $(\alpha|\alpha) = 2(\beta|\beta)$  except in the case of type  $G_2$  where  $(\alpha|\alpha) = 3(\beta|\beta)$ .

In general the root spaces  $\mathfrak{g}_\alpha$  are not necessarily 1-dimensional. In fact they are all 1-dimensional if and only if  $\mathfrak{g}$  is the Lie algebra of a split real form. However the root spaces  $\mathfrak{g}_\alpha$  are 1-dimensional when  $\alpha$  is a long root.

The *open Weyl chamber* corresponding to  $\Sigma^+$  will be denoted

$$\mathfrak{a}^+ = \{X \in \mathfrak{a} \mid \alpha(X) > 0, \forall \alpha \in \Sigma^+\};$$

its closure  $\bar{\mathfrak{a}}^+$  is called the *closed Weyl chamber*. One has also

$$\mathfrak{a}^+ = \{X \in \mathfrak{a} \mid \alpha(X) > 0, \forall \alpha \in \Delta\}.$$

The closed Weyl chamber is the direct sum of the

$$\bar{\mathfrak{a}}_s^+ = \{X \in \mathfrak{a}_s \mid \alpha(X) \geq 0, \forall \alpha \in \Sigma_s^+\} = \bar{\mathfrak{a}}^+ \cap \mathfrak{a}_s.$$

The (relative) interior of  $\bar{\mathfrak{a}}_s^+$  is

$$\mathfrak{a}_s^+ = \{X \in \mathfrak{a}_s \mid \alpha(X) > 0, \forall \alpha \in \Sigma_s^+\},$$

the closure of  $\mathfrak{a}_s^+$  is  $\bar{\mathfrak{a}}_s^+$  and the open Weyl chamber  $\mathfrak{a}^+$  is the direct sum of the open Weyl chambers  $\mathfrak{a}_s^+$ .

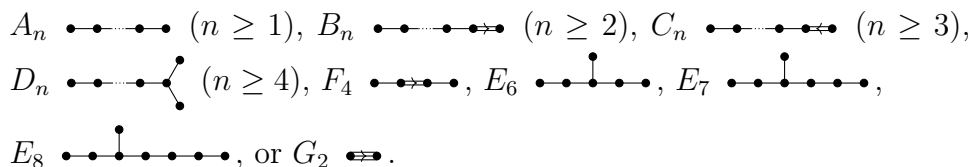
An element of  $\mathfrak{g}$  is said *regular* if it is conjugate to an element of  $\mathfrak{a} \setminus \bigcup_{\alpha \in \Sigma} \ker(\alpha)$ . A regular element is contained in a unique Cartan subspace and also in a unique Weyl chamber.

**2.3. Cartan matrix and Dynkin diagram.** The structure of the root system can be efficiently encoded using the Cartan matrix or the Dynkin diagram.

The *Cartan matrix* of  $\mathfrak{g}$  is the square matrix of size  $\dim \mathfrak{a} = \#\Delta$  whose coefficients  $(A_{\alpha,\beta})_{\alpha,\beta \in \Delta}$  are given by the formula  $A_{\alpha,\beta} = 2(\alpha|\beta)/(\alpha|\alpha)$ .

The Cartan matrix of  $\mathfrak{g}$  is block diagonal with diagonal blocks being the Cartan matrices of  $\mathfrak{g}_s$  (for  $s$  in  $\mathcal{S}$ ) and the Cartan matrices of  $\mathfrak{g}_s$  are irreducible (not block diagonal even up to permutation).

The *Dynkin diagram* of  $\mathfrak{g}$  is the graph whose vertex set is  $\Delta$  and the edges are given by the following recipe: for every  $\alpha \neq \beta$  in  $\Delta$ ,  $A_{\alpha,\beta}A_{\beta,\alpha}$  edges are added between  $\alpha$  and  $\beta$ , an arrow is added from  $\alpha$  to  $\beta$  in the case when  $A_{\alpha,\beta}A_{\beta,\alpha} \neq 0$  and  $(\alpha|\alpha) > (\beta|\beta)$ . The connected components of the Dynkin diagram of  $\mathfrak{g}$  are the Dynkin diagrams of  $\mathfrak{g}_s$  ( $s \in \mathcal{S}$ ). The Dynkin diagram of  $\mathfrak{g}_s$  is among the following diagrams:



We already observed that the root spaces  $\mathfrak{g}_\alpha$  are not necessarily 1-dimensional. Therefore, contrary to the situation for complex simple Lie algebras, the Dynkin diagram alone does not characterize the real Lie algebra, additional information is needed.

**2.4. Non-reduced root systems.** A root system is said to be *reduced* if given any root  $\alpha \in \Sigma$ , the multiples of  $\alpha$  that are in  $\Sigma$  are  $\pm\alpha$ . Most root systems are reduced, and when a simple real Lie algebra has non-reduced type then it is necessarily Hermitian of non-tube type and the corresponding root system is of type  $BC_n$ : if  $\{\alpha_1, \dots, \alpha_n\}$  are the simple roots then  $2\alpha_n$  is also a root.

In this case,  $\mathfrak{g}_{2\alpha_n}$  is again 1-dimensional but  $\mathfrak{g}_{\alpha_n}$  is not, in fact the Lie bracket from  $\mathfrak{g}_{\alpha_n} \times \mathfrak{g}_{\alpha_n}$  to  $\mathfrak{g}_{2\alpha_n}$  is onto.

**2.5. The  $\mathfrak{sl}_2$ -triples.** Even though the root spaces  $\mathfrak{g}_\alpha$  are not necessarily 1-dimensional, their elements lead to embeddings of 3-dimensional simple subalgebras.

For every root  $\alpha \in \Sigma$  and any nonzero element  $X$  in  $\mathfrak{g}_\alpha$ , the Lie bracket  $[\tau(X), X]$  belongs to  $\mathfrak{a}$ ; there is furthermore a (unique) positive multiple  $Y$  of  $X$  such that

$$(Y, \tau(-Y), [\tau(Y), Y])$$

is an  $\mathfrak{sl}_2$ -triple, that is a triple  $(E, F, D)$  such that  $D = [E, F]$ ,  $[D, E] = 2E$ , and  $[D, F] = -2F$ .

The set of elements  $X$  in  $\mathfrak{g}_\alpha$  such that  $(X, \tau(-X), [\tau(X), X])$  is an  $\mathfrak{sl}_2$ -triple is precisely equal to the sphere  $S(\mathfrak{g}_\alpha)$  in  $\mathfrak{g}_\alpha$  defined by the equation  $B(X, \tau(X)) = -2/(\alpha|\alpha)$  (cf. [Kna02, Prop. 6.52, p. 379]) The element  $H_\alpha = [\tau(X), X]$  does not depend on the initial choice of  $X$  in  $S(\mathfrak{g}_\alpha)$ . When  $\alpha, \beta$ , and  $\alpha + \beta$  belong to  $\Sigma$ , one has  $H_{\alpha+\beta} = H_\alpha + H_\beta$ , furthermore  $H_{-\alpha} = -H_\alpha$ .

When  $\alpha$  and  $\beta$  belong to  $\Sigma$ , the suprema

$$p = \sup\{n \in \mathbf{N} \mid \beta - n\alpha \in \Sigma\}$$

and

$$q = \sup\{n \in \mathbf{N} \mid \beta + n\alpha \in \Sigma\}$$

are finite and the subset  $\{\beta + n\alpha\}_{n=-p, \dots, q}$  is contained in  $\Sigma$  and is called the  $\alpha$ -chain containing  $\beta$ . The  $\alpha$ -chain is called *trivial* when  $p = q = 0$ . The significance of the  $\alpha$ -chain  $\{\beta + n\alpha\}_{n=-p, \dots, q}$  is that the direct sum  $\bigoplus_{n=-p}^q \mathfrak{g}_{\beta+n\alpha}$  is invariant by bracketing with elements in  $\mathfrak{g}_\alpha \cup \mathfrak{g}_{-\alpha}$ , in particular it is invariant by the  $\mathfrak{sl}_2$ -triple  $(X, -\tau(X), [\tau(X), X])$  for every  $X$  in  $S(\mathfrak{g}_\alpha)$ .

At different places, we will use basic knowledge about  $\mathfrak{sl}_2(\mathbf{R})$ -modules and the fact that there is, up to isomorphism, a unique  $\mathfrak{sl}_2(\mathbf{R})$ -module of dimension  $n$ . An  $\mathfrak{sl}_2(\mathbf{R})$ -module structure on a vector space  $V$  is equivalently given by an  $\mathfrak{sl}_2$ -triple  $(X, Y, H)$  in  $\mathfrak{gl}(V) = \text{End}(V)$ ; if an element  $v \in V$  satisfies  $X(v) = 0$  and is an eigenvector for  $H$ , it generates an irreducible module  $W$  which is of dimension  $d + 1$  where  $d$  is



the nonnegative integer such that  $H(v) = dv$ ; a basis of  $W$  is then  $\{Y^k(v)\}_{k \in \llbracket 0, d \rrbracket}$ ; the Lie algebra morphism  $\mathfrak{sl}_2 \rightarrow \text{End}(W)$  lifts to an homomorphism  $\text{SL}_2(\mathbf{R}) \rightarrow \text{GL}(W)$ , this homomorphism factors through  $\text{PSL}_2(\mathbf{R})$  if and only if  $d$  is even (and  $W$  is odd dimensional). The element  $\exp(\frac{\pi}{2}(X - Y))$  is an automorphism of  $W$ , of order 2 if  $d$  is even, and that sends the  $H$ -eigenspace with respect to the eigenvalue  $k$  onto the  $H$ -eigenspace with respect to  $-k$ .

**2.6. The Weyl group.** The subgroup  $W \subset \text{GL}(\mathfrak{a}^*)$  of automorphisms of  $\Sigma$  is called the (*restricted*) *Weyl group* of  $G$ . We therefore have an action  $(w, \alpha) \mapsto w(\alpha)$  of  $W$  on  $\Sigma$ . The group  $W$  is a subgroup of the orthogonal group for the scalar product  $(\cdot | \cdot)$  induced by the Killing form  $B$ . It is a finite Coxeter group generated by the orthogonal reflections  $\{s_\alpha\}_{\alpha \in \Delta}$  such that  $s_\alpha(\alpha) = -\alpha$ . The Weyl group  $W$  is (isomorphic to) the direct product of the Weyl groups  $W_s$  of  $G_s$ . The Dynkin diagram determines the Coxeter presentation of the Weyl group. Namely a presentation of  $W$  is given by the following relations  $s_\alpha^2 = e$  ( $\alpha \in \Delta$ ) and  $(s_\alpha s_\beta)^{n_{\alpha, \beta}} = e$  ( $\alpha \neq \beta$ ) where  $n_{\alpha, \beta} = 2, 3, 4, 6$  if the number of edges between  $\alpha$  and  $\beta$  in the Dynkin diagram is respectively 0, 1, 2, 3. The following relations in  $W$  ( $\alpha \neq \beta$ )

$$\begin{aligned} s_\alpha s_\beta &= s_\beta s_\alpha \text{ if } n_{\alpha, \beta} = 2 \\ s_\alpha s_\beta s_\alpha &= s_\beta s_\alpha s_\beta \text{ if } n_{\alpha, \beta} = 3 \\ s_\alpha s_\beta s_\alpha s_\beta &= s_\beta s_\alpha s_\beta s_\alpha \text{ if } n_{\alpha, \beta} = 4 \\ s_\alpha s_\beta s_\alpha s_\beta s_\alpha s_\beta &= s_\beta s_\alpha s_\beta s_\alpha s_\beta s_\alpha \text{ if } n_{\alpha, \beta} = 6; \end{aligned}$$

are called the *braid relations* and play an important role.

Note that the Weyl groups of type  $B_n$  and  $C_n$  are the same since the orientation of the arrows do not contribute to the presentation of the Weyl group.

*Remark 2.1.* When the simple restricted roots are numbered, i.e. when  $\Delta = \{\alpha_1, \alpha_2, \dots, \alpha_r\}$ , we will rather write  $s_i$  instead of  $s_{\alpha_i}$ .

The closed Weyl chamber  $\bar{\mathfrak{a}}^+$  is a fundamental domain for the action of  $W$  on  $\mathfrak{a}$ . It is also the convex cone generated by  $\{H_\alpha\}_{\alpha \in \Delta}$ .

The Weyl group  $W$  is isomorphic to the quotient of the normalizer  $N_K(\mathfrak{a})$  of  $\mathfrak{a}$  in  $K$  by the centralizer  $C_K(\mathfrak{a})$ . On different occasions for an element  $w$  in  $W$  we will denote by  $\dot{w}$  an element of  $N_K(\mathfrak{a})$  representing it. For every  $\alpha$  in  $\Sigma$ , one has  $\text{Ad}(\dot{w})\mathfrak{g}_\alpha = \mathfrak{g}_{w(\alpha)}$ .

By no means the assignment  $w \mapsto \dot{w}$  is an homomorphism. It will sometimes be convenient to have explicit formula for these lifts. For example, when  $\alpha$  is a root, we will take  $\dot{s}_\alpha = \exp(\frac{\pi}{2}(X + \tau(X)))$  for  $X$  in  $S(\mathfrak{g}_\alpha)$  [Kna02, Prop. 6.52]. With this choice, the automorphism  $\text{Ad}(\dot{s}_\alpha)$  of  $\mathfrak{g}$  is of order 2 or 4 according to whether the corresponding morphism from  $\mathfrak{sl}_2(\mathbf{R})$  to  $\mathfrak{g}$  is tangent to a representation of  $\text{PSL}_2(\mathbf{R})$  or not.

The above generating set  $\{s_\alpha\}_{\alpha \in \Delta}$  of  $W$  allow to define the word length function  $\ell: W \rightarrow \mathbf{N}$ . It is well known that  $W$  admits a unique element  $w_\Delta$  of longest length. This element  $w_\Delta$  is of order 2, it is the element of  $W$  (that we now see as acting on  $\mathfrak{a}$ ) sending the Weyl chamber  $\mathfrak{a}^+$  to its opposite  $-\mathfrak{a}^+$ . If  $w_{\Delta_s}$  denotes the longest length

element of  $W_s$ , then the  $w_{\Delta_s}$  pairwise commute and one has  $w_{\Delta} = \prod_{\sigma \in \mathcal{S}} w_{\Delta_s}$ .

The map  $\iota : \alpha \mapsto -w_{\Delta}(\alpha)$  is an involution of  $\Sigma$ , it sends  $\Delta$  to  $\Delta$  and is called the *opposition involution*. If  $\iota_s$  is the opposition involution of  $\Sigma_s$ , one has  $\iota = \prod_{s \in \mathcal{S}} \iota_s$  under the identification  $\Sigma = \prod_{s \in \mathcal{S}} \Sigma_s$ .

Given an element  $w$  in  $W$  of length  $n = \ell(w)$ , an  $n$ -tuple  $(\alpha_1, \dots, \alpha_n)$  in  $\Delta^n$  such that  $w = s_{\alpha_1} \cdots s_{\alpha_n}$  is called a *reduced expression* of  $w$ . We will sometimes say that  $s_{\alpha_1} \cdots s_{\alpha_n}$  is a reduced expression of  $w$ . An important property is that all the reduced expressions of  $w$  can be obtained from one another by successively applying braid relations.

When  $w, w_1$ , and  $w_2$  are elements of  $W$  such that  $w = w_1 w_2$  and  $\ell(w) = \ell(w_1) + \ell(w_2)$ , we will say that  $w_1 w_2$  is a reduced product of  $w$ .

**2.7. Parabolic subgroups.** Let  $\Theta \subset \Delta$  be a subset of the set of simple roots. Setting  $\Sigma_{\Theta}^+ = \Sigma^+ \setminus \text{Span}(\Delta \setminus \Theta)$ , the subspaces

$$\mathfrak{u}_{\Theta} = \sum_{\alpha \in \Sigma_{\Theta}^+} \mathfrak{g}_{\alpha}, \quad \mathfrak{u}_{\Theta}^{\text{opp}} = \sum_{\alpha \in \Sigma_{\Theta}^+} \mathfrak{g}_{-\alpha},$$

are nilpotent Lie subalgebras and the subspace

$$\mathfrak{l}_{\Theta} = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}) \oplus \sum_{\alpha \in \text{Span}(\Delta \setminus \Theta)} \mathfrak{g}_{\alpha} = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}) \oplus \bigoplus_{\alpha \in \text{Span}(\Delta \setminus \Theta) \cap \Sigma^+} (\mathfrak{g}_{\alpha} \oplus \mathfrak{g}_{-\alpha})$$

is a reductive subalgebra.

The *standard parabolic subgroup*  $P_{\Theta}$  associated with  $\Theta \subset \Delta$  is the normalizer in  $G$  of  $\mathfrak{u}_{\Theta}$ ; it is also the normalizer of the Lie algebra  $\mathfrak{p}_{\Theta} = \mathfrak{l}_{\Theta} \oplus \mathfrak{u}_{\Theta}$ ; one has  $\mathfrak{p}_{\Theta} = \text{Lie}(P_{\Theta})$  and  $P_{\Theta}$  is its own normalizer. We also denote by  $P_{\Theta}^{\text{opp}}$  the normalizer in  $G$  of  $\mathfrak{u}_{\Theta}^{\text{opp}}$ ;  $P_{\Theta}^{\text{opp}}$  is called the *standard opposite parabolic subgroup* associated with  $\Theta$ . The longest length element  $w_{\Delta}$  conjugates opposite parabolic and parabolic subgroups:  $\text{Ad}(\dot{w}_{\Delta})\mathfrak{p}_{\Theta}^{\text{opp}} = \mathfrak{p}_{\iota(\Theta)}$ ,  $\text{Ad}(\dot{w}_{\Delta})\mathfrak{p}_{\Theta} = \mathfrak{p}_{\iota(\Theta)}$ ,  $\text{Ad}(\dot{w}_{\Delta})\mathfrak{u}_{\Theta}^{\text{opp}} = \mathfrak{u}_{\iota(\Theta)}$ ,  $\text{Ad}(\dot{w}_{\Delta})\mathfrak{u}_{\Theta} = \mathfrak{u}_{\iota(\Theta)}$ ,  $\dot{w}_{\Delta}P_{\Theta}^{\text{opp}}\dot{w}_{\Delta}^{-1} = P_{\iota(\Theta)}$ , and  $\dot{w}_{\Delta}P_{\Theta}\dot{w}_{\Delta}^{-1} = P_{\iota(\Theta)}$ , where  $\iota$  is the opposition involution  $\alpha \mapsto -w_{\Delta}(\alpha)$ .

The group  $P_{\Theta}$  is the semidirect product of its unipotent radical  $U_{\Theta} = \exp(\mathfrak{u}_{\Theta})$  and the Levi subgroup  $L_{\Theta} = P_{\Theta} \cap P_{\Theta}^{\text{opp}}$ . We denote by  $L_{\Theta}^{\circ}$  the connected component of the identity in  $L_{\Theta}$ . The Lie algebra of  $L_{\Theta}$  is  $\mathfrak{l}_{\Theta}$ . With this convention  $P_{\emptyset} = G$  and  $P_{\Delta}$  is the minimal parabolic subgroup, more generally, if  $\Theta \subset \Psi$ , one has  $P_{\Theta} \supset P_{\Psi}$  and  $L_{\Theta} \supset L_{\Psi}$ .

Setting  $\Theta_s = \Delta_s \cap \Theta$  and adopting similar notation with respect to the group  $G_s$  and the Lie algebra  $\mathfrak{g}_s$ , one has that  $\mathfrak{u}_{\Theta}$  is the direct sum  $\bigoplus_{s \in \mathcal{S}} \mathfrak{u}_{\Theta_s}$  and also that  $\mathfrak{l}_{\Theta} = \bigoplus_{s \in \mathcal{S}} \mathfrak{l}_{\Theta_s}$ ,  $\mathfrak{p}_{\Theta} = \bigoplus_{s \in \mathcal{S}} \mathfrak{p}_{\Theta_s}$ . The (unipotent) Lie group  $U_{\Theta}$  is the product of the Lie groups  $U_{\Theta_s}$ , the Lie group  $L_{\Theta}$  is the almost product of the Lie groups  $L_{\Theta_s}$  and the Lie group  $P_{\Theta}$  is the almost product of the Lie groups  $P_{\Theta_s}$ . Similar statements hold for  $U_{\Theta}^{\text{opp}}$  and  $P_{\Theta}^{\text{opp}}$ .

When  $\Theta_s = \emptyset$ , the group  $G_s$  is included in  $P_{\Theta}$  and this means that  $G_s$  acts trivially on  $G/P_{\Theta}$ . As we would like to work with (virtually) faithful actions of  $G$ , we will rule out this case in Section 3. The Lie algebra  $\mathfrak{l}_{\Theta}$  is stable under the Cartan involution  $\tau$ , the intersection  $\mathfrak{k}_{\Theta} = \mathfrak{l}_{\Theta} \cap \mathfrak{k}$  is a maximal compact subalgebra of  $\mathfrak{l}_{\Theta}$ , and in fact  $\mathfrak{k}$  is the

unique compact maximal subalgebra of  $\mathfrak{g}$  containing  $\mathfrak{k}_\Theta$ . One has

$$(2.1) \quad \mathfrak{k}_\Theta = \mathfrak{m} \oplus \bigoplus_{\alpha \in \text{Span}(\Delta \setminus \Theta) \cap \Sigma^+} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \cap \mathfrak{k},$$

$$(2.2) \quad (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \cap \mathfrak{k} = \{X + \tau(X)\}_{X \in \mathfrak{g}_\alpha}, \quad \forall \alpha \in \Sigma^+.$$

The derived subgroup  $S_\Theta = [L_\Theta, L_\Theta]$  of  $L_\Theta$  is a semisimple real Lie group whose system of restricted roots is given by the Dynkin diagram  $\Delta \setminus \Theta$  (meaning that the vertices in  $\Theta$  are removed as well as the edges incident to those vertices). A Cartan subspace for  $S_\Theta$  is

$$(2.3) \quad \mathfrak{a}_\Theta = \bigoplus_{\alpha \in \Delta \setminus \Theta} \mathbf{R}H_\alpha$$

and a Weyl chamber in this Cartan subspace is  $\sum_{\alpha \in \Delta \setminus \Theta} \mathbf{R}_{>0}H_\alpha$ .

**2.8. The flag variety.** A subgroup of  $G$  conjugated to  $P_\Theta$  (for some  $\Theta \subset \Delta$ ) is a *parabolic subgroup*. A *parabolic subalgebra* of  $\mathfrak{g}$  is a subalgebra conjugated to some  $\mathfrak{p}_\Theta$ , such a subalgebra will sometimes be called of *type*  $\Theta$ . The space of all parabolic subalgebras of type  $\Theta$  is called the *flag variety* (of type  $\Theta$ ) and denoted by  $F_\Theta$ . The space  $F_\Theta$  is a subset of a Grassmanian variety of the real vector space  $\mathfrak{g}$  and is endowed with the induced topology. The flag variety  $F_\Theta$  is compact and the conjugation action of  $G$  on the subalgebras of  $\mathfrak{g}$  induces a continuous and transitive action on  $F_\Theta$ ;  $F_\Theta$  identifies with  $G/P_\Theta$  since  $P_\Theta$  is the stabilizer of  $\mathfrak{p}_\Theta$ ; it also identifies with the space of parabolic subgroups of type  $\Theta$ .

The group  $G$  acts diagonally on  $F_\Theta \times F_{\iota(\Theta)}$ . Since the parabolic subalgebra  $\mathfrak{p}_\Theta^{\text{opp}}$  is conjugate to  $\mathfrak{p}_{\iota(\Theta)} = \text{Ad}(\dot{w}_\Delta)\mathfrak{p}_\Theta^{\text{opp}}$ , the parabolic subalgebra  $\mathfrak{p}_\Theta^{\text{opp}}$  is an element of  $F_{\iota(\Theta)}$ . A pair  $(x, y)$  in  $F_\Theta \times F_{\iota(\Theta)}$  will be called *transverse* if it belongs to the  $G$ -orbit of  $(\mathfrak{p}_\Theta, \mathfrak{p}_\Theta^{\text{opp}})$ . By definition, there is one orbit of transverse pairs; this orbit is isomorphic to  $G/L_\Theta$  and is in fact the unique open  $G$ -orbit in the product  $F_\Theta \times F_{\iota(\Theta)}$ .

In many cases below, the involution  $\iota$  is the identity on  $\Delta$ . In such a case, or more generally under the assumption  $\iota(\Theta) = \Theta$ , the notion of transversality makes sense for pairs of elements in  $F_\Theta$ .

The flag variety  $F_\Theta$  is isomorphic to the product  $\prod_{s \in \mathcal{S}} F_{\Theta_s}$  and this isomorphism is equivariant with respect to the action of  $\prod_{s \in \mathcal{S}} G_s$ . The flag variety  $F_{\iota(\Theta)}$  identifies with  $\prod_{s \in \mathcal{S}} F_{\iota(\Theta_s)}$  and a pair  $(x, y)$  in  $F_\Theta \times F_{\iota(\Theta)}$  is transverse if and only if, for all  $s$ , the  $s$ -component of  $x$  is transverse to the  $s$ -component of  $y$ .

**2.9. The action of  $L_\Theta$  on  $\mathfrak{u}_\Theta$ .** This paragraph introduces the decomposition of the Lie algebras  $\mathfrak{u}_\Theta$  and  $\mathfrak{u}_\Theta^{\text{opp}}$  under the action of the Levi factor  $L_\Theta$ . This decomposition generalizes the root space decomposition; it will be of crucial importance for the definition of  $\Theta$ -positivity (Definition 3.1).

The Levi subgroup  $L_\Theta$  acts via the adjoint action on  $\mathfrak{u}_\Theta$ . Let  $\mathfrak{z}_\Theta$  denote the center of  $\mathfrak{l}_\Theta$  and  $\mathfrak{t}_\Theta = \mathfrak{z}_\Theta \cap \mathfrak{a}$  its intersection with the Cartan subspace. One has

$$\mathfrak{t}_\Theta = \bigcap_{\alpha \in \Delta \setminus \Theta} \ker \alpha,$$

and  $\mathfrak{a}$  is the  $B$ -orthogonal sum of  $\mathfrak{t}_\Theta$  and of  $\mathfrak{a}_\Theta$  (Equation (2.3)). Then  $\mathfrak{u}_\Theta$  decomposes into the weight spaces under the adjoint action of  $\mathfrak{t}_\Theta$ ; for every  $\beta \in \mathfrak{t}_\Theta^*$ , set

$$\mathfrak{u}_\beta := \{N \in \mathfrak{g} \mid \text{ad}(Z)N = \beta(Z)N, \forall Z \in \mathfrak{t}_\Theta\},$$

and denote

$$(2.4) \quad P(\mathfrak{u}_\Theta, \mathfrak{t}_\Theta) = \{\beta \in \mathfrak{t}_\Theta^* \mid \mathfrak{u}_\beta \neq 0\}$$

so that

$$\mathfrak{u}_\Theta = \bigoplus_{\beta \in P(\mathfrak{u}_\Theta, \mathfrak{t}_\Theta)} \mathfrak{u}_\beta.$$

These weight spaces are of course related to those of  $\mathfrak{a}$ :

$$(2.5) \quad \mathfrak{u}_\beta = \bigoplus_{\alpha \in \Sigma, \alpha|_{\mathfrak{t}_\Theta} = \beta} \mathfrak{g}_\alpha,$$

and are invariant under  $L_\Theta^\circ$ .

By a little abuse of notation, for an element  $\beta$  in  $\Sigma$ , we will denote as well by  $\beta$  its restriction to  $\mathfrak{t}_\Theta$  and hence by  $\mathfrak{u}_\beta$  the corresponding weight space. For such a  $\beta$ , one has

$$(2.6) \quad \mathfrak{u}_\beta = \bigoplus_{\substack{\alpha \in \Sigma \\ \alpha - \beta \in \text{Span}(\Delta \setminus \Theta)}} \mathfrak{g}_\alpha.$$

We will make use of the spaces  $\mathfrak{u}_\beta$  particularly when the element  $\beta$  belongs to  $\Theta$ . Thus, when convenient, we will consider  $\Theta$  as a subset of  $P(\mathfrak{u}_\Theta, \mathfrak{t}_\Theta)$ .

The following results are established in [Kos10]:

- Theorem 2.2.** (1) For every  $\beta$  in  $P(\mathfrak{u}_\Theta, \mathfrak{t}_\Theta)$ ,  $\mathfrak{u}_\beta$  is an irreducible representation of  $L_\Theta^\circ$ .  
 (2) For every  $\beta$  and  $\beta'$  in  $P(\mathfrak{u}_\Theta, \mathfrak{t}_\Theta)$ , the relation  $[\mathfrak{u}_\beta, \mathfrak{u}_{\beta'}] = \mathfrak{u}_{\beta+\beta'}$  is satisfied.  
 (3) The Lie algebra  $\mathfrak{u}_\Theta$  is generated by  $\mathfrak{u}_\beta$ ,  $\beta \in \Theta$ . Analogously  $\mathfrak{u}_\Theta^{\text{opp}}$  is generated by  $\mathfrak{u}_{-\beta}$ ,  $\beta \in \Theta$ .

**2.10. Examples.** In these examples, we use classical notation and numbering for the root systems as set for example in [Hel01, Ch. X, §3–4, p. 461–475] or in [Bou07, Ch. VI, Planches and §4, n° 14]

*Split real Lie groups.* Let  $G$  be a split real form, and  $\Theta = \Delta$ . Then  $\mathfrak{t}_\Theta = \mathfrak{a}$ ,  $P(\mathfrak{u}_\Theta, \mathfrak{t}_\Theta) = P(\mathfrak{u}_\Delta, \mathfrak{a}) = \Sigma^+$ , and  $\mathfrak{u}_\beta = \mathfrak{g}_\beta$  for all  $\beta \in \Sigma^+$ .

*Lie groups of Hermitian type.* Let  $G$  be a Lie group of Hermitian type. Then the root system is of type  $C_r$  if  $G$  is of tube type, and of type  $BC_r$  (non-reduced) if  $G$  is not of tube type. Let  $\Delta = \{\alpha_1, \dots, \alpha_r\}$ , and let  $\Theta = \{\alpha_r\}$  so that  $P_\Theta$  is the stabilizer of a point in the Shilov boundary  $F_\Theta$  of the Hermitian symmetric space associated with  $G$ . Then  $\mathfrak{t}_\Theta \simeq \mathbf{R}$ , furthermore  $P(\mathfrak{u}_\Theta, \mathfrak{t}_\Theta) = \{\alpha_r\}$  and  $\mathfrak{u}_\Theta = \mathfrak{u}_{\alpha_r}$  if  $G$  is of tube type, whereas  $P(\mathfrak{u}_\Theta, \mathfrak{t}_\Theta) = \{\alpha_r, 2\alpha_r\}$  and  $\mathfrak{u}_\Theta = \mathfrak{u}_{\alpha_r} \oplus \mathfrak{u}_{2\alpha_r}$  if  $G$  is not of tube type. Note that if  $G$  is not of tube type, there is a unique (up to conjugation) maximal subgroup of tube type, whose Lie algebra is generated by  $\mathfrak{g}_{\pm\alpha_i}$  for  $i < r$  and by  $\mathfrak{g}_{\pm 2\alpha_r}$ .

Let  $G$  be a Hermitian Lie group of tube type. In this case  $\Theta = \{\alpha_r\}$ , and  $\mathfrak{u}_\Theta = \mathfrak{u}_{\alpha_r}$  is Abelian. We now describe  $\mathfrak{u}_{\alpha_r}$  in more detail when

the real rank of  $G$  is 2 or 3. These cases are of particular importance because they also arise when considering the positive structures for indefinite orthogonal groups  $\mathrm{SO}(p, q)$ ,  $2 < p < q$ , and for the exceptional groups with system or restricted roots  $F_4$  (cf. Sections 3.5 and 5.1).

For a Hermitian Lie group of tube type of real rank 2 the roots that are equal to  $\alpha_2$  modulo  $\mathbf{R}\alpha_1$  are  $\alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2 = s_1(\alpha_2)$ , and  $\mathfrak{u}_{\alpha_2} = \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_1 + \alpha_2} \oplus \mathfrak{g}_{2\alpha_1 + \alpha_2}$ , where  $\dim \mathfrak{g}_{\alpha_2} = 1 = \dim \mathfrak{g}_{2\alpha_1 + \alpha_2}$ ; the integer  $a = \dim \mathfrak{g}_{\alpha_1 + \alpha_2}$  is not necessarily equal to 1. For the symplectic group  $\mathrm{Sp}(4, \mathbf{R})$ ,  $a = 1$ , for  $\mathrm{SU}(2, 2)$ ,  $a = 2$ , for  $\mathrm{SO}^*(8)$ ,  $a = 4$ , and for  $\mathrm{SO}(2, m)$ ,  $a = m - 2$  (in fact this last family contains, up to isogeny, the first three examples, cf. [Hel01, p. 519]). In this case,  $\mathfrak{u}_{\alpha_2}$  is equipped with a quadratic form of signature  $(1, a + 1)$  that is invariant under the adjoint action of the Levi factor (see Section 5.1 for more details).

For a Hermitian Lie group of tube type of real rank 3 the roots that are equal to  $\alpha_3$  modulo the span of  $\{\alpha_1, \alpha_2\}$  are  $\alpha_3, \alpha_2 + \alpha_3, 2\alpha_2 + \alpha_3 = s_2(\alpha_3), \alpha_1 + \alpha_2 + \alpha_3 = s_1(\alpha_2 + \alpha_3), \alpha_1 + 2\alpha_2 + \alpha_3 = s_2 s_1(\alpha_2 + \alpha_3), 2\alpha_1 + 2\alpha_2 + \alpha_3 = s_1 s_2(\alpha_3)$ , so  $\mathfrak{u}_{\alpha_3} = \mathfrak{g}_{\alpha_3} \oplus \mathfrak{g}_{\alpha_2 + \alpha_3} \oplus \mathfrak{g}_{\alpha_1 + \alpha_2 + \alpha_3} \oplus \mathfrak{g}_{\alpha_1 + 2\alpha_2 + \alpha_3} \oplus \mathfrak{g}_{2\alpha_2 + \alpha_3} \oplus \mathfrak{g}_{2\alpha_1 + 2\alpha_2 + \alpha_3}$ , where the spaces  $\mathfrak{g}_{\alpha_3}, \mathfrak{g}_{2\alpha_2 + \alpha_3}$ , and  $\mathfrak{g}_{2\alpha_1 + 2\alpha_2 + \alpha_3}$  are 1-dimensional and the spaces  $\mathfrak{g}_{\alpha_1 + \alpha_2 + \alpha_3}, \mathfrak{g}_{\alpha_2 + \alpha_3}$ , and  $\mathfrak{g}_{\alpha_1 + 2\alpha_2 + \alpha_3}$  have dimension  $a$ . For the symplectic group  $\mathrm{Sp}(6, \mathbf{R})$ ,  $a = 1$ , for  $\mathrm{SU}(3, 3)$ ,  $a = 2$ , for  $\mathrm{SO}^*(12)$ ,  $a = 4$ , and for  $E_{7(-25)}$ ,  $a = 8$ . In these cases, the Lie algebra  $\mathfrak{u}_{\alpha_3}$  can be identified with the space of Hermitian  $3 \times 3$  matrices, over the reals ( $a = 1$ ), the complex numbers ( $a = 2$ ), the quaternions ( $a = 4$ ), and the octonions ( $a = 8$ ) respectively.

### 3. $\Theta$ -POSITIVE STRUCTURES

In this section we give the definition of  $\Theta$ -positive structures and their classification. We also prove that the notion of  $\Theta$ -positivity reduces to the case of simple Lie groups so that we will only focus on simple Lie groups later on. The first algebraic properties of  $\Theta$ -positivity are then derived: the existence of a “special” root and of a subgroup of Hermitian tube type (when  $\Theta \neq \Delta$ ), and the commutativity of the  $\mathfrak{u}_\alpha$  ( $\alpha \in \Theta$ ).

**3.1. Definition and characterization.** Recall that a convex cone is *acute* if the only vector space contained in its closure is  $\{0\}$  and is *nontrivial* if it contains a nonzero vector.

**Definition 3.1.** Let  $G$  be a semisimple Lie group with finite center. Let  $\Theta \subset \Delta$  be a subset of simple roots such that  $G$  acts virtually faithfully on the flag variety  $F_\Theta$ . We say that  $G$  admits a  *$\Theta$ -positive structure* if for all  $\beta$  in  $\Theta$  there exists an  $L_\Theta^\circ$ -invariant acute nontrivial convex cone in  $\mathfrak{u}_\beta$ .

Note that such invariant cones must be of nonempty interior since the action of  $L_\Theta^\circ$  on  $\mathfrak{u}_\beta$  is irreducible (Theorem 2.2).

*Remarks 3.2.* • In general it is not possible to request that the cones are invariant under the entire Levi factor  $L_\Theta$ . If the space  $\mathfrak{u}_\beta$  contains an  $L_\Theta^\circ$ -invariant cone, then it contains exactly two  $L_\Theta^\circ$ -invariant cones,  $c_\beta$  and  $-c_\beta$  (cf. point (2) in Proposition 3.6 below). In some cases there is an element in  $L_\Theta$  that interchanges  $c_\beta$  and  $-c_\beta$ , see Remark 5.4 for more details.

- The faithfulness condition says that  $\Theta_s = \Theta \cap \Delta_s$  is nonempty for every factor  $G_s$  of  $G$  (cf. Section 2.7).

**Proposition 3.3.** *Let  $\{G_s\}_{s \in \mathcal{S}}$  be the simple factors of  $G$  and use the notation introduced in Section 2:  $\Theta_s \subset \Theta$ , etc.*

*Then  $G$  admits a  $\Theta$ -positive structure if and only if, for all  $s$  in  $\mathcal{S}$ ,  $G_s$  admits a  $\Theta_s$ -positive structure.*

*Proof.* Let  $\beta$  be in  $\Theta$ . There is thus  $s$  in  $\mathcal{S}$  such that  $\beta$  belongs to  $\Theta_s$ . For every  $t$  in  $\mathcal{S}$  distinct from  $s$ , the factor  $L_{\Theta_t}^\circ$  acts trivially on  $\mathfrak{u}_\beta$ ; therefore  $\mathfrak{u}_\beta$  is an irreducible summand for the action of  $L_{\Theta_s}$ .

This said, the irreducible representations  $\mathfrak{u}_\beta$  of  $L_\Theta^\circ$  ( $\beta \in \Theta$ ) are precisely the irreducible representations  $\mathfrak{u}_\beta$  of  $L_{\Theta_s}^\circ$  ( $s \in \mathcal{S}$ ,  $\beta \in \Theta_s$ ). The equivalence now follows directly from the definition.  $\square$

Due to Proposition 3.3, a positive structure on a semisimple Lie group is completely determined by the induced positive structure on its simple factors. Therefore structural results given in the sequel of the paper (for the unipotent positive semigroup, for the configurations of positive flags, etc.) will be stated only when the group  $G$  is simple but, thanks to the product structure of the flag variety and of the unipotent subgroups, they immediately generalize to the case of semisimple Lie groups.

**We assume from now on that  $G$  is a simple Lie group.**

**Theorem 3.4.** *Let  $G$  be a simple real Lie group, and let  $\Delta$  be its set of simple roots. Let  $\Theta$  be a nonempty subset of  $\Delta$ . Then  $G$  admits a  $\Theta$ -positive structure if and only if the pair  $(G, \Theta)$  belongs to one of the following four cases:*

- (1)  $G$  is a split real form, and  $\Theta = \Delta$ ;
- (2)  $G$  is Hermitian of tube type and of real rank  $r$  and  $\Theta = \{\alpha_r\}$ , where  $\alpha_r$  is the long simple restricted root;
- (3)  $G$  is locally isomorphic to  $\mathrm{SO}(p+1, p+k)$ ,  $p > 1$ ,  $k > 1$  and  $\Theta = \{\alpha_1, \dots, \alpha_p\}$ , where  $\alpha_1, \dots, \alpha_p$  are the long simple restricted roots;
- (4)  $G$  is locally isomorphic to the real form of  $F_4$ ,  $E_6$ ,  $E_7$ , or of  $E_8$  whose system of restricted roots is of type  $F_4$ , and  $\Theta = \{\alpha_1, \alpha_2\}$ , where  $\alpha_1, \alpha_2$  are the long simple restricted roots.

*Remarks 3.5.* • In all cases, one has  $\iota(\Theta) = \Theta$  so that  $F_{\iota(\Theta)} = F_\Theta$  and the notion of transversality makes sense in  $F_\Theta$ . This will be relevant later when introducing positive tuples in  $F_\Theta$  (Section 13).

- When  $G$  is a split real form and  $\Theta = \Delta$ , the positive structure agrees with Lusztig's total positivity in split real Lie groups.
- A split real group of type  $B_n$ ,  $C_n$ , or  $F_4$  appears twice in this list of groups and carries two different positive structures, one corresponding to  $\Theta = \Delta$  and one with respect to a proper subset  $\Theta \subset \Delta$ . The relation between these two different positive structures will be discussed later (Examples 4.3 and Section 13.10).

**3.2. Even and proximal representations.** To prove Theorem 3.4, we recall the notions, for representations of reductive Lie groups, of even representations and of proximal representations and their relation with having an invariant cone. We will use these notions mainly



for representations of the semisimple factor of the Levi components of parabolic subgroups, see Section 3.3 below.

Any (continuous) representation  $\rho: G \rightarrow \mathrm{GL}(V)$  induces a Lie algebra morphism  $\rho_*: \mathfrak{g} \rightarrow \mathrm{End}(V)$ . The restriction of  $\rho_*$  to  $\mathfrak{a}$  admits a weight space decomposition

$$V = \bigoplus_{\lambda \in R(V)} V_\lambda$$

where

$$V_\lambda := \{v \in V \mid \rho_*(H)(v) = \lambda(H)v \text{ for all } H \in \mathfrak{a}\}, \text{ and} \\ R(V) := \{\lambda \in \mathfrak{a}^* \mid V_\lambda \neq \{0\}\}.$$

The representation  $\rho$  will be called *even* if  $\lambda(H_\alpha)$  is an even integer for all  $\lambda \in R(V)$  and all  $\alpha \in \Sigma$  (these numbers are already known to be integers). In classical notation this means the inclusion  $R(V) \subset 2P$  where  $P = \{\lambda \in \mathfrak{a}^* \mid \lambda(H_\alpha) \in \mathbf{Z}, \forall \alpha \in \Sigma\}$  is the weight lattice. Since  $\mathfrak{g}$  is generated by  $\mathfrak{g}_0$  and the root spaces  $\mathfrak{g}_{\pm\alpha}$  ( $\alpha \in \Delta$ ),  $\rho$  is even if and only if  $\lambda(H_\alpha) \in 2\mathbf{Z}$  for every  $\lambda$  in  $R(V)$  and every  $\alpha$  in  $\Delta$ .

The *highest weight* of the representation  $\rho$  (when it is defined) is the greatest element of  $R(V)$  for the (partial) order on  $\mathfrak{a}^*$  defined by  $\lambda \leq \mu \Leftrightarrow \lambda(H_\alpha) \leq \mu(H_\alpha)$  for all  $\alpha \in \Sigma^+$ . An irreducible representation  $\tau$  always admit a highest weight  $\lambda_\tau$  and such an irreducible representation is even if and only if  $\lambda_\tau(H_\alpha) \in 2\mathbf{Z}$  for every  $\alpha \in \Delta$ .

A representation will be called *proximal* if it admits a highest weight and if the corresponding weight space is 1-dimensional.

To prove Theorem 3.4 we make use of the following fact, extracted from [Ben00, Proposition 4.7] that refines Cartan–Helgason’s theorem.

**Proposition 3.6.** (1) *Let  $S$  be a connected semisimple Lie group,  $V$  a real vector space, and  $\pi: S \rightarrow \mathrm{GL}(V)$  an irreducible representation. Then  $S$  preserves an acute closed convex cone  $c$  in  $V$  if and only if the representation  $\pi$  is proximal and even.*  
 (2) *In this case, the union of  $\mathring{c}$  and of  $-\mathring{c}$  is exactly the set of vectors  $v \in V \setminus \{0\}$  whose stabilizers contains a maximal compact subgroup of  $S$ ; likewise a vector  $v \in V \setminus \{0\}$  belongs to  $\mathring{c}$  or to  $-\mathring{c}$  as soon as the Lie algebra of its stabilizer contains a maximal compact subalgebra.*

In order to apply Proposition 3.6 to the semisimple part of a reductive Lie group, we establish the following result.

**Proposition 3.7.** *Let  $L$  be a connected reductive Lie group and let  $S = [L, L]$  be its semisimple part. Let  $\pi: L \rightarrow \mathrm{GL}(V)$  be an irreducible real representation.*

- (1) *Suppose that there is an acute  $\pi(L)$ -invariant cone in  $V$ , then the restriction  $\pi|_S$  is as well irreducible (and hence proximal and even by the previous proposition).*
- (2) *Suppose that  $\pi|_S$  is irreducible, even, and proximal (and hence there is  $c \subset V$  a  $\pi(S)$ -invariant acute closed convex cone by the previous proposition), then the cone  $c$  is equally  $\pi(L)$ -invariant.*

*Proof.* The group  $L$  is the almost direct product of  $Z^\circ$ , the identity component of its center  $Z$ , and of its semisimple part  $S$ .

Let us prove (1). By Schur's lemma, the algebra  $\text{End}_L(V)$  of  $L$ -equivariant endomorphisms of  $V$  is a skew field; in other words,  $V$  is a vector space over a potentially bigger field and the endomorphisms  $\pi(g)$ , for  $g \in L$ , are linear with respect to this field. Since there is a  $\pi(L)$ -invariant acute convex cone, the representation  $\pi$  is proximal [Ben00, Proposition 1.2] in the sense that there is an element  $g$  in  $L$  such that  $\pi(g)$  has a unique eigenvalue of highest modulus: this eigenvalue is real and the corresponding eigenspace is of real dimension 1. As this eigenspace is a vector space under the skew field  $\text{End}_L(V)$ , we deduce that  $\text{End}_L(V) = \mathbf{R}$ . Furthermore, the image  $\pi(Z^\circ)$  is contained in  $\text{End}_L(V)$  and this means that elements  $\pi(g)$ , for  $g \in Z^\circ$ , are homotheties of  $V$ . We deduce from this and the fact that  $L$  is the almost product of  $Z^\circ$  and  $S$ , that a subspace of  $V$  is  $\pi(L)$ -invariant if and only if it is  $\pi(S)$ -invariant. Since  $\pi$  is by hypothesis an irreducible representation of  $L$ , the restriction  $\pi|_S$  is thus an irreducible representation of  $S$ .

We now address (2). We need to show that  $c$  is invariant under the action of  $\pi(Z^\circ)$ . Similarly to above, the elements  $\pi(z)$ , for  $z \in Z^\circ$ , are homotheties of positive dilation so that  $\pi(z)c = c$  for any  $z \in Z^\circ$  since  $c$  is a cone. This implies that  $c$  is  $\pi(L)$ -invariant.  $\square$

**3.3. The proof of the Classification Theorem 3.4.** We apply Proposition 3.7 to the connected Lie group  $L_\Theta^\circ$  and Proposition 3.6.(1) to  $S_\Theta^\circ = [L_\Theta^\circ, L_\Theta^\circ]$  for the irreducible representations  $\pi_\beta: L_\Theta^\circ \rightarrow \text{GL}(\mathfrak{u}_\beta)$ ,  $\beta \in \Theta$ .

Recall that a Cartan subspace for  $S_\Theta^\circ$  is  $\mathfrak{a}_\Theta = \bigoplus_{\alpha \in \Delta \setminus \Theta} \mathbf{R}H_\alpha$ . For this proof it will be more convenient to work with the following Weyl chamber of  $S_\Theta^\circ$

$$\sum_{\alpha \in \Delta \setminus \Theta} -\mathbf{R}_{>0} H_\alpha$$

that is opposite to the natural one, or, what amounts to the same, we shall consider *lowest weights* rather than highest weights of representations (for the natural ordering).

The decomposition of Equation (2.6), p. 17, is

$$\mathfrak{u}_\beta = \mathfrak{g}_\beta \oplus \bigoplus_{\substack{\alpha \in \Sigma, \alpha \neq \beta \\ \alpha - \beta \in \text{Span}(\Delta \setminus \Theta)}} \mathfrak{g}_\alpha.$$

Any  $\alpha$  in the above formula differs from  $\beta$  by a linear combination of the elements in  $\Delta \setminus \Theta$  with nonnegative coefficients. This implies, up to the abuse of considering the above decomposition indexed by the elements in  $\mathfrak{a}_\Theta^*$  rather than by elements in  $\mathfrak{a}^*$ , that the above is the weight space decomposition of the  $S_\Theta^\circ$ -module  $\mathfrak{u}_\beta$ . As a consequence, the lowest weight of this representation is equal to  $\beta$  and the corresponding weight space is  $\mathfrak{g}_\beta$ .

Thus the condition that  $\pi_\beta$  has to be proximal is equivalent to the condition that the root space  $\mathfrak{g}_\beta$  is 1-dimensional. This needs to hold for all  $\beta \in \Theta$ .

The evenness condition of Proposition 3.7 translates into the fact that  $\beta(H_\alpha)$  is even for all  $\beta \in \Theta$  and all  $\alpha \in \Delta \setminus \Theta$  (cf. Section 3.2).

These necessary and sufficient criteria can be easily translated into properties of the Dynkin diagram of the system of restricted roots  $\Sigma$ :

- (1) For all  $\beta$  in  $\Theta$ , the root space  $\mathfrak{g}_\beta$  is 1-dimensional. (*Representation is proximal.*)
- (2) For all  $\beta$  in  $\Theta$ , the node of the Dynkin diagram with label  $\beta$  is either connected to the nodes in  $\Delta \setminus \Theta$  by a double arrow pointing towards  $\Delta \setminus \Theta$ , or it is not connected to  $\Delta \setminus \Theta$  at all. (*Representation is even.*)

Let us consider the two cases,  $\Theta = \Delta$  and  $\Theta \neq \Delta$ .

In the first case, when  $\Theta = \Delta$ , the evenness condition is empty. The multiplicity one condition implies that for every simple root  $\alpha \in \Delta$  the root space  $\mathfrak{g}_\alpha$  is of dimension one. This implies that  $G$  is a split real form.

In the case when  $\Theta \neq \Delta$ , the evenness condition implies that every arrow from  $\Theta$  to  $\Delta \setminus \Theta$  has to be a double arrow, pointing from  $\Theta$  to  $\Delta \setminus \Theta$ . By property of Dynkin diagrams, there can be at most one double arrow in the Dynkin diagram  $\Delta$  [Hel01, Ch. X, Lemma 3.18], and  $\Theta$  must be the set of long simple roots.

Going through the Dynkin diagrams of the system of restricted roots associated with simple real Lie groups (see e.g. [Hel01, Ch. X, § 6 and Table VI p. 532-4]) we can now read off the list of pairs  $(G, \Theta)$ . This completes the proof of Theorem 3.4.

**3.4. The special root.** By the classification of Dynkin diagrams (more accurately, from the fact that a connected Dynkin diagram admits at most one double arrow), we observe that for all pairs  $(G, \Theta)$  admitting a  $\Theta$ -positive structures, both sets  $\Theta$  and  $\Delta \setminus \Theta$  are connected.

When  $\Theta \neq \Delta$ , there is thus a unique root  $\alpha_\Theta$  in  $\Theta$  which is connected to  $\Delta \setminus \Theta$ . The root  $\alpha_\Theta$  will be called the *special root*. For all  $\alpha \in \Theta \setminus \{\alpha_\Theta\}$  we have that  $\mathfrak{u}_\alpha = \mathfrak{g}_\alpha$  and  $\dim \mathfrak{u}_\alpha = 1$ . For  $\alpha_\Theta$ , the vector space

$$\mathfrak{u}_{\alpha_\Theta} = \bigoplus_{\substack{\alpha \in \Sigma_\Theta^+ \\ \alpha|_{\mathfrak{t}_\Theta} = \alpha_\Theta|_{\mathfrak{t}_\Theta}}} \mathfrak{g}_\alpha$$

is of dimension greater than or equal to 2 since there is a least one other root in  $\Sigma$  congruent to  $\alpha_\Theta$  modulo  $\Delta \setminus \Theta$ .

**3.5. The Hermitian tube type subgroup.** Elaborating on the special root and again under the assumption  $\Theta \neq \Delta$ , we notice that the subgraph of the Dynkin diagram with vertices  $\{\alpha_\Theta\} \cup \Delta \setminus \Theta$  is of type  $C_{d+1}$  (where  $d$  is the cardinality of  $\Delta \setminus \Theta$ ). This is the type of the semisimple part  $S_{\Theta \setminus \{\alpha_\Theta\}}$  of the Levi factor  $L_{\Theta \setminus \{\alpha_\Theta\}}$ . One has then the following

**Proposition 3.8.** *Let  $G$  be a simple Lie group admitting a  $\Theta$ -positive structure with  $\Theta \neq \Delta$ . Then the subgroup  $S_{\Theta \setminus \{\alpha_\Theta\}}$  is Hermitian of tube type. Moreover  $\mathfrak{u}_{\alpha_\Theta}$  is contained in the Lie algebra of  $S_{\Theta \setminus \{\alpha_\Theta\}}$  and is the Lie algebra of the unipotent radical of the maximal parabolic subgroup  $P_\Theta \cap S_{\Theta \setminus \{\alpha_\Theta\}}$  of  $S_{\Theta \setminus \{\alpha_\Theta\}}$  associated to the root  $\alpha_\Theta$ ; a Levi factor of this parabolic subgroup of  $S_{\Theta \setminus \{\alpha_\Theta\}}$  is  $L_\Theta \cap S_{\Theta \setminus \{\alpha_\Theta\}}$ .*

Note that the maximal parabolic  $P_\Theta \cap S_{\Theta \setminus \{\alpha_\Theta\}}$  is the stabilizer of a point in the Shilov boundary of the Hermitian symmetric space associated to  $S_{\Theta \setminus \{\alpha_\Theta\}}$ .

*Proof.* Only the statement about  $\mathfrak{u}_{\alpha_\Theta}$  needs to be established.

Let  $\mathfrak{s}$  be the Lie algebra of  $S_{\Theta \setminus \{\alpha_\Theta\}}$ . Clearly,  $\mathfrak{g}_{\alpha_\Theta}$  is contained in  $\mathfrak{s}$ . Furthermore,  $L_\Theta \subset L_{\Theta \setminus \{\alpha_\Theta\}}$  and hence  $S_\Theta$  is contained in  $S_{\Theta \setminus \{\alpha_\Theta\}}$ . Thus the  $S_\Theta$ -module generated by  $\mathfrak{g}_{\alpha_\Theta}$  is contained in  $\mathfrak{s}$ . By Kostant's result (Theorem 2.2) and by Proposition 3.7, the  $S_\Theta$ -module  $\mathfrak{u}_{\alpha_\Theta}$  is irreducible. As  $\mathfrak{u}_{\alpha_\Theta}$  contains  $\mathfrak{g}_{\alpha_\Theta}$ , we obtain that the  $S_\Theta$ -module generated by  $\mathfrak{g}_{\alpha_\Theta}$  is equal to  $\mathfrak{u}_{\alpha_\Theta}$  and therefore that  $\mathfrak{u}_{\alpha_\Theta}$  is contained in  $\mathfrak{s}$ .

The Hermitian tube type Lie group  $S_{\Theta \setminus \{\alpha_\Theta\}}$  has a positive structure associated with the subset  $\Theta_S = \{\alpha_\Theta\}$  of its simple roots; in this case the equality  $\mathfrak{u}_{\Theta_S} = \mathfrak{u}_{\alpha_\Theta}$  holds.  $\square$

### 3.6. The $\mathfrak{u}_\alpha$ are Abelian.

**Proposition 3.9.** *Let  $G$  be a simple Lie group admitting a  $\Theta$ -positive structure. Then, for every  $\alpha$  in  $\Theta$ ,  $\mathfrak{u}_\alpha$  is Abelian.*

*Proof.* When  $\Theta = \Delta$ , all the  $\mathfrak{u}_\alpha$  are 1-dimensional and therefore Abelian.

When  $\Theta \neq \Delta$ , for all  $\alpha$  in  $\Theta \setminus \{\alpha_\Theta\}$ ,  $\mathfrak{u}_\alpha$  is 1-dimensional and Abelian. For the special root  $\alpha_\Theta$  the space  $\mathfrak{u}_{\alpha_\Theta}$  is equal to the unipotent factor of the parabolic subalgebra associated to the Shilov boundary of the Hermitian subgroup (Proposition 3.8) and it is known that this unipotent algebra is Abelian.  $\square$

**3.7. The case of Hermitian Lie groups that are not of tube type.** Note that simple Hermitian Lie groups which are not of tube type do not appear on the list of groups admitting a positive structure. This is because in that case the root system is non-reduced of type  $BC_n$ , and the proximality condition is not satisfied, for every simple root  $\alpha$ , the root space  $\mathfrak{g}_\alpha$  is of dimension  $> 1$ . In this case however, with  $\alpha$  is the simple root such that  $2\alpha$  is also a root, we have at our hand the space  $\mathfrak{u}_{2\alpha}$ , and the representation of  $L_\Theta^\circ$  on  $\mathfrak{u}_{2\alpha}$  is proximal (the root space  $\mathfrak{g}_{2\alpha}$  is 1-dimensional) and even.

## 4. THE NONNEGATIVE AND POSITIVE SEMIGROUPS

In this section, we fix a pair  $(G, \Theta)$  where  $G$  is a simple real Lie group which admits a  $\Theta$ -positive structure; we introduce the nonnegative semigroup  $U_\Theta^{\geq 0}$  and the positive semigroup  $U_\Theta^{> 0}$  in the unipotent radical  $U_\Theta$  of the parabolic subgroup  $P_\Theta$ . Their main properties will be established in Section 10.

For every  $\alpha \in \Theta$  we fix an  $L_\Theta^\circ$ -invariant closed cone  $c_\alpha$  in  $\mathfrak{u}_\alpha$ , and denote by  $\mathring{c}_\alpha$  its interior. For  $\alpha \in \Theta$ , the cone  $\tau(-c_\alpha)$  is contained in  $\mathfrak{u}_{-\alpha}$  ( $\tau$  is the Cartan involution) and will be denoted by  $c_\alpha^{\text{opp}}$ .

### 4.1. The nonnegative semigroup.

**Definition 4.1.** The *unipotent nonnegative semigroup*  $U_\Theta^{\geq 0} \subset U_\Theta$  is the subsemigroup of  $U_\Theta$  generated by  $\bigcup_{\alpha \in \Theta} \exp(c_\alpha)$ . The *opposite unipotent nonnegative semigroup*  $U_\Theta^{\text{opp}, \geq 0}$  is the subsemigroup generated by  $\bigcup_{\alpha \in \Theta} \exp(c_\alpha^{\text{opp}})$ .

We have  $U_\Theta^{\text{opp}, \geq 0} = \tau(U_\Theta^{\geq 0})^{-1}$  (where  $\tau$  denotes also the group homomorphism  $G \rightarrow G$  tangent to the involution  $\tau: \mathfrak{g} \rightarrow \mathfrak{g}$ ). Note that it is not clear from the definition that the unipotent nonnegative semigroup is closed, this will be proven in Section 10.

The unipotent nonnegative semigroups give rise to a semigroup in  $G$ .

**Definition 4.2.** The *nonnegative semigroup*  $G_{\Theta}^{\geq 0} \subset G$  is defined to be the subsemigroup generated by  $U_{\Theta}^{\geq 0}$ ,  $U_{\Theta}^{\text{opp}, \geq 0}$ , and  $L_{\Theta}^{\circ}$ .

*Examples 4.3.* (1) When  $G$  is a split real form, and  $\Theta = \Delta$ , then  $U_{\Theta}^{\geq 0}$  is the nonnegative unipotent semigroup defined by Lusztig in [Lus94], and  $G_{\Theta}^{\geq 0} = G^{\geq 0}$  is the semigroup of totally nonnegative elements defined by Lusztig [Lus94, § 2.2]. In the case when  $G$  is  $\text{SL}_n(\mathbf{R})$ , the nonnegative semigroup  $G_{\Theta}^{\geq 0}$  consists of the matrices, all of whose minors are nonnegative. Similar characterizations for other split real Lie groups in terms of generalized minors exist.

(2) Let  $G$  be a Hermitian Lie group of tube type and of real rank  $r$ , and let  $\Theta = \{\alpha_r\}$ . Then  $U_{\Theta}^{\geq 0} = \exp(c_{\alpha_r}) \subset U_{\Theta}$ , and  $G_{\Theta}^{\geq 0}$  is the contraction semigroup  $G^{\geq 0} \subset G$  [Kou95, Th. 4.9].

In particular note that when  $G$  is the real symplectic group of rank  $r$ , the unique (up to isogeny) rank  $r$  group which is split and of Hermitian type, the nonnegative semigroups  $G_{\Delta}^{\geq 0}$  and  $G_{\{\alpha_r\}}^{\geq 0}$  are different when  $r > 1$ . In this case the nonnegative semigroup  $G_{\{\alpha_r\}}^{\geq 0}$  admits a nice characterization. The symplectic group  $\text{Sp}(2n, \mathbf{R})$  consists of block matrices  $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$  in  $\text{GL}_{2n}(\mathbf{R})$  such that  ${}^tDA - {}^tBC = \text{Id}$ , and the matrices  ${}^tAC$  and  ${}^tDB$  are symmetric.

Such a matrix is then in the nonnegative semigroup  $\text{Sp}(2n, \mathbf{R})^{\geq 0}$  if and only if  $\det(A) > 0$  and that the (necessarily symmetric) matrices  $CA^{-1}$  and  $A^{-1}B$  are semipositive.

(3) For the case when  $G$  is a split real group locally isomorphic to  $\text{SO}(n, n+1)$  with  $n > 1$ , we have two different semigroups, the nonnegative semigroup  $G_{\Theta}^{\geq 0}$ , where  $\Theta = \Delta \setminus \{\alpha_n\}$  and Lusztig's nonnegative semigroup  $G_{\Delta}^{\geq 0}$ . No direct characterizations of the nonnegative semigroup  $G_{\Theta}^{\geq 0}$  are known yet.

(4) Also in the case when  $G$  is the split real form of  $F_4$  we have two different semigroups, the nonnegative semigroup  $G_{\Theta}^{\geq 0}$ , where  $\Theta = \{\alpha_1, \alpha_2\}$  and Lusztig's nonnegative semigroup  $G_{\Delta}^{\geq 0}$ . No direct characterizations of the nonnegative semigroup  $G_{\Theta}^{\geq 0}$  are known yet.

## 4.2. The positive semigroup.

**Definition 4.4.** The interior (relative to  $U_{\Theta}$ ) of the nonnegative semigroup  $U_{\Theta}^{\geq 0}$  is called the *unipotent positive semigroup* and denoted  $U_{\Theta}^{>0}$ . The interior of the opposite nonnegative semigroup  $U_{\Theta}^{\text{opp}, \geq 0}$  is called the *opposite unipotent positive semigroup* and denoted  $U_{\Theta}^{\text{opp}, >0}$ .

It is clear that  $U_{\Theta}^{>0}$  is a semigroup (cf. Lemma 4.6 below), but this definition gives little information about the structure of the positive semigroup. Already in Lusztig's work, the positive unipotent semigroup is not introduced in this way, but is defined given explicit parametrizations of a subset  $U^{>0}$  of  $U$ , which is then shown to be a semigroup and to agree with the interior of  $U^{\geq 0}$ . We will give explicit parametrizations of the unipotent positive semigroup  $U_{\Theta}^{>0}$  in Section 10.

In the case of Hermitian Lie groups of tube type the situation simplifies drastically, because then  $\Theta = \{\alpha_{\Theta}\}$ ,  $U_{\Theta}^{\geq 0} = \exp(c_{\alpha_{\Theta}})$ , and thus  $U_{\Theta}^{>0} = \exp(\overset{\circ}{c}_{\alpha_{\Theta}})$ .

Similarly to the nonnegative semigroup in  $G$ , there is also a positive semigroup in  $G$ .

**Definition 4.5.** The interior of  $G_{\Theta}^{\geq 0}$  is called the *positive semigroup* and denoted  $G_{\Theta}^{> 0}$ .

In the case when  $G = \mathrm{SL}(n, \mathbf{R})$  the semigroup  $G_{\Delta}^{\geq 0}$  (respectively  $G_{\Delta}^{> 0}$ ) is precisely the semigroup of nonnegative (respectively totally positive) matrices. A general geometric description of the semigroups  $G_{\Theta}^{\geq 0}$  and  $G_{\Theta}^{> 0}$  is not known and should be the subject of further investigation. In this article we focus on the nonnegative and positive unipotent semigroups. The nonnegative and positive semigroups  $G_{\Theta}^{\geq 0}$ , respectively  $G_{\Theta}^{> 0}$ , will be further investigated in [GW24] where it is shown that the positive semigroup  $G_{\Theta}^{> 0}$  is the direct product  $U_{\Theta}^{\mathrm{opp}, > 0} \cdot L_{\Theta}^{\circ} \cdot U_{\Theta}^{> 0}$ , and further geometric properties of its elements are established, in particular [GW18, Conjecture 4.10], stating that an element of  $G_{\Theta}^{> 0}$  acts proximally on  $F_{\Theta}$ , will be proved there.

**4.3. Some facts about topological semigroups.** Let us note that semigroups contained in a topological group satisfy a number of immediate properties:

**Lemma 4.6.** *Let  $G$  be a topological group. Let  $U$  be a semigroup contained in  $G$ .*

- (1) *The closure  $\overline{U}$  and the interior  $\overset{\circ}{U}$  are also semigroups.*
- (2) *Let  $\ell$  be in  $G$  such that  $\ell U \ell^{-1} = U$ , then  $\ell \overline{U} \ell^{-1} = \overline{U}$  and  $\ell \overset{\circ}{U} \ell^{-1} = \overset{\circ}{U}$ .*
- (3) *If  $U$  is open, the following inclusions hold  $\overline{U}U \subset U$  and  $U\overline{U} \subset U$ . In the case when furthermore  $\overline{U}$  contains the neutral element, equalities hold:  $\overline{U}U = U = U\overline{U}$ .*
- (4) *If  $U$  is open and if  $e$  belongs to  $\overline{U}$ , then  $U$  is equal to the interior of  $\overline{U}$ .*

*Proof.* Points (1) and (2) are consequences of the continuity of the law in  $G$ .

We now assume that  $U$  is open.

Let  $u$  be in  $\overline{U}$  and  $v$  be in  $U$ . There exists a neighborhood  $\mathcal{N}$  of  $e$  in  $G$  such that  $\mathcal{N}v \subset U$ . Since  $\mathcal{N}^{-1}$  is also a neighborhood of  $e$ ,  $u\mathcal{N}^{-1}$  is a neighborhood of  $u$  and contains an element  $u'$  of  $U$ . The element  $v'$  of  $G$  such that  $uv = u'v'$  belongs to  $\mathcal{N}v$  and hence to  $U$ . As  $U$  is a semigroup,  $u'v'$  is in  $U$  and thus  $uv$  is in  $U$ . This proves  $\overline{U}U \subset U$  and similarly  $U\overline{U} \subset U$  follows. When  $e$  belongs to  $\overline{U}$ , the inclusions  $U \subset \overline{U}U$  and  $U \subset U\overline{U}$  are obvious. This proves point (3).

Suppose that  $e \in \overline{U}$ . Let  $V$  be the interior of  $\overline{U}$ . Since  $U$  is open, one has  $U \subset V$ . Let  $v$  be in  $V$ . Since  $V$  is open, there is a neighborhood  $\mathcal{N}$  of  $e$  such that  $v\mathcal{N} \subset V$ . By the assumption  $e \in \overline{U}$ ,  $U \cap \mathcal{N}^{-1}$  is not empty and let  $g$  be an element in this intersection. Then  $vg^{-1}$  belongs to  $V$  and thus to  $\overline{U}$ ; therefore  $v = (vg^{-1})g$  belongs to  $\overline{U}U = U$ . The equality  $V = U$  is proved.  $\square$

**4.4. Hermitian Lie groups not of tube type.** When  $G$  is a Hermitian Lie group that is not of tube type we saw that  $G$  does not admit a positive structure. However, we can consider the simple root  $\alpha$  such



that  $2\alpha$  is a root. There is an  $L_\Theta^\circ$ -invariant cone  $c_{2\alpha}$  in  $\mathfrak{u}_{2\alpha}$ . We could thus consider the semigroups given by  $\exp(c_{2\alpha})$ , respectively  $\exp(\mathring{c}_{2\alpha})$ . These semigroups are contained in the maximal subgroup of tube type in  $G$ , and thus the discussion reduces to the case of Hermitian Lie groups of tube type. However, several of the properties of the positive semigroups  $U_\Theta^{>0}$  established in Section 10 in fact do not hold for the semigroup  $\exp(\mathring{c}_{2\alpha})$  in Hermitian Lie groups that are not of tube type.

## 5. THE INVARIANT CONES

In this section we fix a pair  $(G, \Theta)$  such that  $G$  is a simple Lie group carrying a  $\Theta$ -positive structure and analyze the invariant cones in  $\mathfrak{u}_\alpha$  ( $\alpha \in \Theta$ ) in more detail. In particular we introduce the notion of a  $\Theta$ -system, that will be used in Section 6 to generate a split subalgebra of  $\mathfrak{g}$  related with the positive structure.

**5.1. The description of the invariant cones.** For every  $\alpha$  in  $\Theta$ ,  $\mathfrak{u}_\alpha$  contains an  $L_\Theta^\circ$ -invariant closed convex cone, which we denote by  $c_\alpha$ . This cone is unique up to the action of  $-\text{Id}$  (cf. Proposition 3.6.(2)). The interior of the cone  $c_\alpha \subset \mathfrak{u}_\alpha$  is denoted by  $\mathring{c}_\alpha$ . We now describe these cones explicitly.

When  $\mathfrak{u}_\alpha \cong \mathbf{R}$  (this happens for all  $\alpha$  in the case when  $\Theta = \Delta$  and for all  $\alpha \neq \alpha_\Theta$  in the case  $\Theta \neq \Delta$ ), the cone  $c_\alpha$  can be identified with the cone  $\mathbf{R}_{\geq 0} \subset \mathbf{R}$ . Thus we now focus on the cone in  $\mathfrak{u}_{\alpha_\Theta}$  (and  $\Theta \neq \Delta$ ).

Let us start with the case when  $G$  is Hermitian of tube type. This means that  $G$  is (up to isogeny)  $\text{Sp}(2n, \mathbf{R})$ ,  $\text{SU}(n, n)$ ,  $\text{SO}^*(4n)$ ,  $\text{SO}(2, q)$ , or the exceptional group  $\text{E}_{7(-25)}$ . The 3 first families of classical groups can be uniformly described: Let  $\mathbf{K}$  be either  $\mathbf{R}$ ,  $\mathbf{C}$ , or  $\mathbf{H}$ , the space  $V = \mathbf{K}^{2n}$  is a right vector space and is endowed with a anti-Hermitian form defined by

$$\omega_{\mathbf{K}}(v, v') = \sum_{i=1}^n \bar{x}_i y'_i - \bar{y}_i x'_i$$

with  $v = (x_1, \dots, x_n, y_1, \dots, y_n)$  and  $v' = (x'_1, \dots, x'_n, y'_1, \dots, y'_n)$ . The subgroup  $G$  in  $\text{GL}_{\mathbf{K}}(V)$  of automorphisms of  $V$  stabilizing  $\omega_{\mathbf{K}}$  is Hermitian of tube type; it is  $\text{Sp}(2n, \mathbf{R})$ ,  $\text{SU}(n, n)$ , or  $\text{SO}^*(4n)$  following that  $\mathbf{K}$  is  $\mathbf{R}$ ,  $\mathbf{C}$ , or  $\mathbf{H}$  respectively. The Levi factor  $L_\Theta$  in this case consists of the elements

$$\ell_A = \begin{pmatrix} A & 0 \\ 0 & {}^t \bar{A}^{-1} \end{pmatrix}, \quad A \in \text{GL}_n(\mathbf{K}),$$

and the unipotent algebra  $\mathfrak{u}_\Theta$  is equal to  $\mathfrak{u}_{\alpha_\Theta}$ . It consists of the matrices

$$u_M = \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}, \quad M \in M_n(\mathbf{K}) \text{ such that } {}^t \bar{M} = M;$$

thus  $\mathfrak{u}_{\alpha_\Theta}$  identifies with the space  $\text{Herm}(n, \mathbf{K})$  of Hermitian matrices over  $\mathbf{K}$ . Since  $\ell_A u_M \ell_A^{-1} = u_{A M {}^t \bar{A}}$ , we get that  $\text{Herm}^{\geq 0}(n, \mathbf{K})$  is the sought for  $L_\Theta$ -invariant cone.

For  $\text{SO}(2, k+1)$ , let us consider the quadratic form  $q_{2, k+1} = -x_1 x_{k+3} + x_2 x_{k+2} - \sum_{j=3}^{k+1} x_j^2$  on  $V = \mathbf{R}^{k+3}$ . The group  $G = \text{SO}(q_{2, k+1})$  is also an

Hermitian Lie group of tube type. The Levi factor  $L_\Theta$  consists of the matrices

$$\ell_{\lambda,g} = \begin{pmatrix} \lambda \det g & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & \lambda^{-1} \end{pmatrix}, \quad \lambda \in \mathbf{R}^*, \quad g \in O(q_{1,k}),$$

where  $q_{1,k}$  is given by restricting  $q_{2,k+1}$  on the codimension 2 space defined by  $x_1 = x_{k+3} = 0$ . The unipotent algebra is composed by the matrices

$$u_v = \begin{pmatrix} 0 & {}^t v J & 0 \\ 0 & 0 & v \\ 0 & 0 & 0 \end{pmatrix}, \quad v \in \mathbf{R}^{k+1},$$

where  $J$  is the matrix of the form  $q_{1,k}$ . One has in this case  $\ell_{\lambda,g} u_v \ell_{\lambda,g}^{-1} = u_{\lambda g v}$  and the cone defined by  $x_2 > 0$ ,  $q_{1,k} > 0$  is invariant by  $L_\Theta^\circ$ .

The exceptional Hermitian simple Lie group of tube type is  $E_{7(-25)}$  (or EVII in Cartan's notation). In this case, the semisimple part of the group  $L_\Theta$  is  $EIV = E_{6(-26)}$  and the action of  $L_\Theta$  on  $\mathfrak{u}_{\alpha_\Theta}$  is the action of the group of isomorphisms of  $\text{Herm}(3, \mathbf{O})$ , in it  $\text{Herm}^{\geq 0}(3, \mathbf{O})$  is a natural invariant cone (cf. [FK94, p. 213]).

We now turn to the general case, i.e. when  $\Theta \neq \Delta$  and the cardinality of  $\Theta$  is  $> 1$ . The description of the cone  $c_{\alpha_\Theta}$  reduces to the Hermitian case by Proposition 3.8. Let us describe concretely what happens in those cases, namely for orthogonal groups of forms of higher signatures and for the groups whose system of restricted roots is  $F_4$ .

When  $G = \text{SO}(p+1, p+k)$ , then (using standard numbering for the simple roots)  $\alpha_\Theta = \alpha_p$ , and  $\mathfrak{u}_{\alpha_\Theta} \cong (\mathbf{R}^{1,k}, q_{1,k})$  with  $c_{\alpha_\Theta} = \{v \in \mathbf{R}^{1,k} \mid v_1 \geq 0, q_{1,k}(v) \geq 0\}$  ( $q_{1,k}$  denotes here the standard quadratic form of signature  $(1, k)$  on  $\mathbf{R}^{1+k}$ ).

When  $G$  is one group in the exceptional family whose system of restricted roots is of type  $F_4$ , then  $\mathfrak{u}_{\alpha_2}$  and the cone  $c_{\alpha_2}$  are given in Table 1.

$G$	$\mathfrak{u}_{\alpha_2}$	$S_\Theta$	$c_{\alpha_2}$
FI = $F_{4(4)}$	$\text{Sym}(3, \mathbf{R})$	$\text{Sp}(6, \mathbf{R})$	$\text{Sym}^{\geq 0}(3, \mathbf{R})$
EII = $E_{6(2)}$	$\text{Herm}(3, \mathbf{C})$	$\text{SU}(3, 3)$	$\text{Herm}^{\geq 0}(3, \mathbf{C})$
EVI = $E_{7(-5)}$	$\text{Herm}(3, \mathbf{H})$	$\text{SO}^*(12)$	$\text{Herm}^{\geq 0}(3, \mathbf{H})$
EIX = $E_{8(-24)}$	$\text{Herm}(3, \mathbf{O})$	$E_{6(-26)}$	$\text{Herm}^{\geq 0}(3, \mathbf{O})$

TABLE 1. The nontrivial cone for groups in the exceptional family

**5.2. Homogeneity.** The explicit description of the invariant cones  $\mathring{c}_\alpha$  for all  $\alpha \in \Theta$  also gives us further information on the action of  $L_\Theta^\circ$  on the product of cones  $\prod_{\alpha \in \Theta} \mathring{c}_\alpha$ . Whereas the representation of  $L_\Theta^\circ$  in  $\text{GL}(\mathfrak{u}_\alpha)$  can have a nontrivial kernel, the kernel of the representation of  $L_\Theta^\circ$  in  $\text{GL}(\prod_{\alpha \in \Theta} \mathfrak{u}_\alpha)$  is reduced to the intersection of the center of  $G$  with  $L_\Theta^\circ$ . Thus we get the following proposition.

**Proposition 5.1.** *Let, for each  $\alpha$  in  $\Theta$ ,  $c_\alpha$  be an  $L_\Theta^\circ$ -invariant acute convex cone in  $\mathfrak{u}_\alpha$  and let  $\mathring{c}_\alpha$  be its interior. Then the diagonal action of  $L_\Theta^\circ$  on  $\prod_{\alpha \in \Theta} \mathring{c}_\alpha$  is transitive and proper. Precisely the stabilizers*

are maximal compact subgroups of  $L_\Theta^\circ$ , therefore  $\prod_{\alpha \in \Theta} \mathring{c}_\alpha$  is a model for the Riemannian symmetric space associated with  $L_\Theta^\circ$ . Furthermore an element in  $\prod_{\alpha \in \Theta} c_\alpha \setminus \prod_{\alpha \in \Theta} \mathring{c}_\alpha$  has a noncompact stabilizer in  $L_\Theta^\circ$ .

*Proof.* When  $\Theta = \Delta$  (and we are thus in the split case), we have that  $L_\Theta^\circ \simeq \mathbf{R}_{>0}^\Theta$ ,  $\prod_{\alpha \in \Theta} \mathring{c}_\alpha \simeq \mathbf{R}_{>0}^\Theta$ , and the action is given by coordinate-wise multiplication. The action of  $L_\Theta^\circ$  on  $\prod_{\alpha \in \Theta} \mathring{c}_\alpha$  is therefore simply transitive. If  $\mathbf{v} = (v_\alpha)_{\alpha \in \Theta}$  belongs to  $\prod_{\alpha \in \Theta} c_\alpha \setminus \prod_{\alpha \in \Theta} \mathring{c}_\alpha$ , it means that one coordinate  $v_\alpha$  of  $\mathbf{v}$  is zero and the  $\alpha$ -th factor of the group  $\mathbf{R}_{>0}^\Theta \simeq L_\Theta^\circ$  is contained in the stabilizer of  $\mathbf{v}$  which is therefore not compact.

Let us investigate now the case  $\Theta \neq \Delta$ . The Lie algebra  $\mathfrak{l}_\Theta$  of  $L_\Theta$  is the direct sum  $\mathfrak{z}_\Theta \oplus \mathfrak{s}_\Theta$  of its center and its semisimple factor. Furthermore  $\mathfrak{z}_\Theta$  is the direct sum of a compact factor  $\mathfrak{m}_\Theta$  and the split factor  $\mathfrak{t}_\Theta$  (cf. Section 2.9). The map  $X \mapsto (\alpha(X))_{\alpha \in \Theta}$  is an isomorphism from  $\mathfrak{t}_\Theta$  to  $\mathbf{R}^\Theta$ . Furthermore a Cartan subspace for  $\mathfrak{s}_\Theta$  is  $\mathfrak{a}_\Theta = \bigoplus_{\alpha \in \Delta \setminus \Theta} \mathbf{R}H_\alpha$ .

The same decomposition holds for the Levi factor  $\mathfrak{l}_{\Theta \setminus \{\alpha_\Theta\}}$  and its semisimple factor  $\mathfrak{s}_{\Theta \setminus \{\alpha_\Theta\}}$ . Therefore  $\mathfrak{t}_{\Theta \setminus \{\alpha_\Theta\}}$  is contained in  $\mathfrak{t}_\Theta$ . Furthermore  $\mathfrak{j} = \mathfrak{l}_\Theta \cap \mathfrak{s}_{\Theta \setminus \{\alpha_\Theta\}}$  is the Levi factor in  $\mathfrak{s}_{\Theta \setminus \{\alpha_\Theta\}}$  corresponding to  $\{\alpha_\Theta\}$  (Proposition 3.8).

We therefore get that, up to a compact factor,  $\mathfrak{l}_{\Theta \setminus \{\alpha_\Theta\}}$  is the direct sum of  $\mathfrak{t}_{\Theta \setminus \{\alpha_\Theta\}}$  and of  $\mathfrak{j}$ . Let us denote by  $T_{\Theta \setminus \{\alpha_\Theta\}}$  and  $J$  the corresponding connected Lie groups. From the construction (and by a similar argument as in the split case) we get that  $T_{\Theta \setminus \{\alpha_\Theta\}}$  acts simply transitively on  $\prod_{\alpha \in \Theta \setminus \{\alpha_\Theta\}} \mathring{c}_\alpha$  and acts trivially on  $\mathring{c}_{\alpha_\Theta}$ . Similarly  $J$  acts trivially on  $\prod_{\alpha \in \Theta \setminus \{\alpha_\Theta\}} \mathring{c}_\alpha$  and it acts transitively on  $\mathring{c}_{\alpha_\Theta}$  with stabilizers being compact maximal subgroups. This implies the result for the action of  $L_\Theta$  on  $\prod_{\alpha \in \Theta} \mathring{c}_\alpha$ .

Now let  $\mathbf{v} = (v_\alpha)_{\alpha \in \Theta}$  belongs to  $\prod_{\alpha \in \Theta} c_\alpha \setminus \prod_{\alpha \in \Theta} \mathring{c}_\alpha$ . Either there is  $\alpha \neq \alpha_\Theta$  such that  $v_\alpha$  is zero and the stabilizer of  $\mathbf{v}$  in  $T_{\Theta \setminus \{\alpha_\Theta\}}$  is not compact, or  $v_{\alpha_\Theta}$  belongs to  $c_{\alpha_\Theta} \setminus \mathring{c}_{\alpha_\Theta}$  and the stabilizer of  $\mathbf{v}$  in  $J$  is not compact (Proposition 3.6). In all cases, the stabilizer of  $\mathbf{v}$  in  $L_\Theta^\circ$  is not compact.  $\square$

**5.3. The group of automorphisms of  $\mathfrak{g}$ .** In order to have stronger (and more natural) transitivity results for the action on the cones (Proposition 5.2 below), we now consider  $\text{Aut}(\mathfrak{g}) \subset \text{GL}(\mathfrak{g})$  the group of automorphisms of the Lie algebra  $\mathfrak{g}$ . It is a Lie group, that is not necessarily connected, and whose Lie algebra is equal to  $\mathfrak{g}$  since  $\mathfrak{g}$  is semisimple. The adjoint action induces a continuous homomorphism  $G \rightarrow \text{Aut}(\mathfrak{g})$ ; this homomorphism has finite kernel and its image is  $\text{Inn}(\mathfrak{g})$  the connected component of the identity in  $\text{Aut}(\mathfrak{g})$ .

By construction, the group  $\text{Aut}(\mathfrak{g})$  acts on the Lie algebra  $\mathfrak{g}$  and also on the subalgebras of  $\mathfrak{g}$ ; this action is orthogonal with respect to the Killing form. Automorphisms of  $\mathfrak{g}$  act on the space of maximal compact subalgebras as well as on the space of Cartan subspaces.

Since the action of  $\text{Inn}(\mathfrak{g})$  is transitive on the Weyl chambers (cf. [Kna02, Theorem 6.51]), one can associate to any element  $g$  of  $\text{Aut}(\mathfrak{g})$  an automorphism  $\phi_g$  of the Dynkin diagram  $D$  of  $\mathfrak{g}$ . These combine to an homomorphism  $\text{Aut}(\mathfrak{g}) \rightarrow \text{Aut}(D)$ . Let  $g$  be in  $\text{Aut}(\mathfrak{g})$ , for every  $\Theta$

contained in  $\Delta$  and for every  $\mathfrak{p}$  in  $F_\Theta$ , i.e.  $\mathfrak{p}$  is a parabolic algebra of type  $\Theta$ , the image  $g(\mathfrak{p})$  is a parabolic subalgebra of type  $\phi_g(\Theta)$ .

The kernel  $\text{Aut}_1(\mathfrak{g})$  of this homomorphism

contains  $\text{Inn}(\mathfrak{g})$ ; it acts on the flag variety  $F_\Theta$  since, for every  $g$  in  $\text{Aut}_1(\mathfrak{g})$  and every  $\mathfrak{p}$  in  $F_\Theta$ ,  $g(\mathfrak{p})$  belongs to  $F_{\phi_g(\Theta)} = F_\Theta$ .

The actions of  $G$  and of  $\text{Aut}_1(\mathfrak{g})$  on  $F_\Theta$  are of course related by the homomorphism  $G \rightarrow \text{Aut}_1(\mathfrak{g})$ .

**5.4. Homogeneity under the group of automorphisms of  $\mathfrak{g}$ .** We prove here that the choices of the cones  $c_\alpha$  in  $\mathfrak{u}_\alpha$  ( $\alpha \in \Theta$ ) are unimportant up to the action of the group  $\text{Aut}_1(\mathfrak{g})$ . Precisely, we have

**Proposition 5.2.** *For every  $\{\varepsilon_\alpha\}_{\alpha \in \Theta}$  in  $\{\pm 1\}^\Theta$ , there is an element  $g$  in  $\text{Aut}_1(\mathfrak{g})$  such that*

- $g$  preserves every factor in the decomposition  $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha$ , in particular it stabilizes  $\mathfrak{u}_\alpha$  for every  $\alpha$  in  $\Theta$ ;
- for every  $\alpha$  in  $\Theta$ , the restriction  $g|_{\mathfrak{u}_\alpha}$  is  $\varepsilon_\alpha \text{Id}$ .

One can even choose  $g$  such that

- the restriction  $g|_{\mathfrak{g}_0}$  is  $+\text{Id}$ , and
- for every  $\alpha$  in  $\Theta$ , there restriction  $g|_{\mathfrak{u}_{-\alpha}}$  is  $\varepsilon_\alpha \text{Id}$ .

*Proof.* Let  $\phi$  be the homomorphism of the additive group  $\text{Span}_{\mathbf{Z}}(\Delta)$  to the multiplicative group  $\{\pm 1\}$ , defined on the basis  $\Delta$  by

- $\phi(\alpha) = \varepsilon_\alpha$  for all  $\alpha$  in  $\Theta$  and  $\phi(\alpha) = 1$  for all  $\alpha$  in  $\Delta \setminus \Theta$ .

Define now  $g$  by  $g|_{\mathfrak{g}_\gamma} = \phi(\gamma) \text{Id}$  for every  $\gamma$  in  $\Sigma \cup \{0\}$ . Thus  $g$  belongs to  $\text{GL}(\mathfrak{g})$ .

- (1) The relations  $\phi(\gamma + \gamma') = \phi(\gamma)\phi(\gamma')$ , for all  $\gamma$  and  $\gamma'$  in  $\Sigma \cup \{0\}$ , imply that  $g$  is an automorphism of  $\mathfrak{g}$ .
- (2) One has  $\phi(0) = 1$  and, for every  $\alpha \in \Theta$  and every  $\gamma$  in  $\Sigma$  that is congruent to  $\pm\alpha$  modulo the span of  $\Delta \setminus \Theta$ ,  $\phi(\gamma) = \varepsilon_\alpha$  and therefore  $g|_{\mathfrak{g}_0} = \text{Id}$  and  $g|_{\pm\mathfrak{u}_\alpha} = \varepsilon_\alpha \text{Id}$ .

By construction  $g$  preserves the root space decomposition; such an automorphism of  $\mathfrak{g}$  automatically preserves the Cartan subspace and all the roots, i.e. it acts as the identity on the Dynkin diagram and belongs to  $\text{Aut}_1(\mathfrak{g})$ . The automorphism  $g$  has hence all the wanted properties.  $\square$

As a corollary of Propositions 5.1 and 5.2 we get

**Corollary 5.3.** *The Levi factor of the parabolic subgroup of  $\text{Aut}_1(\mathfrak{g})$  associated with  $\Theta$  acts transitively on  $\prod_{\alpha \in \Theta} (\dot{c}_\alpha \cup -\dot{c}_\alpha)$ .*

*Remark 5.4.* It is an interesting question to determine the orbits of  $L_\Theta$  on the product  $\prod_{\alpha \in \Theta} (\dot{c}_\alpha \cup -\dot{c}_\alpha)$ . For the cases not corresponding to total positivity (i.e. when  $\Theta \neq \Delta$ ), one has the following result:

- When  $\#\Theta$  is even, the action of  $L_\Theta$  on  $\prod_{\alpha \in \Theta} (\dot{c}_\alpha \cup -\dot{c}_\alpha)$  is transitive,
- When  $\#\Theta$  is odd, this action has two orbits.

**5.5. Construction of  $\Theta$ -systems.** We explain in this paragraph an explicit construction of elements  $E_\alpha$  in  $\mathring{c}_\alpha \cup -\mathring{c}_\alpha$  from “standard” elements presenting the Lie algebra  $\mathfrak{g}$ . This will allow us to construct what we call a  $\Theta$ -system (Definition 5.16). We show in Section 6 that a  $\Theta$ -system generates a split real Lie subalgebra of  $\mathfrak{g}$  and in Section 7 that it determines a special three dimensional simple subalgebra in  $\mathfrak{g}$ , that we call the  $\Theta$ -principal  $\mathfrak{sl}_2$ .

To start we consider the following:

- for each  $\alpha$  in  $\Theta$ , we choose once and for all an element  $X_\alpha$  in  $\mathfrak{g}_\alpha$  such that,  $(X_\alpha, \tau(-X_\alpha), [X_\alpha, \tau(-X_\alpha)])$  is an  $\mathfrak{sl}_2$ -triple (cf. Section 2.5).

Recall that  $X_\alpha$  must satisfy the relation  $B(X_\alpha, \tau(X_\alpha)) = -2/(\alpha|\alpha)$  (i.e.  $X_\alpha \in S(\mathfrak{g}_\alpha)$ , Section 2.5). Since  $\mathfrak{g}_\alpha$  is of dimension 1, the element  $X_\alpha$  is determined uniquely up to sign.

In all cases when  $\Theta = \Delta$  or for  $\alpha \neq \alpha_\Theta$  when  $\Theta \neq \Delta$ , we have  $\mathfrak{u}_\alpha = \mathfrak{g}_\alpha$ , and thus  $X_\alpha$  determines the cone  $\mathring{c}_\alpha = \mathbf{R}_{>0}X_\alpha$ .

When  $\Theta \neq \Delta$  and for  $\alpha = \alpha_\Theta$ , we will use the element  $X_\alpha \in \mathfrak{g}_\alpha$  as a starting point to define, using the action of  $W_{\Delta \setminus \Theta}$  the Weyl group of  $S_\Theta$ , a new element  $E_{\alpha_\Theta}$  (see Theorem 5.12). This element will be contained in  $\mathring{c}_{\alpha_\Theta} \cup -\mathring{c}_{\alpha_\Theta}$  and we will pick the cone so that  $E_\alpha \in \mathring{c}_{\alpha_\Theta}$ .

From the classification (or more precisely from the fact that  $\Theta$  and  $\Delta \setminus \Theta$  are connected) we know that the semisimple Lie group  $S_\Theta$  is of type  $A_d$  where  $d$  is the cardinality of  $\Delta \setminus \Theta$ . We can thus enumerate the roots in  $\Delta \setminus \Theta = \{\chi_1, \dots, \chi_d\}$  so that  $\chi_1$  is connected (by a double arrow) to the special root  $\alpha_\Theta$  and, for all  $i$  in  $\llbracket 1, d-1 \rrbracket$ ,  $\chi_i$  is connected to  $\chi_{i+1}$ . The corresponding reflections in the Weyl group  $W$  will be denoted by  $s_1, \dots, s_d$ .

One has then the following equalities:

$$s_1(\alpha_\Theta) = \alpha_\Theta + 2\chi_1,$$

since  $\alpha_\Theta$  and  $\chi_1$  are connected by a double arrow from  $\alpha_\Theta$  to  $\chi_1$ ;

$$s_i(\alpha_\Theta) = \alpha_\Theta \text{ for all } i > 1,$$

since  $\alpha_\Theta$  and  $\chi_i$  are not connected in the Dynkin diagram; and the following equalities in type  $A_d$ :

$$s_i(\chi_{i-1}) = \chi_{i-1} + \chi_i \text{ for all } i > 1,$$

$$s_i(\chi_{i+1}) = \chi_i + \chi_{i+1} \text{ for all } i < d,$$

$$s_i(\chi_i) = -\chi_i \text{ for } i \geq 1,$$

$$s_i(\chi_j) = \chi_j \text{ if } |i - j| > 1.$$

From this the equalities  $s_i(\chi_{i-1} + \chi_i + \chi_{i+1}) = \chi_{i-1} + \chi_i + \chi_{i+1}$  follow for all  $i$  in  $\llbracket 2, d-1 \rrbracket$ .

**Notation 5.5.** Denote, for all  $k$  in  $\llbracket 0, d \rrbracket$

$$\gamma_k = \alpha_\Theta + 2 \sum_{1 \leq j \leq k} \chi_j.$$

In particular  $\gamma_0 = \alpha_\Theta$ , and the roots  $\gamma_k$  are equal to  $\alpha_\Theta$  modulo the span of  $\chi_1, \dots, \chi_d$ , i.e.  $\mathfrak{g}_{\gamma_k} \subset \mathfrak{u}_{\alpha_\Theta}$  for all  $k$ .

**Lemma 5.6.** *The orbit of the special root  $\alpha_\Theta$  under the group  $W_{\Delta \setminus \Theta}$  is  $\{\gamma_0, \dots, \gamma_d\}$ .*

*Proof.* For all  $i$  in  $\llbracket 2, d \rrbracket$  and all  $k$  in  $\llbracket 0, i-2 \rrbracket$ , one has

$$\begin{aligned} s_i(\gamma_k) &= s_i\left(\alpha_\Theta + 2 \sum_{1 \leq j \leq k} \chi_j\right) \\ &= s_i(\alpha_\Theta) + 2 \sum_{1 \leq j \leq k} s_i(\chi_j) \\ &= \alpha_\Theta + 2 \sum_{1 \leq j \leq k} \chi_j = \gamma_k. \end{aligned}$$

For all  $i$  in  $\llbracket 1, d-1 \rrbracket$  and for all  $k$  in  $\llbracket i+1, d \rrbracket$ , one has

$$\begin{aligned} s_i(\gamma_k) &= s_i\left(\gamma_{i-2} + 2(\chi_{i-1} + \chi_i + \chi_{i+1}) + 2 \sum_{j=i+2}^k \chi_j\right) \\ &= \gamma_{i-2} + 2(\chi_{i-1} + \chi_i + \chi_{i+1}) + 2 \sum_{j=i+2}^k \chi_j = \gamma_k. \end{aligned}$$

Similarly,  $s_i(\gamma_{i-1}) = s_i(\gamma_{i-2} + 2\chi_{i-1}) = \gamma_{i-2} + 2\chi_{i-1} + 2\chi_i = \gamma_i$ , and  $s_i(\gamma_i) = \gamma_{i-1}$  for all  $i$  in  $\llbracket 1, d \rrbracket$ .

From this we directly observe that the reflections  $s_1, \dots, s_d$  stabilize the set  $\{\gamma_0, \dots, \gamma_d\}$  and act transitively on it. This implies the result since  $W_{\Delta \setminus \Theta}$  is generated by  $s_1, \dots, s_d$ .  $\square$

We now establish preliminary results concerning the 3-dimensional Lie subalgebras associated with elements in  $\mathfrak{g}_{\chi_i}$ , and their action on the root spaces  $\mathfrak{g}_{\gamma_k}$ . Recall that for  $X$  an element in the sphere  $S(\mathfrak{g}_{\chi_i})$  of  $\mathfrak{g}_{\chi_i}$ , i.e.  $B(X, \tau(X)) = -2/(\chi_i | \chi_i)$ , the triple  $(X, \tau(-X), [\tau(X), X])$  is an  $\mathfrak{sl}_2$ -triple (cf. Section 2.5), thus the Lie algebra  $\mathfrak{s}$  generated by  $X$  and  $\tau(X)$  is isomorphic to  $\mathfrak{sl}_2(\mathbf{R})$ .

**Lemma 5.7.** *Let  $k$  be in  $\llbracket 0, d \rrbracket$  and let  $i \notin \{k, k+1\}$ . Let  $X$  an element in the sphere  $S(\mathfrak{g}_{\chi_i})$  and let  $\mathfrak{s}$  be the Lie algebra generated by  $X$  and  $\tau(X)$ . The root space  $\mathfrak{g}_{\gamma_k}$  is stable by  $\mathfrak{s}$  and is the trivial  $\mathfrak{s}$ -module.*

*Proof.* Since the  $\chi_i$ -chain containing  $\gamma_k$  is trivial, the subspace  $\mathfrak{g}_{\gamma_k}$  is indeed an  $\mathfrak{s}$ -module (cf. Section 2.5). Since it is 1-dimensional, it is the trivial module.  $\square$

**Lemma 5.8.** *Let  $i$  be in  $\llbracket 1, d \rrbracket$ , and let  $X$  be in  $S(\mathfrak{g}_{\chi_i})$ , so that the Lie algebra  $\mathfrak{s}$  generated by  $X$  and  $\tau(X)$  is isomorphic to  $\mathfrak{sl}_2(\mathbf{R})$ . The  $\mathfrak{s}$ -modules generated by  $\mathfrak{g}_{\gamma_{i-1}}$  and by  $\mathfrak{g}_{\gamma_i}$  coincide and are isomorphic to the 3-dimensional irreducible module of  $\mathfrak{s} \simeq \mathfrak{sl}_2(\mathbf{R})$ .*

*Proof.* Since  $X$  belongs to  $S(\mathfrak{g}_{\chi_i})$ ,  $(X, Y = \tau(-X), H = [X, Y])$  is an  $\mathfrak{sl}_2$ -triple.

The  $\chi_i$ -chain containing  $\gamma_{i-1}$  is  $(\gamma_{i-1}, \gamma_{i-1} + \chi_i, \gamma_{i-1} + 2\chi_i = \gamma_i)$ ; it is also the  $\chi_i$ -chain containing  $\gamma_i$ . This means in particular that

$$\mathfrak{g}_{\gamma_{i-1}} \oplus \mathfrak{g}_{\gamma_{i-1} + \chi_i} \oplus \mathfrak{g}_{\gamma_i}$$

is invariant under the adjoint action of  $\mathfrak{s}$  and that the lowest weight space for this action is the 1-dimensional space  $\mathfrak{g}_{\gamma_{i-1}}$ . Together with the fact that  $\gamma_{i-1}(H) = -2$ , one gets that the  $\mathfrak{s}$ -module generated by  $\mathfrak{g}_{\gamma_{i-1}}$  is the irreducible 3-dimensional  $\mathfrak{s}$ -module (cf. Section 2.5) and contains the line  $\mathfrak{g}_{\chi_i}$ . By irreducibility, the  $\mathfrak{s}$ -module generated by  $\mathfrak{g}_{\chi_i}$  is equal to the  $\mathfrak{s}$ -module generated by  $\mathfrak{g}_{\gamma_{i-1}}$ .  $\square$



The above lemmas enable us to understand the orbit of  $X_{\alpha_\Theta}$  under the subgroup  $\mathcal{W}$  of  $N_K(\mathfrak{a})$  generated by the elements  $\exp(\frac{\pi}{2}(X + \tau(X)))$  where  $X$  varies in  $\bigcup_{i=1}^d S(\mathfrak{g}_{\chi_i})$ ; by [Kna02, Prop. 6.52], for  $X$  in  $S(\mathfrak{g}_{\chi_i})$ , the element  $\exp(\frac{\pi}{2}(X + \tau(X)))$  belongs to  $N_K(\mathfrak{a})$  and represents  $s_i$ .

**Notation 5.9.** For all  $i$  in  $\llbracket 1, d \rrbracket$ , let  $X_i$  be the element in  $S(\mathfrak{g}_{\chi_i})$  so that  $(X_i, \tau(-X_i), [\tau(X_i), X_i])$  is an  $\mathfrak{sl}_2$ -triple and set  $\dot{s}_i = \exp(\frac{\pi}{2}(X_i + \tau(X_i)))$ .

We denote by  $Z_0, \dots, Z_d$  the elements defined by the following equalities  $Z_0 = X_\alpha \in \mathfrak{g}_{\alpha_\Theta}$  and

$$Z_1 = \text{Ad}(\dot{s}_1)Z_0, \quad Z_2 = \text{Ad}(\dot{s}_2)Z_1, \dots, \quad Z_d = \text{Ad}(\dot{s}_d)Z_{d-1}.$$

We denote, for all  $k$  in  $\llbracket 0, d \rrbracket$ ,  $Y_k = -\tau(Z_k)$  and  $W_k = [Z_k, Y_k]$ .

Since  $\text{Ad}(\dot{s}_i)$  commutes with  $\tau$ , one has  $Y_i = \text{Ad}(\dot{s}_i)Y_{i-1}$  and  $W_i = \text{Ad}(\dot{s}_i)W_{i-1}$ .

**Proposition 5.10.** *The elements  $Z_0, \dots, Z_d$  do not depend on the choices of the  $X_i$ .*

*They all belong to  $\mathfrak{u}_{\alpha_\Theta}$ , and are the elements of the orbit of  $Z_0$  under the group  $\mathcal{W}$ .*

*Proof.* For every root  $\alpha$ , one has  $\text{Ad}(\dot{s}_i)\mathfrak{g}_\alpha = \mathfrak{g}_{s_i(\alpha)}$ . Therefore, for all  $k$  in  $\llbracket 0, d \rrbracket$ ,  $Z_k$  belongs to  $\mathfrak{g}_{\gamma_k}$  which is contained in  $\mathfrak{u}_{\alpha_\Theta}$ .

Let us denote, for  $k = 0$  in  $\llbracket 0, d \rrbracket$ , by  $\mathcal{W}_k$  the subgroup of  $N_K(\mathfrak{a})$  generated by the elements  $\exp(\frac{\pi}{2}(X + \tau(X)))$  where  $X$  varies in  $\bigcup_{i=1}^k S(\mathfrak{g}_{\chi_i})$ . The subgroup  $\mathcal{W}_0$  is then the trivial subgroup and  $\mathcal{W}_d = \mathcal{W}$ .

We will establish by induction on  $k$  that the orbit of  $Z_0$  under the action of  $\mathcal{W}_k$  is equal to  $\{Z_0, \dots, Z_k\}$  and that  $Z_k$  does not depend on the choice of  $X_k$ .

This is trivially satisfied for  $k = 0$ . Let us thus assume the result for a given  $k < d$ .

By Lemma 5.7, the action of  $\exp(\frac{\pi}{2}(X + \tau(X)))$  is trivial on  $\mathfrak{g}_{k+1}$  for all  $X$  in  $\bigcup_{i=1}^k S(\mathfrak{g}_{\chi_i})$ . This implies that  $Z_{k+1}$  is fixed by  $\mathcal{W}_k$ . By the same argument,  $\exp(\frac{\pi}{2}(X + \tau(X)))$  fixes the elements  $Z_0, Z_1, \dots, Z_{k-1}$  for all  $X$  in  $S(\mathfrak{g}_{\chi_{k+1}})$ .

Let us now prove that, for all  $X$  in  $S(\mathfrak{g}_{\chi_{k+1}})$ , one has  $\exp(\frac{\pi}{2}(X + \tau(X))) \cdot Z_k = Z_{k+1}$  and  $\exp(\frac{\pi}{2}(X + \tau(X))) \cdot Z_{k+1} = Z_k$ . Let  $\mathfrak{s}$  be the 3-dimensional algebra generated by  $X$  and  $\tau(X)$ .

By Lemma 5.8 and the knowledge of the 3-dimensional irreducible representation of  $\mathfrak{sl}_2(\mathbf{R})$ , the element  $\exp(\frac{\pi}{2}(X + \tau(X)))$  acting on the  $\mathfrak{s}$ -module generated by  $Z_k$  is of order 2 (cf. Section 2.5); this implies that it is enough to establish the identity  $\exp(\frac{\pi}{2}(X + \tau(X))) \cdot Z_k = Z_{k+1}$ . This implies also that the map  $X \mapsto \exp(\frac{\pi}{2}(X + \tau(X))) \cdot Z_k$  is invariant by  $X \mapsto -X$ . Hence this map factors to a continuous map from the projective space  $S(\mathfrak{g}_{\chi_i})/\{\pm 1\}$  to a 0-dimensional sphere  $S(\mathfrak{g}_{\gamma_{k+1}})$  (since  $\mathfrak{g}_{\gamma_{k+1}}$  is 1-dimensional and since  $\exp(\frac{\pi}{2}(X + \tau(X)))$  acts orthogonally with respect to the quadratic form  $B(\cdot, \tau(\cdot))$ ), it is hence the constant map.

This proves that  $Z_{k+1}$  does not depend on  $X_{k+1}$  and also that the orbit of  $Z_0$  under  $\mathcal{W}_{k+1}$  is equal to  $\{Z_0, \dots, Z_k, Z_{k+1}\}$ , hence the induction step.  $\square$

The nontrivial Lie brackets between elements among  $\{Z_k, Y_k, W_k\}$  are zero.

**Lemma 5.11.** (1) For all  $k$  and  $\ell$  in  $\llbracket 0, d \rrbracket$ ,  $[Z_k, Z_\ell] = 0$ ,  $[Y_k, Y_\ell] = 0$ , and  $[W_k, W_\ell] = 0$ .

(2) For all  $k$  and  $\ell$  in  $\llbracket 0, d \rrbracket$ , if  $k \neq \ell$ ,  $[Z_k, Y_\ell] = 0$ ,  $[Z_k, W_\ell] = 0$ , and  $[Y_k, W_\ell] = 0$ .

*Proof.* (1) This follows since  $\mathfrak{u}_{\pm\alpha_\Theta}$  is Abelian and since  $\mathfrak{a}$  is Abelian.

(2) Let us prove that  $[Z_k, Y_\ell] = 0$ . Using the action of  $\tau$  we can assume  $k > \ell$ . The bracket  $[Z_k, Y_\ell]$  belongs to the weight space associated with  $\gamma_k - \gamma_\ell = 2(\chi_{\ell+1} + \cdots + \chi_k)$  which is not a root (since the root system is reduced and  $\chi_{\ell+1} + \cdots + \chi_k$  is a root); this bracket is therefore zero.

The identities  $[Z_k, W_\ell] = 0$  and  $[Y_k, W_\ell] = 0$  follow now using the definition of  $W_\ell$  and the Jacobi identity.  $\square$

From this we deduce

**Theorem 5.12.** The element  $E_{\alpha_\Theta}$  of  $\mathfrak{u}_{\alpha_\Theta}$  defined by

$$E_{\alpha_\Theta} := Z_0 + Z_1 + \cdots + Z_d,$$

belongs to  $\mathring{c}_{\alpha_\Theta} \cup -\mathring{c}_{\alpha_\Theta}$ .

*Proof.* By Proposition 3.6.(2) it is enough to show that the Lie algebra  $\mathfrak{s}$  of the stabilizer of  $Z_0 + \cdots + Z_d$  contains  $\mathfrak{k}_\Theta$ . By Proposition 5.10 this sum is invariant under the action of  $\mathcal{W}$ , therefore  $\mathfrak{s}$  is invariant by  $W_{\Delta \setminus \Theta}$ .

Equation (2.1), p. 16, reads

$$\mathfrak{k}_\Theta = \mathfrak{m} \oplus \bigoplus_{\alpha \in \text{Span}(\Delta \setminus \Theta) \cap \Sigma^+} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \cap \mathfrak{k}.$$

Here  $\text{Span}(\Delta \setminus \Theta) \cap \Sigma$  is the root system of the Lie group  $S_\Theta$  and is of type  $A_d$ . In this situation the corresponding Weyl group  $W_{\Delta \setminus \Theta}$  acts transitively on the roots. Therefore it will be enough to establish the invariance under  $\mathfrak{m} = \mathfrak{z}_\mathfrak{k}(\mathfrak{a})$  and under  $(\mathfrak{g}_{\chi_1} \oplus \mathfrak{g}_{-\chi_1}) \cap \mathfrak{k}$  (since  $\chi_1$  is one of the root of  $S_\Theta$ ).

For all  $k$  in  $\llbracket 0, d \rrbracket$ , the root space  $\mathfrak{g}_{\gamma_k}$  is a 1-dimensional representation of the (compact) Lie algebra  $\mathfrak{m}$  and is thus the trivial representation. Hence the stabilizer of  $Z_k$  contains  $\mathfrak{m}$ . Therefore  $\mathfrak{s}$  contains  $\mathfrak{m}$ .

Let us now establish that  $\mathfrak{s}$  contains

$$(\mathfrak{g}_{\chi_1} \oplus \mathfrak{g}_{-\chi_1}) \cap \mathfrak{k} = \{X + \tau(X)\}_{X \in \mathfrak{g}_{\chi_1}},$$

(cf. Equation (2.2), p. 16). Let  $X$  be an element of  $\mathfrak{g}_{\chi_1}$ , and  $Y = -\tau(X)$ . We want to show that the stabilizer of  $Z_0 + \cdots + Z_d$  contains  $X - Y$ . As the conclusion holds trivially if  $X = 0$ , we can assume that  $X$  is nonzero. Up to multiplying  $X$  by a positive real number, we can assume that  $(X, Y, [X, Y])$  is an  $\mathfrak{sl}_2$ -triple. Denote by  $\mathfrak{f}$  the Lie algebra it generates. By Lemma 5.8, we know that the  $\mathfrak{f}$ -module generated by  $Z_0$  contains  $Z_1 = \text{Ad}(\dot{s}_1)Z_0$  (again with  $\dot{s}_1 = \exp(\frac{\pi}{2}(X - Y))$ ) and is the irreducible 3-dimensional  $\mathfrak{f}$ -module. Explicit knowledge of the 3-dimensional irreducible  $\mathfrak{sl}_2$ -module shows directly that the stabilizer of  $Z_0 + Z_1$  contains  $X - Y$ . By Lemma 5.7, for every  $k$  in  $\llbracket 2, d \rrbracket$ , the

stabilizer of  $Z_k$  contains  $X - Y$ . From this we have the sought for result:  $\{X + \tau(X)\}_{X \in \mathfrak{g}_{X_1}}$  is included in  $\mathfrak{s}$ .  $\square$

*Example 5.13.* We illustrate the construction of the elements above in the example when  $G = \mathrm{Sp}(2n, \mathbf{R})$  and  $\Theta = \{\alpha_n\}$ . In this case  $\mathfrak{u}_{\alpha_\Theta}$  naturally identifies with  $\mathrm{Sym}_n(\mathbf{R})$ , the space of real  $n \times n$  symmetric matrices (cf. Section 5.1) and  $W_{\Delta \setminus \Theta}$  is isomorphic to the symmetric group  $S_n$  acting on  $\mathrm{Sym}_n(\mathbf{R})$  by conjugation by the corresponding permutation matrices and  $d = n - 1$ . We use the standard basis  $(E_{i,j})_{1 \leq i, j \leq n}$  of  $M_n(\mathbf{R})$  (the only nonzero coefficient of  $E_{i,j}$  is in place  $(i, j)$  and is equal to 1) so that  $\{E_{i,i}\}_i \cup \{E_{i,j} + E_{j,i}\}_{i < j}$  is a basis of  $\mathrm{Sym}_n(\mathbf{R})$ . The root space  $\mathfrak{g}_{\alpha_n}$  is equal (as a subspace of  $\mathfrak{u}_{\alpha_\Theta} \simeq \mathrm{Sym}_n(\mathbf{R})$ ) to the line generated by  $E_{1,1}$ . Thus, one can take  $Z_0 = E_{1,1}$  and one has  $Z_1 = E_{2,2}, \dots, Z_i = E_{i+1, i+1}, \dots, Z_{n-1} = E_{n,n}$ . Hence the sum  $E_{\alpha_n} = Z_0 + \dots + Z_{n-1}$  is the identity matrix of  $\mathrm{Sym}_n(\mathbf{R})$ .

As mentioned already, we now fix the invariant cone  $c_{\alpha_\Theta}$  by choosing the cone containing  $E_{\alpha_\Theta}$ .

**Lemma 5.14.** *The intersection of the cone  $\mathring{c}_{\alpha_\Theta}$  with the vector space generated by  $Z_0, \dots, Z_d$  is equal to  $\sum_{i=0}^d \mathbf{R}_{>0} Z_i$  and the intersection of the cone  $c_{\alpha_\Theta}$  with the vector space generated by  $Z_0, \dots, Z_d$  is equal to  $\sum_{i=0}^d \mathbf{R}_{\geq 0} Z_i$ . In particular  $X_{\alpha_\Theta}$  belongs to  $c_{\alpha_\Theta}$ .*

*Proof.* By the action of  $\exp(\mathfrak{a})$ , one finds that  $\sum_{i=0}^d \mathbf{R}_{>0} Z_i$  is contained in  $\mathring{c}_{\alpha_\Theta}$ . To prove the reverse inclusion, it is enough to prove that no element of  $(\sum_{i=0}^d \mathbf{R}_{\geq 0} Z_i) \setminus (\sum_{i=0}^d \mathbf{R}_{>0} Z_i)$  belongs to  $\mathring{c}_{\alpha_\Theta}$ , this follows from Proposition 3.6 since these elements have noncompact stabilizers.  $\square$

*Remark 5.15.* In view of the results in this section, one can go back and forth between the normalization (up to  $\pm 1$ ) of the elements  $X_\alpha$  in  $\mathfrak{g}_\alpha$  ( $\alpha \in \Theta$ ) and the normalization (up to  $\pm \mathrm{Id}$ ) of the cones  $c_\alpha$  ( $\alpha \in \Theta$ ). If the element  $X_\alpha$  is chosen, there is a unique cone  $c_\alpha$  in  $\mathfrak{u}_\alpha$  such that  $c_\alpha$  contains  $X_\alpha$ . Conversely, if the cone  $c_\alpha$  is chosen, there is a unique element  $X_\alpha$  in  $S(\mathfrak{g}_\alpha)$  such that  $X_\alpha$  belongs to  $c_\alpha$ .

Starting with the elements  $Y_\alpha = \tau(-X_\alpha)$ , we analogously determine elements  $F_\alpha$  in  $\mathring{c}_\alpha^{\mathrm{opp}}$ . One has  $F_\alpha = \tau(-E_\alpha)$ .

**Definition 5.16.** Let  $E_\alpha \in \mathring{c}_\alpha$ ,  $F_\alpha \in \mathring{c}_\alpha^{\mathrm{opp}}$  be the elements constructed as above, and set  $D_\alpha := [E_\alpha, F_\alpha]$  for all  $\alpha \in \Theta$ . Then the family  $(E_\alpha, F_\alpha, D_\alpha)_{\alpha \in \Theta}$  is called a  $\Theta$ -system of  $\mathfrak{g}$ .

*Remark 5.17.* In the case when  $\Theta = \Delta$ , i.e. when  $G$  is a split real group, the elements  $\{E_\alpha, F_\alpha, D_\alpha\}_{\alpha \in \Delta}$  are the Chevalley or Cartan–Weyl basis of  $\mathfrak{g}$ .

## 6. SUBALGEBRAS ASSOCIATED TO A $\Theta$ -SYSTEM

In this section we show that a  $\Theta$ -system generates a real split Lie subalgebra  $\mathfrak{h}_\Theta$  in  $\mathfrak{g}$ . Besides being remarkable, this subalgebra will play a crucial role in Section 10.6.

We assume that  $G$  is a simple Lie group admitting a  $\Theta$ -positive structure. We let  $\mathfrak{g}$  be its Lie algebra and  $(E_\alpha, F_\alpha, D_\alpha)_{\alpha \in \Theta}$  a  $\Theta$ -system, with  $E_\alpha$  belonging to  $\mathring{c}_\alpha$  and  $F_\alpha$  belonging to  $\mathring{c}_\alpha^{\mathrm{opp}}$  for  $\alpha$  in  $\Theta$ .

**Theorem 6.1.** *The Lie algebra  $\mathfrak{h}_\Theta \subset \mathfrak{g}$  generated by  $(E_\alpha, F_\alpha)_{\alpha \in \Theta}$  is a split real simple Lie algebra, more precisely  $(E_\alpha, F_\alpha, D_\alpha)_{\alpha \in \Theta}$  are Chevalley generators of  $\mathfrak{h}_\Theta$  and satisfy the Serre relations. When  $G$  is split and  $\Theta = \Delta$ , we have  $\mathfrak{h}_\Delta = \mathfrak{g}$ . If  $G$  is of type  $C_n$  and  $\Theta = \{\alpha_n\}$ , then  $\mathfrak{h}_\Theta$  is of type  $A_1$ . If  $G$  is of type  $B_n$  and  $\Theta = \Delta \setminus \{\alpha_n\}$ , then  $\mathfrak{h}_\Theta$  is of type  $B_{n-1}$ . If  $G$  is of type  $F_4$  and  $\Theta = \{\alpha_1, \alpha_2\}$ , then  $\mathfrak{h}_\Theta$  is of type  $G_2$ .*

*Proof.* Note that the conclusion of the theorem is plainly satisfied in the split case, where  $E_\alpha = X_\alpha$  for every  $\alpha$  in  $\Delta$ . The Lie algebra generated by  $\{X_\alpha, \tau(X_\alpha)\}_{\alpha \in \Delta}$  is then equal to  $\mathfrak{g}$ .

The case when  $\sharp\Theta = 1$  (Hermitian groups) is also easy to deal with. We thus focus on the case when  $\Theta \neq \Delta$  and  $\sharp\Theta > 1$ .

Let  $\{X_i\}_{i \in \llbracket 1, d \rrbracket}$   $\{Z_k, Y_k, W_k\}_{k \in \llbracket 0, d \rrbracket}$  (where  $d$  is the cardinality of  $\Delta \setminus \Theta$ ) be the elements given in Section 5.5 so that  $E_{\alpha_\Theta} = Z_0 + \dots + Z_d$  belongs to the cone  $\hat{c}_{\alpha_\Theta}$ , and for each  $k$  in  $\llbracket 1, d \rrbracket$ , the relations  $Y_k = \text{Ad}(\dot{s}_k)Y_{k-1}$  and  $W_k = \text{Ad}(\dot{s}_k)W_{k-1}$  hold (with  $\dot{s}_k = \exp(\frac{\pi}{2}(X_k + \tau(X_k)))$ ).

For the rest of this proof we will write  $\Theta = \{\alpha_1, \dots, \alpha_p\}$  (with  $p$  the cardinality of  $\Theta$  so that  $p \geq 2$ ) with  $\alpha_p = \alpha_\Theta$  and, for all  $i$  in  $\llbracket 1, p-1 \rrbracket$ ,  $\alpha_i$  and  $\alpha_{i+1}$  are connected in the Dynkin diagram; we will write  $E_i, F_i, D_i$  instead of  $E_{\alpha_i}, F_{\alpha_i}, D_{\alpha_i}$ .

We now prove that  $E_i, F_i, D_i$ , ( $i \in \llbracket 1, p \rrbracket$ ) satisfy the Serre relations (cf. [Bou06, Ch. VIII, § 4, n° 3, Théorème 1], note that we are using here a different normalization for  $\mathfrak{sl}_2$ -triples). This means here the following identities

$$(6.1) \quad [D_i, E_i] = 2E_i, [D_i, F_i] = -2F_i, [E_i, F_i] = D_i, \quad \forall i \in \llbracket 1, p \rrbracket$$

$$(6.2) \quad [E_i, F_j] = 0, \quad \forall i \neq j \in \llbracket 1, p \rrbracket$$

$$(6.3) \quad [E_i, E_j] = 0, [F_i, F_j] = 0, [D_i, E_j] = 0, \text{ and} \\ [D_i, F_j] = 0, \quad \forall i, j \in \llbracket 1, p \rrbracket \text{ with } |i - j| > 1$$

$$(6.4) \quad (\text{ad } E_i)^2 E_j = 0, (\text{ad } F_i)^2 F_j = 0, [D_i, E_j] = -E_j, \text{ and} \\ [D_i, F_j] = F_j, \quad \forall i \in \llbracket 1, p-1 \rrbracket \text{ and } j \in \{i \pm 1\}$$

$$(6.5) \quad [D_p, E_{p-1}] = -(d+1)E_{p-1}, [D_p, F_{p-1}] = (d+1)F_{p-1}$$

$$(6.6) \quad (\text{ad } E_p)^{d+2} E_{p-1} = 0, (\text{ad } F_p)^{d+2} F_{p-1} = 0.$$

(This list seems longer than usual due to the distinguished role of the root  $\alpha_\Theta = \alpha_p$ ).

Equations (6.1) are the facts that the triples  $(E_i, F_i, D_i)$  are  $\mathfrak{sl}_2$ -triples, only the case when  $i = p$  needs more explanation; this case follows from the equalities  $[W_k, Z_j] = 0$  and  $[Y_k, Z_j] = 0$  for all  $k \neq j$  (Lemma 5.11) and the fact that  $(Z_k, Y_k, W_k)$  are  $\mathfrak{sl}_2$ -triples.

Equations (6.2) are inherited from the corresponding identities in  $\mathfrak{g}$ , again only the case when  $i$  or  $j$  is equal to  $p$  needs a comment. Let's consider, for example,  $i = p$ ; one has  $E_p = E_{\alpha_\Theta} = Z_0 + Z_1 + \dots + Z_d$ , and, for all  $k$  in  $\llbracket 1, d \rrbracket$ ,  $Z_k = \text{Ad}(\dot{s}_k)Z_{k-1}$ . Since, from the known identities in  $\mathfrak{g}$ ,  $[Z_0, F_j] = 0$  and since  $\text{Ad}(\dot{s}_k)F_j = F_j$  for all  $k$  in  $\llbracket 1, d \rrbracket$ , one gets that, for all  $k$ ,  $[Z_k, F_j] = \text{Ad}(\dot{s}_k)[Z_{k-1}, F_j] = 0$  and also  $[E_p, F_j] = \sum_{k=0}^d [Z_k, F_j] = 0$ .

Equations (6.3) are also inherited from the corresponding identities in  $\mathfrak{g}$  (with again a special treatment when one of the indices is equal to  $p$ ). Equations (6.4) equally follow from the known equalities in  $\mathfrak{g}$ .

We now prove Equations (6.5). Note first that  $D_p = W_0 + W_1 + \dots + W_d$ . One has, again from identities valid in  $\mathfrak{g}$ ,  $[W_0, E_{p-1}] = -E_{p-1}$ . From the recursive relation  $\text{Ad}(\dot{s}_k)W_{k-1} = W_k$  and since  $\text{Ad}(\dot{s}_k)E_{p-1} = E_{p-1}$  we deduce that  $[W_k, E_{p-1}] = -E_{p-1}$  and, summing over  $k$ ,  $[D_p, E_{p-1}] = -(d+1)E_{p-1}$ . The identity with  $F_{p-1}$  follows by a similar argument (or by applying  $\tau$ ).

Let us now address Equations (6.6). We will prove the seemingly stronger identity  $(\text{ad } E_p)^{d+2} = 0$ . Since  $\text{ad } E_p = \sum_{k=0}^d \text{ad } Z_k$ , one has

$$(\text{ad } E_p)^{d+2} = \sum_{(k_0, \dots, k_{d+1}) \in \llbracket 0, d \rrbracket^{d+2}} \text{ad } Z_{k_0} \text{ad } Z_{k_1} \cdots \text{ad } Z_{k_{d+1}}.$$

Every term in this last sum is zero: indeed the order in a product  $\text{ad } Z_{k_0} \text{ad } Z_{k_1} \cdots \text{ad } Z_{k_{d+1}}$  does not matter (see Proposition 3.9) and, since  $d+2 > d+1$ , at least two indices coincide  $k_m = k_\ell$  so that  $\text{ad } Z_{k_0} \text{ad } Z_{k_1} \cdots \text{ad } Z_{k_{d+1}} = 0$  since  $(\text{ad } Z_{k_\ell})^2 = 0$ . Similar arguments imply the equality  $(\text{ad } F_p)^{d+2} = 0$ .  $\square$

Using the homogeneity under the action of  $L_\Theta^\circ$ , one gets

**Corollary 6.2.** *Let  $V = (V_\alpha)_{\alpha \in \Theta}$  belong to  $\prod_{\alpha \in \Theta} (\dot{c}_\alpha \cup -\dot{c}_\alpha)$ . Then there exists a unique maximal compact subgroup  $K'$  of  $G$  such that  $K' \cap L_\Theta^\circ$  is the stabilizer of  $V$  in  $L_\Theta^\circ$ .*

*Let  $\sigma: \mathfrak{g} \rightarrow \mathfrak{g}$  be Cartan involution associated with  $K'$ . Then the Lie algebra  $\mathfrak{h}_\Theta$  generated by  $\{V_\alpha, \sigma(V_\alpha)\}_{\alpha \in \Theta}$  is isomorphic to a real split Lie algebra of type given by the above theorem.*

*Proof.* Changing one component  $V_\alpha$  of  $V$  to  $-V_\alpha$  does not affect the hypothesis nor the conclusion; hence we can as well assume that  $V$  belongs to  $\prod_{\alpha \in \Theta} \dot{c}_\alpha$ . By the transitivity of the action of  $L_\Theta^\circ$  on  $\prod_{\alpha \in \Theta} \dot{c}_\alpha$  (Proposition 5.1) it is enough to prove the result for one specific element in this product of cones. We can thus assume that, for every  $\alpha$  in  $\Theta$ ,  $V_\alpha = E_\alpha$  (cf. Definition 5.16). We already observed (Theorem 5.12) that the Lie algebra of the stabilizer of  $E = (E_\alpha)_{\alpha \in \Theta}$  contains the Lie algebra of  $K \cap L_\Theta^\circ$ ; therefore (cf. Proposition 5.1) the stabilizer of  $E$  in  $L_\Theta^\circ$  is equal to  $K \cap L_\Theta^\circ$ . The Cartan involution is then  $\tau$ . As  $F_\alpha = -\tau(E_\alpha)$ , Theorem 6.1 implies the wanted conclusion.  $\square$

## 7. POSITIVE $\text{SL}_2$

This section continues the investigation of the special Lie algebraic properties when a simple Lie group  $G$  admits a  $\Theta$ -positive structure; here we stress the existence of a special 3-dimensional subalgebra. We will draw several consequences from this, in particular the existence of a positive circle in the flag variety  $F_\Theta$  (see Section 13.11), that is used in [GLW21, Proposition 2.10].

**7.1. The  $\Theta$ -principal subalgebra.** The split Lie subalgebra  $\mathfrak{h}_\Theta$  constructed in the previous section admits a special subalgebra (rather a conjugacy class of subalgebras), called the *principal*  $\mathfrak{sl}_2$ . In the correspondence with nilpotent elements given by the Jacobson–Morozov theorem, it is the  $\mathfrak{sl}_2$ -subalgebra corresponding to the (conjugacy class of) regular nilpotent element. A regular nilpotent element in  $\mathfrak{h}_\Theta$  is for example  $\sum_{\alpha \in \Theta} E_\alpha$  (where  $E_\alpha$  are given in Section 5.5).

**Definition 7.1.** The  $\Theta$ -principal subalgebra is (the conjugacy class of) the subalgebra of  $\mathfrak{g}$ , isomorphic to  $\mathfrak{sl}_2(\mathbf{R})$  and represented by the principal subalgebra of  $\mathfrak{h}_\Theta$ .

The induced morphism  $\pi_\Theta: \mathfrak{sl}_2(\mathbf{R}) \rightarrow \mathfrak{g}$  will be called  $\Theta$ -principal embedding. Kostant [Kos59, Lemma 5.2] gives formulas for a principal  $\mathfrak{sl}_2$ -triple in term of the Chevalley generators, namely there are positive integers  $q_\alpha$  ( $\alpha \in \Theta$ ) such that

$$E := \sum_{\alpha \in \Theta} q_\alpha^{1/2} E_\alpha, \quad F := \sum_{\alpha \in \Theta} q_\alpha^{1/2} F_\alpha, \quad D := \sum_{\alpha \in \Theta} q_\alpha D_\alpha$$

is a principal  $\mathfrak{sl}_2$ -triple. The following lemma is certainly well known.

**Lemma 7.2.** Let  $H_\Theta$  be the connected subgroup of  $G$  whose Lie algebra is  $\mathfrak{h}_\Theta$ .

Let  $W(H_\Theta)$  be the Weyl group of  $H_\Theta$  (for the choice of Cartan subalgebra  $\mathfrak{c} = \bigoplus_{\alpha \in \Theta} \mathbf{R}H_\alpha$ ). Then  $\dot{s} = \exp(\frac{\pi}{2}(E - F))$  belongs to the normalizer of  $\mathfrak{c}$  in the maximal compact subgroup  $K \cap H_\Theta$ , and it represents the longest length element of  $W(H_\Theta)$ .

*Proof.* One has  $\text{Ad}(s) \cdot D = -D$  (from relation valid in  $\mathfrak{sl}_2(\mathbf{R})$ ). Also  $s$  belongs to  $K \cap H_\Theta$  since  $\tau(E - F) = E - F$  belongs to  $\mathfrak{k}$ .

The regular semisimple element  $D$  belongs to a unique Cartan subalgebra of  $\mathfrak{h}_\Theta$ ; thus  $\text{Ad}(s) \cdot \mathfrak{c}$  is the unique Cartan subalgebra containing  $-D$  and is therefore equal to  $\mathfrak{c}$ . This means that  $s$  belongs to the normalizer of  $\mathfrak{c}$  and represents an element in the Weyl group  $W(H_\Theta)$ .

Moreover  $D$  belongs to a unique Weyl chamber  $\mathfrak{c}^+$  of  $\mathfrak{c}$  and reasoning as above we get that  $\text{Ad}(s) \cdot \mathfrak{c}^+ = -\mathfrak{c}^+$ . Since this equality characterizes the representatives of the longest length element of  $W(H_\Theta)$ , we get the wanted result.  $\square$

- Remarks 7.3.* (1) The Cartan subspace  $\mathfrak{c}$  is equal to  $\mathfrak{a}_{\Delta \setminus \Theta}$ , i.e. to the Cartan subspace of  $S_{\Delta \setminus \Theta}$ .  
 (2) The conclusion of the lemma holds as soon as the element  $D = [E, F]$  is regular.

*Examples 7.4.* When  $G = \text{Sp}(2n, \mathbf{R})$ , let us explain the subgroup corresponding to the  $\Theta$ -principal subalgebra. For the case  $\Theta = \Delta$ , the subgroup is the image of the irreducible representation of  $\text{SL}_2(\mathbf{R})$  of dimension  $2n$ . For the case when  $\Theta = \{\alpha_n\}$ , the subgroup is the image of the diagonal embedding of  $\text{SL}_2(\mathbf{R})$ , i.e. the set of block matrices, with blocks of size  $n$  all scalar multiples of the identity matrix, alternatively there is a natural embedding of  $\text{SL}_2(\mathbf{R})^n$  into  $\text{Sp}(2n, \mathbf{R})$  and the image of the diagonal subgroup of  $\text{SL}_2(\mathbf{R})^n$  by this embedding is the  $\Theta$ -principal  $\text{SL}_2(\mathbf{R})$ .

When  $G = \text{SO}(p+1, p+k)$  and  $\Theta = \{\alpha_1, \dots, \alpha_p\}$ , the group  $H_\Theta$  is  $\text{SO}^\circ(p+1, p)$  (naturally embedded in  $G$ ) and the  $\Theta$ -principal subgroup is the irreducible  $\text{SL}_2(\mathbf{R})$  sitting in  $H_\Theta$ .

*Remark 7.5.* Motivated by the introduction of  $\Theta$ -positivity, Bradlow, Collier, García-Prada, Gothen and Oliveira introduce in [BCGP<sup>+</sup>21] the notion of magical nilpotent elements and of magical  $\mathfrak{sl}_2$ -triples. They further observe that given a complex Lie group and a magical nilpotent element  $E$  there is a canonical real form  $\mathfrak{g}$  associated with  $E$ ,



and this real form admits a  $\Theta$ -positive structure. It is clear from the above construction, that the split real form  $\mathfrak{h}_\Theta$  is the split real subalgebra denoted  $\mathfrak{g}(E)$  in [BCGP<sup>+</sup>21], and the image of the embedding  $\pi_\Theta: \mathfrak{sl}_2 \rightarrow \mathfrak{g}$  is the (real) magical  $\mathfrak{sl}_2$ -triple in  $\mathfrak{g}$ .

## 8. THE $\Theta$ -WEYL GROUP

In this section we introduce the  $\Theta$ -Weyl group, a specific subgroup of the Weyl group of  $G$ . The  $\Theta$ -Weyl group will be crucial in the parametrization of the unipotent positive semigroup (see Section 10). We further introduce the notion of the  $\Theta$ -length on  $W$  which is useful to establish some properties of the  $\Theta$ -Weyl group and is also used in Section 9. We show that the  $\Theta$ -Weyl group normalizes  $W_{\Delta \setminus \Theta}$ .

Let  $W$  be the Weyl group of  $G$ . Recall that the generators corresponding to the simple roots  $\alpha \in \Delta$  are denoted by  $s_\alpha$ , and that  $\ell(w)$  is the length of an element  $w$  with respect to this generating set (Section 2.6).

**8.1. Longest length elements and associated involutions.** We first recall some general properties of Weyl groups and their subgroups. In this paragraph, unless stated otherwise,  $\Theta$  is any subset of  $\Delta$ , not necessarily one that is associated to a  $\Theta$ -positive structure.

For a subset  $F$  of  $\Delta$  the subgroup generated by  $\{s_\alpha\}_{\alpha \in F}$  will be denoted by  $W_F$ , in particular  $W_{\Delta \setminus \Theta}$  is the subgroup generated by  $s_\alpha$  with  $\alpha \in \Delta \setminus \Theta$ , i.e. it is the Weyl group of  $S_\Theta$ . Note that the restriction of  $\ell$  to  $W_F$  is equal to the word length on  $W_F$  equipped with its generating set  $\{s_\alpha\}_{\alpha \in F}$ . We denote by  $w_\Delta$  the longest length element in  $W$ , and by  $w_F$  the longest length element in  $W_F$ , therefore  $w_{\Delta \setminus \Theta}$  is the longest length element in  $W_{\Delta \setminus \Theta}$ .

**Lemma 8.1.** [BT72, Proposition 3.9] *The element  $w_{\max}^\Theta \in W$  defined by the equality*

$$w_\Delta = w_{\max}^\Theta w_{\Delta \setminus \Theta}$$

*satisfies  $\ell(w_\Delta) = \ell(w_{\max}^\Theta) + \ell(w_{\Delta \setminus \Theta})$ . Furthermore  $w_{\max}^\Theta$  is the unique element of minimal length in the coset  $w_\Delta W_{\Delta \setminus \Theta}$ .*

In the case when  $G$  carries a  $\Theta$ -positive structure we want to apply this lemma when we consider the Weyl group associated to the diagram  $\{\alpha_\Theta\} \cup \Delta \setminus \Theta \subset \Delta$ . Applying Lemma 8.1 we then get the following corollary.

**Corollary 8.2.** *If  $G$  admits a  $\Theta$ -positive structure with  $\Theta \neq \Delta$ . We consider the element  $\sigma_{\alpha_\Theta} \in W_{\{\alpha_\Theta\} \cup \Delta \setminus \Theta}$  defined by the equality*

$$w_{\{\alpha_\Theta\} \cup \Delta \setminus \Theta} = \sigma_{\alpha_\Theta} w_{\Delta \setminus \Theta}.$$

*This element satisfies  $\ell(w_{\{\alpha_\Theta\} \cup \Delta \setminus \Theta}) = \ell(\sigma_{\alpha_\Theta}) + \ell(w_{\Delta \setminus \Theta})$ , it is the element of minimal length in the coset  $w_{\{\alpha_\Theta\} \cup \Delta \setminus \Theta} W_{\Delta \setminus \Theta}$ .*

We now establish the following lemma. Applying it to the case when  $G$  admits a  $\Theta$ -positive structure, we get that the above elements are of order 2 (cf. Corollary 8.4).

**Lemma 8.3.** *Let  $W = \langle s_\alpha \rangle_{\alpha \in \Delta}$  be a finite Coxeter group, let  $w_\Delta \in W$  be its longest length element, and let  $\alpha \mapsto \bar{\alpha}$  be the involution of  $\Delta$  defined by the equalities  $w_\Delta s_\alpha w_\Delta^{-1} = s_{\bar{\alpha}}$ . Let also  $\Theta \subset \Delta$  be invariant*

by the involution  $\alpha \mapsto \bar{\alpha}$  and denote by  $w_{\Delta \setminus \Theta}$  the longest length element of the finite Coxeter group  $\langle s_\alpha \rangle_{\alpha \in \Delta \setminus \Theta}$ . Finally set  $w_r = w_\Delta w_{\Delta \setminus \Theta}^{-1}$ . Then

- (1) The elements  $w_\Delta$  and  $w_{\Delta \setminus \Theta}$  are of order 2.
- (2) One has  $w_\Delta w_{\Delta \setminus \Theta} w_\Delta^{-1} = w_{\Delta \setminus \Theta}$ .
- (3) The element  $w_r$  commutes with  $w_\Delta$  and with  $w_{\Delta \setminus \Theta}$  and is of order 2.

*Proof.* Point (1) follows from the fact that the inverse of a reduced expression of  $w_\Delta$  is a reduced expression of  $w_\Delta^{-1}$  and hence uniqueness of the longest length element implies the equality  $w_\Delta^{-1} = w_\Delta$ . Point (2) follows from the fact that if  $s_{\alpha_1} \cdots s_{\alpha_N}$  is a reduced expression of  $w_{\Delta \setminus \Theta}$  then  $s_{\bar{\alpha}_1} \cdots s_{\bar{\alpha}_N}$  is a reduced expression of  $w_\Delta w_{\Delta \setminus \Theta} w_\Delta^{-1}$  and hence the equality  $w_\Delta w_{\Delta \setminus \Theta} w_\Delta^{-1} = w_{\Delta \setminus \Theta}$  holds again from the uniqueness of the longest length element (applied in the subgroup  $\langle s_\alpha \rangle_{\alpha \in \Delta \setminus \Theta}$  this time). Point (3) is now an immediate consequence of (1), (2), and the definition of  $w_r$ .  $\square$

**Corollary 8.4.** *If  $G$  admits a  $\Theta$ -positive structure with  $\Theta \neq \Delta$ , then the elements  $w_\Delta$  and  $w_{\{\alpha_\Theta\} \cup \Delta \setminus \Theta}$  commute, and the elements  $w_{\max}^\Theta$  and  $\sigma_{\alpha_\Theta}$  are of order 2.*

*Proof.* Observe that whenever the Dynkin diagram contains a double arrow, the involution  $\alpha \rightarrow \bar{\alpha}$  is the identity and the hypothesis of Lemma 8.3 are automatically satisfied.

The Dynkin diagram of the root system  $\Delta$  contains a double arrow, hence Lemma 8.3 implies that  $w_{\max}^\Theta$  is of order 2 and, when applied to  $\Theta \setminus \{\alpha_\Theta\}$ , that  $w_\Delta$  and  $w_{\{\alpha_\Theta\} \cup \Delta \setminus \Theta}$  commute.

The Dynkin diagram of the root system  $\{\alpha_\Theta\} \cup \Delta \setminus \Theta$  contains a double arrow. Lemma 8.3 applied to this root system implies that  $\sigma_{\alpha_\Theta}$  is of order 2.  $\square$

**8.2. The  $\Theta$ -length.** We introduce now a function on the Weyl group  $W$  that depends on  $\Theta \subset \Delta$ . This function can be introduced in general, but is particularly useful and has good properties when  $G$  admits a  $\Theta$ -positive structure, because then the sets  $\Theta$  and  $\Delta \setminus \Theta$  are connected by a double arrow (we will assume this starting from Lemma 8.9 below).

For every  $w$  in  $W$ , we denote by  $\ell_\Theta(w)$  the minimal number of occurrences of elements in  $\{s_\alpha\}_{\alpha \in \Theta}$  when  $w$  is written as an expression in the generating set  $\{s_\alpha\}_{\alpha \in \Delta}$ ,

$$(8.1) \quad \ell_\Theta(w) = \min \left\{ k \in \mathbf{N} \mid \exists N \in \mathbf{N}, (\alpha_1, \dots, \alpha_N) \in \Delta^N \text{ with} \right. \\ \left. w = s_{\alpha_1} \cdots s_{\alpha_N} \text{ and } k = \#\{j \in \llbracket 1, N \rrbracket \mid \alpha_j \in \Theta\} \right\}.$$

**Definition 8.5.** The function  $\ell_\Theta: W \rightarrow \mathbf{N}$  is called the  $\Theta$ -length.

One has  $\ell_\Theta(s_\alpha) = 1$  for every  $\alpha$  in  $\Theta$ ,  $\ell_\Theta(s_\alpha) = 0$  for every  $\alpha$  in  $\Delta \setminus \Theta$ , and  $\ell_\Theta(ab) \leq \ell_\Theta(a) + \ell_\Theta(b)$  for every  $a$  and  $b$  in  $W$ .

When  $\Theta = \Delta$ , the  $\Theta$ -length coincides with the length function on  $W$  already introduced.

The  $\Theta$ -length gives a simple characterization of  $W_{\Delta \setminus \Theta}$ .

**Lemma 8.6.** *The subgroup  $W_{\Delta \setminus \Theta}$  is the set of elements of zero  $\Theta$ -length:*

$$W_{\Delta \setminus \Theta} = \{x \in W \mid \ell_\Theta(x) = 0\}.$$

*Proof.* Indeed, for any  $x$  in  $W$ , the following are equivalent:

- $\ell_\Theta(x) = 0$
- there exist  $N \in \mathbf{N}$  and  $\alpha_1, \dots, \alpha_N$  in  $\Delta$  such that  $x = s_{\alpha_1} \cdots s_{\alpha_N}$  and  $\{i \in \llbracket 1, N \rrbracket \mid \alpha_i \in \Theta\} = \emptyset$ ,
- there exist  $N \in \mathbf{N}$  and  $\alpha_1, \dots, \alpha_N$  in  $\Delta \setminus \Theta$  such that  $x = s_{\alpha_1} \cdots s_{\alpha_N}$ ,
- $x$  belongs to  $W_{\Delta \setminus \Theta}$ . □

More is true: the  $\Theta$ -length is  $W_{\Delta \setminus \Theta}$ -invariant.

**Lemma 8.7.** *The function  $\ell_\Theta$  is invariant under the subgroup  $W_{\Delta \setminus \Theta}$ : for every  $w \in W$  and every  $x \in W_{\Delta \setminus \Theta}$  one has  $\ell_\Theta(w) = \ell_\Theta(wx) = \ell_\Theta(wx)$ .*

*Proof.* By symmetry we will prove the equality only for right multiplication by  $W_{\Delta \setminus \Theta}$ . By the subadditivity of  $\ell_\Theta$  and Lemma 8.6, for every  $w \in W$  and every  $x \in W_{\Delta \setminus \Theta}$ , one has  $\ell_\Theta(wx) \leq \ell_\Theta(w)$ .

Applying the above bound to the pair  $(wx, x^{-1})$  replacing  $(w, x)$ , we obtain  $\ell_\Theta((wx)x^{-1}) \leq \ell_\Theta(wx)$  and hence the sought for equality since  $(wx)x^{-1} = w$ . □

A direct reformulation is

**Corollary 8.8.** *The  $\Theta$ -length factors through  $W_{\Delta \setminus \Theta} \backslash W / W_{\Delta \setminus \Theta}$ , the quotient of  $W$  by the left-right action of  $W_{\Delta \setminus \Theta}$ .*

The next result is specific to  $\Theta$ -positivity, more accurately it relies on the fact that  $\Theta$  and  $\Delta \setminus \Theta$  are connected only by double arrows.

**Lemma 8.9.** *Suppose that  $G$  admits a  $\Theta$ -positive structure. The minimum in Equation (8.1) is then achieved on all the reduced expressions: for every  $w \in W$  and for every  $(\alpha_1, \dots, \alpha_{\ell(w)}) \in \Delta^{\ell(w)}$  such that  $w = s_{\alpha_1} \cdots s_{\alpha_{\ell(w)}}$ , one has*

$$\ell_\Theta(w) = \#\{j \in \llbracket 1, \ell(w) \rrbracket \mid \alpha_j \in \Theta\}.$$

*Proof.* For this proof we will denote, for every  $n$  in  $\mathbf{N}$  and every  $(\alpha_1, \dots, \alpha_n) \in \Delta^n$ , by  $\hat{\ell}_\Theta(\alpha_1, \dots, \alpha_n)$  the number of occurrences of elements in  $\Theta$  in this finite sequence:

$$\hat{\ell}_\Theta(\alpha_1, \dots, \alpha_n) = \#\{j \in \llbracket 1, n \rrbracket \mid \alpha_j \in \Theta\}.$$

This function is additive:

$$\hat{\ell}_\Theta(\alpha_1, \dots, \alpha_n, \alpha_{n+1}, \dots, \alpha_m) = \hat{\ell}_\Theta(\alpha_1, \dots, \alpha_n) + \hat{\ell}_\Theta(\alpha_{n+1}, \dots, \alpha_m).$$

By definition, for every element  $w$  of  $W$ ,  $\ell_\Theta(w)$  is the infimum of the  $\hat{\ell}_\Theta(\alpha_1, \dots, \alpha_n)$  for  $(\alpha_1, \dots, \alpha_n)$  varying among the expressions of  $w = s_{\alpha_1} \cdots s_{\alpha_n}$ .

If  $n' \leq n$  and if  $(\alpha'_1, \dots, \alpha'_{n'})$  is obtained from  $(\alpha_1, \dots, \alpha_n)$  by removing some of the entries, then  $\ell_\Theta(\alpha'_1, \dots, \alpha'_{n'}) \leq \hat{\ell}_\Theta(\alpha_1, \dots, \alpha_n)$ .

If  $(\alpha_1, \dots, \alpha_n)$  and  $(\beta_1, \dots, \beta_n)$  differ by a braid relation then

$$\hat{\ell}_\Theta(\alpha_1, \dots, \alpha_n) = \hat{\ell}_\Theta(\beta_1, \dots, \beta_n).$$

Indeed, by the additivity of the function  $\hat{\ell}_\Theta$  on expressions, one needs to check this equality only in the case when  $n$  is the order of  $s_\alpha s_{\alpha'}$  ( $\alpha, \alpha' \in \Delta$ ) and one has  $(\alpha_1, \alpha_2, \dots) = (\alpha, \alpha', \dots)$  and  $(\beta_1, \beta_2, \dots) = (\alpha', \alpha, \dots)$

(precisely  $(\alpha_j, \beta_j) = (\alpha, \alpha')$  for every odd  $j \leq n$  and  $(\alpha_j, \beta_j) = (\alpha', \alpha)$  for every even  $j \leq n$ ). When  $\alpha$  and  $\alpha'$  both belong to  $\Theta$ , we have  $\hat{\ell}_\Theta(\alpha_1, \dots, \alpha_n) = n$  and  $\hat{\ell}_\Theta(\beta_1, \dots, \beta_n) = n$ ; When  $\alpha$  and  $\alpha'$  both belong to  $\Delta \setminus \Theta$ , we have  $\hat{\ell}_\Theta(\alpha_1, \dots, \alpha_n) = 0$  and  $\hat{\ell}_\Theta(\beta_1, \dots, \beta_n) = 0$ . The last case to consider (up to exchanging the 2 sequences) is when  $\alpha$  belongs to  $\Theta$  and  $\alpha'$  belongs to  $\Delta \setminus \Theta$ ; in this case, we know that  $n$  is equal to 4 (since  $\alpha$  and  $\alpha'$  are connected by a double arrow) and we have  $\hat{\ell}_\Theta(\alpha_1, \dots, \alpha_n) = 2$  and  $\hat{\ell}_\Theta(\beta_1, \dots, \beta_n) = 2$ . In every case, the equality  $\hat{\ell}_\Theta(\alpha_1, \dots, \alpha_n) = \hat{\ell}_\Theta(\beta_1, \dots, \beta_n)$  holds.

The conclusions of the lemma follow from these remarks since:

- from every expression  $(\alpha_1, \dots, \alpha_n)$  of  $w$  one can deduce a reduced expression  $(\beta_1, \dots, \beta_m)$  by applying successively a finite number of braid relations or removing subexpressions of the form  $(\alpha, \alpha)$  ( $\alpha \in \Delta$ ) so that  $\hat{\ell}_\Theta(\beta_1, \dots, \beta_m) \leq \hat{\ell}_\Theta(\alpha_1, \dots, \alpha_n)$ , therefore

$$\ell_\Theta(w) = \inf\{\hat{\ell}_\Theta(\beta_1, \dots, \beta_m) \mid (\beta_1, \dots, \beta_m) \text{ is a reduced expr. of } w\},$$

and

- two reduced expressions  $(\beta_1, \dots, \beta_m)$  and  $(\gamma_1, \dots, \gamma_m)$  of  $w$  can be obtained from one another by applying successively a finite number of braid relations so that  $\hat{\ell}_\Theta(\beta_1, \dots, \beta_m) = \hat{\ell}_\Theta(\gamma_1, \dots, \gamma_m)$ .

Thus  $\ell_\Theta(w) = \hat{\ell}_\Theta(\beta_1, \dots, \beta_m)$  for every reduced expression  $(\beta_1, \dots, \beta_m)$  of  $w$ .  $\square$

**8.3. The  $\Theta$ -Weyl group and its generators.** We assume for the rest of this section that  $G$  is a simple Lie group admitting a  $\Theta$ -positive structure.

**Notation 8.10.** We set

- When  $\Theta = \Delta$ ,  $\sigma_\alpha = s_\alpha$  for all  $\alpha$  in  $\Delta$ ;
- When  $\Theta \neq \Delta$ , for every  $\alpha \in \Theta \setminus \{\alpha_\Theta\}$ ,  $\sigma_\alpha = s_\alpha$ ;
- And  $R(\Theta)$  to be equal to  $\{\sigma_\alpha\}_{\alpha \in \Theta}$  (recall that  $\sigma_{\alpha_\Theta}$  is defined in Corollary 8.2).

**Definition 8.11.** We define the  $\Theta$ -Weyl group to be the subgroup  $W(\Theta)$  of  $W$  generated by  $R(\Theta)$ .

We first investigate the Weyl group of the split real form  $\mathfrak{h}_\Theta$  as a subgroup of  $W$ .

**Lemma 8.12.** *Let  $G$  be a simple Lie group admitting a  $\Theta$ -positive structure with  $\Theta \neq \Delta$ . Then the standard generating Coxeter system of the Weyl group  $W(H_\Theta)$  of the split real form  $\mathfrak{h}_\Theta$  is given by*

$$s_\alpha \text{ for all } \alpha \in \Theta \setminus \{\alpha_\Theta\}, \text{ and } w_{\{\alpha_\Theta\} \cup \Delta \setminus \Theta}.$$

*Proof.* Let  $(E_\alpha, F_\alpha, D_\alpha)_{\alpha \in \Theta}$  be the  $\Theta$ -system generating  $\mathfrak{h}_\Theta$ . Then the generating set of  $W(H_\Theta)$  is represented by the elements  $\exp(\frac{\pi}{2}(E_\alpha - F_\alpha))$  for  $\alpha$  varying in  $\Theta$ .

For  $\alpha \neq \alpha_\Theta$ , one has  $(E_\alpha, F_\alpha) = (X_\alpha, -\tau(X_\alpha))$  and  $\exp(\frac{\pi}{2}(E_\alpha - F_\alpha)) = \dot{s}_\alpha$  represents  $s_\alpha$ .

For  $\alpha = \alpha_\Theta$ , one has (with Notation 5.9),  $E_\alpha - F_\alpha = (Z_0 - Y_0) + (Z_1 - Y_1) + \dots + (Z_d - Y_d)$ . Since the elements  $(Z_k - Y_k)$  pairwise commute

(Lemma 5.11),

$$\exp\left(\frac{\pi}{2}(E_\alpha - F_\alpha)\right) = \prod_{k=0}^d \exp\left(\frac{\pi}{2}(Z_k - Y_k)\right)$$

and since, for all  $k$  in  $\llbracket 1, d \rrbracket$  (again with  $\dot{s}_k = \exp(\frac{\pi}{2}(X_k + \tau(X_k)))$ , cf. Notation 5.9),  $Z_k - Y_k = \text{Ad}(\dot{s}_k)(Z_{k-1} - Y_{k-1})$ ,

$$\exp\left(\frac{\pi}{2}(Z_k - Y_k)\right) = \dot{s}_k \exp\left(\frac{\pi}{2}(Z_{k-1} - Y_{k-1})\right) \dot{s}_k^{-1}.$$

Denote  $\dot{s}_0 = \exp(\frac{\pi}{2}(Z_0 - Y_0))$  so that  $\dot{s}_0, \dot{s}_1, \dots, \dot{s}_d$  are lifts of the standard generators  $s_0, s_1, \dots, s_d$  of the Weyl group  $W_{\{\alpha_\Theta\} \cup \Delta \setminus \Theta}$  which is of type  $C_{d+1}$ . Since the longest length element in  $W_{\{\alpha_\Theta\} \cup \Delta \setminus \Theta}$  is the product of  $s_0$  and its conjugate by  $s_1, s_2 s_1, \dots, s_d \cdots s_1$  (cf. Appendix A) and since the above formula shows that  $\exp(\frac{\pi}{2}(E_\alpha - F_\alpha))$  is the product of  $\dot{s}_0$  and its conjugate by  $\dot{s}_1, \dot{s}_2 \dot{s}_1, \dots, \dot{s}_d \cdots \dot{s}_1$ , the result follows.  $\square$

**Corollary 8.13.** *The group  $W(H_\Theta)$  centralizes  $W_{\Delta \setminus \Theta}$ .*

*Proof.* Let  $\alpha$  be in  $\Theta$ . If  $\alpha \neq \alpha_\Theta$ , it means that there are no arrows in the Dynkin diagram between  $\alpha$  and  $\Delta \setminus \Theta$ ; thus  $s_\alpha$  commutes with  $s_\beta$  for every  $\beta$  in  $\Delta \setminus \Theta$ , this means that  $s_\alpha$  centralizes  $W_{\Delta \setminus \Theta}$ .

If  $\alpha = \alpha_\Theta$ , we know that in this situation the element  $w_{\{\alpha_\Theta\} \cup \Delta \setminus \Theta}$  is in the center of  $W_{\{\alpha_\Theta\} \cup \Delta \setminus \Theta}$  since it is  $-\text{Id}$  as an element of  $\text{GL}(\mathfrak{a}^*)$  (cf. Appendix A). Thus the generating set of  $W(H_\Theta)$  is contained in the centralizer of  $W_{\Delta \setminus \Theta}$ , this implies the statement.  $\square$

We next show that the  $\Theta$ -Weyl group is isomorphic to the Weyl group of the split real form  $\mathfrak{h}_\Theta$ .

**Lemma 8.14.** *Let  $G$  be a simple Lie group admitting a  $\Theta$ -positive structure with  $\Theta \neq \Delta$ .*

*There exists then a unique homomorphism  $\phi: W(H_\Theta) \rightarrow W$  such that, for all  $\alpha$  in  $\Theta \setminus \{\alpha_\Theta\}$ ,  $\phi(s_\alpha) = e$  and  $\phi(w_{\{\alpha_\Theta\} \cup \Delta \setminus \Theta}) = w_{\Delta \setminus \Theta}$ .*

*The map  $\psi: W(H_\Theta) \rightarrow W$ ,  $w \mapsto w\phi(w)$  is a group homomorphism, it is one-to-one and its image is equal to the  $\Theta$ -Weyl group  $W(\Theta)$ .*

*Proof.* Uniqueness of  $\phi$  follows from the fact it is defined on a generating set of  $W(H_\Theta)$ . To prove existence, we have to check that  $\phi$  preserves the relations in  $W(H_\Theta)$ , i.e. the fact that the generators are of order 2 and the braid relations. Since  $w_{\Delta \setminus \Theta}$  is of order 2, we just have to check the compatibility with the braid relations (cf. Section 2.6); for  $\alpha$  and  $\beta$  in  $\Theta \setminus \{\alpha_\Theta\}$ , the 2 members of the braid relation involving  $s_\alpha$  and  $s_\beta$  are sent to  $e$  in  $W$ ; when one of  $\alpha$  or  $\beta$  is equal to  $\alpha_\Theta$ , the members of the braid relations have length 2 or 4 and their images under  $\phi$  are both equal to  $w_{\Delta \setminus \Theta}$  (if the length is 2) or both equal to  $w_{\Delta \setminus \Theta}^2 = e$  (if the length is 4). This concludes the existence of  $\phi$ .

Since  $w_{\Delta \setminus \Theta}$  centralizes  $W(H_\Theta)$  (Corollary 8.13), the map  $\psi$  is indeed an homomorphism. The image by  $\psi$  of the generating set of  $W(H_\Theta)$  (cf. Lemma 8.12) is equal to  $R(\Theta)$  and thus the image of  $\psi$  is equal to  $W(\Theta)$ . For all  $w$  in  $W(H_\Theta)$ ,  $\ell_\Theta(\psi(w)) = \ell_\Theta(w)$  since  $\phi(w)$  belongs to  $W_{\Delta \setminus \Theta}$  (Lemma 8.7). Thus the kernel of  $\psi$  is contained in  $W(H_\Theta) \cap W_{\Delta \setminus \Theta}$  and is thus reduced to  $e$  since every nontrivial element of  $W(H_\Theta)$  has positive  $\Theta$ -length (cf. Lemma 8.9).  $\square$

As a corollary we have that  $W(\Theta)$  is a Coxeter group of the same type as the type of the Lie algebra  $\mathfrak{h}_\Theta$ .

**Corollary 8.15.** *Let  $G$  be a simple Lie group admitting a  $\Theta$ -positive structure. Then  $(W(\Theta), R(\Theta))$  is a Coxeter system of the following type:*

- (1) *If  $G$  is a split real form and  $\Theta = \Delta$ ,  $(W(\Theta), R(\Theta))$  has the same type as  $G$ .*
- (2) *If  $G$  is Hermitian of tube type and of real rank  $r$  and if  $\Theta = \{\alpha_r\}$ , then  $(W(\Theta), R(\Theta))$  is of type  $A_1$ .*
- (3) *If  $G$  is locally isomorphic to  $SO(p+1, p+k)$ ,  $p > 0$ ,  $k > 1$ , and  $\Theta = \{\alpha_1, \dots, \alpha_p\}$ , then  $(W(\Theta), R(\Theta))$  is of type  $B_p$ .*
- (4) *If  $G$  is the real form of  $F_4, E_6, E_7$ , or of  $E_8$ , whose reduced root system is of type  $F_4$ , and  $\Theta = \{\alpha_1, \alpha_2\}$ , then  $(W(\Theta), R(\Theta))$  is of type  $G_2$ .*

*Proof.* When  $G$  is a split real form, then  $\Theta = \Delta$ , then  $R(\Theta) = \{s_\alpha\}_{\alpha \in \Delta}$ , hence  $W(\Theta) = W$ .

Let  $G$  be of real rank  $r$  and of Hermitian tube type, and let  $\Theta = \{\alpha_r\} = \{\alpha_\Theta\}$ . Consequently  $W(\Theta)$  is generated by the single order 2 element  $\sigma_{\alpha_\Theta}$ , thus the result.

The other cases follow directly from Lemma 8.14.  $\square$

#### 8.4. The longest length element of $W(\Theta)$ .

**Proposition 8.16.** *Let  $G$  be a simple Lie group admitting a  $\Theta$ -positive structure. Let  $w_\Delta$  be the longest length element in  $W$ ,  $w_{\Delta \setminus \Theta}$  the longest length element in  $W_{\Delta \setminus \Theta}$ , and  $w_{\max}^\Theta = w_\Delta w_{\Delta \setminus \Theta}^{-1}$  so that  $w_{\max}^\Theta$  is the unique element of minimal length in the coset  $w_\Delta W_{\Delta \setminus \Theta}$  (Lemma 8.1). Then the element  $w_{\max}^\Theta$  belongs to  $W(\Theta)$  and is the longest length element of the Coxeter system  $(W(\Theta), R(\Theta))$ .*

*Proof.* We prove the proposition by a case by case analysis, using the computations done in Appendices A and B. We adopt below the notation  $x^y = y^{-1}xy$  in a group, one has thus  $(x^y)^z = x^{yz}$ .

- (1) In the case when  $G$  is a split real simple Lie group there is nothing to prove.
- (2) When  $G$  is a simple Lie group of Hermitian tube type, then  $w_{\max}^\Theta = \sigma_{\alpha_\Theta}$ , which is the longest length element in  $W(\Theta)$  that is of type  $A_1$ .
- (3) When  $G = SO(p+1, p+k)$  with  $p > 0$  and  $k > 1$ , one has  $\Delta = \{\alpha_1, \dots, \alpha_{p+1}\}$ ,  $\Theta = \{\alpha_1, \dots, \alpha_p\}$ , and  $\Delta \setminus \Theta = \{\alpha_{p+1}\}$ .<sup>1</sup> Writing as usual  $s_i$  for  $s_{\alpha_i}$  and for short  $\sigma_p$  for  $\sigma_{\alpha_p}$  (and  $\sigma_i = s_i$  for  $i < p$ ), one has  $w_{\Delta \setminus \Theta} = s_{p+1}$  and a direct calculation gives  $w_{\{\alpha_\Theta\} \cup \Delta \setminus \Theta} = s_p s_{p+1} s_p s_{p+1}$  (so that  $\sigma_p = s_p s_{p+1} s_p$ ). Appendix A gives the following reduced expression for the longest length element  $w_\Delta$ :

$$(8.2) \quad w_\Delta = s_{p+1}^{s_p \cdots s_1} s_{p+1}^{s_p \cdots s_2} \cdots s_{p+1}^{s_p s_{p-1}} s_{p+1}^{s_p} s_{p+1}$$

$$(8.3) \quad w_{\max}^\Theta = s_{p+1}^{s_p \cdots s_1} s_{p+1}^{s_p \cdots s_2} \cdots s_{p+1}^{s_p s_{p-1}} s_{p+1}^{s_p}$$

<sup>1</sup>Observe that the standard numbering used here for the simple roots does not coincide with the ‘‘constructive’’ numbering of the general results proved earlier (cf. the proof of Lemma 8.12 for example), the reindexing being  $i \in \llbracket 1, p+1 \rrbracket \mapsto p+1-i \in \llbracket 0, p \rrbracket$ .



$$\begin{aligned}
&= \sigma_p^{s_{p-1} \cdots s_1} \sigma_p^{s_{p-1} \cdots s_2} \cdots \sigma_p^{s_{p-1}} \sigma_p \\
&= \sigma_p^{\sigma_{p-1} \cdots \sigma_1} \sigma_p^{\sigma_{p-1} \cdots \sigma_2} \cdots \sigma_p^{\sigma_{p-1}} \sigma_p.
\end{aligned}$$

In particular we get that  $w_{\max}^\Theta$  belongs to  $W(\Theta)$  and is the longest length element in the Coxeter system  $(W(\Theta), R(\Theta))$  of type  $B_p$ .

- (4) When  $G$  is the real form of  $F_4, E_6, E_7, E_8$ , whose system of restricted roots is of type  $F_4$ , we have that  $\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ ,  $\Theta = \{\alpha_1, \alpha_2\}$  and  $\Delta \setminus \Theta = \{\alpha_3, \alpha_4\}$ . We write again  $s_i$  for  $s_{\alpha_i}$ ,  $\sigma_2$  for  $\sigma_{\alpha_2}$ , and  $\sigma_1 = s_1$ . By Appendix B,  $\sigma_2 = s_2 s_3 s_4 s_2 s_3 s_2 = (s_4 s_2)^{s_3 s_2}$ ; one has  $w_{\Delta \setminus \Theta} = s_4^{s_3}$  and a direct calculation given in Appendix B gives the following reduced expression for the longest length element

$$(8.4) \quad w_\Delta = (s_4 s_2)^{s_3 s_2} s_1 (s_4 s_2)^{s_3 s_2} s_1 (s_4 s_2)^{s_3 s_2} s_1 s_4^{s_3},$$

thus

$$(8.5) \quad \begin{aligned} w_{\max}^\Theta &= (s_4 s_2)^{s_3 s_2} s_1 (s_4 s_2)^{s_3 s_2} s_1 (s_4 s_2)^{s_3 s_2} s_1 \\ &= \sigma_2 \sigma_1 \sigma_2 \sigma_1 \sigma_2 \sigma_1 \end{aligned}$$

this is precisely the longest length element of  $(W(\Theta), R(\Theta))$  which is of type  $G_2$ .  $\square$

The previous proof also gives the following information about reduced expressions of elements in  $W(\Theta)$ .

**Lemma 8.17.** *Under the hypothesis of Proposition 8.16, let  $x$  be in  $W(\Theta)$ . Let  $\sigma_{\alpha_1} \cdots \sigma_{\alpha_k}$  be a reduced expression of  $x$  in  $(W(\Theta), R(\Theta))$ . Then, for the lengths in  $W$ , one has the equality*

$$\ell(x) = \ell(\sigma_{\alpha_1}) + \cdots + \ell(\sigma_{\alpha_k}),$$

so that  $\sigma_{\alpha_1} \cdots \sigma_{\alpha_k}$  is a reduced product of  $x$  in  $(W, \Delta)$ .

*Proof.* The result is obvious in the split case or in the Hermitian tube type case.

Let  $\mathbf{X} \subset R(\Theta)^k$  be the set of reduced expressions of  $x$  in  $W(\Theta)$ . We will first prove that the map

$$L: \mathbf{X} \rightarrow \mathbf{N} \mid (\sigma_{\alpha_1}, \dots, \sigma_{\alpha_k}) \mapsto \ell(\sigma_{\alpha_1}) + \cdots + \ell(\sigma_{\alpha_k})$$

is constant. Since two reduced expressions are obtained from one another by successive applications of braid relations, it is enough to prove that the map  $L$  coincides on reduced expressions differing by a braid relation; let  $\alpha$  and  $\beta$  be the 2 elements of  $\Theta$  involved in the braid relation; the result is obvious when  $\alpha \neq \alpha_\Theta$  and  $\beta \neq \alpha_\Theta$  since in this case  $\sigma_\alpha = s_\alpha$  and  $\sigma_\beta = s_\beta$ ; the result also holds when  $\alpha$  or  $\beta$  is equal to  $\alpha_\Theta$  since the braid relations (in  $(W(\Theta), R(\Theta))$ ) involving  $\alpha_\Theta$  have the same number of occurrences of  $\alpha_\Theta$  in both members of the braid relation. Hence  $L$  is the constant function.

Let us now prove the result for  $w_{\max}^\Theta$ , the longest length element of the Coxeter group  $W(\Theta)$ . It is therefore enough to find *one* reduced expression  $\sigma_{\alpha_1} \cdots \sigma_{\alpha_r}$  of  $w_{\max}^\Theta$  that is a reduced product of  $w_{\max}^\Theta$  in  $(W, \Delta)$  or, what amounts to the same, such that  $\sigma_{\alpha_1} \cdots \sigma_{\alpha_r} w_{\Delta \setminus \Theta}$  is a reduced product of  $w_\Delta$  in  $W$ . For the case of orthogonal groups, the reduced expression of  $w_\Delta$  with the desired property is given in Equation (8.2) and the prefix producing  $w_{\max}^\Theta$  is given in Equation (8.3). For

the  $F_4$  case the corresponding expressions are given in Equation (8.4) and Equation (8.5).

We now turn back to the element  $x$ . There exists a reduced expression  $\sigma_{\alpha_1} \cdots \sigma_{\alpha_r}$  of  $w_{\max}^\Theta$  such that  $\sigma_{\alpha_1} \cdots \sigma_{\alpha_k}$  is a reduced expression of  $x$ . If the result holds for  $w_{\max}^\Theta$ , one has

$$\begin{aligned} \ell(w_{\max}^\Theta) &\leq \ell(x) + \ell(x^{-1}w_{\max}^\Theta) \leq \ell(\sigma_{\alpha_1}) + \cdots + \ell(\sigma_{\alpha_k}) + \ell(\sigma_{\alpha_{k+1}} \cdots \sigma_{\alpha_r}) \\ &\leq \ell(\sigma_{\alpha_1}) + \cdots + \ell(\sigma_{\alpha_k}) + \ell(\sigma_{\alpha_{k+1}}) + \cdots + \ell(\sigma_{\alpha_r}) = \ell(w_{\max}^\Theta), \end{aligned}$$

and thus all the inequalities here are equalities and in particular  $\ell(x) = \ell(\sigma_{\alpha_1}) + \cdots + \ell(\sigma_{\alpha_k})$  which, together with the first part of the proof, implies the result.  $\square$

### 8.5. Normalizer.

**Proposition 8.18.** *The group  $W(\Theta)$  normalizes  $W_{\Delta \setminus \Theta}$ :*

$$W(\Theta) \subset N_W(W_{\Delta \setminus \Theta}) = \{x \in W \mid xW_{\Delta \setminus \Theta}x^{-1} = W_{\Delta \setminus \Theta}\}.$$

*Proof.* As  $W(\Theta)$  is contained in the group generated by  $W(H_\Theta)$  and  $W_{\Delta \setminus \Theta}$ , this follows from Corollary 8.13.  $\square$

## 9. BRUHAT DECOMPOSITION AND THE INVARIANT CONES

In this section we recall some facts on Bruhat decompositions. We then establish, in the case of  $\Theta$ -positivity, several facts giving a precise control of the Bruhat cells containing the elements  $\exp(v)$  when  $\alpha \in \Theta$  and  $v \in \mathring{c}_\alpha$ . These properties will play a central role in Section 10 when we give a parametrization of the positive unipotent semigroup.

**9.1. Bruhat decomposition.** Let  $G$  be a simple Lie group. We recall here that  $G$  can be decomposed under the left-right multiplication by  $P_\Delta^{\text{opp}} \times P_\Delta^{\text{opp}}$ , where  $P_\Delta^{\text{opp}}$  is the minimal standard opposite parabolic subgroup. These orbits are called the Bruhat cells and are indexed by elements of  $W$ . We also consider decompositions with respect to the action of  $P_\Delta^{\text{opp}} \times P_\Theta^{\text{opp}}$  and  $P_\Theta^{\text{opp}} \times P_\Theta^{\text{opp}}$ , where  $\Theta$  is a subset of  $\Delta$ .

Explicitly, for every  $w$  in  $W \simeq N_K(\mathfrak{a})/C_K(\mathfrak{a})$  (the quotient of the normalizer of  $\mathfrak{a}$  in  $K$  by its centralizer), let  $\dot{w}$  in  $N_K(\mathfrak{a})$  be a representative of  $w$ , the subset

$$P_\Delta^{\text{opp}} \dot{w} P_\Delta^{\text{opp}}$$

depends only of  $w$  (and not on the choice of  $\dot{w}$ ) since  $P_\Delta^{\text{opp}}$  contains the centralizer  $C_K(\mathfrak{a})$ ; it is called the *Bruhat cell indexed by  $w$* ; the following equalities hold

$$P_\Delta^{\text{opp}} \dot{w} P_\Delta^{\text{opp}} = U_\Delta^{\text{opp}} \dot{w} P_\Delta^{\text{opp}} = P_\Delta^{\text{opp}} \dot{w} U_\Delta^{\text{opp}}$$

and we will remove the dots in the further notation.

It is well known that (see e.g. [War72]):

- (1)  $G$  is the disjoint union of the  $P_\Delta^{\text{opp}} w P_\Delta^{\text{opp}}$  for  $w$  varying in  $W$ .
- (2)  $P_\Theta^{\text{opp}}$  is the disjoint union of the  $P_\Delta^{\text{opp}} w P_\Delta^{\text{opp}}$  for  $w$  varying in  $W_{\Delta \setminus \Theta}$ .
- (3) For all  $w_1$  and  $w_2$  in  $W$  such that  $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$ , one has

$$(P_\Delta^{\text{opp}} w_1 P_\Delta^{\text{opp}})(P_\Delta^{\text{opp}} w_2 P_\Delta^{\text{opp}}) = P_\Delta^{\text{opp}} w_1 P_\Delta^{\text{opp}} w_2 P_\Delta^{\text{opp}} = P_\Delta^{\text{opp}} w_1 w_2 P_\Delta^{\text{opp}}.$$

- (4) For all  $\alpha$  in  $\Delta$  and for all  $X$  in  $\mathfrak{g}_\alpha$ , if  $X \neq 0$  then

$$\exp(X) \in P_\Delta^{\text{opp}} s_\alpha P_\Delta^{\text{opp}}.$$

Observe also that  $\dot{s}_\alpha^{-1}$  also belongs to  $P_\Delta^{\text{opp}} s_\alpha P_\Delta^{\text{opp}}$  (simply because  $\dot{s}_\alpha^{-1}$  is a representative of the class of  $s_\alpha$ ).

(5) For every  $\alpha$  in  $\Delta$ ,

$$P_\Delta^{\text{opp}} s_\alpha P_\Delta^{\text{opp}} s_\alpha P_\Delta^{\text{opp}} = P_\Delta^{\text{opp}} \sqcup P_\Delta^{\text{opp}} s_\alpha P_\Delta^{\text{opp}}.$$

(6) For every  $w$  in  $W$ , the  $P_\Delta^{\text{opp}} \times P_\Theta^{\text{opp}}$ -orbit

$$P_\Delta^{\text{opp}} w P_\Theta^{\text{opp}}$$

depends only on the class  $x$  of  $w$  in  $W/W_{\Delta \setminus \Theta}$  and will sometimes be denoted  $P_\Delta^{\text{opp}} x P_\Theta^{\text{opp}}$ .

(7) The group  $G$  is the disjoint union

$$G = \bigsqcup_{x \in W/W_{\Delta \setminus \Theta}} P_\Delta^{\text{opp}} x P_\Theta^{\text{opp}}.$$

(8) Similar notation will be adopted for the action of  $P_\Theta^{\text{opp}} \times P_\Delta^{\text{opp}}$  and of  $P_\Theta^{\text{opp}} \times P_\Theta^{\text{opp}}$ : a double orbit  $P_\Theta^{\text{opp}} w P_\Theta^{\text{opp}}$  depends only on the class  $a$  of  $w$  in  $W_{\Delta \setminus \Theta} \setminus W/W_{\Delta \setminus \Theta}$ , will be denoted  $P_\Theta^{\text{opp}} a P_\Theta^{\text{opp}}$  and

$$G = \bigsqcup_{a \in W_{\Delta \setminus \Theta} \setminus W/W_{\Delta \setminus \Theta}} P_\Theta^{\text{opp}} a P_\Theta^{\text{opp}}.$$

Application of (3) and (5) above gives that, for every  $w_1$  and  $w_2$  in  $W$  and for every  $g$  in  $P_\Delta^{\text{opp}} w_1 P_\Delta^{\text{opp}} w_2 P_\Delta^{\text{opp}}$ , there exist a prefix  $x_1$  of  $w_1$  and a suffix  $x_2$  of  $w_2$  such that  $g$  belongs to  $P_\Delta^{\text{opp}} x_1 x_2 P_\Delta^{\text{opp}}$ . In particular:

**Lemma 9.1.** *Let  $w_1$ ,  $w_2$ , and  $x$  be in  $W$  such that  $P_\Theta^{\text{opp}} x P_\Theta^{\text{opp}} \subset P_\Theta^{\text{opp}} w_1 P_\Theta^{\text{opp}} w_2 P_\Theta^{\text{opp}}$ . Then*

$$\ell_\Theta(x) \leq \ell_\Theta(w_1) + \ell_\Theta(w_2).$$

The following lemma states the equality between some of the orbits for the two groups  $P_\Delta^{\text{opp}}$  and  $P_\Theta^{\text{opp}}$ .

**Lemma 9.2.** *For every  $w$  in  $W(\Theta)$ , one has*

$$P_\Theta^{\text{opp}} w P_\Theta^{\text{opp}} = P_\Delta^{\text{opp}} w P_\Theta^{\text{opp}} = U_\Theta^{\text{opp}} w P_\Theta^{\text{opp}}.$$

*Proof.* Since  $P_\Theta^{\text{opp}} = \bigsqcup_{x \in W_{\Delta \setminus \Theta}} P_\Delta^{\text{opp}} x P_\Delta^{\text{opp}}$ , one has

$$P_\Theta^{\text{opp}} w P_\Theta^{\text{opp}} = \bigcup_{x \in W_{\Delta \setminus \Theta}} P_\Delta^{\text{opp}} x P_\Delta^{\text{opp}} w P_\Theta^{\text{opp}}.$$

It is thus enough to prove the equality for every  $x$  in  $W_{\Delta \setminus \Theta}$

$$P_\Delta^{\text{opp}} x P_\Delta^{\text{opp}} w P_\Theta^{\text{opp}} = P_\Delta^{\text{opp}} w P_\Theta^{\text{opp}}.$$

Since  $x$  belongs to  $W_{\Delta \setminus \Theta}$ , it is a suffix of the longest length element  $w_{\Delta \setminus \Theta}$ . Similarly  $w$  is a prefix of  $w_{\max}^\Theta$ . We deduce from this that  $xw$  is a subword of  $w_{\Delta \setminus \Theta} w_{\max}^\Theta = w_\Delta$  and in particular  $\ell(xw) = \ell(x) + \ell(w)$ . By property (3) above, one has

$$\begin{aligned} P_\Delta^{\text{opp}} x P_\Delta^{\text{opp}} w P_\Theta^{\text{opp}} &= P_\Delta^{\text{opp}} x w P_\Theta^{\text{opp}} \\ &= P_\Delta^{\text{opp}} w (w^{-1} x w) P_\Theta^{\text{opp}} \end{aligned}$$

and, as  $w^{-1} x w$  belongs to  $W_{\Delta \setminus \Theta}$  (Proposition 8.18),

$$= P_\Delta^{\text{opp}} w P_\Theta^{\text{opp}}$$

which is the sought for equality.

Let us prove now the second equality. Since  $U_{\Delta}^{\text{opp}} \subset U_{\Theta}^{\text{opp}} L_{\Theta}^{\circ}$ , it is enough to prove that  $L_{\Theta}^{\circ} w P_{\Theta}^{\text{opp}} = w P_{\Theta}^{\text{opp}}$  which in turn follows from the equalities  $\dot{w}^{-1} L_{\Theta}^{\circ} \dot{w} = L_{\Theta}^{\circ}$  (where  $\dot{w}$  is a representative of  $w$ ). This equality can be proved at the level of the Lie algebra:  $\text{Ad}(\dot{w})\mathfrak{l}_{\Theta} = \mathfrak{l}_{\Theta}$ . Since  $\text{Ad}(\dot{w})$  stabilizes  $\mathfrak{a}$ , it stabilizes also  $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})$  and, thanks to the equality  $\mathfrak{l}_{\Theta} = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}) \oplus \bigoplus_{\alpha \in \text{Span}(\Delta \setminus \Theta) \cap \Sigma} \mathfrak{g}_{\alpha}$  and using  $\text{Ad}(\dot{w})\mathfrak{g}_{\alpha} = \mathfrak{g}_{w(\alpha)}$ , we are reduced to prove that  $w(\text{Span}(\Delta \setminus \Theta) \cap \Sigma) = \text{Span}(\Delta \setminus \Theta) \cap \Sigma$ . Since  $w(\Sigma) = \Sigma$  and since  $W(\Theta)$  is generated by  $R(\Theta)$ , we will just check the equality  $w(\text{Span}(\Delta \setminus \Theta)) = \text{Span}(\Delta \setminus \Theta)$  for  $w$  in  $R(\Theta)$ . The case  $w = s_{\alpha}$ , for  $\alpha \in \Theta \setminus \{\alpha_{\Theta}\}$  is direct since  $s_{\alpha}$  fixes pointwise  $\Delta \setminus \Theta$ . The case  $w = w_{\{\alpha_{\Theta}\} \cup \Delta \setminus \Theta} w_{\Delta \setminus \Theta}$  follows equally since  $w_{\{\alpha_{\Theta}\} \cup \Delta \setminus \Theta}$  induces  $-\text{Id}$  in  $\text{Span}(\Delta \setminus \Theta)$  and since  $w_{\Delta \setminus \Theta}$  induces a permutation of  $\text{Span}(\Delta \setminus \Theta) \cap \Sigma$ .  $\square$

**9.2. Dimensions.** We give here recursive information on the dimensions of the Bruhat cells when the set  $\Theta$  is fixed under the opposition involution or, what amounts to the same, the dimensions of their images in the flag variety  $F_{\iota(\Theta)} = F_{\Theta}$ . This information will be of importance when proving the openness of the parametrizations introduced in Section 10.

Recall that when  $G$  admits a  $\Theta$ -positive structure for a proper subset  $\Theta$  in  $\Delta$ , the opposition involution  $\iota$  is always the identity.

For every  $w$  in  $W(\Theta)$  we will denote by  $C(w)$  the  $P_{\Theta}^{\text{opp}}$ -orbit of  $w \cdot \mathfrak{p}_{\Theta}^{\text{opp}}$  in  $F_{\Theta}$ ;  $C(w)$  is called a *cell* and is isomorphic to the quotient  $P_{\Theta}^{\text{opp}} w P_{\Theta}^{\text{opp}} / P_{\Theta}^{\text{opp}}$ . By Lemma 9.2, it is also the image of the following map

$$\begin{aligned} \mathfrak{u}_{\Theta}^{\text{opp}} &\longrightarrow F_{\Theta} \\ X &\longmapsto \exp(X)w \cdot \mathfrak{p}_{\Theta}^{\text{opp}}. \end{aligned}$$

Recall that  $\mathfrak{u}_{\Theta}^{\text{opp}}$  has the following decomposition  $\mathfrak{u}_{\Theta}^{\text{opp}} = \bigoplus_{\alpha \in \Sigma_{\Theta}^+} \mathfrak{g}_{-\alpha}$ .

For  $w \in W$  we denote

$$\begin{aligned} \Sigma_{w\prec}^+ &= \Sigma^+ \cap w \cdot \Sigma^-, \quad \Sigma_{w\succ}^+ = \Sigma^+ \cap w \cdot \Sigma^+, \quad \text{so that } \Sigma^+ = \Sigma_{w\prec}^+ \sqcup \Sigma_{w\succ}^+, \\ \Sigma_{\Theta, w\prec}^+ &= \Sigma_{w\prec}^+ \cap \Sigma_{\Theta}^+, \quad \Sigma_{\Theta, w\prec}^+ = \Sigma_{w\prec}^+ \cap \Sigma_{\Theta}^+, \quad \text{so that } \Sigma_{\Theta}^+ = \Sigma_{\Theta, w\prec}^+ \sqcup \Sigma_{\Theta, w\prec}^+, \end{aligned}$$

and at the level of Lie algebras

$$\mathfrak{u}_{\Theta, w\prec}^{\text{opp}} = \bigoplus_{\alpha \in \Sigma_{\Theta, w\prec}^+} \mathfrak{g}_{-\alpha}, \quad \mathfrak{u}_{\Theta, w\prec}^{\text{opp}} = \bigoplus_{\alpha \in \Sigma_{\Theta, w\prec}^+} \mathfrak{g}_{-\alpha}, \quad \text{so that } \mathfrak{u}_{\Theta}^{\text{opp}} = \mathfrak{u}_{\Theta, w\prec}^{\text{opp}} \oplus \mathfrak{u}_{\Theta, w\prec}^{\text{opp}},$$

and, for all  $X$  in  $\mathfrak{u}_{\Theta, w\prec}^{\text{opp}}$ ,  $\exp(X)w$  belongs to  $w P_{\Theta}^{\text{opp}}$ .

It is well known that the length of  $w$  is the cardinality of  $\Sigma_{w\prec}^+$  [Bou07, ch. VI, §1, p. 158, cor. 2] and the following equalities hold [BT72, Lemme 3.4]

$$(9.1) \quad \begin{aligned} \Sigma_{s_{\alpha} w\prec}^+ &= \{\alpha\} \sqcup s_{\alpha} \cdot \Sigma_{w\prec}^+ \text{ if } \ell(s_{\alpha} w) = 1 + \ell(w), \\ \Sigma_{x w\prec}^+ &= \Sigma_{x\prec}^+ \sqcup x^{-1} \cdot \Sigma_{w\prec}^+ \text{ if } \ell(xw) = \ell(x) + \ell(w). \end{aligned}$$

**Lemma 9.3.** *Let  $w$  be in  $W(\Theta)$ . The map*

$$\begin{aligned} f_w : \mathfrak{u}_{\Theta, w\prec}^{\text{opp}} &\longrightarrow C(w) \\ X &\longmapsto \exp(X)w \cdot P_{\Theta}^{\text{opp}} \end{aligned}$$

*is a diffeomorphism.*

*Proof.* Clearly this map is  $C^\infty$ . It also results from classical facts about nilpotent Lie algebras that

$$\begin{aligned} \mathfrak{u}_{\Theta, w \prec}^{\text{opp}} \times \mathfrak{u}_{\Theta, w \succ}^{\text{opp}} &\longrightarrow U_\Theta \\ (X, Y) &\longmapsto \exp(X) \exp(Y) \end{aligned}$$

is a diffeomorphism. Since, for all  $Y$  in  $\mathfrak{u}_{\Theta, w \succ}^{\text{opp}}$ ,  $\exp(Y)wP_\Theta^{\text{opp}} = wP_\Theta^{\text{opp}}$ , we have that  $f_w$  is onto. By the very choice of the space  $\mathfrak{u}_{\Theta, w \prec}^{\text{opp}}$ , the map  $f_w$  is a local diffeomorphism at 0. By equivariance with respect to the element  $\exp(\sum_{\alpha \in \Delta} H_\alpha)$  of the Cartan subgroup that acts as a contracting transformation on  $\mathfrak{u}_{\Theta, w \prec}^{\text{opp}}$  we deduce that  $f_w$  is a diffeomorphism.  $\square$

From this we deduce how the dimensions of the cells  $C(w)$  jump:

**Proposition 9.4.** *Let  $w$  be in the Coxeter group  $(W(\Theta), R(\Theta))$  and let  $\alpha$  be in  $\Theta$  such that  $\sigma_\alpha w$  is a reduced product in  $W(\Theta)$ . Then*

$$\dim C(\sigma_\alpha w) - \dim C(w) = \dim \mathfrak{u}_\alpha$$

which means

$$\begin{aligned} \dim C(\sigma_\alpha w) - \dim C(w) &= 1 \text{ if } \alpha \neq \alpha_\Theta, \\ \dim C(\sigma_{\alpha_\Theta} w) - \dim C(w) &= \dim \mathfrak{u}_{\alpha_\Theta} \text{ if } \alpha = \alpha_\Theta. \end{aligned}$$

*Proof.* Denote by  $\delta$  the difference of dimensions. The hypothesis that  $\sigma_\alpha w$  is reduced in  $W(\Theta)$  implies that  $\sigma_\alpha w$  is also a reduced product in  $W$  (Lemma 8.17). The previous lemma, the equation (9.1) (applied with  $x = \sigma_\alpha$ ), and the equalities  $\dim \mathfrak{g}_{-\beta} = \dim \mathfrak{g}_\beta$  imply that

$$\delta = \sum_{\beta \in \Sigma_\Theta^+, \sigma_\alpha^{-1}\beta \in \Sigma^-} \dim \mathfrak{g}_\beta.$$

But Lemma 9.5 below establishes

$$\{\beta \in \Sigma_\Theta^+ \mid \sigma_\alpha^{-1}\beta \in \Sigma^-\} = \{\beta \in \Sigma^+ \mid \beta - \alpha \in \text{Span}(\Delta \setminus \Theta)\},$$

hence

$$\delta = \dim \bigoplus_{\substack{\beta \in \Sigma^+, \\ \beta - \alpha \in \text{Span}(\Delta \setminus \Theta)}} \mathfrak{g}_\beta = \dim \mathfrak{u}_\alpha. \quad \square$$

**Lemma 9.5.** *For all  $\alpha$  in  $\Theta$ , one has*

$$\{\beta \in \Sigma_\Theta^+ \mid \sigma_\alpha^{-1}\beta \in \Sigma^-\} = \{\beta \in \Sigma^+ \mid \beta - \alpha \in \text{Span}(\Delta \setminus \Theta)\}.$$

*Proof.* In the case  $\alpha \neq \alpha_\Theta$ ,  $\sigma_\alpha = s_\alpha$  and it is already known that

$$\Sigma_{s_\alpha \prec}^+ = \{\beta \in \Sigma^+ \mid \sigma_\alpha^{-1}\beta \in \Sigma^-\} = \{\alpha\}.$$

Since  $\alpha$  belongs to  $\Sigma_\Theta^+$ , this proves the equality  $\Sigma_{\Theta, s_\alpha \prec}^+ = \Sigma_{s_\alpha \prec}^+ \cap \Sigma_\Theta^+ = \{\alpha\}$ . Since  $\alpha$  is not connected to  $\Delta \setminus \Theta$ , the only root in the affine subspace  $\alpha + \text{Span}(\Delta \setminus \Theta)$  is  $\alpha$  and this proves the wanted equality.

Let us treat the case  $\alpha = \alpha_\Theta$ . The element  $\sigma_{\alpha_\Theta} = w_{\{\alpha_\Theta\} \cup \Delta \setminus \Theta} w_{\Delta \setminus \Theta}$  belongs to the subgroup  $W_{\{\alpha_\Theta\} \cup \Delta \setminus \Theta}$  of  $W$ .

The set

$$\{\beta \in \Sigma^+ \mid \beta - \alpha_\Theta \in \text{Span}(\Delta \setminus \Theta)\}$$

is contained in  $\text{Span}(\{\alpha_\Theta\} \cup \Delta \setminus \Theta)$  and thus in the root system generated by  $\{\alpha_\Theta\} \cup \Delta \setminus \Theta$ .

From the fact that, for every simple root  $\beta$  in  $\Theta \setminus \{\alpha_\Theta\}$  and for every  $\alpha$  in  $\{\alpha_\Theta\} \cup \Delta \setminus \Theta$ ,  $s_\alpha(\beta) - \beta$  belongs to  $\text{Span}(\{\alpha_\Theta\} \cup \Delta \setminus \Theta)$ , we have that  $w(\Sigma^+ \setminus \text{Span}(\{\alpha_\Theta\} \cup \Delta \setminus \Theta)) = \Sigma^+ \setminus \text{Span}(\{\alpha_\Theta\} \cup \Delta \setminus \Theta)$  for all  $w$  in  $W_{\{\alpha_\Theta\} \cup \Delta \setminus \Theta}$ . In particular the set

$$\{\beta \in \Sigma_\Theta^+ \mid \sigma_{\alpha_\Theta}^{-1}\beta \in \Sigma^-\}$$

must be contained in the root system generated by  $\{\alpha_\Theta\} \cup \Delta \setminus \Theta$ .

Therefore we can and will assume that  $\Delta = \{\alpha_\Theta\} \cup \Delta \setminus \Theta$ , i.e. that we are in the Hermitian tube type case. In this case it is known that the longest length element  $w_{\{\alpha_\Theta\} \cup \Delta \setminus \Theta}$  is  $-\text{Id}$  so that

$$\{\beta \in \Sigma_\Theta^+ \mid \sigma_{\alpha_\Theta}^{-1}\beta \in \Sigma^-\} = \{\beta \in \Sigma_\Theta^+ \mid w_{\Delta \setminus \Theta}\beta \in \Sigma^+\}.$$

Furthermore, since for every  $\beta$  in  $\Sigma_\Theta^+$ ,  $w_{\Delta \setminus \Theta}\beta - \beta$  belongs to  $\text{Span}(\Delta \setminus \Theta)$  (and again since every root is a sum of simple roots with coefficients all of the same sign), this set is equal to  $\Sigma_\Theta^+$ . However, here we are dealing with a root system of type  $C_{d+1}$ , and the equality  $\Sigma_\Theta^+ = \{\beta \in \Sigma^+ \mid \beta - \alpha_\Theta \in \text{Span}(\Delta \setminus \Theta)\}$  is satisfied; indeed it follows from the equality  $\mathbf{u}_\Theta = \mathbf{u}_{\alpha_\Theta}$  at the level of Lie algebras.  $\square$

**9.3. The nontrivial cone and the Bruhat decomposition.** We give now a precise description of the Bruhat cell with respect to the  $P_\Delta^{\text{opp}} \times P_\Theta^{\text{opp}}$  action, containing the image of the cone  $\mathring{c}_{\alpha_\Theta}$  by the exponential map.

We assume in this section that  $\Theta \neq \Delta$  and adopt the notation of the previous parts:  $\alpha_\Theta$  is the special root,  $\Delta \setminus \Theta = \{\chi_1, \dots, \chi_d\}$  (with  $\chi_1$  connected to  $\alpha_\Theta$  in the Dynkin diagram), and the elements  $Z_0, \dots, Z_d$  of the Lie algebra (Section 5.5); the reflection in  $W$  associated with the simple root  $\alpha_\Theta$  is denoted by  $s_0$  and the reflections associated with  $\chi_1, \dots, \chi_d$  are denoted by  $s_1, \dots, s_d$  respectively.

The longest length element in  $W_{\{\alpha_\Theta\} \cup \Delta \setminus \Theta}$  is

$$w_{\{\alpha_\Theta\} \cup \Delta \setminus \Theta} = s_0 s_0^{s_1} \cdots s_0^{s_1 \cdots s_d},$$

again with the notation  $x^y = y^{-1}xy$ , so that  $s_0^{s_1 \cdots s_i} = s_i \cdots s_1 s_0 s_1 \cdots s_i$ , the above is a reduced expression of  $w_{\{\alpha_\Theta\} \cup \Delta \setminus \Theta}$ , and these elements  $s_0^{s_1 \cdots s_i}$  pairwise commute.

Let us introduce also the following elements of  $W$ :

$$(9.2) \quad w_0 = e, \quad w_j = s_0 s_0^{s_1} \cdots s_0^{s_1 \cdots s_{j-1}}, \quad \text{for } j \in \llbracket 1, d+1 \rrbracket$$

and, for every subset  $I$  of  $\llbracket 0, d \rrbracket$ ,

$$(9.3) \quad w_I = \prod_{i \in I} s_0^{s_1 \cdots s_i},$$

so that  $w_\emptyset = e$  is the neutral element of  $W$ ,  $w_{d+1} = w_{\llbracket 0, d \rrbracket} = w_{\{\alpha_\Theta\} \cup \Delta \setminus \Theta}$ , and the expressions (9.2) and (9.3) are reduced. By Lemma 8.9, we have

$$(9.4) \quad \ell_\Theta(w_j) = j, \quad \ell_\Theta(w_I) = \sharp I, \quad \text{for all } j \text{ and for all } I.$$

The  $w_j$  belong to different Bruhat cells:

**Lemma 9.6.** *Let  $j$  and  $k$  be in  $\llbracket 0, d \rrbracket$ . If  $P_\Theta^{\text{opp}} w_j P_\Theta^{\text{opp}} = P_\Theta^{\text{opp}} w_k P_\Theta^{\text{opp}}$ , then  $j = k$ .*



*Proof.* One has  $P_{\Delta}^{\text{opp}} w_j P_{\Delta}^{\text{opp}} \subset P_{\Theta}^{\text{opp}} w_j P_{\Theta}^{\text{opp}}$  and

$$P_{\Theta}^{\text{opp}} w_k P_{\Theta}^{\text{opp}} = \bigcup_{x_1, x_2 \in W_{\Delta \setminus \Theta}} P_{\Delta}^{\text{opp}} x_1 w_k x_2 P_{\Delta}^{\text{opp}}.$$

(The last union is justified by property (2) of Section 9.1 and may not be disjoint.) Hence an equality  $P_{\Theta}^{\text{opp}} w_j P_{\Theta}^{\text{opp}} = P_{\Theta}^{\text{opp}} w_k P_{\Theta}^{\text{opp}}$  implies, and is in fact equivalent to, the existence of  $x_1, x_2$  in  $W_{\Delta \setminus \Theta}$  such that

$$w_j = x_1 w_k x_2.$$

By Lemma 8.7,  $\ell_{\Theta}(x_1 w_k x_2) = \ell_{\Theta}(w_k)$ , hence  $j = \ell_{\Theta}(w_j) = \ell_{\Theta}(w_k) = k$ .  $\square$

We start by determining the Bruhat cells for linear combinations with nonnegative coefficients of the elements  $Z_i$ .

**Lemma 9.7.** *Let  $Y$  be a linear combination with nonnegative coefficients of the  $Z_i$ :  $Y = \sum_{i=0}^d \lambda_i Z_i$ , i.e.  $Y$  belongs to  $c_{\alpha_{\Theta}} \cap \bigoplus \mathbf{R} Z_i$  (Lemma 5.14). Let  $I \subset \llbracket 0, d \rrbracket$  be the set of indices of the nonzero entries in  $(\lambda_0, \dots, \lambda_d)$  (in formula  $I = \{i \in \llbracket 0, d \rrbracket \mid \lambda_i > 0\}$ ). Then*

$$\exp(Y) \text{ belongs to } P_{\Delta}^{\text{opp}} w_I P_{\Delta}^{\text{opp}}.$$

*Proof.* Since the elements  $Z_i$  pairwise commute (Proposition 3.9), we have that

$$\exp(Y) = \exp(\lambda_0 Z_0) \exp(\lambda_1 Z_1) \cdots \exp(\lambda_d Z_d).$$

By property (4) of Section 9.1, if  $\lambda_0 > 0$ , then  $\exp(\lambda_0 Z_0)$  belongs to  $P_{\Delta}^{\text{opp}} s_0 P_{\Delta}^{\text{opp}}$ . For all  $i$ , one has  $Z_i = \text{Ad}(\dot{s}_i \cdots \dot{s}_1) Z_0$  so that  $\exp(\lambda_i Z_i) = \dot{s}_i \cdots \dot{s}_1 \exp(\lambda_i Z_0) \dot{s}_1^{-1} \cdots \dot{s}_i^{-1}$ . Successive applications of the property (3) of Section 9.1 and the fact that  $s_i \cdots s_1 s_0 s_1 \cdots s_i$  is a reduced expression imply that, when  $\lambda_i > 0$ ,  $\exp(\lambda_i Z_i)$  belongs to

$$P_{\Delta}^{\text{opp}} s_i \cdots s_1 s_0 s_1 \cdots s_i P_{\Delta}^{\text{opp}} = P_{\Delta}^{\text{opp}} s_0^{s_1 \cdots s_i} P_{\Delta}^{\text{opp}}.$$

Successive applications again of property (3) and the fact that Equation (9.3) is a reduced expression imply that

$$\exp(Y) \text{ belongs to } P_{\Delta}^{\text{opp}} \left( \prod_{i \in I} s_0^{s_1 \cdots s_i} \right) P_{\Delta}^{\text{opp}},$$

hence the result.  $\square$

We can now determine the Bruhat cell with respect to the left-right action of  $P_{\Delta}^{\text{opp}}$  pcorresponding to elements in the open cone.

**Proposition 9.8.** *Let  $X$  be in the open cone  $\mathring{c}_{\alpha_{\Theta}}$ . Then  $\exp(X)$  belongs to  $P_{\Delta}^{\text{opp}} w_{\{\alpha_{\Theta}\} \cup \Delta \setminus \Theta} P_{\Delta}^{\text{opp}}$ .*

*Proof.* Recall that the element  $E_{\alpha_{\Theta}} = Z_0 + Z_1 + \cdots + Z_d$  belongs to  $\mathring{c}_{\alpha_{\Theta}}$  and that its stabilizer in  $L_{\Theta}^{\circ}$  (for the adjoint action) contains  $K \cap L_{\Theta}^{\circ}$ . Furthermore the action of  $L_{\Theta}^{\circ}$  on  $\mathring{c}_{\alpha_{\Theta}}$  is transitive: there is an element  $g$  in  $L_{\Theta}^{\circ}$  such that  $\text{Ad}(g)E_{\alpha_{\Theta}} = X$ .

The Iwasawa decomposition for  $L_{\Theta}^{\circ}$  states the equality

$$L_{\Theta}^{\circ} = (P_{\Delta}^{\text{opp}} \cap L_{\Theta}^{\circ})(K \cap L_{\Theta}^{\circ}).$$

There are thus elements  $p$  in  $P_{\Delta}^{\text{opp}} \cap L_{\Theta}^{\circ}$  and  $k$  in  $K \cap L_{\Theta}^{\circ}$  such that  $g = pk$ . Since  $\text{Ad}(k)E_{\alpha_{\Theta}} = E_{\alpha_{\Theta}}$ , one has  $\text{Ad}(p)E_{\alpha_{\Theta}} = X$ . By the compatibility of the exponential map with the adjoint action and the conjugation

action, we get  $\exp(X) = p \exp(E_{\alpha_\Theta}) p^{-1}$  so that  $\exp(X)$  and  $\exp(E_{\alpha_\Theta})$  belong to the same Bruhat cell. By Lemma 9.7,  $\exp(E_{\alpha_\Theta})$  belongs to  $P_\Delta^{\text{opp}} w_{[[0,d]]} P_\Delta^{\text{opp}} = P_\Delta^{\text{opp}} w_{\{\alpha_\Theta\} \cup \Delta \setminus \Theta} P_\Delta^{\text{opp}}$ . This concludes that  $\exp(X)$  belongs to  $P_\Delta^{\text{opp}} w_{\{\alpha_\Theta\} \cup \Delta \setminus \Theta} P_\Delta^{\text{opp}}$ .  $\square$

For the elements in the closure of the cone, it will be enough for our purpose to determine their class under the left-right action of  $P_\Theta^{\text{opp}}$ .

**Proposition 9.9.** *Let  $X$  be an element of  $c_{\alpha_\Theta}$ , There is then a unique  $j$  in  $[[0, d]]$  such that*

$$\exp(X) \in P_\Theta^{\text{opp}} w_j P_\Theta^{\text{opp}}.$$

*Proof.* Uniqueness follows from Lemma 9.6.

Since  $c_{\alpha_\Theta}$  is the closure of  $\mathring{c}_{\alpha_\Theta}$ , there exists a sequence  $(X_n)_{n \in \mathbf{N}}$  in  $\mathring{c}_{\alpha_\Theta}$  that converges to  $X$ . For every  $n$  in  $\mathbf{N}$ , let  $g_n$  be an element in  $L_\Theta^\circ$  such that  $X_n = \text{Ad}(g_n) E_{\alpha_\Theta}$ .

The Cartan decomposition in  $L_\Theta^\circ$  gives

$$L_\Theta^\circ = K_\Theta^\circ \exp(\mathfrak{b}^+) K_\Theta^\circ,$$

where  $K_\Theta^\circ = K \cap L_\Theta^\circ$  and  $\mathfrak{b}^+ = \{A \in \mathfrak{a} \mid \chi_i(A) \leq 0, \forall i \in [[1, d]]\}$  is a closed Weyl chamber for the reductive Lie group  $L_\Theta^\circ$ . There exist thus, for all  $n$  in  $\mathbf{N}$ , elements  $k_n, k'_n$  in  $K_\Theta^\circ$ , and  $A_n$  in  $\mathfrak{b}^+$  such that  $g_n = k_n \exp(A_n) k'_n$ . Up to extracting we will assume that the sequence  $(k_n)_{n \in \mathbf{N}}$  is converging and its limit will be denoted  $k_\infty$ . One has therefore, for all  $n$ ,

$$\text{Ad}(\exp(A_n)) E_{\alpha_\Theta} = \text{Ad}(k_n^{-1}) \text{Ad}(g_n) E_{\alpha_\Theta} = \text{Ad}(k_n^{-1}) X_n,$$

so the sequence  $(\text{Ad}(\exp(A_n)) E_{\alpha_\Theta})_{n \in \mathbf{N}}$  is converging and its limit is  $Y = \text{Ad}(k_\infty^{-1}) X$ . As  $K_\Theta^\circ \subset L_\Theta^\circ \subset P_\Theta^{\text{opp}}$ , it will be enough to determine the Bruhat cell of  $\exp(Y)$ .

For all  $n$  in  $\mathbf{N}$  and for all  $i$  in  $[[0, d]]$ , let

$$\lambda_{i,n} = \exp((\alpha_\Theta + 2\chi_1 + \cdots + 2\chi_i)(A_n))$$

so that for all  $n$ , as  $A_n$  belongs to  $\mathfrak{b}^+$ ,  $\lambda_{0,n} \geq \lambda_{1,n} \geq \cdots \geq \lambda_{d,n} > 0$  and

$$\begin{aligned} \sum_{i=0}^d \lambda_{i,n} Z_i &= \sum_{i=0}^d \exp(\text{ad}(A_n)) Z_i \\ &= \exp(\text{ad}(A_n)) \sum_{i=0}^d Z_i = \text{Ad}(\exp(A_n)) E_{\alpha_\Theta}. \end{aligned}$$

This last equality implies that, for all  $i$  in  $[[0, d]]$ , the sequence  $(\lambda_{i,n})_{n \in \mathbf{N}}$  converges in  $\mathbf{R}_{\geq 0}$  and its limit will be denoted  $\lambda_i$ . We have hence

$$Y = \sum_{i=0}^d \lambda_i Z_i,$$

$\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_d \geq 0$ , and the set of indices of the nonzero entries in  $(\lambda_0, \dots, \lambda_d)$  has the form  $[[0, j-1]]$  for some  $j$  in  $[[0, d+1]]$  (this set is  $\emptyset$  when  $j=0$ ). Lemma 9.7 says that  $\exp(Y) \in P_\Delta^{\text{opp}} w_j P_\Delta^{\text{opp}}$  and this implies the wanted result.  $\square$

One can rephrase the previous result:

**Corollary 9.10.** *Let  $Y$  be in  $c_{\alpha_\Theta}$  and let  $w$  be the element of  $W$  such that  $\exp(Y) \in P_\Delta^{\text{opp}} w P_\Delta^{\text{opp}}$ . Then*

- (1)  $\ell_\Theta(w) \leq d + 1$ ;  
(2) if  $\ell_\Theta(w) = d + 1$  then  $Y$  belongs to  $\mathring{c}_{\alpha_\Theta}$  and  $w = w_{\{\alpha_\Theta\} \cup \Delta \setminus \Theta}$ .

**9.4. The Bruhat cells of nonzero elements.** The previous paragraph determines the Bruhat cell of an element of the form  $\exp(X)$  when  $X$  belongs to the cone  $c_{\alpha_\Theta}$ . We now consider the case when  $X$  is only supposed to be a nonzero element of  $\mathfrak{u}_{\alpha_\Theta}$ .

**Lemma 9.11.** *Let  $X$  be a nonzero element of  $\mathfrak{u}_{\alpha_\Theta}$  and let  $x$  be the element of  $W$  such that  $\exp(X) = P_\Delta^{\text{opp}} x P_\Delta^{\text{opp}}$ . Then  $\ell_\Theta(x) > 0$ .*

*Remark 9.12.* The element  $x$  belongs to  $W_{\{\alpha_\Theta\} \cup \Delta \setminus \Theta}$  as  $\exp(X)$  belongs to  $S_{\Theta \setminus \{\alpha_\Theta\}}$ .

*Proof.* One has (Lemma 8.6)  $W_{\Delta \setminus \Theta} = \{x \in W \mid \ell_\Theta(x) = 0\}$ . Furthermore, the following equality holds:

$$\bigcup_{x \in W_{\Delta \setminus \Theta}} P_\Delta^{\text{opp}} x P_\Delta^{\text{opp}} = P_\Theta^{\text{opp}}.$$

Thus the statement will be established if we can prove that  $\exp(X)$  does not belong to  $P_\Theta^{\text{opp}}$ . This last property is a consequence of the fact that the map:

$$\begin{aligned} \mathfrak{u}_\Theta \times P_\Theta^{\text{opp}} &\longrightarrow G \\ (X, g) &\longmapsto \exp(X)g \end{aligned}$$

is an embedding. □

## 10. PARAMETRIZING THE UNIPOTENT POSITIVE SEMIGROUP

In this section, given  $G$  a simple Lie group admitting a  $\Theta$ -positive structure, we give an explicit parametrization of the unipotent positive semigroup  $U_\Theta^{>0}$ . First we give explicit parametrizations of subsets of  $U_\Theta$  associated with a reduced expression of the longest length element in the  $\Theta$ -Weyl group  $W(\Theta)$ . This first step is analogous to the strategy in Lusztig's work on total positivity [Lus94]. In [Lus94] the next step is to show that the image of this parametrization does not depend on the chosen reduced expression, and once this is shown, additional properties of the positive unipotent semigroup are established.

Here, in the next step we first prove some of these crucial topological properties. In particular, we show that the set we parametrize is a connected component of the intersection of the standard unipotent subgroup  $U_\Theta$  with the Bruhat cell  $\Omega_\Theta^{\text{opp}} = P_\Theta^{\text{opp}} w_\Delta P_\Theta^{\text{opp}}$ .

To show that the connected component does not depend on the reduced expression, we make use of the split subgroup  $H_\Theta \subset G$  of type  $W(\Theta)$ .

We fix, for each  $\alpha \in \Theta$ , an  $L_\Theta^0$ -invariant convex cone  $c_\alpha \subset \mathfrak{u}_\alpha$ .

**10.1. Maps associated with reduced expressions in  $W(\Theta)$ .** In this section we define maps from products of the cones to the unipotent group  $U_\Theta$ , which are associated to reduced expressions in the  $\Theta$ -Weyl group.

We consider the element  $w_{\max}^\Theta \in W$ . In Section 8 we saw that  $W(\Theta)$  equipped with its generating set  $R(\Theta)$  (cf. Definition 8.11) is a Coxeter

system of Lie type (cf. Corollary 8.15), and that  $w_{\max}^{\Theta}$  is the longest length element in  $W(\Theta)$ .

We denote by  $N$  the length of  $w_{\max}^{\Theta}$  and by  $\mathbf{W} \subset \Theta^N$  the set of reduced expressions of  $w_{\max}^{\Theta}$ .

For every  $\gamma$  in  $\mathbf{W}$ , the product of cones  $c_{\gamma_1} \times \cdots \times c_{\gamma_N}$  is denoted  $c_{\gamma}$ . We define the map

$$\begin{aligned} F_{\gamma} : c_{\gamma} &\longrightarrow U_{\Theta} \\ (v_1, \dots, v_N) &\longmapsto \exp(v_1) \cdots \exp(v_N). \end{aligned}$$

The image of  $F_{\gamma}$  clearly lies in the nonnegative unipotent semigroup  $U_{\Theta}^{\geq 0}$ . The interior  $\mathring{c}_{\gamma}$  is the product  $\mathring{c}_{\gamma_1} \times \cdots \times \mathring{c}_{\gamma_N}$  and the restriction of  $F_{\gamma}$  to  $\mathring{c}_{\gamma}$  is denoted by  $\mathring{F}_{\gamma}$ . Similar notation,  $c_{\gamma}^{\text{opp}}$ ,  $F_{\gamma}^{\text{opp}}$ , and  $\mathring{F}_{\gamma}^{\text{opp}}$  are adopted for the maps into the opposite unipotent group.

The goal of this section is to show that the image of  $\mathring{F}_{\gamma}$  is independent of  $\gamma$  and coincides with the positive semigroup  $U_{\Theta}^{>0}$ :

**Theorem 10.1.** *For all  $\gamma$  in  $\mathbf{W}$ , the map  $\mathring{F}_{\gamma}$  is a diffeomorphism from  $\mathring{c}_{\gamma}$  onto the unipotent positive semigroup  $U_{\Theta}^{>0}$ , and  $\mathring{F}_{\gamma}^{\text{opp}}$  is a diffeomorphism from  $\mathring{c}_{\gamma}^{\text{opp}}$  onto the opposite unipotent positive semigroup  $U_{\Theta}^{\text{opp}, >0}$ .*

In order to prove Theorem 10.1 we are going to establish several properties of the map  $F_{\gamma}$  and  $\mathring{F}_{\gamma}$  in the following sections.

## 10.2. Equivariance, transversality, injectivity, and openness.

We first note that the map  $F_{\gamma}$  is  $L_{\Theta}^{\circ}$ -equivariant with respect to the adjoint action on  $c_{\gamma}$  and the conjugation action on  $U_{\Theta}$ .

**Lemma 10.2.** *For every  $\gamma$  in  $\mathbf{W}$  the map  $F_{\gamma}$  is  $L_{\Theta}^{\circ}$ -equivariant.*

*Proof.* Let  $\ell$  be in  $L_{\Theta}^{\circ}$  and let  $\mathbf{v} = (v_1, \dots, v_N)$  be in  $c_{\gamma}$ . The action of  $\ell$  on  $\mathbf{v}$  is

$$\ell \cdot \mathbf{v} = (\text{Ad}(\ell)v_1, \dots, \text{Ad}(\ell)v_N)$$

and is again in  $c_{\gamma}$  since the cones  $c_{\alpha}$  are  $L_{\Theta}^{\circ}$ -invariant. The compatibility of the exponential map with the adjoint and conjugation actions implies

$$\begin{aligned} F_{\gamma}(\ell \cdot \mathbf{v}) &= \exp(\text{Ad}(\ell)v_1) \cdots \exp(\text{Ad}(\ell)v_N) \\ &= \ell \exp(v_1) \ell^{-1} \cdots \ell \exp(v_N) \ell^{-1} \\ &= \ell \exp(v_1) \cdots \exp(v_N) \ell^{-1} = \ell F_{\gamma}(\mathbf{v}) \ell^{-1}, \end{aligned}$$

which is the wanted equivariance property.  $\square$

We further establish transversality and injectivity properties of  $\mathring{F}_{\gamma}$ .

**Proposition 10.3.** *For every  $\gamma$  in  $\mathbf{W}$ , the image of the map  $\mathring{F}_{\gamma}$  is contained in the open Bruhat cell  $P_{\Delta}^{\text{opp}} w_{\max}^{\Theta} P_{\Theta}^{\text{opp}} = P_{\Theta}^{\text{opp}} w_{\max}^{\Theta} P_{\Theta}^{\text{opp}} = \Omega_{\Theta}^{\text{opp}}$  and the map*

$$\mathring{c}_{\gamma} \longrightarrow F_{\Theta} \mid \mathbf{v} \longmapsto \mathring{F}_{\gamma}(\mathbf{v}) \cdot \mathfrak{p}_{\Theta}^{\text{opp}}$$

*is injective.*

Note that, since the map  $U_{\Theta} \rightarrow F_{\Theta} \mid u \mapsto u \cdot \mathfrak{p}_{\Theta}^{\text{opp}}$  is an embedding, the injectivity of the above map is equivalent to the injectivity of  $\mathring{F}_{\gamma}$ .

Proposition 10.3 will be proved thanks to an inductive process whose results are of independent interest. For this it will be a little more convenient to have a decreasing numbering for the indices of  $\gamma$ :  $\gamma = (\gamma_N, \gamma_{N-1}, \dots, \gamma_1)$ . With this notation the cone  $c_\gamma$  is the product  $c_{\gamma_N} \times \dots \times c_{\gamma_1}$ .

Set, for all  $j$  in  $\llbracket 1, N \rrbracket$ ,  $x_j = \sigma_{\gamma_j} \cdots \sigma_{\gamma_1}$ , this is an element of  $W(\Theta)$ .

**Proposition 10.4.** *For every  $j$  in  $\llbracket 1, N \rrbracket$ ,*

(1) *The image of the map*

$$\begin{aligned} \mathring{F}_j : \mathring{c}_{\gamma_j} \times \dots \times \mathring{c}_{\gamma_1} &\longrightarrow U_\Theta \\ (X_j, \dots, X_1) &\longmapsto \exp(X_j) \cdots \exp(X_1) \end{aligned}$$

*is contained in  $P_\Delta^{\text{opp}} x_j P_\Theta^{\text{opp}} = P_\Theta^{\text{opp}} x_j P_\Theta^{\text{opp}}$  (the equality follows from Lemma 9.2 since  $x_j$  belongs to  $W(\Theta)$ );*

(2) *The map*

$$\begin{aligned} \mathbf{u}_{\gamma_j} \times \mathring{c}_{\gamma_{j-1}} \times \dots \times \mathring{c}_{\gamma_1} &\longrightarrow \mathbf{F}_\Theta \\ (X_j, \dots, X_1) &\longmapsto \exp(X_j) \cdots \exp(X_1) \cdot \mathbf{p}_\Theta^{\text{opp}} \end{aligned}$$

*is injective.*

*Proof.* We prove first (1) by induction on  $j$  in  $\llbracket 1, N \rrbracket$ . For  $j = 1$  in the proof below  $F_{j-1}(X_{j-1}, \dots, X_1)$  and  $x_{j-1}$  should be replaced by the identity element.

Let  $(X_j, \dots, X_1)$  be in  $\mathring{c}_{\gamma_j} \times \dots \times \mathring{c}_{\gamma_1}$ . If  $\gamma_j$  is not equal to  $\alpha_\Theta$ , then  $\exp(X_j)$  belongs to  $P_\Delta^{\text{opp}} s_{\gamma_j} P_\Delta^{\text{opp}} = P_\Delta^{\text{opp}} \sigma_{\gamma_j} P_\Delta^{\text{opp}} \subset P_\Delta^{\text{opp}} \sigma_{\gamma_j} P_\Theta^{\text{opp}}$  (see Section 9.1). If  $\gamma_j = \alpha_\Theta$ , then  $\sigma_{\gamma_j} = \sigma_{\alpha_\Theta}$  and, by Proposition 9.8,  $\exp(X_j)$  belongs to

$$P_\Delta^{\text{opp}} \sigma_{\alpha_\Theta} w_{\Delta \setminus \Theta} P_\Delta^{\text{opp}} \subset P_\Delta^{\text{opp}} \sigma_{\alpha_\Theta} P_\Theta^{\text{opp}}.$$

By induction (or trivially if  $j = 1$ )  $\mathring{F}_{j-1}(X_{j-1}, \dots, X_1)$  belongs to  $P_\Delta^{\text{opp}} x_{j-1} P_\Theta^{\text{opp}}$ .

We deduce that  $\mathring{F}_j(X_j, \dots, X_1)$  belongs to

$$P_\Delta^{\text{opp}} \sigma_j P_\Theta^{\text{opp}} x_{j-1} P_\Theta^{\text{opp}} = P_\Delta^{\text{opp}} \sigma_{\alpha_\Theta} P_\Delta^{\text{opp}} x_{j-1} P_\Theta^{\text{opp}},$$

where the equality holds by Lemma 9.2 applied to  $x_{j-1}$ ,

$$= P_\Delta^{\text{opp}} \sigma_{\alpha_\Theta} x_{j-1} P_\Theta^{\text{opp}}$$

since  $x_j = \sigma_{\alpha_\Theta} x_{j-1}$  is a reduced product by Lemma 8.17. This concludes the induction step for (1).

We now prove point (2) by induction on  $j$  again. For  $j = 1$ , injectivity follows from the fact that  $\mathbf{u}_\Theta \times P_\Theta^{\text{opp}} \mid (X, g) \mapsto \exp(X)g$  is injective.

Suppose now that  $j \geq 2$  and that the inductive hypothesis has been established up to  $j - 1$ . Let  $(X_j, \dots, X_1)$  and  $(Y_j, \dots, Y_1)$  be in  $\mathbf{u}_{\gamma_j} \times \mathring{c}_{\gamma_{j-1}} \times \dots \times \mathring{c}_{\gamma_1}$  such that

$$F_j(X_j, \dots, X_1) P_\Theta^{\text{opp}} = F_j(Y_j, \dots, Y_1) P_\Theta^{\text{opp}},$$

this means that

(10.1)

$$\begin{aligned} \mathring{F}_{j-1}(X_{j-1}, \dots, X_1) P_\Theta^{\text{opp}} &= \exp(-X_j) \exp(Y_j) \mathring{F}_{j-1}(Y_{j-1}, \dots, Y_1) P_\Theta^{\text{opp}} \\ &= \exp(Y_j - X_j) \mathring{F}_{j-1}(Y_{j-1}, \dots, Y_1) P_\Theta^{\text{opp}}, \end{aligned}$$

where the last equality holds since  $\mathbf{u}_{\gamma_j}$  is Abelian (Proposition 3.9). By point (1) we have that  $\mathring{F}_{j-1}(X_{j-1}, \dots, X_1)$  and  $\mathring{F}_{j-1}(Y_{j-1}, \dots, Y_1)$  belong to  $P_{\Theta}^{\text{opp}} x_{j-1} P_{\Theta}^{\text{opp}}$ . Suppose that  $X_j \neq Y_j$  then (by Lemma 9.11) we have that  $\exp(Y_j - X_j)$  belongs to  $P_{\Delta}^{\text{opp}} x P_{\Delta}^{\text{opp}}$  with  $\ell_{\Theta}(x) > 0$ . Using again the multiplicative properties of Bruhat cells we deduce that the element  $y$  such that

$$\exp(Y_j - X_j) \mathring{F}_{j-1}(Y_{j-1}, \dots, Y_1) \in P_{\Delta}^{\text{opp}} y P_{\Theta}^{\text{opp}}$$

satisfies  $\ell_{\Theta}(y) > \ell_{\Theta}(x_{j-1})$  which is a contradiction with Equation (10.1). Hence  $X_j = Y_j$  and (10.1) together with the induction hypothesis imply that  $X_i = Y_i$  for all  $i$  less than  $j$ .  $\square$

We can now deduce that the maps  $\mathring{F}_{\gamma}$  are open. More precisely, using again the notation of the proposition, and recalling that  $C(x_j) \subset F_{\Theta}$  is the  $U_{\Theta}^{\text{opp}}$ -orbit of the element  $x_j \cdot \mathfrak{p}_{\Theta}^{\text{opp}}$ , we have:

**Corollary 10.5.** *For all  $j$  the map*

$$\begin{aligned} \mathring{c}_{\gamma_j} \times \dots \times \mathring{c}_{\gamma_1} &\longrightarrow C(x_j) \subset F_{\Theta} \\ (X_j, \dots, X_1) &\longmapsto \mathring{F}_j(X_j, \dots, X_1) \cdot \mathfrak{p}_{\Theta}^{\text{opp}} \end{aligned}$$

*is open.*

*Proof.* We know that this map is injective. By Invariance of Domain, it is enough to show that the dimensions of the source and the range coincide. This can be proved again by induction on  $j$ , using the relation  $x_j = \sigma_{\gamma_j} x_{j-1}$  and using Proposition 9.4 which says:

$$\dim C(x_j) - \dim C(x_{j-1}) = \dim \mathbf{u}_{\gamma_j} = \dim c_{\gamma_j}.$$

This is the precise relation needed to show that the induction step holds true.  $\square$

**Corollary 10.6.** *For any  $\gamma$  in  $\mathbf{W}$ , the map  $\mathring{c}_{\gamma} \rightarrow F_{\Theta} \mid \mathbf{v} \mapsto \mathring{F}_{\gamma}(\mathbf{v}) \cdot \mathfrak{p}_{\Theta}^{\text{opp}}$  is open.*

**10.3. Nontransversality at the boundary.** We now establish, that elements in the boundary of  $c_{\gamma}$  are not sent in the open Bruhat cell  $\Omega_{\Theta}^{\text{opp}}$ .

**Proposition 10.7.** *Let  $\mathbf{v}$  be in  $c_{\gamma}$ . If  $\mathbf{v}$  does not belong to  $\mathring{c}_{\gamma}$ , then  $F_{\gamma}(\mathbf{v})$  does not belong to  $P_{\Delta}^{\text{opp}} w_{\max}^{\Theta} P_{\Theta}^{\text{opp}}$ .*

*Proof.* We use again natural numbering for the components of  $\gamma$ :  $\gamma = (\gamma_1, \dots, \gamma_N)$ . We denote  $(v_1, \dots, v_N)$  the components of  $\mathbf{v}$ . Since  $\mathbf{v} \notin \mathring{c}_{\gamma}$ , there exists  $j$  such that  $v_j$  is in  $c_{\alpha_j} \setminus \mathring{c}_{\alpha_j}$ . For all  $i$  in  $\llbracket 1, N \rrbracket$  denote by  $t_i$  an element of  $W$  such that  $\exp(v_i)$  belongs to  $P_{\Theta}^{\text{opp}} t_i P_{\Theta}^{\text{opp}}$ . We have  $\ell_{\Theta}(t_i) \leq \ell_{\Theta}(\sigma_{\alpha_i})$ . Since  $t_j = e$  when  $\alpha_j \neq \alpha_{\Theta}$  or by Corollary 9.10 when  $\alpha_j = \alpha_{\Theta}$ , we have that  $\ell_{\Theta}(t_j) < \ell_{\Theta}(\sigma_{\alpha_j})$ .

Repeated applications of Lemma 9.1 show that the element  $x$  of  $W$  such that  $F_{\gamma}(\mathbf{v})$  belongs to  $P_{\Theta}^{\text{opp}} x P_{\Theta}^{\text{opp}}$  satisfies

$$\ell_{\Theta}(x) \leq \sum_{i=1}^N \ell_{\Theta}(t_i) < \sum_{i=1}^N \ell_{\Theta}(\sigma_{\alpha_i}) = \ell_{\Theta}(x_{\max}^{\Theta}).$$

This implies the result.  $\square$

**Corollary 10.8.** *We have  $F_{\gamma}(c_{\gamma} \setminus \mathring{c}_{\gamma}) \subset U_{\Theta} \setminus \Omega_{\Theta}^{\text{opp}}$ .*

**Corollary 10.9.** *We have  $F_{\gamma}(\mathring{c}_{\gamma}) = F_{\gamma}(c_{\gamma}) \cap \Omega_{\Theta}^{\text{opp}} \subset U_{\Theta}^{\geq 0} \cap \Omega_{\Theta}^{\text{opp}}$ .*



**10.4. Properness.** To prove properness of the maps  $F_\gamma$  we consider the composition of  $\log \circ F_\gamma: c_\gamma \rightarrow \mathfrak{u}_\Theta$  with the projection  $p_\alpha: \mathfrak{u}_\Theta \rightarrow \mathfrak{u}_\alpha$  for  $\alpha$  varying in  $\Theta$ . As the projection  $p_\alpha$  is a Lie algebra homomorphism and by applying the Baker–Campbell–Hausdorff formula, we have for all  $\mathbf{v} = (v_i)_{i \in \llbracket 1, N \rrbracket}$  in  $c_\gamma$

$$(10.2) \quad p_\alpha(\log \circ F_\gamma(\mathbf{v})) = \sum_{\substack{i \in \llbracket 1, N \rrbracket \\ \gamma_i = \alpha}} v_i.$$

**Proposition 10.10.** *The map  $F_\gamma: c_\gamma \rightarrow U_\Theta$  is proper. More precisely, there is a constant  $C \geq 0$  such that, for any  $\mathbf{v} \in c_\gamma$ ,  $\max_i \|v_i\| \leq C \max_{\alpha \in \Theta} \|p_\alpha \circ \log(F_\gamma(\mathbf{v}))\|$ .*

*Proof.* For any acute closed convex cone  $c$  (in a finite dimensional normed real vector space) and for any  $n \geq 1$ , the map  $c^n \rightarrow c \mid (x_1, \dots, x_n) \mapsto x_1 + \dots + x_n$  is Lipschitz: there is constant  $D$  (depending on  $c$  and  $n$ ) such that  $\max \|x_i\| \leq D \|x_1 + \dots + x_n\|$ . Together with the formula (10.2), this implies the precise control on  $\mathbf{v} \mapsto (p_\alpha \circ \log(F_\gamma(\mathbf{v})))_{\alpha \in \Theta}$ .  $\square$

**10.5. Connectedness.** We now prove that the image of  $\mathring{F}_\gamma$  is a connected component of the intersection of  $U_\Theta$  with the open Bruhat cell  $\Omega_\Theta^{\text{opp}}$ .

**Theorem 10.11.** *The set  $F_\gamma(\mathring{c}_\gamma)$  is a connected component of  $\Omega_\Theta^{\text{opp}} \cap U_\Theta$ .*

*Proof.* Corollary 10.5 and the fact that  $U_\Theta \rightarrow C(w_{\max}^\Theta) \mid g \mapsto g \cdot \mathfrak{p}_\Theta^{\text{opp}}$  is a diffeomorphism imply that the set  $F_\gamma(\mathring{c}_\gamma)$  is open in  $U_\Theta$ . Since the map  $F_\gamma|_{c_\gamma}$  is proper, we have that the closure of  $F_\gamma(\mathring{c}_\gamma)$  in  $U_\Theta$  is equal to  $F_\gamma(c_\gamma)$ . Therefore the closure of  $F_\gamma(\mathring{c}_\gamma)$  in  $\Omega_\Theta^{\text{opp}} \cap U_\Theta$  is equal to  $F_\gamma(c_\gamma) \cap \Omega_\Theta^{\text{opp}}$  hence to  $F_\gamma(\mathring{c}_\gamma)$  by Corollary 10.9. Thus  $F_\gamma(\mathring{c}_\gamma)$  is open and closed in  $\Omega_\Theta^{\text{opp}} \cap U_\Theta$ . Since it is connected, it is one connected component.  $\square$

**10.6. Independence on  $\gamma$ .** A priori the connected component  $F_\gamma(\mathring{c}_\gamma)$  could depend on the choice of  $\gamma \in \mathbf{W}$ . We show now that this is not the case. For this it is sufficient that given any two elements  $\gamma, \gamma' \in \mathbf{W}$ , the intersection of  $F_\gamma(\mathring{c}_\gamma)$  and  $F_{\gamma'}(\mathring{c}_{\gamma'})$  is nonempty. To prove this, we first clarify the relation between  $F_\gamma$  and Lusztig's positive unipotent semigroup in the split real subgroup generated by the  $\Theta$ -system  $(E_\alpha, F_\alpha, D_\alpha)_{\alpha \in \Theta}$ .

**10.6.1. Lusztig's positive semigroup.** Recall that the Lie subalgebra  $\mathfrak{h}_\Theta$  generated by the  $\Theta$ -system  $(E_\alpha, F_\alpha, D_\alpha)_{\alpha \in \Theta}$  is a split real Lie algebra of type  $W(\Theta)$ . We denote by  $H_\Theta < G$  the connected Lie subgroup with Lie algebra  $\mathfrak{h}_\Theta$ . Note that the intersection  $\mathfrak{h}_\Theta \cap \mathfrak{p}_\Theta$  is a standard minimal parabolic subalgebra in  $\mathfrak{h}_\Theta$ , and  $\mathfrak{n}_\Theta = \mathfrak{g}_\Theta \cap \mathfrak{u}_\Theta$  is its unipotent radical. We denote by  $N_\Theta$  the corresponding subgroup, one has  $N_\Theta = U_\Theta \cap H_\Theta$ .

The Lie algebra  $\mathfrak{n}_\Theta$  is generated by the elements  $(E_\alpha)_{\alpha \in \Theta}$ . Recall that  $E_\alpha \in \mathring{c}_\alpha$  for all  $\alpha \in \Theta$ , more precisely  $\mathfrak{n}_\Theta \cap \mathring{c}_\alpha = \mathbf{R}_{>0} E_\alpha$ . Given an element  $\gamma \in \mathbf{W}$  we can restrict the map  $F_\gamma$  to  $\mathring{c}_\gamma^H = \prod_{i=1}^N \mathbf{R}_{>0} E_{\gamma_i}$ .

We have

**Proposition 10.12.** *The set  $N_{\Theta}^{>0} = F_{\gamma}(\mathring{c}_{\gamma}^H)$  is the unipotent positive semigroup of  $H_{\Theta}$ . It is independent of the choice of  $\gamma \in \mathbf{W}$ .*

*Sketch of proof.* The proposition follows from Proposition 2.7.(b) in Lusztig's work [Lus94]. Using the material developed above, one can also argue as follows:

The image  $F_{\gamma}(\mathring{c}_{\gamma}^H)$  is a connected component of  $\Omega_{\Theta}^{\text{opp}} \cap N_{\Theta}$ . It is thus enough to show that the images  $F_{\gamma}(\mathring{c}_{\gamma}^H)$  and  $F_{\alpha}(\mathring{c}_{\alpha}^H)$  intersect when  $\alpha$  and  $\gamma$  differ by a braid relation. Focusing on the entries where  $\alpha$  and  $\gamma$  differ, we are reduced to the case of rank 2 split real Lie groups. The situation is obvious when this group is  $\text{SL}_2(\mathbf{R}) \times \text{SL}_2(\mathbf{R})$ .

For the group  $\text{SL}_3(\mathbf{R})$ , denoting  $x_1: \mathbf{R} \rightarrow U_{\Delta} \mid t \mapsto \text{Id} + tE_{1,2}$  and  $x_2: \mathbf{R} \rightarrow U_{\Delta} \mid t \mapsto \text{Id} + tE_{2,3}$  the unipotent 1-parameter groups corresponding to the simple roots  $\alpha_1$  and  $\alpha_2$ , there are here 2 reduced expressions of the longest length element:  $s_1s_2s_1 = s_2s_1s_2$  and the 2 corresponding maps are  $(t_1, t_2, t_3) \mapsto x_1(t_1)x_2(t_2)x_1(t_3)$  and  $(t_1, t_2, t_3) \mapsto x_2(t_1)x_1(t_2)x_2(t_3)$  and one checks that  $x_1(1)x_2(2)x_1(1) = x_2(1)x_1(2)x_2(1)$  showing the result in this case.

For the group  $\text{Sp}_4(\mathbf{R})$ , the unipotent 1-parameter groups are  $x_1: \mathbf{R} \rightarrow U_{\Delta} \mid t \mapsto \text{Id} + t(E_{1,2} + E_{3,4})$  and  $x_2: \mathbf{R} \rightarrow U_{\Delta} \mid t \mapsto \text{Id} + tE_{2,3}$  and one checks the relation  $x_1(1)x_2(2)x_1(3)x_2(1) = x_2(1)x_1(3)x_2(2)x_1(1)$ .

For the case of  $G_2$ , we refer to [BZ97].  $\square$

*Remark 10.13.* Lusztig has an argument to reduce to the simply laced cases and this enables him to treat only the case of  $\text{SL}_3(\mathbf{R})$  where he writes down an explicit formula for the change of coordinates between the images of the maps associated to these two reduced expressions. This is more than what we used here.

As a corollary we obtain the following proposition.

**Proposition 10.14.** *The image  $F_{\gamma}(\mathring{c}_{\gamma})$  is independent of the choice of  $\gamma \in \mathbf{W}$ .*

*Proof.* Since  $E_{\alpha} \in \mathring{c}_{\alpha}$  for all  $\alpha \in \Theta$ , and  $\mathring{c}_{\alpha}$  are convex cones, for any  $\gamma$  in  $\mathbf{W}$  we have that  $F_{\gamma}(\mathring{c}_{\gamma}^H) \subset F_{\gamma}(\mathring{c}_{\gamma})$ . Let  $\gamma, \gamma' \in \mathbf{W}$  be two different reduced expressions of the longest length element in  $W(\Theta)$ , then  $F_{\gamma}(\mathring{c}_{\gamma}) \cap F_{\gamma'}(\mathring{c}_{\gamma'})$  contains  $N_{\Theta}^{>0} = F_{\gamma}(\mathring{c}_{\gamma}^H) = F_{\gamma'}(\mathring{c}_{\gamma'}^H)$ . Therefore  $F_{\gamma}(\mathring{c}_{\gamma})$  and  $F_{\gamma'}(\mathring{c}_{\gamma'})$  have nonempty intersection. Since both sets are connected components of  $\Omega_{\Theta}^{\text{opp}} \cap U_{\Theta}$ , they have to be equal.  $\square$

*Remark 10.15.* We can get in fact precise formulas for the change of coordinates. Two elements  $\gamma$  and  $\gamma'$  differ by a sequence of braid relation. In the split case with  $\Theta = \Delta$  explicit formulas for the change of coordinates for the braid relations are given in [Lus94] for the simply laced case, and in [BZ97] for the general case. In a forthcoming article [GW24] we describe explicit systems of polynomial equations for the general  $\Theta$ -positive case following a similar approach. This gives rise to explicit coordinate changes as well.

**10.7. Parametrizations of the positive unipotent semigroup.** In this section we draw some consequences on the positive and nonnegative unipotent semigroups. In particular we prove that the maps  $\mathring{F}_{\gamma}$  provide parametrizations of  $U_{\Theta}^{>0}$  for any  $\gamma$  in  $\mathbf{W}$ .

By Proposition 10.14, the set  $V = F_\gamma(\mathring{c}_\gamma)$  is independent of the choice of  $\gamma \in \mathbf{W}$ .

We still have to show that this set coincides with  $U_\Theta^{\geq 0}$ . For this we prove first.

**Proposition 10.16.** *The closure  $\overline{V}$  is equal to  $F_\gamma(c_\gamma)$  for every  $\gamma$  in  $\mathbf{W}$ . For all  $\alpha \in \Theta$  and for all  $v_\alpha \in c_\alpha$ ,  $V$  is invariant under left and right multiplication by  $\exp(v_\alpha)$ .*

*Proof.* The properness of  $F_\gamma$  implies the first statement.

We prove only the invariance by left multiplication, the case of right multiplication follows by entirely analogous arguments.

Let  $v_\alpha \in c_\alpha$  for some  $\alpha \in \Theta$ , and let  $u$  be an element in  $V$ . We want to prove that then  $\exp(v_\alpha)u \in V$ . Let us choose  $\gamma \in \mathbf{W}$  such that  $\gamma_1 = \alpha$ . The existence of such a reduced expression  $\gamma$  of the longest length element  $w_{\max}^\Theta$  in  $(W(\Theta), R(\Theta))$  is a classical fact and is established starting from the equality  $w_{\max}^\Theta = \sigma_\alpha(\sigma_\alpha w_{\max}^\Theta)$  and from a reduced expression of  $\sigma_\alpha w_{\max}^\Theta$ , indeed since  $w_{\max}^\Theta$  is the longest length element of  $W(\Theta)$ , the element  $\sigma_\alpha w_{\max}^\Theta$  is of smaller length and  $\sigma_\alpha(\sigma_\alpha w_{\max}^\Theta)$  is a reduced product. Let also  $\mathbf{v}$  be in  $\mathring{c}_\gamma$  such that  $F_\gamma(\mathbf{v}) = u$ . Then we have

$$\begin{aligned} \exp(v_\alpha)u &= \exp(v_\alpha) \exp(v_1) \cdots \exp(v_N) \\ &= \exp(v_\alpha + v_1) \cdots \exp(v_N), \end{aligned}$$

where we used that  $\gamma$  starts with  $\alpha$  and that  $\mathfrak{u}_\alpha$  is Abelian. Since  $c_\alpha$  is a convex cone,  $v_1 \in \mathring{c}_\alpha$  and  $v_\alpha \in c_\alpha$  we have that  $v_\alpha + v_1 \in \mathring{c}_\alpha$ . Therefore  $\exp(v_\alpha) \cdot u$  belongs to  $F_\gamma(\mathring{c}_\gamma) = V$ .  $\square$

As a consequence of this proposition we obtain

**Corollary 10.17.** (1) *The inclusions  $U_\Theta^{\geq 0}V \subset V$ , and  $VU_\Theta^{\geq 0} \subset V$  hold.*

(2) *The inclusions  $U_\Theta^{\geq 0}\overline{V} \subset \overline{V}$ , and  $\overline{V}U_\Theta^{\geq 0} \subset \overline{V}$  hold.*

(3)  *$V$  is a semigroup.*

(4) *One has  $\overline{V} = U_\Theta^{\geq 0}$ .*

(5) *One has  $V = U_\Theta^{\geq 0}$ .*

*Proof.* The first point (1) is a direct consequence of Proposition 10.16 and the fact that the semigroup  $U_\Theta^{\geq 0}$  is generated by  $\bigcup_{\alpha \in \Theta} \exp(c_\alpha)$ .

By taking closure in  $uV \subset V$  (for  $u$  in  $U_\Theta^{\geq 0}$ ), we get point (2).

Since  $V \subset U_\Theta^{\geq 0}$ , point (1) implies that  $VV \subset V$ , i.e. that  $V$  is a semigroup.

One has  $\overline{V} = F_\gamma(c_\gamma) \subset U_\Theta^{\geq 0}$  and, using point (2) with the fact that  $e$  belongs to  $\overline{V}$ , we also have  $U_\Theta^{\geq 0} \subset \overline{V}$ . This proves point (4).

Point (5) now follows from Lemma 4.6.(4).  $\square$

We can now record the result proved so far with:

**Theorem 10.18.** *For any  $\gamma \in \mathbf{W}$ , the map  $\mathring{F}_\gamma: \mathring{c}_\gamma \rightarrow U_\Theta$  gives a parametrization of the positive unipotent semigroup  $U_\Theta^{\geq 0}$ .*

And furthermore we proved also:

**Corollary 10.19.** (1) *The inclusions  $U_\Theta^{\geq 0}U_\Theta^{\geq 0} \subset U_\Theta^{\geq 0}$ , and  $U_\Theta^{\geq 0}U_\Theta^{\geq 0} \subset U_\Theta^{\geq 0}$  hold.*

- (2) The nonnegative unipotent semigroup  $U_{\Theta}^{\geq 0}$  is equal to the closure of  $U_{\Theta}^{> 0}$ , in particular it is closed.
- (3) The nonnegative unipotent semigroup  $U_{\Theta}^{\geq 0}$  is the image  $F_{\gamma}(c_{\gamma})$  for every  $\gamma$  in  $\mathbf{W}$ . In particular, every element in  $U_{\Theta}^{\geq 0}$  can be written as a finite product (of length at most  $N$ ) of elements of the form  $\exp(v_{\alpha})$  with  $v_{\alpha} \in c_{\alpha}$ .
- (4) The semigroup  $U_{\Theta}^{\geq 0}$  is equal to the intersection of  $U_{\Theta}^{\geq 0}$  with  $\Omega_{\Theta}^{\text{opp}}$ .

*Remark 10.20.* Note that  $F_{\gamma}$  is in general not injective on  $c_{\gamma}$ , and thus, point (3) of Corollary 10.19 does not give a parametrization of  $U_{\Theta}^{\geq 0}$ . However, we expect that the restriction of  $F_{\gamma}$  to appropriate subsets of  $c_{\gamma}$  will give a parametrization of  $U_{\Theta}^{\geq 0}$ . We describe this in detail for the orthogonal group  $G = \text{SO}(3, q)$ ,  $q \geq 4$  in Section 11.

**10.8. Tangent cone of the semigroup.** We now determine the tangent cone at  $e$  of the semigroup  $U_{\Theta}^{\geq 0}$ . For this we introduce some notation.

Let  $d$  be the degree of nilpotency of the Lie algebra  $\mathfrak{u}_{\Theta}$  (i.e. the iterated Lie brackets of  $d + 1$  elements of  $\mathfrak{u}_{\Theta}$  is always zero). Let  $\mathfrak{e}_{N,d}$  be the free degree  $d$  nilpotent real Lie algebra generated by elements  $e_1, \dots, e_N$ . Then  $\mathfrak{e}_{N,d}$  is a finite dimensional graded Lie algebra whose degree 1 component is equal to  $\bigoplus_{i=1}^N \mathbf{R}e_i$ .

Let  $\gamma$  be in  $\mathbf{W}$ .

For any  $\mathbf{v} = (v_1, \dots, v_N)$  in  $\mathfrak{u}_{\gamma_1} \times \dots \times \mathfrak{u}_{\gamma_N}$  the unique Lie algebra morphism  $\mathfrak{e}_{N,d} \rightarrow \mathfrak{u}_{\Theta}$  sending, for each  $i$  in  $\llbracket 1, N \rrbracket$ ,  $e_i$  to  $v_i$  will be denoted  $\Psi_{\mathbf{v}}$ . The Lie algebra  $\mathfrak{u}_{\Theta}$  is also graded (with degree 1 component equal to  $\bigoplus_{\alpha \in \Theta} \mathfrak{u}_{\alpha}$ ) and the map  $\Psi_{\mathbf{v}}$  is in fact a graded morphism.

The simply connected Lie group with Lie algebra  $\mathfrak{e}_{N,d}$  will be denoted  $E_{N,d}$ . It is well known that  $\exp: \mathfrak{e}_{N,d} \rightarrow E_{N,d}$  is a diffeomorphism whose inverse will be denoted by  $\log$  and the map  $\mathfrak{e}_{N,d} \times \mathfrak{e}_{N,d} \rightarrow \mathfrak{e}_{N,d} \mid (X, Y) \mapsto \log(\exp(X)\exp(Y))$  is given by the Baker–Campbell–Hausdorff formula.

Let us introduce the following element of  $\mathfrak{e}_{N,d}$

$$\Gamma := \log(\exp(e_1) \cdots \exp(e_N)).$$

Then, for every  $\mathbf{v}$  in  $\mathfrak{u}_{\gamma_1} \times \dots \times \mathfrak{u}_{\gamma_N}$ , the following holds

$$\log(\exp(v_1) \cdots \exp(v_N)) = \Psi_{\mathbf{v}}(\Gamma).$$

The Baker–Campbell–Hausdorff formula implies that the degree 1 component of  $\Gamma$  is equal to

$$(10.3) \quad \Gamma_1 = \sum_{i=1}^N e_i;$$

thus  $\Gamma - \Gamma_1$  the sum of components of higher degree.

From the fact that  $\Psi_{\mathbf{v}}$  preserves the degree, we deduce:

**Lemma 10.21.** *Let  $\|\cdot\|$  be a norm on  $\mathfrak{u}_{\Theta}$ . There is a constant  $C \geq 0$  such that, for every  $\mathbf{v}$  in  $\mathfrak{u}_{\gamma_1} \times \dots \times \mathfrak{u}_{\gamma_N}$ ,*

- (1) *The degree 1 component of  $\log(\exp(v_1) \cdots \exp(v_N))$  is equal to  $\psi_{\mathbf{v}}(\Gamma_1) = \sum_{i=1}^N v_i$*

- (2) We have  $\log(\exp(v_1) \cdots \exp(v_N)) - \sum_{i=1}^N v_i = \psi_{\mathbf{v}}(\Gamma - \Gamma_1)$  and
- $$\|\psi_{\mathbf{v}}(\Gamma - \Gamma_1)\| \leq C \max_i \|v_i\|^2.$$

The projection  $\mathfrak{u}_{\Theta} \rightarrow \bigoplus_{\alpha \in \Theta} \mathfrak{u}_{\alpha}$  according to the decomposition  $\mathfrak{u}_{\Theta} = (\bigoplus_{\alpha \in \Theta} \mathfrak{u}_{\alpha}) \oplus (\bigoplus_{\alpha \in P(\mathfrak{u}_{\Theta}, \mathfrak{t}_{\Theta}) \setminus \Theta} \mathfrak{u}_{\alpha})$  is denoted by  $p_1$ .

**Proposition 10.22.** *Let  $c := \bigoplus_{\alpha \in \Theta} c_{\alpha} \subset \mathfrak{u}_{\Theta}$ . Then  $c$  is the tangent cone at  $e$  of the semigroup  $U_{\Theta}^{\geq 0}$ , precisely*

- (1) *There is a constant  $C$  such that, for every  $X$  in  $\log(U_{\Theta}^{\geq 0}) \subset \mathfrak{u}_{\Theta}$ ,  $p_1(X)$  belongs to  $c$  and  $\|X - p_1(X)\| \leq C\|X\|^2$ .*
- (2) *For any continuous map  $\gamma: [0, 1] \rightarrow U_{\Theta}^{\geq 0}$ , if  $\gamma$  is differentiable at 0 then  $\gamma'(0)$  belongs to  $c$ .*
- (3) *For all  $Y$  in  $c$ , there is  $X$  in  $\log(U_{\Theta}^{\geq 0})$  such that  $p_1(X) = Y$ .*
- (4) *For all  $Y$  in  $c$ , the map  $\gamma: \mathbf{R}_{\geq 0} \rightarrow U_{\Theta} \mid t \mapsto \exp(tX)$  is of class  $C^{\infty}$ , contained in  $U_{\Theta}^{\geq 0}$  and its derivative at 0 is equal to  $Y$ .*

*Proof.* The first item (1) is a direct consequence of the estimates given in point (2) of Lemma 10.21 and in Proposition 10.10.

The second item (2) is a consequence of the first one.

The third item (3) follows easily from the formula for the map  $\mathbf{v} \mapsto p_1 \circ \log(F_{\gamma}(\mathbf{v}))$  (cf. Section 10.4).

For item (4), it is classical that this map is  $C^{\infty}$  and that its derivative at 0 is  $Y$ . Let  $\gamma$  be in  $\mathbf{W}$  and let also  $\mathbf{v}$  be in  $c_{\gamma}$  be such that  $p_1 \circ \log(F_{\gamma}(\mathbf{v})) = Y$ . Let  $t$  be in  $\mathbf{R}_{\geq 0}$ . For all  $n$  in  $\mathbf{N}_{>0}$ ,  $t\mathbf{v}/n$  belongs to  $c_{\gamma}$  so that

$$F_{\gamma}(t\mathbf{v}/n) \text{ and } F_{\gamma}(t\mathbf{v}/n)^n$$

belong to  $U_{\Theta}^{\geq 0}$ . Since the sequence  $(F_{\gamma}(t\mathbf{v}/n)^n)_{n>0}$  converges to  $\exp(tY)$  and since  $U_{\Theta}^{\geq 0}$  is closed, we deduce that  $\exp(tY)$  belongs to  $U_{\Theta}^{\geq 0}$ .  $\square$

*Remark 10.23.* The tangent cone at  $e$  of  $U_{\Theta}^{\geq 0}$  is open if and only if  $\Theta$  consists of one element, i.e. if and only if the Lie group  $G$  is Hermitian of tube type. Otherwise the tangent cone is of positive codimension. We refer to Figure 1 (Section 12.4) for a visualization of this phenomenon in the case of  $\mathrm{SL}_3(\mathbf{R})$ . A related phenomenon has already been observed in the case of total positivity in [Lus98] where the 1-parameter semigroups contained in the unipotent positive semigroup are determined.

As a consequence we also obtain a proof of

**Theorem 10.24.** [GW18, Conjecture 4.11] *Let  $v = \sum_{\alpha \in \Theta} v_{\alpha}$  with  $v_{\alpha} \in \mathring{c}_{\alpha}$  for all  $\alpha \in \Theta$ . Then  $\exp(v) \in U_{\Theta}^{\geq 0}$ . Conversely, if  $v \in \mathfrak{u}_{\Theta}$  with  $\exp(tv) \in U_{\Theta}^{\geq 0}$  for all  $t$  in  $(0, \varepsilon)$  (for some  $\varepsilon > 0$ ), then  $v = \sum_{\alpha \in \Theta} v_{\alpha}$  with  $v_{\alpha} \in \mathring{c}_{\alpha}$  for all  $\alpha \in \Theta$ .*

*Proof.* Let us prove the first statement. Let  $E = \sum_{\alpha \in \Theta} q_{\alpha}^{1/2} E_{\alpha}$  be the element of the  $\Theta$ -principal  $\mathfrak{sl}_2$ -triple  $(E, F, D)$  given in Section 7. Since  $L_{\Theta}^{\circ}$  acts transitively on  $\prod_{\alpha \in \Theta} \mathring{c}_{\alpha}$  (Proposition 5.1), there exists  $\ell \in L_{\Theta}^{\circ}$  such that  $v = \mathrm{Ad}(\ell)E$ . Since  $\exp: \mathfrak{u}_{\Theta} \rightarrow U_{\Theta}$  is equivariant with respect to  $L_{\Theta}^{\circ}$ , and  $U_{\Theta}^{\geq 0}$  is  $L_{\Theta}^{\circ}$ -invariant, it is enough to show that  $\exp(E) \in U_{\Theta}^{\geq 0}$ . It is clear that  $\exp(E)$  belongs to  $U_{\Theta}^{\geq 0}$ . Let  $a$  and  $b$  be the points  $\mathfrak{p}_{\Theta}$  and  $\mathfrak{p}_{\Theta}^{\mathrm{opp}}$  of  $F_{\Theta}$ . We need to prove that  $\exp(E) \cdot b$  is

transverse to  $b$  (Corollary 10.19.(4)). From equality valid in  $\mathrm{SL}_2(\mathbf{R})$ , one has  $\exp(E) \cdot b = \exp(F)\dot{s} \cdot b$  with  $\dot{s} = \exp(\frac{\pi}{2}(E - F))$ . Lemma 7.2 implies that  $\dot{s} \cdot b = a$  and thus the wanted transversality.

Conversely, let  $v$  be in  $\mathfrak{u}_\Theta$  such that  $\exp(tv)$  belongs to  $U_\Theta^{>0}$  for all  $t$  in  $(0, \varepsilon)$ . This implies first that  $v$  belongs to the tangent cone at  $e$  to  $U_\Theta^{\geq 0}$  and therefore  $v = \sum_{\alpha \in \Theta} v_\alpha$  with  $v_\alpha \in c_\alpha$  for all  $\alpha$  in  $\Theta$ . Suppose by contradiction that there exists  $\alpha$  in  $\Theta$  such that  $v_\alpha$  does not belong to  $\mathring{c}_\alpha$ . By Proposition 5.1, the stabilizer of  $v$  in  $L_\Theta$  is not compact; this would imply that for all  $t$  the centralizer of  $\exp(tv)$  in  $L_\Theta$  is not compact, in contradiction with the properness of the action of  $L_\Theta$  on the positive unipotent semigroup.  $\square$

*Remark 10.25.* As  $U_\Theta^{>0}$  is a semigroup, the hypothesis in the theorem is equivalent to having  $\exp(tv) \in U_\Theta^{>0}$  for all  $t$  in  $\mathbf{R}_{>0}$ .

## 11. THE ORTHOGONAL GROUPS

In this section we discuss in a bit more detail the case when  $G = \mathrm{SO}(3, q)$ ,  $q \geq 4$ . The description of the  $\Theta$ -positive structure for general orthogonal groups  $\mathrm{SO}(p, q)$ ,  $q > p > 2$  restricts basically to this case and the description of the positive structure for  $\mathrm{SL}_3(\mathbf{R})$ , see also the discussion in [GW18, Section 4.5]. For the case when  $G = \mathrm{SO}(3, q)$  ( $q \geq 4$ ), we also provide a parametrization of the nonnegative semigroup  $U_\Theta^{\geq 0}$ .

**11.1. The positive semigroup.** We realize  $\mathrm{SO}(3, q) = \mathrm{SO}(b_Q)$ , where  $b_Q$  is the nondegenerate symmetric bilinear form of signature  $(3, q)$  on

$\mathbf{R}^{3+q}$  given by  $b_Q(v, w) = {}^t v Q w$ , with  $Q = \begin{pmatrix} 0 & 0 & K \\ 0 & J & 0 \\ -K & 0 & 0 \end{pmatrix}$ ,  $K =$

$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ , and  $J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -\mathrm{Id}_{q-3} & 0 \\ 1 & 0 & 0 \end{pmatrix}$ . We denote by  $b_J$  the form

$b_J(x, y) = \frac{1}{2} {}^t x J y$  and set  $q_J(x) = b_J(x, x)$ . Note that  $b_J$  is a nondegenerate symmetric bilinear form of signature  $(1, q-2)$  on the corresponding subspace of  $\mathbf{R}^{q+3}$ .

We choose the Cartan subspace  $\mathfrak{a} \subset \mathfrak{so}(3, q)$  to be the intersection of the set of diagonal matrices with  $\mathfrak{so}(3, q)$ . Concretely  $\mathfrak{a}$  is the space of diagonal matrices  $\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, 0, \dots, 0, -\lambda_3, -\lambda_2, -\lambda_1)$ , with  $\lambda_1, \lambda_2$ , and  $\lambda_3$  varying in  $\mathbf{R}$ . Denote  $e_i: \mathfrak{a} \rightarrow \mathbf{R}$  the linear form that associates to a diagonal matrix its  $i$ -th diagonal coefficient. The set of (restricted) roots is  $\{\pm e_i \pm e_j\}_{1 \leq i < j \leq 3} \cup \{\pm e_i\}_{1 \leq i \leq 3}$ ; a standard choice for the set of positive roots is  $\{e_i \pm e_j\}_{1 \leq i < j \leq 3} \cup \{e_i\}_{1 \leq i \leq 3}$  and the set of simple roots  $\Delta = \{\alpha_1, \alpha_2, \alpha_3\}$  is given by  $\alpha_i = e_i - e_{i+1}$ , for  $i = 1, 2$ , and  $\alpha_3 = e_3$ .

We have  $\Theta = \{\alpha_1, \alpha_2\}$ . Furthermore  $\mathfrak{u}_{\alpha_1} \cong \mathbf{R}$  and  $\mathfrak{u}_{\alpha_2} \cong \mathbf{R}^{1, q-2}$ . Then the closed cones identify with  $c_1 \cong \mathbf{R}_{\geq 0}$  and  $c_2 \cong c^{1, q-2} = \{x \in \mathbf{R}^{1, q-2} \mid q_J(x) \geq 0, x_1 \geq 0\}$  and the closed cone  $\mathring{c}_1 \cong \mathbf{R}_{>0}$ , and  $\mathring{c}_2 \cong \mathring{c}^{1, q-2} = \{x \in \mathbf{R}^{1, q-2} \mid q_J(x) > 0, x_1 > 0\}$ .

The Weyl group  $W$  is isomorphic to  $\{\pm 1\}^3 \rtimes S_3$ , the group of signed permutation matrices. It is presented by the generators  $s_1, s_2, s_3$  associated with the simple roots, with the relations  $s_i^2 = (s_1 s_2)^3 = (s_1 s_3)^2 = (s_2 s_3)^4 = e$ . The group  $W(\Theta)$  is generated by  $\sigma_1 = s_1$



and  $\sigma_2 = s_2 s_3 s_2$ . It is a Weyl group of type  $B_2$  and the longest length element is  $w_{\max}^\Theta = \sigma_1 \sigma_2 \sigma_1 \sigma_2 = \sigma_2 \sigma_1 \sigma_2 \sigma_1$ .

Using that the map from  $\mathfrak{u}_{\alpha_1} \times \mathfrak{u}_{\alpha_2} \times \mathfrak{u}_{\alpha_1 + \alpha_2} \times \mathfrak{u}_{2\alpha_1 + \alpha_2}$  to  $U_\Theta$  given by the product of exponentials is a diffeomorphism, every element in  $U_\Theta$  can be uniquely written as a matrix

$$U(a, e_1, e_2, c) = \begin{pmatrix} 1 & a & {}^t(ae_1 - e_2)J & c + aq_J(e_1) & -ac - 2ab_J(e_1, e_2) - q_J(e_2) \\ & 1 & {}^t e_1 J & q_J(e_1) & -c - 2b_J(e_1, e_2) \\ & & \text{Id}_{q-1} & e_1 & e_2 \\ & & & 1 & a \\ & & & & 1 \end{pmatrix}.$$

The exponential maps  $\mathbf{R} \rightarrow U_\Theta$  and  $\mathbf{R}^{q-1} \rightarrow U_\Theta$  are denoted  $x_1, x_2$  and given by

$$\begin{aligned} x_1(s) &= U(s, 0, 0, 0) \\ x_2(v) &= U(0, v, 0, 0). \end{aligned}$$

Thus, setting  $\gamma = (\alpha_1, \alpha_2, \alpha_1, \alpha_2)$ , the positive semigroup  $U_\Theta^{>0}$  in this case is

$$U_\Theta^{>0} = F_\gamma(\dot{c}_1 \times \dot{c}_2 \times \dot{c}_1 \times \dot{c}_2),$$

i.e. all matrices, that can be written as  $x_1(s_1)x_2(v_1)x_1(s_2)x_2(v_2)$ , with  $s_1, s_2 \in \mathbf{R}_{>0}$  and  $v_1, v_2 \in \dot{c}^{1, q-2}$ .

Concretely, one has:

$$\begin{aligned} x_1(s_1)x_2(v_1)x_1(s_2)x_2(v_2) &= \\ &U(s_1 + s_2, v_1 + v_2, s_2 v_1, -s_2(q_J(v_1) + 2b_J(v_1 + v_2, v_2))). \end{aligned}$$

A matrix  $U(a, e_1, e_2, c)$  hence belongs to  $U_\Theta^{>0}$  if and only if

$$\begin{aligned} -c - 2b_J(e_1, e_2) &> 0, \quad -ac - 2ab_J(e_1, e_2) - q_J(e_2) > 0 \\ e_2 &\in \dot{c}^{1, q-2}, \quad q_J(e_2)e_1 + (c + 2b_J(e_1, e_2))e_2 \in \dot{c}^{1, q-2}. \end{aligned}$$

When we use the other reduced expression of the longest length element  $w_{\max}^\Theta = \sigma_2 \sigma_1 \sigma_2 \sigma_1$ , and consider elements  $x_2(w_2)x_1(t_2)x_2(w_1)x_1(t_1)$  with  $t_1, t_2 \in \mathbf{R}_{>0}$  and  $w_1, w_2 \in \dot{c}^{1, q-2}$ , we parametrize as well the positive semigroup  $U_\Theta^{>0}$ .

The equation

$$(11.1) \quad x_1(s_1)x_2(v_1)x_1(s_2)x_2(v_2) = x_2(w_2)x_1(t_2)x_2(w_1)x_1(t_1),$$

determines then the change between the two parametrizations; comparing the entries of the corresponding matrices, we get the following relations

$$(11.2) \quad s_1 + s_2 = t_1 + t_2$$

$$(11.3) \quad v_1 + v_2 = w_1 + w_2$$

$$(11.4) \quad s_1(v_1 + v_2) + s_2 v_2 = t_2 w_1$$

$$(11.5) \quad s_1 q_J(v_1 + v_2) + s_2 q_J(v_2) = t_2 q_J(w_1),$$

$$(11.6) \quad s_2 v_1 = t_2 w_2 + t_1(w_1 + w_2),$$

$$(11.7) \quad s_2 q_J(v_1) = t_2 q_J(w_2) + t_1 q_J(w_1 + w_2)$$

$$(11.8) \quad s_1 s_2 q_J(v_1) = t_2 t_1 q_J(w_1)$$

*Remarks 11.1.* (1) Since the maps  $x_1, x_2$  have some compatibility with transposition (namely with  $D$  the block diagonal matrix with blocks  $\text{Id}_2, J, \text{Id}_2$ , for all  $v$  in  $\mathbf{R}^{q-1}$ ,  ${}^t x_2(v) = x_2(Jv) = Dx_2(v)D^{-1}$  and, for all  $s$  in  $\mathbf{R}$ ,  ${}^t x_1(s) = x_1(s) = Dx_1(s)D^{-1}$ ), Equation (11.1) above is equivalent to

$$x_1(t_1)x_2(w_1)x_1(t_2)x_2(w_2) = x_2(v_2)x_1(s_2)x_2(v_1)x_1(s_1).$$

Hence the chosen numbering makes the relation between  $(s_i, v_i)$  and  $(t_i, w_i)$  more symmetric.

- (2) Equations (11.2)–(11.5) are the same than the ones obtained by Berenstein and Zelevinsky [BZ97, p. 140] for  $\text{Sp}_4(\mathbf{R})$ , i.e. for a Weyl group of type  $C_2 = B_2$ , where the squares in their formulas there are replaced by the quadratic form  $q_J$  here.
- (3) Equations (11.2)–(11.5) determine the others: indeed multiplying (11.2)×(11.3) and subtracting (11.4) gives (11.6); the scalar product of (11.3) with the difference of (11.4) and (11.6) is the difference of (11.5) and (11.7); applying  $q_J$  to (11.4) and subtracting the product of (11.2) and (11.5) gives (11.8).

These equations can be solved as follow: the ratio of (11.8) and (11.5) gives  $t_1$ ; the ratio of  $q_J(11.4)$  and (11.5) gives  $t_2$ ; once  $t_2$  is determined, (11.4) gives  $w_1$ ; once  $w_1$  is determined (11.3) gives  $w_2$ . Explicitely we have

$$\begin{aligned} t_1 &= \frac{s_1 s_2 q_J(v_1)}{s_1 q_J(v_1 + v_2) + s_2 q_J(v_2)}, \quad t_2 = \frac{q_J(s_1(v_1 + v_2) + s_2 v_2)}{s_1 q_J(v_1 + v_2) + s_2 q_J(v_2)}, \\ w_1 &= \frac{s_1 q_J(v_1 + v_2) + s_2 q_J(v_2)}{q_J(s_1(v_1 + v_2) + s_2 v_2)} (s_1(v_1 + v_2) + s_2 v_2), \\ w_2 &= \frac{s_2}{q_J(s_1(v_1 + v_2) + s_2 v_2)} \left( s_2 q_J(v_2) v_1 + \right. \\ &\quad \left. s_1 (q_J(v_1 + v_2) v_1 - q_J(v_1)(v_1 + v_2)) \right). \end{aligned}$$

Note that even though the formula for  $w_2$  contains a minus sign, the following lemma implies that  $w_2$  belongs to  $\mathring{c}^{1,q-2}$ .

**Lemma 11.2.** *For all  $v_1$  and  $v_2$  in  $\mathring{c}^{1,q-2}$ , the element*

$$a(v_1, v_2) = q_J(v_1 + v_2)v_1 - q_J(v_1)(v_1 + v_2)$$

*belongs to  $\mathring{c}^{1,q-2}$ .*

*Proof.* We calculate  $q_J(a(v_1, v_2))$ . This gives

$$q_J(a(v_1, v_2)) = q_J(v_1 + v_2)q_J(v_1)q_J(v_2).$$

Thus  $a(v_1, v_2)$  belongs to  $\mathring{c}^{1,q-2} \sqcup -\mathring{c}^{1,q-2}$ . Since  $a(v_1, v_1) = 2q_J(v_1)v_1$  a connectedness argument shows that, for all  $v_1$  and  $v_2$  in  $\mathring{c}^{1,q-2}$ ,  $a(v_1, v_2)$  belongs to  $\mathring{c}^{1,q-2}$ .  $\square$

**11.2. The nonnegative semigroup.** By point (3) of Corollary 10.19, the nonnegative unipotent semigroup is characterized as

$$U_{\Theta}^{\geq 0} = F_{\gamma}(c_1 \times c_2 \times c_1 \times c_2),$$

i.e. every element in  $U_{\Theta}^{\geq 0}$  can be written as

$$x_1(s_1)x_2(v_1)x_1(s_2)x_2(v_2),$$

with  $s_1, s_2 \in \mathbf{R}_{\geq 0}$  and  $v_1, v_2 \in c^{1,q-2}$ .

However this does not give a parametrization of  $U_{\Theta}^{\geq 0}$  since one can easily check that  $F_{\gamma}$  is not injective. For example take  $v_1, v_2 \in c^{1,q-2}$  and  $s_1 = s_2 = 0$ . Then

$$x_1(s_1)x_2(v_1)x_1(s_2)x_2(v_2) = x_2(v_1)x_2(v_2) = x_2(v_1 + v_2),$$

hence  $F_{\gamma}(0, v_1, 0, v_2) = F_{\gamma}(0, v_1 + v_2, 0, 0)$ .

Giving a parametrization of  $U_{\Theta}^{\geq 0}$  is thus more subtle. Already in the case of split real groups, the maps  $F_{\gamma}$  (defined here on  $\mathbf{R}_{\geq 0}^N$ ) are not injective (for the same reason than in the example above). A way to remedy to this non-injectivity in this split case has been found by Lusztig in [Lus98, Corollary 2.8]; Starting from the fact that the nonnegative unipotent semigroup  $U^{\geq 0}$  is the disjoint union of subsets  $U^{>0}(w) = U^{\geq 0} \cap P_{\Delta}^{\text{opp}} w P_{\Delta}^{\text{opp}}$ , where  $w$  varies over the elements of the Weyl group  $W$ . Then parametrizations of  $U^{>0}(w)$  are obtained from reduced expressions of  $w$ , similarly to the parametrizations of  $U^{>0}$ . Of course Lusztig showed first that the image of these parametrizations does not depend on the choice of reduced expression. Lusztig showed first that this image does not depend on the choice of reduced expression.

In the parametrization of the positive unipotent semigroup  $U_{\Theta}^{>0}$  the role of the Weyl group  $W$  in the split case is replaced by the Weyl group  $W(\Theta)$ , so one might hope to get a decomposition of  $U_{\Theta}^{\geq 0}$  into a disjoint union of parametrized sets, indexed by elements in  $W(\Theta)$ . The discussion in Section 9.3 shows that this can not work already in the case of Hermitian Lie groups. In order to describe a decomposition of  $U_{\Theta}^{\geq 0}$  we have to consider a different object than the  $\Theta$ -Weyl group and to replace it by the double quotient  $W_{\Delta \setminus \Theta} \backslash W / W_{\Delta \setminus \Theta}$ .

We describe a decomposition of the nonnegative unipotent semigroup for  $\text{SO}(3, q)$  and a parameterization of it, using  $W_{\Delta \setminus \Theta} \backslash W / W_{\Delta \setminus \Theta}$ . It will be of interest to explore this further in the general case.

Let us first consider the Bruhat decomposition of  $G$  with respect to the action of  $P_{\Theta}^{\text{opp}} \times P_{\Theta}^{\text{opp}}$ :  $G = \bigsqcup_{a \in W_{\Delta \setminus \Theta} \backslash W / W_{\Delta \setminus \Theta}} P_{\Theta}^{\text{opp}} a P_{\Theta}^{\text{opp}}$ . The next lemma determines representatives of  $W_{\Delta \setminus \Theta} \backslash W / W_{\Delta \setminus \Theta}$ .

**Lemma 11.3.** *The following provides a list of smallest length representatives for the 16 classes in  $W_{\Delta \setminus \Theta} \backslash W / W_{\Delta \setminus \Theta}$ :*

- (1)  $e$
- (2)  $s_1$
- (3)  $s_2$
- (4)  $(s_2 s_3 s_2)$
- (5)  $s_2 s_1$
- (6)  $(s_2 s_3 s_2) s_1$
- (7)  $s_1 s_2$
- (8)  $s_1 (s_2 s_3 s_2)$
- (9)  $(s_2 s_3) s_1 s_2$
- (10)  $s_2 s_1 (s_2 s_3 s_2)$
- (11)  $(s_2 s_3 s_2) s_1 s_2$
- (12)  $(s_2 s_3 s_2) s_1 (s_2 s_3 s_2)$
- (13)  $s_1 s_2 s_1$
- (14)  $s_1 (s_2 s_3 s_2) s_1$

$$(15) \quad s_1(s_2s_3s_2)s_1s_2$$

$$(16) \quad s_1(s_2s_3s_2)s_1(s_2s_3s_2).$$

*Proof.* Here the group  $W$  can be realized as the group of signed permutation  $3 \times 3$ -matrices and the subgroup  $W_{\Delta \setminus \Theta}$  is the subgroup (isomorphic to  $\mathbf{Z}/2\mathbf{Z}$ ) of diagonal matrices whose only possibly nontrivial diagonal coefficient is in the third position. The element  $s_1$  corresponds to the transposition (12),  $s_2$  corresponds to the transposition (23), and  $s_3$  is the nontrivial element in  $W_{\Delta \setminus \Theta}$ .

A direct calculation shows then the result.  $\square$

In the present situation the flag variety  $F_\Theta$  can be realized as the space of partial flags in  $(\mathbf{R}^{3+q}, b_Q)$  consisting of an isotropic line and an isotropic 2-plane containing it. For concreteness the canonical basis of  $\mathbf{R}^{3+q}$  used so far will be denoted

$$(e_1, e_2, e_3, g_1, g_2, \dots, g_{q-3}, f_3, f_2, f_1).$$

Then  $E^+ = (\mathbf{R}e_1, \mathbf{R}e_1 \oplus \mathbf{R}e_2)$  and  $E^- = (\mathbf{R}f_1, \mathbf{R}f_1 \oplus \mathbf{R}f_2)$  are partial isotropic flags,  $P_\Theta = \text{Stab}(E^+)$ , and  $P_\Theta^{\text{opp}} = \text{Stab}(E^-)$ . More generally, given a pair of linearly independent vectors  $(v_1, v_2)$  in  $\mathbf{R}^{3+q}$ , the flag determined by  $(v_1, v_2)$  is denoted  $E_{(v_1, v_2)} = (\mathbf{R}v_1, \mathbf{R}v_1 \oplus \mathbf{R}v_2)$ ; this flag depends only on the lines  $\mathbf{R}v_1$  and  $\mathbf{R}v_2$ , it belongs to  $F_\Theta$  if and only if the plane  $\mathbf{R}v_1 \oplus \mathbf{R}v_2$  is isotropic with respect to  $b_Q$ .

**Lemma 11.4.** *The  $P_\Theta^{\text{opp}}$ -orbits in  $F_\Theta$  are in one-to-one correspondence with  $P_\Theta^{\text{opp}} \times P_\Theta^{\text{opp}}$ -orbits in  $G$  and hence with the 16 classes in  $W_{\Delta \setminus \Theta} \backslash W/W_{\Delta \setminus \Theta}$ . For each class, we give a flag in  $F_\Theta$  belonging to the corresponding orbit:*

- (1)  $e: E_{(f_1, f_2)} = E^-$
- (2)  $s_1: E_{(f_2, f_1)}$
- (3)  $s_2: E_{(f_1, f_3)}$
- (4)  $(s_2s_3s_2): E_{(f_1, e_2)}$
- (5)  $s_2s_1: E_{(f_3, f_1)}$
- (6)  $(s_2s_3s_2)s_1: E_{(e_2, f_1)}$
- (7)  $s_1s_2: E_{(f_2, f_3)}$
- (8)  $s_1(s_2s_3s_2): E_{(f_2, e_1)}$
- (9)  $(s_2s_3)s_1s_2: E_{(f_3, e_2)}$
- (10)  $s_2s_1(s_2s_3s_2): E_{(f_3, e_1)}$
- (11)  $(s_2s_3s_2)s_1s_2: E_{(e_2, f_3)}$
- (12)  $(s_2s_3s_2)s_1(s_2s_3s_2): E_{(e_2, e_1)}$
- (13)  $s_1s_2s_1: E_{(f_3, f_2)}$
- (14)  $s_1(s_2s_3s_2)s_1: E_{(e_1, f_2)}$
- (15)  $s_1(s_2s_3s_2)s_1s_2: E_{(e_1, f_3)}$
- (16)  $s_1(s_2s_3s_2)s_1(s_2s_3s_2): E_{(e_1, e_2)} = E^+.$

*Proof.* The flag  $E^- = E_{(f_1, f_2)}$  corresponds to the class of the trivial element. A flag belonging to the orbit  $P_\Theta^{\text{opp}}[w]P_\Theta^{\text{opp}}/P_\Theta^{\text{opp}} \subset G/P_\Theta^{\text{opp}} = F_\Theta$ , associated with  $w$  in  $W$ , is then  $\dot{w} \cdot E^-$  where  $\dot{w}$  is a lift of  $W$  to  $\text{SO}(3, q)$ .

For the explicit calculation, we can choose

- For the lift of  $s_1$  the matrix that exchanges  $e_1$  and  $e_2$ , exchanges  $f_1$  and  $f_2$ , and fixes the other basis vectors.

- For the lift of  $s_2$  the matrix that exchanges  $e_2$  and  $e_3$ , exchanges  $f_2$  and  $f_3$ , and fixes the other basis vectors.
- For the lift of  $s_3$  the matrix that sends  $e_3$  to  $f_3$ , sends  $f_3$  to  $-e_3$ , and fixes the other basis vectors.  $\square$

For every  $x$  in  $W_{\Delta \setminus \Theta} \setminus W/W_{\Delta \setminus \Theta}$ , define  $U_{\Theta}^{>0}(x)$  to be the intersection of  $P_{\Theta}^{\text{opp}} x P_{\Theta}^{\text{opp}}$  with  $U_{\Theta}^{\geq 0}$ . Let  $F_{\gamma}$  be as above the map from  $c_1 \times c_2 \times c_1 \times c_2$  to  $U_{\Theta} \subset \text{SO}(3, q)$ . For each reduced expression  $w$  listed in Lemma 11.3 we define  $F_w$  to be the restriction of  $F_{\gamma}$  to the set  $D_w$  where

- (1)  $D_e = \{0\} \times \{0\} \times \{0\} \times \{0\}$ ,
- (2)  $D_{s_1} = \mathbf{R}_{>0} \times \{0\} \times \{0\} \times \{0\}$ ,
- (3)  $D_{s_2} = \{0\} \times (\partial c_2 \setminus \{0\}) \times \{0\} \times \{0\}$ ,
- (4)  $D_{s_2 s_3 s_2} = \{0\} \times \dot{c}_2 \times \{0\} \times \{0\}$ ,
- (5)  $D_{s_2 s_1} = \{0\} \times (\partial c_2 \setminus \{0\}) \times \mathbf{R}_{>0} \times \{0\}$ ,
- (6)  $D_{(s_2 s_3 s_2) s_1} = \{0\} \times \dot{c}_2 \times \mathbf{R}_{>0} \times \{0\}$ ,
- (7)  $D_{s_1 s_2} = \mathbf{R}_{>0} \times (\partial c_2 \setminus \{0\}) \times \{0\} \times \{0\}$ ,
- (8)  $D_{s_1 (s_2 s_3 s_2)} = \mathbf{R}_{>0} \times \dot{c}_2 \times \{0\} \times \{0\}$ ,
- (9)  $D_{(s_2 s_3) s_1 s_2} = \{(0, v_1, s_1, v_2) \mid s_1 > 0, v_1, v_2 \in \partial c_2, b_J(v_1, v_2) \neq 0\}$ ,
- (10)  $D_{s_2 s_1 (s_2 s_3 s_2)} = \{0\} \times (\partial c_2 \setminus \{0\}) \times \mathbf{R}_{>0} \times \dot{c}_2$ ,
- (11)  $D_{(s_2 s_3 s_2) s_1 s_2} = \{0\} \times \dot{c}_2 \times \mathbf{R}_{>0} \times (\partial c_2 \setminus \{0\})$ ,
- (12)  $D_{(s_2 s_3 s_2) s_1 (s_2 s_3 s_2)} = \{0\} \times \dot{c}_2 \times \mathbf{R}_{>0} \times \dot{c}_2$ ,
- (13)  $D_{s_1 s_2 s_1} = \mathbf{R}_{>0} \times (\partial c_2 \setminus \{0\}) \times \mathbf{R}_{>0} \times \{0\}$ ,
- (14)  $D_{s_1 (s_2 s_3 s_2) s_1} = \mathbf{R}_{>0} \times \dot{c}_2 \times \mathbf{R}_{>0} \times \{0\}$ ,
- (15)  $D_{s_1 (s_2 s_3 s_2) s_1 s_2} = \mathbf{R}_{>0} \times \dot{c}_2 \times \mathbf{R}_{>0} \times (\partial c_2 \setminus \{0\})$ ,
- (16)  $D_{s_1 (s_2 s_3 s_2) s_1 (s_2 s_3 s_2)} = \mathbf{R}_{>0} \times \dot{c}_2 \times \mathbf{R}_{>0} \times \dot{c}_2$ .

**Proposition 11.5.** *For every  $w$  in the list of Lemma 11.3 the map*

$$\begin{aligned} D_w &\longrightarrow F_{\Theta} \\ z &\longmapsto F_w(z) \cdot E^{-} \end{aligned}$$

is injective; the image of  $F_w$  is  $U_{\Theta}^{>0} \cap P_{\Theta}^{\text{opp}} w P_{\Theta}^{\text{opp}} = U_{\Theta}^{>0}(x)$  with  $x = [w]$ . The nonnegative semigroup  $U_{\Theta}^{\geq 0}$  is the disjoint union the  $U_{\Theta}^{>0}(x)$ .

*Proof.* The map  $(s_1, v_1, s_2, v_2) \mapsto F_{\gamma}(s_1, v_1, s_2, v_2) \cdot E^{-}$  is completely determined by the last two columns of the matrix  $F_{\gamma}(s_1, v_1, s_2, v_2)$  that is

$$\begin{pmatrix} s_1 q_J(v_1 + v_2) + s_2 q_J(v_2) & s_1 s_2 q_J(v_1) \\ q_J(v_1 + v_2) & s_2 q_J(v_1) \\ v_1 + v_2 & s_2 v_1 \\ 1 & s_1 + s_2 \\ 0 & 1 \end{pmatrix}.$$

We can now check by case by case consideration that the statement holds.  $\square$

## 12. INVARIANT UNIPOTENT SEMIGROUPS

In Section 13 we will consider a more geometric approach to positivity in terms of triples in flag varieties. In order to connect this with the algebraic approach we took so far, we will make use of the results of the present section where we show that essentially any  $L_{\Theta}^{\circ}$ -invariant semigroup  $U^{+}$  in  $U_{\Theta}$  arises from a  $\Theta$ -positive structure.

We say that a semigroup  $U$  in a group  $G$  is *sharp* if its only invertible element is the neutral element (i.e. if  $g \in U^+$  and  $g^{-1} \in U^+$  then  $g = e$ ).

**Theorem 12.1.** *Let  $G$  be a connected semisimple Lie group, and let  $U_\Theta$  be a standard unipotent subgroup of  $G$  and  $L_\Theta$  be the corresponding standard Levi subgroup. Suppose that there is  $U^+ \subset U_\Theta$  such that*

- (1)  $U^+$  is closed and of nonempty interior;
- (2)  $U^+$  is  $L_\Theta^\circ$ -invariant;
- (3)  $U^+$  is a semigroup;
- (4)  $U^+$  is sharp.

*Then  $G$  admits a  $\Theta$ -positive structure; for all  $\alpha$  in  $\Theta$ , there is a unique  $L_\Theta^\circ$ -invariant closed nontrivial convex cone  $c_\alpha \subset \mathfrak{u}_\alpha$  such that  $\exp(c_\alpha) \subset U^+$ . Defining the nonnegative unipotent semigroup with these cones (Definition 4.1), the semigroup  $U^+$  contains the semigroup  $U_\Theta^{\geq 0}$ .*

We treat now the case of a semigroup that is equal to the intersection of  $U_\Theta$  with the open Bruhat cell:

**Theorem 12.2.** *Let  $G$  be a connected semisimple Lie group, and let  $U_\Theta$  be a standard unipotent subgroup of  $G$  and  $\Omega_\Theta^{\text{opp}}$  the open Bruhat cell with respect to the opposite parabolic group  $P_\Theta^{\text{opp}}$ . Suppose that there is  $V^+ \subset U_\Theta$  such that*

- (1)  $V^+$  is a connected component of  $U_\Theta \cap \Omega_\Theta^{\text{opp}}$ .
- (2)  $V^+$  is a semigroup.

*Then  $G$  admits a  $\Theta$ -positive structure; for every  $\alpha$  in  $\Theta$ , there is a unique  $L_\Theta^\circ$ -invariant closed nontrivial convex cone  $c_\alpha \subset \mathfrak{u}_\alpha$  such that  $\exp(c_\alpha) \subset \overline{V^+}$ . Defining the positive unipotent semigroup with these cones (Definition 4.4), the semigroup  $V^+$  is equal to  $U_\Theta^{\geq 0}$ .*

As a corollary of these theorems, we have

**Corollary 12.3.** *Let  $U^+$  be a closed  $L_\Theta^\circ$ -invariant semigroup in  $U_\Theta$ . Assume that the interior of  $U^+$  is nonempty and contained in the open Bruhat cell with respect to  $P_\Theta^{\text{opp}}$ , then the group  $G$  has a  $\Theta$ -positive structure and (with the right choice of cones)  $U^+ = U_\Theta^{\geq 0}$ .*

Section 12.1 derives a number of structural results common to both theorems, then Section 12.2 addresses the proof of Theorem 12.1 and Section 12.3 addresses the proof of Theorem 12.2. Section 12.4 gives examples of semigroups that satisfy the hypothesis of Theorem 12.1 and are not equal to  $U_\Theta^{\geq 0}$ .

**12.1. The cones associated with  $U^+$ .** We work here with  $G$ ,  $\Theta$ , and  $U^+$  as in Theorem 12.1 but satisfying only the hypothesis (1), (2), and (3) of this theorem. We first note that these three hypothesis are produced by Theorem 12.2:

**Lemma 12.4.** *Let  $(G, \Theta, V^+)$  satisfying the hypothesis of Theorem 12.2. Then  $U^+ = \overline{V^+}$  is a closed  $L_\Theta^\circ$ -invariant semigroup of nonempty interior, i.e. it satisfies the hypothesis (1), (2), and (3) of Theorem 12.1.*

*Proof.* In view of Lemma 4.6, it is enough to prove that the semigroup  $V^+$  is  $L_\Theta^\circ$ -invariant. Indeed  $U_\Theta$  and  $\Omega_\Theta^{\text{opp}}$  are invariant by  $L_\Theta^\circ$ , and since  $L_\Theta^\circ$  is connected, every connected component of  $U_\Theta \cap \Omega_\Theta^{\text{opp}}$  is invariant by  $L_\Theta^\circ$ .  $\square$



For all  $\alpha$  in  $\Theta$  we introduce

$$k_\alpha = p_\alpha(\log(U^+))$$

where, as above  $p_\alpha: \mathfrak{u}_\Theta \rightarrow \mathfrak{u}_\alpha$  is the  $L_\Theta$ -equivariant projection.

We will first show that

**Proposition 12.5.** *Let  $X$  be an element of  $\mathfrak{u}_\alpha$ . Then  $X$  belongs to  $k_\alpha$  if and only if  $\exp(X)$  belongs to  $U^+$ .*

*Proof.* Let us define the following sequence  $(A_n)_{n \in \mathbf{N}}$  in  $\mathfrak{a}$ : for all  $n \in \mathbf{N}$ ,  $A_n$  is the element of  $\mathfrak{a}$  defined by the equalities

$$\alpha(A_n) = 0, \gamma(A_n) = 0 \quad \forall \gamma \in \Delta \setminus \Theta \quad \text{and} \quad \gamma(A_n) = -n \quad \forall \gamma \in \Theta \setminus \{\alpha\}.$$

One has then, for all  $X \in \mathfrak{u}_\alpha$

$$\text{ad}(A_n)X = 0$$

and for all  $\beta$  in  $P(\mathfrak{u}_\Theta, \mathfrak{t}_\Theta) \setminus \{\alpha\}$  and for all  $Y$  in  $\mathfrak{u}_\beta$

$$\text{ad}(A_n)Y = -s_\beta n Y$$

for some  $s_\beta > 0$ . Recall that  $\mathfrak{u}_\beta$  is the sum of the  $\mathfrak{g}_\delta$  for  $\alpha$  in  $\Sigma \cap (\beta + \text{Span}(\Delta \setminus \Theta))$  so that  $\delta(A_n) = \beta(A_n)$ , see Section 2.9.

The sequence  $(g_n = \exp(A_n))_{n \in \mathbf{N}}$  belongs to  $\exp(\mathfrak{a}) \subset L_\Theta^\circ$  and, for every  $X$  in  $\mathfrak{u}_\Theta$ , using the decomposition  $X = \sum X_\beta$  according to the direct sum  $\mathfrak{u}_\Theta = \bigoplus_\beta \mathfrak{u}_\beta$ , one has, for all  $n$  in  $\mathbf{N}$ ,

$$\begin{aligned} \text{Ad}(g_n)X &= \exp(\text{ad}(A_n))X_\alpha + \sum_{\beta \neq \alpha} \exp(\text{ad}(A_n))X_\beta \\ &= X_\alpha + \sum_{\beta \neq \alpha} e^{-s_\beta n} X_\beta. \end{aligned}$$

Thus the sequence  $(\text{Ad}(g_n) \cdot X)_{n \in \mathbf{N}}$  converges to  $X_\alpha = p_\alpha(X)$ .

Let now  $X_\alpha$  be in  $k_\alpha$ . There exists then  $X \in \mathfrak{u}_\Theta$  such that  $\exp(X)$  belongs to  $U^+$  and  $X_\alpha = p_\alpha(X)$ . By the above, the sequence

$$g_n \exp(X) g_n^{-1} = \exp(\text{Ad}(g_n)X)$$

converges to  $\exp(X_\alpha)$ . Since  $U^+$  is  $L_\Theta^\circ$ -invariant and closed, one has  $\exp(X_\alpha)$  belongs to  $U^+$ .

Conversely, let  $X$  be in  $\mathfrak{u}_\alpha$  such that  $\exp(X)$  belongs to  $U^+$ , then  $p_\alpha(\log(\exp(X))) = p_\alpha(X) = X$  belongs to  $k_\alpha$ .  $\square$

From this we deduce:

**Corollary 12.6.** *The set  $k_\alpha$  is a closed  $L_\Theta^\circ$ -invariant convex cone.*

*Proof.* By the previous proposition  $k_\alpha = \log^{-1}(U^+ \cap \exp(\mathfrak{u}_\alpha))$ . Since  $U^+ \cap \exp(\mathfrak{u}_\alpha)$  is closed and  $L_\Theta^\circ$ -invariant, this implies that  $k_\alpha$  is closed and  $L_\Theta^\circ$ -invariant.

Let  $X \in k_\alpha$  and let  $t > 0$ . Let  $A \in \mathfrak{a}$  be defined by  $\alpha(A) = \log t$  and  $\gamma(A) = 0$  for all  $\gamma \in \Delta \setminus \{\alpha\}$ . Then

$$tX = \exp(\text{ad}(A))X = \text{Ad}(\exp(A))X$$

belongs to  $k_\alpha$  since  $\exp(A)$  belongs to  $L_\Theta^\circ$  and  $k_\alpha$  is  $L_\Theta^\circ$ -invariant. This means that  $k_\alpha$  is a cone.

Let  $X$  and  $Y$  be in  $k_\alpha$ . Then  $\exp(X)\exp(Y)$  belongs to  $U^+$ . From the Baker–Campbell–Hausdorff formula,

$$p_\alpha \circ \log(\exp(X)\exp(Y)) = X + Y$$

belongs to  $k_\alpha$ . This is the property that the cone  $k_\alpha$  is convex.  $\square$

The cones  $k_\alpha$  are nontrivial:

**Lemma 12.7.** *For every  $\alpha$  in  $\Theta$ , the cone  $k_\alpha$  is not reduced to  $\{0\}$ .*

*Proof.* Suppose that  $k_\alpha = \{0\}$ . Then  $U^+$  would be contained in  $\{g \in U_\Theta \mid p_\alpha(\log(g)) = 0\}$  and would be of empty interior, contrary to the assumptions.  $\square$

**12.2. The positive structure.** We now turn to the proof of Theorem 12.1, i.e. we furthermore assume property (4).

The last point to make the relation with Definition 3.1 is that the cones  $k_\alpha$  are acute:

**Lemma 12.8.** *For every  $\alpha$  in  $\Theta$ , the cone  $k_\alpha$  contains no line.*

*Proof.* Let  $X$  in  $\mathfrak{u}_\alpha$  be such that  $X$  and  $-X$  belong to  $k_\alpha$ . Then  $g = \exp(X)$  belongs to  $U^+$  as well as  $g^{-1} = \exp(-X)$ . Thus  $g = e$  (since by assumption the semigroup  $U^+$  is sharp) and  $X = \log(g) = 0$ . This means that the convex cone  $k_\alpha$  contains no line.  $\square$

Thus  $G$  has a  $\Theta$ -positive structure and one can introduce the semigroup  $U_\Theta^{\geq 0}$  with the following choice of invariant cones: for all  $\alpha$  in  $\Theta$  set  $c_\alpha = k_\alpha$ . Proposition 12.5 and the semigroup property obviously imply the inclusion  $U_\Theta^{\geq 0} \subset U^+$ . This concludes the proof of Theorem 12.1.

**12.3. The cones associated with  $V^+$  and the induced positive structure.** We now turn to the proof of Theorem 12.2. Applying the results of Section 12.1 to  $U^+ = \overline{V}^+$  (cf. Lemma 12.4), we get that, for every  $\alpha$  in  $\Theta$

$$k_\alpha = p_\alpha(\log(\overline{V}^+))$$

is a closed convex, nonzero,  $L_\Theta^\circ$ -invariant cone and that, for every  $X$  in  $\mathfrak{u}_\alpha$ ,  $\exp(X)$  belongs to  $\overline{V}^+$  if and only if  $X$  belongs to  $k_\alpha$ .

Applying Lemma 4.6, we get  $V^+\overline{V}^+ = V^+$ , and  $\overline{V}^+V^+ = V^+$  and that  $V^+$  is the interior of its closure  $\overline{V}^+$ .

We also have

**Lemma 12.9.** *For all  $\alpha$  in  $\Theta$ ,  $k_\alpha$  is an acute convex cone in  $\mathfrak{u}_\alpha$ .*

*Proof.* Since  $k_\alpha$  is a closed convex cone and is  $L_\Theta^\circ$ -invariant, the subspace of maximal dimension contained in  $k_\alpha$  is  $L_\Theta^\circ$ -invariant. Since the action of  $L_\Theta^\circ$  on  $\mathfrak{u}_\alpha$  is irreducible (Theorem 2.2), this subspace is either  $\{0\}$  or  $\mathfrak{u}_\alpha$ . Hence we have to exclude the case when  $k_\alpha = \mathfrak{u}_\alpha$ .

Suppose, by contradiction, that  $k_\alpha = \mathfrak{u}_\alpha$ . We then have  $\exp \mathfrak{u}_\alpha \subset \overline{V}^+$ . Recall that the Lie algebra  $\mathfrak{u}_\Theta$  admits the following decomposition

$$\mathfrak{u}_\Theta = \mathfrak{u}_\alpha \oplus \mathfrak{u}_{\hat{\alpha}}, \text{ where } \mathfrak{u}_{\hat{\alpha}} = \bigoplus_{\beta \in P(\mathfrak{u}_\Theta, \mathfrak{t}_\Theta) \setminus \{\alpha\}} \mathfrak{u}_\beta$$

and that the map

$$\mathfrak{u}_\alpha \times \mathfrak{u}_{\hat{\alpha}} \longrightarrow U_\Theta$$

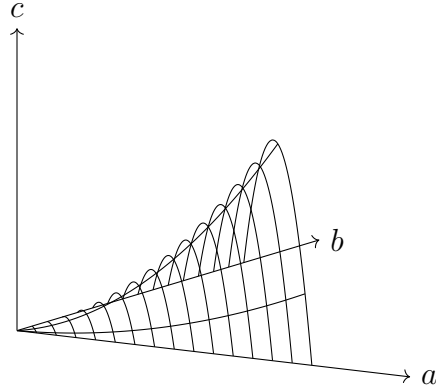


FIGURE 1. The semigroup  $U^{\geq 0} = U_1$  in the matrix coordinates is contained in the corner  $a \geq 0, b \geq 0, c \geq 0$  and bounded by the surface  $c = ab$  on which a few curves have been drawn.

$$(X, Y) \mapsto \exp(X) \exp(Y)$$

is a diffeomorphism.

Let now  $x$  be an element of  $V^+$ . There exists thus  $(X, Y)$  in  $\mathfrak{u}_\alpha \times \mathfrak{u}_\alpha$  such that  $x = \exp(X) \exp(Y)$ . The equality  $k_\alpha = \mathfrak{u}_\alpha$  implies that  $\exp(-X)$  belongs to  $\overline{V}^+$  and  $\exp(Y) = \exp(-X)x$  belongs to  $\overline{V}^+V^+ = V^+$ . But this element  $\exp(Y)$  does not belong to  $\Omega_\Theta^{\text{opp}}$ , in contradiction with the fact that  $V^+$  is contained in  $\Omega_\Theta^{\text{opp}}$ .  $\square$

As a consequence the group  $G$  has a  $\Theta$ -positive structure and we can choose  $c_\alpha = k_\alpha$  in the construction of  $U_\Theta^{\geq 0}$ . The last point is to notice the equality  $V^+ = U_\Theta^{\geq 0}$ . As in Section 12.2 we have  $U_\Theta^{\geq 0} \subset \overline{V}^+$  so that  $U_\Theta^{\geq 0} \subset \overline{V}^+$  and  $U_\Theta^{\geq 0} \subset V^+$  by Lemma 4.6. Since these 2 open sets are connected components of  $U_\Theta \cap \Omega_\Theta^{\text{opp}}$ , they are indeed equal.

**12.4. Semigroups bigger than  $U_\Theta^{\geq 0}$ .** We end this section with examples of closed semigroups contained in the unipotent standard subgroup of  $\text{SL}_3(\mathbf{R})$  that are bigger than the semigroup  $U^{\geq 0}$  of totally nonnegative unipotent matrices.

For each  $r \in \mathbf{R}$  denote by  $U_r$  the set of matrices

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, \text{ with } a, b \geq 0 \text{ and } 0 \leq c \leq rab.$$

*Remark 12.10.* The semigroup  $U^{\geq 0}$  is  $U_1$ . The matrix with  $a = b = c = 1$  is not in  $\Omega_\Delta^{\text{opp}}$  and belongs to the interior of  $U_r$  as soon as  $r > 1$ ; therefore, for  $r > 1$ ,  $\overset{\circ}{U}_r$  is not contained in  $\Omega_\Theta^{\text{opp}}$ .

**Lemma 12.11.** *For all  $r \geq 1$ ,  $U_r$  is a closed sharp semigroup invariant by conjugation by diagonal matrices with positive coefficients.*

*Proof.* Closedness and invariance are easily checked. To check the semigroup property, we calculate the product of two elements of  $U_r$ :

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a' & c' \\ 0 & 1 & b' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a'' & c'' \\ 0 & 1 & b'' \\ 0 & 0 & 1 \end{pmatrix}$$

then  $a'' = a + a' \geq 0$ ,  $b'' = b + b' \geq 0$  and  $c'' = c + c' + ab' \geq 0$ , thus

$$ra''b'' - c'' = (rab - c) + (ra'b' - c') + ra'b + (r - 1)ab' \geq 0,$$

and the product belongs to  $U_r$ . The above formula shows that if an element

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

is invertible in  $U_r$  then necessarily  $a = b = 0$  and for such an element the condition of being in  $U_r$  says  $0 \leq c \leq 0$ , i.e.  $c = 0$ . This proves that  $U_r$  is sharp.  $\square$

### 13. POSITIVITY IN THE FLAG VARIETY $F_\Theta$

In this section we first consider  $G$  having a  $\Theta$ -positive structure and use the positive unipotent semigroups  $U_\Theta^{>0}$  to define positive  $n$ -tuples in the flag variety  $F_\Theta$ . The key notion is the one of positive triples of flags, which can be defined using  $U_\Theta^{>0}$  or  $U_\Theta^{\text{opp}, >0}$ . To express the properties of positive triples, we use the language of ‘‘diamonds’’. In the simplest flag variety  $\mathbf{RP}^1$  diamonds are intervals. The notion of diamond was coined in [LT23] in the case when  $G$  is  $\text{SO}(2, n)$ , in which case diamonds actually look like diamonds. In the general case, diamonds have a slightly more complicated geometry, but can still thought of as generalized intervals. The language of diamonds, which is also used in [GLW21], captures well all the necessary geometric properties of positive triples and quadruples needed in applications.

We also prove the reverse direction: a family of diamonds gives rise to a  $\Theta$ -positive structure and is produced by the  $\Theta$ -positivity. This thus gives a geometric characterization of  $\Theta$ -positivity.

In the case of split real groups and Lusztig’s total positivity, the positive structure on the flag variety  $F_\Delta$  has been described in [Lus98] and the notion and key properties of positive  $n$ -tuples have been established by Fock and Goncharov in [FG06].

**13.1. A first diamond.** We will denote by  $\mathcal{O} \subset F_\Theta$  the set of points transverse to  $\mathfrak{p}_\Theta$  and by  $\mathcal{O}^{\text{opp}}$  the set of points transverse to  $\mathfrak{p}_\Theta^{\text{opp}}$ . These sets are open and diffeomorphic to the unipotent Lie groups  $U_\Theta$  and  $U_\Theta^{\text{opp}}$ ; indeed the maps  $U_\Theta \rightarrow \mathcal{O} \mid u \mapsto u \cdot \mathfrak{p}_\Theta^{\text{opp}}$  and  $U_\Theta^{\text{opp}} \rightarrow \mathcal{O}^{\text{opp}} \mid u \mapsto u \cdot \mathfrak{p}_\Theta$  are diffeomorphisms. Since  $w_\Delta \cdot \mathfrak{p}_\Theta = \mathfrak{p}_\Theta^{\text{opp}}$  and  $w_\Delta \cdot \mathfrak{p}_\Theta^{\text{opp}} = \mathfrak{p}_\Theta$ , one has also  $\mathcal{O} = \Omega_\Theta \cdot \mathfrak{p}_\Theta$  and  $\mathcal{O}^{\text{opp}} = \Omega_\Theta^{\text{opp}} \cdot \mathfrak{p}_\Theta^{\text{opp}}$ , where  $\Omega_\Theta = P_\Theta w_\Delta P_\Theta$  and  $\Omega_\Theta^{\text{opp}} = P_\Theta^{\text{opp}} w_\Delta P_\Theta^{\text{opp}}$ . A diamond will be a connected component of  $\mathcal{O} \cap \mathcal{O}^{\text{opp}}$ , see Section 13.2.

We first show that  $U_\Theta^{>0}$  or  $U_\Theta^{\text{opp}, >0}$  gives rise to the same connected component, hence the same diamond.

**Proposition 13.1.** *One has the equality, in  $F_\Theta$ ,*

$$U_\Theta^{>0} \cdot \mathfrak{p}_\Theta^{\text{opp}} = U_\Theta^{\text{opp}, >0} \cdot \mathfrak{p}_\Theta.$$

*More precisely these sets are equal to the same connected component of  $\mathcal{O} \cap \mathcal{O}^{\text{opp}}$ .*

*Proof.* By Corollary 10.19.(4) we have that  $U_\Theta^{>0} \cdot \mathfrak{p}_\Theta^{\text{opp}}$  is a connected component of  $\mathcal{O} \cap \mathcal{O}^{\text{opp}}$ . Equally,  $U_\Theta^{\text{opp}, >0} \cdot \mathfrak{p}_\Theta$  is a connected component

of  $\mathcal{O} \cap \mathcal{O}^{\text{opp}}$ . Hence we only need to see that these sets intersect. This can be obtained along the same lines than Theorem 10.24 using the equality  $\exp(E) \cdot \mathfrak{p}_\Theta^{\text{opp}} = \exp(F) \cdot \mathfrak{p}_\Theta$  (with  $(E, F, D)$  the  $\Theta$ -principal  $\mathfrak{sl}_2$ -triple, Section 7).  $\square$

**13.2. Axiomatic of diamonds.** In this section and the next one, we do not assume that  $G$  has a  $\Theta$ -positive structure. We will prove however that the definition below forces the presence of a  $\Theta$ -positive structure.

We introduce now the expected properties for positive triples in  $F_\Theta$ . It will be a little easier to express these properties in terms of “diamonds” that are, fixing  $a$  and  $b$  in  $F_\Theta$ , the connected components of the set  $\{x \in F_\Theta \mid (a, x, b) \text{ is a positive triple}\}$  (cf. below Definition 13.14).

Let  $\Theta \subset \Delta$  be a subset invariant by the opposition involution  $\iota: \alpha \mapsto -w_\Delta \cdot \alpha$  (so that  $F_{\iota(\Theta)} = F_\Theta$  and we can speak of transverse pairs in  $F_\Theta$ , cf. Section 2.8). For a point  $a$  in  $F_\Theta$ , let us denote by  $\mathcal{O}_a$  the (open) subset of  $F_\Theta$  whose points are those transverse to  $a$  and by  $P_a$  its stabilizer in  $G$ .

**Definition 13.2.** A family of diamonds in  $F_\Theta$  is a family  $\mathcal{F}$  of triples  $(D, a, b)$  where  $D$  is a subset of  $F_\Theta$  and  $a, b$  belong to  $F_\Theta$  (and will be called the *extremities* of the diamond) such that

- (1) For every  $(D, a, b)$  in  $\mathcal{F}$ ,  $a$  is transverse to  $b$  and  $D$  is a connected component of  $\mathcal{O}_a \cap \mathcal{O}_b$ ;
- (2) For every  $(D, a, b)$  in  $\mathcal{F}$ ,  $(D, b, a)$  belongs to  $\mathcal{F}$ ;
- (3) For every  $(D, a, b)$  in  $\mathcal{F}$ , and for every  $g$  in  $\text{Aut}_1(\mathfrak{g})$ ,  $(g \cdot D, g \cdot a, g \cdot b)$  belongs to  $\mathcal{F}$ ;
- (4) For every  $(D, a, b)$  in  $\mathcal{F}$ , and for every  $x$  in  $D$ , there exists a unique diamond  $(D', a, x)$  in  $\mathcal{F}$  such that  $D'$  is contained in  $D$ .

The symmetry condition (2) together with the last condition imply also

- (5) For every  $(D, a, b)$  in  $\mathcal{F}$ , and for every  $x$  in  $D$ , there exists a unique diamond  $(D', x, b)$  in  $\mathcal{F}$  such that  $D'$  is contained in  $D$ .

*Remark 13.3.* It is natural to ask that the notion of diamond (or the notion of positive triple to come later) is invariant under all the automorphisms of the flag variety  $F_\Theta$ . This is why we require invariance under the group  $\text{Aut}_1(\mathfrak{g})$  in the definition and not only under  $G$ .

We can immediately note that:

- For all  $a$  and  $b$  in  $F_\Theta$ , with  $a$  transverse to  $b$ , there is a least one diamond with extremities  $a$  and  $b$  (since the group  $\text{Aut}_1(\mathfrak{g})$  acts transitively on the space of pairs of transverse points), and there are finitely many such diamonds (since  $\mathcal{O}_a \cap \mathcal{O}_b$  has finitely many components);
- For every diamond  $(D, a, b)$  the set  $D$  is invariant under the Lie group  $L_{a,b}^\circ$  that is the neutral component of  $L_{a,b} = P_a \cap P_b$ .

**13.3. From diamonds to  $\Theta$ -positivity.** We assume in this subsection that there is a family of diamonds  $\mathcal{F}$  in  $F_\Theta$ . We immediately note that this family is produced from a semigroup in  $U_\Theta$ .

**Proposition 13.4.** Let  $a$  and  $b$  be respectively the elements  $\mathfrak{p}_\Theta$  and  $\mathfrak{p}_\Theta^{\text{opp}}$  of  $F_\Theta$ . Let  $(D, a, b)$  be a diamond in  $\mathcal{F}$  with extremities  $a$  and  $b$ .

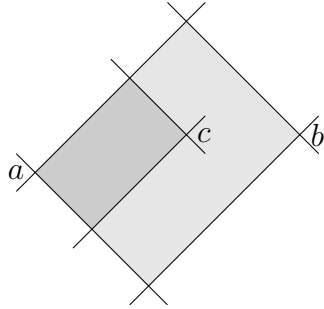


FIGURE 2. The diamonds in the case of the group  $\text{SO}(2, 2)$ . The point  $c$  is contained in the diamond with extremities  $a$  and  $b$  (the light gray region); the gray region is the diamond with extremities  $a$  and  $c$  contained in the first diamond. In the general case, the diamonds are curved as shown in Figure 1, which pictures the tip of a diamond in the case of the group  $\text{SL}_3(\mathbf{R})$ .

Then the set

$$V^+ := \{v \in U_\Theta \mid v \cdot b \in D\}$$

is a connected component of  $U_\Theta \cap \Omega_\Theta^{\text{opp}}$  and is a semigroup.

*Proof.* The fact that  $V^+$  is a connected component of  $U_\Theta \cap \Omega_\Theta^{\text{opp}}$  follows directly from Condition (1) in Definition 13.2.

Proving that  $V^+$  is a semigroup amounts to show that, for all  $v$  in  $V^+$ ,  $vV^+ \subset V^+$ . This inclusion can be phrased in terms of diamonds: let  $x = v \cdot b$  so that  $x$  belongs to  $D$  and there is a unique diamond  $(D', a, x)$  with extremities  $a$  and  $x$  with  $D' \subset D$ ; with this notation, one wants to prove the equality  $D' = v \cdot D$ .

Let  $\ell_s = \exp(sX)$  ( $s \in \mathbf{R}$ ) be the 1-parameter subgroup associated with the element  $X$  of  $\mathfrak{a}$  such that  $\alpha(X) = -1$  for all  $\alpha$  in  $\Delta$ . Define  $(v_t)_{t \in [0,1]}$  and  $(x_t)_{t \in [0,1]}$  by the equalities  $v_0 = e$ ,  $v_t = \ell_{\log t} v \ell_{\log t}^{-1}$  for  $t > 0$ , and  $x_t = v_t \cdot b$ . By the choice of  $X$ , these paths are continuous and  $x_0 = b$ ,  $x_1 = x$ .

For all positive  $t$ ,  $D'_t = \ell_{\log t} \cdot D'$  is the diamond with extremities  $a$  and  $x_t$  contained in  $\ell_{\log t} \cdot D = D$ ; i.e.  $D'_t$  is the connected component of  $\mathcal{O}_a \cap \mathcal{O}_{x_t}$  contained in  $D$ . From the convergence of  $x_t$  to  $b$ , we deduce that  $D'_t$  converges to  $D$  as  $t$  tends to 0. Stated differently, setting  $D'_0 = D$ , the family  $(D'_t)_{t \in [0,1]}$  is continuous. This implies that the family  $(v_t^{-1} \cdot D'_t)_{t \in [0,1]}$  of diamonds with extremities  $a$  and  $b$  is continuous. Since there are finitely many connected components in  $\mathcal{O}_a \cap \mathcal{O}_b$ , we deduce that this family is constant equal to  $D$ . In particular  $D = v_1^{-1} \cdot D'_1 = v^{-1} \cdot D'$  which is the sought for equality.  $\square$

As a consequence of Proposition 13.4, we can apply Theorem 12.2 and thus  $G$  admits a  $\Theta$ -positive structure. Let us then fix, for every  $\alpha$  in  $\Theta$ , a nonzero acute  $L_\Theta^\circ$ -invariant closed convex cone  $c_\alpha$  in  $\mathfrak{u}_\alpha$ . We already noticed that there are exactly two such cones, namely  $c_\alpha$  and  $-c_\alpha$ . To take into account all the possible choices of cones, let us introduce, for every  $\varepsilon = (\varepsilon_\alpha)_{\alpha \in \Theta}$  in  $\{\pm 1\}^\Theta$  the nonnegative unipotent semigroup  $U_\Theta^{\varepsilon \geq 0}$  generated by  $\bigcup_{\alpha \in \Theta} \exp(\varepsilon_\alpha c_\alpha)$  and the positive unipotent semigroup  $U_\Theta^{\varepsilon > 0}$  that is the interior (relative to  $U_\Theta$ ) of  $U_\Theta^{\varepsilon \geq 0}$ .



Theorem 12.2 gives the following corollary

**Corollary 13.5.** *The diamonds in  $\mathcal{F}$  with extremities  $a = \mathfrak{p}_\Theta$  and  $b = \mathfrak{p}_\Theta^{\text{opp}}$  are exactly the triples  $(U_\Theta^{\varepsilon > 0} \cdot b, a, b)$  for  $\varepsilon$  varying in  $\{\pm 1\}^\Theta$ .*

*Proof.* Indeed Proposition 13.4 and Theorem 12.2 prove that every diamond with extremities  $a$  and  $b$  has this form. The fact that all signs  $\varepsilon$  are achieved is a consequence of the invariance under  $\text{Aut}_1(\mathfrak{g})$  and that there are elements in this group exchanging the cones (Proposition 5.2).  $\square$

In fact the transitivity observed in Proposition 5.2 implies that there is exactly one orbit of diamonds under the action of  $\text{Aut}_1(\mathfrak{g})$ .

**Corollary 13.6.** *Let  $a$  and  $b$  be respectively  $\mathfrak{p}_\Theta$  and  $\mathfrak{p}_\Theta^{\text{opp}}$ . Then the diamonds are exactly the  $(g \cdot U_\Theta^>0 \cdot b, g \cdot a, g \cdot b)$  for  $g$  varying in  $\text{Aut}_1(\mathfrak{g})$ .*

**13.4. From  $\Theta$ -positivity to diamonds.** We now consider the reverse direction of Corollary 13.6. Namely, we assume that  $G$  has a  $\Theta$ -positive structure and consider the family  $\mathcal{F}$  consisting of the  $\text{Aut}_1(\mathfrak{g})$ -orbit of  $(U_\Theta^>0 \cdot b, a, b)$  where  $a$  and  $b$  are respectively  $\mathfrak{p}_\Theta$  and  $\mathfrak{p}_\Theta^{\text{opp}}$ .

Equivalently, the family  $\mathcal{F}$  can be defined by taking the union of the  $G$ -orbits of  $(U_\Theta^{\varepsilon > 0} \cdot b, a, b)$  for  $\varepsilon$  varying in  $\{\pm 1\}^\Theta$ .

**Proposition 13.7.** *The above family  $\mathcal{F}$  is a family of diamonds in  $F_\Theta$ .*

*Proof.* We need to check the conditions of Definition 13.2. Condition (1) follows from the corresponding property of  $U_\Theta^>0$  (see Proposition 13.1). Condition (2) is a consequence of the equality  $w_\Delta U_\Theta^>0 w_\Delta^{-1} = U_\Theta^{\text{opp}, > 0}$  and of Proposition 13.1. Condition (3) is there by construction. The existence part in Condition (4) follows from the fact that  $U_\Theta^>0$  is a semigroup: indeed let  $(D, a, b)$  be in  $\mathcal{F}$  and let  $x$  be in  $D$ , by equivariance we can assume that  $a = \mathfrak{p}_\Theta$ ,  $b = \mathfrak{p}_\Theta^{\text{opp}}$  and  $D = U_\Theta^>0 \cdot b$ ; let  $u$  be the element of  $U_\Theta^>0$  such that  $x = u \cdot b$  and set  $D' = uU_\Theta^>0$ , since  $(D', a, x) = u \cdot (D, a, b)$ ,  $D'$  is a diamond and the inclusion  $D' \subset D$  follows from the fact that  $U_\Theta^>0$  is a semigroup.

Let us now address the uniqueness in Condition (4). With the notation introduced in this condition, we can (by  $\text{Aut}_1(\mathfrak{g})$ -invariance) assume that  $a = \mathfrak{p}_\Theta$ ,  $b = \mathfrak{p}_\Theta^{\text{opp}}$ , and  $D = U_\Theta^>0 \cdot b$ . Let  $u$  be the element of  $U_\Theta^>0$  such that  $x = u \cdot b$  and let  $(D', a, x)$  be a diamond in  $\mathcal{F}$  such that  $D' \subset D$ . Then  $(u^{-1}D', a, b)$  is a diamond with extremities  $a$  and  $b$ .

There is thus an element  $\ell$  in  $\text{Aut}_1(\mathfrak{g})$  fixing  $a$  and  $b$  and such that  $u^{-1} \cdot D' = \ell \cdot D$ . The element  $\ell$  stabilizes all the spaces  $\mathfrak{u}_\alpha$  (for  $\alpha$  in  $\Theta$ ) and thus sends, for every  $\alpha$  in  $\Theta$ , the cone  $c_\alpha$  to  $\varepsilon_\alpha c_\alpha$  for some  $\varepsilon_\alpha$  in  $\{\pm 1\}$ . Setting  $\varepsilon = (\varepsilon_\alpha)_{\alpha \in \Theta}$  one has thus  $u^{-1} \cdot D' = U_\Theta^{\varepsilon > 0} \cdot b$  and the inclusion  $D' \subset D$  can be rewritten as  $uU_\Theta^{\varepsilon > 0} \subset U_\Theta^>0$ .

The sought for uniqueness is now equivalently expressed in the equalities  $\varepsilon_\alpha = 1$  for all  $\alpha$  in  $\Theta$ . Suppose by contradiction that there exists  $\alpha$  with  $\varepsilon_\alpha = -1$ . Then, using for example the parametrization of the semigroups, one can construct an element  $v$  in  $U_\Theta^{\varepsilon > 0}$  such that  $p_\alpha(\log(uv)) = p_\alpha(\log u) + p_\alpha(\log v) = 0$  (where again  $p_\alpha: \mathfrak{u}_\Theta \rightarrow \mathfrak{u}_\alpha$  is the projection on the factor  $\mathfrak{u}_\alpha$ ); this is incompatible with the fact that  $uv$  belongs to  $U_\Theta^>0$  since we should have that  $p_\alpha(\log(uv))$  belongs to  $\mathring{c}_\alpha$ .  $\square$

Observe that, for  $\varepsilon = (-1)_{\alpha \in \Theta}$  (i.e., for all  $\alpha$  in  $\Theta$ ,  $\varepsilon_\alpha = -1$ ), one has  $U_\Theta^{\varepsilon > 0} = (U_\Theta^{> 0})^{-1}$ . In particular  $((U_\Theta^{> 0})^{-1} \cdot b, a, b)$  is also in  $\mathcal{F}$ .

**13.5.  $U$ -pinnings.** We explain here the relation between the diamonds introduced here and the definition that is used in [GLW21, Definition 2.4].

Denote, for every  $a$  in  $F_\Theta$  by  $U_a$  the unipotent radical of  $P_a$  (so that  $U_a = U_\Theta$  when  $a$  is  $\mathfrak{p}_\Theta$ ). The group  $U_a$  is completely determined by its Lie algebra and will be also identified with a subgroup of  $\text{Aut}_1(\mathfrak{g})$  (namely the unipotent radical of the stabilizer of  $a$  in  $\text{Aut}_1(\mathfrak{g})$ ). We will call a  $U$ -pinning of  $U_a$  any homomorphism  $s_a: U_\Theta \rightarrow U_a$  induced by the map  $x \mapsto gxg^{-1}$  where  $g$  is an element of  $\text{Aut}_1(\mathfrak{g})$  such that  $g \cdot \mathfrak{p}_\Theta = a$ .

**Lemma 13.8.** *Let  $a$  and  $b$  be two transverse points of  $F_\Theta$ . Then the diamonds with extremities  $a$  and  $b$  are exactly the triples*

$$(s_a(U_\Theta^{> 0}) \cdot b, a, b)$$

where  $s_a$  runs through the  $U$ -pinnings of  $U_a$ .

*Proof.* Indeed every such triple is a diamond and the family defined is invariant by the action of  $\text{Aut}_1(\mathfrak{g})$ . Hence the result holds by the transitivity observed in Corollary 13.6.  $\square$

**13.6. Opposite diamond.** The notion of  $U$ -pinning allows us to introduce the notion of the opposite of a diamond. Let  $(D, a, b)$  be a diamond and let  $s_a: U_\Theta \rightarrow U_a$  be a  $U$ -pinning such that  $D = s_a(U_\Theta^{> 0}) \cdot b$  then the triple  $(s_a(U_\Theta^{> 0})^{-1} \cdot b, a, b)$  is a diamond that is called *opposite* to  $(D, a, b)$ .

**Lemma 13.9.** *There is a unique diamond  $(D', a, b)$  opposite to  $(D, a, b)$ .*

*Proof.* Let  $(D', a, b)$  be a diamond opposite to  $(D, a, b)$  defined by a  $U$ -pinning  $s_a$ . We set  $V = \{u \in U_a \mid u \cdot b \in D\}$  and  $V' = \{u \in U_a \mid u \cdot b \in D'\}$ . Then  $V$  and  $V'$  uniquely determine  $D$  and  $D'$ . From the definition we have,  $V = s_a(U_\Theta^{> 0})$

$$V' = s_a((U_\Theta^{> 0})^{-1}) = (s_a(U_\Theta^{> 0}))^{-1} = V^{-1}.$$

This proves that  $V'$  does not depend on the choice of  $s_a$  but only on  $V$  and in turn that  $D'$  depends only on  $D$  and not on  $s_a$ .  $\square$

The diamond opposite to  $(D, a, b)$  will be denoted  $(D^\vee, a, b)$ . One obviously has  $(D^\vee)^\vee = D$ . Uniqueness implies equivariance:

**Corollary 13.10.** *If  $D$  is a diamond, and if  $g$  belongs to  $\text{Aut}_1(\mathfrak{g})$ , then the opposite of the diamond  $g \cdot D$  is  $g \cdot D^\vee$ :  $(g \cdot D)^\vee = g \cdot D^\vee$ .*

**Lemma 13.11.** *Let  $(D, a, b)$  be a diamond. Then every  $x$  in  $D$  and every  $y$  in  $D^\vee$  are transverse.*

*Proof.* By invariance we can assume that  $a = \mathfrak{p}_\Theta$ ,  $b = \mathfrak{p}_\Theta^{\text{opp}}$  and  $D = U_\Theta^{> 0} \cdot b$ . There are then  $u$  and  $v$  in  $U_\Theta^{> 0}$  such that  $x = u \cdot b$  and  $y = v^{-1} \cdot b$ . Then the pair  $(x, y)$  is in the same orbit as  $(vu \cdot b, b)$ . Since  $vu$  belongs to  $U_\Theta^{> 0}$ ,  $vu \cdot b$  belongs to  $D$  and is transverse to  $b$ , thus  $x$  is transverse to  $y$ .  $\square$

The opposition of diamonds reverses inclusion:

**Lemma 13.12.** *Let  $(D, a, b)$  be a diamond and  $x$  belong to  $D$ . Let  $(D', a, x)$  be the diamond contained in  $D$ . Then the opposite diamond  $D'^\vee$  contains  $D^\vee$ .*

*Proof.* We can assume  $a = \mathfrak{p}_\Theta$  and  $b = \mathfrak{p}_\Theta^{\text{opp}}$  and  $D = U_\Theta^{>0} \cdot b$ . Let  $u$  be in  $U_\Theta^{>0}$  such that  $x = u \cdot b$  so that  $D' = u \cdot D$ . One thus has  $D^\vee = (U_\Theta^{>0})^{-1} \cdot b$  and  $D'^\vee = u \cdot D^\vee$ , that is  $u^{-1} \cdot D'^\vee = D^\vee$ . The sought for inclusion is therefore equivalent to  $u^{-1}(U_\Theta^{>0})^{-1} \subset (U_\Theta^{>0})^{-1}$  which is a consequence of the fact that  $U_\Theta^{>0}$  is a semigroup.  $\square$

The last lemma implies

**Corollary 13.13.** *The set  $D^\vee$  consists of those  $x$  in  $F_\Theta$  transverse to  $b$  and such that there is a diamond  $(D', a, x)$  containing  $D$ .*

**13.7. Positive triples in  $F_\Theta$ .** We now use the family of diamonds to define positive triples of flags.

**Definition 13.14.** A triple  $(f_1, f_2, f_3) \in (F_\Theta)^3$  is *positive* if  $f_1$  is transverse to  $f_3$  and if there exists a diamond with extremities  $f_1$  and  $f_3$  that contains  $f_2$ .

The next proposition collects the main properties of positive triples.

- Proposition 13.15.** (1) [Invariance] *For every  $(f_1, f_2, f_3)$  and every  $g$  in  $\text{Aut}_1(\mathfrak{g})$ , the triple  $(f_1, f_2, f_3)$  is positive if and only if the triple  $(g \cdot f_1, g \cdot f_2, g \cdot f_3)$  is positive.*
- (2) *A triple  $(f_i)_{i \in \mathbf{Z}/3\mathbf{Z}}$  is positive if and only if, for all  $i \neq j$  in  $\mathbf{Z}/3\mathbf{Z}$  there exists a diamond  $D_{i,j}$  with extremities  $f_i$  and  $f_j$  with  $D_{j,i} = D_{i,j}^\vee$  and  $f_k$  belongs to  $D_{i,j}$  for all  $(i, k, j)$  cyclically ordered (i.e.  $j - k = k - i = 1$  in  $\mathbf{Z}/3\mathbf{Z}$ ).*
- (3) [Permutation] *For every permutation  $\sigma \in S_3$  and every  $(f_1, f_2, f_3)$ , the triple  $(f_1, f_2, f_3)$  is positive if and only if  $(f_{\sigma(1)}, f_{\sigma(2)}, f_{\sigma(3)})$  is positive.*
- (4) *A triple is positive if and only if it is in the orbit (under  $\text{Aut}_1(\mathfrak{g})$ ) of  $(\mathfrak{p}_\Theta, u \cdot \mathfrak{p}_\Theta^{\text{opp}}, \mathfrak{p}_\Theta^{\text{opp}})$  for some  $u$  in  $U_\Theta^{>0}$ .*
- (5) *A triple is positive if and only if it is in the orbit (under  $\text{Aut}_1(\mathfrak{g})$ ) of  $(\mathfrak{p}_\Theta, u^{-1} \cdot \mathfrak{p}_\Theta^{\text{opp}}, \mathfrak{p}_\Theta^{\text{opp}})$  for some  $u$  in  $U_\Theta^{>0}$ .*
- (6) [Component] *The set of positive triples is a union of connected components of the space of pairwise transverse triples, in particular it is open in  $(F_\Theta)^3$ .*
- (7) [Properness] *The group  $G$  acts properly on the space of positive triples. In particular stabilizers are compact.*

*Proof.* (1) This property follows from the invariance of the family of diamonds.

(2) If  $(f_1, f_2, f_3)$  satisfies these conditions, then  $f_2$  belongs to  $D_{1,3}$ , a diamond with extremities  $f_1$  and  $f_3$  and the triple is positive.

Conversely, if the triple  $(f_1, f_2, f_3)$  is positive, there exists a diamond  $D_{1,3}$  with extremities  $f_1$  and  $f_3$  and containing  $f_2$ . Define  $D_{1,2}$  to be the unique diamond with extremities  $f_1$  and  $f_2$  contained in  $D_{1,3}$  and  $D_{2,3}$  to be the unique diamond with extremities  $f_2$  and  $f_3$  contained in  $D_{1,3}$ . For  $i > j$  define  $D_{i,j} = D_{j,i}^\vee$ . Then the wanted membership properties of the  $f_k$  in the  $D_{i,j}$  follow from the choices made and from Corollary 13.13.

- (3) The characterization of the previous point is clearly invariant by permutation, hence the result.
- (4) This follows from the definition and from the fact that  $(U_{\Theta}^{>0} \cdot b, a, b)$  is a diamond (again  $a = \mathfrak{p}_{\Theta}$  and  $b = \mathfrak{p}_{\Theta}^{\text{opp}}$ ).
- (5) This property follows from the previous one, the invariance by the transposition (23), and the invariance by  $\text{Aut}_1(\mathfrak{g})$ .
- (6) This is a consequence of the connectedness properties of diamonds.
- (7) The map  $\mathcal{F} \rightarrow \mathbb{F}_{\Theta}^{2*}$  (where  $\mathbb{F}_{\Theta}^{2*}$  is the space of transverse pairs) is continuous and equivariant. The sought for properness is then equivalent to the properness of the action of  $L_{\Theta}^{\circ}$  on the semigroup  $U_{\Theta}^{>0}$ ; this properness is in turn a consequence of the fact that the parametrizations are  $L_{\Theta}^{\circ}$ -equivariant (Lemma 10.2) and from the already known properness of  $L_{\Theta}^{\circ}$  on the product of cones (Proposition 5.1).  $\square$

*Remark 13.16.* In resonance with Remark 5.4, we can establish (when  $\Theta \neq \Delta$ ) that the space of positive triples has two connected components when  $\sharp\Theta$  is odd and one connected component when  $\sharp\Theta$  is even.

Point (6) can be used to prove other characterizations of positive triples, for example

**Corollary 13.17.** *A triple is positive if and only if it is in the  $\text{Aut}_1(\mathfrak{g})$ -orbit of a triple of the form  $(vu \cdot b, v \cdot b, b)$  where  $b = \mathfrak{p}_{\Theta}^{\text{opp}}$  and  $u, v$  belong to  $U_{\Theta}^{>0}$ .*

*Proof.* Let us prove first that any such triple is positive. By property of the semigroup  $U_{\Theta}^{>0}$ , the flags in  $(vu \cdot b, v \cdot b, b)$  are pairwise transverse. Let  $X$  be the element of  $\mathfrak{a}$  such that  $\alpha(X) = -1$  for all  $\alpha$  in  $\Delta$  and let  $\ell_s = \exp(sX)$  ( $s \in \mathbf{R}$ ) be the associated 1-parameter subgroup contained in the Cartan subspace. Then the family  $\{(v\ell_s u \ell_s^{-1} \cdot b, v \cdot b, b)\}_{s \geq 0}$  consists of pairwise transverse triples and converges, as  $s \rightarrow \infty$ , to the positive triple  $(a, v \cdot b, b)$  (where  $a = \mathfrak{p}_{\Theta}$ ). This implies (thanks to point (6) of the above proposition) that  $(vu \cdot b, v \cdot b, b)$  is positive.

Let us now prove the reverse statement. Any positive triple is in the orbit of  $(a, w \cdot b, b)$  for some  $w$  in  $U_{\Theta}^{>0}$ . The diamond with extremities  $a$  and  $b$  and containing  $w \cdot b$  is our first diamond  $D = U_{\Theta}^{>0} \cdot b = U_{\Theta}^{\text{opp}, >0} \cdot a$ . For  $x \in U_{\Theta}^{\text{opp}, >0}$ , one has  $x \cdot (a, w \cdot b, b) = (x \cdot a, xw \cdot b, b)$ . The element  $x \cdot a$  belongs to  $D$  and hence is of the form  $r \cdot b$  for some  $r$  in  $U_{\Theta}^{>0}$ .

The element  $xw \cdot b$  belongs also to  $D$ : indeed there is  $z \in U_{\Theta}^{\text{opp}, >0}$  such that  $w \cdot b = z \cdot a$ ; hence  $xw \cdot b = (xz) \cdot a$  and  $xz$  belongs to  $U_{\Theta}^{\text{opp}, >0}$  thus  $xw \cdot b$  belongs to  $U_{\Theta}^{\text{opp}, >0} \cdot a = D$ . There is therefore  $v$  in  $U_{\Theta}^{>0}$  such that  $xw \cdot b = v \cdot b$ .

We finally prove that the element  $u = v^{-1}r$  belongs to  $U_{\Theta}^{>0}$ . For this, note first that  $u$  belongs to  $\Omega_{\Theta}^{\text{opp}}$  since  $v^{-1}r \cdot b$  is transverse to  $b$ ; choose  $(w_t)_{t \in [0,1]}$  a continuous path satisfying  $w_0 = e$ ,  $w_1 = w$  and  $w_t \in U_{\Theta}^{>0}$  for all  $t$  in  $(0, 1]$  (such a path can be constructed thanks to a parametrization  $F_{\gamma}$ ). The path  $(v_t)_{t \in [0,1]}$  in  $U_{\Theta}$  defined by the equality  $v_t \cdot b = xw_t \cdot b$  is continuous and satisfies  $v_0 = e$ ,  $v_1 = v$  and  $v_t^{-1}r$  belongs to  $\Omega_{\Theta}^{\text{opp}}$  for all  $t$ . Hence  $u = v^{-1}r = v_1^{-1}r$  and  $r = v_0^{-1}r$  belongs to the same connected component of  $U_{\Theta} \cap \Omega_{\Theta}^{\text{opp}}$ . Since  $U_{\Theta}^{>0}$  is a connected component of that intersection and since  $r$  belongs to  $U_{\Theta}^{>0}$ , we obtain that  $u$  belongs to  $U_{\Theta}^{>0}$ , as announced.  $\square$

13.7.1. *Hermitian Lie groups.* When  $G$  is a Hermitian Lie group of tube type, and  $\Theta = \{\alpha_\Theta\}$ , the flag variety  $F_\Theta$  coincides with the Shilov boundary of the symmetric space of  $G$ , and a triple is positive in the sense here if and only if it has maximal or minimal Maslov index. Note that for Hermitian Lie groups that are not of tube type, the Maslov index of a triple of points in the Shilov boundary is still defined, and one can look at triples of maximal (or minimal) Maslov index. In fact, it is shown in [BIW10], that every maximal triple is contained in the Shilov boundary of a unique maximal Hermitian symmetric subspace of tube type. One important difference between the tube type and non tube type situation is that in the first case, the space of pairwise transverse triples in the Shilov boundary has several connected components, and the triples of maximal or minimal Maslov index form connected components, and thus give rise to diamonds, whereas in the non-tube type situation the space of pairwise transverse triples in the Shilov boundary is connected see [BIW07, Section 3], and thus the space of triples of maximal or minimal Maslov index does not form connected components.

13.8. **Positive quadruples.** We investigate here properties of positive quadruples.

**Definition 13.18.** A quadruple  $(a, x, b, y)$  in  $F_\Theta$  is said to be *positive* if there exists a diamond  $(D, a, b)$  such that  $x$  belongs to  $D$  and  $y$  belongs to  $D^\vee$ .

**Proposition 13.19.** (1) Let  $(a, x, b, y)$  be a quadruple in  $F_\Theta$  and let  $g$  be in  $\text{Aut}_1(\mathfrak{g})$ . Then  $(a, x, b, y)$  is positive if and only if  $(g \cdot a, g \cdot x, g \cdot b, g \cdot y)$  is positive.

(2) A quadruple is positive if and only if it is in the  $\text{Aut}_1(\mathfrak{g})$ -orbit of  $(a, u \cdot b, b, v^{-1} \cdot b)$  where  $a = \mathfrak{p}_\Theta$ ,  $b = \mathfrak{p}_\Theta^{\text{opp}}$  and  $u, v$  belong to  $U_\Theta^{>0}$ .

(3) A quadruple is positive if and only if it is in the  $\text{Aut}_1(\mathfrak{g})$ -orbit of  $(a, vu \cdot b, v \cdot b, b)$  where  $a = \mathfrak{p}_\Theta$ ,  $b = \mathfrak{p}_\Theta^{\text{opp}}$  and  $u, v$  belong to  $U_\Theta^{>0}$ .

(4) Let  $(a, x, b, y)$  be a quadruple in  $F_\Theta$ . Then  $(a, x, b, y)$  is positive if and only if there is a diamond  $(D, a, y)$  such that  $b$  belongs to  $D$  and  $x$  belongs to  $D'$ , the unique diamond with extremities  $a$  and  $b$  contained in  $D$ .

(5) A quadruple is positive if and only if it is in the  $\text{Aut}_1(\mathfrak{g})$ -orbit of  $(a, x \cdot a, xy \cdot a, b)$  where  $a = \mathfrak{p}_\Theta$ ,  $b = \mathfrak{p}_\Theta^{\text{opp}}$  and  $x, y$  belong to  $U_\Theta^{\text{opp}, >0}$ .

(6) Let  $(a, x, b, y)$  be a quadruple in  $F_\Theta$ . Then  $(a, x, b, y)$  is positive if and only if there is a diamond  $(D, a, y)$  such that  $x$  belongs to  $D$  and  $b$  belongs to  $D'$ , the unique diamond with extremities  $x$  and  $y$  contained in  $D$ .

(7) A quadruple  $(f_i)_{i \in \mathbf{Z}/4\mathbf{Z}}$  is positive if and only if, for all  $i \neq j$  in  $\mathbf{Z}/4\mathbf{Z}$  there exists a diamond  $D_{i,j}$  with extremities  $f_i$  and  $f_j$  with  $D_{j,i} = D_{i,j}^\vee$  and  $f_k$  belongs to  $D_{i,j}$  for all  $(i, k, j)$  cyclically ordered (i.e.  $(k - i, j - k)$  is  $(1, 1)$ ,  $(1, 2)$ , or  $(2, 1)$ ).

(8) The space of positive quadruples is invariant under the dihedral group  $D_4 \subset S_4$  (i.e. the group generated by the 4-cycle  $(1, 2, 3, 4)$  and the double transposition  $(1, 4)(2, 3)$ ). Namely for  $\sigma \in D_4$  and a quadruple  $(f_1, f_2, f_3, f_4)$  then  $(f_1, f_2, f_3, f_4)$  is positive if and only if  $(f_{\sigma(1)}, f_{\sigma(2)}, f_{\sigma(3)}, f_{\sigma(4)})$  is positive.



- Proof.* (1) This follows from the invariance of the family of diamonds and the equivariance of the map  $D \mapsto D^\vee$  (Corollary 13.10).  
 (2) This follows from the definition and from the fact that, up to the action of  $\text{Aut}_1(\mathfrak{g})$ , we can assume  $a = \mathfrak{p}_\Theta$ ,  $b = \mathfrak{p}_\Theta^{\text{opp}}$ , and  $D = U_\Theta^{>0} \cdot b$ .  
 (3) Applying the element  $v \in U_\Theta^{>0}$  to the quadruple  $(a, u \cdot b, b, v^{-1}b)$  gives the wanted result.  
 (4) This is the previous characterization stated in terms of diamonds.  
 (5) This follows by the same arguments as for (3) above, using this time the equality  $U_\Theta^{\text{opp}, >0} \cdot a = U_\Theta^{>0} \cdot b$ .  
 (6) This is the previous characterization stated in terms of diamonds.  
 (7) If a quadruple satisfies the stated condition, then the condition in Definition 13.18 is obviously satisfied and the quadruple is positive.

Conversely, suppose that  $(f_1, f_2, f_3, f_4)$  is positive. By definition, there is a diamond  $D_{1,3}$  with extremities  $f_1$  and  $f_3$ , containing  $f_2$  and such that  $f_4$  belongs to  $D_{3,1} := D_{1,3}^\vee$ . We define  $D_{1,2}$  to be the diamond contained in  $D_{1,3}$  and with extremities  $f_1$  and  $f_2$  and similarly  $D_{2,3}$  is the diamond contained in  $D_{1,3}$  with extremities  $f_2$  and  $f_3$ . The characterization of point (4) gives also a diamond  $D_{1,4}$  with extremities  $f_1$  and  $f_4$  and containing  $f_2$  and  $f_3$ . This diamond can be used to define the diamond  $D_{2,4}$  and  $D_{3,4}$ . The other diamonds are defined thanks to the requirement  $D_{j,i} = D_{i,j}^\vee$ . All the wanted memberships are satisfied by construction and by Corollary 13.13 except possibly that  $f_3$  belongs to  $D_{2,4}$ . However this membership follows from the characterization established in point (6).

- (8) The invariance under the dihedral group follows from the previous point.  $\square$

Based on point (3) above one has

**Lemma 13.20.** *Let  $a$  and  $b$  be the elements  $\mathfrak{p}_\Theta$  and  $\mathfrak{p}_\Theta^{\text{opp}}$  of  $\mathbb{F}_\Theta$  and let  $v$  be in  $U_\Theta^{>0}$  and  $x = v \cdot b$ . For  $y \in \mathbb{F}_\Theta$ , the quadruple  $(a, y, x, b)$  is positive if and only if there is  $u$  in  $U_\Theta^{>0}$  such that  $y = vu \cdot b$ .*

Diamonds associated with positive quadruples are properly contained one in another:

**Lemma 13.21.** *Let  $(a, x, b, y)$  be a positive quadruple and let  $(D, a, y)$  be the diamond containing  $x$  and  $b$  and let  $(D', x, b)$  be the diamond contained in  $D$ . Then the closure  $\overline{D'}$  is contained in  $D$ .*

*Proof.* It will be enough to prove that this closure is contained in the intersection  $\mathcal{O}_a \cap \mathcal{O}_y$ . By symmetry, we need only to prove the inclusion into  $\mathcal{O}_y$ . We can assume that  $a = \mathfrak{p}_\Theta$ ,  $y = \mathfrak{p}_\Theta^{\text{opp}}$  and  $b = v \cdot y$ ,  $x = vu \cdot y$  with  $u, v$  in  $U_\Theta^{>0}$ . Any point in  $\overline{D'}$  is of the form  $vw \cdot y$  with  $w \in U_\Theta^{>0}$  (and  $w^{-1}u$  must also belong to  $U_\Theta^{>0}$ ). In particular  $vw$  belongs to  $U_\Theta^{>0}$  (Corollary 10.19.(1)) and  $vw \cdot y$  is transverse to  $y$  as wanted.  $\square$

Similarly to what has been established for triples we note

**Lemma 13.22.** *The space of positive quadruples is a union of connected components of the space of pairwise transverse quadruples.*



**13.9. Positive tuples.** Let  $n$  be an integer  $\geq 3$ . We introduce here the positive  $n$ -tuples in generalizing point (2) of Proposition 13.15 or point (7) of Proposition 13.19. Positive tuples of flags have been also extensively studied in [GLW21].

**Definition 13.23.** A  $n$ -tuple  $(f_i)_{i \in \mathbf{Z}/n\mathbf{Z}}$  of elements of  $F_\Theta$  is *positive* if, for all  $i \neq j$  in  $\mathbf{Z}/n\mathbf{Z}$ , there exists a diamond  $D_{i,j}$  with extremities  $f_i$  and  $f_j$  with  $D_{j,i} = D_{i,j}^\vee$  and  $f_k$  belongs to  $D_{i,j}$  for all  $(i, k, j)$  cyclically ordered (i.e. there are integers  $a, b$ , and  $c$  representing  $i, j$ , and  $k$  respectively and such that  $a < b < c < a + n$ ).

Note that the diamonds are uniquely determined by the properties that  $f_{i+1}$  belongs to  $D_{i,j}$  (when  $j \notin \{i, i+1\}$ ) and  $D_{i,i+1} = D_{i+1,i}^\vee$ .

From the definition, it is obvious that

**Lemma 13.24.** (1) *The set of positive  $n$ -tuples is invariant by the dihedral group  $D_n$  (i.e. by cyclic permutations and by the permutation  $i \leftrightarrow n+1-i$ ).*

(2) *Any subconfiguration of a positive  $n$ -tuple consisting of  $m$  elements is a positive  $m$ -tuple.*

Here is one (among many) characterizations of positive tuples.

**Lemma 13.25.** *A  $n$ -tuple is positive if and only if it is in the  $\text{Aut}_1(\mathfrak{g})$ -orbit of*

$$(a, u_1 \cdots u_{n-3} u_{n-2} \cdot b, u_1 \cdots u_{n-3} \cdot b, \dots, u_1 \cdot b, b),$$

where  $a = \mathfrak{p}_\Theta$ ,  $b = \mathfrak{p}_\Theta^{\text{opp}}$ , and, for all  $i$  in  $\llbracket 1, n-2 \rrbracket$ ,  $u_i$  belongs to  $U_\Theta^{>0}$ .

*Proof.* Let  $(f_1, \dots, f_n)$  be a positive tuple. Up to the  $\text{Aut}_1(\mathfrak{g})$  action, we can assume that  $f_1 = a$ ,  $f_n = b$  and  $f_{n-1} = u_1 \cdot b$ . Repeated applications of Lemma 13.20 give the sequence  $(u_2, \dots, u_{n-2})$ .

Conversely let  $u_1, \dots, u_{n-2}$  be in  $U_\Theta^{>0}$  and let  $(f_1, \dots, f_n)$  be

$$(a, \dots, u_1 \cdots u_i \cdot b, \dots, b).$$

For all  $i < j$  in  $\llbracket 1, n \rrbracket$ , let  $D_{i,j}$  be the unique diamond with extremities  $f_i$  and  $f_j$  contained in  $D = U_\Theta^{>0} \cdot b$  and  $D_{j,i} = D_{i,j}^\vee$ . The facts that, for  $i < k < j$  in  $\llbracket 1, n \rrbracket$ ,  $f_k$  belongs to  $D_{i,j}$  come from Corollary 13.17, the other cases come from Corollary 13.13.  $\square$

Restated in terms of diamonds, the lemma says it is enough to check that some sub-quadruples are positive:

**Proposition 13.26.** *Let  $(f_1, \dots, f_n)$  be in  $(F_\Theta)^n$ . Then  $(f_1, \dots, f_n)$  is positive if and only if, for every  $i = 2, \dots, n-2$ , the quadruple  $(f_1, f_i, f_{i+1}, f_n)$  is positive.*

We also note that

**Lemma 13.27.** *The set of positive  $n$ -tuples is a union of connected components of the space of pairwise transverse  $n$ -tuples.*

**13.10. Compatibility of positive structures.** As observed before (see Remarks 3.5 and Examples 4.3), if  $G$  is a split real group whose Dynkin diagram has a double arrow, then  $G$  carries two positive structures, one with respect to  $\Delta$  and one with respect to a proper subset  $\Theta \subset \Delta$ . Both positive structures give rise to a notion of positive triples

in the flag varieties  $F_\Delta$ , respectively  $F_\Theta$ . Here we show that these two different notions are compatible. This concerns the following groups up to isogeny:

- the symplectic Lie group  $\mathrm{Sp}(2n, \mathbf{R})$ ,  $n \geq 2$ ;
- the orthogonal groups  $\mathrm{SO}(n, n+1)$ ,  $n \geq 3$ ;
- the real split Lie group of type  $F_4$ .

Note that in this case, as explained in Section 5.5, the choice of root vectors  $X_\alpha$  for all  $\alpha \in \Delta$  determine the cones  $c_\alpha$  for  $\alpha \in \Theta$ . Let  $U_\Delta^{>0}$ ,  $U_\Delta^{\mathrm{opp}, >0}$ , and  $U_\Theta^{>0}$ ,  $U_\Theta^{\mathrm{opp}, >0}$  be the corresponding semigroups. The natural projection  $\pi: F_\Delta \rightarrow F_\Theta$  is  $\mathrm{Aut}_1(\mathfrak{g})$ -equivariant and  $\pi(\mathfrak{p}_\Delta) = \mathfrak{p}_\Theta$  and  $\pi(\mathfrak{p}_\Delta^{\mathrm{opp}}) = \mathfrak{p}_\Theta^{\mathrm{opp}}$ . This projection behaves well with respect to the notion of positivity introduced:

**Proposition 13.28.** *Let  $(a, x, b)$  be a positive triple in  $F_\Delta$ . Then  $(\pi(a), \pi(x), \pi(b))$  is a positive triple in  $F_\Theta$ .*

*Proof.* The positive semigroup  $U_\Delta^{>0}$  is determined here by elements  $X_\alpha$  generating  $\mathfrak{g}_\alpha$  for  $\alpha$  in  $\Delta$ . These elements can be used to determine the semigroup  $U_\Theta^{>0}$ , i.e. we choose the cones  $c_\alpha \subset \mathfrak{u}_\alpha$  ( $\alpha \in \Theta$ ) by the convention that  $X_\alpha \in c_\alpha$  (Remark 5.15).

First observe that, by the transitivity of the action of  $G$  on the space of transverse pairs and since  $(\pi(\mathfrak{p}_\Delta), \pi(\mathfrak{p}_\Delta^{\mathrm{opp}})) = (\mathfrak{p}_\Theta, \mathfrak{p}_\Theta^{\mathrm{opp}})$ , the image by  $\pi$  of every transverse pair in  $F_\Delta$  is a transverse pair in  $F_\Theta$ .

Let now  $(a, x, b)$  be a positive triple in  $F_\Delta$ . By the transitivity of the action of  $\mathrm{Aut}_1(\mathfrak{g})$ , we can assume that  $a = \mathfrak{p}_\Delta$ ,  $b = \mathfrak{p}_\Delta^{\mathrm{opp}}$  and  $x = u \cdot b$  for  $u \in U_\Delta^{>0}$ . Hence  $\pi(x) = u \cdot \pi(b)$  and we need to show that  $u \cdot \pi(b) \in U_\Theta^{>0} \cdot \pi(b)$ . Since  $\pi(x)$  is transverse to  $\pi(b)$  it is enough to prove that  $u \cdot \pi(b) \in U_\Theta^{\geq 0} \cdot \pi(b)$ .

For this, we will prove the inclusions  $u \cdot (U_\Theta^{\geq 0} \cdot \pi(b)) \subset U_\Theta^{\geq 0} \cdot \pi(b)$  for  $u$  in the generating set  $\bigcup_{\alpha \in \Delta} \exp(\mathbf{R}_{\geq 0} X_\alpha)$  of  $U_\Delta^{\geq 0}$ . Let thus  $\alpha$  be in  $\Delta$ ,  $t \geq 0$ , and  $u = \exp(tX_\alpha)$ . When  $\alpha$  belongs to  $\Theta$ , then  $u$  belongs to  $U_\Theta^{\geq 0}$  and the inclusion comes from the fact that  $U_\Theta^{\geq 0}$  is a semigroup. When  $\alpha$  belongs to  $\Delta \setminus \Theta$ , then  $u$  belongs to  $L_\Theta^\circ$  and the inclusion comes from the fact that  $L_\Theta^\circ$  normalizes  $U_\Theta^{\geq 0}$  and fixes  $\pi(b)$ .  $\square$

**13.11. Positive maps.** We extend the notion of positive tuples to the notion of positive maps. For this let us denote by  $\Lambda$  a set equipped with a cyclic ordering, i.e. there is a notion of “ $y$  being between  $x$  and  $z$ ” satisfying a number of natural properties that are a bit similar to the properties requested for diamonds, cf. [Hum16] for example (typically  $\Lambda$  is a subset of the circle). This means that there is a subset  $\Lambda^{3+}$  of  $\Lambda^3$  consisting of cyclically ordered triples. There is therefore a notion of cyclically ordered  $n$ -tuples in  $\Lambda$  for every  $n \geq 3$ .

**Definition 13.29.** A map  $f: \Lambda \rightarrow F_\Theta$  is said to be *positive* if the image by  $f$  of every cyclically ordered  $n$ -tuple is a positive  $n$ -tuple.

Proposition 13.26 implies

**Lemma 13.30.** *Suppose  $\Lambda$  is  $\geq 4$ . A map  $f$  is positive if and only if it sends every cyclically ordered quadruple to a positive quadruple.*



(with respect to this particular matrix realization of the group) are the  $e_i$  mapping a diagonal matrix to its  $i$ -th diagonal entry ( $i$  in  $\llbracket 1, p+1 \rrbracket$ ). We will use this to identify  $\mathfrak{a}$  with  $\mathbf{R}^{p+1}$  and describe the Weyl group and its elements in  $\mathrm{GL}_{p+1}(\mathbf{R})$ .

The roots are

$$\{\pm e_i\}_{i \in \llbracket 1, p+1 \rrbracket} \cup \{\pm e_j \pm e_k\}_{j, k \in \llbracket 1, p+1 \rrbracket, j < k}.$$

For the lexicographic order the positive roots are

$$\{e_i\}_{i \in \llbracket 1, p+1 \rrbracket} \cup \{e_j \pm e_k\}_{j, k \in \llbracket 1, p+1 \rrbracket, j < k}$$

and the simple roots are  $\alpha_1 = e_1 - e_2$ ,  $\alpha_2 = e_2 - e_3$ ,  $\dots$ ,  $\alpha_p = e_p - e_{p+1}$ ,  $\alpha_{p+1} = e_{p+1}$ .

The Weyl group  $W$  naturally identifies with  $(\mathbf{Z}/2\mathbf{Z})^{p+1} \rtimes S_{p+1}$  acting on  $\mathbf{R}^{p+1}$  by permuting the coordinates and changing their signs. Its generators  $s_1, \dots, s_{p+1}$  associated with  $\alpha_1, \dots, \alpha_{p+1}$  are the transformations:

$$\begin{aligned} s_1 &: (x_1, x_2, \dots, x_{p+1}) \longmapsto (x_2, x_1, \dots, x_{p+1}), \\ s_2 &: (x_1, x_2, x_3, \dots, x_{p+1}) \longmapsto (x_1, x_3, x_2, \dots, x_{p+1}), \\ &\dots, \\ s_p &: (x_1, \dots, x_p, x_{p+1}) \longmapsto (x_1, \dots, x_{p+1}, x_p) \\ \text{and } s_{p+1} &: (x_1, x_2, \dots, x_{p+1}) \longmapsto (x_1, x_2, \dots, -x_{p+1}). \end{aligned}$$

Furthermore the longest length element of  $W$  is  $-\mathrm{Id}_{p+1}$  as it exchanges the Weyl chamber with its opposite.

Let  $\Theta = \{\alpha_1, \dots, \alpha_p\}$ . With the notation of Section 3.4,  $\alpha_\Theta = \alpha_p$ , the longest length element in  $W_{\Delta \setminus \Theta} = \langle s_{p+1} \rangle$  is  $s_{p+1}$  itself and the subgroup of  $W$  generated by  $s_p$  and  $s_{p+1}$  is isomorphic to  $(\mathbf{Z}/2\mathbf{Z})^2 \rtimes S_2$  (the Weyl group of  $\mathrm{SO}(2, 2+k)$ ). Its longest length element is  $s_p s_{p+1} s_p s_{p+1}$  so that the element  $\sigma_p = \sigma_\Theta$  is  $s_p s_{p+1} s_p$ . Seen as an element of  $\mathrm{GL}_{p+1}(\mathbf{R})$ ,  $\sigma_p$  is the transformation  $(x_1, \dots, x_p, x_{p+1}) \mapsto (x_1, \dots, -x_p, x_{p+1})$ .

Recall that the group  $W(\Theta)$  is the subgroup of  $W$  generated by  $s_1, \dots, s_{p-1}$  and  $\sigma_p$ . As all these generators fix the last coordinate, we can identify  $W(\Theta)$  with a subgroup of  $\mathrm{GL}_p(\mathbf{R})$ . With this identification in mind and the above description, it is apparent that this subgroup is the Weyl group of type  $B_p$  and its longest length element is, in this geometric realization,  $-\mathrm{Id}_p$ .

Turning back to subgroups of  $\mathrm{GL}_{p+1}(\mathbf{R})$ , the longest length element of  $W(\Theta)$  is the transformation

$$(x_1, \dots, x_p, x_{p+1}) \mapsto (-x_1, \dots, -x_p, x_{p+1}).$$

Since the longest length element of  $W$  is  $-\mathrm{Id}_{p+1}$  and the longest length element of  $W_{\Delta \setminus \Theta}$  is  $s_{p+1}$  one gets as well that  $w_{\max}^\Theta$  is

$$(x_1, \dots, x_p, x_{p+1}) \mapsto (-x_1, \dots, -x_p, x_{p+1})$$

establishing, for the type  $B_{p+1}$ , the fact that  $w_{\max}^\Theta$  belongs to  $W(\Theta)$  (Proposition 8.16).

Written as products of generators, these elements are: (we use the notation  $x^y = y^{-1}xy$  so that  $(x^y)^z = x^{yz}$  and  $\sigma_p = s_{p+1}^{s_p}$ )

$$\begin{aligned} w_\Delta &= s_{p+1}^{s_p \cdots s_1} s_{p+1}^{s_p \cdots s_2} \cdots s_{p+1}^{s_p} \cdots s_{p+1} \\ w_{\max}^\Theta &= s_{p+1}^{s_p \cdots s_1} s_{p+1}^{s_p \cdots s_2} \cdots s_{p+1}^{s_p} \end{aligned}$$

$$= \sigma_p^{s_{p-1} \cdots s_1} \sigma_p^{s_{p-1} \cdots s_2} \cdots \sigma_p^{s_{p-1}} \cdot \sigma_p,$$

The above equalities are established noting that  $s_{p+1}^{s_p \cdots s_k}$  is

$$(x_1, \dots, x_k, \dots, x_{p+1}) \mapsto (x_1, \dots, -x_k, \dots, x_{p+1}).$$

One can also verify that the above decompositions are reduced: for example, the length of  $w_\Delta$  is the dimension of the complete flag variety for the split group  $\mathrm{SO}(p+1, p+2)$  and is thus equal to  $(p+1)^2$ ; this number coincides with the length of the above product.

#### APPENDIX B. LONGEST LENGTH ELEMENT IN $F_4$

In this section we determine a reduced expression of the longest length element in the Weyl group associated to a root system of type  $F_4$ .

The root system  $F_4$  is intimately related with the lattice  $\Lambda$  of  $\mathbf{R}^4$  generated by  $\mathbf{Z}^4$  and the element  $\frac{1}{2}(1, 1, 1, 1)$ . An alternative description of  $\Lambda$  is the set of elements in  $\mathbf{R}^4$  all of whose coordinates have the same remainder, 0 or  $1/2$ , modulo 1.

The elements of  $F_4$  are the elements of  $\Lambda$  whose Euclidean norms are 1 or  $\sqrt{2}$ . They can be explicitly listed: let  $(e_i)_{i \in \llbracket 1, 4 \rrbracket}$  be the canonical basis of  $\mathbf{R}^4$ , then

$$F_4 = \{ \pm e_i \}_{i \in \llbracket 1, 4 \rrbracket} \cup \{ \pm e_k \pm e_\ell \}_{k, \ell \in \llbracket 1, 4 \rrbracket, k < \ell} \cup \left\{ \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4) \right\}.$$

The positive roots are (using colexicographic order)

$$\{ e_i \}_{i \in \llbracket 1, 4 \rrbracket} \cup \{ \pm e_k + e_\ell \}_{k, \ell \in \llbracket 1, 4 \rrbracket, k < \ell} \cup \left\{ \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 + e_4) \right\},$$

and the simple roots are

$$\alpha_1 = -e_2 + e_3, \quad \alpha_2 = -e_1 + e_2, \quad \alpha_3 = e_1, \quad \alpha_4 = \frac{1}{2}(-e_1 - e_2 - e_3 + e_4).$$

The Weyl group is the subgroup of  $\mathrm{GL}_4(\mathbf{R})$  generated by the symmetries  $s_1, \dots, s_4$  associated with  $\alpha_1, \dots, \alpha_4$ . The transformation  $s_i$  is  $x \mapsto x - 2 \frac{\langle \alpha_i, x \rangle}{\langle \alpha_i, \alpha_i \rangle} \alpha_i$ . In matrix coordinates

$$s_4 = \frac{1}{2} \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

The element  $s_3$  is  $(x_1, x_2, x_3, x_4) \mapsto (-x_1, x_2, x_3, x_4)$ . Finally  $s_2$  and  $s_1$  are respectively  $(x_1, x_2, x_3, x_4) \mapsto (x_2, x_1, x_3, x_4)$  and  $(x_1, x_2, x_3, x_4) \mapsto (x_1, x_3, x_2, x_4)$ .

The relevant subset of the simple roots in this situation is  $\Theta = \{ \alpha_1, \alpha_2 \}$  and the root  $\alpha_\Theta$  is  $\alpha_2$ . The subgroup  $W_{\Delta \setminus \Theta}$  is generated by  $s_3$  and  $s_4$  and is isomorphic to  $S_3$ , its longest length element is  $s_3 s_4 s_3$ .

The subgroup generated by the symmetries indexed by  $\Delta \setminus \Theta$  and  $\alpha_\Theta$  is the group generated by  $\{ s_2, s_3, s_4 \}$  and is isomorphic to the Weyl group  $C_3$ , which is the same as the Weyl group  $B_3$  (with the reindexation  $1 \mapsto 4, 2 \mapsto 3, 3 \mapsto 2$  with respect to the previous appendix). Its longest length element is  $s_2 s_2^{s_3} s_2^{s_3 s_4} = s_2 s_3 s_2 s_3 s_4 s_3 s_2 s_3 s_4$ . Hence the element  $\sigma_2 = \sigma_\Theta$  is:

$$\sigma_2 = s_2 s_3 s_2 s_3 s_4 s_3 s_2 (s_3 s_4 s_4 s_3) s_4$$

$$\begin{aligned}
 &= s_2 s_3 s_2 s_3 s_4 s_3 (s_2 s_4) \\
 &= s_2 s_3 s_2 (s_3 s_4 s_3 s_4) s_2 \\
 &= s_2 s_3 s_2 s_4 s_3 s_2 = s_2 s_3 s_4 s_2 s_3 s_2.
 \end{aligned}$$

And the last two expressions are reduced (this can be deduced from the fact below that  $s_2 s_3 s_4 s_2 s_3 s_2$  is a subword of a reduced expression of the longest length element).

One can calculate  $\sigma_2$  in  $\mathrm{GL}_4(\mathbf{R})$ :

$$\sigma_2 = \frac{1}{2} \begin{pmatrix} -1 & -1 & -1 & 1 \\ -1 & -1 & 1 & -1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \end{pmatrix},$$

as well as (recall that  $\sigma_1 = s_1$ )

$$\sigma_1 \sigma_2 = \frac{1}{2} \begin{pmatrix} -1 & -1 & -1 & 1 \\ -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & -1 \\ 1 & -1 & 1 & 1 \end{pmatrix}.$$

Its square is

$$(\sigma_1 \sigma_2)^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

showing that  $\sigma_1 \sigma_2$  is of order 6 and that  $W(\Theta)$  is of type  $G_2$ .

Finally, the following holds

$$\begin{aligned}
 s_3 s_4 s_3 &= \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix} \text{ and} \\
 (\sigma_1 \sigma_2)^3 &= \frac{1}{2} \begin{pmatrix} -1 & -1 & -1 & 1 \\ -1 & -1 & 1 & -1 \\ -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 \end{pmatrix} = -(s_3 s_4 s_3).
 \end{aligned}$$

We thus have

$$\begin{aligned}
 (\sigma_1 \sigma_2)^3 s_3 s_4 s_3 &= (\sigma_2 \sigma_1)^3 s_3 s_4 s_3 = -\mathrm{Id}_4 \\
 &= s_2 s_3 s_2 s_4 s_3 s_2 s_1 \cdot s_2 s_3 s_2 s_4 s_3 s_2 s_1 \cdot s_2 s_3 s_2 s_4 s_3 s_2 s_1 \cdot s_3 s_4 s_3
 \end{aligned}$$

which is the longest length element of  $F_4$  (since it sends the Weyl chamber to its opposite) and a reduced decomposition of it (since the length of this decomposition is equal to 24, the number of positive roots). This shows Proposition 8.16 in this case too.

## REFERENCES

- [AABC<sup>+</sup>19] Marta Aparicio-Arroyo, Steven Bradlow, Brian Collier, Oscar García-Prada, Peter B. Gothen, and André Oliveira, *SO(p, q)-Higgs bundles and higher Teichmüller components*, *Invent. Math.* **218** (2019), no. 1, 197–299.



- [ABR<sup>+</sup>22] Daniele Alessandrini, Arkady Berenstein, Vladimir Retakh, Eugen Rogozinnikov, and Anna Wienhard, *Symplectic groups over noncommutative algebras*, Sel. Math., New Ser. **28** (2022), no. 4, 119, Id/No 82.
- [AGRW19] Daniele Alessandrini, Olivier Guichard, Eugen Rogozinnikov, and Anna Wienhard, *Noncommutative coordinates for symplectic representations*, 2019, to appear Memoirs of the AMS.
- [BCGP<sup>+</sup>21] Steve Bradlow, Brian Collier, Oscar García-Prada, Peter Gothen, and André Oliveira, *A general Cayley correspondence and higher Teichmüller spaces*, arXiv/2021.09377, 2021.
- [BCL20] Martin Bridgeman, Richard Canary, and François Labourie, *Simple length rigidity for Hitchin representations*, Adv. Math. **360** (2020), 61.
- [Ben00] Yves Benoist, *Automorphismes des cônes convexes*, Invent. Math. **141** (2000), no. 1, 149–193.
- [BGL<sup>+</sup>24] Jonas Beyrer, Olivier Guichard, François Labourie, Beatrice Pozzetti, and Anna Wienhard, *Positivity, cross-ratios and the Collar Lemma*, in preparation, 2024.
- [BILW05] Marc Burger, Alessandra Iozzi, François Labourie, and Anna Wienhard, *Maximal representations of surface groups: Symplectic Anosov structures*, Pure Appl. Math. Q. **1** (2005), no. 3, Special Issue: In memory of Armand Borel. Part 2, 543–590.
- [BIW07] Marc Burger, Alessandra Iozzi, and Anna Wienhard, *Hermitian symmetric spaces and Kähler rigidity*, Transformation Groups. **12** (2007), 5–32.
- [BIW09] ———, *Tight homomorphisms and Hermitian symmetric spaces*, Geom. Funct. Anal. **19** (2009), no. 3, 678–721.
- [BIW10] ———, *Surface group representations with maximal Toledo invariant*, Ann. of Math. (2) **172** (2010), no. 1, 517–566.
- [Bou06] Nicolas Bourbaki, *Éléments de mathématique. Groupes et algèbres de Lie. Chapitres 7 et 8*, reprint of the 1975 original ed., Berlin: Springer, 2006.
- [Bou07] ———, *Éléments de mathématique. Groupes et algèbres de Lie. Chapitres 4 à 6*, reprint of the 1968 original ed., Berlin: Springer, 2007.
- [BP21] Jonas Beyrer and Beatrice Pozzetti, *Positive surface group representations in  $PO(p, q)$* , arXiv:2106.14725, 2021.
- [BR18] Arkady Berenstein and Vladimir Retakh, *Noncommutative marked surfaces*, Adv. Math. **328** (2018), 1010–1087.
- [BT72] Armand Borel and Jacques Tits, *Compléments à l'article: "Groupes réductifs"*, Publ. Math. Inst. Hautes Étud. Sci. **41** (1972), 253–276.
- [BZ97] Arkady Berenstein and Andrei Zelevinsky, *Total positivity in Schubert varieties*, Comment. Math. Helv. **72** (1997), no. 1, 128–166.
- [Col20] Brian Collier,  *$SO(n, n + 1)$ -surface group representations and Higgs bundles*, Ann. Sci. ENS (4) **53** (2020), no. 6, 1561–1616.
- [FG06] Vladimir Fock and Alexander Goncharov, *Moduli spaces of local systems and higher Teichmüller theory*, Publ. Math. Inst. Hautes Études Sci. **103** (2006), 1–211.
- [FK94] Jacques Faraut and Adam Korányi, *Analysis on symmetric cones*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1994, Oxford Science Publications.
- [GK35] Felix Gantmacher and Mark Krein, *Sur les matrices oscillatoires.*, C. R. Acad. Sci., Paris **201** (1935), 577–579.
- [GLW21] Olivier Guichard, François Labourie, and Anna Wienhard, *Positivity and representations of surface groups*, arXiv/2106.14584, 2021.
- [Gui05] Olivier Guichard, *Une dualité pour les courbes hyperconvexes*, Geom. Dedicata **112** (2005), 141–164.
- [GW18] Olivier Guichard and Anna Wienhard, *Positivity and higher Teichmüller theory*, European congress of mathematics. Proceedings of

- the 7th ECM (7ECM) congress, Berlin, Germany, July 18–22, 2016, Zürich: European Mathematical Society (EMS), 2018, pp. 289–310.
- [GW24] ———, *Generalizing Lusztig's total positivity II*, in preparation, 2024.
- [Hel01] Sigurdur Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Graduate Studies in Mathematics, vol. 34, American Mathematical Society, Providence, RI, 2001, Corrected reprint of the 1978 original.
- [Hit92] Nigel J. Hitchin, *Lie groups and Teichmüller space*, *Topology* **31** (1992), no. 3, 449–473.
- [Hun16] Edward V. Huntington, *A set of independent postulates for cyclic order*, *Proc. Natl. Acad. Sci. USA* **2** (1916).
- [Kna02] Anthony W. Knaapp, *Lie groups beyond an introduction*, second ed., *Progress in Mathematics*, vol. 140, Birkhäuser Boston Inc., Boston, MA, 2002.
- [Kos59] Bertram Kostant, *The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group*, *Amer. J. Math.* **81** (1959), 973–1032.
- [Kos10] Bertram Kostant, *Root systems for Levi factors and Borel-de Siebenthal theory*, *Symmetry and spaces*. In Honor of Gerry Schwarz on the occasion of his 60th birthday, Basel: Birkhäuser, 2010, pp. 129–152.
- [Kou95] Khalid Koufany, *Semi-groupe de Lie associé à un cône symétrique*, *Ann. Inst. Fourier (Grenoble)* **45** (1995), no. 1, 1–29.
- [KR22] Clarence Kineider and Eugen Rogozinnikov, *On partial Abelianization of framed local systems*, arXiv/2212.08393, 2022.
- [Lab06] François Labourie, *Anosov flows, surface groups and curves in projective space*, *Invent. Math.* **165** (2006), no. 1, 51–114.
- [LT23] François Labourie and Jérémy Toulisse, *Quasicircles and quasiperiodic surfaces in pseudo-hyperbolic spaces*, *Invent. Math.* **233** (2023), no. 1, 81–168.
- [Lus94] George Lusztig, *Total positivity in reductive groups*, *Lie theory and geometry*, *Progr. Math.*, vol. 123, Birkhäuser Boston, Boston, MA, 1994, pp. 531–568.
- [Lus98] ———, *Total positivity in partial flag manifolds*, *Represent. Theory* **2** (1998), 70–78 (electronic).
- [PSW23] Beatrice Pozzetti, Andrés Sambarino, and Anna Wienhard, *Anosov representations with Lipschitz limit set*, *Geom. Topol.* **27** (2023), no. 8, 3303–3360.
- [Sch30] Isaac Schoenberg, *Über variationsvermindernde lineare Transformationen*, *Math. Z.* **32** (1930), no. 1, 321–328.
- [War72] Garth Warner, *Harmonic analysis on semi-simple Lie groups. I*, *Die Grundlehren der mathematischen Wissenschaften*, Springer-Verlag, New York, 1972, Vol 188.
- [Whi52] Anne Whitney, *A reduction theorem for totally positive matrices*, *J. Analyse Math.* **2** (1952), 88–92.
- [Wie18] Anna Wienhard, *An invitation to higher Teichmüller theory*, *Proceedings of the international congress of mathematicians, ICM 2018, Rio de Janeiro, Brazil, August 1–9, 2018. Volume II. Invited lectures*, Hackensack, NJ: World Scientific; Rio de Janeiro: Sociedade Brasileira de Matemática (SBM), 2018, pp. 1013–1039.

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