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Olivier Guichard, Anna Wienhard

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# GENERALIZING LUSZTIG'S TOTAL POSITIVITY

OLIVIER GUICHARD AND ANNA WIENHARD

ABSTRACT. We introduce the notion of  $\Theta$ -positivity in real simple Lie groups. This notion at the same time generalizes Lusztig's total positivity in split real Lie groups and invariant orders in Lie groups of Hermitian type. We show that there are four families of Lie groups which admit  $\Theta$ -positive structures, and investigate basic properties of  $\Theta$ -positivity.

## 1. INTRODUCTION

The theory of totally positive matrices arose in the beginning of the 20th century through work of Schoenberg [Sch30] and Gantmacher and Krein [GK35]. Total positivity has since become an important concept in several mathematical fields. In the 1990's the theory has been generalized widely by Lusztig [Lus98] who introduced the total positive semigroup of a general split real semisimple Lie group. Lusztig's total positivity plays an important role in representation theory, cluster algebras, and has many relations to other areas in mathematics as well as in theoretical physics. In this article we introduce a generalization of Lusztig's total positivity in the context of real semisimple Lie groups  $G$  that are not necessarily split. We call this generalization  $\Theta$ -positivity because it depends on the choice of a flag variety associated with  $G$  which is determined by a subset  $\Theta$  of the set of simple roots. The notion of  $\Theta$ -positivity generalizes at the same time Lusztig's total positivity (when the group is split) as well as Lie semigroups of Lie groups of Hermitian type, which are related to bi-invariant orders and causal structures. We give a classification of simple Lie groups admitting a

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$\Theta$ -positive structure. Besides split real Lie groups and Hermitian Lie groups of tube type, two further families admit a  $\Theta$ -positive structure: the groups locally isomorphic to indefinite orthogonal groups  $SO(p, q)$ , with  $2 \leq p < q$ , and an exceptional family, whose reduced root system is of type  $F_4$  (cf. Theorem 1.1 below). To our knowledge, for these two families no positive structure was known before.

Our interest in  $\Theta$ -positivity arose from higher rank Teichmüller theory, in particular from trying to find a unifying framework that explains the similarities between Hitchin representations and maximal representations. The notion of  $\Theta$ -positivity provides such a unifying systematic framework. It also leads to several conjectures regarding higher rank Teichmüller spaces, some of which have been formulated in [GW18] and [Wie18], and partly proven in [Col20, AABC<sup>+</sup>19, BCGP<sup>+</sup>21], [GLW21], and [BP21]. However, the reach of  $\Theta$ -positivity goes far beyond higher rank Teichmüller theory. For example, as we will discuss in a bit more detail below,  $\Theta$ -positivity suggests that Hermitian Lie groups of tube type should be considered as groups of type  $A_1$  over non-commutative algebras. Something, which has been made precise in [ABR<sup>+</sup>21] for most classical groups.

Before we describe the notion of  $\Theta$ -positivity and our results in more detail, let us note that we described the notion of  $\Theta$ -positivity several years ago in a survey paper [GW18] with only few proofs and a hands on description of the notion for indefinite orthogonal groups  $SO(p, q)$ ,  $2 \leq p < q$ . Several of the properties we described there have been already used by other people. This article now finally gives the foundation of  $\Theta$ -positivity. We focus here on the structure of unipotent subgroups in  $G$  and on  $\Theta$ -positivity in generalized flag varieties. In particular, we provide all the background needed to introduce positive configurations in flag varieties, positive maps or positive representations of surface groups. We give all the background and prove the results used in [Col20], [AABC<sup>+</sup>19, BCGP<sup>+</sup>21], in [BP19], [BP21], as well as all results used in [GLW21]. Ideas from this paper and [GLW21] were also used in [BCL20]. In a forthcoming second foundational article we will focus on the finer structure of  $\Theta$ -positivity, including braid relations, Gauß decomposition theorems, and geometric properties of positive elements in  $G$ .

We now describe the results of the paper in more detail.

**1.1.  $\Theta$ -positive structures.** Totally positive matrices are defined by requiring all minors to be positive. Anne Whitney showed in [Whi52] that the semigroup of totally positive, or more general totally non-negative matrices can be generated by an explicit set of simple matrices. This reduction theorem allowed the generalization of total positivity to all simple split real Lie groups [Lus98]. The key building block in

Lusztig's approach is the construction of a closed non-negative semigroup  $U^{\geq 0}$  in the unipotent radical  $U$  of the minimal parabolic subgroup  $P$  of a split real Lie group  $G$ . This semigroup is constructed as follows. The Lie algebra  $\mathfrak{u}$  can be described as the sum of the root spaces  $\mathfrak{g}_\alpha$  for all positive roots  $\alpha$ . Since  $G$  is split, any root space  $\mathfrak{g}_\alpha$  is of dimension 1 and hence can be identified with  $\mathbf{R}$ . Given the set of simple positive roots  $\Delta$ , we consider the map  $x_\alpha: \mathfrak{g}_\alpha \cong \mathbf{R} \rightarrow U$ ,  $s \mapsto \exp(s)$ . The non-negative semigroup  $U^{\geq 0}$  is the semigroup generated by  $x_\alpha(\mathbf{R}_{\geq 0})$ ,  $\alpha \in \Delta$ .

Turning to our situation, when  $G$  is a simple real Lie group, which is not necessarily split. We consider a subset  $\Theta \subset \Delta$  of the set of simple roots. This defines a standard parabolic subgroup  $P_\Theta$  in  $G$ . (We choose the convention so that  $P_\Delta$  is the minimal parabolic subgroup of  $G$ .) The group  $P_\Theta$  is a semidirect product of a reductive Lie group  $L_\Theta$  and its unipotent radical  $U_\Theta$ . The Lie algebra  $\mathfrak{u}_\Theta$  of  $U_\Theta$  carries a natural action by  $L_\Theta$  and can hence be decomposed into its  $L_\Theta$ -irreducible pieces. For every simple root  $\alpha \in \Theta$  there is an irreducible piece  $\mathfrak{u}_\alpha$ , which is now in general not of dimension 1. When  $G$  is split and  $\Delta = \Theta$  these would be precisely the root spaces  $\mathfrak{g}_\alpha$ .

We say that  $G$  admits a  $\Theta$ -positive structure if it satisfies the following property. For every  $\alpha \in \Theta$  there exists an acute convex cone  $c_\alpha \subset \mathfrak{u}_\alpha$  that is invariant under the action of  $L_\Theta^\circ$  on  $\mathfrak{u}_\alpha$ .

We prove

**Theorem 1.1** (see Theorem 3.4). *A simple Lie group  $G$  admits a  $\Theta$ -positive structure if and only if the pair  $(G, \Theta)$  belongs to the following list:*

- (1)  $G$  is a split real form, and  $\Theta = \Delta$ ;
- (2)  $G$  is Hermitian of tube type and of real rank  $r$  and  $\Theta = \{\alpha_r\}$ , where  $\alpha_r$  is the long simple restricted root;
- (3)  $G$  is locally isomorphic to  $\mathrm{SO}(p+1, p+k)$ ,  $p > 1$ ,  $k > 1$  and  $\Theta = \{\alpha_1, \dots, \alpha_p\}$ , where  $\alpha_1, \dots, \alpha_p$  are the long simple restricted roots;
- (4)  $G$  is the real form of  $F_4$ ,  $E_6$ ,  $E_7$ , or of  $E_8$  whose reduced root system is of type  $F_4$ , and  $\Theta = \{\alpha_1, \alpha_2\}$ , where  $\alpha_1, \alpha_2$  are the long simple restricted roots.

The cones  $c_\alpha \subset \mathfrak{u}_\alpha$ ,  $\alpha \in \Theta$  allow to define the nonnegative semigroup  $U_\Theta^{\geq 0}$  to be the sub-semigroup of  $U_\Theta$  generated by elements  $\exp(v)$ , with  $v \in c_\alpha$  for some  $\alpha \in \Theta$ .

The reader might find this a rather ad-hoc construction, but in fact we show that the existence of a  $\Theta$ -positive structure can be characterized geometrically in terms of positive triples in flag varieties. To give this geometric characterization the following theorem that relates the existence of semigroups to the existence of  $\Theta$ -positive structures plays a crucial role.

**Theorem 1.2** (see Theorem 9.1). *Let  $G$  be a connected simple Lie group. Suppose that there is  $U^+ \subset U_\Theta$  such that  $U^+$  is a closed  $L_\Theta^\circ$ -invariant semigroup of non-empty interior, which contains no nontrivial invertible element. Then  $G$  admits a  $\Theta$ -positive structure and the semigroup  $U^+$  contains the semigroup  $U_\Theta^{\geq 0}$ .*

*If we further assume that the interior of  $U^+$  is contained in the open Bruhat cell with respect to  $P_\Theta^{\text{opp}}$ , then  $U^+ = U_\Theta^{\geq 0}$ .*

Theorem 9.2 is a variant of this result that turns out to be useful to relate  $\Theta$ -positivity with positivity in flag varieties.

### 1.2. The unipotent positive semigroup and the $\Theta$ -Weyl group.

Whereas the definition of the nonnegative unipotent semigroup  $U^{\geq 0}$  in Lusztig's work is straight forward, the definition of the positive unipotent semigroup  $U^{>0}$  is a bit more tricky and involves the Weyl group  $W$ . Recall that the Weyl group is generated by elements  $s_\alpha$ ,  $\alpha \in \Delta$ . There is a unique longest element  $w_0 \in W$ . To construct the positive unipotent semigroup  $U^{>0}$  one fixes a reduced expression of the longest word  $w_0 = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_k}}$ . Making this choice Lusztig defines a map  $F_{w_0}: \mathfrak{g}_{\alpha_{i_1}} \times \cdots \times \mathfrak{g}_{\alpha_{i_k}} \rightarrow U$ ,  $(s_1, \dots, s_k) \mapsto \exp(s_1) \cdots \exp(s_k)$ . The image of  $\mathbf{R}_{>0}^k \subset \mathfrak{g}_{\alpha_{i_1}} \times \cdots \times \mathfrak{g}_{\alpha_{i_k}}$  under this map  $F_{w_0}$  is the positive unipotent semigroup. A key point is of course to show that this image is independent of the choice of the reduced expression. In [Lus98] this is done by giving explicit formulas for the change of coordinates given by a braid relation,  $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ , which are positive rational functions (see Berenstein–Zelevinsky [BZ97] for explicit formulas for the braid relations in the case of non simply laced Dynkin diagrams). Since any two reduced expressions of  $w_0$  are related by a sequence of braid relations, this thus proves that the image  $F_{w_0}(\mathbf{R}^k)$  is independent of the chosen reduced expression.

In order to define the positive unipotent semigroup  $U_\Theta^{\geq 0}$  when  $G$  admits a  $\Theta$ -positive structure, the role played by the Weyl group in Lusztig's total positivity is replaced by what we call the  $\Theta$ -Weyl group  $W(\Theta)$ , a subgroup of  $W$  generated by elements  $\sigma_\alpha$  for all  $\alpha \in \Theta$ . For all  $\alpha \in \Theta$  which are not connected to  $\Delta \setminus \Theta$ ,  $\sigma_\alpha$  is just the reflection  $s_\alpha$ . For the unique  $\alpha \in \Theta$  which is connected to  $\Delta \setminus \Theta$ ,  $\sigma_\alpha$  is a specific word in the subgroup generated by  $\{\alpha\} \cup \Delta \setminus \Theta$ . It might a priori be a bit surprising that this subgroup  $W(\Theta)$  of  $W$  is natural isomorphic to the Weyl group of a different root system. Even more, there is a natural embedding of a split real Lie group of type  $W(\Theta)$  into  $G$ . The  $\Theta$ -Weyl group  $W(\Theta)$  now plays the same role as the Weyl group in order to parametrize and define the positive unipotent semigroup.

**Theorem 1.3** (see Theorem 8.1). *Given a reduced expression  $\sigma_{\gamma_1} \cdots \sigma_{\gamma_n}$  of the longest element  $w_{\max}^\Theta \in W(\Theta)$ , the map  $F: c_{\gamma_1} \times \cdots \times c_{\gamma_n} \rightarrow U_\Theta$ , defined as the product of the exponential map on each factor, is proper with the following properties*

- (1)  $F|_{\dot{c}_{\gamma_1} \times \dots \times \dot{c}_{\gamma_n}}$  is open and injective.
- (2)  $F(\dot{c}_{\gamma_1} \times \dots \times \dot{c}_{\gamma_n})$  is contained in the open Bruhat cell  $\Omega_{\Theta}^{\text{opp}}$  with respect to  $P_{\Theta}^{\text{opp}}$ . In fact, it is a connected component of  $\Omega_{\Theta}^{\text{opp}} \cap U_{\Theta}$ .
- (3) The image  $F(\dot{c}_{\gamma_1} \times \dots \times \dot{c}_{\gamma_n})$  is independent of the reduced expression.

We set  $U_{\Theta}^{>0} := F(\dot{c}_{\gamma_1} \times \dots \times \dot{c}_{\gamma_n})$  and call it the positive unipotent semigroup. This is justified by the following theorem.

**Theorem 1.4** (see Corollary 8.16). *The set  $U_{\Theta}^{>0}$  is a semigroup, invariant by conjugation by  $L_{\Theta}^{\circ}$ . The closure of  $U_{\Theta}^{>0}$  is the non-negative semigroup  $U_{\Theta}^{\geq 0}$ , and  $U_{\Theta}^{>0}$  is the interior of  $U_{\Theta}^{\geq 0}$ . Furthermore,  $U_{\Theta}^{>0}U_{\Theta}^{>0} \subset U_{\Theta}^{>0}$  and  $U_{\Theta}^{>0}U_{\Theta}^{\geq 0} \subset U_{\Theta}^{>0}$ .*

The proof of Theorem 1.4 relies on a fine understanding of the image of the map  $F$  relative to the Bruhat decompositions of  $G$  with respect to  $P_{\Theta}$  and with respect to  $P_{\Delta}$ .

Of course the analogous construction can be made to define the non-negative semigroup  $U_{\Theta}^{\text{opp}, \geq 0}$  and the positive semigroup  $U_{\Theta}^{\text{opp}, > 0}$  in  $U_{\Theta}^{\text{opp}}$ .

With this we define the  $\Theta$ -nonnegative semigroup  $G_{\Theta}^{\geq 0}$  to be the semigroup of  $G$  generated by  $U_{\Theta}^{\text{opp}, \geq 0}$ ,  $U_{\Theta}^{\geq 0}$ , and  $L_{\Theta}^{\circ}$ , and the  $\Theta$ -positive semigroup  $G_{\Theta}^{>0}$  to be the semigroup of  $G$  generated by  $U_{\Theta}^{\text{opp}, > 0}$ ,  $U_{\Theta}^{>0}$ , and  $L_{\Theta}^{\circ}$ .

*Remark 1.5.* Note that to prove these results, we do not need to establish the explicit change of coordinates with respect to changing the reduced expression of  $w_{\max}^{\Theta}$  by a braid relation. However, to get a finer understanding, it is in fact interesting to write down explicit braid relations. In this paper we will only shortly discuss the braid relations in the case that  $G = \text{SO}(p, q)$  derived from explicit matrix equations. The more general treatment of braid relations involves  $\Theta$ -versions of universal enveloping algebras, adapting the strategy in [BZ97]. This will appear in a forthcoming paper in which we investigate the finer properties of the positive semigroup  $G_{\Theta}^{>0}$ .

**1.3. Positivity on flag varieties.** We now turn our attention to the flag variety  $F_{\Theta} \cong G/P_{\Theta}$ . There is a unique flag  $f_{\Theta} \in F_{\Theta}$  fixed by  $P_{\Theta}$  and a unique flag  $f_{\Theta}^{\text{opp}} \in F_{\Theta}$  fixed by  $P_{\Theta}^{\text{opp}}$ . It is well known that the set of flags that are transverse to  $f_{\Theta}$  can be parametrized by  $U_{\Theta}$ . The positive unipotent semigroup  $U_{\Theta}^{>0}$  thus allows us to introduce the notion of positive  $n$ -tuple of flags in  $F_{\Theta}$ .

An  $n$ -tuple of flags  $(f_0, f_1, \dots, f_{n-2}, f_{\infty})$  is said to be  $\Theta$ -positive if there exist an element  $g \in G$  and  $u_1, \dots, u_{n-2} \in U_{\Theta}^{>0}$  such that  $g \cdot (f_{\Theta}^{\text{opp}}, u_1 \cdot f_{\Theta}^{\text{opp}}, \dots, u_i \cdot \dots \cdot u_1 \cdot f_{\Theta}^{\text{opp}}, \dots, u_{n-2} \cdot \dots \cdot u_1 \cdot f_{\Theta}^{\text{opp}}, f_{\Theta}) = (f_0, f_1, \dots, f_{n-2}, f_{\infty})$ .

A consequence of Theorem 1.4 is

**Theorem 1.6.** *The set of flags  $U_{\Theta}^{>0} \cdot f_{\Theta}^{\text{opp}}$  is a connected component of the set of flags that are transverse to  $f_{\Theta}$  and to  $f_{\Theta}^{\text{opp}}$ . The stabilizer in  $G$  of a  $\Theta$ -positive triple is compact.*

Let us emphasize at this point, that in order to define  $\Theta$ -positivity we start with a rather algebraic condition — the existence of invariant cones which leads to the existence of semigroups in the unipotent group  $U_{\Theta}$ . However, investigating the induced notion of positive triples in the flag variety  $F_{\Theta}$  we obtain a geometric characterization in terms of positive triples of flags, which we show to be equivalent to the algebraic definition.

*Remark 1.7.* The set of  $\Theta$ -positive triples will in general not be connected, but can consist of several connected components of the set of pairwise transverse triples of flags. In Section 10 we introduce the notion of diamonds which turns out to be very helpful to keep track of positivity of triples and more general  $n$ -tuples of flags. Diamonds can be considered as generalized intervals, see also [GLW21].

Note that the split real Lie groups of type  $B_n$ ,  $C_n$  and  $F_4$  admit two  $\Theta$ -positive structures, one for  $\Theta = \Delta$  and one for  $\Theta$  being a strict subset of  $\Delta$ . We prove that the natural projection  $F_{\Delta} \rightarrow F_{\Theta}$  maps positive  $n$ -tuples to positive  $n$ -tuples (Section 10.10).

**1.4. The  $\Theta$ -principal  $\mathfrak{sl}(2)$  and positive circles.** For split real Lie groups it is well known that they contain a distinguished conjugacy class of three-dimensional simple subalgebras, the principal  $\mathfrak{sl}(2)$ . This principal  $\mathfrak{sl}(2)$  plays an important role in the construction of the Hitchin fibration on the moduli space of Higgs bundles, and in particular for the Hitchin section and consequently the Hitchin component [Hit92]. When  $G$  admits a  $\Theta$ -positive structure, we determine a specific split real subalgebra  $\mathfrak{g}_{\Theta}$  in the Lie algebra  $\mathfrak{g}$  of  $G$ , as well as a principal three dimensional simple subalgebra in  $\mathfrak{g}_{\Theta}$ . We call this special  $\mathfrak{sl}(2)$  the  $\Theta$ -principal  $\mathfrak{sl}(2)$ . This  $\Theta$ -principal  $\mathfrak{sl}(2)$  gives rise to a positive circle in the flag variety  $F_{\Theta}$ .

In Section 11.2 we state the explicit relation between the  $\Theta$ -principal  $\mathfrak{sl}(2)$  and the magical triples considered in [BCGP<sup>+</sup>21].

**1.5. Positive representations and higher rank Teichmüller spaces.**

Lusztig's total positivity on the one hand and positive semigroups in Hermitian Lie groups of tube type on the other play an important role in higher rank Teichmüller theory. Both, Hitchin components and spaces of maximal representations, have been characterized as positive representations [Lab06, Gui05, FG06, BILW05, BIW03]. At the same time, total positivity plays an important role in the construction of cluster coordinates by Fock and Goncharov [FG06].

As  $\Theta$ -positivity realizes total positivity in split real Lie groups and Lie semigroups in Hermitian Lie groups of tube type as two incarnations

of the same concept, it provides an interesting systematic framework for higher rank Teichmüller spaces. This is based on the notion of positive  $n$ -tuples in the flag variety  $F_\Theta$ , which allows us to introduce the notion of positive representations.

Let  $\pi_1(S)$  be the fundamental group of an oriented surface of negative Euler characteristic. Then the boundary  $\partial_\infty\pi_1(S)$  carries a natural partial ordering. Given a simple Lie group  $G$  with a  $\Theta$ -positive structure, we call a representation  $\rho: \pi_1(S) \rightarrow G$   $\Theta$ -positive if there exists a  $\rho$ -equivariant map from  $\partial_\infty\pi_1(S)$  to the flag variety  $F_\Theta$ , which sends every positive four-tuple of points in  $\partial_\infty\pi_1(S)$  to a positive four-tuple of flags in  $F_\Theta$ .

In [GW18] we conjectured that the set of positive representations form higher rank Teichmüller spaces, i.e. it is open and closed and consists entirely of discrete and faithful representations. Using several of the results obtained in this paper, this conjecture has been proven to a large extent in [GLW21] and [BP21], relying partly on [Col20, BCGP<sup>+</sup>21].

However,  $\Theta$ -positivity suggest an even deeper connection between the different families of higher rank Teichmüller spaces, including extensions of Fock–Goncharov’s construction of cluster coordinates to spaces of positive representations. The expectation is that the cluster coordinates for positive representations into the Lie group  $G$  with respect to its  $\Theta$ -positive structure provide examples of non-commutative cluster algebras of type  $W(\Theta)$ . This has been proven for the symplectic group (as a Hermitian Lie group of tube type) in [ABR<sup>+</sup>21], where appropriate non-commutative coordinates give a geometric realization of the non-commutative cluster algebra of type  $A_1$  introduced in [BR18].

For further conjectures and questions regarding  $\Theta$ -positivity, we refer the reader to [Wie18].

## 2. STRUCTURE OF PARABOLIC SUBGROUPS

In this section we recall background on the structure of semisimple Lie groups and their parabolic subgroups.

Note that unless explicitly stated otherwise all vector spaces, Lie algebras, Lie groups, and their representations are defined over  $\mathbf{R}$ .

**2.1. Cartan involution.** Let  $G$  be a connected real simple Lie group with finite center and denote by  $\mathfrak{g}$  its Lie algebra. Let  $\mathfrak{k}$  be the Lie algebra of a maximal compact subgroup  $K < G$ . Then  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{k}^\perp$  where  $\mathfrak{k}^\perp$  is the orthogonal of  $\mathfrak{k}$  with respect to the Killing form  $B$  on  $\mathfrak{g}$ . We denote by  $\tau$  the *Cartan involution* with respect to  $\mathfrak{k}$ , it is the Lie algebra automorphism  $\tau: \mathfrak{g} \rightarrow \mathfrak{g}$  whose fixed point set is  $\mathfrak{k}$  and which is an involution; namely  $\tau|_{\mathfrak{k}} = \text{id}$  and  $\tau|_{\mathfrak{k}^\perp} = -\text{id}$ .

**2.2. Restricted roots.** We now choose a maximal abelian subspace  $\mathfrak{a}$  contained in  $\mathfrak{k}^\perp$ ;  $\mathfrak{a}$  is called a Cartan subspace. We denote by  $\Sigma =$

$\Sigma(\mathfrak{g}, \mathfrak{a})$  the system of restricted roots, i.e.  $\Sigma \subset \mathfrak{a}^*$  is the set of nonzero weights for the adjoint action of  $\mathfrak{a}$  on  $\mathfrak{g}$ ; the corresponding weight spaces are denoted  $\mathfrak{g}_\alpha$  ( $\alpha \in \Sigma$ ). When  $\alpha \in \mathfrak{a}^*$ , the corresponding weight space is

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g} \mid \text{ad}(H)(X) = \alpha(H)X \text{ for all } H \in \mathfrak{a}\}$$

and is called a root space when  $\alpha \in \Sigma$ . The weight space  $\mathfrak{g}_0$  is the centralizer  $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})$  of  $\mathfrak{a}$  in  $\mathfrak{g}$ . Also  $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}) = \mathfrak{m} \oplus \mathfrak{a}$ , where  $\mathfrak{m} = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a}) = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}) \cap \mathfrak{k}$  is the centralizer of  $\mathfrak{a}$  in  $\mathfrak{k}$  and the sum is orthogonal with respect to the Killing form  $B$ . Therefore we have a  $B$ -orthogonal decomposition  $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha$ .

For every root  $\alpha \in \Sigma$ , the intersection of  $\mathfrak{g}_\alpha$  and  $\mathfrak{k}$  is trivial (reduced to  $\{0\}$ ) and one has  $\tau(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$ .

We choose  $\Sigma^+ \subset \Sigma$  a set of positive roots, and denote by  $\Sigma^-$  the set of negative roots. The set of simple roots is denoted by  $\Delta \subset \Sigma^+$ . The (open) Weyl chamber corresponding to this choice will be denoted  $\mathfrak{a}^+ = \{X \in \mathfrak{a} \mid \alpha(X) > 0, \forall \alpha \in \Sigma^+\}$ ; its closure  $\bar{\mathfrak{a}}^+$  is called the closed Weyl chamber.

When  $\alpha$  and  $\beta$  belong to  $\Sigma$ , the supremums  $p = \sup\{n \in \mathbf{N} \mid \beta - n\alpha \in \Sigma\}$  and  $q = \sup\{n \in \mathbf{N} \mid \beta + n\alpha \in \Sigma\}$  are finite and the subset  $\{\beta + n\alpha\}_{n=-p, \dots, q}$  is contained in  $\Sigma$  and is called the  $\alpha$ -chain containing  $\beta$ . The  $\alpha$ -chain is called *trivial* when  $p = q = 0$ .

The Killing form induces, by restriction and duality, a Euclidean scalar product  $(\cdot, \cdot)$  on  $\mathfrak{a}^*$ . It is well known that either all the roots have the same norm or the set of simple roots is the disjoint union of two nonempty subsets  $\Delta_s$  and  $\Delta_l$  with the following properties: all the elements of  $\Delta_s$  have the same norm, all the elements of  $\Delta_l$  have the same norm, and the norm of elements in  $\Delta_l$  is greater than the norm of elements in  $\Delta_s$ . Elements of  $\Delta_l$  are called *long roots*, elements of  $\Delta_s$  are called *short roots*. For  $\alpha \in \Delta_l$  and  $\beta \in \Delta_s$ , one has in fact  $(\alpha, \alpha) = 2(\beta, \beta)$  except in the case of type  $G_2$  where  $(\alpha, \alpha) = 3(\beta, \beta)$ .

Note that in general the root spaces  $\mathfrak{g}_\alpha$  are not necessarily 1-dimensional. In fact they are all 1-dimensional if and only if  $\mathfrak{g}$  is the Lie algebra of a split real form.

**2.3. The  $\mathfrak{sl}_2$ -triples.** Even though the root spaces for a semisimple real Lie algebra are not necessarily 1-dimensional, vectors in the root spaces lead to embeddings of three dimensional simple subalgebras.

For every root  $\alpha \in \Sigma$  and any non-zero element  $X$  in  $\mathfrak{g}_\alpha$ , one has  $B(X, \tau(X)) > 0$  and the Lie bracket  $[X, \tau(X)]$  belongs to  $\mathfrak{a}$ ; there is furthermore a (unique) positive multiple  $X_\alpha$  of  $X$  such that  $(X_\alpha, Y_\alpha := \tau(X_\alpha), H_\alpha := [X_\alpha, \tau(X_\alpha)])$  is a  $\mathfrak{sl}_2$ -triple (i.e. a triple  $(E, F, D)$  such that  $D = [E, F]$ ,  $[D, E] = 2E$ , and  $[D, F] = -2F$ ). The element  $H_\alpha = [X_\alpha, \tau(X_\alpha)]$  does not depend on the initial choice of  $X$  in  $\mathfrak{g}_\alpha$ . Clearly  $H_{-\alpha} = -H_\alpha$ . In particular, for any  $\alpha \in \Sigma$  we obtain a morphism  $\pi_\alpha: \mathfrak{sl}_2(\mathbf{R}) \rightarrow \mathfrak{g}$ .

Note that when  $\alpha$ ,  $\beta$ , and  $\alpha + \beta$  belong to  $\Sigma$ , one has  $H_{\alpha + \beta} = H_\alpha + H_\beta$ .

**2.4. Representations and weights.** Any (continuous) representation  $\rho: G \rightarrow \mathrm{GL}(V)$  induces a Lie algebra morphism  $\rho_*: \mathfrak{g} \rightarrow \mathrm{End}(V)$ . The restriction of  $\rho_*$  to  $\mathfrak{a}$  admits a weight decomposition

$$V = \bigoplus_{\lambda \in R(V)} V_\lambda$$

where

$$V_\lambda := \{v \in V \mid \rho_*(H)(v) = \lambda(H)v \text{ for all } H \in \mathfrak{a}\}$$

$$R(V) := \{\lambda \in \mathfrak{a}^* \mid V_\lambda \neq \{0\}\}.$$

The representation  $\rho$  will be called *even* if  $\lambda(H_\alpha)$  is an even integer for all  $\lambda \in R(V)$  and  $\alpha \in \Sigma$  (these numbers are already known to be integers); note that it is enough to check this for  $\lambda \in R(V)$  and  $\alpha \in \Delta$ . In classical terms this means the inclusion  $R(V) \subset 2P$  where  $P = \{\lambda \in \mathfrak{a}^* \mid \lambda(H_\alpha) \in \mathbf{Z}, \forall \alpha \in \Sigma\}$  is the weight lattice.

The *highest weight* of the representation  $\rho$  (when it is defined) is the greatest element of  $R(V)$  for the (partial) order on  $\mathfrak{a}^*$  defined by  $\lambda \leq \mu \Leftrightarrow \lambda(H_\alpha) \leq \mu(H_\alpha)$  for all  $\alpha \in \Sigma^+$ . Irreducible representations always admit a highest weight  $\lambda_{\max}$  and such an irreducible representation is even if and only if  $\lambda_{\max}(H_\alpha) \in 2\mathbf{Z}$  for every  $\alpha \in \Delta$ .

A representation will be called *proximal* if it admits a highest weight and if the corresponding weight space is 1-dimensional. (In this article we will use these notions mainly for representations of the semisimple factor of the Levi components of parabolic subgroups, see below.)

**2.5. The Weyl group.** The subgroup  $W \subset \mathrm{GL}(\mathfrak{a}^*)$  of automorphisms of  $\Sigma$  is called the (*restricted*) *Weyl group*. It is a finite Coxeter group generated by hyperplanes reflections  $\{s_\alpha\}_{\alpha \in \Delta}$  such that  $s_\alpha(\alpha) = -\alpha$ .

When the simple restricted roots are numbered, i.e. when  $\Delta = \{\alpha_1, \alpha_2, \dots, \alpha_r\}$ , we will rather write  $s_i$  instead of  $s_{\alpha_i}$ . The group  $W$  is a subgroup of the orthogonal group associated with the Euclidean scalar product on  $\mathfrak{a}^*$  induced by the Killing form  $B$ . The closed Weyl chamber  $\bar{\mathfrak{a}}^+$  is a fundamental domain for the action of  $W$  on  $\mathfrak{a}$ . The Weyl chamber is also the convex cone generated by  $\{H_\alpha\}_{\alpha \in \Delta}$ .

The Weyl group  $W$  is isomorphic to the quotient of the normalizer  $N_K(\mathfrak{a})$  of  $\mathfrak{a}$  in  $K$  by the centralizer  $C_K(\mathfrak{a})$ . In order to define an action of the Weyl group (and of its subgroups) on the Lie algebra  $\mathfrak{g}$  and its subspaces, we choose elements  $\dot{w}$  in  $N_K(\mathfrak{a})$  lifting the elements  $w$  of  $W$ . For  $\alpha$  in  $\Sigma$ , one way to choose the lift  $\dot{s}_\alpha$  is to use the morphism  $\pi_\alpha: \mathfrak{sl}_2(\mathbf{R}) \rightarrow \mathfrak{g}$  (cf. Section 2.3). We can take  $\dot{s}_\alpha = \exp(\pm \frac{\pi}{2}(X_\alpha - Y_\alpha))$ . With this choice, the automorphism  $\mathrm{Ad}(\dot{s}_\alpha)$  of  $\mathfrak{g}$  is of order 2 or 4.

The above generating set  $\{s_\alpha\}_{\alpha \in \Delta}$  of  $W$  permits the definition of a word length function  $\ell: W \rightarrow \mathbf{N}$ . It is well known that  $W$  admits a unique element  $w_\Delta$  of longest length. This element  $w_\Delta$  is of order 2, it

is the element of  $W$  (that we now see as acting on  $\mathfrak{a}$ ) sending the Weyl chamber  $\mathfrak{a}^+$  to its opposite  $-\mathfrak{a}^+$ .

**2.6. Parabolic subgroups.** Let  $\Theta \subset \Delta$  be a subset of the set of simple roots. We set

$$\mathfrak{u}_\Theta = \sum_{\alpha \in \Sigma_\Theta^+} \mathfrak{g}_\alpha, \quad \mathfrak{u}_\Theta^{\text{opp}} = \sum_{\alpha \in \Sigma_\Theta^+} \mathfrak{g}_{-\alpha}$$

where  $\Sigma_\Theta^+ = \Sigma^+ \setminus \text{Span}(\Delta \setminus \Theta)$ , and we set

$$\mathfrak{l}_\Theta = \mathfrak{z}_\mathfrak{g}(\mathfrak{a}) \oplus \sum_{\alpha \in \text{Span}(\Delta \setminus \Theta)} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) = \mathfrak{z}_\mathfrak{g}(\mathfrak{a}) \oplus \bigoplus_{\alpha \in \text{Span}(\Delta \setminus \Theta) \cap \Sigma^+} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}).$$

The *standard parabolic subgroup*  $P_\Theta$  associated with  $\Theta \subset \Delta$  is the normalizer in  $G$  of  $\mathfrak{u}_\Theta$ ; it is also the normalizer of the Lie algebra  $\mathfrak{p}_\Theta = \mathfrak{l}_\Theta \oplus \mathfrak{u}_\Theta$ ; one has  $\mathfrak{p}_\Theta = \text{Lie}(P_\Theta)$  and  $P_\Theta$  is its own normalizer. We also denote by  $P_\Theta^{\text{opp}}$  the normalizer in  $G$  of  $\mathfrak{u}_\Theta^{\text{opp}}$ .

The group  $P_\Theta$  is the semidirect product of its unipotent radical  $U_\Theta := \exp(\mathfrak{u}_\Theta)$  and the Levi subgroup  $L_\Theta = P_\Theta \cap P_\Theta^{\text{opp}}$ . We denote by  $L_\Theta^\circ$  the connected component of the identity in  $L_\Theta$ . The Lie algebra of  $L_\Theta$  is  $\mathfrak{l}_\Theta$ . With this convention  $P_\emptyset = G$  and  $P_\Delta$  is the minimal parabolic subgroup.

The Lie algebra  $\mathfrak{l}_\theta$  is stable under the Cartan involution  $\tau$  and is the Lie algebra of  $L_\Theta$ . The intersection  $\mathfrak{k}_\theta = \mathfrak{l}_\theta \cap \mathfrak{k}$  is a maximal compact subalgebra of  $\mathfrak{l}_\theta$ ; one has

$$(2.1) \quad \mathfrak{k}_\theta = \mathfrak{m} \oplus \bigoplus_{\alpha \in \text{Span}(\Delta \setminus \Theta) \cap \Sigma^+} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \cap \mathfrak{k},$$

$$(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \cap \mathfrak{k} = \{X - \tau(X)\}_{X \in \mathfrak{g}_\alpha}.$$

The commutator subgroup  $S_\Theta = [L_\Theta, L_\Theta]$  of  $L_\Theta$  is a semisimple real Lie group whose restricted root system is given by the Dynkin diagram<sup>1</sup>  $\Delta \setminus \Theta$ . A Cartan subspace for  $S_\Theta$  is

$$\mathfrak{a}_\Theta = \bigoplus_{\alpha \in \Delta \setminus \Theta} \mathbf{R}H_\alpha$$

and a Weyl chamber in this Cartan subspace is  $\sum_{\alpha \in \Delta \setminus \Theta} \mathbf{R}_{>0}H_\alpha$ .

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<sup>1</sup>This is a disadvantage of the convention, chosen in this paper, to index the standard parabolic subgroups that the complement of  $\Theta$  has to be taken here; the other classical indexing convention (that amounts to exchanging  $\Theta$  and  $\Delta \setminus \Theta$  in the indexation) will make the Dynkin diagram associated with the semisimple part more transparent but in return the decomposition of the unipotent radical  $\mathfrak{u}_\Theta$  will be less natural.

**2.7. The flag variety.** A subgroup of  $G$  conjugated to  $P_\Theta$  (for some  $\Theta \subset \Delta$ ) is a *parabolic subgroup*. A *parabolic subalgebra* of  $\mathfrak{g}$  is a subalgebra conjugated to some  $\mathfrak{p}_\Theta$ , such a subalgebra will sometimes be called of *type*  $\Theta$ . The space of all parabolic subalgebra of type  $\Theta$  is called the *flag variety* (of type  $\Theta$ ) and denoted by  $F_\Theta$ . The space  $F_\Theta$  is a subset of a Grassmanian variety on the real vector space  $\mathfrak{g}$  and is endowed with the induced topology. The flag variety  $F_\Theta$  is compact and the conjugation action of  $G$  on the subalgebras of  $\mathfrak{g}$  induces a continuous and homogenous action on  $F_\Theta$  that naturally identifies with  $G/P_\Theta$  because  $P_\Theta$  is the stabilizer of  $\mathfrak{p}_\Theta$ ; it also naturally identifies with the space of parabolic subgroups of type  $\Theta$ .

Since the parabolic subalgebra  $\mathfrak{p}_\Theta^{\text{opp}}$  is conjugate to  $\dot{w}_\Delta \mathfrak{p}_\Theta^{\text{opp}} \dot{w}_\Delta^{-1} = \mathfrak{p}_{\iota(\Theta)}$  where  $\iota: \Delta \rightarrow \Delta$  is the involution  $\alpha \mapsto -w_\Delta \cdot \alpha$ , the parabolic subalgebra  $\mathfrak{p}_\Theta^{\text{opp}}$  is also an element of  $F_{\iota(\Theta)}$ . In many cases below, the involution  $\iota$  is the identity on  $\Delta$ .

The group  $G$  acts diagonally on  $F_\Theta \times F_{\iota(\Theta)}$ . A pair  $(x, y)$  in  $F_\Theta \times F_{\iota(\Theta)}$  will be called *transverse* if it belongs to the  $G$ -orbit of  $(\mathfrak{p}_\Theta, \mathfrak{p}_\Theta^{\text{opp}})$ . By definition, there is one orbit of transverse pairs and this orbit is isomorphic to  $G/L_\Theta$ .

**2.8. The group of automorphisms of  $\mathfrak{g}$ .** We will denote by  $\text{Aut}(\mathfrak{g}) \subset \text{GL}(\mathfrak{g})$  the group of automorphisms of the Lie algebra  $\mathfrak{g}$ . It is a Lie group, that is not necessarily connected, and whose Lie algebra is equal to  $\mathfrak{g}$  since  $\mathfrak{g}$  is semisimple.

By construction the group  $\text{Aut}(\mathfrak{g})$  acts on the Lie algebra  $\mathfrak{g}$  and also on the flag variety  $F_\Theta$  as this flag variety identifies with a space of subalgebras of  $\mathfrak{g}$ .

Equally the adjoint action induces a morphism  $G \rightarrow \text{Aut}(\mathfrak{g})$  and the actions of  $G$  and of  $\text{Aut}(\mathfrak{g})$  on  $F_\Theta$  are related by this morphism.

**2.9. The action of  $L_\Theta$  on  $\mathfrak{u}_\Theta$ .** The Levi subgroup  $L_\Theta$  acts via the adjoint action on  $\mathfrak{u}_\Theta$ . Let  $\mathfrak{z}_\Theta$  denote the center of  $\mathfrak{l}_\Theta$  and  $\mathfrak{t}_\Theta = \mathfrak{z}_\Theta \cap \mathfrak{a}$  its intersection with the Cartan subspace. One has

$$\mathfrak{t}_\Theta = \bigcap_{\alpha \in \Delta \setminus \Theta} \ker \alpha,$$

and  $\mathfrak{a}$  is the  $B$ -orthogonal sum of  $\mathfrak{a}_\Theta$  and of  $\mathfrak{t}_\Theta$ . Then  $\mathfrak{u}_\Theta$  decomposes into the weight spaces under the adjoint action of  $\mathfrak{t}_\Theta$ ; for every  $\beta \in \mathfrak{t}_\Theta^*$ , set

$$\mathfrak{u}_\beta := \{N \in \mathfrak{g} \mid \text{ad}(Z)N = \beta(Z)N, \forall Z \in \mathfrak{t}_\Theta\}.$$

These weight spaces are of course related to those of  $\mathfrak{a}$ :

$$(2.2) \quad \mathfrak{u}_\beta = \sum_{\alpha \in \Sigma, \alpha|_{\mathfrak{t}_\Theta} = \beta} \mathfrak{g}_\alpha,$$

and are invariant under  $L_\Theta^\circ$ .

By a little abuse of notation, for an element  $\beta$  in  $\Sigma$ , we will denote as well by  $\beta$  its restriction to  $\mathfrak{t}_\Theta$  and hence by  $\mathfrak{u}_\beta$  the corresponding weight space. For such a  $\beta$ , one has

$$\mathfrak{u}_\beta = \sum_{\substack{\alpha \in \Sigma \\ \alpha - \beta \in \text{Span}(\Delta \setminus \Theta)}} \mathfrak{g}_\alpha.$$

This notation will be particularly used for the elements  $\beta$  of  $\Theta$ .

The following results are established in [Kos10]:

**Theorem 2.1.** *Each of the  $\mathfrak{u}_\beta$  is an irreducible representation of  $L_\Theta^\circ$ . The relation  $[\mathfrak{u}_\beta, \mathfrak{u}_{\beta'}] = \mathfrak{u}_{\beta+\beta'}$  is satisfied, and  $\mathfrak{u}_\Theta$  is generated as a Lie algebra by  $\mathfrak{u}_\beta$ ,  $\beta \in \Theta$ . Analogously  $\mathfrak{u}_\Theta^{\text{opp}}$  is generated by  $\mathfrak{u}_{-\beta}$ ,  $\beta \in \Theta$ .*

**2.10. Examples.** In these examples, we use classical notation and numbering for the root systems as set for example in [Hel01, Ch. X, §3–4, p. 461–475] or in [Bou68, Ch. VI, Planches and §4, n° 14]

- (1) Let  $G$  be a split real form, and  $\Theta = \Delta$ . Then  $\mathfrak{t}_\Theta = \mathfrak{a}$  and  $\mathfrak{u}_\beta = \mathfrak{g}_\beta$  for all  $\beta \in \Sigma^+$ .
- (2) Let  $G$  be a Lie group of Hermitian type. Then the root system is of type  $C_r$  if  $G$  is of tube type, and of type  $BC_r$  (non-reduced) if  $G$  is not of tube type. Let  $\Delta = \{\alpha_1, \dots, \alpha_r\}$ , and let  $\Theta = \{\alpha_r\}$  so that  $P_\Theta$  is the stabilizer of a point in the Shilov boundary  $F_\Theta$  of the Hermitian symmetric space associated with  $G$ . Then  $\mathfrak{u}_\Theta = \mathfrak{u}_{\alpha_r}$  if  $G$  is of tube type, and  $\mathfrak{u}_\Theta = \mathfrak{u}_{\alpha_r} \oplus \mathfrak{u}_{2\alpha_r}$  if  $G$  is not of tube type.
- (3) Let  $G$  be a Hermitian Lie group of tube type. In this case  $\Theta = \{\alpha_r\}$ , and  $\mathfrak{u}_\Theta = \mathfrak{u}_{\alpha_r}$  is abelian. We describe  $\mathfrak{u}_{\alpha_r}$  in more detail when the real rank of  $G$  is 2 or 3.

For a Hermitian Lie group of tube type of real rank 2 the roots that are equal to  $\alpha_2$  modulo  $\mathbf{R}\alpha_1$  are  $\alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2 = s_1(\alpha_2)$ , and  $\mathfrak{u}_{\alpha_2} = \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_1 + \alpha_2} \oplus \mathfrak{g}_{2\alpha_1 + \alpha_2}$ , where  $\dim \mathfrak{g}_{\alpha_2} = 1 = \dim \mathfrak{g}_{2\alpha_1 + \alpha_2}$ ; the integer  $a = \dim \mathfrak{g}_{\alpha_1 + \alpha_2}$  is not necessarily equal to 1. For the symplectic group  $\text{Sp}(4, \mathbf{R})$ ,  $a = 1$ , for  $\text{SU}(2, 2)$ ,  $a = 2$ , for  $\text{SO}^*(8)$ ,  $a = 4$ , and for  $\text{SO}(2, m)$ ,  $a = m - 2$  (in fact this last family contains, up to isogeny, the three first examples, cf. [Hel01, p. 519]). In this case,  $\mathfrak{u}_{\alpha_2}$  is equipped with a quadratic form of signature  $(1, a + 1)$  that is invariant under the adjoint action of the Levi factor.

For a Hermitian Lie group of tube type of real rank 3 the roots that are equal to  $\alpha_3$  modulo the span of  $\{\alpha_1, \alpha_2\}$  are  $\alpha_3, \alpha_2 + \alpha_3, 2\alpha_2 + \alpha_3 = s_2(\alpha_3), \alpha_1 + \alpha_2 + \alpha_3 = s_1(\alpha_2 + \alpha_3), \alpha_1 + 2\alpha_2 + \alpha_3 = s_2s_1(\alpha_2 + \alpha_3), 2\alpha_1 + 2\alpha_2 + \alpha_3 = s_1s_2(\alpha_3)$ , so  $\mathfrak{u}_{\alpha_3} = \mathfrak{g}_{\alpha_3} \oplus \mathfrak{g}_{\alpha_2 + \alpha_3} \oplus \mathfrak{g}_{\alpha_1 + \alpha_2 + \alpha_3} \oplus \mathfrak{g}_{\alpha_1 + 2\alpha_2 + \alpha_3} \oplus \mathfrak{g}_{2\alpha_2 + \alpha_3} \oplus \mathfrak{g}_{2\alpha_1 + 2\alpha_2 + \alpha_3}$ , where  $\dim \mathfrak{g}_{\alpha_3} = 1 = \dim \mathfrak{g}_{2\alpha_2 + \alpha_3} = \dim \mathfrak{g}_{2\alpha_1 + 2\alpha_2 + \alpha_3}$ , and  $\dim \mathfrak{g}_{\alpha_1 + \alpha_2 + \alpha_3} = \dim \mathfrak{g}_{\alpha_2 + \alpha_3} = \dim \mathfrak{g}_{\alpha_1 + 2\alpha_2 + \alpha_3} =: a$ . For the symplectic group  $\text{Sp}(6, \mathbf{R})$ ,  $a = 1$ , for  $\text{SU}(3, 3)$ ,  $a = 2$ , for  $\text{SO}^*(12)$ ,  $a = 4$ , and for  $E_{7(-25)}$ ,  $a = 8$ . In these cases, the Lie algebra  $\mathfrak{u}_{\alpha_3}$  can be identified with the space of

Hermitian  $3 \times 3$  matrices, over the reals ( $a = 1$ ), the complex numbers ( $a = 2$ ), the quaternions ( $a = 4$ ), and the octonions ( $a = 8$ ) respectively.

### 3. $\Theta$ -POSITIVE STRUCTURES

In this section we give the definition of  $\Theta$ -positive structures and their classification.

#### 3.1. Definition and characterization.

**Definition 3.1.** Let  $G$  be a semisimple Lie group with finite center. Let  $\Theta \subset \Delta$  be a subset of simple roots. We say that  $G$  admits a  $\Theta$ -positive structure if for all  $\beta \in \Theta$  there exists an  $L_\Theta^\circ$ -invariant acute nontrivial convex cone in  $\mathfrak{u}_\beta$ .

Recall that a convex cone is said *acute* if the only vector space contained in its closure is  $\{0\}$  and a cone is said nontrivial if it is not contained in  $\{0\}$ . Note that such invariant cones must be of nonempty interior since the action of  $L_\Theta^\circ$  on  $\mathfrak{u}_\beta$  is irreducible (Theorem 2.1).

*Remark 3.2.* In general it is not possible to request that the cones are invariant under the entire Levi factor  $L_\Theta$ . If the space  $\mathfrak{u}_\beta$  contains an  $L_\Theta^\circ$ -invariant cone, then it contains exactly two  $L_\Theta^\circ$ -invariant cones,  $c_\beta$  and  $-c_\beta$ , that might be exchanged by  $L_\Theta$ .

*Remark 3.3.* A semisimple Lie group  $G$  is the almost product of its simple factors; the subgroup  $L_\Theta^\circ$  is equally the almost product of the corresponding Levi factors and the decomposition of the Lie algebra  $\mathfrak{u}_\Theta$  under the action of  $L_\Theta^\circ$  is also compatible with this product structure.

This implies that it is enough to classify Lie groups  $G$  admitting a  $\Theta$ -positive structure under the additional hypothesis that  $G$  is a simple Lie group. Furthermore the different structural results given in the sequel of the paper (for the positive unipotent semigroup, for the configurations of positive flags, etc.) will be stated only when the group  $G$  is simple but immediately generalize to the case of semisimple Lie groups.

**Theorem 3.4.** *Let  $G$  be a simple real Lie group, and let  $\Delta$  be the set of positive roots. Then  $G$  admits a  $\Theta$ -positive structure if and only if the pair  $(G, \Theta)$  belongs to one of the following four cases:*

- (1)  $G$  is a split real form, and  $\Theta = \Delta$ ;
- (2)  $G$  is Hermitian of tube type and of real rank  $r$  and  $\Theta = \{\alpha_r\}$ , where  $\alpha_r$  is the long simple restricted root;
- (3)  $G$  is locally isomorphic to  $\mathrm{SO}(p+1, p+k)$ ,  $p > 1$ ,  $k > 1$  and  $\Theta = \{\alpha_1, \dots, \alpha_p\}$ , where  $\alpha_1, \dots, \alpha_p$  are the long simple restricted roots;
- (4)  $G$  is the real form of  $F_4$ ,  $E_6$ ,  $E_7$ , or of  $E_8$  whose reduced root system is of type  $F_4$ , and  $\Theta = \{\alpha_1, \alpha_2\}$ , where  $\alpha_1, \alpha_2$  are the long simple restricted roots.

To prove this theorem we make use of the following fact, extracted from [Ben00, Proposition 4.7] that refines Cartan–Helgason’s theorem.

**Proposition 3.5.** *Let  $S$  be a connected semisimple Lie group,  $V$  a real vector space, and  $\pi: S \rightarrow \mathrm{GL}(V)$  an irreducible representation. Then  $S$  preserves an acute closed convex cone  $c$  in  $V$  if and only if the representation  $\pi$  is proximal and even.*

*In this case, the union of  $\mathring{c}$  and of  $-\mathring{c}$  is exactly the set of vectors  $v \in V$  whose stabilizers is a maximal compact subgroup of  $S$ ; more precisely a vector  $v \in V$  belongs to  $\mathring{c}$  or to  $-\mathring{c}$  as soon as the Lie algebra of its stabilizer contains a maximal compact subalgebra.*

We shall need to understand a version of this proposition for representations of reductive groups.

**Proposition 3.6.** *Let  $L$  be a connected reductive Lie group and let  $S = [L, L]$  be its semisimple part. Let  $\pi: L \rightarrow \mathrm{GL}(V)$  be an irreducible real representation.*

- (1) *Suppose that there is an acute  $\pi(L)$ -invariant cone in  $V$ , then the restriction  $\pi|_S$  is as well irreducible (and hence proximal and even by the previous proposition).*
- (2) *Suppose that  $\pi|_S$  is irreducible, even, and proximal (and hence there is  $c \subset V$  a  $\pi(S)$ -invariant convex cone by the previous proposition), then the cone  $c$  is equally  $\pi(L)$ -invariant.*

*Proof.* The group  $L$  is the almost direct product of  $Z^\circ$ , the identity component of its center  $Z$ , and of its semisimple part  $S$ . Let us denote by  $\mathfrak{a}^+$  a Weyl chamber in a Cartan subspace of the Lie algebra of  $S$ .

Let us prove (1). By Schur’s lemma, the algebra  $\mathrm{End}_L(V)$  of  $L$ -equivariant endomorphisms of  $V$  is a skew field and is hence isomorphic to  $\mathbf{R}$ ,  $\mathbf{C}$  or  $\mathbf{H}$ ; in other words,  $V$  is a vector space over a potentially bigger field and the endomorphisms  $\pi(g)$ , for  $g \in L$ , are linear with respect to this field. Since there is a  $\pi(L)$ -invariant acute convex cone, the representation  $\pi$  is proximal [Ben00, Proposition 1.2] in the sense that there is an element  $g$  in  $L$  such that  $\pi(g)$  has a unique eigenvalue of highest modulus: this eigenvalue is real and the corresponding eigenspace is of real dimension 1. As this eigenspace is invariant by  $\mathrm{End}_L(V)$  we deduce that  $\mathrm{End}_L(V) = \mathbf{R}$ . Furthermore, the image  $\pi(Z^\circ)$  is included in  $\mathrm{End}_L(V)$  and this means that elements  $\pi(g)$ , for  $g \in Z^\circ$ , are homotheties of  $V$ . We deduce from this and the fact that  $L$  is the almost product of  $Z^\circ$  and  $S$ , that a subspace of  $V$  is  $\pi(L)$ -invariant if and only if it is  $\pi(S)$ -invariant. Since  $\pi$  is by hypothesis an irreducible representation of  $L$ , the restriction  $\pi|_S$  is thus an irreducible representation of  $S$ .

We now address (2). We need to show that  $c$  is invariant under the action of  $\pi(Z^\circ)$ . Similarly as above, we can prove that the elements  $\pi(z)$ , for  $z \in Z^\circ$ , are homotheties of positive dilation so that  $\pi(z)c = c$  for any  $z \in Z^\circ$  since  $c$  is a cone. This implies that  $c$  is  $\pi(L)$ -invariant.  $\square$

*Proof of Theorem 3.4.* We apply Proposition 3.6 to the connected Lie group  $L_\Theta^\circ$  and Proposition 3.5 to  $S_\Theta^\circ = [L_\Theta^\circ, L_\Theta^\circ]$  for the irreducible representations  $\pi_\beta: L_\Theta^\circ \rightarrow \mathrm{GL}(\mathfrak{u}_\beta)$ ,  $\beta \in \Theta$ .

Recall that a Cartan subspace for  $S_\Theta^\circ$  is  $\mathfrak{a}_\Theta = \bigoplus_{\alpha \in \Delta \setminus \Theta} \mathbf{R}H_\alpha$ . For this proof it will be more convenient to work with the following Weyl chamber of  $S_\Theta^\circ$

$$\sum_{\alpha \in \Delta \setminus \Theta} -\mathbf{R}_{>0} H_\alpha$$

that is opposite to the natural one, or, what amounts to the same, we shall rather consider *lowest weights* and not highest weights of representations (for the natural ordering).

The decomposition of Equation (2.2), p. 11 is

$$\mathfrak{u}_\beta = \mathfrak{g}_\beta \oplus \bigoplus_{\substack{\alpha \in \Sigma, \alpha \neq \beta \\ \alpha - \beta \in \mathrm{Span}(\Delta \setminus \Theta)}} \mathfrak{g}_\alpha.$$

By properties of the root system  $\Sigma$ , any  $\alpha$  in the above formula differs from  $\beta$  by a linear combination of the elements in  $\Delta \setminus \Theta$  with nonnegative coefficients. This implies that (up to the abuse consisting of considering the above decomposition indexed by the elements in  $\mathfrak{a}_\Theta^*$  rather than by elements in  $\mathfrak{a}^*$ ) the above is the weight space decomposition of the  $S_\Theta^\circ$ -module  $\mathfrak{u}_\beta$ . As a consequence, the lowest weight of this representation is equal to  $\beta$  and the corresponding weight space is  $\mathfrak{g}_\beta$ .

Thus the condition that  $\pi_\beta$  has to be proximal is equivalent to the condition that the root space  $\mathfrak{g}_\beta$  is one-dimensional (and this should hold for all  $\beta \in \Theta$ ).

The evenness condition translates into the fact that  $\beta(H_\alpha)$  is even for all  $\beta \in \Theta$  and all  $\alpha \in \Delta \setminus \Theta$  (cf. Section 2.4).

These necessary and sufficient criteria can be easily translated into properties of the Dynkin diagram of the system of restricted roots  $\Sigma$ :

- (1)  $\forall \beta \in \Theta$ , the root space  $\mathfrak{g}_\beta$  is one-dimensional. (*Representation is proximal.*)
- (2)  $\forall \beta \in \Theta$ , the node of the Dynkin diagram with label  $\beta$  is either connected to the nodes in  $\Delta \setminus \Theta$  by a double arrow pointing towards  $\Delta \setminus \Theta$ , or it is not connected to  $\Delta \setminus \Theta$  at all. (*Representation is even.*)

Thus either we are in the case  $\Theta = \Delta$  and the multiplicity one condition implies that  $G$  is a split real form. Or we are in the case where  $\Theta \neq \Delta$  in which case every arrow from  $\Theta$  to  $\Delta \setminus \Theta$  has to be a double arrow. In this case there is exactly one double arrow in the Dynkin diagram  $\Delta$  and  $\Theta$  must be the set of long simple roots.

Going through the Dynkin diagrams of the system of restricted roots for all simple real Lie groups (see e.g. [Hel01, Ch. X, § 6 and Table VI p. 532-4]) we deduce the above list of pairs  $(G, \Theta)$ .  $\square$

By the classification of Dynkin diagrams (more accurately, from the fact that a connected Dynkin diagram admits at most one double arrow—cf. [Hel01, Ch. X, Lemma 3.18]), we observe that for all pairs  $(G, \Theta)$  admitting a  $\Theta$ -positive structures, both sets  $\Theta$  and  $\Delta \setminus \Theta$  are connected. When  $\Theta \neq \Delta$ , there is thus a unique root,  $\alpha_\Theta \in \Theta$  which is connected to  $\Delta \setminus \Theta$ . The root  $\alpha_\Theta$  will be called the *special root*. For all  $\alpha \in \Theta \setminus \{\alpha_\Theta\}$  we have that  $\mathfrak{u}_\alpha = \mathfrak{g}_\alpha$  and  $\dim \mathfrak{u}_\alpha = 1$ . For  $\alpha_\Theta$ , the vector space

$$\mathfrak{u}_{\alpha_\Theta} = \sum_{\substack{\alpha \in \Sigma_\Theta^+ \\ \alpha|_{\mathfrak{t}_\Theta} = \alpha_\Theta|_{\mathfrak{t}_\Theta}}} \mathfrak{g}_\alpha$$

is of dimension greater than or equal to 2 since there is a least one other root in  $\Sigma$  congruent to  $\alpha_\Theta$  modulo  $\Delta \setminus \Theta$ .

**3.2. The invariant cones.** In this section we describe the invariant cones in  $\mathfrak{u}_\alpha$  ( $\alpha \in \Theta$ ) in more detail.

For every  $\alpha \in \Theta$ , we fix an  $L_\Theta^\circ$ -invariant closed convex cone, which we denote by  $c_\alpha \subset \mathfrak{u}_\alpha$ . This cone is unique up to the action of  $-\text{id}$  (cf. Proposition 3.5). The interior of the cone  $c_\alpha \subset \mathfrak{u}_\alpha$ , which is denoted by  $\mathring{c}_\alpha$  is a homogeneous space under the action of  $L_\Theta^\circ$ . It is an acute cone. For all  $\alpha \in \Theta \setminus \{\alpha_\Theta\}$  we have  $\mathfrak{u}_\alpha \cong \mathbf{R}$ , and the cone  $c_\alpha$  can be identified with the cone  $\mathbf{R}_{\geq 0} \subset \mathbf{R}$ . The cones  $c_{\alpha_\Theta}$  can be explicitly described as well.

When  $G$  is of Hermitian of tube type, the cone  $c_{\alpha_\Theta} \subset \mathfrak{u}_{\alpha_\Theta}$  can be identified with one of the cones listed in the following table:

$G$	$r$	$\mathfrak{u}_{\alpha_\Theta}$	$c_{\alpha_\Theta}$
$\text{Sp}(2n, \mathbf{R})$	$n$	$\text{Sym}(n, \mathbf{R})$	$\text{Sym}^{\geq 0}(n, \mathbf{R})$
$\text{SU}(n, n)$	$n$	$\text{Herm}(n, \mathbf{C})$	$\text{Herm}^{\geq 0}(n, \mathbf{C})$
$\text{SO}^*(4n)$	$n$	$\text{Herm}(n, \mathbf{H})$	$\text{Herm}^{\geq 0}(n, \mathbf{H})$
$\text{EVII} = \text{E}_{7(-25)}$	3	$\text{Herm}(3, \mathbf{O})$	$\text{Herm}^{\geq 0}(3, \mathbf{O})$
$\text{SO}(2, 1+k)$	2	$(\mathbf{R}^{1,k}, q_{1,k})$	$\{v \in \mathbf{R}^{1,k} \mid v_1 \geq 0, q_{1,k}(v) \geq 0\}$

TABLE 1. The cones in the Hermitian cases;  $r$  is the real rank of  $G$

When  $G = \text{SO}(p+1, p+k)$ , then  $\mathfrak{u}_{\alpha_i} \cong \mathbf{R}$  for  $1 \leq i \leq p-1$ ,  $\alpha_\Theta = \alpha_p$ , and  $\mathfrak{u}_{\alpha_\Theta} \cong (\mathbf{R}^{1,k}, q_{1,k})$  with  $c_{\alpha_\Theta} = \{v \in \mathbf{R}^{1,k} \mid v_1 \geq 0, q_{1,k}(v) \geq 0\}$  ( $q_{1,k}$  denotes the standard quadratic form of signature  $(1, k)$  on  $\mathbf{R}^{1+k}$ ).

When  $G$  is one group in the exceptional family with restricted Dynkin diagram of type  $F_4$ , then  $\mathfrak{u}_{\alpha_1} \cong \mathbf{R}$ , and  $\mathfrak{u}_{\alpha_2}$  and the cone  $c_{\alpha_2}$  are given by the following:

**3.2.1. Homogeneity.** The explicit description of the invariant cones easily implies the following proposition; this statement could also be established with Lie algebra techniques in the spirit of the next paragraph and independently of the classification.

$G$	$\mathfrak{u}_{\alpha_2}$	$c_{\alpha_2}$
FI = $F_{4(4)}$	$\text{Sym}(3, \mathbf{R})$	$\text{Sym}^{\geq 0}(3, \mathbf{R})$
EII = $E_{6(2)}$	$\text{Herm}(3, \mathbf{C})$	$\text{Herm}^{\geq 0}(3, \mathbf{C})$
EVI = $E_{7(-5)}$	$\text{Herm}(3, \mathbf{H})$	$\text{Herm}^{\geq 0}(3, \mathbf{H})$
EIX = $E_{8(-24)}$	$\text{Herm}(3, \mathbf{O})$	$\text{Herm}^{\geq 0}(3, \mathbf{O})$

TABLE 2. The nontrivial cone for groups in the exceptional family

**Proposition 3.7.** *Let, for each  $\alpha$  in  $\Theta$ ,  $c_\alpha$  be an  $L_\Theta^\circ$ -invariant acute convex cone in  $\mathfrak{u}_\alpha$  and let  $\mathring{c}_\alpha$  be its interior. Then the diagonal action of  $L_\Theta^\circ$  on  $\prod_{\alpha \in \Theta} \mathring{c}_\alpha$  is transitive and proper. More precisely the stabilizers are maximal compact subgroups of  $L_\Theta^\circ$  so that  $\prod_{\alpha \in \Theta} \mathring{c}_\alpha$  is a model of the symmetric space associated to  $L_\Theta^\circ$ .*

3.2.2. *Homogeneity under the group of automorphisms of  $\mathfrak{g}$ .* We prove here that the choices of the cones  $c_\alpha$  in  $\mathfrak{u}_\alpha$  ( $\alpha \in \Theta$ ) is unimportant up to the action of the group  $\text{Aut}(\mathfrak{g})$ . Precisely, we have

**Proposition 3.8.** *For every  $\{\varepsilon_\alpha\}_{\alpha \in \Theta}$  in  $\{\pm 1\}^\Theta$ , there is an element  $g$  in  $\text{Aut}(\mathfrak{g})$  such that*

- for every  $\alpha$  in  $\Theta$ , the restriction  $g|_{\mathfrak{u}_\alpha}$  is  $\varepsilon_\alpha \text{id}$ .

One can even choose  $g$  such that

- the restriction  $g|_{\mathfrak{g}_0}$  is  $+\text{id}$ , and
- for every  $\alpha$  in  $\Theta$ , there restriction  $g|_{\mathfrak{u}_{-\alpha}}$  is  $\varepsilon_\alpha \text{id}$ .

*Proof.* Let us try to find  $g$  defined by  $g|_{\mathfrak{g}_\gamma} = \mu_\gamma \text{id}$  for every  $\gamma$  in  $\Sigma \cup \{0\}$  where  $\mu_\gamma = \pm 1$ . The family  $\{\mu_\gamma\}$  should satisfy the following properties:

- (1) For all  $\gamma$  and  $\gamma'$  in  $\Sigma \cup \{0\}$  such that  $\gamma + \gamma'$  belongs to  $\Sigma \cup \{0\}$ ,  $\mu_{\gamma+\gamma'} = \mu_\gamma \mu_{\gamma'}$  (so that  $g$  is indeed an automorphism of  $\mathfrak{g}$ ).
- (2) One has  $\mu_0 = 1$  and, for every  $\alpha \in \Theta$  and every  $\gamma$  in  $\Sigma$  that is congruent to  $\pm\alpha$  modulo the span of  $\Delta \setminus \Theta$ ,  $\mu_\gamma = \varepsilon_\alpha$ .

For this we will define  $\{\mu_\gamma\}_{\gamma \in \Sigma \cup \{0\}}$  to be the restriction to  $\Sigma \cup \{0\}$  of a group homomorphism  $\phi: \text{Span}_{\mathbf{Z}}(\Delta) \rightarrow \{\pm 1\}$ . The property (1) is then obviously satisfied as well as the equality  $\mu_0 = \phi(0) = 1$  and the property (2) will be satisfied as well if  $\phi$  is chosen so that

- $\phi(\alpha) = \varepsilon_\alpha$  for all  $\alpha$  in  $\Theta$  and  $\phi(\alpha) = 1$  for all  $\alpha$  in  $\Delta \setminus \Theta$ .  $\square$

As a corollary of the two previous propositions we get

**Corollary 3.9.** *The Levi factor of the parabolic subgroup of  $\text{Aut}(\mathfrak{g})$  associated with  $\Theta$  acts transitively on  $\prod_{\alpha \in \Theta} (\mathring{c}_\alpha \cup -\mathring{c}_\alpha)$ .*

*Remark 3.10.* It is an interesting question to determine the orbits of  $L_\Theta$  on the product  $\prod_{\alpha \in \Theta} (\mathring{c}_\alpha \cup -\mathring{c}_\alpha)$ . For the cases not corresponding to total positivity (i.e. when  $\Theta \neq \Delta$ ), one has the following result:

- When  $\#\Theta$  is even, the action of  $L_\Theta$  on  $\prod_{\alpha \in \Theta} (\dot{c}_\alpha \cup -\dot{c}_\alpha)$  is transitive,
- When  $\#\Theta$  is odd, this action has two orbits.

**3.3. Putting a hand on elements in the cones.** We explain in this paragraph an explicit construction of elements in the cones  $\dot{c}_\alpha$  from “standard” elements presenting the Lie algebra  $\mathfrak{g}$ . This will allow us to construct what we call a  $\Theta$ -base, which gives rise to a split real Lie algebra  $\mathfrak{g}_\Theta < \mathfrak{g}$  (see Section 5) and a special  $\mathfrak{sl}_2$ -triple in  $\mathfrak{g}$  (see Section 11). To start we consider the following:

- for each  $\alpha \in \Theta$ , we choose once for all an element  $X_\alpha$  in  $\mathfrak{g}_\alpha$  such that, setting  $X_{-\alpha} = \tau(X_\alpha)$  and  $H_\alpha = [X_\alpha, X_{-\alpha}]$ , the triple  $(X_\alpha, X_{-\alpha}, H_\alpha)$  is an  $\mathfrak{sl}_2$ -triple (cf. Section 2.3).

Since  $\mathfrak{g}_\alpha$  is of dimension 1, the element  $X_\alpha$  is determined uniquely up to sign.

For  $\alpha \neq \alpha_\Theta$  we have  $\mathfrak{u}_\alpha = \mathfrak{g}_\alpha$ , and thus (up to sign) we have that  $X_\alpha \in \dot{c}_\alpha$ . For  $\alpha = \alpha_\Theta$ , we use the element  $Z_0 := X_\alpha \in \mathfrak{g}_\alpha$  as a starting point to define a new element  $X_{\alpha_\Theta} \in \dot{c}_{\alpha_\Theta}$ , using the action of the Weyl group of  $S_\Theta$ .

From the classification result (Theorem 3.4) we know that the semi-simple Lie group  $S_\Theta$  is of type  $A_d$  where  $d$  is the cardinality of  $\Delta \setminus \Theta$ . In other words, we can enumerate  $\Delta \setminus \Theta = \{\chi_1, \dots, \chi_d\}$  so that  $\chi_1$  is connected (by a double arrow) to the root  $\alpha_\Theta$  and, for all  $i = 1, \dots, d-1$ ,  $\chi_i$  is connected to  $\chi_{i+1}$ . The corresponding reflections in the Weyl group  $W$  will be denoted by  $s_1, \dots, s_d$ .

One has thus the following equalities:

$$\begin{aligned} s_1(\alpha_\Theta) &= \alpha_\Theta + 2\chi_1, & s_i(\alpha_\Theta) &= \alpha_\Theta \text{ for } i > 1, \\ s_i(\chi_{i-1}) &= \chi_{i-1} + \chi_i, & s_{i-1}(\chi_i) &= \chi_i + \chi_{i-1} \text{ for } i > 1, \\ s_i(\chi_i) &= -\chi_i \text{ for } i \geq 1, \\ s_i(\chi_j) &= \chi_j \text{ if } |i - j| > 1. \end{aligned}$$

**Lemma 3.11.** *The elements in the orbit of the special root  $\alpha_\Theta$  under the group  $W_{\Delta \setminus \Theta} = \langle s_1, \dots, s_d \rangle$  are the following:*

$$\begin{aligned} \gamma_0 &= \alpha_\Theta \\ \gamma_1 &= \alpha_\Theta + 2\chi_1 = s_1(\alpha_\Theta) \\ \gamma_2 &= \alpha_\Theta + 2\chi_1 + 2\chi_2 = s_2 s_1(\alpha_\Theta) \\ &\dots \\ \gamma_i &= \alpha_\Theta + 2\chi_1 + \dots + 2\chi_i = s_i \dots s_1(\alpha_\Theta) \\ &\dots \\ \gamma_d &= \alpha_\Theta + 2\chi_1 + \dots + 2\chi_d = s_d \dots s_1(\alpha_\Theta). \end{aligned}$$

*In particular, those roots are equal to  $\alpha_\Theta$  modulo the span of  $\chi_1, \dots, \chi_d$ , i.e.  $\mathfrak{g}_{\gamma_i} \subset \mathfrak{u}_{\alpha_\Theta}$  for all  $i$ .*

*Proof.* From the identities given before the lemma, one deduces

$$\begin{aligned} s_i(\gamma_k) &= \gamma_k \text{ when } |k - i| > 2 \\ s_i(\gamma_{i-1}) &= \gamma_i, \quad s_i(\gamma_i) = \gamma_{i-1} \text{ for } i = 1, \dots, d \\ s_i(\gamma_{i+1}) &= \gamma_{i+1} \text{ for } i = 1, \dots, d - 1. \end{aligned}$$

From this we directly observe that the reflections  $s_1, \dots, s_d$  stabilize the set  $\{\gamma_0, \dots, \gamma_d\}$  and act transitively on it, hence the result since  $W_{\Delta \setminus \Theta}$  is generated by  $s_1, \dots, s_d$ .  $\square$

Recall (see Section 2) that we chose elements  $\dot{s}_i$  of  $G$  lifting the elements  $s_i$  of  $W$ .

**Proposition 3.12.** *The elements  $Z_0 = X_\alpha \in \mathfrak{g}_{\alpha_\Theta}$  and*

$$Z_1 = \text{Ad}(\dot{s}_1)Z_0, \quad Z_2 = \text{Ad}(\dot{s}_2)Z_1, \dots, \quad Z_d = \text{Ad}(\dot{s}_d)Z_{d-1}$$

*do not depend on the choices of the lifts  $\dot{s}_i$ . They all belong to  $\mathfrak{u}_{\alpha_\Theta}$ , and are the elements of the orbit of  $Z_0$  under the group generated by  $\{\dot{s}_1, \dots, \dot{s}_d\}$ .*

*Proof.* We use the notation introduced in the previous lemma.

By naturality one has that  $\text{Ad}(\dot{s}_i)\mathfrak{g}_\alpha = \mathfrak{g}_{s_i(\alpha)}$ . Therefore, for all  $k = 0, \dots, d$ ,  $Z_k$  belongs to  $\mathfrak{g}_{\gamma_k}$  which is contained in  $\mathfrak{u}_{\alpha_\Theta}$ .

For each  $i = 1, \dots, d$ , let  $X_i$  in  $\mathfrak{g}_{\chi_i}$  be such that, setting  $Y_i = \tau(X_i)$  and  $H_i = [X_i, Y_i]$ , the triple  $(X_i, Y_i, H_i)$  is an  $\mathfrak{sl}_2$ -triple. Let also  $\mathfrak{s}_i$  be the subalgebra of  $\mathfrak{g}$  generated by  $X_i, H_i$ , and  $Y_i$ .

Among  $\{\gamma_0, \gamma_1, \dots, \gamma_d\}$ , the only elements contained in a nontrivial  $\chi_i$ -chain are  $\gamma_{i-1}$  and  $\gamma_i$ . Furthermore, the chain containing them is  $\gamma_{i-1}, \gamma_{i-1} + \chi_i, \gamma_{i-1} + 2\chi_i = \gamma_i$ . This means in particular that

$$\mathfrak{g}_{\gamma_{i-1}} \oplus \mathfrak{g}_{\gamma_{i-1} + \chi_i} \oplus \mathfrak{g}_{\gamma_i}$$

is invariant under the adjoint action of  $\mathfrak{s}_i$  and that the lowest weight space for this action is the 1-dimensional space  $\mathfrak{g}_{\gamma_{i-1}}$ . Together with the fact that  $\gamma_{i-1}(H_i) = -2$ , one gets that the  $\mathfrak{s}_i$ -module  $W_i$  generated by  $\mathfrak{g}_{\gamma_{i-1}}$  is the irreducible 3-dimensional  $\mathfrak{s}_i$ -module.

Since furthermore  $\text{Ad}(\dot{s}_i) = \exp(\pm \frac{\pi}{2} \text{ad}(X_i - Y_i))$  (where the choice of the lift  $\dot{s}_i$  influences the sign and could be the source of undeterminacy) and since the action of  $\exp(\pi(X_i - Y_i))$  is trivial on the irreducible 3-dimensional module, we deduce (by induction on  $i$ ) that  $Z_i$  does not depend on the choice of the lifts  $\dot{s}_j$ . Also the relation  $\text{Ad}(\dot{s}_i)Z_i = Z_{i-1}$  follows and, with a similar but simpler argument,  $\text{Ad}(\dot{s}_i)Z_k = Z_k$  for  $k \neq i - 1, i$ . From this the statement about the orbit of  $Z_0$  follows readily.  $\square$

From this proposition we deduce

**Theorem 3.13.** *With the notation of Proposition 3.12, the element  $E_{\alpha_\Theta}$  of  $\mathfrak{u}_{\alpha_\Theta}$  defined by*

$$E_{\alpha_\Theta} := Z_0 + Z_1 + \dots + Z_d,$$

which is the sum of the elements in the orbit of  $Z_0 \in \mathfrak{g}_{\alpha_\Theta}$  under the action of the group  $W_{\Delta \setminus \Theta}$ , belongs to the  $L_\Theta^\circ$ -invariant open cone  $\hat{c}_\alpha$ .

*Proof.* By Proposition 3.5 it is enough to show that the Lie algebra of the stabilizer of  $Z_0 + \dots + Z_d$  contains  $\mathfrak{k}_\Theta$ . We already know that this sum is invariant under the (lifted) action of the Weyl group  $W_{\Delta \setminus \Theta}$ .

From this already known invariance and from the fact that the Lie group  $S_\Theta$  is of type  $A_d$  and since in type  $A$  all the roots are in the same Weyl group orbit, it is enough to establish the invariance by  $\{X - \tau(X) \mid X \in \mathfrak{g}_{\chi_1}\}$  and the invariance by  $\mathfrak{m} = \mathfrak{z}_\mathfrak{k}(\mathfrak{a})$  (cf. Equation (2.1), p. 10).

The root space  $\mathfrak{g}_{\alpha_\Theta}$  is a 1-dimensional representation of the (compact) Lie algebra  $\mathfrak{m}$  and is thus the trivial representation. Hence the stabilizer of  $Z_0$  contains  $\mathfrak{m}$ . By equivariance under the Weyl group  $W_{\Delta \setminus \Theta}$  we deduce that, for all  $k = 0, \dots, d$ , the stabilizer of  $Z_k$  contains  $\mathfrak{m}$ . Therefore the stabilizer of  $Z_0 + \dots + Z_d$  contains  $\mathfrak{m}$ .

Let now  $X$  be an element of  $\mathfrak{g}_{\chi_1}$ , and let  $Y = \tau(X)$ . We want to show that the stabilizer of  $Z_0 + \dots + Z_d$  contains  $X - Y$ . As the conclusion holds trivially if  $X = 0$ , we can assume that  $X$  is non zero. Up to multiplying  $X$  by a positive real number, we can assume that  $(X, Y, [X, Y])$  is an  $\mathfrak{sl}_2$ -triple. Denote by  $\mathfrak{s}$  the Lie algebra it generates. By Proposition 3.12 and the analysis performed in its proof we know that the  $\mathfrak{s}$ -module generated by  $Z_0$  contains  $Z_1 = \text{Ad}(\dot{s}_1)Z_0$  and is the irreducible 3-dimensional  $\mathfrak{s}$ -module. Explicit knowledge of the 3-dimensional irreducible  $\mathfrak{sl}_2$ -module shows directly that the stabilizer of  $Z_0 + Z_1$  contains  $X - Y$ . By similar arguments, for every  $k = 2, \dots, d$ , the stabilizer of  $Z_k$  contains  $X - Y$ . From this we have the sought for result:  $\{X - \tau(X) \mid X \in \mathfrak{g}_{\chi_1}\}$  is included in the stabilizer of  $Z_0 + \dots + Z_d$ .  $\square$

Note that the elements  $Z_i$  pairwise commute.

**Lemma 3.14.** *For every  $i, j$  in  $\{0, \dots, d\}$ , the Lie bracket  $[Z_i, Z_j]$  is zero.*

*Proof.* For every  $i, j \in \{0, \dots, d\}$  the non zero weight  $\gamma_i + \gamma_j$  is not a root, thus  $\mathfrak{g}_{\gamma_i + \gamma_j} = \{0\}$ . Since  $[Z_i, Z_j]$  belongs to  $\mathfrak{g}_{\gamma_i + \gamma_j}$ , the conclusion follows.  $\square$

*Example 3.15.* We illustrate the construction of the elements above in the example when  $G = \text{Sp}(2n, \mathbf{R})$  and  $\Theta = \{\alpha_n\}$ . In this case  $\mathfrak{u}_{\alpha_\Theta}$  naturally identifies with  $\text{Sym}_n(\mathbf{R})$ , the space of real  $n \times n$  symmetric matrices and  $W_{\Delta \setminus \Theta}$  is isomorphic to the symmetric group  $S_n$  acting on  $\text{Sym}_n(\mathbf{R})$  by conjugation by the corresponding permutation matrices and  $d = n - 1$ . We use the standard basis  $(E_{i,j})_{1 \leq i, j \leq n}$  of  $M_n(\mathbf{R})$  (the only non zero coefficient of  $E_{i,j}$  is in place  $(i, j)$  and is equal to 1) so that  $\{E_{i,i}\}_i \cup \{E_{i,j} + E_{j,i}\}_{i < j}$  is a basis of  $\text{Sym}_n(\mathbf{R})$ . The root space  $\mathfrak{g}_{\alpha_n}$  is equal (as a subspace of  $\mathfrak{u}_{\alpha_\Theta} \simeq \text{Sym}_n(\mathbf{R})$ ) to the line generated by  $E_{1,1}$ . Thus, one can take  $Z_0 = E_{1,1}$  and one

has  $Z_1 = E_{2,2}, \dots, Z_i = E_{i+1,i+1}, \dots, Z_{n-1} = E_{n,n}$ . Hence the sum  $E_{\alpha_n} = Z_0 + \dots + Z_{n-1}$  is the identity matrix of  $\text{Sym}_n(\mathbf{R})$ .

Note that if  $E_\alpha \in \mathring{c}_\alpha$  is the element determined by choosing  $X_\alpha \in \mathfrak{g}_\alpha$ , then the element in  $\mathfrak{u}_\alpha$  determined by choosing  $-X_\alpha \in \mathfrak{g}_\alpha$  is  $-E_\alpha \in -\mathring{c}_\alpha$ . In completely analogous way, the choice of the element  $Y_\alpha \in \mathfrak{g}_{-\alpha}$  determines an element  $F_\alpha \in \mathring{c}_\alpha^{\text{opp}}$ .

**Definition 3.16.** Let  $E_\alpha \in \mathring{c}_\alpha, F_\alpha \in \mathring{c}_\alpha^{\text{opp}}$  be the elements constructed as above, and set  $D_\alpha := [E_\alpha, F_\alpha]$  for all  $\alpha \in \Theta$ . Then the family  $(E_\alpha, F_\alpha, D_\alpha)_{\alpha \in \Theta}$  is called a  $\Theta$ -base of  $\mathfrak{g}$ .

*Remark 3.17.* In the case when  $\Theta = \Delta$ , i.e. when  $G$  is a split real group, the elements  $\{E_\alpha, F_\alpha, D_\alpha\}_{\alpha \in \Delta}$  are up to normalization the Chevalley or Cartan–Weyl basis of  $\mathfrak{g}$ .

#### 4. THE $\Theta$ -WEYL GROUP

This section starts the first investigation of the specific algebraic properties that arise when a simple Lie group  $G$  admits a  $\Theta$ -positive structure. Here we focus on the  $\Theta$ -Weyl group, a specific subgroup of the Weyl group of  $G$  whose generators are chosen with respect to the Bruhat decomposition (cf. Section 7). The  $\Theta$ -Weyl group will be crucial in the parametrization of the unipotent positive semigroup (cf. Section 8).

Let  $W$  be the Weyl group of  $G$ . Recall that the generators corresponding to the simple roots  $\alpha \in \Delta$  are denoted by  $s_\alpha$ , and that  $\ell(w)$  is the length of an element  $w$  with respect to this generating set.

**4.1. Longest elements and some involutions.** Let again  $W_{\Delta \setminus \Theta}$  be the subgroup generated by  $s_\alpha$  with  $\alpha \in \Delta \setminus \Theta$  (for a subset  $F$  of  $\Delta$  the subgroup generated by  $\{s_\alpha\}_{\alpha \in F}$  will be denoted by  $W_F$ ). We denote by  $w_\Delta$  the longest length element in  $W$ , and by  $w_{\Delta \setminus \Theta}$  the longest length element in  $W_{\Delta \setminus \Theta}$ .

**Lemma 4.1.** [BT87, Proposition 3.9] *The element  $w_{\max}^\Theta \in W$  defined by the equality*

$$w_\Delta = w_{\max}^\Theta w_{\Delta \setminus \Theta}$$

*satisfies  $\ell(w_\Delta) = \ell(w_{\max}^\Theta) + \ell(w_{\Delta \setminus \Theta})$ . Furthermore  $w_{\max}^\Theta$  is the unique element of minimal length in the coset  $w_\Delta W_{\Delta \setminus \Theta}$ .*

We want to apply this lemma when we consider the Weyl group associated to the diagram  $\{\alpha_\Theta\} \cup \Delta \setminus \Theta \subset \Delta$ . The Weyl group  $W_{\{\alpha_\Theta\} \cup \Delta \setminus \Theta}$  is naturally a subgroup of  $W$ . Applying Lemma 4.1 we then get the following corollary.

**Corollary 4.2.** *The element  $\sigma_{\alpha_\Theta} \in W_{\{\alpha_\Theta\} \cup \Delta \setminus \Theta}$  defined by the equality*

$$w_{\{\alpha_\Theta\} \cup \Delta \setminus \Theta} = \sigma_{\alpha_\Theta} w_{\Delta \setminus \Theta}$$

satisfies  $\ell(w_{\{\alpha_\Theta\} \cup \Delta \setminus \Theta}) = \ell(\sigma_{\alpha_\Theta}) + \ell(w_{\Delta \setminus \Theta})$ , it is the element of minimal length in the coset  $w_{\{\alpha_\Theta\} \cup \Delta \setminus \Theta} W_{\Delta \setminus \Theta}$ .

*Remark 4.3.* Note that the Dynkin diagram  $\{\alpha_\Theta\} \cup \Delta \setminus \Theta$  is of type  $C_{d+1}$  where  $d$  is the cardinality of  $\Delta \setminus \Theta$ .

We now establish that the above elements are of order 2:

**Lemma 4.4.** (1) The elements  $w_{\max}^\Theta$  and  $\sigma_{\alpha_\Theta}$  are of order 2.

(2) Let  $W = \langle s_\alpha \rangle_{\alpha \in \Delta}$  be a finite Coxeter group, let  $w_\Delta \in W$  be its longest length element, and let  $\alpha \mapsto \bar{\alpha}$  be the involution of  $\Delta$  defined by the equalities  $w_\Delta s_\alpha w_\Delta^{-1} = s_{\bar{\alpha}}$ . Let also  $\Theta \subset \Delta$  be invariant by the involution  $\alpha \mapsto \bar{\alpha}$  and denote by  $w_{\Delta \setminus \Theta}$  the longest length element of the finite Coxeter group  $\langle s_\alpha \rangle_{\alpha \in \Delta \setminus \Theta}$ . Finally set  $w_r = w_\Delta w_{\Delta \setminus \Theta}^{-1}$ . Then

(a) The elements  $w_\Delta$  and  $w_{\Delta \setminus \Theta}$  are of order 2.

(b) One has  $w_\Delta w_{\Delta \setminus \Theta} w_\Delta^{-1} = w_{\Delta \setminus \Theta}$ .

(c) The element  $w_r$  is of order 2 and commutes with  $w_\Delta$  and with  $w_{\Delta \setminus \Theta}$ .

*Proof.* For the item (1) we observe that in those cases, the involution  $\alpha \mapsto \bar{\alpha}$  is the identity (this is always the case when a Dynkin diagram has a multiple arrow), hence the result follows from item (2).

Let us prove (2). Point (2a) is classical and follows from the fact that the inverse of a reduced expression of  $w_\Delta$  is a reduced expression of  $w_\Delta^{-1}$  and hence uniqueness of the longest length element implies the equality  $w_\Delta^{-1} = w_\Delta$ . Point (2b) follows from the fact that if  $s_{\alpha_1} \cdots s_{\alpha_N}$  is a reduced expression of  $w_{\Delta \setminus \Theta}$  then  $s_{\bar{\alpha}_1} \cdots s_{\bar{\alpha}_N}$  is a reduced expression of  $w_\Delta w_{\Delta \setminus \Theta} w_\Delta^{-1}$  and hence the equality  $w_\Delta w_{\Delta \setminus \Theta} w_\Delta^{-1} = w_{\Delta \setminus \Theta}$  holds again from the uniqueness of the longest length element. Point (2c) is now an immediate consequence of (2a) and (2b) and the definition of  $w_r$ .  $\square$

#### 4.2. The $\Theta$ -Weyl group and its generators.

**Definition 4.5.** We define the  $\Theta$ -Weyl group to be the subgroup  $W(\Theta)$  of  $W$  generated by  $s_\alpha$  for all  $\alpha \in \Theta \setminus \{\alpha_\Theta\}$ , and  $\sigma_{\alpha_\Theta}$ ; this generating set is denoted by  $R(\Theta)$ .

**Notation 4.6.** In order to have more uniform notation later, we set, for every  $\alpha \in \Theta \setminus \{\alpha_\Theta\}$ ,  $\sigma_\alpha = s_\alpha$ . Hence  $R(\Theta) = \{\sigma_\alpha\}_{\alpha \in \Theta}$

Observe that when  $\Theta = \Delta$  (hence when  $\alpha_\Theta$  is not defined) one has  $W(\Theta) = W$ .

The next proposition shows that  $W(\Theta)$  is a Coxeter group, even more it is isomorphic to a Weyl group of a different root system.

**Proposition 4.7.** Let  $G$  be a simple Lie group admitting a  $\Theta$ -positive structure. Then  $(W(\Theta), R(\Theta))$  is a Coxeter system of the following type:

- (1) If  $G$  is a split real form and  $\Theta = \Delta$ ,  $(W(\Theta), R(\Theta))$  has the same type as  $G$ .
- (2) If  $G$  is of Hermitian tube type and of real rank  $r$  and if  $\Theta = \{\alpha_r\}$ , then  $(W(\Theta), R(\Theta))$  is of type  $A_1$ .
- (3) If  $G$  is locally isomorphic to  $\mathrm{SO}(p+1, p+k)$ ,  $p > 0$ ,  $k > 1$ , and  $\Theta = \{\alpha_1, \dots, \alpha_p\}$ , then  $(W(\Theta), R(\Theta))$  is of type  $B_p$ .
- (4) If  $G$  is the real form of  $F_4, E_6, E_7$ , or of  $E_8$ , whose reduced root system is of type  $F_4$ , and  $\Theta = \{\alpha_1, \alpha_2\}$ , then  $(W(\Theta), R(\Theta))$  is of type  $G_2$ .

*Proof.* The proof follows from a case by case consideration.

- (1) When  $G$  is a split real form, then  $\Theta = \Delta$ , then  $R(\Theta) = \{s_\alpha\}_{\alpha \in \Delta}$ , hence  $W(\Theta) = W$ .
- (2) Let  $G$  be of real rank  $r$  and of Hermitian tube type, and let  $\Theta = \{\alpha_r\} = \{\alpha_\Theta\}$ . Consequently  $W(\Theta)$  is generated by the single order 2 element  $\sigma_{\alpha_\Theta}$ , thus the result.
- (3) When  $G$  is locally isomorphic to  $\mathrm{SO}(p+1, p+k)$ ,  $p > 0$ ,  $k > 1$ , and  $\Theta = \{\alpha_1, \dots, \alpha_p\}$ , one has  $\alpha_\Theta = \alpha_p$ , and  $\Delta \setminus \Theta = \{\alpha_{p+1}\}$ . The group  $W_{\{\alpha_\Theta\} \cup \Delta \setminus \Theta}$  is the group generated by  $s_p$  and  $s_{p+1}$  (recall that we write  $s_i$  instead of  $s_{\alpha_i}$ ) and its longest length element is  $w_{\alpha_\Theta \cup \Delta \setminus \Theta} = s_p s_{p+1} s_p s_{p+1}$ . Since  $w_{\Delta \setminus \Theta} = s_{p+1}$  and  $\sigma_{\alpha_\Theta} = w_{\alpha_\Theta \cup \Delta \setminus \Theta} w_{\Delta \setminus \Theta}^{-1}$  we get that  $\sigma_{\alpha_\Theta} = s_p s_{p+1} s_p$ . Appendix A identifies  $W_{\{\alpha_\Theta \cup \Delta \setminus \Theta\}}$  (and its generating set) with the group of signed permutation matrices of size  $p+1$  (and its standard generating set); this appendix also determines the pair  $(W(\Theta), R(\Theta))$  which naturally identify with the group of signed permutation matrices of size  $p$  (and its standard generating set). This shows the result.
- (4) When  $G$  is the real form of  $F_4, E_6, E_7, E_8$ , whose restricted root system is of type  $F_4$ , then  $\Theta = \{\alpha_1, \alpha_2\}$  with  $\alpha_\Theta = \alpha_2$ , and  $\Delta \setminus \Theta = \{\alpha_3, \alpha_4\}$ . We write  $s_i$  for  $s_{\alpha_i}$  and  $\sigma_2$  for  $\sigma_{\alpha_2}$ . The following identities are proved in Appendix B:  $\sigma_2 = s_2 s_3 s_4 s_2 s_3 s_2$  and  $s_1 \sigma_2 s_1 \sigma_2 s_1 \sigma_2 = \sigma_2 s_1 \sigma_2 s_1 \sigma_2 s_1$  from which we deduce that  $W(\Theta)$  is of type  $G_2$ .  $\square$

### 4.3. The longest element of $W(\Theta)$ .

**Proposition 4.8.** *Let  $G$  be a simple Lie group admitting a  $\Theta$ -positive structure. Let  $w_\Delta$  be the longest length element in  $W$ ,  $w_{\Delta \setminus \Theta}$  the longest length element in  $W_{\Delta \setminus \Theta}$ , and  $w_{\max}^\Theta = w_\Delta w_{\Delta \setminus \Theta}^{-1}$  so that  $w_{\max}^\Theta$  is the unique element of minimal length in the coset  $w_\Delta W_{\Delta \setminus \Theta}$  (Lemma 4.1). Then*

- (1) The elements  $w_{\max}^\Theta$  and  $w_{\Delta \setminus \Theta}$  commute:  $w_{\max}^\Theta w_{\Delta \setminus \Theta} = w_{\Delta \setminus \Theta} w_{\max}^\Theta$ .
- (2) The element  $w_{\max}^\Theta$  belongs to  $W(\Theta)$  and is the longest length element of the Coxeter system  $(W(\Theta), R(\Theta))$ .

*Proof.* Property (1) was already noticed in Lemma 4.4.

We prove the property (2) by a case by case analysis.

- (1) In the case when  $G$  is a split real simple Lie group there is nothing to prove.
- (2) When  $G$  is a simple Lie group of Hermitian tube type, then  $w_{\max}^{\Theta} = \sigma_{\alpha_{\Theta}}$ , which is the longest length element in  $W(\Theta)$  that is of type  $A_1$ .
- (3) When  $G = \mathrm{SO}(p+1, p+k)$ ,  $p > 0$ ,  $k > 1$ ,  $\Delta = \{\alpha_1, \dots, \alpha_{p+1}\}$ ,  $\Theta = \{\alpha_1, \dots, \alpha_p\}$  and  $\Delta \setminus \Theta = \{\alpha_{p+1}\}$ . Writing as usual  $s_i$  for  $s_{\alpha_i}$  and for short  $\sigma_p$  for  $\sigma_{\alpha_p}$  (and  $\sigma_i = s_i$  for  $i < p$ ), one has  $w_{\Delta \setminus \Theta} = s_{p+1}$  and a direct calculation performed in Appendix A gives the following reduced expression for the longest length element  $w_{\Delta}$  (we adopt here the notation  $x^y = y^{-1}xy$  in a group, one has thus  $(x^y)^z = x^{yz}$  and  $\sigma_p = s_{p+1}^{s_p}$ ):

$$(4.1) \quad w_{\Delta} = s_{p+1}^{s_p \cdots s_1} s_{p+1}^{s_p \cdots s_2} \cdots s_{p+1}^{s_p s_{p-1}} s_{p+1}^{s_p} s_{p+1}$$

$$(4.2) \quad \begin{aligned} w_{\max}^{\Theta} &= s_{p+1}^{s_p \cdots s_1} s_{p+1}^{s_p \cdots s_2} \cdots s_{p+1}^{s_p s_{p-1}} s_{p+1}^{s_p} \\ &= \sigma_p^{s_{p-1} \cdots s_1} \sigma_p^{s_{p-1} \cdots s_2} \cdots \sigma_p^{s_{p-1}} \sigma_p \\ &= \sigma_p^{\sigma_{p-1} \cdots \sigma_1} \sigma_p^{\sigma_{p-1} \cdots \sigma_2} \cdots \sigma_p^{\sigma_{p-1}} \sigma_p. \end{aligned}$$

In particular we get that  $w_{\max}^{\Theta}$  belongs to  $W(\Theta)$  and is the longest length element of the Coxeter system  $(W(\Theta), R(\Theta))$  of type  $B_p$ .

- (4) When  $G$  is the real form of  $F_4, E_6, E_7, E_8$ , whose reduced root system is of type  $F_4$ , we have that  $\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ ,  $\Theta = \{\alpha_1, \alpha_2\}$  and  $\Delta \setminus \Theta = \{\alpha_3, \alpha_4\}$ . We write again  $s_i$  for  $s_{\alpha_i}$ ,  $\sigma_2$  for  $\sigma_{\alpha_2}$ , and  $\sigma_1 = s_1$ . By Appendix B,  $\sigma_2 = s_2 s_3 s_4 s_2 s_3 s_2 = (s_4 s_2)^{s_2 s_3}$ ; one has  $w_{\Delta \setminus \Theta} = s_4^{s_3}$  and a direct calculation given in Appendix B gives the following reduced expression for the longest length element

$$(4.3) \quad w_{\Delta} = (s_4 s_2)^{s_2 s_3} s_1 (s_4 s_2)^{s_2 s_3} s_1 (s_4 s_2)^{s_2 s_3} s_1 s_4^{s_3},$$

thus

$$(4.4) \quad \begin{aligned} w_{\max}^{\Theta} &= (s_4 s_2)^{s_2 s_3} s_1 (s_4 s_2)^{s_2 s_3} s_1 (s_4 s_2)^{s_2 s_3} s_1 \\ &= \sigma_2 \sigma_1 \sigma_2 \sigma_1 \sigma_2 \sigma_1 \end{aligned}$$

this is precisely the longest length element of  $(W(\Theta), R(\Theta))$  which is of type  $G_2$ .  $\square$

The previous proof also gives the following information about reduced expressions of elements in  $W(\Theta)$ .

**Lemma 4.9.** *Let  $x$  be in  $W(\Theta)$ . Let  $\sigma_{\alpha_1} \cdots \sigma_{\alpha_k}$  be a reduced expression of  $x$  in  $(W(\Theta), R(\Theta))$ . For each  $i$  between 1 and  $k$  let  $n_i = \ell(\sigma_{\alpha_1}) + \cdots + \ell(\sigma_{\alpha_i})$  (and set  $n_0 = 0$ ) and let  $s_{\beta_{n_{i-1}+1}} \cdots s_{\beta_{n_i}}$  be a reduced expression of  $\sigma_{\alpha_i}$  in  $(W, \Delta)$ . Then*

$$s_{\beta_1} \cdots s_{\beta_{n_k}}$$

*is a reduced expression of  $x$  in  $(W, \Delta)$ .*

*Remark 4.10.* Note that, when  $\alpha_i \neq \alpha_{\Theta}$ , one has  $\sigma_{\alpha_i} = s_{\alpha_i}$  and  $n_i = n_{i-1} + 1$ .

*Proof.* Since  $\sigma_{\alpha_1} \cdots \sigma_{\alpha_k}$  is a prefix of a reduced expression of the longest length element  $w_{\max}^{\Theta}$  of  $(W(\Theta), R(\Theta))$  and since prefixes of reduced expressions in  $(W, \Delta)$  are again reduced expressions, it is enough to prove the result for  $x = w_{\max}^{\Theta}$ .

Rephrasing the question, it is enough to find a prefix of a reduced expression of  $w_{\Delta}$  and, in it, find successive expressions of elements that belong to  $R(\Theta)$  that form together a reduced expression (in  $(W(\Theta), R(\Theta))$ ) of  $w_{\max}^{\Theta}$ . We proceed now with a case by case analysis.

The result is obvious in the split case or in the Hermitian tube type case. For the case of orthogonal groups, the reduced expression of  $w_{\Delta}$  is found in Equation (4.1) and the prefix producing  $w_{\max}^{\Theta}$  is in Equation (4.2). For the  $F_4$  case the meaningful equations are (4.3) and (4.4).  $\square$

The  $\Theta$ -Weyl group  $W(\Theta)$  and the reduced expression of its longest element will be important to parametrize the positive semigroup. Before we describe this, we will emphasize the existence of a real split subgroup of  $G$  that is of type  $W(\Theta)$  (Section 5).

**4.4. The  $\Theta$ -length.** We introduce now a function on the Weyl group  $W$  that depends on  $\Theta \subset \Delta$  and that has good properties when  $G$  admits a  $\Theta$ -positive structure.

**Definition 4.11.** The  $\Theta$ -length is the biggest subadditive function

$$\ell_{\Theta}: W \rightarrow \mathbf{N}$$

such that  $\ell_{\Theta}(s_{\alpha}) = 1$  for every  $\alpha$  in  $\Theta$  and  $\ell_{\Theta}(s_{\alpha}) = 0$  for every  $\alpha$  in  $\Delta \setminus \Theta$ .

More concretely, for every  $w$  in  $W$ ,  $\ell_{\Theta}(w)$  is the minimal number of occurrences of elements in  $\{s_{\alpha}\}_{\alpha \in \Theta}$  when  $w$  is written as an expression in the generating set  $\{s_{\alpha}\}_{\alpha \in \Delta}$ ; in formula this gives

$$(4.5) \quad \ell_{\Theta}(w) = \min\{k \in \mathbf{N} \mid \exists N \in \mathbf{N}, (\alpha_1, \dots, \alpha_N) \in \Delta^N \text{ with} \\ w = s_{\alpha_1} \cdots s_{\alpha_N} \text{ and } k = \#\{j \leq N \mid \alpha_j \in \Theta\}\}.$$

Furthermore, for every function  $f: W \rightarrow \mathbf{N}$  such that  $f(ab) \leq f(a) + f(b)$  for every  $a, b$  in  $W$  (that is,  $f$  is subadditive) and such that  $f(s_{\alpha}) = 1$  if  $\alpha \in \Theta$  and  $f(s_{\alpha}) = 0$  if  $\alpha \in \Delta \setminus \Theta$ , then  $f \leq \ell_{\Theta}$ .

When  $\Theta = \Delta$ , the  $\Theta$ -length coincides with the length function on  $W$  already introduced.

**Lemma 4.12.** *The function  $\ell_{\Theta}$  is invariant under the subgroup  $W_{\Delta \setminus \Theta}$  generated by  $\{s_{\alpha}\}_{\alpha \in \Delta \setminus \Theta}$ : for every  $w \in W$  and every  $x \in W_{\Delta \setminus \Theta}$  one has  $\ell_{\Theta}(w) = \ell_{\Theta}(xw) = \ell_{\Theta}(wx)$ .*

*Proof.* By symmetry we will prove the equality only for right multiplication by  $W_{\Delta \setminus \Theta}$ .

Let us show first that, for every  $w \in W$  and every  $x \in W_{\Delta \setminus \Theta}$ ,

$$\ell_{\Theta}(wx) \leq \ell_{\Theta}(w).$$

Let  $N \in \mathbf{N}$  and  $(\alpha_1, \dots, \alpha_N) \in \Delta^N$  be such that  $w = s_{\alpha_1} \cdots s_{\alpha_N}$  and  $\ell_{\Theta}(w) = \#\{j \leq N \mid \alpha_j \in \Theta\}$ . Let also  $(\alpha_{N+1}, \dots, \alpha_M) \in (\Delta \setminus \Theta)^{M-N}$  be such that  $x = s_{\alpha_{N+1}} \cdots s_{\alpha_M}$ . With this, one has  $wx = s_{\alpha_1} \cdots s_{\alpha_M}$  and  $\#\{j \leq M \mid \alpha_j \in \Theta\} = \ell_{\Theta}(w)$  and thus  $\ell_{\Theta}(wx) \leq \ell_{\Theta}(w)$ .

Applying the above bound to the pair  $(wx, x^{-1})$  replacing  $(w, x)$ , we obtain  $\ell_{\Theta}((wx)x^{-1}) \leq \ell_{\Theta}(wx)$  and hence the sought for equality since  $(wx)x^{-1} = w$ .  $\square$

A direct reformulation is

**Corollary 4.13.** *The  $\Theta$ -length factors through the quotient*

$$W_{\Delta \setminus \Theta} \backslash W / W_{\Delta \setminus \Theta}$$

of  $W$  by the left-right action of  $W_{\Delta \setminus \Theta}$ .

**Lemma 4.14.** *Suppose that  $G$  admits a  $\Theta$ -positive structure. The minimum in Equation (4.5) is then achieved on reduced expressions; more precisely, for every  $w \in W$  and for every  $(\alpha_1, \dots, \alpha_{\ell(w)}) \in \Delta^{\ell(w)}$  such that  $w = s_{\alpha_1} \cdots s_{\alpha_{\ell(w)}}$ , one has*

$$\ell_{\Theta}(w) = \#\{j \leq \ell(w) \mid \alpha_j \in \Theta\}.$$

*Proof.* For this proof we will denote, for every  $N \in \mathbf{N}$  and every  $(\alpha_1, \dots, \alpha_N) \in \Delta^N$ , by  $\hat{\ell}_{\Theta}(\alpha_1, \dots, \alpha_N)$  the number of occurrences of elements in  $\Theta$  in this finite sequence:

$$\hat{\ell}_{\Theta}(\alpha_1, \dots, \alpha_N) = \#\{j \leq N \mid \alpha_j \in \Theta\}.$$

This function is additive:

$$\hat{\ell}_{\Theta}(\alpha_1, \dots, \alpha_N, \alpha_{N+1}, \dots, \alpha_M) = \hat{\ell}_{\Theta}(\alpha_1, \dots, \alpha_N) + \hat{\ell}_{\Theta}(\alpha_{N+1}, \dots, \alpha_M).$$

For an element  $w$  of  $W$ , one has, by definition, that  $\ell_{\Theta}(w)$  is the infimum of the  $\hat{\ell}_{\Theta}(\alpha_1, \dots, \alpha_N)$  for  $(\alpha_1, \dots, \alpha_N)$  varying among the expressions of  $w$ :  $w = s_{\alpha_1} \cdots s_{\alpha_N}$ .

If  $N' \leq N$  and  $(\alpha'_1, \dots, \alpha'_{N'})$  is obtained from  $(\alpha_1, \dots, \alpha_N)$  by removing some of the entries, then  $\hat{\ell}_{\Theta}(\alpha'_1, \dots, \alpha'_{N'}) \leq \hat{\ell}_{\Theta}(\alpha_1, \dots, \alpha_N)$ .

If  $(\alpha_1, \dots, \alpha_N)$  and  $(\beta_1, \dots, \beta_N)$  differ by a braid relation then

$$\hat{\ell}_{\Theta}(\alpha_1, \dots, \alpha_N) = \hat{\ell}_{\Theta}(\beta_1, \dots, \beta_N).$$

Indeed, by the additivity of the function  $\hat{\ell}_{\Theta}$  on expressions, one needs to check this equality only in the case when  $N = m_{\alpha, \alpha'}$  is the order of  $s_{\alpha} s_{\alpha'}$  ( $\alpha, \alpha' \in \Delta$ ) and one has  $(\alpha_j, \beta_j) = (\alpha, \alpha')$  for every odd  $j \leq N$  and  $(\alpha_j, \beta_j) = (\alpha', \alpha)$  for every even  $j \leq N$ . When  $\alpha$  and  $\alpha'$  both belong to  $\Theta$ , we have  $\hat{\ell}_{\Theta}(\alpha_1, \dots, \alpha_N) = N$  and  $\hat{\ell}_{\Theta}(\beta_1, \dots, \beta_N) = N$ ; When  $\alpha$  and  $\alpha'$  both belong to  $\Delta \setminus \Theta$ , we have  $\hat{\ell}_{\Theta}(\alpha_1, \dots, \alpha_N) = 0$  and  $\hat{\ell}_{\Theta}(\beta_1, \dots, \beta_N) = 0$ . The last case to consider (up to exchanging the 2

sequences) is when  $\alpha$  belongs to  $\Theta$  and  $\alpha'$  belongs to  $\Delta \setminus \Theta$ ; in this case, we know that  $N = m_{\alpha, \alpha'}$  is even (cf. Theorem 3.4) and we have  $\hat{\ell}_\Theta(\alpha_1, \dots, \alpha_N) = N/2$  and  $\hat{\ell}_\Theta(\beta_1, \dots, \beta_N) = N/2$ . In every case, the equality  $\hat{\ell}_\Theta(\alpha_1, \dots, \alpha_N) = \hat{\ell}_\Theta(\beta_1, \dots, \beta_N)$  holds.

The conclusions of the lemma follow from these remarks since:

- from every expression of  $w$  one can deduce a reduced expression by applying successively a finite number of braid relations or removing subexpressions of the form  $(\alpha, \alpha)$  ( $\alpha \in \Delta$ ), and
- two reduced expressions of  $w$  can be obtained from one another by applying successively a finite number of braid relations.  $\square$

The  $\Theta$ -length gives a simple characterization of  $W_{\Delta \setminus \Theta}$ .

**Lemma 4.15.** *The subgroup  $W_{\Delta \setminus \Theta}$  is the set of elements of zero  $\Theta$ -length:*

$$W_{\Delta \setminus \Theta} = \{x \in W \mid \ell_\Theta(x) = 0\}.$$

*Proof.* Indeed, for any  $x$  in  $W$ , the following are equivalent:

- $\ell_\Theta(x) = 0$
- there exist  $N \in \mathbf{N}$  and  $\alpha_1, \dots, \alpha_N$  in  $\Delta$  such that  $x = s_{\alpha_1} \cdots s_{\alpha_N}$  and  $\{i \in \{1, \dots, N\} \mid \alpha_i \in \Theta\} = \emptyset$ ,
- there exist  $N \in \mathbf{N}$  and  $\alpha_1, \dots, \alpha_N$  in  $\Delta \setminus \Theta$  such that  $x = s_{\alpha_1} \cdots s_{\alpha_N}$ ,
- $x$  belongs to  $W_{\Delta \setminus \Theta}$ .  $\square$

#### 4.5. Normalizer.

**Proposition 4.16.** *The group  $W(\Theta)$  normalizes  $W_{\Delta \setminus \Theta}$ :*

$$W(\Theta) \subset N_W(W_{\Delta \setminus \Theta}) = \{x \in W \mid xW_{\Delta \setminus \Theta}x^{-1} = W_{\Delta \setminus \Theta}\}.$$

*Proof.* Since  $N = N_W(W_{\Delta \setminus \Theta})$  is a subgroup and since  $W(\Theta)$  is generated by  $R(\Theta)$ , it is enough to prove that  $R(\Theta)$  is contained in  $N$ .

Let  $\alpha$  be in  $\Theta$ . If  $\alpha \neq \alpha_\Theta$ , it means that there are no arrows in the Dynkin diagram between  $\alpha$  and  $\Delta \setminus \Theta$ ; thus  $\sigma_\alpha = s_\alpha$  commutes with  $s_\beta$  for every  $\beta$  in  $\Delta \setminus \Theta$ , this means that  $\sigma_\alpha$  centralizes  $W_{\Delta \setminus \Theta}$  and in particular  $\sigma_\alpha$  belongs to  $N$ .

If  $\alpha = \alpha_\Theta$ , then  $\sigma_\alpha = w_{\{\alpha_\Theta\} \cup \Delta \setminus \Theta} w_{\Delta \setminus \Theta}$  (cf. Corollary 4.2). It is therefore enough to notice that  $w_{\{\alpha_\Theta\} \cup \Delta \setminus \Theta}$  belongs to  $N$ . However we know that in this situation the element  $w_{\{\alpha_\Theta\} \cup \Delta \setminus \Theta}$  is in the center of  $W_{\{\alpha_\Theta\} \cup \Delta \setminus \Theta}$ , in particular it centralizes  $W_{\Delta \setminus \Theta}$  and belongs to  $N$ .  $\square$

### 5. THE SPLIT GROUP OF TYPE $W(\Theta)$

This section continues the investigation of the special algebraic properties arising from a  $\Theta$ -positive structure. Here show the existence of a real split Lie subalgebra on  $\mathfrak{g}$ . Besides being remarkable, this subalgebra will play a crucial role in some arguments (Section 8.7).

In Section 3.3 we constructed a  $\Theta$ -base  $(E_\alpha, F_\alpha, D_\alpha)_{\alpha \in \Theta}$ , with  $E_\alpha$  belonging to  $\mathring{c}_\alpha$  and  $F_\alpha$  belonging to  $\mathring{c}_\alpha^{\text{opp}}$  for  $\alpha \in \Theta$ . The aim of this section is to prove that the Lie subalgebra  $\mathfrak{g}_\Theta$  generated by elements of the  $\Theta$ -base  $(E_\alpha, F_\alpha, D_\alpha)_{\alpha \in \Theta}$  is a split real Lie algebra of type  $W(\Theta)$ . More precisely one has

**Theorem 5.1.** *Let  $V = (V_\alpha)_{\alpha \in \Theta}$  belong to  $\prod_{\alpha \in \Theta} (\mathring{c}_\alpha \cup -\mathring{c}_\alpha)$ . Then there exists a maximal compact subgroup  $H$  of  $G$  such that  $H \cap L_\Theta^\circ$  is the stabilizer of  $V$  in  $L_\Theta^\circ$ .*

*For every such  $H$ , let  $\sigma: \mathfrak{g} \rightarrow \mathfrak{g}$  be the associated Cartan involution. Then the Lie algebra generated by  $\{V_\alpha, \sigma(V_\alpha)\}_{\alpha \in \Theta}$  is isomorphic to the real split Lie algebra of type  $W(\Theta)$ .*

*Proof.* Changing one component  $V_\alpha$  of  $V$  to  $-V_\alpha$  does not affect the hypothesis nor the conclusion; hence we can as well assume that  $V$  belongs to  $\prod_{\alpha \in \Theta} \mathring{c}_\alpha$ .

Note that the conclusion of the theorem is plainly satisfied in the split case for which one has  $W(\Theta) = W$  and  $V_\alpha = E_\alpha$  for every  $\alpha$  in  $\Delta$  (in this case, one chooses  $H$  to be the maximal compact subgroup  $K$  of Section 2 so that  $\sigma$  is the Cartan involution fixed in Section 2.1). The Lie algebra generated by  $\{V_\alpha, \sigma(V_\alpha)\}_{\alpha \in \Delta}$  is then equal to  $\mathfrak{g}$ .

The case when  $\sharp\Theta = 1$  (Hermitian groups) is also easy to deal with.

By the transitivity of the action of  $L_\Theta^\circ$  on  $\prod_{\alpha \in \Theta} \mathring{c}_\alpha$  (Proposition 3.7) it is enough to prove the result for one specific element in this product of cones.

We can thus assume that, for every  $\alpha$  in  $\Theta$ ,  $V_\alpha = E_\alpha$  (cf. Definition 3.16). We already observed (Theorem 3.13) that the Lie algebra of the stabilizer of  $E = (E_\alpha)_{\alpha \in \Theta}$  contains the Lie algebra of  $K \cap L_\Theta^\circ$ ; therefore (cf. Proposition 3.5) the stabilizer of  $E$  in  $L_\Theta^\circ$  is equal to  $K \cap L_\Theta^\circ$ . The Cartan involution associated with the chosen compact subgroup  $K$  of Section 2 is denoted by  $\tau$ . As above (Definition 3.16) we set  $F_\alpha = \tau(E_\alpha)$  and  $D_\alpha = [E_\alpha, F_\alpha]$  ( $\alpha \in \Theta$ ). We will prove the conclusion with  $H = K$  and  $\sigma = \tau$ .

Let also  $Z_0, \dots, Z_d$  (where  $d = \sharp\Delta \setminus \Theta$ ) be the elements given in Section 3.3 and Theorem 3.13 so that  $E_{\alpha_\Theta} = Z_0 + \dots + Z_d$  belongs to the cone  $\mathring{c}_{\alpha_\Theta}$ , and for each  $k = 0, \dots, d$ , setting  $Y_k = \tau(Z_k)$  and  $D_k = [Z_k, Y_k]$ , the triple  $(Z_k, Y_k, D_k)$  is an  $\mathfrak{sl}_2$ -triple. We observe that, since the lifts  $\mathring{s}_k$  belong to  $K$  and since  $Z_k = \text{Ad}(\mathring{s}_k)Z_{k-1}$ , the relations  $Y_k = \text{Ad}(\mathring{s}_k)Y_{k-1}$  and  $D_k = \text{Ad}(\mathring{s}_k)D_{k-1}$  hold.

For the rest of this proof we will write  $\Theta = \{\alpha_1, \dots, \alpha_p\}$  (with  $p$  the cardinality of  $\Theta$ ) with  $\alpha_p = \alpha_\Theta$  and, for all  $i = 1, \dots, p-1$ ,  $\alpha_i$  and  $\alpha_{i+1}$  are connected in the Dynkin diagram.

In order to deduce that the Lie algebra generated by  $\{E_\alpha, F_\alpha\}_{\alpha \in \Theta}$  is of type  $W(\Theta)$ , it then suffices to prove that the family

$$E_\alpha, F_\alpha, D_\alpha, \quad (\alpha \in \Theta)$$

satisfies the Serre's relations. This means here that the following identities must be satisfied (cf. [Bou06, Ch. VIII, § 4, n° 3, Théorème 1], note that we are using here a different normalization for  $\mathfrak{sl}_2$ -triples)

$$(5.1) \quad [D_{\alpha_i}, E_{\alpha_i}] = 2E_{\alpha_i}, [D_{\alpha_i}, F_{\alpha_i}] = -2F_{\alpha_i}, \text{ and} \\ [E_{\alpha_i}, F_{\alpha_i}] = D_{\alpha_i}, \forall i \in \{1, \dots, p\}$$

$$(5.2) \quad [E_{\alpha_i}, F_{\alpha_j}] = 0, \forall i \neq j \in \{1, \dots, p\}$$

$$(5.3) \quad [E_{\alpha_i}, E_{\alpha_j}] = 0, [F_{\alpha_i}, F_{\alpha_j}] = 0, [D_{\alpha_i}, E_{\alpha_j}] = 0, \text{ and} \\ [D_{\alpha_i}, F_{\alpha_j}] = 0, \forall i, j \in \{1, \dots, p\} \text{ with } |i - j| > 1$$

$$(5.4) \quad (\text{ad } E_{\alpha_i})^2 E_{\alpha_j} = 0, (\text{ad } F_{\alpha_i})^2 F_{\alpha_j} = 0, [D_{\alpha_i}, E_{\alpha_j}] = -E_{\alpha_j}, \text{ and} \\ [D_{\alpha_i}, F_{\alpha_j}] = F_{\alpha_j}, \forall i \in \{1, \dots, p-1\} \text{ and } j \in \{i \pm 1\}$$

$$(5.5) \quad [D_{\alpha_p}, E_{\alpha_{p-1}}] = -(d+1)E_{\alpha_{p-1}}, [D_{\alpha_p}, F_{\alpha_{p-1}}] = (d+1)F_{\alpha_{p-1}}$$

$$(5.6) \quad (\text{ad } E_{\alpha_p})^{d+2} E_{\alpha_{p-1}} = 0, (\text{ad } F_{\alpha_p})^{d+2} F_{\alpha_{p-1}} = 0.$$

Equations (5.1) are the fact that the triple  $(E_{\alpha_i}, F_{\alpha_i}, D_{\alpha_i})$  is an  $\mathfrak{sl}_2$ -triple. Equations (5.2) are inherited from the corresponding identities in  $\mathfrak{g}$ , only the case when  $i$  or  $j$  is equal to  $p$  needs a comment. Let's consider, for example,  $i = p$ ; one has  $E_{\alpha_p} = E_{\alpha_\Theta} = Z_0 + Z_1 + \dots + Z_d$ , and, for all  $k$  in  $\{1, \dots, d\}$ ,  $Z_k = \text{Ad}(\dot{s}_k)Z_{k-1}$ . Since, from the known identities in  $\mathfrak{g}$ ,  $[Z_0, F_{\alpha_j}] = 0$  and since  $\text{Ad}(\dot{s}_k)F_{\alpha_j} = F_{\alpha_j}$  for all  $k = 1, \dots, d$ , one gets that, for all  $k$ ,  $[Z_k, F_{\alpha_j}] = \text{Ad}(\dot{s}_k)[Z_{k-1}, F_{\alpha_j}] = 0$  and also  $[E_{\alpha_p}, F_{\alpha_j}] = \sum_{k=0}^d [Z_k, F_{\alpha_j}] = 0$ .

Equations (5.3) are also inherited from the corresponding identities in  $\mathfrak{g}$  (with again a special treatment when one of the indices is equal to  $p$ ). Equations (5.4) equally follow from the known equalities in  $\mathfrak{g}$ .

We now prove Equations (5.5). Note first that  $D_{\alpha_p} = D_0 + D_1 + \dots + D_d$ . One has, again from identities valid in  $\mathfrak{g}$ ,  $[D_0, E_{\alpha_{p-1}}] = -E_{\alpha_{p-1}}$ . From the recursive relation  $\text{Ad}(\dot{s}_k)D_{k-1} = D_k$  we deduce that  $[D_k, E_{\alpha_{p-1}}] = -E_{\alpha_{p-1}}$  and, summing over  $k$ ,  $[D_{\alpha_p}, E_{\alpha_{p-1}}] = -(d+1)E_{\alpha_{p-1}}$ . The identity with  $F_{\alpha_{p-1}}$  follows by a similar argument (or by applying  $\tau$ ).

Let us now address Equations (5.6). We will prove the seemingly stronger identity  $(\text{ad } E_{\alpha_p})^{d+2} = 0$ . Since  $\text{ad } E_{\alpha_p} = \sum_{k=0}^d \text{ad } Z_k$ , one has

$$(\text{ad } E_{\alpha_p})^{d+2} = \sum_{k_0, \dots, k_{d+1}} \text{ad } Z_{k_0} \text{ad } Z_{k_1} \cdots \text{ad } Z_{k_{d+1}},$$

where each index  $k_m$  runs into  $\{0, \dots, d\}$ . Every term in this last sum is zero: indeed the order in a product  $\text{ad } Z_{k_0} \text{ad } Z_{k_1} \cdots \text{ad } Z_{k_{d+1}}$  does not matter (see Lemma 3.14) and, since  $d+2 > d+1$ , at least two indices coincide  $k_m = k_\ell$  so that  $\text{ad } Z_{k_0} \text{ad } Z_{k_1} \cdots \text{ad } Z_{k_{d+1}} = 0$  since  $(\text{ad } Z_{k_\ell})^2 = 0$ . The equality  $(\text{ad } F_{\alpha_p})^{d+2} = 0$  is a consequence of similar arguments.  $\square$

## 6. THE NONNEGATIVE AND POSITIVE SEMIGROUPS

Let  $G$  be a simple real Lie group which admits a  $\Theta$ -positive structure. In this section we introduce the nonnegative semigroup  $U_{\Theta}^{\geq 0}$  and the positive semigroup  $U_{\Theta}^{> 0}$  in the unipotent radical  $U_{\Theta}$  of the parabolic subgroup determined by  $\Theta$ .

For every  $\alpha \in \Theta$  we fix an  $L_{\Theta}^{\circ}$ -invariant closed cone, denoted by  $c_{\alpha} \subset \mathfrak{u}_{\alpha}$ , and denote by  $\mathring{c}_{\alpha}$  its interior. For  $\alpha \in \Theta$ , the cone  $\tau(c_{\alpha})$  contained in  $\mathfrak{u}_{-\alpha}$  ( $\tau$  is the Cartan involution) will be denoted by  $c_{\alpha}^{\text{opp}}$ .

### 6.1. The nonnegative semigroup.

**Definition 6.1.** The *unipotent nonnegative semigroup*  $U_{\Theta}^{\geq 0} \subset U_{\Theta}$  is the subsemigroup of  $U_{\Theta}$  generated by  $\exp(c_{\alpha})$ ,  $\alpha \in \Theta$ . The *opposite nonnegative semigroup*  $U_{\Theta}^{\text{opp}, \geq 0} \subset U_{\Theta}^{\text{opp}}$  is the subsemigroup generated by  $\exp(c_{\alpha}^{\text{opp}})$ ,  $\alpha \in \Theta$ .

The unipotent nonnegative semigroups give rise to a semigroup in  $G$ .

**Definition 6.2.** The *nonnegative semigroup*  $G_{\Theta}^{\geq 0} \subset G$  is defined to be the subsemigroup generated by  $U_{\Theta}^{\geq 0}$ ,  $U_{\Theta}^{\text{opp}, \geq 0}$ , and  $L_{\Theta}^{\circ}$ .

*Examples 6.3.* (1) When  $G$  is a split real form, and  $\Theta = \Delta$ , then  $U_{\Theta}^{\geq 0}$  is the nonnegative unipotent semigroup defined by Lusztig in [Lus94], and  $G_{\Theta}^{\geq 0} = G^{\geq 0}$  is the semigroup of totally nonnegative elements defined by Lusztig [Lus94, § 2.2].

(2) Let  $G$  be a Hermitian Lie group of tube type and  $\Theta = \{\alpha_r\}$ . Then  $U_{\Theta}^{\geq 0} = \exp(c_{\alpha}) < U_{\Theta}$ , and  $G_{\Theta}^{\geq 0}$  is the contraction semigroup  $G^{\succ 0} \subset G$  [Kou95, Th. 4.9].

In particular note that when  $G$  is the real symplectic group, the unique (up to isogeny) group which is split and of Hermitian type, the nonnegative semigroups  $G_{\Delta}^{\geq 0}$  and  $G_{\{\alpha_r\}}^{\geq 0}$  are different.

(3) Also in the case when  $G$  is the split real form of  $F_4$  we have two different semigroups, the nonnegative semigroup  $G_{\Theta}^{\geq 0}$ , where  $\Theta = \{\alpha_3, \alpha_4\}$  and Lusztig's nonnegative semigroup  $G_{\Delta}^{\geq 0}$ .

**6.2. The positive semigroup.** To introduce the positive unipotent semigroup we will follow the strategy of Lusztig in [Lus94]. More precisely, we will give an explicit parametrization of a subset of  $U_{\Theta}$ , which a priori depends on the choice of a reduced expression of the longest element in the  $\Theta$ -Weyl group  $W(\Theta)$ . In a second step we will then show that this subset is independent of this choice and is in fact a semigroup.

Consider the element  $w_{\max}^{\Theta} \in W$ . In Section 4 we saw that  $W(\Theta)$  equipped with its generating system  $R(\Theta)$  (cf. Definition 4.5) is a Coxeter system of Lie type (cf. Proposition 4.7), and that  $w_{\max}^{\Theta}$  is the longest length element in  $W(\Theta)$ .

We denote by  $N$  the length of  $w_{\max}^{\Theta}$  and by  $\mathbf{W} \subset \Theta^N$  the set of tuples  $\gamma = (\gamma_1, \dots, \gamma_N)$  such that  $w_{\max}^{\Theta} = \sigma_{\gamma_1} \cdots \sigma_{\gamma_N}$ , i.e.  $(\sigma_{\gamma_1}, \dots, \sigma_{\gamma_N})$  is a reduced expression of  $w_{\max}^{\Theta}$ .

For every  $\gamma$  in  $\mathbf{W}$ , the product of cones  $c_{\gamma_1} \times \cdots \times c_{\gamma_N}$  is denoted  $c_\gamma$ . We define the map

$$\begin{aligned} F_\gamma: c_\gamma &\longrightarrow U_\Theta \\ (v_1, \dots, v_N) &\longmapsto \exp(v_1) \cdots \exp(v_N). \end{aligned}$$

We first note that those maps are compatible with the adjoint and conjugation actions of  $L_\Theta^\circ$ :

**Lemma 6.4.** *For every  $\gamma$  in  $\mathbf{W}$  the map  $F_\gamma$  is  $L_\Theta^\circ$ -equivariant.*

*Proof.* Let  $\ell$  be in  $L_\Theta^\circ$  and let  $\mathbf{v} = (v_1, \dots, v_N)$  be in  $c_\gamma$ . The action of  $\ell$  on  $\mathbf{v}$  is

$$\ell \cdot \mathbf{v} = (\text{Ad}(\ell)v_1, \dots, \text{Ad}(\ell)v_N)$$

and is again in  $c_\gamma$  since the cones  $c_\alpha$  are  $L_\Theta^\circ$ -invariant. The compatibility of the exponential map with the adjoint and conjugation actions implies

$$\begin{aligned} F_\gamma(\ell \cdot \mathbf{v}) &= \exp(\text{Ad}(\ell)v_1) \cdots \exp(\text{Ad}(\ell)v_N) \\ &= \ell \exp(v_1) \ell^{-1} \cdots \ell \exp(v_N) \ell^{-1} \\ &= \ell \exp(v_1) \cdots \exp(v_N) \ell^{-1} = \ell F_\gamma(\mathbf{v}) \ell^{-1}, \end{aligned}$$

which is the wanted equivariance property.  $\square$

The interior  $\mathring{c}_\gamma$  is the product  $\mathring{c}_{\gamma_1} \times \cdots \times \mathring{c}_{\gamma_N}$  and the restriction of  $F_\gamma$  to  $\mathring{c}_\gamma$  is denoted by  $\mathring{F}_\gamma$ . Similar notation,  $c_\gamma^{\text{opp}}$ ,  $F_\gamma^{\text{opp}}$ , and  $\mathring{F}_\gamma^{\text{opp}}$  are adopted for the maps into the opposite unipotent group.

We will prove in Section 8 that the image of  $\mathring{F}_\gamma$  is independent of  $\gamma$  and that it is a semigroup (similar statements hold for the maps  $\mathring{F}_\gamma^{\text{opp}}$ ).

This then allows us to make the following definition

**Definition 6.5.** The common image of the maps  $\mathring{F}_\gamma$  (for  $\gamma$  in  $\mathbf{W}$ ) is the *unipotent positive semigroup*  $U_\Theta^{>0}$  of  $U_\Theta$ . The *opposite unipotent positive semigroup*  $U_\Theta^{\text{opp}, >0}$  in  $U_\Theta^{\text{opp}}$  is the common image of the maps  $\mathring{F}_\gamma^{\text{opp}}$ .

The unipotent positive semigroups  $U_\Theta^{>0}$ ,  $U_\Theta^{\text{opp}, >0}$  give rise to a positive semigroup in  $G$ .

**Definition 6.6.** The *positive semigroup*  $G_\Theta^{>0} \subset G$  is defined to be the subsemigroup generated by  $U_\Theta^{>0}$ ,  $U_\Theta^{\text{opp}, >0}$ , and  $L_\Theta^\circ$ .

## 7. BRUHAT DECOMPOSITION AND CONES

The first properties of the maps  $\mathring{F}_\gamma$  that will be obtained in Section 8 are injectivity and transversality. In order to achieve this, we will need a precise control of the Bruhat cells containing the elements  $\exp(v)$  when  $\alpha \in \Theta$  and  $v \in \mathring{c}_\alpha$ .

**7.1. Bruhat decomposition.** We recall here that  $G$  can be decomposed under the left-right multiplication by  $P^{\text{opp}} \times P^{\text{opp}}$ . These orbits are sometimes called the Bruhat cells and are indexed by elements of  $W$ .

Explicitely, for every  $w$  in  $W \simeq N_K(\mathfrak{a})/C_K(\mathfrak{a})$  (the quotient of the normalizer of  $\mathfrak{a}$  in  $K$  by its centralizer), let  $\dot{w}$  in  $N_K(\mathfrak{a})$  be a representative of  $w$ , the subset

$$P^{\text{opp}}\dot{w}P^{\text{opp}}$$

depends only of  $w$  (and not on the choice of  $\dot{w}$ ) since  $P^{\text{opp}}$  contains the centralizer  $C_K(\mathfrak{a})$  and is called the *Bruhat cell indexed by  $w$* ; the following equalities hold

$$P^{\text{opp}}\dot{w}P^{\text{opp}} = U^{\text{opp}}\dot{w}P^{\text{opp}} = P^{\text{opp}}\dot{w}U^{\text{opp}}$$

and we will (abusively) remove the dots in the further notation.

It is well known that:

- (1)  $G$  is the disjoint union of the  $P^{\text{opp}}wP^{\text{opp}}$  for  $w$  varying in  $W$ .
- (2)  $P_{\Theta}^{\text{opp}}$  is the disjoint union of the  $P^{\text{opp}}wP^{\text{opp}}$  for  $w$  varying in  $W_{\Delta \setminus \Theta}$ .
- (3) For all  $w_1$  and  $w_2$  in  $W$  such that  $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$ , one has

$$(P^{\text{opp}}w_1P^{\text{opp}})(P^{\text{opp}}w_2P^{\text{opp}}) = P^{\text{opp}}w_1P^{\text{opp}}w_2P^{\text{opp}} = P^{\text{opp}}w_1w_2P^{\text{opp}}.$$

- (4) For all  $\alpha$  in  $\Delta$  and for all  $X$  in  $\mathfrak{g}_{\alpha}$ , if  $X \neq 0$  then

$$\exp(X) \in P^{\text{opp}}s_{\alpha}P^{\text{opp}}.$$

Observe also that  $\dot{s}_{\alpha}^{-1}$  also belongs to  $P^{\text{opp}}s_{\alpha}P^{\text{opp}}$  (simply because  $\dot{s}_{\alpha}^{-1}$  is a representative of the class of  $s_{\alpha}$ ).

- (5) For every  $\alpha$  in  $\Delta$ ,

$$P^{\text{opp}}s_{\alpha}P^{\text{opp}}s_{\alpha}P^{\text{opp}} = P^{\text{opp}}\bigsqcup P^{\text{opp}}s_{\alpha}P^{\text{opp}}.$$

- (6) For every  $w$  in  $W$ , the  $P^{\text{opp}} \times P_{\Theta}^{\text{opp}}$ -orbit

$$P^{\text{opp}}wP_{\Theta}^{\text{opp}}$$

depends only on the class  $[w]$  of  $w$  in  $W/W_{\Delta \setminus \Theta}$  and will sometimes be denoted  $P^{\text{opp}}[w]P_{\Theta}^{\text{opp}}$ .

- (7) The group  $G$  is the disjoint union

$$G = \bigsqcup_{x \in W/W_{\Delta \setminus \Theta}} P^{\text{opp}}xP_{\Theta}^{\text{opp}}.$$

- (8) Similar notation will be adopted for the action of  $P_{\Theta}^{\text{opp}} \times P^{\text{opp}}$  and of  $P_{\Theta}^{\text{opp}} \times P_{\Theta}^{\text{opp}}$ : a double orbit  $P_{\Theta}^{\text{opp}}wP_{\Theta}^{\text{opp}}$  depends only on the class  $[w]$  of  $w$  in  $W_{\Delta \setminus \Theta} \setminus W/W_{\Delta \setminus \Theta}$  and

$$G = \bigsqcup_{[w] \in W_{\Delta \setminus \Theta} \setminus W/W_{\Delta \setminus \Theta}} P_{\Theta}^{\text{opp}}wP_{\Theta}^{\text{opp}}.$$

Application of (3) and (5) above gives that, for every  $w_1$  and  $w_2$  in  $W$  and for every  $g$  in  $P^{\text{opp}}w_1P^{\text{opp}}w_2P^{\text{opp}}$ , there exist a prefix  $x_1$  of  $w_1$  and a suffix  $x_2$  of  $w_2$  such that  $g$  belongs to  $P^{\text{opp}}x_1x_2P^{\text{opp}}$ . In particular (cf. Lemma 4.14) the  $\Theta$ -length of  $x = x_1x_2$  is less or equal to  $\ell_\Theta(w_1) + \ell_\Theta(w_2)$ . Using the fact that  $W_{\Delta \setminus \Theta}$  consists of elements of zero  $\Theta$ -length together with point (2) above we deduce:

**Lemma 7.1.** *Let  $w_1$ ,  $w_2$ , and  $x$  be in  $W$  such that  $P_\Theta^{\text{opp}}xP_\Theta^{\text{opp}} \subset P_\Theta^{\text{opp}}w_1P_\Theta^{\text{opp}}w_2P_\Theta^{\text{opp}}$ . Then*

$$\ell_\Theta(x) \leq \ell_\Theta(w_1) + \ell_\Theta(w_2).$$

The following lemma states the equality between some of the orbits for the two groups  $P^{\text{opp}}$  and  $P_\Theta^{\text{opp}}$ .

**Lemma 7.2.** *For every  $w$  in  $W(\Theta)$ , one has*

$$P_\Theta^{\text{opp}}wP_\Theta^{\text{opp}} = P^{\text{opp}}wP_\Theta^{\text{opp}}.$$

*Proof.* Since  $P_\Theta^{\text{opp}} = \bigsqcup_{x \in W_{\Delta \setminus \Theta}} P^{\text{opp}}xP^{\text{opp}}$ , one has

$$P_\Theta^{\text{opp}}wP_\Theta^{\text{opp}} = \bigcup_{x \in W_{\Delta \setminus \Theta}} P^{\text{opp}}xP^{\text{opp}}wP_\Theta^{\text{opp}}.$$

It is thus enough to prove the equality, for every  $x$  in  $W_{\Delta \setminus \Theta}$

$$P^{\text{opp}}xP^{\text{opp}}wP_\Theta^{\text{opp}} = P^{\text{opp}}wP_\Theta^{\text{opp}}.$$

Since  $x$  belongs to  $W_{\Delta \setminus \Theta}$ , it is a suffix of the longest element  $w_{\Delta \setminus \Theta}$ . Similarly  $w$  is a prefix of  $w_{\max}^\Theta$ . We deduce from this that  $xw$  is a subword of  $w_{\Delta \setminus \Theta}w_{\max}^\Theta = w_\Delta$  and in particular  $\ell(xw) = \ell(x) + \ell(w)$ . By property (3) above, one has

$$\begin{aligned} P^{\text{opp}}xP^{\text{opp}}wP_\Theta^{\text{opp}} &= P^{\text{opp}}xwP_\Theta^{\text{opp}} \\ &= P^{\text{opp}}w(w^{-1}xw)P_\Theta^{\text{opp}} \end{aligned}$$

and, as  $w^{-1}xw$  belongs to  $W_{\Delta \setminus \Theta}$  (Proposition 4.16)

$$= P^{\text{opp}}wP_\Theta^{\text{opp}}$$

which is the sought for equality.  $\square$

**7.2. Dimensions.** We give here a recursive information on the dimensions of the Bruhat cells or, what amounts to the same, the dimensions of their images in the flag variety  $F_{\iota(\Theta)} = F_\Theta$  (recall that the involution  $\iota$  is always the identity when  $\Theta \neq \Delta$ ). A complete calculation of the dimensions is possible but will not be performed here.

For every  $w$  in  $W$  we will denote  $C(w)$  the  $P_\Theta^{\text{opp}}$ -orbit of  $\mathfrak{p}_\Theta = w \cdot \mathfrak{p}_\Theta^{\text{opp}}$  in  $F_\Theta$ ;  $C(w)$  is isomorphic to the quotient  $P_\Theta^{\text{opp}}wP_\Theta^{\text{opp}}/P_\Theta^{\text{opp}}$ . It is also the image of the following map

$$\begin{aligned} \mathfrak{u}_\Theta &\longrightarrow F_\Theta \\ X &\longmapsto \exp(X)w \cdot \mathfrak{p}_\Theta^{\text{opp}}. \end{aligned}$$

Recall that  $\mathfrak{u}_\Theta^{\text{opp}}$  has the following decomposition

$$\mathfrak{u}_\Theta^{\text{opp}} = \bigoplus_{\alpha \in \Sigma_\Theta^+} \mathfrak{g}_{-\alpha}.$$

For  $w \in W$  we denote

$$\Sigma_{w\prec}^+ = \Sigma^+ \cap w \cdot \Sigma^-, \quad \Sigma_{w\succ}^+ = \Sigma^+ \cap w \cdot \Sigma^+,$$

so that  $\Sigma^+ = \Sigma_{w\prec}^+ \sqcup \Sigma_{w\succ}^+$ ,

$$\Sigma_{\Theta, w\prec}^+ = \Sigma_{w\prec}^+ \cap \Sigma_\Theta^+, \quad \Sigma_{\Theta, w\succ}^+ = \Sigma_{w\succ}^+ \cap \Sigma_\Theta^+,$$

so that  $\Sigma_\Theta^+ = \Sigma_{\Theta, w\prec}^+ \sqcup \Sigma_{\Theta, w\succ}^+$ , and at the level of Lie algebras

$$\mathfrak{u}_{\Theta, w\prec}^{\text{opp}} = \bigoplus_{\alpha \in \Sigma_{\Theta, w\prec}^+} \mathfrak{g}_{-\alpha}, \quad \mathfrak{u}_{\Theta, w\succ}^{\text{opp}} = \bigoplus_{\alpha \in \Sigma_{\Theta, w\succ}^+} \mathfrak{g}_{-\alpha},$$

so that  $\mathfrak{u}_\Theta^{\text{opp}} = \mathfrak{u}_{\Theta, w\prec}^{\text{opp}} \oplus \mathfrak{u}_{\Theta, w\succ}^{\text{opp}}$  and, for all  $X$  in  $\mathfrak{u}_{\Theta, w\succ}^{\text{opp}}$ ,  $\exp(X)w$  belongs to  $wP_\Theta^{\text{opp}}$ .

It is well known that the length of  $w$  is the cardinality of  $\Sigma_{w\prec}^+$  and the following equalities hold

$$(7.1) \quad \begin{aligned} \Sigma_{s_\alpha w\prec}^+ &= \{\alpha\} \sqcup s_\alpha \cdot \Sigma_{w\prec}^+ \text{ if } \ell(s_\alpha w) = 1 + \ell(w) \\ \Sigma_{xw\prec}^+ &= \Sigma_{x\prec}^+ \sqcup x^{-1} \cdot \Sigma_{w\prec}^+ \text{ if } \ell(xw) = \ell(x) + \ell(w) \end{aligned}$$

**Lemma 7.3.** *Let  $w$  be in  $W$ . The map*

$$\begin{aligned} f_w : \mathfrak{u}_{\Theta, w\prec}^{\text{opp}} &\longrightarrow C(w) \\ X &\longmapsto \exp(X)w \cdot P_\Theta^{\text{opp}} \end{aligned}$$

*is a diffeomorphism.*

*Proof.* Clearly this map is  $C^\infty$ . It also results from classical facts about nilpotent Lie algebras that

$$\begin{aligned} \mathfrak{u}_{\Theta, w\prec}^{\text{opp}} \times \mathfrak{u}_{\Theta, w\succ}^{\text{opp}} &\longrightarrow U_\Theta \\ (X, Y) &\longmapsto \exp(X) \exp(Y) \end{aligned}$$

is a diffeomorphism. Since, for all  $Y$  in  $\mathfrak{u}_{\Theta, w\succ}^{\text{opp}}$ ,  $\exp(Y)wP_\Theta^{\text{opp}} = wP_\Theta^{\text{opp}}$ , we have that  $f_w$  is onto. By the very choice of the space  $\mathfrak{u}_{\Theta, w\prec}^{\text{opp}}$ , this map is a local diffeomorphism at 0. By equivariance with respect to the element  $\exp(\sum_{\alpha \in \Delta} H_\alpha)$  of the Cartan subspace that acts as a contracting transformation on  $\mathfrak{u}_{\Theta, w\prec}^{\text{opp}}$  we deduce that  $f_w$  is a diffeomorphism.  $\square$

From this we deduce how the dimensions of the cells  $C(w)$  jump:

**Proposition 7.4.** *Let  $w$  be in the Coxeter group  $(W(\Theta), R(\Theta))$  and let  $\alpha$  be in  $\Theta$  such that  $\sigma_\alpha w$  is a reduced expression in  $W(\Theta)$ . Then*

$$\dim C(\sigma_\alpha w) - \dim C(w) = \dim \mathfrak{u}_\alpha$$

which means

$$\begin{aligned} \dim C(\sigma_\alpha w) - \dim C(w) &= 1 \text{ if } \alpha \neq \alpha_\Theta, \\ \dim C(\sigma_{\alpha_\Theta} w) - \dim C(w) &= \dim \mathfrak{u}_{\alpha_\Theta} \text{ if } \alpha = \alpha_\Theta. \end{aligned}$$

*Proof.* Denote by  $\delta$  the difference of dimensions. The hypothesis implies that  $\sigma_\alpha w$  is also a reduced expression in  $W$ . By the previous lemma, the equation (7.1) (applied with  $x = \sigma_\alpha$ ), and the equalities  $\dim \mathfrak{g}_{-\beta} = \dim \mathfrak{g}_\beta$  we deduce that

$$\delta = \sum_{\beta \in \Sigma_\Theta^+, \sigma_\alpha^{-1}\beta \in \Sigma^-} \dim \mathfrak{g}_\beta.$$

But Lemma 7.5 below establishes

$$\{\beta \in \Sigma_\Theta^+ \mid \sigma_\alpha^{-1}\beta \in \Sigma^-\} = \{\beta \in \Sigma^+ \mid \beta - \alpha \in \text{Span}(\Delta \setminus \Theta)\},$$

hence

$$\delta = \dim \bigoplus_{\substack{\beta \in \Sigma^+, \\ \beta - \alpha \in \text{Span}(\Delta \setminus \Theta)}} \mathfrak{g}_\beta = \dim \mathfrak{u}_\alpha. \quad \square$$

**Lemma 7.5.** *Let  $\alpha$  in  $\Theta$ . If  $\alpha \neq \alpha_\Theta$ , then*

$$\{\beta \in \Sigma_\Theta^+ \mid \sigma_\alpha^{-1}\beta \in \Sigma^-\} = \{\alpha\} = \{\beta \in \Sigma^+ \mid \beta - \alpha \in \text{Span}(\Delta \setminus \Theta)\},$$

and if  $\alpha = \alpha_\Theta$

$$\{\beta \in \Sigma_\Theta^+ \mid \sigma_{\alpha_\Theta}^{-1}\beta \in \Sigma^-\} = \{\beta \in \Sigma^+ \mid \beta - \alpha_\Theta \in \text{Span}(\Delta \setminus \Theta)\}.$$

*Proof.* In the first case,  $\sigma_\alpha = s_\alpha$  and it is already known that

$$\{\beta \in \Sigma_\Theta^+ \mid \sigma_\alpha^{-1}\beta \in \Sigma^-\} = \Sigma_{s_\alpha \prec}^+ = \{\alpha\}.$$

Since  $\alpha$  belongs to  $\Sigma_\Theta^+$ , this proves the equality  $\Sigma_{\Theta, s_\alpha \prec}^+ = \Sigma_{s_\alpha \prec}^+ \cap \Sigma_\Theta^+ = \{\alpha\}$ . Since  $\alpha$  is not connected to  $\Delta \setminus \Theta$ , the only root in the affine subspace  $\alpha + \text{Span}(\Delta \setminus \Theta)$  is  $\alpha$  and this proves the equality  $\{\alpha\} = \{\beta \in \Sigma^+ \mid \beta - \alpha \in \text{Span}(\Delta \setminus \Theta)\}$ .

Let us treat the case  $\alpha = \alpha_\Theta$ . The element  $\sigma_{\alpha_\Theta} = w_{\{\alpha_\Theta\} \cup \Delta \setminus \Theta} w_{\Delta \setminus \Theta}$  belongs to the subgroup  $W_{\{\alpha_\Theta\} \cup \Delta \setminus \Theta} = \langle s_\beta \rangle_{\beta \in \{\alpha_\Theta\} \cup \Delta \setminus \Theta}$  of  $W$ .

Since every positive root is a sum with nonnegative coefficients of simple roots, we have that the set

$$\{\beta \in \Sigma^+ \mid \beta - \alpha_\Theta \in \text{Span}(\Delta \setminus \Theta)\}$$

is contained in  $\text{Span}(\{\alpha_\Theta\} \cup \Delta \setminus \Theta)$  and thus in the root system generated by  $\{\alpha_\Theta\} \cup \Delta \setminus \Theta$  since every positive root  $\gamma$  is contained in a chain of roots  $(\gamma_0, \dots, \gamma_k)$  with  $\gamma_k = \gamma$ ,  $\gamma_0 = 0$  and  $\gamma_i - \gamma_{i-1} \in \Delta$  for every  $i = 1, \dots, k$ .

For this reason and from the fact that, for every simple root  $\beta$  in  $\Theta \setminus \{\alpha_\Theta\}$  and for every  $\alpha$  in  $\{\alpha_\Theta\} \cup \Delta \setminus \Theta$ ,  $s_\alpha(\beta) - \beta$  belongs to  $\text{Span}(\{\alpha_\Theta\} \cup \Delta \setminus \Theta)$ , we have that the set

$$\{\beta \in \Sigma_\Theta^+ \mid \sigma_{\alpha_\Theta}^{-1}\beta \in \Sigma^-\}$$

is contained in the root system generated by  $\{\alpha_\Theta\} \cup \Delta \setminus \Theta$ .

Therefore we can and will assume that  $\Delta = \{\alpha_\Theta\} \cup \Delta \setminus \Theta$ , i.e. that we are in the Hermitian tube type case. In this case it is known that the longest length element  $w_{\{\alpha_\Theta\} \cup \Delta \setminus \Theta}$  is  $-\text{id}$  so that

$$\{\beta \in \Sigma_\Theta^+ \mid \sigma_{\alpha_\Theta}^{-1}\beta \in \Sigma^-\} = \{\beta \in \Sigma_\Theta^+ \mid w_{\Delta \setminus \Theta}\beta \in \Sigma^+\}.$$

Furthermore, since for every  $\beta$  in  $\Sigma_\Theta^+$ ,  $w_{\Delta \setminus \Theta}\beta - \beta$  belongs to  $\text{Span}(\Delta \setminus \Theta)$  (and again since every root is a sum of simple roots with coefficients all of the same sign), this set is equal to  $\Sigma_\Theta^+$ . However in this  $C_{d+1}$  type, the equality  $\Sigma_\Theta^+ = \{\beta \in \Sigma^+ \mid \beta - \alpha_\Theta \in \text{Span}(\Delta \setminus \Theta)\}$  is satisfied. Indeed a standard description of this root system is  $\{\pm e_i \pm e_j\}_{1 \leq i < j \leq d+1} \cup \{\pm 2e_i\}_{1 \leq i \leq d+1}$  (as a subset of  $\mathbf{R}^{d+1}$  with its standard basis  $(e_1, \dots, e_{d+1})$ ) with positive roots being  $\{e_i \pm e_j\}_{1 \leq i < j \leq d+1} \cup \{2e_i\}_{1 \leq i \leq d+1}$  and simple roots  $\alpha_1 = e_1 - e_2, \dots, \alpha_d = e_d - e_{d+1}$ , and  $\alpha_{d+1} = 2e_{d+1}$ . One has here  $\Theta = \{\alpha_\Theta\}$  and  $\alpha_\Theta = \alpha_{d+1}$ , the span of  $\Delta \setminus \Theta$  is  $\{x \in \mathbf{R}^{d+1} \mid x_1 + \dots + x_{d+1} = 0\}$ . Thus the positive roots in this span are  $\{e_i \pm e_j\}_{1 \leq i < j \leq d+1}$  so that  $\Sigma_\Theta^+$  is  $\{2e_i\}_{1 \leq i \leq d+1}$ ; and the roots equal to  $\alpha_{d+1}$  modulo this span are  $2e_i$  for  $i = 1, \dots, d+1$ . This proves the announced equality.  $\square$

**7.3. The nontrivial cone and the Bruhat decomposition.** We explain here the precise understanding of the Bruhat cells containing the image of the cone  $\mathring{c}_{\alpha_\Theta}$  by the exponential map.

We assume in this section that  $\Theta \neq \Delta$  and adopt the notation of the previous parts:  $\alpha_\Theta$  is the special root,  $\Delta \setminus \Theta = \{\chi_1, \dots, \chi_d\}$  (with  $\chi_1$  connected to  $\alpha_\Theta$  in the Dynkin diagram), and the elements  $Z_0, \dots, Z_d$  of the Lie algebra (Section 3.3). The reflection in  $W$  associated with  $\alpha_\Theta$  is denoted by  $s_0$  and the reflections associated with  $\chi_1, \dots, \chi_d$  are denoted by  $s_1, \dots, s_d$  respectively.

We know that the longest element in  $W_{\{\alpha_\Theta\} \cup \Delta \setminus \Theta}$  is

$$w_{\{\alpha_\Theta\} \cup \Delta \setminus \Theta} = s_0 s_0^{s_1} \cdots s_0^{s_1 \cdots s_d},$$

(with the notation  $x^y = y^{-1}xy$  so that<sup>2</sup>  $s_0^{s_1 \cdots s_i} = s_i \cdots s_1 s_0 s_1 \cdots s_i$ ) and that the above is a reduced expression of  $w_{\{\alpha_\Theta\} \cup \Delta \setminus \Theta}$ .

Let us introduce also the following elements of  $W$ :

$$(7.2) \quad w_j = s_0 s_0^{s_1} \cdots s_0^{s_1 \cdots s_{j-1}} \text{ for } j = 0, \dots, d+1$$

and, for every subset  $I$  of  $\{0, \dots, d\}$

$$(7.3) \quad w_I = \prod_{i \in I} s_0^{s_1 \cdots s_i},$$

so that  $w_0 = w_\emptyset = e$  is the neutral element of  $W$ ,  $w_{d+1} = w_{\{0, \dots, d\}} = w_{\{\alpha_\Theta\} \cup \Delta \setminus \Theta}$ , and the above are reduced expressions. Therefore, by

<sup>2</sup>Note that these elements  $s_0^{s_1 \cdots s_i}$  pairwise commute.

Lemma 4.14, we have that

$$(7.4) \quad \ell_{\Theta}(w_j) = j, \ell_{\Theta}(w_I) = \sharp I, \forall j \text{ and } \forall I.$$

We start by determining the Bruhat cells for linear combination of the elements  $Z_i$  with nonnegative coefficients.

**Lemma 7.6.** *Let  $Y$  be a linear combination with nonnegative coefficients of the  $Z_i$ :  $Y = \sum_{i=0}^d \lambda_i Z_i$ , i.e.  $Y$  belongs to  $c_{\alpha_{\Theta}} \cap \bigoplus \mathbf{R}Z_i$ . Let  $I \subset \{0, \dots, d\}$  be the indices of the nonzero entries in  $(\lambda_0, \dots, \lambda_d)$  (in formula  $I = \{i \leq d \mid \lambda_i > 0\}$ ). Then*

$$\exp(Y) \text{ belongs to } P^{\text{opp}} w_I P^{\text{opp}}.$$

*Proof.* Since the elements  $Z_i$  pairwise commute, we have that

$$\exp(Y) = \exp(\lambda_0 Z_0) \exp(\lambda_1 Z_1) \cdots \exp(\lambda_d Z_d).$$

By property (4) of Section 7.1, if  $\lambda_0 > 0$ , then  $\exp(\lambda_0 Z_0)$  belongs to  $P^{\text{opp}} s_0 P^{\text{opp}}$ . For all  $i$ , one has  $Z_i = \text{Ad}(\dot{s}_i \cdots \dot{s}_1) Z_0$  so that  $\exp(\lambda_i Z_i) = \dot{s}_i \cdots \dot{s}_1 \exp(\lambda_i Z_0) \dot{s}_1^{-1} \cdots \dot{s}_i^{-1}$ . Recursive application of the property (3) of Section 7.1 and the fact that  $s_i \cdots s_1 s_0 s_1 \cdots s_i$  is a reduced expression imply that, when  $\lambda_i > 0$ ,  $\exp(\lambda_i Z_i)$  belongs to

$$P^{\text{opp}} s_i \cdots s_1 s_0 s_1 \cdots s_i P^{\text{opp}} = P^{\text{opp}} s_0^{s_1 \cdots s_i} P^{\text{opp}}.$$

Recursive application again of property (3) and the fact that Equation (7.3) is a reduced expression imply that

$$\exp(Y) \text{ belongs to } P^{\text{opp}} \prod_{i: \lambda_i > 0} s_0^{s_1 \cdots s_i} P^{\text{opp}},$$

hence the result.  $\square$

We can now determine the Bruhat cell corresponding to elements in the open cone.

**Proposition 7.7.** *Let  $X$  be in the open cone  $\mathring{c}_{\alpha_{\Theta}}$ . Then  $\exp(X)$  belongs to  $P^{\text{opp}} w_{\{\alpha_{\Theta}\} \cup \Delta \setminus \Theta} P^{\text{opp}}$ .*

*Proof.* Recall that the element  $E_{\alpha_{\Theta}} = Z_0 + Z_1 + \cdots + Z_d$  belongs to  $\mathring{c}_{\alpha_{\Theta}}$  and that its stabilizer in  $L_{\Theta}^{\circ}$  (for the adjoint action) contains  $K \cap L_{\Theta}^{\circ}$ . Furthermore the action of  $L_{\Theta}^{\circ}$  on  $\mathring{c}_{\alpha_{\Theta}}$  is transitive: there is an element  $g$  in  $L_{\Theta}^{\circ}$  such that  $\text{Ad}(g)E_{\alpha_{\Theta}} = X$ .

The Iwasawa decomposition for  $L_{\Theta}^{\circ}$  states the equality

$$L_{\Theta}^{\circ} = (P^{\text{opp}} \cap L_{\Theta}^{\circ})(K \cap L_{\Theta}^{\circ}).$$

There are thus elements  $p$  in  $P^{\text{opp}} \cap L_{\Theta}^{\circ}$  and  $k$  in  $K \cap L_{\Theta}^{\circ}$  such that  $g = pk$ . Since  $\text{Ad}(k)E_{\alpha_{\Theta}} = E_{\alpha_{\Theta}}$ , one has  $\text{Ad}(p)E_{\alpha_{\Theta}} = X$ . By the compatibility of the exponential map with the adjoint action and the conjugation action, we get  $\exp(X) = p \exp(E_{\alpha_{\Theta}}) p^{-1}$  so that  $\exp(X)$  and  $\exp(E_{\alpha_{\Theta}})$  belong to the same Bruhat cell. By Lemma 7.6,  $\exp(E_{\alpha_{\Theta}})$  belongs to  $P^{\text{opp}} w_{\{0, \dots, d\}} P^{\text{opp}} = P^{\text{opp}} w_{\{\alpha_{\Theta}\} \cup \Delta \setminus \Theta} P^{\text{opp}}$ . This concludes that  $\exp(X)$  belongs to  $P^{\text{opp}} w_{\{\alpha_{\Theta}\} \cup \Delta \setminus \Theta} P^{\text{opp}}$ .  $\square$

For the elements in the closure of the cone, it will be enough for our purpose to determine their class under the left-right action of  $P_\Theta^{\text{opp}}$ .

**Proposition 7.8.** *Let  $X$  be an element of  $c_{\alpha_\Theta}$ , There is then a unique  $j$  in  $\{0, \dots, d\}$  such that*

$$\exp(X) \in P_\Theta^{\text{opp}} w_j P_\Theta^{\text{opp}}.$$

*Proof.* Uniqueness follows from the observation: for  $j, k$  in  $\{0, \dots, d\}$

$$P_\Theta^{\text{opp}} w_j P_\Theta^{\text{opp}} \subset P_\Theta^{\text{opp}} w_j P_\Theta^{\text{opp}} \text{ and}$$

$$P_\Theta^{\text{opp}} w_k P_\Theta^{\text{opp}} = \bigcup_{x_1, x_2 \in W_{\Delta \setminus \Theta}} P_\Theta^{\text{opp}} x_1 w_k x_2 P_\Theta^{\text{opp}}.$$

(The last union is justified by property (2) of Section 7.1 and may contain more than once the same Bruhat cell.) Hence an equality  $P_\Theta^{\text{opp}} w_j P_\Theta^{\text{opp}} = P_\Theta^{\text{opp}} w_k P_\Theta^{\text{opp}}$  implies (and is in fact equivalent to) the existence of  $x_1, x_2$  in  $W_{\Delta \setminus \Theta}$  such that

$$w_j = x_1 w_k x_2.$$

Hence  $j = \ell_\Theta(w_j) = \ell_\Theta(x_1 w_k x_2) = \ell_\Theta(w_k)$  (by Lemma 4.12) =  $k$  and  $w_j = w_k$ .

Consider now an element  $X$  in  $c_{\alpha_\Theta}$ . Since  $c_{\alpha_\Theta}$  is the closure of  $\mathring{c}_{\alpha_\Theta}$ , there exists a sequence  $(X_n)_{n \in \mathbf{N}}$  in  $\mathring{c}_{\alpha_\Theta}$  that converges to  $X$ . For every  $n$  in  $\mathbf{N}$ , let  $g_n$  be an element in  $L_\Theta^\circ$  such that  $X_n = \text{Ad}(g_n)E_{\alpha_\Theta}$ .

The Cartan decomposition in  $L_\Theta^\circ$  gives

$$L_\Theta^\circ = K_\Theta^\circ \exp(\mathfrak{b}^+) K_\Theta^\circ,$$

where  $K_\Theta^\circ = K \cap L_\Theta^\circ$  and  $\mathfrak{b}^+ = \{A \in \mathfrak{a} \mid \chi_i(A) \leq 0, \forall i = 1, \dots, d\}$  is a closed Weyl chamber for the reductive Lie group  $L_\Theta^\circ$ . There exist thus, for all  $n$  in  $\mathbf{N}$ , elements  $k_n, k'_n$  in  $K_\Theta^\circ$ , and  $A_n$  in  $\mathfrak{b}^+$  such that  $g_n = k_n \exp(A_n) k'_n$ . Up to extracting we will assume that the sequence  $(k_n)_{n \in \mathbf{N}}$  is converging and its limit will be denoted  $k_\infty$ . One has therefore, for all  $n$ ,

$$\text{Ad}(\exp(A_n))E_{\alpha_\Theta} = \text{Ad}(k_n^{-1}) \text{Ad}(g_n)E_{\alpha_\Theta} = \text{Ad}(k_n^{-1})X_n,$$

so the sequence  $(\text{Ad}(\exp(A_n))E_{\alpha_\Theta})_{n \in \mathbf{N}}$  is converging and its limit is  $Y = \text{Ad}(k_\infty^{-1})X$ . As  $K_\Theta^\circ \subset P_\Theta^{\text{opp}}$ , it will be enough to determine the Bruhat cell of  $\exp(Y)$ .

For all  $n$  in  $\mathbf{N}$  and for all  $i = 0, \dots, d$ , let

$$\lambda_{i,n} = \exp((\alpha_\Theta + 2\chi_1 + \dots + 2\chi_i)(A_n))$$

so that, for all  $n$ ,  $\lambda_{0,n} \geq \lambda_{1,n} \geq \dots \geq \lambda_{d,n} > 0$  and

$$\begin{aligned} \sum_{i=0}^d \lambda_{i,n} Z_i &= \sum_{i=0}^d \exp(\text{ad}(A_n)) Z_i \\ &= \exp(\text{ad}(A_n)) \sum_{i=0}^d Z_i = \text{Ad}(\exp(A_n)) E_{\alpha_\Theta}. \end{aligned}$$

This last equality implies that the sequences  $(\lambda_{i,n})_{n \in \mathbf{N}}$  ( $i = 0, \dots, d$ ) converge in  $\mathbf{R}_{\geq 0}$  and their limits will be denoted  $\lambda_i$  ( $i = 0, \dots, d$ ). We have hence

$$Y = \sum_{i=0}^d \lambda_i Z_i$$

and  $\lambda_0 \geq \lambda_1 \geq \dots \geq \lambda_d \geq 0$  and the set of indices of the nonzero elements in  $(\lambda_0, \dots, \lambda_d)$  has the form  $\{0, \dots, j-1\}$  for some  $j$  in  $\{0, \dots, d+1\}$  (this set is  $\emptyset$  when  $j = 0$ ). Lemma 7.6 says that  $\exp(Y) \in P^{\text{opp}} w_j P^{\text{opp}}$  and this implies the wanted result.  $\square$

*Remark 7.9.* Building on the Iwasawa decomposition (instead of the Cartan decomposition) one can in fact establish that the element  $\exp(Y)$  belongs to the Bruhat cell corresponding to  $w_I$  for some subset  $I$  of  $\{0, \dots, d\}$ .

**Corollary 7.10.** *Let  $Y$  be in  $c_{\alpha_\Theta}$  and let  $w$  be the element of  $W$  such that  $\exp(Y) \in P^{\text{opp}} w P^{\text{opp}}$ . Then*

- (1)  $\ell_\Theta(w) \leq d+1$ ;
- (2) if  $\ell_\Theta(w) = d+1$  then  $Y$  belongs to  $\mathring{c}_{\alpha_\Theta}$  and  $w = w_{\{\alpha_\Theta\} \cup \Delta \setminus \Theta}$ .

**7.4. The Bruhat cells of non-zero elements.** The previous paragraph determines the Bruhat cell of elements of the form  $\exp(X)$  when  $X$  belongs to the cone  $c_{\alpha_\Theta}$ . We now consider the case when  $X$  is only supposed to be a non-zero element of  $\mathfrak{u}_{\alpha_\Theta}$ .

**Lemma 7.11.** *Let  $X$  be a non-zero element of  $\mathfrak{u}_{\alpha_\Theta}$  and let  $x$  be the element of  $W$  such that  $\exp(X) = P^{\text{opp}} x P^{\text{opp}}$ . Then  $\ell_\Theta(x) > 0$ .*

*Remark 7.12.* Note that the element  $x$  belongs to  $W_{\{\alpha_\Theta\} \cup \Delta \setminus \Theta}$ .

*Proof.* One has (Lemma 4.15)

$$W_{\Delta \setminus \Theta} = \{x \in W \mid \ell_\Theta(x) = 0\}.$$

Furthermore, the following equality holds:

$$\bigcup_{x \in W_{\Delta \setminus \Theta}} P^{\text{opp}} x P^{\text{opp}} = P_\Theta^{\text{opp}}.$$

Thus the statement will be established if we can prove that  $\exp(X)$  does not belong to  $P_\Theta^{\text{opp}}$ . This last property is a consequence of the fact that the map:

$$\begin{aligned} \mathfrak{u}_\Theta \times P_\Theta^{\text{opp}} &\longrightarrow G \\ (X, g) &\longmapsto \exp(X)g \end{aligned}$$

is an embedding.  $\square$

## 8. THE UNIPOTENT POSITIVE SEMIGROUP

This section addresses the properties of the maps  $F_\gamma$  (see Section 6.2) enabling among other things the definition of the unipotent positive semigroup.

**8.1. The main statement.** We denote by  $\Omega_{\Theta}^{\text{opp}} = P_{\Theta}^{\text{opp}} w_{\Delta} P_{\Theta}^{\text{opp}}$  the open Bruhat cell with respect to  $P_{\Theta}^{\text{opp}}$  and by  $\Omega_{\Theta} = P_{\Theta} w_{\Delta} P_{\Theta}$  the open Bruhat cell with respect to  $P_{\Theta}$ . Recall that  $\mathbf{W}$  denotes the set of reduced expressions of the longest element in  $W(\Theta)$ .

**Theorem 8.1.** (1) For any  $\gamma$  in  $\mathbf{W}$ , the image of the map  $\mathring{F}_{\gamma}$  is contained in  $\Omega_{\Theta}^{\text{opp}} \cap U_{\Theta}$ .

(2) For any  $\gamma$  in  $\mathbf{W}$ , the map  $\mathring{c}_{\gamma} \rightarrow \mathbb{F}_{\Theta} \mid \mathbf{v} \mapsto \mathring{F}_{\gamma}(\mathbf{v}) \cdot \mathfrak{p}_{\Theta}^{\text{opp}}$  is injective.

(3) For any  $\gamma$  in  $\mathbf{W}$ , the map  $\mathring{c}_{\gamma} \rightarrow \mathbb{F}_{\Theta} \mid \mathbf{v} \mapsto \mathring{F}_{\gamma}(\mathbf{v}) \cdot \mathfrak{p}_{\Theta}^{\text{opp}}$  is open.

(4) We have  $F_{\gamma}(c_{\gamma} \setminus \mathring{c}_{\gamma}) \subset U_{\Theta} \setminus \Omega_{\Theta}^{\text{opp}}$ .

(5) We have  $F_{\gamma}(\mathring{c}_{\gamma}) = F_{\gamma}(c_{\gamma}) \cap \Omega_{\Theta}^{\text{opp}}$ .

(6) The map  $F_{\gamma}: c_{\gamma} \rightarrow U_{\Theta}$  is proper.

(7) The image of  $\mathring{F}_{\gamma}$  is a connected component of  $\Omega_{\Theta}^{\text{opp}} \cap U_{\Theta}$ .

(8) The image of  $\mathring{F}_{\gamma}$  does not depend on  $\gamma$ .

Analogous properties hold for  $F_{\gamma}^{\text{opp}}$ .

Property 8 in this theorem justifies Definition 6.5, other consequences will be drawn in Section 8.8. Theorem 8.1 will be proved in Sections 8.2–8.4 and 8.6–8.7 below. Section 8.5 gives an explicit description of the tangent cone of the positive unipotent semigroup.

**8.2. Transversality, injectivity, and openness of the maps  $F_{\gamma}$ .** We prove here points (1) and (2) of Theorem 8.1, that is:

**Proposition 8.2.** For every  $\gamma$  in  $\mathbf{W}$ , the image of the map  $\mathring{F}_{\gamma}$  is contained in  $P_{\Theta}^{\text{opp}} w_{\max}^{\Theta} P_{\Theta}^{\text{opp}} = P_{\Theta}^{\text{opp}} w_{\max}^{\Theta} P_{\Theta}^{\text{opp}} = \Omega_{\Theta}^{\text{opp}}$  and the map

$$\mathring{c}_{\gamma} \longrightarrow \mathbb{F}_{\Theta} \mid \mathbf{v} \longmapsto \mathring{F}_{\gamma}(\mathbf{v}) \cdot \mathfrak{p}_{\Theta}^{\text{opp}}$$

is injective.

*Remark 8.3.* Note that, since the map  $U_{\Theta} \rightarrow \mathbb{F}_{\Theta} \mid u \mapsto u \cdot \mathfrak{p}_{\Theta}^{\text{opp}}$  is an embedding, the injective of the above map is equivalent to the injectivity of  $\mathring{F}_{\gamma}$ .

The result will be proved thanks to an inductive process whose results have their own interest. For this it will be a little more convenient to have a decreasing numbering for the indices of  $\gamma$ :  $\gamma = (\gamma_N, \gamma_{N-1}, \dots, \gamma_1)$  where  $N$  is the length of  $w_{\max}^{\Theta}$  in  $(W(\Theta), R(\Theta))$ . With this notation the cone  $c_{\gamma}$  is the product  $c_{\gamma_N} \times \dots \times c_{\gamma_1}$  and the map  $F_{\gamma}$  is

$$\begin{aligned} c_{\gamma} &\longrightarrow U_{\Theta} \\ (X_N, \dots, X_1) &\longmapsto \exp(X_N) \cdots \exp(X_1). \end{aligned}$$

(Recall that  $\mathring{F}_{\gamma}$  is the restriction of  $F_{\gamma}$  to  $\mathring{c}_{\gamma}$ .)

**Proposition 8.4.** Let, for  $j = 1, \dots, N$ ,  $x_j = \sigma_{\gamma_j} \cdots \sigma_{\gamma_1}$ . Then for every  $j$  in  $\{1, \dots, N\}$

(1) *The image of the map*

$$\begin{aligned} \mathring{F}_j: \mathring{c}_{\gamma_j} \times \cdots \times \mathring{c}_{\gamma_1} &\longrightarrow U_{\Theta} \\ (X_j, \dots, X_1) &\longmapsto \exp(X_j) \cdots \exp(X_1). \end{aligned}$$

is contained in  $P^{\text{opp}} x_j P_{\Theta}^{\text{opp}} = P_{\Theta}^{\text{opp}} x_j P_{\Theta}^{\text{opp}}$  (the equality follows from Lemma 7.2 since  $x_j$  belongs to  $W(\Theta)$ );

(2) *The map*

$$\begin{aligned} \mathbf{u}_{\gamma_j} \times \mathring{c}_{\gamma_{j-1}} \times \cdots \times \mathring{c}_{\gamma_1} &\longrightarrow \mathbf{F}_{\Theta} \\ (X_j, \dots, X_1) &\longmapsto \exp(X_j) \cdots \exp(X_1) \cdot \mathfrak{p}_{\Theta}^{\text{opp}} \end{aligned}$$

is injective.

*Proof.* We prove first (1) by induction on  $j$  in  $\{1, \dots, N\}$ . For  $j = 1$  in the proof below  $F_{j-1}(X_{j-1}, \dots, X_1)$  and  $x_{j-1}$  should be replaced by the identity element.

Let  $(X_j, \dots, X_1)$  be in  $\mathring{c}_{\gamma_j} \times \cdots \times \mathring{c}_{\gamma_1}$ . Suppose first that  $\gamma_j$  is not equal to  $\alpha_{\Theta}$ . Then (Section 7.1)  $\exp(X_j)$  belongs to  $P^{\text{opp}} \sigma_{\gamma_j} P^{\text{opp}} = P^{\text{opp}} s_{\gamma_j} P^{\text{opp}}$ . Since  $x_j = s_{\gamma_j} \sigma_{\gamma_{j-1}} \cdots \sigma_{\gamma_1} = s_{\gamma_j} x_{j-1}$  is a subword of a reduced expression of  $w_{\Delta}$ , we have for the lengths

$$\ell(s_{\gamma_j} x_{j-1}) = \ell(s_{\gamma_j}) + \ell(x_{j-1})$$

thus

$$P^{\text{opp}} s_{\gamma_j} P^{\text{opp}} x_{j-1} P_{\Theta}^{\text{opp}} = P^{\text{opp}} x_j P_{\Theta}^{\text{opp}}.$$

By induction  $\mathring{F}_{j-1}(X_{j-1}, \dots, X_1)$  belongs to  $P^{\text{opp}} x_{j-1} P_{\Theta}^{\text{opp}}$  and since

$$\mathring{F}_j(X_j, \dots, X_1) = \exp(X_j) \mathring{F}_{j-1}(X_{j-1}, \dots, X_1),$$

we can conclude that  $\mathring{F}_j(X_j, \dots, X_1)$  belongs to  $P^{\text{opp}} x_j P_{\Theta}^{\text{opp}}$ .

We now treat the case when  $\gamma_j = \alpha_{\Theta}$ , thus  $\sigma_{\gamma_j} = \sigma_{\alpha_{\Theta}}$ . By Proposition 7.7  $\exp(X_j)$  belongs to

$$P^{\text{opp}} \sigma_{\alpha_{\Theta}} w_{\Delta \setminus \Theta} P^{\text{opp}} \subset P^{\text{opp}} \sigma_{\alpha_{\Theta}} P_{\Theta}^{\text{opp}}.$$

We deduce that  $\mathring{F}_j(X_j, \dots, X_1)$  belongs to

$$\begin{aligned} P^{\text{opp}} \sigma_{\alpha_{\Theta}} P_{\Theta}^{\text{opp}} x_{j-1} P_{\Theta}^{\text{opp}} &= P^{\text{opp}} \sigma_{\alpha_{\Theta}} P^{\text{opp}} x_{j-1} P_{\Theta}^{\text{opp}} \\ &= P^{\text{opp}} \sigma_{\alpha_{\Theta}} x_{j-1} P_{\Theta}^{\text{opp}} \end{aligned}$$

where we applied Lemma 7.2 to  $x_{j-1}$  and the fact that  $x_j = \sigma_{\alpha_{\Theta}} x_{j-1}$  is also a reduced expression.

We now prove point (2) by induction on  $j$  again. For  $j = 1$ , injectivity follows from the fact that  $\mathbf{u}_{\Theta} \times P_{\Theta}^{\text{opp}} \mid (X, g) \mapsto \exp(X)g$  is injective.

Suppose now that  $j \geq 2$  and that the inductive hypothesis has been established up to  $j - 1$ . Let  $(X_j, \dots, X_1)$  and  $(Y_j, \dots, Y_1)$  be in  $\mathbf{u}_{\gamma_j} \times \mathring{c}_{\gamma_{j-1}} \times \cdots \times \mathring{c}_{\gamma_1}$  such that

$$F_j(X_j, \dots, X_1) P_{\Theta}^{\text{opp}} = F_j(Y_j, \dots, Y_1) P_{\Theta}^{\text{opp}},$$

this means that

$$(8.1) \quad \mathring{F}_{j-1}(X_{j-1}, \dots, X_1)P_{\Theta}^{\text{opp}} = \exp(Y_j - X_j)\mathring{F}_{j-1}(Y_{j-1}, \dots, Y_1)P_{\Theta}^{\text{opp}}.$$

By point (1) we have that  $\mathring{F}_{j-1}(X_{j-1}, \dots, X_1)$  and  $\mathring{F}_{j-1}(Y_{j-1}, \dots, Y_1)$  belong to  $P_{\Theta}^{\text{opp}}x_{j-1}P_{\Theta}^{\text{opp}}$ . Suppose that  $X_j \neq Y_j$  then (by Lemma 7.11 or by point (4) of Section 7.1) we have that  $\exp(Y_j - X_j)$  belongs to  $P^{\text{opp}}xP^{\text{opp}}$  with  $\ell_{\Theta}(x) > 0$ . Using again the multiplicative properties of Bruhat cells we deduce that the element  $y$  such that

$$\exp(Y_j - X_j)\mathring{F}_{j-1}(Y_{j-1}, \dots, Y_1) \in P^{\text{opp}}yP^{\text{opp}}$$

satisfies  $\ell_{\Theta}(y) > \ell_{\Theta}(x_{j-1})$  which is a contradiction with Equality (8.1). Hence  $X_j = Y_j$  and the equality (8.1) together with the induction hypothesis imply that  $X_i = Y_i$  for all  $i$  less than  $j$ .  $\square$

With the notation of the proposition, we can now deduce point (3) of Theorem 8.1. Recall that  $C(x_j) \subset \mathbb{F}_{\Theta}$  is the  $P_{\Theta}^{\text{opp}}$ -orbit of the element  $x_j \cdot \mathfrak{p}_{\Theta}^{\text{opp}}$ . We have:

**Corollary 8.5.** *For all  $j$  the map*

$$\begin{aligned} \mathring{c}_{\gamma_j} \times \dots \times \mathring{c}_{\gamma_1} &\longrightarrow C(x_j) \subset \mathbb{F}_{\Theta} \\ (X_j, \dots, X_1) &\longmapsto \mathring{F}_j(X_j, \dots, X_1) \cdot \mathfrak{p}_{\Theta}^{\text{opp}} \end{aligned}$$

is open.

*Proof.* We know that this map is injective. By Invariance of Domain, it is enough to show that the dimensions of the source and the range coincide. This can be proved again by induction on  $j$ , using the relation  $x_j = \sigma_{\gamma_j}x_{j-1}$  and Proposition 7.4 which says in this context:

$$\dim C(x_j) - \dim C(x_{j-1}) = \dim \mathfrak{u}_{\gamma_j} = \dim c_{\gamma_j}.$$

This is the precise relation to show that the induction step holds true.  $\square$

**8.3. Nontransversality at the boundary.** We now prove point (4) of Theorem 8.1:

**Proposition 8.6.** *Let  $\mathbf{v}$  be in  $c_{\gamma}$ . If  $\mathbf{v}$  does not belong to  $\mathring{c}_{\gamma}$ , then  $F_{\gamma}(\mathbf{v})$  does not belong to  $P^{\text{opp}}w_{\max}^{\Theta}P_{\Theta}^{\text{opp}}$ .*

*Proof.* We denote  $(v_N, \dots, v_1)$  the components of  $\mathbf{v}$ , at least one of them  $v_j$  is in  $c_{\alpha_j} \setminus \mathring{c}_{\alpha_j}$ . For all  $i$  between 1 and  $N$  denote by  $t_i$  an element of  $W$  such that  $\exp(v_i)$  belongs to  $P_{\Theta}^{\text{opp}}t_iP_{\Theta}^{\text{opp}}$ . We have, for all  $i$ ,  $\ell_{\Theta}(t_i) \leq \ell_{\Theta}(\sigma_{\alpha_i})$  and, since  $t_j = e$  when  $\alpha_j \neq \alpha_{\Theta}$  or by Corollary 7.10 when  $\alpha_j = \alpha_{\Theta}$ , we have that  $\ell_{\Theta}(t_j) < \ell_{\Theta}(\sigma_{\alpha_j})$ .

Repeated applications of Lemma 7.1 show that the element  $x$  of  $W$  such that  $F_{\gamma}(\mathbf{v})$  belongs to  $P_{\Theta}^{\text{opp}}xP_{\Theta}^{\text{opp}}$  satisfies

$$\ell_{\Theta}(x) \leq \sum_{i=1}^N \ell_{\Theta}(t_i)$$

$$< \sum_{i=1}^N \ell_{\Theta}(\sigma_{\alpha_i}) = \ell_{\Theta}(x_{\max}^{\Theta}).$$

This implies the result.  $\square$

The point (5) of Theorem 8.1 is now an immediate consequence of (1) and (4) in that theorem.

**8.4. Properness.** To prove properness of the maps  $F_{\gamma}$  we will obtain a formula for the composition of  $\log \circ F_{\gamma}: c_{\gamma} \rightarrow \mathfrak{u}_{\Theta}$  with the projection  $\mathfrak{u}_{\Theta} \rightarrow \bigoplus_{\alpha \in \Theta} \mathfrak{u}_{\alpha}$ . For this a few notations will be useful. We will use again natural numbering for the components of  $\gamma$ :  $\gamma = (\gamma_1, \dots, \gamma_N)$ .

Let  $d$  be the degree of nilpotency of the Lie algebra  $\mathfrak{u}_{\Theta}$  (i.e. the iterated Lie brackets of  $d + 1$  elements of  $\mathfrak{u}_{\Theta}$  is always zero). Let  $\mathfrak{e}_{N,d}$  be the free degree  $d$  nilpotent real Lie algebra generated by elements  $e_1, \dots, e_N$ . Then  $\mathfrak{e}_{N,d}$  is a finite dimensional graded Lie algebra whose degree 1 component is equal to  $\bigoplus_{i=1}^N \mathbf{R}e_i$ .

For any  $\mathbf{v} = (v_1, \dots, v_N)$  in  $\prod_{i=1}^N \mathfrak{u}_{\gamma_i}$  the unique Lie algebra morphism  $\mathfrak{e}_{N,d} \rightarrow \mathfrak{u}_{\Theta}$  sending, for each  $i = 1, \dots, N$ ,  $e_i$  to  $v_i$  will be denoted  $\Psi_{\mathbf{v}}$ . The Lie algebra  $\mathfrak{u}_{\Theta}$  is also graded (with degree 1 component equal to  $\bigoplus_{\alpha \in \Theta} \mathfrak{u}_{\alpha}$ ) and the map  $\Psi_{\mathbf{v}}$  is in fact a graded morphism.

The simply connected Lie group with Lie algebra  $\mathfrak{e}_{N,d}$  will be denoted  $E_{N,d}$ . It is well known that  $\exp: \mathfrak{e}_{N,d} \rightarrow E_{N,d}$  is a diffeomorphism whose inverse will be denoted by  $\log$  and the map  $\mathfrak{e}_{N,d} \times \mathfrak{e}_{N,d} \rightarrow \mathfrak{e}_{N,d} \mid (X, Y) \mapsto \log(\exp(X)\exp(Y))$  is given by the Baker–Campbell–Hausdorff formula. (In fact, we could have used directly this formula.)

Let us introduce the following element of  $\mathfrak{e}_{N,d}$

$$\Gamma := \log(\exp(e_1) \cdots \exp(e_N)).$$

Then, for every  $\mathbf{v}$  in  $\prod_{i=1}^N \mathfrak{u}_{\gamma_i}$ , the following holds

$$\log(\exp(v_1) \cdots \exp(v_N)) = \Psi_{\mathbf{v}}(\Gamma).$$

The Baker–Campbell–Hausdorff formula implies that the degree 1 component of  $\Gamma$  is equal to

$$(8.2) \quad \Gamma_1 = \sum_{i=1}^N e_i;$$

and we will denote  $\Gamma_{\geq 2} = \Gamma - \Gamma_1$  the sum of components of higher degree.

From the fact that  $\Psi_{\mathbf{v}}$  preserves the degree, we deduce:

**Lemma 8.7.** *Let  $\|\cdot\|$  be a norm on  $\mathfrak{u}_{\Theta}$ . There is a constant  $C \geq 0$  such that, for every  $\mathbf{v}$  in  $\prod_{i=1}^N \mathfrak{u}_{\gamma_i}$ ,*

(1) the degree 1 component of  $\log(\exp(v_1) \cdots \exp(v_N))$  is equal to

$$\log(\exp(v_1) \cdots \exp(v_N))_1 = \sum_{i=1}^N v_i,$$

(2) and, denoting

$$\begin{aligned} \log(\exp(v_1) \cdots \exp(v_N))_{\geq 2} = \\ \log(\exp(v_1) \cdots \exp(v_N)) - \log(\exp(v_1) \cdots \exp(v_N))_1 \end{aligned}$$

the sum of components of higher degree, we have

$$\|\log(\exp(v_1) \cdots \exp(v_N))_{\geq 2}\| \leq C \max_i \|v_i\|^2.$$

We can now deduce:

**Corollary 8.8.** *The map  $F_\gamma: c_\gamma \rightarrow U_\Theta$  is proper. More precisely, there is a constant  $C \geq 0$  such that, for any  $\mathbf{v} \in c_\gamma$ ,  $\max_i \|v_i\| \leq C \|\log(F_\gamma(\mathbf{v}))_1\|$ .*

*Proof.* It is enough to establish that the map

$$\begin{aligned} c_\gamma &\longrightarrow \bigoplus_{\alpha \in \Theta} \mathfrak{u}_\alpha \\ \mathbf{v} &\longmapsto \log(F_\gamma(\mathbf{v}))_1 \end{aligned}$$

is proper. But this map is

$$\begin{aligned} c_\gamma &\longrightarrow \bigoplus_{\alpha \in \Theta} c_\alpha \\ (v_1, \dots, v_N) &\longmapsto \left( \sum_{i: \gamma_i = \alpha} v_i \right)_{\alpha \in \Theta}. \end{aligned}$$

This last application is clearly proper since, for any acute closed convex cone  $c$  (in a finite dimensional normed real vector space) and for any  $n \geq 1$ , the map  $c^n \rightarrow c \mid (x_1, \dots, x_n) \mapsto x_1 + \cdots + x_n$  is proper. Furthermore there is constant  $D$  (depending on  $c$  and  $n$ ) such that  $\max \|x_i\| \leq D \|x_1 + \cdots + x_n\|$  and this implies the precise control on  $\mathbf{v} \mapsto \log(F_\gamma(\mathbf{v}))_1$ .  $\square$

**8.5. Tangent cone of the semigroup.** The techniques introduced in the previous section enable the determination of the tangent cone at  $e$  of the semigroup  $U_\Theta^{\geq 0}$ :

**Corollary 8.9.** *Let  $c := \bigoplus_{\alpha \in \Theta} c_\alpha \subset \mathfrak{u}_\Theta$ . Then  $c$  is the tangent cone at  $e$  of the semigroup  $U_\Theta^{\geq 0}$ , precisely*

(1) *There is a constant  $C$  such that, for every  $X$  in  $\log(U_\Theta^{\geq 0}) \subset \mathfrak{u}_\Theta$ , let  $X_1 \in \bigoplus \mathfrak{u}_\alpha$  be its degree 1 component, then  $X_1$  belongs to  $c$  and  $\|X - X_1\| \leq C \|X\|^2$ .*

- (2) For any continuous map  $\gamma: [0, 1] \rightarrow U_{\Theta}^{\geq 0}$ , if  $\gamma$  is differentiable at 0 then  $\gamma'(0)$  belongs to  $c$
- (3) For all  $Y$  in  $c$ , there is  $X$  in  $\log(U_{\Theta}^{\geq 0})$  such that  $X_1 = Y$ .
- (4) For all  $Y$  in  $c$ , the map  $\gamma: \mathbf{R}_{\geq 0} \rightarrow U_{\Theta} \mid t \mapsto \exp(tX)$  is of class  $C^{\infty}$ , contained in  $U_{\Theta}^{\geq 0}$  and its derivative at 0 is equal to  $Y$ .

*Remark 8.10.* The proof below will use that the semigroup  $U_{\Theta}^{\geq 0}$  is closed and that it is the image of  $F_{\gamma}$ ; this will be established later in Section 8.8 (and without appealing to Corollary 8.9).

*Proof.* The first item (1) is a direct consequence of the estimates given in Corollary 8.8 and in point (2) of Lemma 8.7.

The second item (2) is a consequence of the first one.

The third item (3) follows easily from the formula for the map  $\mathbf{v} \mapsto \log(F_{\gamma}(\mathbf{v}))_1$  (cf. proof of Corollary 8.8).

For item (4), it is very classical that this map is  $C^{\infty}$  and that its derivative at 0 is  $Y$ . Let  $\gamma$  be in  $\mathbf{W}$  and let also  $\mathbf{v}$  be in  $c_{\gamma}$  be such that  $\log(F_{\gamma}(\mathbf{v}))_1 = Y$ . Let  $t$  be in  $\mathbf{R}_{\geq 0}$ . For all  $n \in \mathbf{N}_{>0}$ ,  $t\mathbf{v}/n$  belongs to  $c_{\gamma}$  so that

$$F_{\gamma}(t\mathbf{v}/n) \text{ and } F_{\gamma}(t\mathbf{v}/n)^n$$

belong to  $U_{\Theta}^{\geq 0}$ . Since the sequence  $(F_{\gamma}(t\mathbf{v}/n)^n)_{n>0}$  converges to  $\exp(tY)$  and since  $U_{\Theta}^{\geq 0}$  is closed, we deduce that  $\exp(tY)$  belongs to  $U_{\Theta}^{\geq 0}$ .  $\square$

**8.6. Connectedness.** We now prove point (7) of Theorem 8.1:

**Corollary 8.11.** *The set  $F_{\gamma}(\dot{c}_{\gamma})$  is a connected component of  $\Omega_{\Theta}^{\text{opp}} \cap U_{\Theta}$ .*

*Proof.* Corollary 8.5 and the fact that  $U_{\Theta} \rightarrow C(w_{\max}^{\Theta}) \mid g \mapsto g \cdot \mathfrak{p}_{\Theta}^{\text{opp}}$  is a diffeomorphism imply that the set  $F_{\gamma}(\dot{c}_{\gamma})$  is open in  $U_{\Theta}$ . Since the map  $F_{\gamma}|_{c_{\gamma}}$  is proper, we have that the closure of  $F_{\gamma}(\dot{c}_{\gamma})$  in  $U_{\Theta}$  is equal to  $F_{\gamma}(c_{\gamma})$ . Therefore the closure of  $F_{\gamma}(\dot{c}_{\gamma})$  in  $\Omega_{\Theta}^{\text{opp}} \cap U_{\Theta}$  is equal to  $F_{\gamma}(c_{\gamma}) \cap \Omega_{\Theta}^{\text{opp}}$  hence to  $F_{\gamma}(\dot{c}_{\gamma})$  by point (5). Thus  $F_{\gamma}(\dot{c}_{\gamma})$  is open and closed in  $\Omega_{\Theta}^{\text{opp}} \cap U_{\Theta}$ . Since it is connected, it is one connected component.  $\square$

**8.7. Independence on  $\gamma$ .** A priori the connected component  $F_{\gamma}(\dot{c}_{\gamma})$  could depend on the choice of  $\gamma \in \mathbf{W}$ . We show now that this is not the case. For this we consider the split real Lie algebra  $\mathfrak{g}_{\Theta}$  of  $\mathfrak{g}$  of type  $W(\Theta)$  determined by a  $\Theta$ -base  $(E_{\alpha}, F_{\alpha}, D_{\alpha})_{\alpha \in \Theta}$ . Note that the intersection  $\mathfrak{g}_{\Theta} \cap \mathfrak{p}_{\Theta}$  is a standard minimal parabolic subalgebra, and  $\mathfrak{n}_{\Theta} = \mathfrak{g}_{\Theta} \cap \mathfrak{u}_{\Theta}$  is its unipotent radical. We denote by  $N_{\Theta} < U_{\Theta}$  the corresponding subgroup.

The Lie algebra  $\mathfrak{n}_{\Theta}$  is generated by the elements  $(E_{\alpha})_{\alpha \in \Theta}$ . Recall that  $E_{\alpha} \in \dot{c}_{\alpha}$  for all  $\alpha \in \Theta$ .

Given an element  $\gamma \in \mathbf{W}$  we can restrict the map  $F_{\gamma}$  to  $\prod_i \mathbf{R}_{>0} E_{\gamma_i}$ . Lusztig showed

**Proposition 8.12** ([Lus94, Prop. 2.7 (b)]). *The image of  $N_{\Theta}^{>0} = F_{\gamma}(\prod_{i=1}^N \mathbf{R}_{>0} E_{\gamma_i})$  is independent of the choice of  $\gamma \in \mathbf{W}$ .*

As a corollary we obtain the following proposition, which is point (8) of Theorem 8.1.

**Proposition 8.13.** *The image of  $F_{\gamma}(\mathring{c}_{\gamma})$  is independent of the choice of  $\gamma \in \mathbf{W}$ .*

*Proof.* Since  $E_{\alpha} \in \mathring{c}_{\alpha}$  for all  $\alpha \in \Theta$ , and  $\mathring{c}_{\alpha}$  are convex cones, for any  $\gamma \in \mathbf{W}$  we have that  $F_{\gamma}(\prod_{i=1}^N \mathbf{R}_{>0} E_{\gamma_i}) \subset F_{\gamma}(\mathring{c}_{\gamma})$ . Let  $\gamma, \gamma' \in \mathbf{W}$  be two different reduced expressions of the longest word in  $W(\Theta)$ , then  $F_{\gamma}(\mathring{c}_{\gamma}) \cap F_{\gamma'}(\mathring{c}_{\gamma'})$  contains  $N_{\Theta}^{>0} = F_{\gamma}(\prod_{i=1}^N \mathbf{R}_{>0} E_{\gamma_i}) = F_{\gamma'}(\prod_{i=1}^N \mathbf{R}_{>0} E_{\gamma'_i})$ . Therefore  $F_{\gamma}(\mathring{c}_{\gamma})$  and  $F_{\gamma'}(\mathring{c}_{\gamma'})$  have non empty intersection. Since both sets are connected components of  $\Omega_{\Theta}^{\text{opp}} \cap U_{\Theta}$ , they have to be equal.  $\square$

*Remark 8.14.* Note that we can get in fact precise formulas for the change of coordinates if two elements  $\gamma$  and  $\gamma'$  differ by a braid relation. This is done by considering appropriate notions of universal enveloping algebras for  $U_{\Theta}$  and deriving explicit systems of polynomial equations from relations in the universal enveloping algebra, as has been done for the split real case by Berenstein and Zelevinsky [BZ97]. These explicit formulas for the braid relations, as well as a more detailed investigation of the positive semigroup  $G_{\Theta}^{>0}$  will appear in a forthcoming article [GW22].

**8.8. Consequences.** In this section we draw some consequences from the results of the previous sections and the explicit parametrizations by  $F_{\gamma}$ , not only for the positive semigroup, but also for the non-negative semigroup.

As defined in Section 6.5, we set  $U_{\Theta}^{>0} := \mathring{F}_{\gamma}(\mathring{c}_{\gamma}) \subset U_{\Theta}$ . By Proposition 8.13 this set is independent of the choice of  $\gamma \in \mathbf{W}$ . But we still have to show that  $U_{\Theta}^{>0}$  is indeed a semigroup. For this we prove a slightly stronger statement.

**Proposition 8.15.** *For all  $\alpha \in \Theta$  and for all  $v_{\alpha} \in c_{\alpha}$ , the set  $U_{\Theta}^{>0}$  is invariant under left and right multiplication by  $\exp(v_{\alpha})$ .*

*Proof.* We prove only the invariance by left multiplication, the case of right multiplication follows by entirely similar arguments.

Let  $v_{\alpha} \in c_{\alpha}$  for some  $\alpha \in \Theta$ , and let  $u$  be an element in  $U_{\Theta}^{>0}$ . We want to prove that then  $\exp(v_{\alpha}) \cdot u \in U_{\Theta}^{>0}$ . Let us choose  $\gamma \in \mathbf{W}$  such that  $\gamma_1 = \alpha$ , and  $\mathbf{v} \in \mathring{c}_{\gamma}$  such that  $F_{\gamma}(\mathbf{v}) = u$  (the existence of such reduced expression  $\gamma$  of the longest length element  $w_{\max}^{\Theta}$  in  $(W(\Theta), R(\Theta))$  is a classical fact and is established starting from the equality  $w_{\max}^{\Theta} = \sigma_{\alpha}(\sigma_{\alpha} w_{\max}^{\Theta})$  and from a reduced expression of  $\sigma_{\alpha} w_{\max}^{\Theta}$ ). Then we have

$$\exp(v_{\alpha}) \cdot u = \exp(v_{\alpha}) \exp(v_1) \cdots \exp(v_N)$$

$$= \exp(v_\alpha + v_1) \cdots \exp(v_N),$$

where we used that  $\gamma$  starts with  $\alpha$ . Since  $c_\alpha$  is a convex cone,  $v_1 \in \mathring{c}_\alpha$  and  $v_\alpha \in c_\alpha$  we have that  $v_\alpha + v_1 \in \mathring{c}_\alpha$ . Therefore  $\exp(v_\alpha) \cdot u$  belongs to  $F_\gamma(\mathring{c}_\gamma) = U_\Theta^{\geq 0}$ .  $\square$

As a direct consequence of this proposition we obtain

- Corollary 8.16.** (1) The inclusions  $U_\Theta^{\geq 0} U_\Theta^{\geq 0} \subset U_\Theta^{\geq 0}$ , and  $U_\Theta^{\geq 0} U_\Theta^{\geq 0} \subset U_\Theta^{\geq 0}$  hold.
- (2) The subset  $U_\Theta^{\geq 0} \subset U_\Theta$  is a semigroup that is invariant by conjugation by  $L_\Theta^\circ$ .
- (3) The closure  $\overline{U_\Theta^{\geq 0}}$  is a semigroup, invariant by conjugation by  $L_\Theta^\circ$ .
- (4) The semigroup  $\overline{U_\Theta^{\geq 0}}$  is the image  $F_\gamma(c_\gamma)$  for every  $\gamma$  in  $\mathbf{W}$ .
- (5) The semigroup  $\overline{U_\Theta^{\geq 0}}$  is equal to the nonnegative semigroup  $U_\Theta^{\geq 0}$ , in particular the non-negative semigroup  $U_\Theta^{\geq 0}$  is closed and every element in it can be written as a finite product (of length at most  $N$ ) of elements of the form  $\exp(v_\alpha)$  with  $v_\alpha \in c_\alpha$ .
- (6) The semigroup  $U_\Theta^{\geq 0}$  is equal to the intersection of  $U_\Theta^{\geq 0}$  with  $\Omega_\Theta^{\text{opp}}$ .

*Proof.* The first point (1) is a direct consequence of Proposition 8.15 and the fact that the semigroup  $U_\Theta^{\geq 0}$  is generated by  $\exp(c_\alpha)$  (for  $\alpha$  in  $\Theta$ ).

Point (2) follows from (1), from the (obvious) inclusion  $U_\Theta^{\geq 0} \subset U_\Theta^{\geq 0}$ , and from the equivariance property of the maps  $F_\gamma$  (Lemma 6.4). Since, for every  $\gamma$  in  $\mathbf{W}$ , the map  $F_\gamma$  is proper, its image is closed and is equal to the closure of  $F_\gamma(\mathring{c}_\gamma) = U_\Theta^{\geq 0}$ . The closure of a semigroup is a semigroup and again by  $L_\Theta^\circ$ -equivariance, this concludes points (3) and (4).

We now have that  $\overline{U_\Theta^{\geq 0}}$  is a semigroup containing the elements  $\exp(v)$  for every  $\alpha$  in  $\Theta$  and  $v$  in  $c_\alpha$ . Hence this semigroup contains  $U_\Theta^{\geq 0}$ . Conversely, since  $\overline{U_\Theta^{\geq 0}}$  is the image of  $c_\gamma$  by  $F_\gamma$ , it is also contained in  $U_\Theta^{\geq 0}$  hence the equality of point (5); the other statements in that point follow from the already proven ones.

Since the map  $\mathring{F}_\gamma$  is open (cf. Corollary 8.5), its image  $U_\Theta^{\geq 0}$  is contained in the interior of  $U_\Theta^{\geq 0}$ . Conversely let  $u$  be in the interior of  $U_\Theta^{\geq 0}$ . There is thus a neighborhood  $V$  of the identity in  $G$  such that, for every  $v$  in  $V$ ,  $v^{-1}u$  belongs to  $U_\Theta^{\geq 0}$ . It is however clear that the image of  $\mathring{F}_\gamma$  contains elements in  $V$ . Let  $v$  be such an element. Then  $v$  belongs to  $U_\Theta^{\geq 0}$  and the equality  $u = v(v^{-1}u)$  (together with point (1)) shows that  $u$  belongs to  $U_\Theta^{\geq 0}$ . This concludes point (3).

Point (6) follows from the corresponding properties established for the parametrizations  $F_\gamma$  (points 4 and 5 of Theorem 8.1).  $\square$

*Remark 8.17.* Note that  $F_\gamma|_{c_\gamma}$  is in general not injective, and thus, point (5) of Corollary 8.16 does not give a parametrization of  $U_\Theta^{\geq 0}$ . We

discuss the structure of  $U_{\Theta}^{\geq 0}$  in the case when  $G = \mathrm{SO}(3, q)$ ,  $q \geq 4$  in more detail, see Section 8.9.2.

**8.9. The orthogonal groups.** In this subsection we discuss in a bit more detail the case when  $G = \mathrm{SO}(3, q)$ ,  $q \geq 4$ . The description of the  $\Theta$ -positive structure for general orthogonal groups  $\mathrm{SO}(p, q)$ ,  $q > p > 2$  restricts basically to this case and the description of the positive structure for  $\mathrm{SL}_3(\mathbf{R})$ , see also the discussion in [GW18]. For the case when  $G = \mathrm{SO}(3, q)$ ,  $q \geq 4$  we in particular also provide a parametrization of the non-negative semigroup  $U_{\Theta}^{\geq 0}$ .

8.9.1. *The positive semigroup.* We realize  $\mathrm{SO}(3, q) = \mathrm{SO}(b_Q)$ , where  $b_Q$  is the non-degenerate symmetric bilinear form of signature  $(3, q)$

on  $\mathbf{R}^{3+q}$  given by  $b_Q(v, w) = {}^t v Q w$ , with  $Q = \begin{pmatrix} 0 & 0 & K \\ 0 & J & 0 \\ -K & 0 & 0 \end{pmatrix}$ ,  $K =$

$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ , and  $J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -\mathrm{id}_{q-3} & 0 \\ 1 & 0 & 0 \end{pmatrix}$ . We denote by  $b_J$  the form

$b_J(x, y) = \frac{1}{2} {}^t x J y$  and set  $q_J(x) = b_J(x, x)$ . Note that  $b_J$  is a non-degenerate symmetric bilinear form of signature  $(1, q-2)$  on the corresponding subspace of  $\mathbf{R}^{q+3}$ .

We choose the Cartan subspace  $\mathfrak{a} \subset \mathfrak{so}(3, q)$  to be the intersection of the set of diagonal matrices with  $\mathfrak{so}(3, q)$ . Denote  $e_i: \mathfrak{a} \rightarrow \mathbf{R}$  the linear form that associates to a diagonal matrix its  $i$ -th diagonal coefficient. The set of (restricted) roots is  $\{\pm e_i \pm e_j\}_{1 \leq i < j \leq 3} \cup \{\pm e_i\}_{1 \leq i \leq 3}$ ; a classical choice for the set of positive roots is  $\{e_i \pm e_j\}_{1 \leq i < j \leq 3} \cup \{e_i\}_{1 \leq i \leq 3}$  and the set of simple roots  $\Delta = \{\alpha_1, \alpha_2, \alpha_3\}$  is given by  $\alpha_i = e_i - e_{i+1}$ , for  $i = 1, 2$ , and  $\alpha_3 = e_3$ .

We have  $\Theta = \{\alpha_1, \alpha_2\}$ . Furthermore  $\mathfrak{u}_{\alpha_1} \cong \mathbf{R}$  and  $\mathfrak{u}_{\alpha_2} \cong \mathbf{R}^{1, q-2}$ . Then the cone  $\mathring{c}_1 \cong \mathbf{R}_{>0}$ , and  $\mathring{c}_2 \cong \mathring{c}^{1, q-2} = \{x \in \mathbf{R}^{1, q-2} \mid q_J(x) > 0, x_1 > 0\}$ . Note that the closed cones identify with  $c_1 \cong \mathbf{R}_{\geq 0}$  and  $c_2 \cong c^{1, q-2} = \{x \in \mathbf{R}^{1, q-2} \mid q_J(x) \geq 0, x_1 \geq 0\}$ .

The Weyl group  $W$  is isomorphic to  $\{\pm 1\}^3 \rtimes S_3$ , the group of signed permutation matrices. It is presented by the generators  $s_1, s_2, s_3$  associated with the simple roots, with the relations  $s_i^2 = (s_1 s_2)^3 = (s_1 s_3)^2 = (s_2 s_3)^4 = e$ . The group  $W(\Theta)$  is generated by  $\sigma_1 = s_1$  and  $\sigma_2 = s_2 s_3 s_2$ . It is a Weyl group of type  $B_2$  and the longest word is  $w_{\max}^{\Theta} = \sigma_1 \sigma_2 \sigma_1 \sigma_2 = \sigma_2 \sigma_1 \sigma_2 \sigma_1$ .

The exponential maps  $\mathbf{R} \rightarrow U_\Theta$  and  $\mathbf{R}^{q-1} \rightarrow U_\Theta$  are denoted  $x_1, x_2$  and given by

$$x_1(s) = \exp \begin{pmatrix} 0 & s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & s \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & s & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \text{id}_{q-1} & 0 & 0 \\ 0 & 0 & 0 & 1 & s \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$x_2(v) = \exp \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & {}^t v & 0 & 0 \\ 0 & 0 & 0 & Jv & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & {}^t v & q_J(v) & 0 \\ 0 & 0 & \text{id}_{q-1} & Jv & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Thus, the positive semigroup  $U_\Theta^{>0}$  in this case is

$$U_\Theta^{>0} = F_{\sigma_1 \sigma_2 \sigma_1 \sigma_2}(\dot{c}_1 \times \dot{c}_2 \times \dot{c}_1 \times \dot{c}_2),$$

i.e. all matrices, that can be written as  $x_1(s_1)x_2(v_1)x_1(s_2)x_2(v_2)$ , with  $s_1, s_2 \in \mathbf{R}_{>0}$  and  $v_1, v_2 \in \dot{c}^{1, q-2}$ .

When we use the other reduced expression of the longest element  $w_{\max}^\Theta = \sigma_2 \sigma_1 \sigma_2 \sigma_1$ , and consider elements  $x_2(w_2)x_1(t_2)x_2(w_1)x_1(t_1)$  with  $t_1, t_2 \in \mathbf{R}_{>0}$  and  $w_1, w_2 \in \dot{c}^{1, q-2}$ , we parametrize as well the positive semigroup  $U_\Theta^{>0}$ .

The equation

$$(8.3) \quad x_1(s_1)x_2(v_1)x_1(s_2)x_2(v_2) = x_2(w_2)x_1(t_2)x_2(w_1)x_1(t_1),$$

determines then the change between these two parametrizations; comparing the entries of the corresponding matrices, we get the following relations

$$(8.4) \quad s_1 + s_2 = t_1 + t_2$$

$$(8.5) \quad v_1 + v_2 = w_1 + w_2$$

$$(8.6) \quad s_1(v_1 + v_2) + s_2 v_2 = t_2 w_1$$

$$(8.7) \quad s_1 q_J(v_1 + v_2) + s_2 q_J(v_2) = t_2 q_J(w_1),$$

$$(8.8) \quad s_2 v_1 = t_2 w_2 + t_1(w_1 + w_2),$$

$$(8.9) \quad s_2 q_J(v_1) = t_2 q_J(w_2) + t_1 q_J(w_1 + w_2)$$

$$(8.10) \quad s_1 s_2 q_J(v_1) = t_2 t_1 q_J(w_1)$$

*Remark 8.18.* Since the maps  $x_1, x_2$  have some compatibility with transposition (namely, for all  $v$  in  $\mathbf{R}^{q-1}$ ,  ${}^t x_2(v) = x_2(Jv) = Dx_2(v)D^{-1}$  where  $D$  is the bloc diagonal matrix with blocs  $\text{id}_2, J, \text{id}_2$ , and, for all  $s$  in  $\mathbf{R}$ ,  ${}^t x_1(s) = x_1(s) = Dx_1(s)D^{-1}$ ), Equation (8.3) above is equivalent to

$$x_1(t_1)x_2(w_1)x_1(t_2)x_2(w_2) = x_2(v_2)x_1(s_2)x_2(v_1)x_1(s_1).$$

Hence the chosen numbering will make the relation between  $(s_i, v_i)$  and  $(t_i, w_i)$  more symmetric.

We observe that Equations (8.4)–(8.7) determine the others: indeed multiplying (8.4)×(8.5) and subtracting (8.6) gives (8.8); the scalar product of (8.5) with the difference of (8.6) and (8.8) is the difference of (8.7) and (8.9); applying  $q_J$  to (8.6) and subtracting the product of (8.4) and (8.7) gives (8.10).

These equations can be solved as follow: the ratio of (8.10) and (8.7) gives  $t_1$ ; the ratio of  $q_J(8.6)$  and (8.7) gives  $t_2$ ; once  $t_2$  is determined, (8.6) gives  $w_1$ ; once  $w_1$  is determined (8.5) gives  $w_2$ . Explicitely we have

$$\begin{aligned} t_1 &= \frac{s_1 s_2 q_J(v_1)}{s_1 q_J(v_1 + v_2) + s_2 q_J(v_2)} \\ t_2 &= \frac{q_J(s_1(v_1 + v_2) + s_2 v_2)}{s_1 q_J(v_1 + v_2) + s_2 q_J(v_2)} \\ w_1 &= \frac{s_1 q_J(v_1 + v_2) + s_2 q_J(v_2)}{q_J(s_1(v_1 + v_2) + s_2 v_2)} (s_1(v_1 + v_2) + s_2 v_2) \\ w_2 &= \frac{s_2}{q_J(s_1(v_1 + v_2) + s_2 v_2)} \left( s_2 q_J(v_2) v_1 + \right. \\ &\quad \left. s_1 (q_J(v_1 + v_2) v_1 - q_J(v_1)(v_1 + v_2)) \right) \end{aligned}$$

Note that even though the formula for  $w_2$  contains a minus sign, the following lemma implies that  $w_2$  belongs to  $\mathring{c}_2$ .

**Lemma 8.19.** *For all  $v_1$  and  $v_2$  in  $\mathring{c}_2$ , the element*

$$a(v_1, v_2) = q_J(v_1 + v_2)v_1 - q_J(v_1)(v_1 + v_2)$$

*belongs to  $\mathring{c}_2$ .*

*Proof.* We calculate  $q_J(a(v_1, v_2))$ . This gives

$$q_J(a(v_1, v_2)) = q_J(v_1 + v_2)g_J(v_1)q_J(v_2).$$

Thus  $a(v_1, v_2)$  belongs to  $\mathring{c}_2 \sqcup -\mathring{c}_2$ . Since  $a(v_1, v_1) = 2q_J(v_1, v_1)v_1$  a connectedness argument shows that, for all  $v_1$  and  $v_2$  in  $\mathring{c}_2$ ,  $a(v_1, v_2)$  belongs to  $\mathring{c}_2$ .  $\square$

*Remark 8.20.* As explained the parametrization of the positive unipotent semigroup depend on the choice of a reduced expression of the longest word in  $W(\Theta)$ . In order to get explicit formulas for changes of coordinates it is sufficient to derive such formulas for braid relations of the form  $(\sigma_i \sigma_j)^{m_{ij}} = 1$  among the generators of  $W(\Theta)$ . For Lusztig's total positivity in split real groups formulas for the braid relations in the simply laced cases were determined by Lusztig [Lus98], and for the non-simply laced cases by Berenstein and Zelevinsky [BZ97]. The above gives explicit formulas for the braid relations in the  $SO(p, q)$  cases. The formulas for general braid relations for  $\Theta$ -positivity will be derived in [GW22].

8.9.2. *The non-negative semigroup.* Note that by point (5) of Corollary 8.16, the non-negative unipotent semigroup is characterized as

$$U_{\Theta}^{\geq 0} = F_{\sigma_1 \sigma_2 \sigma_1 \sigma_2}(c_1 \times c_2 \times c_1 \times c_2),$$

i.e. every element in  $U_{\Theta}^{\geq 0}$  can be written as

$$x_1(s_1)x_2(v_1)x_1(s_2)x_2(v_2),$$

with  $s_1, s_2 \in \mathbf{R}_{\geq 0}$  and  $v_1, v_2 \in c^{1, q-2}$ .

However this does not give a parametrization of  $U_{\Theta}^{\geq 0}$  since one can easily check that  $F_{\sigma_1 \sigma_2 \sigma_1 \sigma_2}$  is not injective. For example take  $v_1, v_2 \in c^{1, q-2}$  and  $s_1 = s_2 = 0$ . Then

$$x_1(s_1)x_2(v_1)x_1(s_2)x_2(v_2) = x_2(v_1)x_2(v_2) = x_2(v_1 + v_2),$$

hence  $F_{\sigma_1 \sigma_2 \sigma_1 \sigma_2}(0, v_1, 0, v_2) = F_{\sigma_1 \sigma_2 \sigma_1 \sigma_2}(0, v_1 + v_2, 0, 0)$ .

Giving a parametrization of  $U_{\Theta}^{\geq 0}$  is thus more subtle. Already in the case of split real groups, the maps  $F_{\gamma}$  (defined here on  $(\mathbf{R}_{\geq 0})^N$ ) are not injective (for the same reason than in the example above). A solution to this non-injectivity in this split case has been found by Lusztig in [Lus98, Corollary 2.8]; he showed that the non-negative unipotent semigroup  $U^{\geq 0}$  can be written as a disjoint union of subsets  $U^{>0}(w)$ , where  $w$  varies over the elements of the Weyl group  $W$  and  $U^{>0}(w)$  is the intersection of  $U^{\geq 0}$  with  $P_{\Delta}^{\text{opp}} w P_{\Delta}^{\text{opp}}$ . Then a parameterization of  $U^{>0}(w)$  is obtained as follows: a reduced expression  $\gamma = s_{i_1} \cdots s_{i_k}$  of  $w$  is chosen and the map  $\mathfrak{g}_{i_1} \times \cdots \times \mathfrak{g}_{i_k} \rightarrow U_{\Delta}$ ,  $(v_1, \dots, v_k) \mapsto \exp(v_1) \cdots \exp(v_k)$  is denoted  $F_{\gamma}$ ; then  $U^{>0}(w) = F_{\gamma}(\mathring{c}_{i_1} \times \cdots \times \mathring{c}_{i_k})$ . Lusztig showed first that this image does not depend of the choice of reduced expression. He introduced a monoidal structure on  $W$ , where given  $w_1$  and  $w_2$ , the element  $w_1 * w_2$  is determined by  $U^{>0}(w_1)U^{>0}(w_2) = U^{>0}(w_1 * w_2)$ .

In the parametrization of the positive unipotent semigroup  $U_{\Theta}^{\geq 0}$  the role of the Weyl group  $W$  in the split case is replaced by the Weyl group  $W(\Theta)$ , so one might hope to get a decomposition of  $U_{\Theta}^{\geq 0}$  into a disjoint union of parametrized sets, indexed by elements in  $W(\Theta)$ . However, we already saw in the case when  $\Theta = \{\alpha_{\Theta}\}$  that this does not work (cf. the discussion in Section 7.3). In order to describe a decomposition of  $U_{\Theta}^{\geq 0}$  we have to consider a different object and replace  $W$  by  $W_{\Delta \setminus \Theta} \backslash W / W_{\Delta \setminus \Theta}$  (which for  $\Theta = \Delta$  is obviously equal to  $W$ ).

We describe a decomposition of the nonnegative unipotent semigroup for  $\text{SO}(3, q)$  and a parameterization of it, using  $W_{\Delta \setminus \Theta} \backslash W / W_{\Delta \setminus \Theta}$ . It will be of interest to explore this further in the general case, and investigate the monoidal structure this gives.

In order to give a parametrization of  $U_{\Theta}^{\geq 0}$ , let us first consider the Bruhat decomposition of  $G$  with respect to the action of  $P_{\Theta}^{\text{opp}} \times P_{\Theta}^{\text{opp}}$ :  $G = \bigsqcup_{[w] \in W_{\Delta \setminus \Theta} \backslash W / W_{\Delta \setminus \Theta}} P_{\Theta}^{\text{opp}} w P_{\Theta}^{\text{opp}}$ . The next lemma determines the number of orbits in this decomposition.

**Lemma 8.21.** *The following provides a list of smallest length representatives for the 16 classes in  $W_{\Delta \setminus \Theta} \backslash W / W_{\Delta \setminus \Theta}$ : (1)  $e$  (2)  $s_1$  (3)  $s_2$  (4)  $(s_2 s_3 s_2)$  (5)  $s_2 s_1$  (6)  $(s_2 s_3 s_2)$*

*Proof.* Here the group  $W$  can be realized as the group of signed permutation  $3 \times 3$ -matrices and the subgroup  $W_{\Delta \setminus \Theta}$  is the subgroup (isomorphic to  $\mathbf{Z}/2\mathbf{Z}$ ) of diagonal matrices whose only possibly nontrivial diagonal coefficient is in the third position. The element  $s_1$  corresponds to the transposition (12),  $s_2$  corresponds to the transposition (23), and  $s_3$  is the nontrivial element in  $W_{\Delta \setminus \Theta}$ .

A direct calculation shows then the result.  $\square$

In the present situation the flag variety  $F_{\Theta}$  can be realized as the space of partial flags in  $\mathbf{R}^{3,q}$  consisting of an isotropic line contained in an isotropic 2-plane. For concreteness the canonical basis of  $\mathbf{R}^{3,q}$  with respect to which  $b_Q$  is given by the matrix  $Q$  above will be denoted

$$(e_1, e_2, e_3, g_1, g_2, \dots, g_{q-3}, f_3, f_2, f_1).$$

Then  $E^+ = (\mathbf{R}e_1, \mathbf{R}e_1 \oplus \mathbf{R}e_2)$  and  $E^- = (\mathbf{R}f_1, \mathbf{R}f_1 \oplus \mathbf{R}f_2)$  are partial isotropic flags, and  $P_{\Theta} = \text{Stab}(E^+)$  and  $P_{\Theta}^{\text{opp}} = \text{Stab}(E^-)$ . More generally, given a pair of vectors  $(v_1, v_2)$  in  $\mathbf{R}^{3,q}$ , the flag determined by  $(v_1, v_2)$  is denoted  $E_{(v_1, v_2)} = (\mathbf{R}v_1, \mathbf{R}v_1 \oplus \mathbf{R}v_2)$ ; this flag depends only on the lines  $\mathbf{R}v_1$  and  $\mathbf{R}v_2$ .

**Lemma 8.22.** *For each double  $P_{\Theta}^{\text{opp}}$  orbit in  $G$ , the following gives a flag in  $F_{\Theta}$  representing the corresponding  $P_{\Theta}^{\text{opp}}$  orbit in that quotient:*

- (1)  $e: E_{(f_1, f_2)}$  (2)  $s_1: E_{(f_2, f_1)}$  (3)  $s_2: E_{(f_1, f_3)}$
- (4)  $(s_2 s_3 s_2): E_{(f_1, e_2)}$  (5)  $s_2 s_1: E_{(f_3, f_1)}$  (6)  $(s_2 s_3 s_2) s_1: E_{(e_2, f_1)}$
- (7)  $s_1 s_2: E_{(f_2, f_3)}$  (8)  $s_1 (s_2 s_3 s_2): E_{(f_2, e_1)}$  (9)  $(s_2 s_3) s_1 s_2: E_{(f_3, e_2)}$
- (10)  $s_2 s_1 (s_2 s_3 s_2): E_{(f_3, e_1)}$  (11)  $(s_2 s_3 s_2) s_1 s_2: E_{(e_2, f_3)}$
- (12)  $(s_2 s_3 s_2) s_1 (s_2 s_3 s_2): E_{(e_2, e_1)}$  (13)  $s_1 s_2 s_1: E_{(f_3, f_2)}$
- (14)  $s_1 (s_2 s_3 s_2) s_1: E_{(e_1, f_2)}$  (15)  $s_1 (s_2 s_3 s_2) s_1 s_2: E_{(e_1, f_3)}$
- (16)  $s_1 (s_2 s_3 s_2) s_1 (s_2 s_3 s_2): E_{(e_1, e_2)}$ .

*Proof.* The flag  $E^- = E_{(f_1, f_2)}$  corresponds to the class of the trivial element. The flag corresponding to a double class  $P_{\Theta}^{\text{opp}}[w]P_{\Theta}^{\text{opp}}$ , associated with  $w$  in  $W$ , is then  $\dot{w} \cdot F^-$  where  $\dot{w}$  is a lift of  $W$  to  $\text{SO}(3, q)$ .

For the explicit calculation, we can choose

- For the lift of  $s_1$  the matrix that exchanges  $e_1$  and  $e_2$ , exchanges  $f_1$  and  $f_2$ , and fixes the other basis vectors.
- For the lift of  $s_2$  the matrix that exchanges  $e_2$  and  $e_3$ , exchanges  $f_2$  and  $f_3$ , and fixes the other basis vectors.
- For the lift of  $s_3$  the matrix that sends  $e_3$  to  $f_3$ , sends  $f_3$  to  $-e_3$ , and fixes the other basis vectors.  $\square$

For every  $x$  in  $W_{\Delta \setminus \Theta} \backslash W / W_{\Delta \setminus \Theta}$ , define  $U_{\Theta}^{\geq 0}(x)$  to be the intersection of  $P_{\Theta}^{\text{opp}} x P_{\Theta}^{\text{opp}}$  with  $U_{\Theta}^{\geq 0}$ . Let  $\gamma = (\sigma_1, \sigma_2, \sigma_1, \sigma_2)$  and  $F_{\gamma}$  be the map

$$\mathbf{R} \times \mathbf{R}^{q-1} \times \mathbf{R} \times \mathbf{R}^{q-1} \longrightarrow \text{SO}(3, q)$$

$$(s_1, v_1, s_2, v_2) \longmapsto x_1(s_1)x_2(v_1)x_1(s_2)x_2(v_2).$$

For each of the reduced expression  $w$  listed in Lemma 8.21 we define  $F_w$  to be the restriction of  $F_\gamma$  to the set  $D_w$  where

- (1)  $D_e = \{0\} \times \{0\} \times \{0\} \times \{0\}$ ,
- (2)  $D_{s_1} = \mathbf{R}_{>0} \times \{0\} \times \{0\} \times \{0\}$ ,
- (3)  $D_{s_2} = \{0\} \times (\partial c_2 \setminus \{0\}) \times \{0\} \times \{0\}$ ,
- (4)  $D_{s_2 s_3 s_2} = \{0\} \times \dot{c}_2 \times \{0\} \times \{0\}$ ,
- (5)  $D_{s_2 s_1} = \{0\} \times (\partial c_2 \setminus \{0\}) \times \mathbf{R}_{>0} \times \{0\}$ ,
- (6)  $D_{(s_2 s_3 s_2) s_1} = \{0\} \times \dot{c}_2 \times \mathbf{R}_{>0} \times \{0\}$ ,
- (7)  $D_{s_1 s_2} = \mathbf{R}_{>0} \times (\partial c_2 \setminus \{0\}) \times \{0\} \times \{0\}$ ,
- (8)  $D_{s_1 (s_2 s_3 s_2)} = \mathbf{R}_{>0} \times \dot{c}_2 \times \{0\} \times \{0\}$ ,
- (9)  $D_{(s_2 s_3) s_1 s_2} = \{(0, v_1, s_1, v_2) \mid s_1 > 0, v_1, v_2 \in \partial c_2, b_J(v_1, v_2) \neq 0\}$ ,
- (10)  $D_{s_2 s_1 (s_2 s_3 s_2)} = \{0\} \times (\partial c_2 \setminus \{0\}) \times \mathbf{R}_{>0} \times \dot{c}_2$ ,
- (11)  $D_{(s_2 s_3 s_2) s_1 s_2} = \{0\} \times \dot{c}_2 \times \mathbf{R}_{>0} \times (\partial c_2 \setminus \{0\})$ ,
- (12)  $D_{(s_2 s_3 s_2) s_1 (s_2 s_3 s_2)} = \{0\} \times \dot{c}_2 \times \mathbf{R}_{>0} \times \dot{c}_2$ ,
- (13)  $D_{s_1 s_2 s_1} = \mathbf{R}_{>0} \times (\partial c_2 \setminus \{0\}) \times \mathbf{R}_{>0} \times \{0\}$ ,
- (14)  $D_{s_1 (s_2 s_3 s_2) s_1} = \mathbf{R}_{>0} \times \dot{c}_2 \times \mathbf{R}_{>0} \times \{0\}$ ,
- (15)  $D_{s_1 (s_2 s_3 s_2) s_1 s_2} = \mathbf{R}_{>0} \times \dot{c}_2 \times \mathbf{R}_{>0} \times (\partial c_2 \setminus \{0\})$ ,
- (16)  $D_{s_1 (s_2 s_3 s_2) s_1 (s_2 s_3 s_2)} = \mathbf{R}_{>0} \times \dot{c}_2 \times \mathbf{R}_{>0} \times \dot{c}_2$ .

**Proposition 8.23.** *For every  $w$  in the list of Lemma 8.21 the map*

$$\begin{aligned} D_w &\longrightarrow \mathbf{F}_\Theta \\ z &\longmapsto F_w(z) \cdot F^- \end{aligned}$$

is injective; the image of  $F_w$  is  $U_\Theta^{\geq 0} \cap P_\Theta^{\text{opp}} w P_\Theta^{\text{opp}} = U_\Theta^{\geq 0}(x)$  with  $x = [w]$ . The nonnegative semigroup  $U_\Theta^{\geq 0}$  is the disjoint union the  $U_\Theta^{\geq 0}(x)$ .

*Proof.* The map  $(s_1, v_1, s_2, v_2) \mapsto F_\gamma(s_1, v_1, s_2, v_2) \cdot F^-$  is equivalently given by the last two columns of the matrix  $F_\gamma(s_1, v_1, s_2, v_2)$  that is

$$\begin{pmatrix} s_1 q_J(v_1 + v_2) + s_2 q_J(v_2) & s_1 s_2 q_J(v_1) \\ q_J(v_1 + v_2) & s_2 q_J(v_1) \\ v_1 + v_2 & s_2 v_1 \\ 1 & s_1 + s_2 \\ 0 & 1 \end{pmatrix}.$$

We can now check by case by case consideration that the statement holds.  $\square$

## 9. INVARIANT UNIPOTENT SEMIGROUPS

In order to define  $\Theta$ -positivity we took a very algebraic approach. In this section we will show that essentially any  $L_\Theta^\circ$  invariant semigroup  $U^+$  in  $U_\Theta$  arise from a  $\Theta$ -positive structure. This gives the two following theorems characterizing  $\Theta$ -positivity. These results will be useful to make the connection with a geometric characterization of  $\Theta$ -positivity (Section 10).

**Theorem 9.1.** *Let  $G$  be a connected semisimple Lie group, and let  $U_\Theta$  be a standard unipotent subgroup of  $G$  and  $L_\Theta$  be the corresponding standard Levi subgroup. Suppose that there is  $U^+ \subset U_\Theta$  such that*

- (1)  $U^+$  is closed and of nonempty interior;
- (2)  $U^+$  is  $L_\Theta^\circ$ -invariant;
- (3)  $U^+$  is a semigroup;
- (4)  $U^+$  contains no nontrivial invertible element (if  $g \in U^+$  and  $g^{-1} \in U^+$  then  $g = e$ ).

*Then  $G$  admits a  $\Theta$ -positive structure and the semigroup  $U^+$  contains the semigroup  $U_\Theta^{\geq 0}$ .*

*If we further assume that*

- (5) *the interior of  $U^+$  is contained in  $\Omega_\Theta^{\text{opp}}$ ,*

*then  $U^+ = U_\Theta^{\geq 0}$ .*

In fact, when condition (5) holds the sharpness of the semigroup (condition (4)) can be relaxed.

**Theorem 9.2.** *Let  $G$  be a connected semisimple Lie group, and let  $U_\Theta$  be a standard unipotent subgroup of  $G$  and  $\Omega_\Theta^{\text{opp}}$  the open Bruhat cell with respect to the parabolic group  $P_\Theta^{\text{opp}}$ . Suppose that there is  $V^+ \subset U_\Theta$  such that*

- (1)  $V^+$  is a connected component of  $U_\Theta \cap \Omega_\Theta^{\text{opp}}$ .
- (2)  $V^+$  is a semigroup;

*Then  $G$  admits a  $\Theta$ -positive structure and the semigroup  $V^+$  is equal to  $U_\Theta^{\geq 0}$ .*

Sections 9.1–9.2 address the proof of Theorem 9.1 and Section 9.3 addresses the proof of Theorem 9.2. Section 9.4 gives examples of semigroups that satisfy the hypothesis of Theorem 9.1 and are not equal to  $U_\Theta^{\geq 0}$ .

**9.1. The cones associated with  $U^+$ .** With the notation of Theorem 9.1 (and the general notation introduced in Section 2), for all  $\alpha$  in  $\Theta$  we introduce

$$k_\alpha = p_\alpha(\log(U^+))$$

where  $p_\alpha: \mathfrak{u}_\Theta \rightarrow \mathfrak{u}_\alpha$  is the projection coming from the decomposition  $\mathfrak{u}_\Theta = \bigoplus_\beta \mathfrak{u}_\beta$ .

We will first show that

**Proposition 9.3.** *Let  $X_\alpha$  be an element of  $\mathfrak{u}_\alpha$ . Then  $X_\alpha$  belongs to  $k_\alpha$  if and only if  $\exp(X_\alpha)$  belongs to  $U^+$ .*

*Proof.* Let us define the following sequence  $(A_n)_{n \in \mathbf{N}}$  in  $\mathfrak{a}$ ; for all  $n \in \mathbf{N}$ ,  $A_n$  is the element of  $\mathfrak{a}$  defined by the equalities

$$\alpha(A_n) = 0, \gamma(A_n) = 0 \quad \forall \gamma \in \Delta \setminus \Theta \quad \text{and} \quad \gamma(A_n) = -n \quad \forall \gamma \in \Theta \setminus \{\alpha\}.$$

One has then, for all  $X_\alpha \in \mathfrak{u}_\alpha$

$$\mathrm{ad}(A_n)X_\alpha = 0$$

and for all  $\beta$  not congruent to  $\alpha$  modulo the span of  $\Delta \setminus \Theta$  and for all  $Y$  in  $\mathfrak{u}_\beta$

$$\mathrm{ad}(A_n)X_\beta = -s_\beta n X_\beta$$

for some  $s_\beta > 0$ . The sequence  $(g_n = \exp(A_n))_{n \in \mathbf{N}}$  belongs to  $\exp(\mathfrak{a}) \subset L_\Theta^\circ$  and, for every  $X$  in  $\mathfrak{u}_\Theta$ , the sequence

$$\mathrm{Ad}(g_n) \cdot X = \exp(\mathrm{ad}(A_n))X$$

converges to  $X_\alpha := p_\alpha(X)$ ; indeed using the decomposition  $X = \sum X_\beta$  according to the direct sum  $\mathfrak{u}_\Theta = \bigoplus_\beta \mathfrak{u}_\beta$ , one has

$$\begin{aligned} \mathrm{Ad}(g_n) \cdot X &= \exp(\mathrm{ad}(A_n))X_\alpha + \sum_{\beta \neq \alpha} \exp(\mathrm{ad}(A_n))X_\beta \\ &= X_\alpha + \sum_{\beta \neq \alpha} e^{-s_\beta n} X_\beta. \end{aligned}$$

Let now  $X_\alpha$  be in  $k_\alpha$ . There exists then  $X \in \mathfrak{u}_\Theta$  such that  $\exp(X)$  belongs to  $U^+$  and  $X_\alpha = p_\alpha(X)$ . By the above, the sequence

$$g_n \exp(X) g_n^{-1} = \exp(\mathrm{Ad}(g_n)X)$$

converges to  $\exp(X_\alpha)$ . Since  $U^+$  is  $L_\Theta^\circ$ -invariant and closed, one has  $\exp(X_\alpha)$  belongs to  $U^+$ .

Conversely if  $\exp(X_\alpha)$  belongs to  $U^+$ , then

$$p_\alpha(\log(\exp(X_\alpha))) = p_\alpha(X_\alpha) = X_\alpha$$

belongs to  $k_\alpha$ . □

From this we deduce:

**Corollary 9.4.** *The set  $k_\alpha$  is a closed  $L_\Theta^\circ$ -invariant convex cone.*

*Proof.* By the previous proposition  $k_\alpha = \log^{-1}(U^+ \cap \exp(\mathfrak{u}_\alpha))$ . Since  $U^+ \cap \exp(\mathfrak{u}_\alpha)$  is closed and  $L_\Theta^\circ$ -invariant, this implies that  $k_\alpha$  is closed and  $L_\Theta^\circ$ -invariant.

Let  $X_\alpha \in k_\alpha$  and let  $t > 0$ . Let  $A \in \mathfrak{a}$  be defined by  $\alpha(A) = \log t$  and  $\gamma(A) = 0$  for all  $\gamma \in \Delta \setminus \{\alpha\}$ . Then

$$tX_\alpha = \exp(\mathrm{ad}(A))X_\alpha = \mathrm{Ad}(\exp(A))X_\alpha$$

belongs to  $k_\alpha$  since  $\exp(A)$  belongs to  $L_\Theta^\circ$ . This means that  $k_\alpha$  is a cone.

Let  $X_\alpha$  and  $Y_\alpha$  be in  $k_\alpha$ . Then  $\exp(X_\alpha)\exp(Y_\alpha)$  belongs to  $U^+$ . Thus, from the Baker–Campbell–Hausdorff formula,

$$p_\alpha \circ \log(\exp(X_\alpha)\exp(Y_\alpha)) = X_\alpha + Y_\alpha$$

belongs to  $k_\alpha$ . This is the property that  $k_\alpha$  is convex. □

**9.2. The positive structure.** The last point to make the relation with Definition 3.1 is that  $k_\alpha$  are nontrivial and acute:

**Lemma 9.5.** *For every  $\alpha$  in  $\Theta$ , the cone  $k_\alpha$  is not reduced to  $\{0\}$  and contains no line.*

*Proof.* Suppose that  $k_\alpha = \{0\}$ . Then  $U^+$  would be contained in  $\{g \in U_\Theta \mid p_\alpha(\log(g)) = 0\}$  and would be of empty interior, contrary to the assumptions.

Let now  $X$  in  $\mathfrak{u}_\alpha$  be such that  $X$  and  $-X$  belong to  $k_\alpha$ . Then  $g = \exp(X)$  belongs to  $U^+$  as well as  $g^{-1} = \exp(-X)$ . Thus  $g = e$  (since  $U^+$  contains no nontrivial invertible element) and  $X = \log(g) = 0$ . This means that the convex cone  $k_\alpha$  contains no line.  $\square$

Thus  $G$  has a  $\Theta$ -positive structure and one can introduce the semi-group  $U_\Theta^{\geq 0}$  with the following choice  $c_\alpha = k_\alpha$  of invariant cones. With this choice one has obviously the inclusion  $U_\Theta^{\geq 0} \subset U^+$ . This concludes the proof of Theorem 9.1 except for the last point, but this point will be a direct consequence of Theorem 9.2 that we will prove in the next section.

**9.3. The cones associated with  $V^+$  and the induced positive structure.** We now turn to the proof of Theorem 9.2. The strategy is similar to the above and we introduce, for every  $\alpha$  in  $\Theta$

$$k_\alpha = p_\alpha(\log(\overline{V}^+)).$$

Observe that  $\overline{V}^+$  is a semigroup.

It is easy to show that

**Proposition 9.6.** *Let  $X_\alpha$  be an element of  $\mathfrak{u}_\alpha$ . Then  $X_\alpha$  belongs to  $k_\alpha$  if and only if  $\exp(X_\alpha)$  belongs to  $\overline{V}^+$ .*

Furthermore the  $L_\Theta^\circ$ -invariance is a consequence of the hypothesis here:

**Lemma 9.7.** *The semigroup  $V^+$  is  $L_\Theta^\circ$ -invariant.*

*Proof.* Indeed  $U_\Theta$  and  $\Omega_\Theta^{\text{opp}}$  are invariant by  $L_\Theta^\circ$ , hence every connected component of  $U_\Theta \cap \Omega_\Theta^{\text{opp}}$  is invariant by  $L_\Theta^\circ$ .  $\square$

From the results already established we deduce as above

**Lemma 9.8.** *For every  $\alpha$  in  $\Theta$ ,  $k_\alpha$  is a closed convex  $L_\Theta^\circ$ -invariant cone in  $\mathfrak{u}_\alpha$  and is not reduced to  $\{0\}$ .*

**Lemma 9.9.** *The following inclusions hold*

$$V^+\overline{V}^+ \subset V^+, \text{ and } \overline{V}^+V^+ \subset V^+.$$

*Remark 9.10.* Equalities hold in fact since the neutral element belongs to  $\overline{V}^+$ .

*Proof.* Let  $x$  be an element of  $V^+\overline{V}^+$ . There exist thus  $u \in V^+$  and  $v \in \overline{V}^+$  such that  $x = uv$ . Let  $(v_n)$  be a sequence in  $V^+$  converging to  $v$ . Then the sequence defined by, for all  $n \in \mathbf{N}$ ,  $u_n = xv_n^{-1} = uvv_n^{-1}$  converges to  $u$ . There exists hence  $n_0$  such that, for all  $n > n_0$ ,  $u_n$  belongs to  $V^+$ . For such  $n$ , the equality  $x = u_nv_n$  shows that  $x$  belongs to the semigroup  $V^+$ .

The other inclusion follows by similar arguments.  $\square$

**Corollary 9.11.** *The open set  $V^+$  is the interior of its closure  $\overline{V}^+$ .*

*Proof.* Since  $V^+$  is open and contained in  $\overline{V}^+$ , one has  $V^+ \subset \overset{\circ}{\overline{V}^+}$ .

Let  $(y_n)$  be a sequence in  $V^+$  converging to the neutral element (the existence of such sequence is insured, for example, by the  $L_\Theta^\circ$ -invariance).

Let  $x$  be an element of  $\overset{\circ}{\overline{V}^+}$ . Since the sequence  $(y_n^{-1}x)$  converges to  $x$ , there exists  $n_0$  such that, if  $n > n_0$ , then  $y_n^{-1}x$  belongs to  $\overset{\circ}{\overline{V}^+}$ . For such an integer  $n$ , the equality  $x = y_n(y_n^{-1}x)$  together with Lemma 9.9 imply that  $x$  belongs to  $V^+$ .  $\square$

**Corollary 9.12.** *For all  $\alpha$  in  $\Theta$ ,  $k_\alpha$  is an acute convex cone in  $\mathfrak{u}_\alpha$ .*

*Proof.* Since  $k_\alpha$  is a closed convex cone and is  $L_\Theta^\circ$ -invariant, the subspace of maximal dimension contained in  $k_\alpha$  is  $L_\Theta^\circ$ -invariant. Since the action of  $L_\Theta^\circ$  on  $\mathfrak{u}_\alpha$  is irreducible (cf. Theorem 2.1), this subspace is either  $\{0\}$  or  $\mathfrak{u}_\alpha$ . Hence we have to exclude the case when  $k_\alpha = \mathfrak{u}_\alpha$ .

Suppose, by contradiction, that  $k_\alpha = \mathfrak{u}_\alpha$ . We then have (Proposition 9.6)  $\exp \mathfrak{u}_\alpha \subset \overline{V}^+$ . Recall that the Lie algebra  $\mathfrak{u}_\Theta$  admits the following decomposition

$$\mathfrak{u}_\Theta = \mathfrak{u}_\alpha \oplus \mathfrak{u}_{\hat{\alpha}}, \text{ where } \mathfrak{u}_{\hat{\alpha}} = \bigoplus_{\beta \neq \alpha} \mathfrak{u}_\beta$$

and that the map

$$\begin{aligned} \mathfrak{u}_\alpha \oplus \mathfrak{u}_{\hat{\alpha}} &\longrightarrow U_\Theta \\ (X, Y) &\longmapsto \exp(X)\exp(Y) \end{aligned}$$

is a diffeomorphism.

Let now  $x$  be an element of  $V^+$ . There exist thus  $X \in \mathfrak{u}_\alpha$  and  $Y \in \mathfrak{u}_{\hat{\alpha}}$  such that  $x = \exp(X)\exp(Y)$ . The equality  $k_\alpha = \mathfrak{u}_\alpha$  implies that  $\exp(-X)$  belongs to  $\overline{V}^+$  and, by the Lemma 9.9,  $\exp(Y) = \exp(-X)x$  belongs to  $V^+$ . But this element  $\exp(Y)$  does not belong to  $\Omega_\Theta^{\text{opp}}$ , in contradiction with the fact that  $V^+$  is contained in  $\Omega_\Theta^{\text{opp}}$ .  $\square$

As a consequence the group  $G$  has a  $\Theta$ -positive structure and we can choose  $c_\alpha = k_\alpha$  in the construction of  $U_\Theta^{>0}$ . The last point is to notice

**Lemma 9.13.** *One has  $V^+ = U_\Theta^{>0}$ .*

*Proof.* These two open sets are connected components of  $U_\Theta \cap \Omega_\Theta^{\text{opp}}$  and must intersect; they are then equal.  $\square$

**9.4. Semigroups bigger than  $U_{\Theta}^{\geq 0}$ .** We end this section with an example of semigroups contained in the unipotent standard subgroup of  $\mathrm{SL}_3(\mathbf{R})$  that are bigger than the semigroup of totally positive unipotent matrices.

For each  $r \in \mathbf{R}$  denote by  $U_r$  the set of matrices

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

with  $a, b \geq 0$  and  $0 \leq c \leq rab$ .

*Remark 9.14.* The semigroup of totally nonnegative unipotent matrices is  $U_1$ .

**Lemma 9.15.** *For all  $r \geq 1$ ,  $U_r$  is a closed semigroup without nontrivial invertible element and invariant by conjugation by diagonal matrices with positive coefficients.*

*Proof.* Closedness and invariance are easily checked. To check the semigroup property, we calculate the product of two elements of  $U_r$ :

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a' & c' \\ 0 & 1 & b' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a'' & c'' \\ 0 & 1 & b'' \\ 0 & 0 & 1 \end{pmatrix}$$

then  $a'' = a + a' \geq 0$ ,  $b'' = b + b' \geq 0$  and  $c'' = c + c' + ab' \geq 0$ , thus

$$ra''b'' - c'' = (rab - c) + (ra'b' - c') + ra'b + (r - 1)ab' \geq 0,$$

and the product belongs to  $U_r$ . The above formula shows that if an element

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

is invertible in  $U_r$  then necessarily  $a = b = 0$  and for such an element the condition of being in  $U_r$  says  $0 \leq c \leq 0$ , i.e.  $c = 0$ . This proves the wanted properties for  $U_r$ .  $\square$

## 10. POSITIVITY IN THE FLAG VARIETY $F_{\Theta}$

In this section we use the positive unipotent semigroups to give a notion of positivity in the flag variety  $F_{\Theta}$ .

**10.1. A first diamond.** We assume that  $G$  has a  $\Theta$ -positive structure.

The positive semigroups  $U_{\Theta}^{>0}$  and  $U_{\Theta}^{\mathrm{opp}, >0}$  will enable us to define “diamonds” (cf. Section 10.4 below) in the flag variety  $F_{\Theta}$ .

We will denote by  $\mathcal{O} \subset F_{\Theta}$  the set of points transverse to  $\mathfrak{p}_{\Theta}$  and by  $\mathcal{O}^{\mathrm{opp}}$  the set of points transverse to  $\mathfrak{p}_{\Theta}^{\mathrm{opp}}$ . These sets are open and diffeomorphic to the unipotent Lie groups  $U_{\Theta}$  and  $U_{\Theta}^{\mathrm{opp}}$ ; indeed the maps  $U_{\Theta} \rightarrow \mathcal{O} \mid u \mapsto u \cdot \mathfrak{p}_{\Theta}$  and  $U_{\Theta}^{\mathrm{opp}} \rightarrow \mathcal{O}^{\mathrm{opp}} \mid u \mapsto u \cdot \mathfrak{p}_{\Theta}$  are

diffeomorphisms. Since  $w_\Delta \cdot \mathfrak{p}_\Theta = \mathfrak{p}_\Theta^{\text{opp}}$  and  $w_\Delta \cdot \mathfrak{p}_\Theta^{\text{opp}} = \mathfrak{p}_\Theta$ , one has also  $\mathcal{O} = \Omega_\Theta \cdot \mathfrak{p}_\Theta$  and  $\mathcal{O}^{\text{opp}} = \Omega_\Theta^{\text{opp}} \cdot \mathfrak{p}_\Theta^{\text{opp}}$ .

Our first result is

**Proposition 10.1.** *One has the equality, in  $F_\Theta$ ,*

$$U_\Theta^{>0} \cdot \mathfrak{p}_\Theta^{\text{opp}} = U_\Theta^{\text{opp}, >0} \cdot \mathfrak{p}_\Theta.$$

*More precisely these sets are equal to the same connected component of  $\mathcal{O} \cap \mathcal{O}^{\text{opp}}$ .*

*Proof.* Point (7) of Theorem 8.1 implies that  $U_\Theta^{>0} \cdot \mathfrak{p}_\Theta^{\text{opp}}$  is a connected component of  $\mathcal{O} \cap \mathcal{O}^{\text{opp}}$ . Equally,  $U_\Theta^{\text{opp}, >0} \cdot \mathfrak{p}_\Theta$  is a connected component of  $\mathcal{O} \cap \mathcal{O}^{\text{opp}}$ . Hence we only need to see that these sets intersect. This can be obtained exactly along the same lines than Proposition 8.13 using the split Lie subalgebra  $\mathfrak{g}_\Theta$  and Lusztig's work. Indeed [Lus94, Th. 8.7] precisely implies that the two above sets intersect.  $\square$

**10.2. Axiomatic of diamonds.** In this section and the next one, we do not assume that  $G$  has a  $\Theta$ -positive structure. We will prove however that the definition below forces the presence of a  $\Theta$ -positive structure.

We introduce now the expected properties that one wants for positive triples in  $F_\Theta$ . It will be a little easier to express these properties in terms of ‘‘diamonds’’ that are, fixing  $a$  and  $b$  in  $F_\Theta$ , the connected components of the set  $\{x \in F_\Theta \mid (a, x, b) \text{ is a positive triple}\}$  (cf. below Definition 10.14).

Let  $\Theta \subset \Delta$  be a subset invariant by the involution  $\iota: \alpha \mapsto -w_\Delta \cdot \alpha$  (so that  $F_{\iota(\Theta)} = F_\Theta$  and we can speak of transverse pairs in  $F_\Theta$ ). For a point  $a$  in  $F_\Theta$ , let us denote by  $\mathcal{O}_a$  the (open) subset of  $F_\Theta$  whose points are those transverse to  $a$ .

**Definition 10.2.** *A family of diamonds in  $F_\Theta$  is a family  $\mathcal{F}$  of triples, called diamonds,  $(D, a, b)$  where  $D$  is a subset of  $F_\Theta$  and  $a, b$  belong to  $F_\Theta$  (and will be called the extremities of the diamond) such that*

- (1) For every  $(D, a, b)$  in  $\mathcal{F}$ ,  $a$  is transverse to  $b$  and  $D$  is a connected component of  $\mathcal{O}_a \cap \mathcal{O}_b$ ;
- (2) For every  $(D, a, b)$  in  $\mathcal{F}$ ,  $(D, b, a)$  belongs to  $\mathcal{F}$ ;
- (3) For every  $(D, a, b)$  in  $\mathcal{F}$ , and for every  $g$  in  $\text{Aut}(\mathfrak{g})$ ,  $(g \cdot D, g \cdot a, g \cdot b)$  belongs to  $\mathcal{F}$ ;
- (4) For every  $(D, a, b)$  in  $\mathcal{F}$ , and for every  $x$  in  $D$ , there exists a unique diamond  $(D', a, x)$  in  $\mathcal{F}$  such that  $D'$  is contained in  $D$ .

The symmetry condition (2) together with the last condition imply also

- (5) For every  $(D, a, b)$  in  $\mathcal{F}$ , and for every  $x$  in  $D$ , there exists a unique diamond  $(D', x, b)$  in  $\mathcal{F}$  such that  $D'$  is contained in  $D$ .

*Remark 10.3.* It is natural to ask that the notion of diamond (or the notion of positive triple to come later) is invariant under all the automorphisms of the flag variety  $F_\Theta$ . This is why we require invariance under the group  $\text{Aut}(\mathfrak{g})$  in the definition and not only under  $G$ .

We can immediately note that:

- For all  $a$  and  $b$  in  $F_\Theta$ , with  $a$  transverse to  $b$ , there is a least one diamond with extremities  $a$  and  $b$  (simply since the group  $\text{Aut}(\mathfrak{g})$  acts transitively on the space of pairs of transverse points), and there are finitely many such diamonds (since  $\mathcal{O}_a \cap \mathcal{O}_b$  has finitely many components);
- For every diamond  $(D, a, b)$  the set  $D$  is invariant under the connected Lie group  $L_{a,b}^\circ$  that is the neutral component of the stabilizer (in  $G$ ) of the pairs  $(a, b)$ .

**10.3. From diamonds to  $\Theta$ -positivity.** We assume in this subsection that there is a family of diamonds  $\mathcal{F}$  in  $F_\Theta$ . We immediately note that this family is produced from a semigroup in  $U_\Theta$ .

**Proposition 10.4.** *Let  $a$  and  $b$  be respectively the elements  $\mathfrak{p}_\Theta$  and  $\mathfrak{p}_\Theta^{\text{opp}}$  of  $F_\Theta$ . Let  $(D, a, b)$  be a diamond in  $\mathcal{F}$  with extremities  $a$  and  $b$ . Then the set*

$$V^+ := \{v \in U_\Theta \mid v \cdot b \in D\}$$

*is a connected component of  $U_\Theta \cap \Omega_\Theta^{\text{opp}}$  and is a semigroup.*

*Proof.* The fact that  $V^+$  is a connected component of  $U_\Theta \cap \Omega_\Theta^{\text{opp}}$  follows directly from the conditions on the family of diamonds  $\mathcal{F}$ .

Proving that  $V^+$  is a semigroup amounts to show the inclusion, for all  $v$  in  $V^+$ ,  $vV^+ \subset V^+$ . This inclusion can be phrased in term of diamonds: let  $x = v \cdot b$  so that  $x$  belongs to  $D$  and there is a unique diamond  $(D', a, x)$  with extremities  $a$  and  $x$  with  $D' \subset D$ ; with this notation, one wants to prove the equality  $D' = v \cdot D$ .

Let  $\ell_s = \exp(sX)$  ( $s \in \mathbf{R}$ ) be the 1-parameter subgroup associated with  $X$ , the element of  $\mathfrak{a}$  such that  $\alpha(X) = -1$  for all  $\alpha$  in  $\Delta$ . Define  $(x_t)_{t \in [0,1]}$  and  $(v_t)_{t \in [0,1]}$  by the equalities  $x_t = v_t \cdot b$ ,  $v_0 = e$ , and  $v_t = \ell_{\log t} v \ell_{\log t}^{-1}$  for  $t > 0$ . By the choice of  $X$ , the path  $(v_t)_{t \in [0,1]}$  is continuous and  $x_0 = b$ ,  $x_1 = x$ .

For all positive  $t$ ,  $D'_t = \ell_{\log t} \cdot D'$  is the diamond with extremities  $a$  and  $x_t$  contained in  $D$ ; i.e.  $D'_t$  is the connected component of  $\mathcal{O}_a \cap \mathcal{O}_{x_t}$  contained in  $D$ . From the convergence of  $x_t$  to  $b$ , we deduce that  $D'_t$  converges to  $D$  as  $t$  tends to 0. Stated differently, setting  $D'_0 = D$ , the family  $(D'_t)_{t \in [0,1]}$  is continuous. This implies that the family  $(v_t^{-1} \cdot D'_t)_{t \in [0,1]}$  of diamonds with extremities  $a$  and  $b$  is continuous. Since there are finitely many connected components in  $\mathcal{O}_a \cap \mathcal{O}_b$ , we deduce that this family is constant equal to  $D$ . In particular  $D = v_1^{-1} \cdot D'_1 = v^{-1} \cdot D'$  which is the sought for equality.  $\square$

This means that Theorem 9.2 applies and that  $G$  admits a  $\Theta$ -positive structure. Let us then fix, for every  $\alpha$  in  $\Theta$ , a nonzero acute  $L_\Theta^\circ$ -invariant closed convex cone  $c_\alpha$  in  $\mathfrak{u}_\alpha$ . We already noticed that there are exactly two such cones, namely  $c_\alpha$  and  $-c_\alpha$ . To take into account all the possible choices of cones, let us introduce, for every  $\varepsilon = (\varepsilon_\alpha)_{\alpha \in \Theta}$

in  $\{\pm 1\}^\Theta$  the nonnegative unipotent semigroup  $U_\Theta^{\varepsilon \geq 0}$  generated by  $\exp(\varepsilon_\alpha c_\alpha)$  (for  $\alpha$  varying in  $\Theta$ ) and the positive unipotent semigroup  $U_\Theta^{\varepsilon > 0}$  that is the interior (relative to  $U_\Theta$ ) of  $U_\Theta^{\varepsilon \geq 0}$ .

Applying Theorem 9.2 and the fact that the cones  $c_\alpha$  are determined up to sign, we get the following corollary

**Corollary 10.5.** *Let  $a$  and  $b$  be respectively  $\mathfrak{p}_\Theta$  and  $\mathfrak{p}_\Theta^{\text{opp}}$ . Then the diamonds in  $\mathcal{F}$  with extremities  $a$  and  $b$  are exactly the triples  $(U_\Theta^{\varepsilon > 0} \cdot b, a, b)$  for  $\varepsilon$  varying in  $\{\pm 1\}^\Theta$ .*

*Proof.* Indeed Proposition 10.4 and Theorem 9.2 prove that every diamond with extremities  $a$  and  $b$  has this form. The fact that all signs  $\varepsilon$  are possible is a consequence of the invariance under  $\text{Aut}(\mathfrak{g})$  and that there are elements in this group exchanging the cones (Proposition 3.8).  $\square$

In fact the transitivity observed in Proposition 3.8 implies that there is exactly one orbit of diamonds under the action of  $\text{Aut}(\mathfrak{g})$ .

**Corollary 10.6.** *Let  $a$  and  $b$  be  $\mathfrak{p}_\Theta$  and  $\mathfrak{p}_\Theta^{\text{opp}}$ . Then  $(D, a, b) = (U_\Theta^>0 \cdot b, a, b)$  is a diamond and the diamonds are exactly the  $(g \cdot D, g \cdot a, g \cdot b)$  for  $g$  varying in  $\text{Aut}(\mathfrak{g})$ .*

**10.4. From  $\Theta$ -positivity to diamonds.** We now consider the reverse direction of Corollary 10.6. Namely, we assume that  $G$  has a  $\Theta$ -positive structure and consider the family  $\mathcal{F}$  consisting of the  $\text{Aut}(\mathfrak{g})$ -orbit of the triple  $(U_\Theta^>0 \cdot b, a, b)$  where  $a$  and  $b$  are respectively  $\mathfrak{p}_\Theta$  and  $\mathfrak{p}_\Theta^{\text{opp}}$ .

Equivalently, the family  $\mathcal{F}$  can be defined by taking the  $G$ -orbits of the diamonds  $(U_\Theta^{\varepsilon > 0} \cdot b, a, b)$  for  $\varepsilon$  varying in  $\{\pm 1\}^\Theta$ .

**Proposition 10.7.** *The family  $\mathcal{F}$  is a family of diamonds in  $F_\Theta$ .*

*Proof.* We need to check the conditions of Definition 10.2. Condition (1) follows from the corresponding property of  $U_\Theta^>0$  (point (7) of Theorem 8.1) (cf. also Proposition 10.1). Condition (2) is a consequence of the equality  $w_\Delta U_\Theta^>0 w_\Delta^{-1} = U_\Theta^{\text{opp}, > 0}$  and of Proposition 10.1. Condition (3) is there by construction of the family  $\mathcal{F}$ . The existence part in Condition (4) follows directly from the fact that  $U_\Theta^>0$  is a semigroup.

Let us now address the uniqueness in Condition (4). With the notation introduced in this condition, we can (by  $\text{Aut}(\mathfrak{g})$ -invariance) assume that  $a = \mathfrak{p}_\Theta$ ,  $b = \mathfrak{p}_\Theta^{\text{opp}}$ , and  $D = U_\Theta^>0 \cdot b$ . Let  $u$  be the element of  $U_\Theta^>0$  such that  $x = u \cdot b$  and let  $(D', a, x)$  be a diamond in  $\mathcal{F}$  such that  $D' \subset D$ . Then  $(u^{-1}D', a, b)$  is a diamond with extremities  $a$  and  $b$ .

There is thus an element  $\ell$  in  $\text{Aut}(\mathfrak{g})$  fixing  $a$  and  $b$  and such that  $u^{-1} \cdot D' = \ell \cdot D$ . The element  $\ell$  stabilizes all the spaces  $\mathfrak{u}_\alpha$  (for  $\alpha$  in  $\Theta$ ) and thus sends, for every  $\alpha$  in  $\Theta$ , the cone  $c_\alpha$  to  $\varepsilon_\alpha c_\alpha$  for some  $\varepsilon_\alpha$  in  $\{\pm 1\}$ . Setting  $\varepsilon = \{\varepsilon_\alpha\}_{\alpha \in \Theta}$  one has thus  $u^{-1} \cdot D' = U_\Theta^{\varepsilon > 0} \cdot b$  and the inclusion  $D' \subset D$  can be rewritten as  $uU_\Theta^{\varepsilon > 0} \subset U_\Theta^>0$ .

The sought for uniqueness is now equivalently expressed in the equalities  $\varepsilon_\alpha = 1$  for all  $\alpha$  in  $\Theta$ . Suppose by contradiction that there exists  $\alpha$  with  $\varepsilon_\alpha = -1$ . Then, using for example the parametrization of the semigroups, one can construct an element  $v$  in  $U_\Theta^{\varepsilon > 0}$  such that  $\pi_\alpha(\log(uv)) = \pi_\alpha(\log u) + \pi_\alpha(\log v) = 0$  (where again  $\pi_\alpha: \mathfrak{u}_\Theta \rightarrow \mathfrak{u}_\alpha$  is the projection on the factor  $\mathfrak{u}_\alpha$ ); this is incompatible with the fact that  $uv$  belongs to  $U_\Theta^{> 0}$  since we should have that  $\pi_\alpha(\log(uv))$  belongs to  $\mathfrak{c}_\alpha$ .  $\square$

**10.5.  $U$ -pinnings.** We make the relation with the diamonds introduced here and the definition that was used in [GLW21, Definition 2.4].

Denote, for every element  $a$  in  $F_\Theta$  by  $P_a$  its stabilizer in  $G$  and by  $U_a$  the unipotent radical of  $P_a$  (so that  $U_a = U_\Theta$  when  $a$  is  $\mathfrak{p}_\Theta$ ). The group  $U_a$  is completely determined by its Lie algebra and will be also identified with a subgroup of  $\text{Aut}(\mathfrak{g})$  (namely the unipotent radical of the stabilizer of  $a$  in  $\text{Aut}(\mathfrak{g})$ ). We will call a  $U$ -pinning of  $U_a$  any homomorphism  $s_a: U_\Theta \rightarrow U_a$  induced by the map  $x \mapsto gxg^{-1}$  where  $g$  is an element of  $\text{Aut}(\mathfrak{g})$  such that  $g \cdot \mathfrak{p}_\Theta = a$ .

**Lemma 10.8.** *Let  $a$  and  $b$  be two transverse points of  $F_\Theta$ . Then the diamonds with extremities  $a$  and  $b$  are exactly the triples*

$$(s_a(U_\Theta^{> 0}) \cdot b, a, b)$$

where  $s_a$  runs through the  $U$ -pinnings of  $U_a$ .

*Proof.* Indeed every triple in the statement of the lemma is a diamond and the family defined is invariant by the action of  $\text{Aut}(\mathfrak{g})$ . Hence the result by the transitivity observed in Corollary 10.6.  $\square$

**10.6. Opposite diamond.** The notion of  $U$ -pinning allows us to introduce the notion of the opposite of a diamond. Let  $(D, a, b)$  be a diamond and let  $s_a: U_\Theta \rightarrow U_a$  be a  $U$ -pinning such that  $D = s_a(U_\Theta^{> 0}) \cdot b$  then the triple  $(s_a(U_\Theta^{> 0})^{-1} \cdot b, a, b)$  is a diamond that is called *opposite* to  $(D, a, b)$ .

**Lemma 10.9.** *There is a unique diamond  $(D', a, b)$  opposite to  $(D, a, b)$ .*

*Proof.* Set  $V = \{u \in U_a \mid u \cdot b \in D\}$  and  $V' = \{u \in U_a \mid u \cdot b \in D'\}$ . Then from the definition

$$V' = s_a((U_\Theta^{> 0})^{-1}) = (s_a(U_\Theta^{> 0}))^{-1} = V^{-1} = \{u^{-1}\}_{u \in V},$$

this proves uniqueness.  $\square$

The diamond opposite to  $(D, a, b)$  will be denoted  $(D^\vee, a, b)$ . One obviously has  $(D^\vee)^\vee = D$ . Uniqueness directly implies

**Corollary 10.10.** *If  $D$  is a diamond, and if  $g$  belongs to  $\text{Aut}(\mathfrak{g})$ , then the opposite of the diamond  $g \cdot D$  is  $g \cdot D^\vee$ :  $(g \cdot D)^\vee = g \cdot D^\vee$ .*

**Lemma 10.11.** *Let  $(D, a, b)$  be a diamond. Then every  $x$  in  $D$  and every  $y$  in  $D^\vee$  are transverse.*

*Proof.* By invariance we can assume that  $a = \mathfrak{p}_\Theta$ ,  $b = \mathfrak{p}_\Theta^{\text{opp}}$  and  $D = U_\Theta^{>0} \cdot b$ . There are then  $u$  and  $v$  in  $U_\Theta^{>0}$  such that  $x = u \cdot b$  and  $y = v^{-1} \cdot b$ . Then the pair  $(x, y)$  is in the same orbit that  $(vu \cdot b, b)$ . Since  $vu$  belongs to  $U_\Theta^{>0}$ ,  $vu \cdot b$  belongs to  $D$  and is transverse to  $b$ , thus  $x$  is transverse to  $y$ .  $\square$

The opposition of diamonds reverses inclusion:

**Lemma 10.12.** *Let  $(D, a, b)$  be a diamond and  $x$  belong to  $D$ . Let  $(D', a, x)$  be the diamond contained in  $D$ . Then the opposite diamond  $D'^\vee$  contains  $D^\vee$ .*

*Proof.* We can assume  $a = \mathfrak{p}_\Theta$  and  $b = \mathfrak{p}_\Theta^{\text{opp}}$  and  $D = U_\Theta^{>0} \cdot b$ . Let  $u$  be in  $U_\Theta^{>0}$  such that  $x = u \cdot b$  so that  $D' = u \cdot D$ . One thus has  $D^\vee = (U_\Theta^{>0})^{-1} \cdot b$  and  $D'^\vee = u \cdot D^\vee$  or  $u^{-1} \cdot D'^\vee = D^\vee$ . The sought for inclusion is therefore equivalent to  $u^{-1}(U_\Theta^{>0})^{-1} \subset (U_\Theta^{>0})^{-1}$  which is a consequence of the fact that  $U_\Theta^{>0}$  is a semigroup.  $\square$

The last lemma implies

**Corollary 10.13.** *The set  $D^\vee$  consists of those  $x$  in  $\mathbf{F}_\Theta$  transverse to  $b$  and such that there is a diamond  $(D', a, x)$  containing  $D$ .*

**10.7. Positive triples in  $\mathbf{F}_\Theta$ .** We now use the family of diamonds to define positive triples of flags.

**Definition 10.14.** A triple  $(f_1, f_2, f_3) \in (\mathbf{F}_\Theta)^3$  is *positive* if  $f_1$  is transverse to  $f_3$  and if there exists a diamond with extremities  $f_1$  and  $f_3$  that contains  $f_2$ .

The next proposition collects the main properties of positive triples.

- Proposition 10.15.** (1) [Invariance] *for every  $(f_1, f_2, f_3)$  and every  $g$  in  $\text{Aut}(\mathfrak{g})$ , the triple  $(f_1, f_2, f_3)$  is positive if and only if the triple  $(g \cdot f_1, g \cdot f_2, g \cdot f_3)$  is positive.*
- (2) *A triple  $(f_i)_{i \in \mathbf{Z}/3\mathbf{Z}}$  is positive if and only if, for all  $i \neq j$  in  $\mathbf{Z}/3\mathbf{Z}$  there exists a diamond  $D_{i,j}$  with extremities  $f_i$  and  $f_j$  with  $D_{j,i} = D_{i,j}^\vee$  and  $f_k$  belongs to  $D_{i,j}$  for all  $(i, k, j)$  cyclically ordered (i.e.  $j - k = k - i \neq 0$  in  $\mathbf{Z}/3\mathbf{Z}$ ).*
- (3) [Permutation] *for every permutation  $\sigma \in S_3$  and every  $(f_1, f_2, f_3)$ , the triple  $(f_1, f_2, f_3)$  is positive if and only  $(f_{\sigma(1)}, f_{\sigma(2)}, f_{\sigma(3)})$  is positive.*
- (4) *A triple is positive if and only if it is in the orbit (under  $\text{Aut}(\mathfrak{g})$ ) of  $(\mathfrak{p}_\Theta, u \cdot \mathfrak{p}_\Theta^{\text{opp}}, \mathfrak{p}_\Theta^{\text{opp}})$  for some  $u$  in  $U_\Theta^{>0}$ .*
- (5) *A triple is positive if and only if it is in the orbit (under  $\text{Aut}(\mathfrak{g})$ ) of  $(\mathfrak{p}_\Theta, u^{-1} \cdot \mathfrak{p}_\Theta^{\text{opp}}, \mathfrak{p}_\Theta^{\text{opp}})$  for some  $u$  in  $U_\Theta^{>0}$ .*
- (6) [Component] *the set of positive triples is a union of connected components of the space of pairwise transverse triples, in particular it is open in  $(\mathbf{F}_\Theta)^3$ .*

(7) [Properness] *The group  $G$  acts properly on the space of positive triples. In particular stabilizers are compact.*

*Proof.* (1) This property follows from the invariance of the family of diamonds.

(2) If  $(f_1, f_2, f_3)$  satisfies these conditions, then  $f_2$  belongs to  $D_{1,3}$ , a diamond with extremities  $f_1$  and  $f_3$  and the triple is positive.

Conversely, if the triple  $(f_1, f_2, f_3)$  is positive, there exists a diamond  $D_{1,3}$  with extremities  $f_1$  and  $f_3$  and containing  $f_2$ . Define  $D_{1,2}$  to be the unique diamond with extremities  $f_1$  and  $f_2$  contained in  $D_{1,3}$  and  $D_{2,3}$  to be the unique diamond with extremities  $f_2$  and  $f_3$  contained in  $D_{1,3}$ . For  $i > j$  define  $D_{i,j} = D_{j,i}^\vee$ . Then the wanted membership properties of the  $f_k$  in the  $D_{i,j}$  follow from the choices made and from Corollary 10.13.

(3) The characterization of the previous point is clearly invariant by permutation, hence the result.

(4) This follows from the definition and from the fact that  $(U_\Theta^{>0} \cdot b, a, b)$  is a diamond (again  $a = \mathfrak{p}_\Theta$  and  $b = \mathfrak{p}_\Theta^{\text{opp}}$ ).

(5) This property follows from the previous one and the invariance by the transposition (23).

(6) This is a consequence of the connectedness properties of diamonds.

(7) The map  $\mathcal{F} \rightarrow \mathbb{F}_\Theta^{2*}$  (where  $\mathbb{F}_\Theta^{2*}$  is the space of transverse pairs) is continuous and equivariant. The sought for properness is then equivalent to the properness of the action of  $L_\Theta^\circ$  onto the semigroup  $U_\Theta^{>0}$ ; this properness is in turn a consequence of the fact that the parameterizations (cf. Section 6.2) are  $L_\Theta^\circ$ -equivariant and from the already known properness of  $L_\Theta^\circ$  on the product of cones (Proposition 3.7).  $\square$

*Remark 10.16.* In resonance with Remark 3.10, we can establish (when  $\Theta \neq \Delta$ ) that the space of positive triples has two connected components when  $\sharp\Theta$  is odd and one connected component when  $\sharp\Theta$  is even.

Point (6) can be used to prove other characterizations of positive triples, for example

**Corollary 10.17.** *A triple is positive if and only if it is in the  $|\text{Aut}(\mathfrak{g})$ -orbit of a triple of the form  $(vu \cdot b, v \cdot b, b)$  where  $b = \mathfrak{p}_\Theta^{\text{opp}}$  and  $u, v$  belong to  $U_\Theta^{>0}$ .*

*Proof.* Let us prove first that any such triple is positive. By property of the semigroup  $U_\Theta^{>0}$ , the flags in  $(vu \cdot b, v \cdot b, b)$  are pairwise transverse. Let  $\ell_s = \exp(sX)$  ( $s \in \mathbf{R}$ ) be the 1-parameter subgroup, contained in the Cartan subspace, where  $X$  is the element of  $\mathfrak{a}$  such that  $\alpha(X) = -1$  for all  $\alpha$  in  $\Delta$ . Then the family  $\{(v\ell_s u \ell_s^{-1} \cdot b, v \cdot b, b)\}_{s \geq 0}$  consists of pairwise transverse triples and converges, as  $s \rightarrow \infty$ , to the positive triple  $(a, v \cdot b, b)$  (where  $a = \mathfrak{p}_\Theta$ ). This implies (thanks to point (6) of the above proposition) that  $(vu \cdot b, v \cdot b, b)$  is positive.

Let now prove the reverse statement. Any positive triple is in the orbit of  $(a, w \cdot b, b)$  for some  $w$  in  $U_{\Theta}^{>0}$ . The diamond with extremities  $a$  and  $b$  and containing  $w \cdot b$  is our first diamond  $D = U_{\Theta}^{>0} \cdot b = U_{\Theta}^{\text{opp}, >0} \cdot a$ . For  $x \in U_{\Theta}^{\text{opp}, >0}$ , one has  $x \cdot (a, w \cdot b, b) = (x \cdot a, xw \cdot b, b)$ . The element  $x \cdot a$  belongs to  $D$  and hence is of the form  $r \cdot b$  for some  $r$  in  $U_{\Theta}^{>0}$ .

The element  $xw \cdot b$  belongs also to  $D$ : indeed there is  $z \in U_{\Theta}^{\text{opp}, >0}$  such that  $w \cdot b = z \cdot a$ ; hence  $xw \cdot b = (xz) \cdot a$  and  $xz$  belongs to  $U_{\Theta}^{\text{opp}, >0}$  thus  $xw \cdot b$  belongs to  $U_{\Theta}^{\text{opp}, >0} \cdot a = D$ . There is therefore  $v$  in  $U_{\Theta}^{>0}$  such that  $xw \cdot b = v \cdot b$ .

We finally prove that the element  $u = v^{-1}r$  belongs to  $U_{\Theta}^{>0}$ . For this, note first that  $u$  belongs to  $\Omega_{\Theta}^{\text{opp}}$  since  $v^{-1}r \cdot b$  is transverse to  $b$ ; choose  $(w_t)_{t \in [0,1]}$  a continuous path satisfying  $w_0 = e$ ,  $w_1 = w$  and  $w_t \in U_{\Theta}^{>0}$  for all  $t$  in  $]0, 1[$  (such a path can be constructed thanks to a parametrization  $F_{\gamma}$ ). The path  $(v_t)_{t \in [0,1]}$  in  $U_{\Theta}$  defined by the equality  $v_t \cdot b = xw_t \cdot b$  is continuous and satisfies  $v_0 = e$ ,  $v_1 = v$  and  $v_t^{-1}r$  belongs to  $\Omega_{\Theta}^{\text{opp}}$  for all  $t$ . Hence  $u = v^{-1}r = v_1^{-1}r$  and  $r = v_0^{-1}r$  belongs to the same connected component of  $U_{\Theta} \cap \Omega_{\Theta}^{\text{opp}}$ . Since  $U_{\Theta}^{>0}$  is a connected component of that intersection and since  $r$  belongs to  $U_{\Theta}^{>0}$ , we obtain that  $u$  belongs to  $U_{\Theta}^{>0}$ , as announced.  $\square$

**10.8. Positive quadruples.** We investigate here properties of positive quadruples.

**Definition 10.18.** A quadruple  $(a, x, b, y)$  in  $\mathbf{F}_{\Theta}$  is said *positive* if there exists a diamond  $(D, a, b)$  such that  $x$  belongs to  $D$  and  $y$  belongs to  $D^{\vee}$ .

**Proposition 10.19.** (1) Let  $(a, x, b, y)$  be a quadruple in  $\mathbf{F}_{\Theta}$  and let  $g$  be in  $\text{Aut}(\mathfrak{g})$ . Then  $(a, x, b, y)$  is positive if and only if  $(g \cdot a, g \cdot x, g \cdot b, g \cdot y)$  is positive.

(2) A quadruple is positive if and only if it is in the  $\text{Aut}(\mathfrak{g})$ -orbit of  $(a, u \cdot b, b, v^{-1} \cdot b)$  where  $a = \mathfrak{p}_{\Theta}$ ,  $b = \mathfrak{p}_{\Theta}^{\text{opp}}$  and  $u, v$  belong to  $U_{\Theta}^{>0}$ .

(3) A quadruple is positive if and only if it is in the  $\text{Aut}(\mathfrak{g})$ -orbit of  $(a, vu \cdot b, v \cdot b, b)$  where  $a = \mathfrak{p}_{\Theta}$ ,  $b = \mathfrak{p}_{\Theta}^{\text{opp}}$  and  $u, v$  belong to  $U_{\Theta}^{>0}$ .

(4) Let  $(a, x, b, y)$  be a quadruple in  $\mathbf{F}_{\Theta}$ . Then  $(a, x, b, y)$  is positive if and only if there is a diamond  $(D, a, y)$  such that  $b$  belongs to  $D$  and  $x$  belongs to  $D'$ , the unique diamond with extremities  $a$  and  $b$  contained in  $D$ .

(5) A quadruple is positive if and only if it is in the  $\text{Aut}(\mathfrak{g})$ -orbit of  $(a, x \cdot a, xy \cdot a, b)$  where  $a = \mathfrak{p}_{\Theta}$ ,  $b = \mathfrak{p}_{\Theta}^{\text{opp}}$  and  $x, y$  belong to  $U_{\Theta}^{\text{opp}, >0}$ .

(6) Let  $(a, x, b, y)$  be a quadruple in  $\mathbf{F}_{\Theta}$ . Then  $(a, x, b, y)$  is positive if and only if there is a diamond  $(D, a, y)$  such that  $x$  belongs to  $D$  and  $b$  belongs to  $D'$ , the unique diamond with extremities  $x$  and  $y$  contained in  $D$ .

(7) A quadruple  $(f_i)_{i \in \mathbf{Z}/4\mathbf{Z}}$  is positive if and only if, for all  $i \neq j$  in  $\mathbf{Z}/4\mathbf{Z}$  there exists a diamond  $D_{i,j}$  with extremities  $f_i$  and  $f_j$  with  $D_{j,i} = D_{i,j}^{\vee}$  and  $f_k$  belongs to  $D_{i,j}$  for all  $(i, k, j)$  cyclically ordered.

- (8) *The space of positive quadruples is invariant under the dihedral group  $D_4 \subset S_4$  (i.e. the group generated by the 4-cycle  $(1, 2, 3, 4)$  and the double transposition  $(1, 4)(2, 3)$ ). Precisely for  $\sigma \in D_4$  and a quadruple  $(f_1, f_2, f_3, f_4)$  then  $(f_1, f_2, f_3, f_4)$  is positive if and only if  $(f_{\sigma(1)}, f_{\sigma(2)}, f_{\sigma(3)}, f_{\sigma(4)})$  is positive.*

*Proof.* (1) This follows from the invariance of the family of diamonds and the equivariance of opposition (Corollary 10.10).

- (2) This follows from the definition and from the fact that, up to the action of  $\text{Aut}(\mathfrak{g})$ , we can assume  $a = \mathfrak{p}_\Theta$ ,  $b = \mathfrak{p}_\Theta^{\text{opp}}$  and that  $D = U_\Theta^{>0} \cdot b$ .
- (3) Applying the element  $v \in U_\Theta^{>0}$  to the quadruple  $(a, u \cdot b, b, v^{-1}b)$  gives the wanted result.
- (4) This is the previous characterization stated in term of diamonds.
- (5) This follows as (3) above using this time the equality  $U_\Theta^{\text{opp}, >0} \cdot a = U_\Theta^{>0} \cdot b$ .
- (6) This is the previous characterization stated in term of diamonds.
- (7) If a quadruple satisfies the stated condition, then the condition of the definition is obviously satisfied and the quadruple is positive.

Conversely, suppose that  $(f_1, f_2, f_3, f_4)$  is positive. By definition, there is a diamond  $D_{1,3}$  with extremities  $f_1$  and  $f_3$ , containing  $f_2$  and such that  $f_4$  belongs to  $D_{3,1} := D_{1,3}^\vee$ . We define  $D_{1,2}$  to be the diamond contained in  $D_{1,3}$  and with extremities  $f_1$  and  $f_2$  and similarly  $D_{2,3}$  is the diamond contained in  $D_{1,3}$  with extremities  $f_2$  and  $f_3$ . The characterization of point 4 gives also a diamond  $D_{1,4}$  with extremities  $f_1$  and  $f_4$  and containing  $f_2$  and  $f_3$ . This diamond can be used to define the diamond  $D_{2,4}$  and  $D_{3,4}$ . The other diamonds are defined thanks to the requirement  $D_{j,i} = D_{i,j}^\vee$ . All the wanted memberships are satisfied by construction and by Corollary 10.13 except possibly that  $f_3$  belongs to  $D_{2,4}$ . However this membership follows from the characterization established in point (6).

- (8) The permutation invariance follows from the previous point.  $\square$

Based on point (3) above one has

**Lemma 10.20.** *Let  $a$  and  $b$  be the elements  $\mathfrak{p}_\Theta$  and  $\mathfrak{p}_\Theta^{\text{opp}}$  of  $\mathbb{F}_\Theta$  and let  $v$  be in  $U_\Theta^{>0}$  and  $x = v \cdot b$ . For  $y \in \mathbb{F}_\Theta$ , the quadruple  $(a, y, x, b)$  is positive if and only if there is  $u$  in  $U_\Theta^{>0}$  such that  $y = vu \cdot b$ .*

Diamonds associated with positive quadruples are properly contained one in another:

**Lemma 10.21.** *Let  $(a, x, b, y)$  be a positive quadruple and let  $(D, a, y)$  be the diamond containing  $x$  and  $b$  and let  $(D', x, b)$  be the diamond contained in  $D$ . Then the closure  $\overline{D'}$  is contained in  $D$ .*

*Proof.* It will be enough to prove that this closure is contained in the intersection  $\mathcal{O}_a \cap \mathcal{O}_y$ . By symmetry, we need only to prove the inclusion into  $\mathcal{O}_y$ . We can assume that  $a = \mathfrak{p}_\Theta$ ,  $y = \mathfrak{p}_\Theta^{\text{opp}}$  and  $b = v \cdot y$ ,  $x = vu \cdot y$  with  $u, v$  in  $U_\Theta^{>0}$ . Any point in  $\overline{D'}$  is of the form  $vw \cdot y$  with  $w \in U_\Theta^{>0}$  (and  $w^{-1}u$  must also belong to  $U_\Theta^{>0}$ ). In particular  $vw$  belongs to  $U_\Theta^{>0}$  (Corollary 8.16.(1)) and  $vw \cdot y$  is transverse to  $y$  as wanted.  $\square$

Similarly to what as been established for triples we note

**Lemma 10.22.** *The space of positive quadruples is a union of connected components of the space  $F_\Theta^{4*}$  of pairwise transverse quadruples.*

**10.9. Positive tuples.** Let  $n$  be an integer  $\geq 3$ . We introduce here the positive  $n$ -tuples in generalizing point (2) of Proposition 10.15 or point (7) of Proposition 10.19.

**Definition 10.23.** A  $n$ -tuple  $(f_i)_{i \in \mathbf{Z}/n\mathbf{Z}}$  of elements of  $F_\Theta$  is *positive* if, for all  $i \neq j$  in  $\mathbf{Z}/n\mathbf{Z}$ , there exists a diamond  $D_{i,j}$  with extremities  $f_i$  and  $f_j$  with  $D_{j,i} = D_{i,j}^\vee$  and  $f_k$  belongs to  $D_{i,j}$  for all  $(i, k, j)$  cyclically ordered.

Note that the diamonds are uniquely determined by the properties that  $f_{i+1}$  belongs to  $D_{i,j}$  (when  $j \notin \{i, i+1\}$ ) and  $D_{i,i+1} = D_{i+1,i}^\vee$ .

From the definition, it is obvious that

**Lemma 10.24.** (1) *The set of positive  $n$ -tuples is invariant by the dihedral group  $D_n$  (i.e. by cyclic permutations and by the permutation  $i \leftrightarrow n+1-i$ ).*

(2) *Any subconfiguration of a positive tuple is a positive tuple.*

Here is one (among many) characterization of positive tuples.

**Lemma 10.25.** *A  $n$ -tuple is positive if and only if it is in the  $\text{Aut}(\mathfrak{g})$ -orbit of*

$$(a, u_1 \cdots u_{n-3} u_{n-2} \cdot b, u_1 \cdots u_{n-3} \cdot b, \dots, u_1 \cdot b, b),$$

where  $a = \mathfrak{p}_\Theta$ ,  $b = \mathfrak{p}_\Theta^{\text{opp}}$ , and, for all  $i = 1, \dots, n-2$ ,  $u_i$  belongs to  $U_\Theta^{>0}$ .

*Proof.* Let  $(f_1, \dots, f_n)$  be a positive tuple. Up to the  $\text{Aut}(\mathfrak{g})$  action, we can assume that  $f_1 = a$ ,  $f_n = b$  and  $f_{n-1} = u_1 \cdot b$ . Repeated applications of 10.20 give the sequence  $(u_2, \dots, u_{n-2})$ .

Conversely let  $u_1, \dots, u_{n-2}$  be in  $U_\Theta^{>0}$  and let  $(f_1, \dots, f_n)$  be

$$(a, \dots, u_1 \cdots u_i \cdot b, \dots, b).$$

For all  $i < j$  in  $\{1, \dots, n\}$ , let  $D_{i,j}$  be the unique diamond with extremities  $f_i$  and  $f_j$  contained in  $D = U_\Theta^{>0} \cdot b$  and  $D_{j,i} = D_{i,j}^\vee$ . The facts that  $f_k$  belongs to  $D_{i,j}$  (for  $i < k < j$  in  $\{1, \dots, n\}$ ) come from Corollary 10.17, the other cases come from Corollary 10.13.  $\square$

Restated in term of diamonds, the lemma says it is enough to check that some sub-4-tuples are positive:

**Proposition 10.26.** *Let  $(f_1, \dots, f_n)$  be in  $(F_\Theta)^n$ . Then  $(f_1, \dots, f_n)$  is positive if and only if, for every  $i = 2, \dots, n-2$ , the quadruple  $(f_1, f_i, f_{i+1}, f_n)$  is positive.*

We also note that

**Lemma 10.27.** *The set of positive  $n$ -tuples is a union of connected components of the space of pairwise transverse  $n$ -tuples.*

**10.10. Compatibility of positive structures.** It sometimes happens that a Lie group admits positive structures with respect to two different flag varieties. This can happen only when the Lie group is split over  $\mathbf{R}$  and if its Dynkin diagram has a double arrow. This concerns the following groups up to isogeny:

- the symplectic Lie group  $\mathrm{Sp}(2n, \mathbf{R})$ ;
- the orthogonal groups  $\mathrm{SO}(n, n+1)$ ;
- the real split Lie group of type  $F_4$ .

Let  $G$  be a split real group that also admits a positive structure for  $\Theta \neq \Delta$ . We denote by  $B = P_\Delta$  the Borel subgroup and by  $P_\Theta$  the parabolic subgroup associated with  $\Theta$ ; their Lie algebra will be denoted  $\mathfrak{b} = \mathfrak{p}_\Delta$  and  $\mathfrak{p}_\Theta$  respectively; similar notation will be adopted for the standard opposite parabolic subgroups and algebras. We have  $B < P_\Theta$ . Note that in this case, as explained in Section 3.3, a pinning in the sense of Chevalley determines a  $\Theta$ -base. Let  $U^{>0}$ ,  $U^{\mathrm{opp}, >0}$ , and  $U_\Theta^{>0}$ ,  $U_\Theta^{\mathrm{opp}, >0}$  be the corresponding semigroups. The natural projection  $\pi: F_\Delta \rightarrow F_\Theta$  is  $\mathrm{Aut}(\mathfrak{g})$ -equivariant and  $\pi(\mathfrak{b}) = \mathfrak{p}_\Theta$  and  $\pi(\mathfrak{b}^{\mathrm{opp}}) = \mathfrak{p}_\Theta^{\mathrm{opp}}$ . This projection behaves well with respect to the notion of positivity introduced:

**Proposition 10.28.** *Let  $(a, x, b)$  be a positive triple in  $F_\Delta$ . Then  $(\pi(a), \pi(x), \pi(b))$  is a positive triple in  $F_\Theta$ .*

*Proof.* The positive semigroup  $U^{>0}$  is determined here by elements  $X_\alpha$  generating  $\mathfrak{g}_\alpha$  for  $\alpha$  in  $\Delta$ . These elements can be used to fix the semigroup  $U_\Theta^{>0}$ , i.e. to fix the cones  $c_\alpha \subset \mathfrak{u}_\alpha$  ( $\alpha \in \Theta$ ): for  $\alpha \neq \alpha_\Theta$ ,  $\mathfrak{u}_\alpha = \mathfrak{g}_\alpha$  and we let  $c_\alpha = \mathbf{R}_{\geq 0} X_\alpha$ , for  $\alpha = \alpha_\Theta$  we fix  $c_\alpha \subset \mathfrak{u}_\alpha$  by the property that  $X_\alpha \in c_\alpha$ .

First observe that, by the transitivity of the action of  $G$  on the space of transverse pairs and since  $(\pi(\mathfrak{b}), \pi(\mathfrak{b}^{\mathrm{opp}})) = (\mathfrak{p}_\Theta, \mathfrak{p}_\Theta^{\mathrm{opp}})$ , the image by  $\pi$  of every transverse pairs in  $F_\Delta$  is a transverse pair in  $F_\Theta$ .

Let us prove the proposition. By the transitivity of the action of  $\mathrm{Aut}(\mathfrak{g})$ , we can assume that  $a = \mathfrak{b}$ ,  $b = \mathfrak{b}^{\mathrm{opp}}$  and  $x = u \cdot b$  for  $u \in U^{>0}$ . Hence  $\pi(x) = u \cdot \pi(b)$  and we need to show that  $u \cdot \pi(b) \in U_\Theta^{>0} \cdot \pi(b)$ . Since  $\pi(x)$  is transverse to  $\pi(b)$  it is enough to prove that  $u \cdot \pi(b) \in U_\Theta^{\geq 0} \cdot \pi(b)$ .

Let us prove thus that  $U^{\geq 0} \cdot (U_\Theta^{\geq 0} \cdot \pi(b)) \subset U_\Theta^{\geq 0} \cdot \pi(b)$ . Since  $U^{\geq 0}$  is the semigroup generated by the  $\exp(X)$  where  $X = tX_\alpha \in \mathfrak{g}_\alpha$  for all  $t \geq 0$

and  $\alpha \in \Delta$ , it is enough to show the inclusion  $\exp(X) \cdot (U_{\Theta}^{\geq 0} \cdot \pi(b)) \subset U_{\Theta}^{\geq 0} \cdot \pi(b)$  for such an  $X$ . When  $\alpha$  belongs to  $\Theta$ , then  $\exp(X)$  belongs to  $U_{\Theta}^{\geq 0}$  and the inclusion comes from the fact that  $U_{\Theta}^{\geq 0}$  is a semigroup. When  $\alpha$  belongs to  $\Delta \setminus \Theta$ , then  $\exp(X)$  belongs to  $L_{\Theta}^{\circ}$  and the inclusion comes from the fact that  $L_{\Theta}^{\circ}$  normalizes  $U_{\Theta}^{\geq 0}$  and fixes  $\pi(b)$ .  $\square$

**10.11. Positive maps.** We extend the notion of positive tuples to the notion of positive maps. For this let us denote by  $\Lambda$  a set equipped with a cyclic ordering (typically  $\Lambda$  is a subset of the circle). This means that there is a subset  $\Lambda^{3+}$  of  $\Lambda^3$  consisting of cyclically oriented triples. There is therefore a notion of cyclically oriented  $n$ -tuples in  $\Lambda$ .

**Definition 10.29.** A map  $f: \Lambda \rightarrow F_{\Theta}$  is said *positive* if the image by  $f$  of every cyclically oriented  $n$ -tuple is a positive  $n$ -tuple.

Proposition 10.26 implies immediately

**Lemma 10.30.** *A map  $f$  is positive if and only if it sends every cyclically oriented quadruple to a positive quadruple.*

## 11. POSITIVE $SL_2$

In this section we will show the Lie algebra of a simple Lie group  $G$  admitting a  $\Theta$ -positive structure admits a special 3-dimensional subalgebra. We will draw several consequences from this, in particular the existence of a positive circle in  $F_{\Theta}$ , that is used in [GLW21, Proposition 2.9].

**11.1. The  $\Theta$ -principal subalgebra.** The split Lie subalgebra  $\mathfrak{g}_{\Theta}$  admits a special subalgebra (rather a conjugacy class of subalgebras), called the *principal  $\mathfrak{sl}_2$* . In the correspondence with nilpotent elements given by the Jacobson–Morozov theorem, it is the  $\mathfrak{sl}_2$ -subalgebra corresponding to a regular nilpotent element (again it is rather the conjugacy class that makes intrinsic sense). A regular nilpotent element is for example  $\sum_{\alpha \in \Theta} E_{\alpha}$  (where  $E_{\alpha}$  are given in Section 3.3).

**Definition 11.1.** The  *$\Theta$ -principal subalgebra* is (the conjugacy class of) the subalgebra of  $\mathfrak{g}$ , isomorphic to  $\mathfrak{sl}_2(\mathbf{R})$  and represented by the principal subalgebra of  $\mathfrak{g}_{\Theta}$ .

The induced morphism  $\pi_{\Theta}: \mathfrak{sl}_2(\mathbf{R}) \rightarrow \mathfrak{g}$  will be called  *$\Theta$ -principal embedding*.

*Example 11.2.* When  $G = Sp(2n, \mathbf{R})$  let us explain the subgroup corresponding to the  $\Theta$ -principal subalgebra. For the case  $\Theta = \Delta$ , the subgroup is the image of the irreducible representation of  $SL_2(\mathbf{R})$  of dimension  $2n$ . For the case when  $\Theta = \{\alpha_n\}$ , the subgroup is the set of bloc matrices, with blocs of size  $n$  all scalar multiples of the identity matrix.

**11.2. The  $\Theta$ -principal  $\mathfrak{sl}_2$ -triple.** Recall from Section 3.3, that the choice, for all  $\alpha \in \Theta$ , of elements  $X_\alpha$  in  $\mathfrak{g}_\alpha$  such that  $\{X_\alpha, X_{-\alpha} = \tau(X_\alpha), H_\alpha = [X_\alpha, X_{-\alpha}]\}$  form an  $\mathfrak{sl}_2$ -triple, determine a family called a  $\Theta$ -base  $\{E_\alpha, F_\alpha, D_\alpha\}_{\alpha \in \Theta}$  and generating  $\mathfrak{g}_\Theta$ .

Let  $\Phi$  denote the simple roots of  $\mathfrak{g}_\Theta$  (where the Cartan subalgebra is chosen to be the span of the  $D_\alpha$ ). By the choice of the  $\Theta$ -base,  $\Theta$  is naturally identified with the simple roots in  $\Phi$  and the associated set of positive roots will be denoted  $\Phi^+$ . For every  $\alpha$  in  $\Phi$ , we will also denote by  $(E_\alpha, F_\alpha, D_\alpha)$  an  $\mathfrak{sl}_2$ -triple in  $\mathfrak{g}_\Theta$  corresponding to the root  $\alpha$ .

Let us introduce

$$D = \sum_{\alpha \in \Phi^+} D_\alpha,$$

there are positive integers  $q_\alpha$  ( $\alpha \in \Theta$ ) such that

$$D = \sum_{\alpha \in \Theta} q_\alpha D_\alpha.$$

and we can introduce also

$$E = \sum_{\alpha \in \Theta} q_\alpha^{1/2} E_\alpha, \quad F = \sum_{\alpha \in \Theta} q_\alpha^{1/2} F_\alpha.$$

**Lemma 11.3** ([Kos59, Lemma 5.2]). *The triple  $(E, F, D)$  is an  $\mathfrak{sl}_2$ -triple and it generates the principal 3-dimensional subalgebra of  $\mathfrak{g}_\Theta$ .*

**Definition 11.4.** The triple  $(E, F, D)$  (and its conjugates) will be called a  $\Theta$ -principal  $\mathfrak{sl}_2$ -triple.

For Lusztig's total positivity in split real Lie groups, it is well-known [Lus98, Proposition 5.9.(a)] that for  $X = \sum_{\alpha \in \Delta} k_\alpha X_\alpha$  with  $k_\alpha > 0$  for all  $\alpha \in \Delta$  one has that  $\exp(X)$  is contained in the positive unipotent semigroup. This directly implies

**Lemma 11.5.** *The element  $\exp(E)$  belongs to  $N_\Theta^{>0}$ , where  $N_\Theta^{>0}$  is the totally positive unipotent semigroup in  $N_\Theta = G_\Theta \cap U_\Theta$ .*

From this lemma, we deduce the following corollary, which is of independent interest. It generalizes [Lus98, Proposition 5.9.(a)].

**Corollary 11.6.** *Let  $Z = \sum_{\alpha \in \Theta} Z_\alpha$ , with  $Z_\alpha \in \mathring{c}_\alpha$  for all  $\alpha \in \Theta$ . Then  $\exp(Z)$  belongs to  $U_\Theta^{>0}$ .*

*Proof.* Since  $L_\Theta^\circ$  acts transitively on  $\Pi_{\alpha \in \Theta} \mathring{c}_\alpha$  (Proposition 3.7), there exists  $\ell \in L_\Theta^\circ$  such that  $Z = \text{Ad}(\ell)E$ . Since  $\exp: \mathfrak{u}_\Theta \rightarrow U_\Theta$  is equivariant with respect to  $L_\Theta^\circ$ , and  $U_\Theta^{>0}$  is  $L_\Theta^\circ$ -invariant, Lemma 11.5 (together with the formula defining  $E$ ) implies the claim.  $\square$

The  $\Theta$ -principal embedding  $\pi_\Theta: \mathfrak{sl}_2 \rightarrow \mathfrak{g}$  induces an homomorphism  $\pi_\Theta: \text{SL}_2(\mathbf{R}) \rightarrow G$  and hence an action of  $\text{SL}_2(\mathbf{R})$  on the flag variety  $F_\Theta$ .

**Lemma 11.7.** *The stabilizer of  $\mathfrak{p}_\Theta \in F_\Theta$  in  $SL_2(\mathbf{R})$  is the standard Borel subgroup  $B$  (whose Lie algebra is  $\mathfrak{b} = \mathbf{R}E \oplus \mathbf{R}D$ ).*

*Proof.* Since  $\pi_\Theta(E)$  and  $\pi_\Theta(D)$  belongs to  $\mathfrak{p}_\Theta$ , it is clear that the Lie algebra of this stabilizer contains  $\mathfrak{b}$ . Since the  $SL_2(\mathbf{R})$ -orbit of  $\mathfrak{p}_\theta$  is not trivial, the Lie algebra of the stabilizer must be equal to  $\mathfrak{b}$ . Hence the stabilizer is either  $B$  or its neutral component  $B^\circ$ . We thus need to prove that  $-\text{id} = \exp(\pi(E - F))$  belongs to the stabilizer.

It is well known that the element  $x = \exp(\frac{\pi}{2}(E - F))$  of  $SL_2(\mathbf{R})$  is sent by  $\pi_\Theta$  to (a representative of) the longest length element of  $G_\Theta$ . One has hence (cf. Proposition 4.8) that  $\pi_\Theta(x) \cdot \mathfrak{p}_\Theta = w_\Delta \cdot \mathfrak{p}_\Theta = \mathfrak{p}_\Theta^{\text{opp}}$  and  $\pi_\Theta(x) \cdot \mathfrak{p}_\Theta^{\text{opp}} = \mathfrak{p}_\Theta$  which implies the sought for equality:  $\pi_\Theta(x^2) \cdot \mathfrak{p}_\Theta = \mathfrak{p}_\Theta$ .  $\square$

From this, identifying  $\mathbb{P}(\mathbf{R}^2)$  with  $SL_2(\mathbf{R})/B$ , we obtain an equivariant embedding  $\mathbb{P}(\mathbf{R}^2) \rightarrow F_\Theta$ . As a direct consequence of Lemma 11.5 (and since the cyclically ordered tuples of  $\mathbb{P}(\mathbf{R}^2)$  are well understood) we obtain the following

**Proposition 11.8.** *The embedding  $\mathbb{P}(\mathbf{R}^2)$  into  $F_\Theta$  induced from the Lie algebra homomorphism  $\pi_\Theta: \mathfrak{sl}_2(\mathbf{R}) \rightarrow \mathfrak{g}$  is a positive circle.*

*Remark 11.9.* When  $\Theta \neq \Delta$ , there are sometimes more than one embedding of  $\mathfrak{sl}_2$  that give rise to a positive circle in  $F_\Theta$ . For example when  $G$  is a split real group, the principal 3-dimensional subalgebra of  $\mathfrak{g}$  determines a positive circle in  $F_\Theta$  for any  $\Theta$  such that  $G$  admits a  $\Theta$ -positive structure (cf. Section 10.10).

*Remark 11.10.* Motivated by the introduction of  $\Theta$ -positivity, Bradlow, Collier, García-Prada, Gothen and Oliveira introduce in [BCGP<sup>+</sup>21] the notion of magical nilpotent elements and of magical  $\mathfrak{sl}_2$ -triples. They further observe that given a complex Lie group and a magical nilpotent element  $e$  there is a canonical real form  $\mathfrak{g}$  associated with  $e$ , and this real form admits a  $\Theta$ -positive structure. It is clear from the above construction, that the split real form  $\mathfrak{g}_\Theta$  is the split real subalgebra denoted  $\mathfrak{g}(e)$  in [BCGP<sup>+</sup>21], and the image of the embedding  $\pi_\Theta: \mathfrak{sl}_2 \rightarrow \mathfrak{g}$  is the (real) magical  $\mathfrak{sl}_2$ -triple in  $\mathfrak{g}$ .

## APPENDIX A. LONGEST LENGTH ELEMENT IN $B_{p+1}$

In this section we determine a reduced expression of the longest element in the Weyl group associated to a root system of type  $B_{p+1}$ .

This is the type of the system of restricted roots of the groups  $SO(p+1, p+k)$  ( $p > 0, k > 1$ ). A choice of quadratic form which makes the

calculations a little easier is given by the matrix  $Q = \begin{pmatrix} 0 & 0 & K \\ 0 & J & 0 \\ {}^tK & 0 & 0 \end{pmatrix}$ ,

where  $K = \begin{pmatrix} & & & (-1)^p \\ & & \ddots & \\ & & 1 & \\ -1 & & & \end{pmatrix}$ , and  $J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -\text{id}_{k-1} & 0 \\ 1 & 0 & 0 \end{pmatrix}$ .

With this choice, a Cartan subspace  $\mathfrak{a}$  of  $\mathfrak{so}(p+1, p+k)$  is its intersection with the space of diagonal matrices. A natural basis of  $\mathfrak{a}^*$  (with respect to this particular matrix realisation of the group) are the  $e_i$  ( $i = 1, \dots, p+1$ ) mapping a diagonal matrix to its  $i$ -th diagonal entry. We will use this to identify  $\mathfrak{a}$  with  $\mathbf{R}^{p+1}$  and describe the Weyl group and its elements in  $\text{GL}_{p+1}(\mathbf{R})$ .

The roots are

$$\{\pm e_i\}_{i \in \{1, \dots, p+1\}} \cup \{\pm e_j \pm e_k\}_{j, k \in \{1, \dots, p+1\}, j < k}.$$

For the lexicographic order the positive roots are

$$\{e_i\}_{i \in \{1, \dots, p+1\}} \cup \{e_j \pm e_k\}_{j, k \in \{1, \dots, p+1\}, j < k}$$

and the simple roots are  $\alpha_1 = e_1 - e_2$ ,  $\alpha_2 = e_2 - e_3$ ,  $\dots$ ,  $\alpha_p = e_p - e_{p+1}$ ,  $\alpha_{p+1} = e_{p+1}$ .

The Weyl group  $W$  naturally identifies with  $(\mathbf{Z}/2\mathbf{Z})^{p+1} \rtimes S_{p+1}$  acting on  $\mathbf{R}^{p+1}$  by permuting the coordinates and changing their signs. Its generators  $s_1, \dots, s_{p+1}$  associated with  $\alpha_1, \dots, \alpha_{p+1}$  are the transformations:

$$\begin{aligned} s_1 &: (x_1, x_2, \dots, x_{p+1}) \mapsto (x_2, x_1, \dots, x_{p+1}), \\ s_2 &: (x_1, x_2, x_3, \dots, x_{p+1}) \mapsto (x_1, x_3, x_2, \dots, x_{p+1}), \\ &\dots, \\ s_p &: (x_1, \dots, x_p, x_{p+1}) \mapsto (x_1, \dots, x_{p+1}, x_p) \\ \text{and } s_{p+1} &: (x_1, x_2, \dots, x_{p+1}) \mapsto (x_1, x_2, \dots, -x_{p+1}). \end{aligned}$$

Furthermore the longest length element of  $W$  is  $-\text{id}_{p+1}$  as it must exchange the Weyl chamber with its opposite.

Let  $\Theta = \{1, \dots, p\}$ . With the notation of Section 4  $\alpha_\Theta = \alpha_p$ , the longest length element in  $W_{\Delta \setminus \Theta} = \langle s_{p+1} \rangle$  is  $s_{p+1}$  itself and the subgroup of  $W$  generated by  $s_p$  and  $s_{p+1}$  is isomorphic to  $(\mathbf{Z}/2\mathbf{Z})^2 \rtimes S_2$  (the Weyl group of  $\text{SO}(2, 2+k)$ ). Its longest length element is  $s_p s_{p+1} s_p s_{p+1} = s_{p+1} s_p s_{p+1} s_p$  so that the element  $\sigma_p = \sigma_\Theta$  is  $s_p s_{p+1} s_p$ . Seen as an element of  $\text{GL}_{p+1}(\mathbf{R})$ ,  $\sigma_p$  is the transformation  $(x_1, \dots, x_p, x_{p+1}) \mapsto (x_1, \dots, -x_p, x_{p+1})$

Recall that the group  $W(\Theta)$  is the subgroup of  $W$  generated by  $s_1, \dots, s_{p-1}$  and  $\sigma_p$ . As all these generators fix the last coordinate, we can identify  $W(\Theta)$  as a subgroup of  $\text{GL}_p(\mathbf{R})$ . With this identification in mind and the above description, it is apparent that this subgroup

is the Weyl group of type  $B_p$  and its longest length element is, in this geometric realization,  $-\text{id}_p$ .

Turning back to subgroups of  $\text{GL}_{p+1}(\mathbf{R})$ , the longest length element of  $W(\Theta)$  is the transformation

$$(x_1, \dots, x_p, x_{p+1}) \mapsto (-x_1, \dots, -x_p, x_{p+1}).$$

Since the longest length element of  $W$  is  $-\text{id}_{p+1}$  and the longest length element of  $W_{\Delta \setminus \Theta}$  is  $s_{p+1}$  one gets as well that  $w_{\max}^\Theta$  is

$$(x_1, \dots, x_p, x_{p+1}) \mapsto (-x_1, \dots, -x_p, x_{p+1})$$

establishing, for the type  $B_{p+1}$ , the equality of Proposition 4.8.

Written as products of generators, these elements are: (we use the notation  $x^y = y^{-1}xy$  so that  $(x^y)^z = x^{yz}$  and  $\sigma_p = s_{p+1}^{s_p}$ )

$$\begin{aligned} w_\Delta &= s_{p+1}^{s_p \cdots s_1} s_{p+1}^{s_p \cdots s_2} \cdots s_{p+1}^{s_p} \cdots s_{p+1} \\ w_{\max}^\Theta &= s_{p+1}^{s_p \cdots s_1} s_{p+1}^{s_p \cdots s_2} \cdots s_{p+1}^{s_p} \\ &= \sigma_p^{s_{p-1} \cdots s_1} \sigma_p^{s_{p-1} \cdots s_2} \cdots \sigma_p^{s_{p-1}} \cdots \sigma_p, \end{aligned}$$

which gives an other proof of the fact that  $w_{\max}^\Theta$  is the longest length element in  $W(\Theta)$ . The above equalities are easy to establish noting that  $s_{p+1}^{s_p \cdots s_k}$  is  $(x_1, \dots, x_k, \dots, x_{p+1}) \mapsto (x_1, \dots, -x_k, \dots, x_{p+1})$ .

One can also verify that the above decompositions are reduced: for example, the length of  $w_\Delta$  is the dimension of the complete flag variety for the split group  $\text{SO}(p+1, p+2)$  and is thus equal to  $(p+1)^2$ ; this number matches the length of the above product.

#### APPENDIX B. LONGEST LENGTH ELEMENT IN $F_4$

In this section we determine a reduced expression of the longest element in the Weyl group associated to a root system of type  $F_4$ .

The root system  $F_4$  is intimately related with the lattice  $\Lambda$  of  $\mathbf{R}^4$  generated by  $\mathbf{Z}^4$  and the element  $\frac{1}{2}(1, 1, 1, 1)$ . An alternative description of  $\Lambda$  is the set of elements in  $\mathbf{R}^4$  all of whose coordinates have the same remainder, 0 or  $1/2$ , modulo 1.

The elements of  $F_4$  are the elements of  $\Lambda$  whose Euclidean norms are 1 or  $\sqrt{2}$ . They can be explicitly listed: let  $(e_i)_{i=1, \dots, 4}$  the canonical basis of  $\mathbf{R}^4$ , then

$$F_4 = \{ \pm e_i \}_{i \in \{1, \dots, 4\}} \{ \pm e_k \pm e_\ell \}_{k, \ell \in \{1, \dots, 4\}, k < \ell} \left\{ \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4) \right\}.$$

The positive roots are (using colexicographic order)

$$\{ e_i \}_{i \in \{1, \dots, 4\}} \{ \pm e_k + e_\ell \}_{k, \ell \in \{1, \dots, 4\}, k < \ell} \left\{ \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 + e_4) \right\},$$

and the simple roots are

$$\alpha_1 = -e_2 + e_3, \quad \alpha_2 = -e_1 + e_2, \quad \alpha_3 = e_1, \quad \alpha_4 = \frac{1}{2}(-e_1 - e_2 - e_3 + e_4).$$

The Weyl group is the subgroup of  $\mathrm{GL}_4(\mathbf{R})$  generated by the symmetries  $s_1, \dots, s_4$  associated with  $\alpha_1, \dots, \alpha_4$ . The transformation  $s_i$  is  $x \mapsto x - 2 \frac{\langle \alpha_i, x \rangle}{\langle \alpha_i, \alpha_i \rangle} \alpha_i$ . In matrix coordinates

$$s_4 = \frac{1}{2} \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

The element  $s_3$  is  $(x_1, x_2, x_3, x_4) \mapsto (-x_1, x_2, x_3, x_4)$ . Finally  $s_2$  and  $s_1$  are respectively  $(x_1, x_2, x_3, x_4) \mapsto (x_2, x_1, x_3, x_4)$  and  $(x_1, x_2, x_3, x_4) \mapsto (x_1, x_3, x_2, x_4)$ .

The relevant subset of the simple roots in this situation is  $\Theta = \{\alpha_1, \alpha_2\}$  and the root  $\alpha_\Theta$  is  $\alpha_2$ . The subgroup  $W_{\Delta \setminus \Theta}$  is generated by  $s_3$  and  $s_4$  and is isomorphic to  $S_3$ , its longest length element is  $s_3^{s_4} = s_4 s_3 s_4 = s_3 s_4 s_3 = s_4^{s_3}$ .

The subgroup generated by the symmetries indexed by  $\Delta \setminus \Theta$  and  $\alpha_\Theta$  is the group generated by  $\{s_2, s_3, s_4\}$  and is isomorphic to the Weyl group  $B_3$  (with the reindexation  $1 \mapsto 4, 2 \mapsto 3, 3 \mapsto 2$  with respect to the previous appendix). Its longest length element is  $s_2 s_2^{s_3} s_2^{s_3 s_4} = s_2 s_3 s_2 s_3 s_4 s_3 s_2 s_3 s_4$ . Hence the element  $\sigma_2 = \sigma_\Theta$  is:

$$\begin{aligned} \sigma_2 &= s_2 s_2^{s_3} s_2^{s_3 s_4} s_4 s_3 s_4 \\ &= s_2 s_3 s_2 s_3 s_4 s_3 s_2 (s_3 s_4 s_4 s_3) s_4 \\ &= s_2 s_3 s_2 s_3 s_4 s_3 (s_2 s_4) \\ &= s_2 s_3 s_2 (s_3 s_4 s_3 s_4) s_2 \\ &= s_2 s_3 s_2 s_4 s_3 s_2 \\ &= s_2 s_3 s_4 s_2 s_3 s_2. \end{aligned}$$

And the last two expressions are reduced (this can be deduced from the fact below that  $s_2 s_3 s_4 s_2 s_3 s_2$  is a subword of a reduced expression of the longest length element).

One can calculate  $\sigma_2$  in  $\mathrm{GL}_4(\mathbf{R})$ :

$$\sigma_2 = \frac{1}{2} \begin{pmatrix} -1 & -1 & -1 & 1 \\ -1 & -1 & 1 & -1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \end{pmatrix},$$

as well as (recall that  $\sigma_1 = s_1$ )

$$\sigma_1 \sigma_2 = \frac{1}{2} \begin{pmatrix} -1 & -1 & -1 & 1 \\ -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & -1 \\ 1 & -1 & 1 & 1 \end{pmatrix}.$$

Its square is

$$(\sigma_1\sigma_2)^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

showing that  $\sigma_1\sigma_2$  is of order 6 and that  $W(\Theta)$  is of type  $G_2$ .

Finally, the following holds

$$s_3s_4s_3 = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix} \quad \text{and}$$

$$(\sigma_1\sigma_2)^3 = \frac{1}{2} \begin{pmatrix} -1 & -1 & -1 & 1 \\ -1 & -1 & 1 & -1 \\ -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 \end{pmatrix} = -{}^t(s_3s_4s_3).$$

We thus have

$$\begin{aligned} (\sigma_1\sigma_2)^3 s_3s_4s_3 &= (\sigma_2\sigma_1)^3 s_3s_4s_3 = -\text{id}_4 \\ &= s_2s_3s_2s_4s_3s_2s_1 \cdot s_2s_3s_2s_4s_3s_2s_1 \cdot s_2s_3s_2s_4s_3s_2s_1 \cdot s_3s_4s_3 \end{aligned}$$

which is the longest length element of  $F_4$  (since it sends the Weyl chamber to its opposite) and a reduced decomposition of it (since the length of this decomposition is equal to 24, the number of positive roots). This shows the identities in Proposition 4.8 in this case too.

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UNIVERSITÉ DE STRASBOURG, IRMA, 7 RUE DESCARTES, 67000 STRASBOURG, FRANCE

*Email address:* `olivier.guichard@math.unistra.fr`

RUPRECHT-KARLS UNIVERSITÄT HEIDELBERG, MATHEMATISCHES INSTITUT, IM NEUENHEIMER FELD 288, 69120 HEIDELBERG, GERMANY  
HITS gGMBH, HEIDELBERG INSTITUTE FOR THEORETICAL STUDIES, SCHLOSS-WOLFSBRUNNENWEG 35, 69118 HEIDELBERG, GERMANY

*Email address:* `wienhard@mathi.uni-heidelberg.de`