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Extended Abstract : Viability and Invariance of Systems on Metric Spaces

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Abstract: In some applied models (as for instance of flocking or of the crowd control) it is more natural to deal with elements of a metric space (as for instance a family of subsets of a vector space endowed with the Hausdorff metric) rather than with vectors of a normed vector space. We consider a generalized control system on a metric space and investigate necessary and sufficient conditions for viability and invariance of proper subsets, describing state constraints. As examples of application we study controlled continuity equations on the metric space of probability measures, endowed with the Wasserstein distance, and controlled morphological systems on the space of nonempty compact subsets of the Euclidean space endowed with the Hausdorff metric. We also provide sufficient conditions for the existence and uniqueness of contingent solutions to the Hamilton-Jacobi-Bellman equation on a proper metric space.

Keywords: mutational control system; viability and invariance; optimal control; Wasserstein space; Hamilton-Jacobi inequalities.
MSC 2020: 34A06, 49L12, 49Q22.

Dynamical systems *under state constraints* are ubiquitous in the literature since a long while. Indeed, in many applied fields, as economics, finance, demography, medical sciences, aerospace, sustainable development, robotics, etc., the models do involve pointwise constraints on trajectories. Viability theory is an area of mathematics that studies evolutions of trajectories of dynamical systems under state constraints and many related questions. This theory is also a helpful tool for investigation of some classical questions arising in control. Indeed, various problems of control theory can be linked to viability and invariance properties of appropriately chosen sets, as for instance the optimal synthesis problem can be related to the *viability retroaction map* on the epigraph of the value function arising in optimal control [10], solutions of Hamilton-Jacobi-Bellman equations – to functions having viable/invariant epigraph/hypograph under extended control systems [11], optimal trajectories – to viable trajectories on the epigraph of the value function for an extended control system [11], stabilising controls – to the viability retroaction map on the epigraph of a Lyapunov function [12]. Viability theory is well investigated in the finite dimensional framework and Hilbert spaces, see for instance [4, 5, 8] and the bibliographies contained therein.

In the recent years, there is an increasing interest in control problems stated on metric spaces, cf. [2, 9, 13, 14]. The main goal of the present paper is to extend the viability and invariance theorems to the framework of proper subsets of a metric space on which a subset of transitions is fixed.

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As an application, we discuss the Hamilton-Jacobi inequalities on a proper metric space. We show that under some technical assumptions the value function of the mutational Mayer optimal control problem is the unique contingent solution satisfying a prescribed final time condition. Two examples of applications are provided as well.

1. PRELIMINARIES AND BASIC DEFINITIONS

Let (E, d) be a metric space (with the metric d) and denote by $B(x, r)$ the closed ball centered at $x \in E$ with radius $r \geq 0$. Recall that a subset $K \subset E$ is called proper if $K \cap B(x, r)$ is compact for any $(x, r) \in K \times \mathbb{R}_+$. The distance between two nonempty subsets K, M of E is defined by $\mathrm{dist}(K, M) := \inf_{k\in K,\, m\in M} d(k, m)$. Note that $\mathrm{dist}(K, M)$ measures the proximity of sets and it is not a distance function on subsets of E. Obviously, it is smaller than the Hausdorff distance.

We first recall some definitions and notations from [3, 15].

A map $\mathcal{V} : [0, 1] \times E \to E$ is called transition on (E, d) if:

- $\forall\, x \in E,\ \mathcal{V}(0, x) = x$;
- $\forall\, x \in E,\ \forall\, t,\ h \in [0, 1[$ with $t + h \leq 1$, $\mathcal{V}(t + h, x) = \mathcal{V}(h,\ \mathcal{V}(t, x))$;
- $\alpha(\mathcal{V}) := \sup\limits_{x,y\in E; x\neq y} \limsup\limits_{h\to 0+} \dfrac{d(\mathcal{V}(h, x), \mathcal{V}(h, y)) - d(x, y)}{h\, d(x, y)} < +\infty$;
- $\beta(\mathcal{V}) := \sup\limits_{x\in E} \limsup\limits_{h\to 0+} \dfrac{d(x,\ \mathcal{V}(h, x))}{h} < +\infty$.

For any transitions $\mathcal{V}_1$, $\mathcal{V}_2$ on (E, d), define the pseudo distance

$$d_\Lambda(\mathcal{V}_1, \mathcal{V}_2) := \sup_{x\in E} \limsup_{h\to 0+} \frac{1}{h}\, d(\mathcal{V}_1(h, x),\ \mathcal{V}_2(h, x)).$$

Let $\Theta(E)$ be a fixed nonempty set of transitions, $T > 0$ and $x(\cdot) : [0, T] \to E$. For $t \in [0, T[$, the set

$$\overset{\circ}{x}(t) := \{\mathcal{V} \in \Theta(E) \mid \lim_{h \to 0+} \frac{1}{h} d\Big(\mathcal{V}(h, x(t)), x(t+h)\Big) = 0\}$$

is called mutation of $x(\cdot)$ at time t (relative to $\Theta(E)$).

A mapping $x(\cdot) : [0, T] \to E$ is called primitive of $\mathcal{V} : [0, T] \to \Theta(E)$ if it is Lipschitz and $\overset{\circ}{x}(t) \ni \mathcal{V}(t)$ for a.e. $t \in [0, T]$. The reverse sign $\ni$ reflects the fact that $\overset{\circ}{x}(t)$ may be multivalued, while $\mathcal{V}(t)$ is single-valued.

For a nonempty subset K of E and $x \in K$, the set

$$\overset{\circ}{T}_K(x) := \{\mathcal{V} \in \Theta(E) \mid \liminf_{h \to 0+} \frac{1}{h} \mathrm{dist}(\mathcal{V}(h, x), K) = 0\}$$

is called the contingent transition set to K at x.

Let $T > 0$, $W : [0, T] \times E \to \mathbb{R} \cup \{+\infty\}$ and (t, x) be in the domain of W with $t < T$. Below, $\mathbf{0} : [0, 1] \times \mathbb{R} \to \mathbb{R}$ and $\mathbf{1} : [0, 1] \times \mathbb{R} \to \mathbb{R}$ denote the transitions on $\mathbb{R}$ defined by $\mathbf{0}(h, t) = t$, $\mathbf{1}(h, t) = t + h$, for all $h \in [0, 1]$, $t \in \mathbb{R}$. For any transition $\mathcal{V} \in \Theta(E)$, the contingent directional derivative of W at (t, x) in the direction $(\mathbf{1}, \mathcal{V})$ is defined by

$$\overset{\circ}{D}_{\uparrow} W(t, x)(\mathbf{1}, \mathcal{V}) = \lim_{\varepsilon \to 0+} \inf_{\substack{h \in]0, \varepsilon] \\ y \in B(\mathcal{V}(h, x), \varepsilon h)}} \frac{W(t + h, y) - W(t, x)}{h}$$

As in [6], it is not difficult to link directional derivatives to the contingent transition set to the epigraph of W at (t, x).

Let (U, d_U) be a complete separable metric space of control parameters and define the set of admissible controls by

$$\mathcal{U} := \{u(\cdot) : [0, \infty) \to U \mid u(\cdot) \text{ is Lebesgue measurable}\}.$$

2. WEAK INVARIANCE, VIABILITY AND INVARIANCE

Consider a map $f : E \times U \to \Theta(E)$. f is called continuous if for any $(x_0, u_0) \in E \times U$ and $\varepsilon > 0$, there exists some $\delta > 0$ such that for all $(x, u) \in E \times U$,

$$d(x, x_0) + d_U(u, u_0) < \delta \implies d_\Lambda(f(x, u), f(x_0, u_0)) < \varepsilon.$$

$f(\cdot, u)$ is said to be uniformly Lipschitz in u, if there is a constant $L > 0$ such that

$$d_\Lambda(f(x, u),\ f(y, u)) \le L\, d(x, y), \quad \forall\, x, y \in E,\ \forall\, u \in U.$$

For any $x_0 \in E$ consider the mutational control system

$$[S] \quad \overset{\circ}{x}(s) \ni f(x(s),\ u(s)) \text{ a.e., } x(0) = x_0,\ u(\cdot) \in \mathcal{U}.$$

A Lipschitz mapping $x(\cdot) : [0, T] \to E$ is said to be a solution to [S] on $[0, T]$ for some $T > 0$, if there exists a control $u(\cdot) \in \mathcal{U}$ such that $\overset{\circ}{x}(t) \ni f(x(t), u(t))$ a.e. in $[0, T]$ and $x(0) = x_0$. If $f(\cdot, u)$ is uniformly Lipschitz in u, then, by [15, Proposition 21, p. 41] and the Gronwall lemma, to every control $u(\cdot) \in \mathcal{U}$ corresponds at most one solution of [S]. Below we always assume that f is continuous and that

$$\sup_{x \in E, u \in U} \alpha(f(x, u)) < +\infty; \quad \sup_{x \in E, u \in U} \beta(f(x, u)) < +\infty.$$

We also assume that for each $x_0 \in E$ and $\bar{u} \in U$ there exists a solution to $\overset{\circ}{x} \ni f(x, \bar{u})$, $x(0) = x_0$ defined on $[0, 1]$. By [15, Theorem 20, p.40] it is always the case when E is proper. For more general metric spaces such existence depends on the choice of $\Theta(E)$ and f.

For any $x_0 \in E$ and $t \ge 0$, denote by $\mathcal{S}_t(x_0)$ the set of all solutions to the mutational control system $[S]$ defined on $[0, t]$ and by $R(t; x_0) := \{x(t) \in E \mid x(\cdot) \in \mathcal{S}_t(x_0)\}$, the reachable set from x_0 at time t.

Consider the controlled mutational equation

$$\overset{\circ}{x}(s) \ni f(x(s),\ u(s)) \text{ a.e. } s \ge 0,\ u(\cdot) \in \mathcal{U} \tag{1}$$

and let $K \subset E$ be a proper nonempty set. K is called weakly invariant under (1) if for every $x_0 \in K$ we have $R(t; x_0) \cap K \neq \emptyset$ for any $t \ge 0$, see [17].

K is called viable under (1) if for any $x_0 \in K$, there exists a solution $x(\cdot)$ of (1) with $x(0) = x_0$ satisfying $x(t) \in K$ for all $t \ge 0$.

K is called invariant under (1) if every solution $x(\cdot)$ of (1) with $x(0) \in K$ satisfies $x(t) \in K$ for all $t \ge 0$.

Clearly any viable set is weakly invariant. To illustrate that weak invariance does not yield viability, consider the two dimensional control system $x' = u(t)$ in $\mathbb{R}^2$ with $U = \{0\} \times \{-1, 1\}$ and the set K equal to the unit sphere in $\mathbb{R}^2$. Then every trajectory of this system staring in K leaves it immediately. At the same time $x_0 \in R(t; x_0)$ for any $x_0 \in K$ and $t \ge 0$. Therefore K is weakly invariant.

Proposition 1. (Necessary condition for weak invariance). Assume that (U, d_U) is compact and $K \subset E$ is weakly invariant. If $x \in K$ is so that for a sequence $h_i \to 0+$

$$\sup_{y \in R(h_i; x)} \inf_{u \in U} d\Big(y, f(x, u)(h_i, x)\Big) = o(h_i), \tag{2}$$

then there exists $u \in U$ satisfying $f(x, u) \in \overset{\circ}{T}_K(x)$. In particular, if for every $x \in K$ we can find $h_i \to 0+$ satisfying (2), then $f(x, U) \cap \overset{\circ}{T}_K(x) \neq \emptyset$ for every $x \in K$.

Theorem 2. (Sufficient condition for weak invariance). Assume that $f(\cdot, u)$ uniformly Lipschitz in u, that $R(t; x_0)$ compact for all $x_0 \in K$ and $t > 0$ and that for any $x \in K$, $f(x, U) \cap \overset{\circ}{T}_K(x) \neq \emptyset$. Then K is weakly invariant under (1).

Theorem 3. (Sufficient condition for viability). Under the assumptions of Theorem 2 suppose that for each $x_0 \in K$ and $t > 0$, the set $\mathcal{S}_t(x_0)$ is closed in the metric of uniform convergence. Then K is viable under (1).

Under the assumptions of Theorem 2, it can be shown, using the Ascoli-Arzelà Theorem 47.1 from [16], that every sequence $\{x_n(\cdot)\}_n \subset \mathcal{S}_T(x_0)$ has a subsequence converging uniformly to a Lipschitz function $x : [0, T] \to E$. Clearly, $x(t) \in R(t; x_0)$ for all $t \in [0, T]$. This does not yield however that $x(\cdot) \in \mathcal{S}_T(x_0)$. When E is a locally compact, complete metric space, using a more sophisticated construction as in [17, Theorem of Barbashin], it is even possible to get a Lipschitz mapping $x : [0, T] \to E$ satisfying $x(t_2) \in R(t_2; t_1, x(t_1))$ for all $0 \le t_1 < t_2 \le T$. Still this does not imply that $x(\cdot) \in \mathcal{S}_T(x_0)$. To illustrate that the assumptions in the above two theorems are not equivalent, consider a compact subset $U \subset \mathbb{R}^n$ and the control system $x' = u(t) \in U$, $x(0) = x_0$. It is well known that its reachable sets are compact, while, in general, $\mathcal{S}_T(x_0)$ is not closed in the metric of uniform convergence.

Theorem 4. (Mutational invariance). Suppose that $f(\cdot, u)$ is uniformly Lipschitz in u. Then K is invariant under (1) if and only if $f(x, U) \subset \overset{\circ}{T}_K(x)$ for each $x \in K$.

3. HAMILTON-JACOBI INEQUALITIES

In this section, (E, d) is a proper metric space. Given an extended lower semicontinuous cost function $g : E \to \mathbb{R} \cup \{+\infty\}$ and $x_0 \in E$, we associate to it the Mayer optimal control problem

$[P]$ minimize $g(x(1))$

over all the solutions of the mutational control system [S] defined on $[0, 1]$.

Consider the mutational equation

$$\overset{\circ}{z}(\cdot) \ni f(z(\cdot), u(\cdot)), \quad u(\cdot) \in \mathcal{U} \tag{3}$$

and define the value function $V : [0, 1] \times E \to \mathbb{R} \cup \{+\infty\}$ by : for all $t \in [0, 1]$ and $x \in E$, $V(t, x) := \inf\{g(z(1)) \mid$

$z(\cdot)$ is a solution to (3) on $[t, 1]$, $z(t) = x\} \in \mathbb{R} \cup \{\pm\infty\}$.

Theorem 5. Assume that for any $x \in E$, the set $\mathcal{S}_1(x)$ is closed in the metric of uniform convergence. Then for any $t_0 \in [0, 1]$ and $x \in E$ there exist a control $u(\cdot) \in \mathcal{U}$ and a solution z to (3) with $z(t_0) = x$ defined on $[t_0, 1]$ and satisfying $V(t_0, x) = g(z(1))$.

Theorem 6. Assume that $f(\cdot, u)$ is uniformly Lipschitz in u and that for any $x \in E$, there exist $h_i \to 0+$ satisfying (2). Then, the value function V verifies the boundary condition $V(1, \cdot) = g(\cdot)$ and the following contingent inequalities:

- for any (t, x) in the domain of V with $t < 1$, $\sup_{u \in U} \overset{\circ}{D}_{\uparrow}(-V)(t, x)(\mathbf{1}, f(x, u)) \leq 0$.
- for any (t, x) in the domain of V with $t < 1$, $\overset{\circ}{D}_{\uparrow} V(t, x)(\mathbf{1}, f(x, u)) \leq 0$ for some $u \in U$.

A continuous map $w : [0, 1] \times E \to \mathbb{R}$ is called a contingent solution to the mutational Hamilton-Jacobi equation (associated with $[P]$, $[S]$) if it satisfies the boundary condition $w(1, \cdot) = g(\cdot)$ and the above two contingent inequalities with V replaced by w.

Theorem 7. Assume that $f(\cdot, u)$ is uniformly Lipschitz in u. If g is continuous, then V is continuous. Furthermore, if g is locally Lipschitz, then V is locally Lipschitz.

Theorem 8. Let $g : E \to \mathbb{R}$ be continuous and $f(\cdot, u)$ be uniformly Lipschitz in u. Assume that for any $x \in E$, the set $\mathcal{S}_1(x)$ is closed in the metric of uniform convergence and there exist $h_i \to 0+$ satisfying (2). Then V is the unique continuous contingent solution to the mutational Hamilton-Jacobi equation.

4. EXAMPLES

In this section we discuss two examples where the general results of previous sections do apply. We endow the space $Lip(\mathbb{R}^N, \mathbb{R}^N)$ of all bounded Lipschitz continuous functions $F : \mathbb{R}^N \to \mathbb{R}^N$ with the topology of local uniform convergence. For any $F \in Lip(\mathbb{R}^N, \mathbb{R}^N)$, denote by $Lip\, F$ the smallest Lipschitz constant of F and set $||F||_\infty := \sup_{x \in \mathbb{R}^N} |F(x)|$. For $F : \mathbb{R}^N \to \mathbb{R}^N$ and $K \subset \mathbb{R}^N$ define $(Id + F)(K) := \{x + F(x) \mid x \in K\}$.

4.1 Morphological control problem

Consider the metric space $\mathcal{K}(\mathbb{R}^N)$ of nonempty compact subsets of $\mathbb{R}^N$ supplied with the Pompeiu-Hausdorff distance: for all $K_1, K_2 \in \mathcal{K}(\mathbb{R}^N)$

$$d_H(K_1, K_2) := \max\left\{\max_{x \in K_1} \operatorname{dist}(x, K_2),\ \max_{x \in K_2} \operatorname{dist}(x, K_1)\right\}.$$

Then $(\mathcal{K}(\mathbb{R}^N),\ d_H)$ is a proper metric space, see for instance [15, Proposition 47, p.57].

For any $F : [0, \infty) \to Lip(\mathbb{R}^N, \mathbb{R}^N)$, $t \geq 0$ and $K_0 \in \mathcal{K}(\mathbb{R}^N)$, the set $\mathcal{V}_{F(\cdot)}(t, K_0) := \{x(t) \mid$

$x(\cdot) \in W^{1,1}\big([0, t], \mathbb{R}^N\big)$, $x'(s) = F(s)(x(s))$ a.e., $x(0) \in K_0\}$

is called the reachable set at time t of the system governed by $F(\cdot)(\cdot)$ from the initial condition K_0. The subset of transitions is given by

$$\Theta(\mathcal{K}(\mathbb{R}^N)) = \{\mathcal{V}_F \mid F \in Lip(\mathbb{R}^N, \mathbb{R}^N)\}.$$

Let $f : \mathcal{K}(\mathbb{R}^N) \times U \to Lip(\mathbb{R}^N, \mathbb{R}^N)$ be continuous with

$$\sup_{u \in U,\, K \in \mathcal{K}(\mathbb{R}^N)} \Big(Lip\ f(K, u) + ||f(K, u)||_\infty\Big) < +\infty\ . \tag{4}$$

Consider the morphological control system

$$[M] \qquad \overset{\circ}{K}(\cdot) \ni \mathcal{V}_{f(K(\cdot), u(\cdot))},\ u(\cdot) \in \mathcal{U}.$$

A map $K(\cdot) : [0, T] \to \mathcal{K}(\mathbb{R}^N)$, where $T > 0$, is called a solution to [M] if $K(\cdot)$ is Lipschitz continuous with respect to d_H and for some $u(\cdot) \in \mathcal{U}$ and for a.e. $t \in [0, T]$

$$\lim_{h \to 0+} \frac{1}{h} \cdot d_H\Big(\mathcal{V}_{f(K(t), u(t))}(h, K(t)), K(t + h)\Big) = 0.$$

Setting $F(t) := f(K(t), u(t))$, it follows that $K(\cdot)$ is the mutational primitive of $\mathcal{V}_{F(\cdot)}$ on $[0, T]$. The results from [15, pp. 388, 113, 74 and 24] imply

Proposition 9. For any $K_0 \in \mathcal{K}(\mathbb{R}^N)$, $T > 0$ and $u(\cdot) \in \mathcal{U}$, there exists a solution $K(\cdot)$ to $[M]$ on $[0, T]$ with $K(0) = K_0$ and for every time $t \in [0, T]$, $K(t)$ coincides with the reachable set of the differential equation

$$x'(\tau) = f\big(K(\tau), u(\tau)\big)\big(x(\tau)\big), \quad x(0) \in K_0.$$

Moreover, if $f(\cdot, u)$ is uniformly Lipschitz in u w.r.t. the metric on $Lip(\mathbb{R}^N, \mathbb{R}^N)$ generated by $\|\cdot\|_\infty$, then the solution to [M] with the initial condition K_0 is unique.

We shall need the following assumptions:

$$(\mathbf{H1}) \quad \begin{cases} (4) \text{ holds and } (U, d_U) \text{ is a compact metric space;} \\ f(\cdot, u) \text{ is uniformly Lipschitz in } u \text{ w.r.t. } \|\cdot\|_\infty; \\ f(K, U) \text{ is convex for every } \ K \in \mathcal{K}(\mathbb{R}^N). \end{cases}$$

Theorem 10. Assume **(H1)** and consider a closed nonempty subset $\Omega \subset \mathcal{K}(\mathbb{R}^N)$. Then Ω is viable under [M] if and only if for each $K \in \Omega$, there exists some $u \in U$ satisfying

$$\liminf_{h \to 0+} \frac{1}{h} \operatorname{dist}((Id + hf(K, u))(K), \Omega) = 0.$$

Furthermore, Ω is invariant under [M] if and only if the above equality holds true for each $K \in \Omega$ and any $u \in U$.

4.2 Control system in a Wasserstein space

Denote by $\mathcal{P}(\mathbb{R}^N)$ the family of all Borel probability measures on $\mathbb{R}^N$ endowed with the narrow topology.
For any $\emptyset \neq K \subseteq \mathbb{R}^N$, denote by $\mathcal{P}(K) \subseteq \mathcal{P}(\mathbb{R}^N)$ the set of all Borel probability measures with the support contained in K and by $\mathcal{P}_c(\mathbb{R}^N)$ the subset of all Borel probability measures with a compact support.

We first recall some notions in $\mathcal{P}(\mathbb{R}^N)$, see for instance [1].

For any $\mu \in \mathcal{P}(\mathbb{R}^N)$ and a Borel map $T : \mathbb{R}^N \to \mathbb{R}^N$, let $T_\#\mu \in \mathcal{P}(\mathbb{R}^N)$ denote the pushforward of μ through T:

$$T_\#\mu(B) := \mu(T^{-1}(B)) \text{ for any Borel set } B \subset \mathbb{R}^N.$$

Denote $\mathcal{P}_2(\mathbb{R}^N) := \{\mu \in \mathcal{P}(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} |x|^2 \, \mathrm{d}\mu(x) < +\infty\}$ and by W_2 the Wasserstein distance on $\mathcal{P}_2(\mathbb{R}^N)$, see [1]. The space $(\mathcal{P}_2(\mathbb{R}^N), W_2)$ is called the Wasserstein space of order 2. It is well known that $(\mathcal{P}_2(\mathbb{R}^N), W_2)$ is complete and separable. Furthermore, for any nonempty compact $K \subset \mathbb{R}^N$, the set $\mathcal{P}(K)$ is compact in $(\mathcal{P}_2(\mathbb{R}^N), W_2)$.

Consider a continuous map $f : \mathcal{P}_2(\mathbb{R}^N) \times U \to Lip(\mathbb{R}^N, \mathbb{R}^N)$ such that $f(\cdot, u)$ is uniformly Lipschitz in u w.r.t. the metric on $Lip(\mathbb{R}^N, \mathbb{R}^N)$ generated by $\|\cdot\|_\infty$ and the controlled continuity equation

$$[C] \qquad \partial_t \mu(t) + \nabla(f(\mu(t), u(t)) \cdot \mu(t)) = 0, \quad u(\cdot) \in \mathcal{U}.$$

Given $T > 0$, an absolutely continuous map $\mu(\cdot) : [0, T] \to \mathcal{P}_2(\mathbb{R}^N)$ is a solution to [C] on $[0, T]$ if for some $u(\cdot) \in \mathcal{U}$ it solves

$$\partial_t \mu(t) + \nabla(f(\mu(t), u(t)) \cdot \mu(t)) = 0 \tag{5}$$

on $[0, T]$ in the sense of distributions.

The existence and the representation of solutions of the non-local continuity equation (5) for every initial condition in $\mathcal{P}_c(\mathbb{R}^N)$ were investigated in [7]. We shall assume:

$$(\mathbf{H2}) \quad \begin{cases} A_2 := \sup_{u \in U,\, \mu \in \mathcal{P}_2(\mathbb{R}^N)} Lip\, f(\mu, u) < \infty; \\ \rho_2 := \sup_{u \in U,\, \mu \in \mathcal{P}_2(\mathbb{R}^N)} ||f(\mu, u)||_\infty < +\infty; \\ (U, d_U) \text{ is a compact metric space;} \\ f(\mu, U) \text{ is convex } \forall\, \mu \in \mathcal{P}_2(\mathbb{R}^N). \end{cases}$$

Under assumptions (H2), $\mathcal{P}_c(\mathbb{R}^N)$ is invariant by solutions of [C], see [7].

Let $g \in Lip(\mathbb{R}^N, \mathbb{R}^N)$ and for any $\mu_0 \in \mathcal{P}_c(\mathbb{R}^N)$ and $h \in [0, 1]$, define $\mathcal{V}_g(h, \mu_0) := \mu(h)$, where $\mu(\cdot) : [0, 1] \to \mathcal{P}_2(\mathbb{R}^N)$ is the distributional solution to the continuity equation

$$\partial_t \mu(t) + \nabla(g \cdot \mu(t)) = 0, \quad \mu(0) = \mu_0. \tag{6}$$

Then $\mathcal{V}_g : [0, 1] \times \mathcal{P}_c(\mathbb{R}^N) \to \mathcal{P}_c(\mathbb{R}^N)$ is a transition on $(\mathcal{P}_c(\mathbb{R}^N), W_2)$. For any $t > 0$ and $\mu_0 \in \mathcal{P}_c(\mathbb{R}^N)$, consider the sets $\mathcal{S}_t^C(\mu_0)$ of solutions to [C] on $[0, t]$ with $\mu(0) = \mu_0\}$ and

$$R^C(t, \mu_0) := \{\mu(t) \mid \mu \in \mathcal{S}_t^C(\mu_0)\}.$$

It follows from [7] that assumption **(H2)** implies that for any $T > 0$ and $\mu_0 \in \mathcal{P}_c(\mathbb{R}^N)$ the set $\mathcal{S}_T^C(\mu_0)$ is compact in the metric of uniform convergence.

Proposition 11. Assume **(H2)**. Then, there exists $k > 0$ such that for any $\mu_0 \in \mathcal{P}_c(\mathbb{R}^N)$, any $h_n \to 0+$ and any $\mu(h_n) \in R^C(h_n, \mu_0)$ we can find $u_n \in U$ satisfying

$$W_2\Big(\mu(h_n), \mathcal{V}_{f(\mu_0, u_n)}(h_n, \mu_0)\Big) \le k h_n^2$$

and $W_2\Big((Id + f(\mu_0, u_n))_\#\mu_0, \mathcal{V}_{f(\mu_0, u_n)}(h_n, \mu_0)\Big) \le A_2 \rho_2 h_n^2$.

The above results together with those from Section 3 allow to deduce the following viability and invariance theorem.

Theorem 12. Assume **(H2)** and let $\Omega \subset \mathcal{P}_c(\mathbb{R}^N)$ be nonempty and proper. Then Ω is viable under [C] if and only if for each $\mu \in \Omega$, there exists some $u \in U$ satisfying

$$\liminf_{h \to 0+} \frac{1}{h} \operatorname{dist}((Id + hf(\mu, u))_\#\mu, \Omega) = 0.$$

Furthermore, Ω is invariant under [C] if and only if the above equality holds true for each $\mu \in \Omega$ and any $u \in U$.

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