

# Nonsmooth Modal Analysis of a Rectangular Plate in Unilateral Contact

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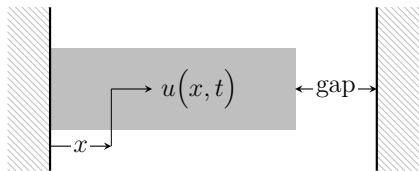
McGill University

Tuesday 19<sup>th</sup> July, 2022

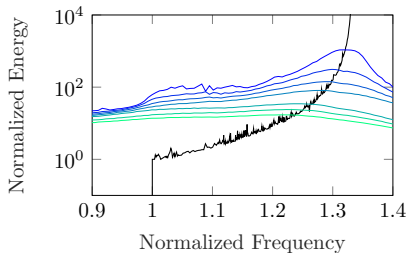


# Nonsmooth modal analysis

- Nonsmooth modal analysis: extension of linear modal analysis to nonsmooth systems
- Nonsmooth normal modes: invariant manifolds composed of periodic solutions to the autonomous system



(a) Bar in unilateral contact



(b) Nonsmooth mode (black) and forced response diagrams (blue) for nonsmooth bar. Every point on the plot represents a periodic solution

Figure 1: Nonsmooth modal analysis of the one dimensional bar<sup>1</sup>

- **Nonsmooth modal analysis of multidimensional systems remains a challenging topic**

<sup>1</sup>David Urman, Mathias Legrand. Nonsmooth Modal Analysis of a Varying Cross-Sectional Area Bar in Unilateral Contact. Preprint, 2022.

## Signorini Problem: Rectangular Plate in Unilateral Contact

## Governing Equations

- Deformable mechanics → **Linear Elasticity**

$$\ddot{\mathbf{u}}(\mathbf{x}, t) - \operatorname{div}(\boldsymbol{\sigma}(\bar{\mathbf{u}}(\mathbf{x}, t))) = \mathbf{0}, \quad \mathbf{x} \in \Omega \quad (1)$$

- Unilateral contact → **Signorini complementarity conditions**

$$\bar{\mathbf{u}}(\mathbf{x}, t) \cdot \mathbf{n} - g \leq 0 \perp \sigma_n(\bar{\mathbf{u}}) \leq 0 \quad \mathbf{x} \in \Gamma_C, \quad \sigma_n(\bar{\mathbf{u}}) = \mathbf{n}^\top \boldsymbol{\sigma}(\bar{\mathbf{u}}) \mathbf{n} \quad (2)$$

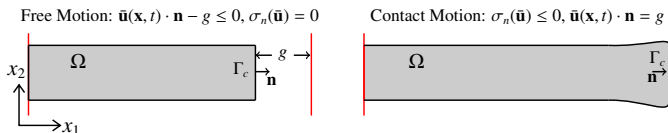


Figure 2: Two phases of the Signorini problem

- Governing equations are non-smooth<sup>2</sup>
- Determine periodic solutions to the Signorini problem for detection of nonsmooth modes
- Problem:** Conventional schemes do not allow for autonomous periodic solutions with lasting contact

<sup>2</sup>V. A. Yastrebov. *Numerical Methods in Contact Mechanics*. John Wiley & Sons, 2013.

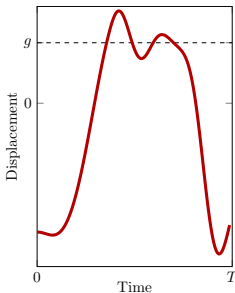
# Motivation

## FEM Numerical Schemes

- **Nitsche's Method**<sup>3</sup>
- Newton impact law
- Nodal Boundary Method

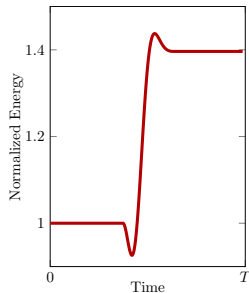
## Conserved Properties of True Solution

- ✗ Impenetrability of the obstacle
- ✗ Lasting contact phase
- ✓ Allows autonomous periodic solutions



(a) Nitsche solution

(b) Displacement of red point in (a)



(c) Total energy

Figure 3: Nitsche's method for 9 elements and quadratic shape functions

<sup>3</sup>F. Chouly and Y. Renard. Explicit Verlet time-integration for a Nitsche-based approximation of elastodynamic contact problems. *Advanced Modeling and Simulation in Engineering Sciences*, 5(31), 2018.

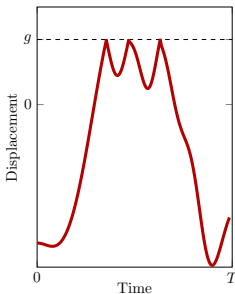
# Motivation

## FEM Numerical Schemes

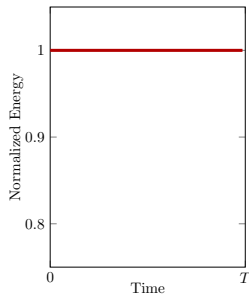
- Nitsche's Method
- **Newton impact law:**  $v(t^-) = -ev(t^+)$ .  
 $e = 1$ , fully elastic impact<sup>4</sup>
- Nodal Boundary Method

## Conserved Properties of True Solution

- ✓ Impenetrability of the obstacle
- ✗ Lasting contact phase (Chattering)
- ✓ Allows autonomous periodic solutions



(a)  $e = 1$  solution



(b) Displacement of red point in (a)

(c) Total energy

Figure 4: Newton's impact law  $e = 1$  for 9 elements and quadratic shape functions

<sup>4</sup>V. Acary, et al. An introduction to Siconos, Technical Report RT-0340, INRIA, 2019.

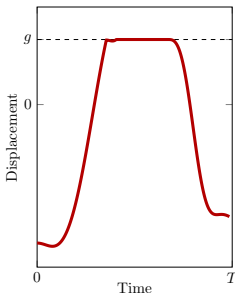
# Motivation

## FEM Numerical Schemes

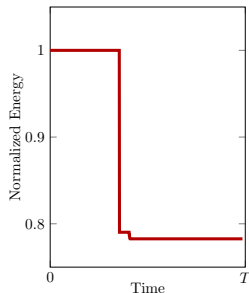
- Nitsche's Method
- **Newton impact law:**  $v(t^-) = -e v(t^+)$   
 $e = 0$ , dissipative impact<sup>5</sup>
- Nodal Boundary Method

## Conserved Properties of True Solution

- ✓ Impenetrability of the obstacle
- ✓ Lasting contact phase
- ✗ Allows autonomous periodic solutions



(a)  $e = 0$  solution



(b) Displacement of red point in (a)

(c) Total energy

Figure 5: Newton's impact law  $e = 0$  for 9 elements and quadratic shape functions

<sup>5</sup>V. Acary, et al. An introduction to Siconos, Technical Report RT-0340, INRIA, 2019.

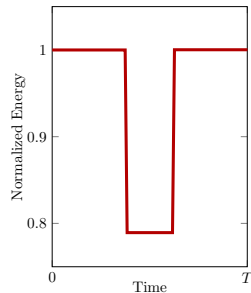
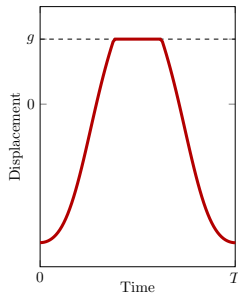
# Motivation

## FEM Numerical Schemes

- Nitsche's Method
- Newton impact law
- **Nodal Boundary Method**

## Conserved Properties of True Solution

- ✓ Impenetrability of the obstacle
- ✓ Lasting contact phase
- ✓ Allows autonomous periodic solutions



(a) NBM solution

(b) Displacement of red point in (a)

(c) Total energy

Figure 6: Nodal Boundary Method for 9 elements and quadratic shape functions

# Prelude to NBM

## Finite Element Approximation

- Governing equation

$$\mathbf{w}^\top (\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t)) - \mathbf{w}_c^\top \mathbf{f}_c = 0, \quad \bar{\mathbf{u}}(\mathbf{x}, t) \approx \mathbf{P}(\mathbf{x})\mathbf{u}(t) \quad (3)$$

- Partition: "c" - contact nodes "o" - internal nodes

$$\mathbf{u} = \begin{bmatrix} \mathbf{u}_o \\ \mathbf{u}_c \end{bmatrix} \quad \mathbf{w} = \begin{bmatrix} \mathbf{w}_o \\ \mathbf{w}_c \end{bmatrix} \quad (4)$$

- The Signorini conditions

$$\mathbf{f}_c \leq \mathbf{0} \perp \mathbf{u}_c - \mathbf{1}g \leq \mathbf{0} \quad (5)$$

- Equation (3) and (5) do not admit a unique solution given initial conditions

## Treatment of Contact Conditions

- Nitsche:  $\mathbf{f}_c$  is a non-smooth force in stress and displacement on  $\Gamma_C$
- Newton impact law:  $\mathbf{f}_c$  is an impulse force satisfying the impact law
- **Nodal Boundary Method:**

$\mathbf{f}_c$  is described via FE approximation of stress on boundary, and the behaviour of contact nodes is defined by the shape functions exclusively (5)



## Treatment of Signorini conditions in NBM

## Satisfying the Signorini conditions

- **Contact force** in NBM = FEM stress at  $\Gamma_C$

$$\mathbf{f}_c(\mathbf{u}_o, \mathbf{u}_c) = \int_{\Gamma_C} \mathbf{P}(\mathbf{x})\mathbf{S}(\mathbf{x})\mathbf{u}(t)d\mathbf{x} = \mathbf{F}_{co}\mathbf{u}_o + \mathbf{F}_{cc}\mathbf{u}_c, \quad \sigma_n(\bar{\mathbf{u}})\mathbf{n} \approx \mathbf{S}(\mathbf{x})\mathbf{u}(t) \quad (6)$$

- **Signorini condition** in NBM:

$$\mathbf{u}_c \leq \mathbf{1}g \perp \mathbf{F}_{cc}\mathbf{u}_c \leq -\mathbf{F}_{co}\mathbf{u}_o \quad (7)$$

and admits a unique solution  $\mathbf{u}_c$  given  $\mathbf{u}_o$

- **Solution to Signorini condition:**

$$\mathbf{u}_c = \mathbf{A}(\mathbf{u}_o, g)\mathbf{u}_o + \mathbf{d}(\mathbf{u}_o, g) \quad (8)$$

$\mathbf{A}$  and  $\mathbf{d}$  are piecewise constant quantities in  $\mathbf{u}_o$

- The form of  $\mathbf{A}$  and  $\mathbf{d}$  depend on the state of the contact nodes (contact ON or OFF)
- **NBM Shape functions satisfying Signorini conditions:**

$$\bar{\mathbf{u}}(\mathbf{x}, t) \approx \mathbf{P}(\mathbf{x})\mathbf{u}(t) = \mathbf{P}(\mathbf{x}) \begin{pmatrix} \mathbf{u}_o(t) \\ \mathbf{u}_c(t) \end{pmatrix} = \mathbf{P} \left( \begin{bmatrix} \mathbf{I} \\ \mathbf{A}(\mathbf{u}_o, g) \end{bmatrix} \mathbf{u}_o(t) + \begin{pmatrix} \mathbf{0} \\ \mathbf{d}(\mathbf{u}_o, g) \end{pmatrix} \right) \quad (9)$$

## Illustration of NBM

Figure 7: Contact configurations in simple NBM model: 2 elements, displacement in a single axis

- Signorini problem in NBM for model

$$\begin{pmatrix} u_{2x}(t) \\ u_{4x}(t) \end{pmatrix} \leq \begin{pmatrix} g \\ g \end{pmatrix} \perp \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{pmatrix} u_{2x}(t) \\ u_{4x}(t) \end{pmatrix} \leq \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{pmatrix} u_{1x}(t) \\ u_{3x}(t) \end{pmatrix}, \quad \mathbf{f}_C = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \left( \begin{pmatrix} u_{2x}(t) \\ u_{4x}(t) \end{pmatrix} - \begin{pmatrix} u_{1x}(t) \\ u_{3x}(t) \end{pmatrix} \right) \quad (10)$$

- The matrices  $\mathbf{A}$  and  $\mathbf{d}$  are formulated from the different solutions of the linear complementarity problem
- **Example: Homogeneous Neumann conditions in NBM**  $u_{1x}(t) \leq g, u_{3x}(t) \leq g$

$$\begin{pmatrix} u_{2x}(t) \\ u_{4x}(t) \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} u_{1x}(t) \\ u_{3x}(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{d} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Figure 8: Contact configurations in simple NBM model: 2 elements, displacement in a single axis

- Signorini problem in NBM for model

$$\begin{pmatrix} u_{2x}(t) \\ u_{4x}(t) \end{pmatrix} \leq \begin{pmatrix} g \\ g \end{pmatrix} \perp \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{pmatrix} u_{2x}(t) \\ u_{4x}(t) \end{pmatrix} \leq \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{pmatrix} u_{1x}(t) \\ u_{3x}(t) \end{pmatrix}, \quad \mathbf{f}_c = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \left( \begin{pmatrix} u_{2x}(t) \\ u_{4x}(t) \end{pmatrix} - \begin{pmatrix} u_{1x}(t) \\ u_{3x}(t) \end{pmatrix} \right) \quad (11)$$

- The matrices  $\mathbf{A}$  and  $\mathbf{d}$  are formulated from the different solutions of the linear complementarity problem
- The quantities  $\mathbf{A}$  and  $\mathbf{d}$  represent different **contact configurations**
- **Switching method**: contact configurations are switched depending on  $\mathbf{u}_o(t)$

## NBM Ordinary Differential Equation

Galerkin-Bubnov Procedure<sup>6</sup>

- NBM Shape functions used to approximate PDE solution via Galerkin-Bubnov

$$\mathbf{M}_A(\mathbf{u}_o(t), g)\ddot{\mathbf{u}}_o(t) + \mathbf{K}_A(\mathbf{u}_o(t), g)\mathbf{u}_o(t) + \mathbf{K}_d(\mathbf{u}_o(t), g) = \mathbf{0} \quad (12)$$

$$\mathbf{M}_A = \begin{bmatrix} \mathbf{I} & \mathbf{A}^\top \end{bmatrix} \mathbf{M} \begin{pmatrix} \mathbf{I} \\ \mathbf{A} \end{pmatrix}, \quad \mathbf{K}_A = \begin{bmatrix} \mathbf{I} & \mathbf{A}^\top \end{bmatrix} \mathbf{K} \begin{pmatrix} \mathbf{I} \\ \mathbf{A} \end{pmatrix}, \quad \mathbf{K}_d = \begin{bmatrix} \mathbf{I} & \mathbf{A}^\top \end{bmatrix} \mathbf{K} \begin{pmatrix} \mathbf{0} \\ \mathbf{d} \end{pmatrix} \quad (13)$$

(a) NBM solution

(b)  $e = 0$  solution(c)  $e = 1$  solution

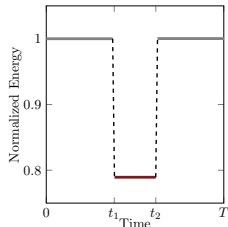
Figure 9: Comparison of solutions obtained via time marching for 100 elements, quadratic shape functions

<sup>6</sup>V. A. Krysko et al. Nonlinear vibration and characteristics of flexible plate-strips with non-symmetric boundary conditions, *Communications in Nonlinear Science and numerical Simulation*. 11:95-124, 2006.

# Notes on NBM

## Energy Behaviour

- The energy in NBM is different for every contact configuration



(a) Inactive contact,  
 $t \in [0, t_1]$

(b) Active contact,  
 $t \in (t_1, t_2)$

(c) Inactive contact,  
 $t \in (t_2, T]$

(d) Total energy

Figure 10: Nodal Boundary Method for 9 elements and quadratic shape functions

- Energy jumps reduce in amplitude with increasing number of elements

Dirac delta in  $\ddot{\mathbf{u}}_c$

Similarity to Mass Redistribution Method (MRM)

# Notes on NBM

## Energy Behaviour

- The energy in NBM is different for every contact configuration
- Energy jumps reduce with increasing number of elements

## Dirac delta in $\ddot{\mathbf{u}}_c$

- Switching method neglects existence of a dirac delta in  $\ddot{\mathbf{u}}_c(t)$
- Investigated solutions of NBM show convergence to motions by other schemes

## Similarity to Mass Redistribution Method (MRM)<sup>7</sup>

- MRM eliminates the mass on the contact boundary and results in ODE on  $\mathbf{u}_o(t)$
- MRM allows for continuous contact phases and impenetrability
- ✗ Disadvantage of MRM: a new mass matrix must be constructed for every FEM model<sup>8</sup>

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<sup>7</sup>H. B. Khenous et al. Mass redistribution method for finite element contact problems in elastodynamics, *European Journal of Mechanics - A/Solids*. 27:918-932, 2008.

<sup>8</sup>C. D. Monjaraz et al. A massless boundary component mode synthesis method for elastodynamic contact problems, *Computers and Structures*. 260:106698, 2021.

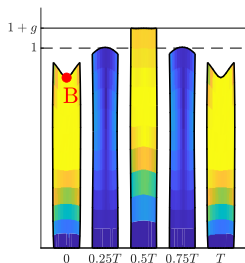
# Periodic Solution via HBM

## Harmonic Balance Method (HBM)

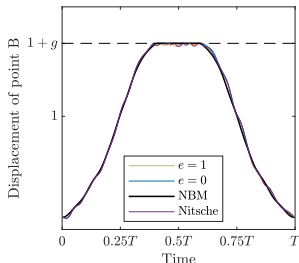
- Approximation of  $\mathbf{u}_o(t)$  in ODE (12)

$$\mathbf{u}_o(t) \approx \sum_{i=0}^H \cos(i\omega t) \mathbf{a}_i, \quad \omega = \frac{2\pi}{T} \quad (14)$$

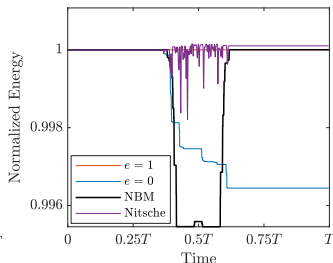
- Find  $\omega$  and  $\mathbf{a}_i$  via Galerkin-method using Newton-Raphson solver
- Accuracy increases with large  $H$  to true periodic solution of the ODE



(a) Plate in NBM



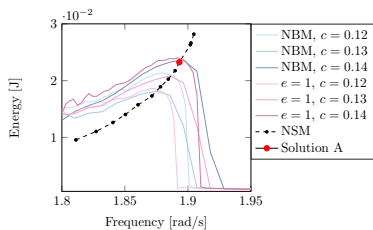
(b) Displacement of point B



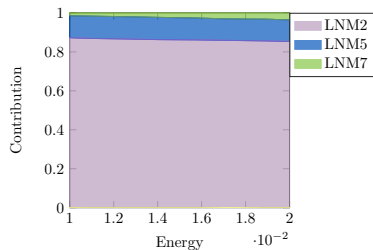
(c) Total Energy

Figure 11: Periodic solution obtained for  $H = 40$  for the fixed plate problem

## Nonsmooth modal analysis - Symmetric Mode



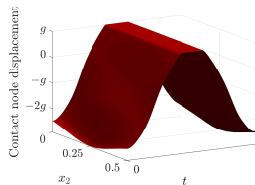
(a) Forced response and backbone curve for a symmetric nonsmooth mode (NSM)



(b) Contributions of linear normal modes (LNM) in NSM

(c) NBM motion at point A

(d)  $e = 1$  motion at point A

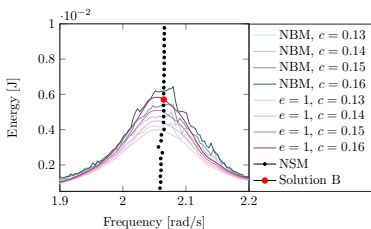


(e) Contact nodes at solution A

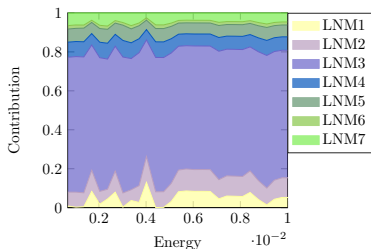
Figure 12: Nonsmooth modal analysis of a  $[0, 1] \times [0, 0.5]$  plate with  $g = 0.1$ . 100 finite elements. 10 harmonics used for generation of NSM



# Nonsmooth modal analysis - Vibration in Tangential Direction



(a) Forced response and backbone curve for a nonsmooth mode (NSM) moving in  $x_2$

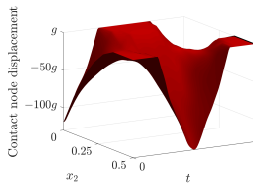


(b) Contributions of linear normal modes (LNM) in NSM

(c) NBM motion at point B

(d)  $e = 1$  motion at point B

(e) Contact nodes at solution B



**Figure 13:** Nonsmooth modal analysis of a  $[0, 1] \times [0, 0.5]$  plate with  $g = 0.001$ . 49 finite elements. 10 harmonics used for generation of NSM

# Conclusions

## The NBM-HBM for periodic solutions

- The NBM solves the Signorini problem by switching between shape functions
- NBM solutions exhibit a lasting contact phase and no penetration (similarly to  $e = 0$ )
- Periodic solutions exist for NBM and can be found via HBM
- Nonsmooth Modal Analysis of the Signorini problem via NBM

## Future Work

- Comparison of NBM against Nitsche-HBM and MRM-HBM
- Nonsmooth modal analysis on other configurations in 2D and triangular elements
- Nonsmooth modal analysis in three dimensions