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Identification of the spatial component in the source term of a space-time fractional diffusion equation

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Abstract

This paper is concerned with an inverse source problem for a space-time fractional diffusion equation. The aim is to reconstruct the spatial component in the source term from partially observed data. The considered ill-posed inverse source problem is formulated as a minimization one. The existence and the stability of the minimization system are discussed. An iterative process is developed for identifying the unknown source term. The efficiency and the accuracy of the algorithm are shown by several numerical experiments.

Keywords

fractional derivative ; inverse source problem ; minimization problem ; Numerical reconstruction algorithm

I INTRODUCTION

Fractional diffusion equations (FDEs), which are obtained by replacing the classical derivative (in time and /or space) by a generalized derivative of fractional order, have been applied to explain various physical phenomena (see [3] and references therein). Specifically, FDEs have been used to describe anomalous diffusion phenomena due to nonlocal property of fractional order derivatives.

In this article, we investigate an inverse problem for a fractional diffusion equation, that is the identification of the spatial component in the source term of a space-time fractional diffusion equation. Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$, be an open bounded domain with a sufficiently smooth boundary and $T > 0$, we consider the following space-time fractional diffusion equation

$$\partial_t^\alpha u(x, t) + (-\Delta)^s u(x, t) = f(x) g(t), \quad x \in \Omega, \quad t \in (0, T), \quad (1)$$

subject to the boundary condition

$$u(x, t) = 0 \quad \text{in } \Omega^c \times (0, T), \quad \text{where } \Omega^c = \mathbb{R}^d \setminus \Omega, \quad (2)$$

with initial condition

$$u(x, 0) = 0 \quad \text{in } \Omega. \quad (3)$$

In equation (1), the operator ∂_t^α represents the Caputo fractional derivative of order $0 < \alpha < 1$ and is defined as

$$\partial_t^\alpha w(t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{w'(\tau)}{(t-\tau)^\alpha} d\tau,$$

where $\Gamma(\cdot)$ is the usual Gamma function. For $0 < s < 1$, $(-\Delta)^s$ denotes the fractional Laplace operator, defined as

$$(-\Delta)^s w(x) = \frac{2^{2s} s \Gamma(s + \frac{d}{2})}{\pi^{d/2} \Gamma(1-s)} \text{P.V.} \int_{\mathbb{R}^d} \frac{w(x) - w(y)}{|x-y|^{d+2s}} dy,$$

where "P.V." denotes the principal value of the integral:

$$\text{P.V.} \int_{\mathbb{R}^d} \frac{w(x) - w(y)}{|x-y|^{d+2s}} dy = \lim_{\varepsilon \downarrow 0} \int_{\{y \in \mathbb{R}^d, |y-x| > \varepsilon\}} \frac{w(x) - w(y)}{|x-y|^{d+2s}} dy.$$

An extra condition usually termed as an over-specified condition is considered for determination of the solution of the inverse considered problem for (1)-(3) and is given by

$$u(x, t) = \mathcal{V}_{\text{obs}}^\delta \quad \text{in } \omega \times (0, T),$$

where $\omega \subset \Omega$ be an arbitrarily chosen open subdomain.

Recently, inverse source problems for FDEs have been considered by many researchers. For space fractional diffusion equations, identification of a source term is considered in [7, 13], the considered problem has application in water contamination. On the other hand, inverse source problems for time-fractional diffusion equations have been extensively studied. Yamamoto et al. [8] proved the uniqueness result by using the Duhamel principle. Wang et al. [4] determined the space-dependent source term from the final time data. Ben Salah [11] utilized the topological gradient method to identify the shape and location of the source term. However, to the best of our knowledge, there are rare work on inverse source problems for the space-time fractional diffusion equations. Muhammad et al. [9] used the eigenfunction expansion method to prove the uniqueness of the inverse source problem. Tatar and Ulusoy [6] have proved the uniqueness of inverse source problem using a final overdetermining data $u(x, T)$. The main difference from these studies is that we will examine both theoretical and numerical aspects related to the considered inverse problem.

It is well known that the considered inverse source problem mentioned above is an ill-posed problem in the sense of Hadamard [1]. To deal with this ill-posedness, in this paper, we will utilize a classical Tikhonov regularization method. Hence, the reconstruction of the source term can be reformulated as a minimization one. After that, the existence, uniqueness and stability of the optimization problem have been discussed. In the numerical aspect, we will propose an efficient and accurate iterative reconstruction algorithm. The main idea of our approach is to characterize the minimizer by a variational equation with the help of the adjoint state, which results in the iterative thresholding algorithm. The efficiency and accuracy of the proposed algorithm are illustrated by some numerical results.

This paper is organized as follows: In the next section we reformulate the inverse problem as a regularized minimization one. Existence, uniqueness and stability for the minimization problem are discussed in Section III. Section IV is concerned with the proposed reconstruction approach. An optimality condition is established in Section 4.1. The main steps of the derived numerical algorithm are described in Section 4.2. In Section V, some numerical results are presented to show the efficiency and the accuracy of the proposed method. Finally, we give a conclusion in Section VI.

II INVERSE PROBLEM FORMULATION

Let $\Omega \subset \mathbb{R}^d$ ($d \geq 1$) be an open bounded domain and $T > 0$ be a fixed final time. The inverse problem that we consider consists in identifying the source term f in the following initial-boundary value problem

$$\begin{cases} \partial_t^\alpha u(f) + (-\Delta)^s u(f) = f(x) g(t) & \text{in } \Omega \times (0, T) \\ u(f) = 0 & \text{in } \Omega^c \times (0, T) \\ u(f) = 0 & \text{in } \Omega \times \{0\}. \end{cases} \quad (4)$$

from an internal observation data $\mathcal{V}_{\text{obs}}^\delta$, measured on $\omega \times (0, T)$ where $\omega \subset \Omega$ be a given subdomain.

It is well known that this inverse problem is ill-posed in the sense of Hadamard [1]. To deal with the ill-posedness, we shall utilize a classical Tikhonov regularization methodology as that in [2, 8, 12]. Consequently, the considered inverse source problem may be reformulated and modeled by the following regularized minimization problem:

$$\min_{f \in L^2(\Omega)} \mathcal{K}(f), \quad \mathcal{K}(f) := \|u(f) - \mathcal{V}_{\text{obs}}^\delta\|_{L^2(\omega \times (0, T))}^2 + \beta \|f\|_{L^2(\Omega)}^2 \quad (\mathcal{P}_{\min})$$

where $u(f)$ is solution to (4) and $\beta > 0$ is the regularization parameter.

III MATHEMATICAL ASPECTS

This section is concerned with a mathematical analysis of the minimization problem (\mathcal{P}_{\min}).

3.1 Preliminaries

In this subsection, we first set up notations, and review some of standard facts on the fractional calculus. For $s \in (0, 1)$, the usual fractional Sobolev space $H^s(\Omega)$ is defined as

$$H^s(\Omega) = \left\{ v \in L^2(\Omega) : \int_{\Omega} \int_{\Omega} \frac{|v(x) - v(y)|^2}{|x - y|^{d+2s}} dx dy < \infty \right\}.$$

It is well-known that $H^s(\Omega)$ is a Hilbert space endowed with the norm $\|\cdot\|_{H^s(\Omega)}$, defined by

$$\|v\|_{H^s(\Omega)} = \left(\|v\|_{L^2(\Omega)}^2 + |v|_{H^s(\Omega)}^2 \right)^{\frac{1}{2}},$$

where $| \cdot |_{H^s(\Omega)}$ denotes the Aronszajn-Slobodeckij seminorm, given by

$$|v|_{H^s(\mathcal{D})}^2 = \int_{\Omega} \int_{\Omega} \frac{|v(x) - v(y)|^2}{|x - y|^{d+2s}} dx dy.$$

Moreover, let us introduce the fractional space $\tilde{H}^s(\Omega)$ of order $s \in (0, 1)$, which is defined as $\bar{\Omega}$

$$\tilde{H}^s(\Omega) := \{v \in H^s(\mathbb{R}^d) : \text{supp } v \subset \bar{\Omega}\}.$$

The following theorem gives us the regularity of the weak solution of the forward problem (4).

Theorem III.1: [10, Theorem 3.2]

Let $0 < \alpha, s < 1$, $f \in L^2(\Omega)$ and $g \in L^2(0, T)$ be given. Then, problem (4) admits a unique solution $u \in L^2(0, T; \tilde{H}^s(\Omega) \cap H^{s+\gamma}(\Omega))$ such that $\partial_t^\alpha u \in L^2(\Omega \times (0, T))$. Further, there exists a constant $C > 0$ such that

$$\|\partial_t^\alpha u\|_{L^2(\Omega \times (0, T))} + \|u\|_{L^2(0, T; H^{s+\gamma}(\Omega))} \leq C \|f\|_{L^2(\Omega)},$$

here $\gamma := \min\{s, 1/2 - \varepsilon\}$ with $\varepsilon > 0$ arbitrarily small.

3.2 Analysis of the minimization problem

In this subsection, we will discuss some questions related to the minimization problem (\mathcal{P}_{\min}) . We start by showing the unique existence result in Theorem III.2. The stability question is proved in Theorem III.3.

Theorem III.2: [13, Theorem 3.5]

Let $\mathcal{V}_{\text{obs}}^\delta \in L^2(\omega \times (0, T))$ be an observed data, measured on a given sub-domain $\omega \subset \Omega$. Then, the minimization problem (\mathcal{P}_{\min}) admits a unique solution $f^* \in L^2(\Omega)$.

Next, we justify the stability of the minimization problem (\mathcal{P}_{\min}) with respect to the perturbation in observation data.

Theorem III.3: [13, Corollary 3.6]

Let $\mathcal{V}_\ell^\delta \subset L^2(\omega \times (0, T))$ be a sequence such that $\mathcal{V}_\ell^\delta \rightarrow \mathcal{V}_{\text{obs}}^\delta$ in $L^2(0, T; L^2(\omega))$ as $\ell \rightarrow \infty$ and $\{f^\ell\}$ be a sequence of minimizers of problems

$$\min_{f \in L^2(\Omega)} \mathcal{K}_\ell(f), \quad \mathcal{K}_\ell(f) := \|u(f) - \mathcal{V}_\ell^\delta\|_{L^2(\omega \times (0, T))}^2 + \beta \|f\|_{L^2(\Omega)}^2, \quad \ell = 1, 2, \dots$$

Then $\{f^\ell\}$ converges strongly in $L^2(\Omega)$ to the minimizer of (\mathcal{P}_{\min}) .

IV RECONSTRUCTION APPROACH

This section is devoted to develop a numerical reconstruction approach for solving the minimization problem (\mathcal{P}_{\min}) . It is based on an optimality condition which is obtained with the help of the corresponding adjoint problem of (4).

4.1 Optimality condition

In the following proposition, we will derive a first-order optimality condition which consists to provide a new characterization of the unknown source term f^* .

Proposition IV.1: [13, Theorem 3.6]

Let f^* be the solution to the minimization problem (\mathcal{P}_{\min}) . Then, the source term f^* satisfies the following optimality condition

$$\int_0^T g(t) z(f^*)(\cdot, t) dt + \beta f^* = 0, \quad (5)$$

where $z(f^*)$ is the associated adjoint state, solution to the following system:

$$\begin{cases} D_t^\alpha z + (-\Delta)^s z = \chi_\omega (u(f^*) - \mathcal{V}_{\text{obs}}^\delta) & \text{in } \Omega \times (0, T) \\ z = 0 & \text{in } \Omega^c \times (0, T) \\ J_{T-}^{1-\alpha} z = 0 & \text{in } \Omega \times \{T\} \end{cases} \quad (6)$$

where χ_ω denotes the characterization function of ω , the operator J_{T-}^α stands the right Riemann-Liouville fractional integral which is defined by

$$J_{T-}^\alpha z(\cdot, t) = \frac{1}{\Gamma(\alpha)} \int_t^T \frac{z(\cdot, \tau)}{(\tau - t)^{1-\alpha}} d\tau,$$

and D_t^α is the backward Riemann-Liouville fractional derivative

$$D_{T^-}^\alpha z(\cdot, t) := \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^T \frac{z'(\cdot, \tau)}{(\tau-t)^\alpha} d\tau.$$

4.2 Reconstruction algorithm

Following the same line as that in [2, 12, 13], the variational equation (5) can be solved by the following iterative thresholding algorithm

$$f^{k+1} = \frac{\mathcal{M}}{\mathcal{M} + \beta} f^k - \frac{\tau}{\mathcal{M} + \beta} \int_0^T g(t) z(f^k)(\cdot, t) dt, \quad k = 0, 1, \dots, \quad (7)$$

where $\mathcal{M} > 0$ is a tuning parameter for the convergence. The choice of \mathcal{M} should be larger than the operator norm of the forward operator

$$\begin{aligned} \mathcal{B} : L^2(\Omega) &\longrightarrow L^2(\omega \times (0, T)) \\ f &\longmapsto u(f)|_{\omega \times (0, T)} \end{aligned}$$

It means, this choice can be achieved by choosing

$$\|\mathcal{B}\| \leq \mathcal{M}. \quad (8)$$

For more details, one can refer to [2, 12].

We are now ready to propose the iterative thresholding algorithm for the reconstruction.

Algorithm 1: Iterative reconstruction algorithm.

Initialisation :

- Choose an initial source term f^0 .
- Fix a tuning constant $\mathcal{M} > 0$ according to (8) and a regularization parameter $\beta > 0$.
- Choose an error tolerance $\rho > 0$ and a fixed observation sub-domain $\omega \subset \Omega$.
- Give an observed data $\mathcal{V}_{\text{obs}}^\delta$ with a fixed level of noise δ (can be obtained from an exact solution f_{exact}).

Set $k = 0$

- (i) Solve problems (4) and (6), respectively, with $f = f^0$.
- (ii) Determine the term f^1 by using the relation (7).
- (iii) Compute the error estimate $\text{err}(0)$ by using the following error function

$$\text{err}(k) = \frac{\|f^{k+1} - f^k\|_{L^2(\Omega)}}{\|f^k\|_{L^2(\Omega)}}.$$

Set $k = 1$

while $\text{err}(k-1) > \rho$

- (i) Solve problems (4) and (6), respectively, with $f = f^k$.
- (ii) Determine the term f^{k+1} by using the relation (7).
- (iii) Compute the error estimate $\text{err}(k)$.

$k \longleftarrow k + 1$.

end while

In the next section, we shall present several numerical experiments by applying Algorithm 1 to solve the minimization problem (\mathcal{P}_{min}).

V NUMERICAL EXPERIMENTS

In this section, we present several numerical examples that demonstrate the efficiency of the proposed approach. The aim is the reconstruction of the spatial component in the source term for the considered space-time fractional diffusion system by applying Algorithm 1.

5.1 Numerical implementation

Without loss of generality, in the numerical implementation, we set

$$\Omega = (-1, 1), \quad T = 1.$$

To solve problems (4) and (6), we utilize an approximation method based on the combination of the finite difference approximation method [5] and the \mathbb{P}_1 finite element discretization [10, 13]. The noisy observation data $\mathcal{V}_{\text{obs}}^\delta$ is generated by adding a random perturbation, i.e.

$$\mathcal{V}_{\text{obs}}(x, t) = (1 + \delta \text{rand}(-1, 1))u(f_{\text{exact}})(x, t), \quad (x, t) \in \omega \times (0, T).$$

where f_{exact} is the true solution, $\text{rand}(-1, 1)$ denotes the uniformly distributed random number in $[-1, 1]$ and $\delta \geq 0$ is the noise level. The known temporal component g , the parameters β and ρ are fixed as : $g(t) = 1 + t^2$, $\beta = 10^{-4}$ and $\rho = 10^{-4}$, respectively.

5.2 Reconstruction results

Throughout this subsection, we denote by \mathcal{I}_n the iteration number and by err the relative error between the exact solution and the reconstructed one.

5.2.1 Effect of the derivative orders

In this test, we aim to present the influence of the derivative orders on the numerical performance. In doing so, we fix $\delta = 0\%$ and $\mathcal{M} = 2$, and we apply Algorithm 1 with exact function $f_{\text{exact}}(x) = \cos(\pi x) + x^2 + 3$. In order to appear the effect of (α, s) on the reconstruction results, we consider the following error function

$$\text{err}_\alpha(s) = \frac{\|f^{\text{rec},s} - f_{\text{exact}}\|_{L^2(\Omega)}}{\|f_{\text{exact}}\|_{L^2(\Omega)}}, \quad \text{for } s \in \{0.05, \dots, 0.95\}.$$

Figure 1 represents the variation of the error function err_α for three values of α . From Figure 1,

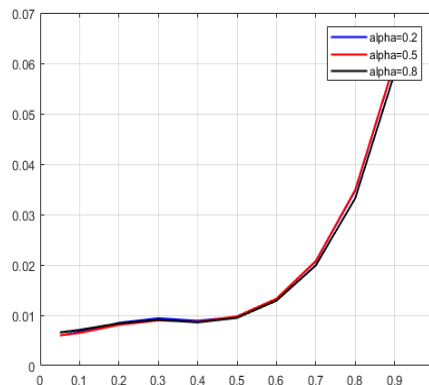


Figure 1: Variation of the functions err_α with respect to fractional-Laplace order s .

one can remark that the time-fractional order α is not significant and Algorithm 1 works slightly better when the fractional-order s is small.

δ	0.5%	1%	2%	4%	8%
err	1.44%	1.55%	1.95%	3.10%	6.16%
\mathcal{I}_n	28	28	28	29	32

Table 1: The relative errors and the iteration numbers with respect to the noise level.

5.2.2 Effect of the solution regularity

This test is concerned with the identification of source terms with different regularities. More precisely, we fix $\alpha = 0.4$, $s = 0.6$, $\mathcal{M} = 0.5$, $\delta = 2\%$. and apply Algorithm 1 to reconstruct the following functions

$$f_{\text{exact}}^{\text{sm}}(x) = \sin(\pi x) \quad \text{and} \quad f_{\text{exact}}^{\text{n-sm}}(x) = \begin{cases} 0.25 & \text{if } -1 \leq x < -0.5 \\ 0.5 & \text{if } -0.5 \leq x < 0.5 \\ 0.75 & \text{if } 0.5 \leq x \leq 1 \end{cases}$$

In Figure 2, we illustrate the true source terms with the corresponding reconstructions, and the iteration steps \mathcal{I}_n and the relative errors are shown in the caption. From Figure 2, one can

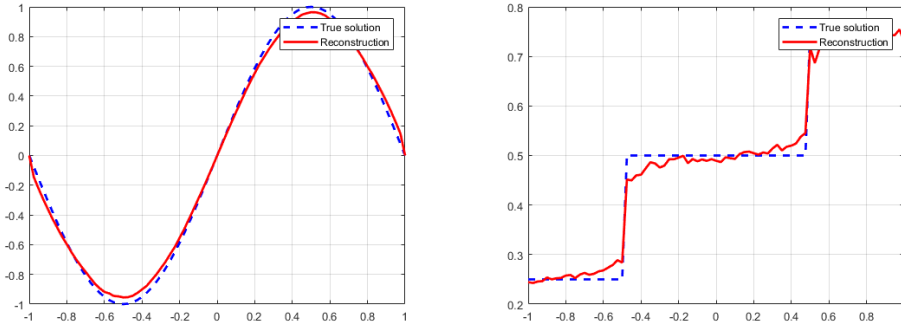


Figure 2: True source terms and their reconstructions. Left: $f_{\text{exact}}^{\text{sm}}$, $\mathcal{I}_n = 37$, $err = 3.15\%$. Right: $f_{\text{exact}}^{\text{n-sm}}$, $\mathcal{I}_n = 67$, $err = 5.85\%$.

observe that Algorithm 1 works better on the reconstruction of a smooth function.

5.2.3 Effect of the noisy data

In this test, we fix $\alpha = 0.8$, $s = 0.1$, $\mathcal{M} = 0.1$ and the true solution $f_{\text{exact}}(x) = \cos(\pi x) + x^2 + 3$. Our aim is to test the numerical performance of Algorithm 1 with various choices of the noise level δ to see their influences on the reconstructions. The choices of δ in the tests and the corresponding numerical performances are listed in Table 1. From Table 1, one can observe that the relative error and iteration steps increase with larger noise level as expected.

5.2.4 Effect of the subdomain ω

This test, we examine the influence of the subdomain ω on the accuracy of the reconstruction results. In doing so, we fix $\mathcal{M} = 1$, $\delta = 2\%$, $\alpha = 0.4$ and $s = 0.7$ and we apply Algorithm 1 to identify the true source term $f_{\text{exact}}(x) = 2 - x$ from an observation data $\mathcal{V}_{\text{obs}}^\delta$ measured on $\omega_\zeta \times (0, T)$ where the subdomain ω_ζ is defined by

$$[-1, -1 + \zeta] \cup [1 - \zeta, 1], \quad \zeta \in [0, 1].$$

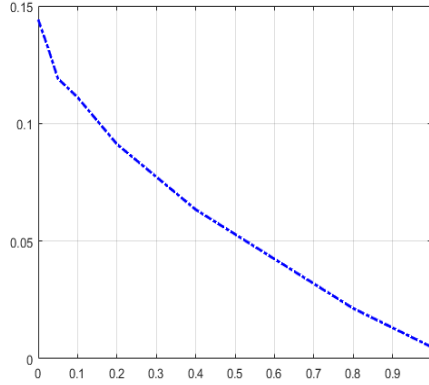


Figure 3: Variation of the error err with respect to ζ .

From Figure 3 shows the variation of the relative error with respect to the parameter ζ . Figure 3, one can observe that the relative error decreases with the increase of the parameter ζ . However, the efficiency of the algorithm increases as the size of the subdomain ω_ζ increases.

VI CONCLUSION AND REFERENCES

In this paper, we investigate an inverse space-dependent source problem for a space-time fractional diffusion equation by internal noisy measured data. In the theoretical part, the inverse problem is formulated as a regularized minimization one. The existence, uniqueness and stability questions related to the minimization problem have been examined. In the numerical aspect, the minimization problem has been characterized by a variational equation with the help of the adjoint problem (6). Based on this characterization, an efficient and accurate iterative reconstruction algorithm has been developed and implemented. Then several numerical tests are implemented to show the influence of some parameters on the efficiency and accuracy of Algorithm 1.

REFERENCES

- [1] J. Hadamard. “Sur les problèmes aux dérivées partielles et leur signification physique”. In: *Princeton university bulletin* (1902), pages 49–52.
- [2] I. Daubechies, M. Defrise, and C. De Mol. “An iterative thresholding algorithm for linear inverse problems with a sparsity constraint”. In: *Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences* 57.11 (2004), pages 1413–1457.
- [3] A. Kilbas Anatoliui, H. M. Srivastava, and J. J. Trujillo. *Theory and applications of fractional differential equations*. Volume 204. elsevier, 2006.
- [4] W. Wang, M. Yamamoto, and B. Han. “Numerical method in reproducing kernel space for an inverse source problem for the fractional diffusion equation”. In: *Inverse Problems* 29.9 (2013), page 095009.
- [5] B. Jin, R. Lazarov, Y. Liu, and Z. Zhou. “The Galerkin finite element method for a multi-term time-fractional diffusion equation”. In: *Journal of Computational Physics* 281 (2015), pages 825–843.
- [6] S. Tatar and S. Ulusoy. “An inverse source problem for a one-dimensional space–time fractional diffusion equation”. In: *Applicable Analysis* 94.11 (2015), pages 2233–2244.

- [7] Y. Zhang, M. M. Meerschaert, and R. M. Neupauer. “Backward fractional advection dispersion model for contaminant source prediction”. In: *Water Resources Research* 52.4 (2016), pages 2462–2473.
- [8] D. Jiang, Z. Li, Y. Liu, and M. Yamamoto. “Weak unique continuation property and a related inverse source problem for time-fractional diffusion-advection equations”. In: *Inverse Problems* 33.5 (2017), page 055013.
- [9] M. Ali, S. Aziz, and S. A. Malik. “Inverse source problem for a space-time fractional diffusion equation”. In: *Fractional Calculus and Applied Analysis* 21.3 (2018), pages 844–863.
- [10] G. Acosta, F. M. Bersetche, and J. P. Borthagaray. “Finite element approximations for fractional evolution problems”. In: *Fractional Calculus and Applied Analysis* 22.3 (2019), pages 767–794.
- [11] M. Ben Salah. “Topological sensitivity method for reconstruction of the spatial component in the source term of a time-fractional diffusion equation”. In: *Ricerche di Matematica* (2021), pages 1–26.
- [12] M. BenSalah and M. Hassine. “Inverse source problem for a diffusion equation involving the fractional spectral Laplacian”. In: *Mathematical Methods in the Applied Sciences* 44.1 (2021), pages 917–936.
- [13] M. BenSaleh and H. Maatoug. “Inverse source problem for a space-time fractional diffusion equation”. In: *Ricerche di Matematica* (2021), pages 1–33.