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A side-sensitive double sampling \bar{X} monitoring scheme with estimated process parameters

Motsepa C.M.¹, *Malela-Majika J.-C.¹, Castagliola P.², Shongwe S.C.¹.

Abstract

In statistical process monitoring (SPM), most of the monitoring schemes are designed assuming that the process parameters for the underlying distribution are known (i.e. Case K). In a variety of contexts, it has been shown that when the parameters used to design the control limits are unknown (i.e. Case U), this greatly affects the monitoring schemes properties. Hence, in this paper, we study the parameter estimation effect of the side-sensitive double sampling (SSDS) \bar{X} monitoring scheme for detecting changes in the process mean when distribution design parameters are estimated from an in-control retrospective sample. A thorough investigation is conducted using the unconditional run-length properties (i.e. average, standard deviation and percentiles), average sample size (ASS) and average number of observations to signal (ANOS) through exact integral formulas and simulations. In addition, the average extra quadratic loss (AEQL), average ratio of the average run-length (ARARL) and performance comparison index (PCI) are used to quantify the run-length of the SSDS scheme from an overall performance perspective. Comparisons with other established monitoring schemes when parameters are unknown indicate that the SSDS scheme has a better overall performance. An illustrative example is also given to facilitate the design and implementation of the new scheme. An additional section briefly discussing the synthetic version of the SSDS scheme is also provided.

Keywords: Double sampling scheme; side-sensitive double sampling scheme; overall performance measures; estimated process parameters; Phase I; Phase II.

1. Introduction

One of the key objectives of statistical process monitoring (SPM) is to detect any irregularity in a process as quickly as possible. Control charts are used to identify the causes of variation in the process. Two sources of variation can be distinguished in SPM, namely the common (or chance) causes and assignable (or special) causes of variation. Unlike the assignable causes, common causes cannot be avoided. When the process runs in the presence of common causes only, the process is considered to be in-control (IC). Otherwise, the process is said to be out-of-control (OOC). It is well-known that a double sampling (DS) chart (or scheme) is one of the most powerful tools used in SPM to detect any variation (or change) in the process as soon as possible. The DS \bar{X} scheme was introduced by Daudin (1992) in an effort to improve the detection abilities of a standard Shewhart scheme in detecting small and moderate shifts in the

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process mean. The majority of literature on DS schemes is based on the assumption of known IC process parameters (i.e. Case K), see for example: Costa (1994), Irianto and Shinozaki (1998), He, Grigoryan and Sigh (2002), Carot, Jabaloyes and Carot (2002), Khoo et al. (2011), Lee, Torng and Liao (2012), Khoo et al. (2013a), Costa and Machado (2015), Costa (2017), You (2017), Chong et al. (2018), Haq and Khoo (2018, 2019), Malela-Majika and Rapoo (2019), Malela-Majika (2019), etc. Although there is a lots of research work based on Case K, in practice, the process parameters are generally unknown (i.e. Case U).

The Case U scenario requires monitoring schemes to be applied in a two-phase approach, i.e. Phase I and Phase II (see Jensen et al. (2006) and Psarakis, Vyniou and Castagliola (2013) for a review of parameter estimation effect articles). In Phase I, monitoring schemes are implemented retrospectively in order to estimate the distribution parameters using an IC reference sample. However, in Phase II, monitoring schemes are implemented prospectively to continuously monitor any departures from an IC state using the parameters estimated in Phase I. There have been number of articles that investigated the Case U scenario in the context of DS schemes, see for example, Khoo et al. (2013b), Teoh et al. (2013, 2014, 2015, 2016a, 2016b), Castagliola, Oprime and Khoo (2017), You et al. (2016), You (2018) and, Lee and Khoo (2019). These latter articles studied the DS scheme in Case U for a variety of design criterion and contexts, e.g. optimized DS scheme based on unconditional average run-length (*ARL*), unconditional median run-length, unconditional standard deviation of the run-length (*SDRL*), unconditional expected *ARL* (*EARL*), minimization of the average sample size (*ASS*), average number of observations to signal (*ANOS*), etc.

It is worth noting that all the articles on Case U DS schemes mentioned in the previous paragraph, used the non-side-sensitive design discussed in Daudin (1992). More recently, Malela-Majika, Motsepa and Graham (2019) showed that the basic DS \bar{X} scheme can be divided into two main groups, i.e. non-side-sensitive and side-sensitive designs. Since the abovementioned articles have thoroughly discussed the design and implementation of the non-side-sensitive DS (hereafter NSSDS) \bar{X} scheme; in this paper, we investigate the design and implementation of the side-sensitive DS (hereafter SSDS) \bar{X} scheme using the (5th, 25th, 50th, 75th, 95th) percentiles, *ARL*, *SDRL*, *ASS*, *ANOS*, the average extra quadratic loss (*AEQL*), performance comparison index (*PCI*) and average ratio of the average run-length (*ARARL*) metrics.

The rest of this paper is structured as follows: Section 2 presents the operation, design and run-length properties of the SSDS \bar{X} scheme in Case K and Case U; while, Section 3, presents the

overall performance metrics. The optimization model is given in Section 4. Section 5 assesses the IC, OOC performance of the SSDS \bar{X} scheme and compares their overall performance with the NSSDS \bar{X} scheme and other established monitoring schemes in Case U. In Section 6, a case study is given using real-life data to demonstrate the implementation and design of the Case U SSDS \bar{X} scheme. A synthetic version of the proposed SSDS \bar{X} scheme is briefly discussed in Section 7. Finally, some concluding remarks and recommendations are given in Section 8.

2. The SSDS \bar{X} monitoring scheme

2.1. Design of the SSDS \bar{X} scheme with known process parameters

Charting regions and charting statistics in Stage 1 and Stage 2

Assume that the observations, i.e. Y_{ij} , of the quality characteristic \bar{X} are independently and identically distributed (i.i.d.) from a $N(\mu_0, \sigma_0)$ distribution, where μ_0 and σ_0 represent the IC mean and the IC standard deviation, respectively. The SSDS \bar{X} chart is a two-stage monitoring scheme with mutually exclusive regions as shown in Figure 1: $A = (-L_1, L_1)$, $B^+ = [L_1, L)$, $B^- = (-L, -L_1]$, $C = (-\infty, -L] \cup [L, +\infty)$ in Stage 1, where L_1 and L (with $L \geq L_1 > 0$) are the Stage 1 warning and control limits, respectively; and $D^+ = [L_2, +\infty)$, $D^- = (-\infty, L_2)$, $E^- = (-\infty, -L_2]$, $E^+ = (-L_2, +\infty)$, where L_2 (with $L_2 > 0$) is the Stage 2 control limit.

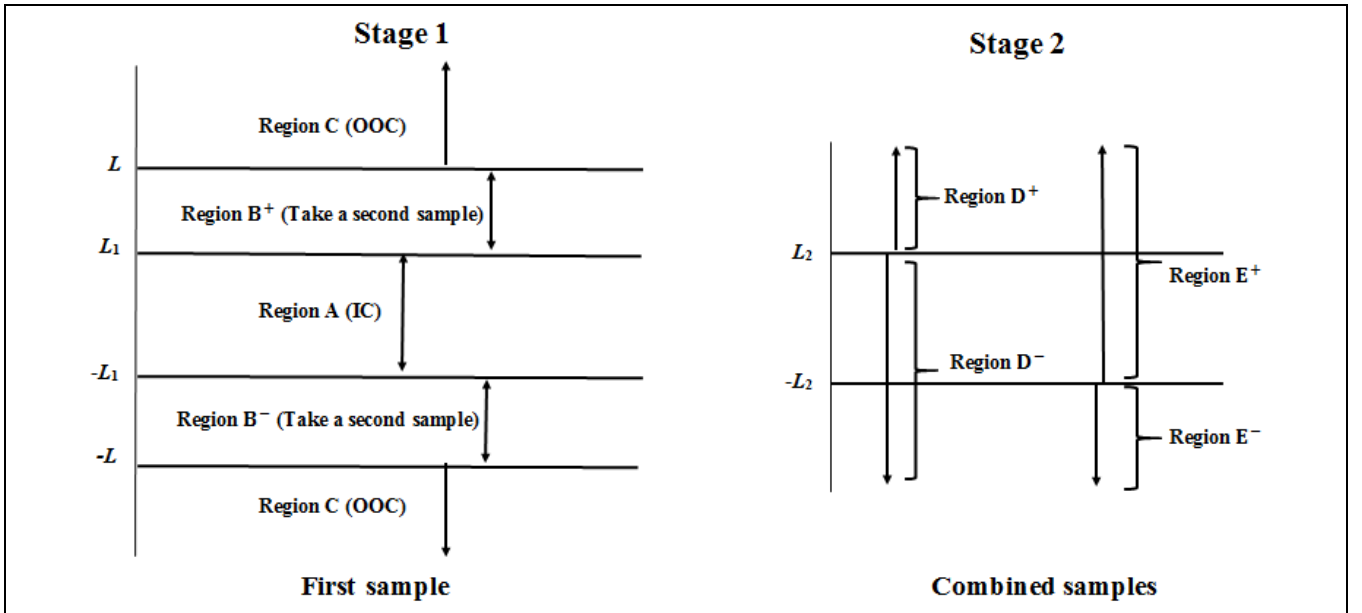


Figure 1. The charting regions of the SSDS \bar{X} scheme for Stage 1 and Stage 2

From the Y_{ij} observations, a first subgroup sample of size n_1 is collected at the i^{th} sampling time (denoted as Y_{1ij} , $i = 1, 2, \dots$, and $j = 1, 2, \dots, n_1$). If the standardized charting statistic

based on the first sample falls on Region B^- or B^+ , then a second subgroup sample of size n_2 (where $n_2 \geq n_1$) is also collected at the i^{th} sampling time (denoted as Y_{2ij} , $i = 1, 2, \dots$, and $j = 1, 2, \dots, n_2$). Then the SSDS \bar{X} scheme uses these two stages to decide whether the process is IC or OOC, and each stage's charting statistic is as follows.

Stage 1: Let $\bar{Y}_{1i} = \sum_{j=1}^{n_1} Y_{1ij}/n_1$ be the mean of the first sample of subgroup size n_1 at the i^{th} sampling time. Hence, in Case K, the standardized statistic for the first sample at the i^{th} sampling time is then given by

$$Z_{1i} = \frac{\bar{Y}_{1i} - \mu_0}{\sigma_0/\sqrt{n_1}} \quad (1)$$

where $\bar{Y}_{1i} \sim N(\mu_0 + \delta\sigma_0, \frac{\sigma_0}{\sqrt{n_1}})$ and $\delta = |\mu_1 - \mu_0|/\sigma_0$ represents the magnitude of the standardized mean shift with the OOC mean μ_1 ($\mu_1 = \mu_0 + \delta\sigma_0$), so that $\delta = 0$ means that the process is IC. In this case, Z_{1i} follows a standard normal distribution (i.e. $Z_{1i} \sim N(0,1)$). However, when $\delta \neq 0$, the process is OOC and $Z_{1i} \sim N(\delta, 1)$.

Stage 2: At the i^{th} sampling time of the second sample, the sample mean, i.e. $\bar{Y}_{2i} = \sum_{j=1}^{n_2} Y_{2ij}/n_2$, and the combined sample mean, i.e. $\bar{Y}_i = (n_1\bar{Y}_{1i} + n_2\bar{Y}_{2i})/(n_1 + n_2)$ are calculated, respectively. Hence, in Case K, the standardized charting statistic for the combined samples at the i^{th} sampling time is then given by

$$Z_i = \frac{\bar{Y}_i - \mu_0}{\sigma_0/\sqrt{n_1 + n_2}} \quad (2)$$

where $\bar{Y}_i \sim N(\mu_0 + \delta\sigma_0, \frac{\sigma_0}{\sqrt{n_1 + n_2}})$. When the process is IC, $Z_i \sim N(0, 1)$ since $\delta = 0$ and when the process is OOC, $Z_i \sim N(\delta, 1)$.

That is, there are two distinct standardized charting statistics (i.e. Z_{1i} and Z_i) used during Stage 1 and Stage 2 (if needed), respectively (see Figure 1).

Run-length properties

Let P_{0k} represents the probability that the process is regarded as IC at stage k , where $k = 1, 2$.

Then, $P_0 = P_{01} + P_{02}$ is the probability that the process is IC, where:

$$P_{01} = P(Z_{1i} \in A) = \Phi[L_1 + \delta\sqrt{n_1}] - \Phi[-L_1 + \delta\sqrt{n_1}], \quad (3)$$

and

$$\begin{aligned} P_{02} &= P[Z_{1i} \in B^+ \text{ and } Z_i \in D^-] + P[Z_{1i} \in B^- \text{ and } Z_i \in E^+] \\ &= \int_{Z_{1i} \in B^{++}} \left\{ \Phi[cL_2 + rc\delta - z\sqrt{n_1/n_2}] \right\} \phi(z) dz \\ &\quad + \int_{Z_{1i} \in B^{--}} \left\{ 1 - \Phi[-cL_2 + rc\delta - z\sqrt{n_1/n_2}] \right\} \phi(z) dz \end{aligned} \quad (4)$$

where $\Phi(\cdot)$ and $\phi(\cdot)$ are the c.d.f. (cumulative distribution function) and p.d.f. (probability density function) of the standard normal random variable, respectively; $r^2 = n_1 + n_2$, $c = r/\sqrt{n_2}$, $B^{++} = (L_1 + \delta\sqrt{n_1}, L + \delta\sqrt{n_1}]$ and $B^{--} = [-L + \delta\sqrt{n_1}, -L_1 + \delta\sqrt{n_1})$. Hence,

$$P_0 = \Phi[L_1 + \delta\sqrt{n_1}] - \Phi[-L_1 + \delta\sqrt{n_1}] + \int_{Z_{1i} \in B^{++}} \left\{ \Phi[cL_2 + rc\delta - z\sqrt{n_1/n_2}] \right\} \phi(z) dz \\ + \int_{Z_{1i} \in B^{--}} \left\{ 1 - \Phi[-cL_2 + rc\delta - z\sqrt{n_1/n_2}] \right\} \phi(z) dz. \quad (5)$$

Given that the SSDS \bar{X} scheme is a Shewhart-type one, its run-length (RL) follows a geometric distribution. Therefore, the c.d.f. of the RL distribution (denoted $F_{RL}(\ell)$) is obtained as

$$F_{RL}(\ell) = P(RL \leq \ell) = 1 - P_0^\ell \quad (6)$$

where $\ell \in \{1, 2, 3, \dots\}$. Then, the $(100\rho)^{th}$ percentile of the RL distribution, ℓ_ρ , is given by

$$P(RL \leq \ell_\rho - 1) \leq \rho \text{ and } P(RL \leq \ell_\rho) > \rho. \quad (7)$$

It follows that, the ARL , $SDRL$ and ASS at each sampling time are given by

$$ARL = \frac{1}{1 - P_0}, \quad (8)$$

$$SDRL = \frac{\sqrt{P_0}}{1 - P_0}, \quad (9)$$

$$ASS = n_1 + n_2 P_2, \quad (10)$$

respectively, where $P_2 = P(Z_{1i} \in B^- \cup B^+)$ is the probability of taking the second sample, and it is given by

$$P_2 = \Phi(L + \delta\sqrt{n_1}) - \Phi(L_1 + \delta\sqrt{n_1}) + \Phi(-L_1 + \delta\sqrt{n_1}) - \Phi(-L + \delta\sqrt{n_1}).$$

2.2. Design of the SSDS \bar{X} control chart with estimated process parameters

2.2.1 Phase I and Phase II operation of the SSDS \bar{X} monitoring scheme

Phase I parameter estimation

Since the IC process parameters, μ_0 and σ_0 , are usually unknown they have to be estimated from m Phase I subgroup samples, each of size n , i.e. $\{X_{ij}\}_{j=1,2,\dots,n}^{i=1,2,\dots,m}$. The estimated IC process parameters, $\hat{\mu}_0$ and $\hat{\sigma}_0$, are given by

$$\hat{\mu}_0 = \frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n X_{ij} \quad (11)$$

and

$$\hat{\sigma}_0 = \sqrt{\frac{1}{m(n-1)} \sum_{i=1}^m \sum_{j=1}^n (X_{ij} - \bar{X}_i)^2}, \quad (12)$$

respectively, where $\bar{X}_i = \sum_{j=1}^n X_{ij}/n$.

Phase II charting statistics and operation procedure: Stage 1 and Stage 2

Let Y_{ij} be the Phase II observations from i.i.d. $N(\mu_1, \sigma_0)$, where μ_1 is the OOC mean (i.e. $\mu_1 = \mu_0 + \delta\sigma_0$) with $\delta = |\mu_1 - \mu_0|/\sigma_0$ the magnitude of the standardized mean shift from μ_0 to μ_1 . In Phase II of the SSDS \bar{X} monitoring scheme, there are two distinct standardized charting statistics in Case U (i.e. \hat{Z}_{1i} and \hat{Z}_i , shown below) used during Stage 1 and Stage 2, respectively (see Figure 1).

Stage 1: Similarly as in Case K, $\bar{Y}_{1i} = \sum_{j=1}^{n_1} Y_{1ij}/n_1$; however, the standardized statistic for the first sample at the i^{th} sampling time in Case U is given by

$$\hat{Z}_{1i} = \frac{\bar{Y}_{1i} - \hat{\mu}_0}{\hat{\sigma}_0/\sqrt{n_1}}. \quad (13)$$

Stage 2: Similarly, at the i^{th} sampling time of the second sample, the sample mean is given by $\bar{Y}_{2i} = \sum_{j=1}^{n_2} Y_{2ij}/n_2$, so that the combined sample mean is given by $\bar{Y}_i = (n_1\bar{Y}_{1i} + n_2\bar{Y}_{2i})/(n_1 + n_2)$. Thus, the standardized charting statistic in Case U at the i^{th} sampling time is given by

$$\hat{Z}_i = \frac{\bar{Y}_i - \hat{\mu}_0}{\hat{\sigma}_0/\sqrt{n_1 + n_2}} \quad (14)$$

Thus, based on the description above, the operational procedure of the Case U SSDS \bar{X} scheme is given as follows:

1. From the IC retrospective data with m samples, estimate the IC mean and standard deviation of the process using Equations (11) and (12), respectively.
2. In the prospective phase, take a sample of size n_1 and calculate the sample mean \bar{Y}_{1i} at the i^{th} sampling time at Stage 1.
3. If $\hat{Z}_{1i} \in A$, the process is considered as IC.
4. If $\hat{Z}_{1i} \in C$, the process is said to be OOC and then the necessary corrective action must be taken to find and remove the assignable causes.
5. If $\hat{Z}_{1i} \in B^- \cup B^+$, take a second sample of size n_2 ($n_2 \geq n_1$) and calculate the sample mean \bar{Y}_{2i} at the i^{th} sampling time of the second sample.
6. At the i^{th} sampling time, calculate the combined sample mean \bar{Y}_i and then \hat{Z}_i .
7. The process is declared IC if:

- (a) If $\hat{Z}_{1i} \in B^+$ and $\hat{Z}_i \in D^-$, or
- (b) If $\hat{Z}_{1i} \in B^-$ and $\hat{Z}_i \in E^+$

However, the process is declared OOC:

- (c) If $\hat{Z}_{1i} \in B^+$ and $\hat{Z}_i \in D^+$, or
- (d) If $\hat{Z}_{1i} \in B^-$ and $\hat{Z}_i \in E^-$.

In essence, if the plotting statistic falls in region B^+ (region B^-) in Stage 1, then it can only fall in regions D^- or D^+ (regions E^+ or E^-) only, in Stage 2, respectively. Conversely, if in Stage 1, $\hat{Z}_{1i} \in B^+$, then in Stage 2, we have $\hat{Z}_i \notin \{E^+, E^-\}$. Similarly, if in Stage 1, $\hat{Z}_{1i} \in B^-$, then in Stage 2, we have $\hat{Z}_i \notin \{D^-, D^+\}$. The flow chart illustrating the steps involved in the operation of the Case U SSDS \bar{X} monitoring scheme is shown in Figure 2.

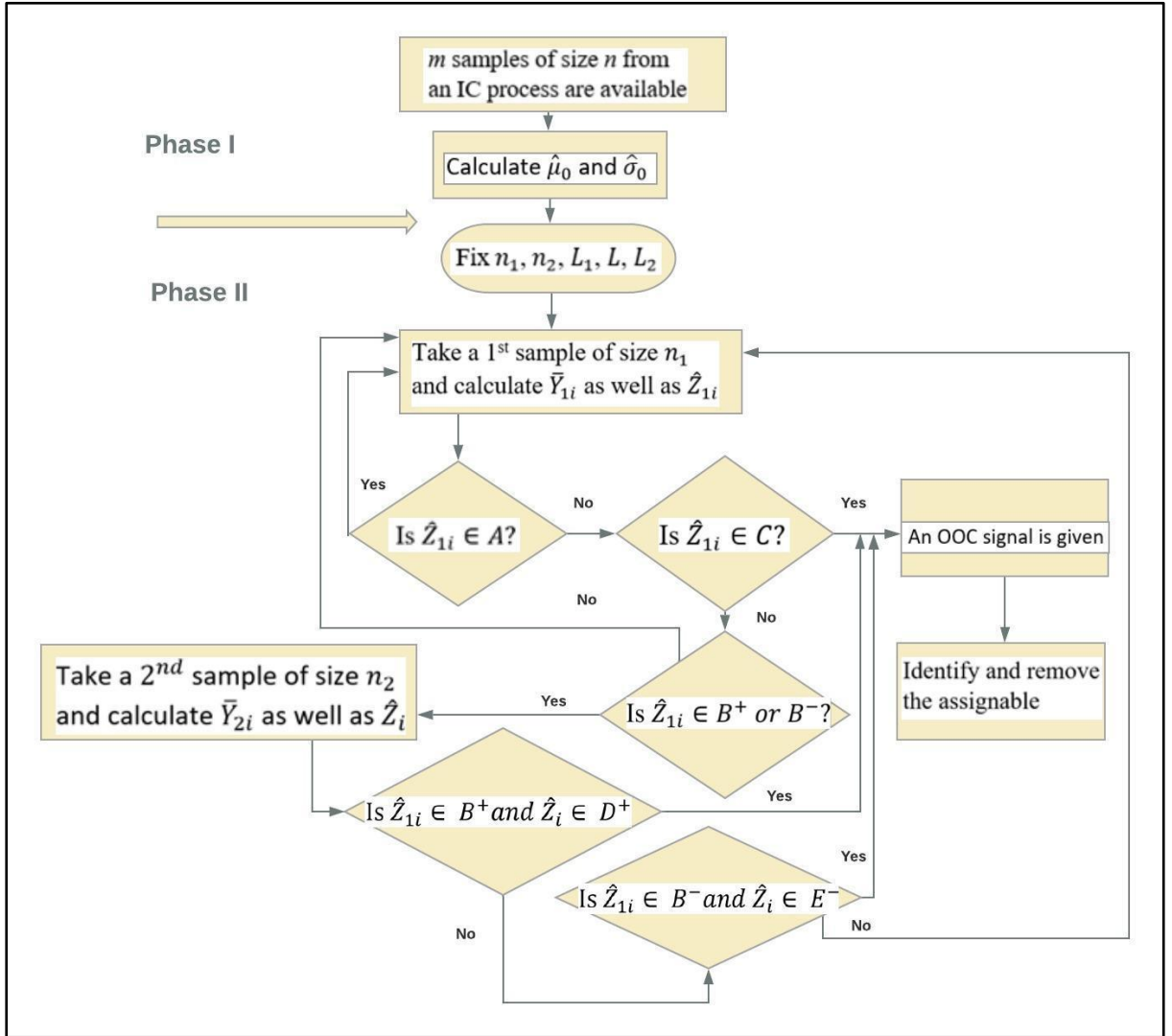


Figure 2. Flow chart for the proposed SSDS \bar{X} monitoring scheme

2.2.2 Unconditional run-length properties of the SSDS \bar{X} scheme

In order to calculate the unconditional RL properties, we need to first derive the conditional ones, see Jensen et al. (2006). Hence, the conditional c.d.f. of \hat{Z}_{1i} , given $\hat{\mu}_0$ and $\hat{\sigma}_0$ is defined as

$$F_{\hat{Z}_{1i}}(z|\hat{\mu}_0, \hat{\sigma}_0) = \Phi\left(U\sqrt{\frac{n_1}{mn}} + Vz - \delta\sqrt{n_1}\right). \quad (15)$$

where $U = (\hat{\mu}_0 - \mu_0)\frac{\sqrt{mn}}{\sigma_0}$ and $V = \hat{\sigma}_0/\sigma_0$. Consequently, the conditional p.d.f. of \hat{Z}_{1i} , given $\hat{\mu}_0$ and $\hat{\sigma}_0$ is given by

$$f_{\hat{Z}_{1i}}(z|\hat{\mu}_0, \hat{\sigma}_0) = V\phi\left(U\sqrt{\frac{n_1}{mn}} + Vz - \delta\sqrt{n_1}\right). \quad (16)$$

Since $\hat{\mu}_0 \sim N(\mu_0, \frac{\sigma_0^2}{mn})$, then $U \sim N(0, 1)$ so that the p.d.f. of the random variable U is simply,

$$f_U(u) = \phi(u). \quad (17)$$

Zhang et al. (2011) used the fact that $V^2 = (\hat{\sigma}_0/\sigma_0)^2$ has a gamma distribution with parameters $m(n-1)/2$ and $2/[m(n-1)]$ to show that the p.d.f. of V is defined as

$$f_v(v|m, n) = 2vf_\gamma\left[v^2\left|\frac{m(n-1)}{2}, \frac{2}{m(n-1)}\right|\right], \quad (18)$$

where $f_\gamma(\cdot)$ is the p.d.f. of the gamma distribution with parameters $\frac{m(n-1)}{2}$ and $\frac{2}{m(n-1)}$. For more details on the conditional RL properties for DS schemes, readers are referred to You et al. (2016).

Next, to derive the unconditional c.d.f. of the RL of the proposed monitoring scheme, we need to first derive the unconditional probability of the IC process. Let \hat{P}_{0k} denote the probability that the process with estimated parameters remains IC at the sampling stage k (with $k = \{1, 2\}$), i.e. a Case U extension of Equation (5). Then, the probability that the process is IC is given by

$$\hat{P}_0 = \hat{P}_{01} + \hat{P}_{02} \quad (19)$$

where,

$$\begin{aligned} \hat{P}_{01} &= \Phi\left(U\sqrt{\frac{n_1}{mn}} + VL_1 - \delta\sqrt{n_1}\right) - \Phi\left(U\sqrt{\frac{n_1}{mn}} - VL_1 - \delta\sqrt{n_1}\right) \\ \hat{P}_{02} &= \int_{Z \in B^{++}} \hat{P}_D^- f_{\hat{Z}_{1i}}(z|\hat{\mu}_0, \hat{\sigma}_0) dz + \int_{Z \in B^{--}} \hat{P}_E^+ f_{\hat{Z}_{1i}}(z|\hat{\mu}_0, \hat{\sigma}_0) dz \end{aligned} \quad (20)$$

with

$$\hat{P}_D^- = \Phi\left[U\sqrt{\frac{n_2}{mn}} + V\left(\frac{L_2\sqrt{n_1+n_2} - z\sqrt{n_1}}{\sqrt{n_2}}\right) - \delta\sqrt{n_2}\right]$$

and

$$\hat{P}_{E^+} = 1 - \Phi \left[U \sqrt{\frac{n_2}{mn}} - V \left(\frac{L_2 \sqrt{n_1 + n_2} - z \sqrt{n_1}}{\sqrt{n_2}} \right) - \delta \sqrt{n_2} \right].$$

Then, the unconditional c.d.f. of the SSDS \bar{X} monitoring scheme for Case U is given by

$$F_{RL}(\ell) = \int_{-\infty}^{+\infty} \int_0^{+\infty} (1 - \hat{P}_0^\ell) f_U(u) f_V(v) dv du, \quad (21)$$

where $\ell \in \{1, 2, 3, \dots\}$, $f_U(u)$ and $f_V(v)$ are defined in Equations (17) and (18), respectively.

Therefore, the unconditional *ARL* and *SDRL* of the proposed SSDS \bar{X} monitoring scheme with estimated process parameters are given by

$$ARL = \int_{-\infty}^{+\infty} \int_0^{+\infty} \left(\frac{1}{1 - \hat{P}_0} \right) f_U(u) f_V(v) dv du \quad (22)$$

and

$$SDRL = \left[\int_{-\infty}^{+\infty} \int_0^{+\infty} \left(\frac{1 + \hat{P}_0}{1 - \hat{P}_0} \right) f_U(u) f_V(v) dv du - ARL^2 \right]^{1/2}. \quad (23)$$

The Case U ASS is given by

$$ASS = \int_{-\infty}^{+\infty} \int_0^{+\infty} (n_1 + n_2 \hat{P}_2) f_U(u) f_V(v) dv du \quad (24)$$

where \hat{P}_2 is the probability of taking the second sample, which is given by $\hat{P}_2 = P(\hat{Z}_{1i} \in B^- \cup B^+ | \hat{\mu}_0, \hat{\sigma}_0)$, or simply,

$$\begin{aligned} \hat{P}_2 = & \Phi \left(U \sqrt{\frac{n_1}{mn}} + VL - \delta \sqrt{n_1} \right) - \Phi \left(U \sqrt{\frac{n_1}{mn}} + VL_1 - \delta \sqrt{n_1} \right) \\ & + \Phi \left(U \sqrt{\frac{n_1}{mn}} - VL_1 - \delta \sqrt{n_1} \right) - \Phi \left(U \sqrt{\frac{n_1}{mn}} - VL - \delta \sqrt{n_1} \right). \end{aligned} \quad (25)$$

Then, the *ANOS* is given by

$$ANOS = \int_{-\infty}^{+\infty} \int_0^{+\infty} (n_1 + n_2 \hat{P}_2) \left(\frac{1}{1 - \hat{P}_0} \right) f_U(u) f_V(v) dv du. \quad (26)$$

Since the *ANOS* depends on the *ASS* and *ARL* values, a larger *ANOS* value implies that either the monitoring scheme is inefficient and/or the cost of inspection is higher.

3. Measures of the overall performance

The *ARL* (see Equation (22)) is defined as the average number of samples required before an OOC signal is issued in the process. It is well-known that the *RL* distribution of a monitoring scheme is generally highly right-skewed in Case U; see for example Jones, Champ and Rigdon (2004). As a result, many researchers prefer to use more meaningful performance measures

(such as the percentiles of the *RL* which includes the median run-length (*MRL*)) to better evaluate the performance of the schemes. Furthermore, the *ARL* has been widely criticized by many authors, see for example, Wu et al. (2008) and Machado and Costa (2014). The *ARL* was simply criticized because of its ineffectiveness in assessing the overall performance, especially when the aim of the study is to assess the performance of a monitoring scheme over a range of shifts. Several authors have revealed that if a monitoring scheme is designed based on one specific size of a mean shift, it will perform poorly when the actual size of the shift is significantly different from the assumed size (see Reynolds and Lou (2010), Ryu, Wan and Kim (2010), Machado and Costa (2014) and Shongwe, Malela-Majika and Rapoo (2019)). Therefore, many researchers have recommended the use of a quality loss function (QLF) instead of the *ARL* to assess the performance of a monitoring scheme. A QLF describes the relationship between the shift size and the quality impact. The average extra quadratic loss (*AEQL*) is an alternative measure of the *ARL* used to assess the overall performance of a monitoring scheme for a range of shifts. Therefore, when the aim of a study is to measure the overall performance of a scheme over a range of shifts (say, $0 \leq \delta \leq 2.5$), the objective function can be defined in terms of the *AEQL* given by

$$AEQL = \int_{\delta_{min}}^{\delta_{max}} \int_{-\infty}^{+\infty} \int_0^{+\infty} W(\delta) \left(\frac{1}{1 - \hat{P}_0} \right) f(\delta) f_U(u) f_V(v) dv du d\delta \quad (27)$$

where δ_{min} and δ_{max} are the lower and upper boundary of the range of shifts under consideration and $W(\delta)$ (with $W(\delta) = \delta^2$) represents the weight function associated with δ . Since it is generally assumed that all location shifts (mean shifts) occur with equal probability; hence, a uniform distribution of δ is implied, i.e. $f(\delta) = 1/(\delta_{max} - \delta_{min})$.

In order to measure the relative effectiveness of two different schemes, Wu et al. (2008) suggested the use of the *PCI*, which is the ratio between the *AEQL* of a competing monitoring scheme and the *AEQL* of the benchmark scheme under the same settings. In this paper, the proposed scheme is used as the benchmark. The *PCI* is then defined by

$$PCI = \frac{AEQL}{AEQL^*} \quad (28)$$

where $AEQL^*$ is the *AEQL* of the benchmark scheme. In addition to the *AEQL* and the *PCI*, many authors also suggested the use of the *ARARL* to measure the overall performance of a benchmark scheme against other competing schemes; see Wu et al. (2008). The *ARARL* is given by

$$ARARL = \frac{1}{\delta_{max} - \delta_{min}} \sum_{\delta=\delta_{min}}^{\delta_{max}} \frac{ARL(\delta)}{ARL^*(\delta)} \quad (29)$$

where ARL^* is the ARL of the benchmark scheme. Note that, if the PCI and/or $ARARL$ is larger than one, the competing scheme will produce larger ARL s over the range of shifts under consideration, which means that the benchmark scheme outperforms the competing scheme for that particular range; otherwise, if PCI is less than one, then the competing scheme is more sensitive than the benchmark scheme.

4. Bi-objective model of the proposed SSDS \bar{X} monitoring scheme

There are three control limits L_1, L and L_2 and two sample sizes n_1 and n_2 that need to be specified for a specific ASS_0 in order to design the SSDS \bar{X} monitoring scheme. The efficiency of the proposed SSDS \bar{X} scheme depends on the combination $(m, n_1, n_2, L_1, L, L_2)$. There are two main steps in the optimal design of the proposed scheme: Firstly, the nominal IC ARL (ARL_0) is set to a high desired value, such as 370.4 or 500; secondly, the combination that yields an ARL_0 as close as possible to the nominal ARL_0 value and the smallest OOC ARL (ARL_δ) for a given mean shift δ and a minimum $AEQL$ value is considered to be the optimal combination. Therefore, the optimization model is presented as follows:

$$(L_1^*, L^*, L_2^*) = \underset{m, n_1, n_2, L_1, L, L_2}{\operatorname{argmin}} (ARL_1, AEQL) \quad (30)$$

subjects to

$$ARL_0 = \tau \quad (31)$$

and

$$ASS_0 = \xi, \quad (32)$$

where ASS_0 represents the expected IC ASS , ξ is the prespecified ASS_0 value and τ represents the nominal ARL_0 value. Note that the ASS_0 and OOC ASS (ASS_δ) are used because the sample size is not fixed in advance (it can be n_1 or $n_1 + n_2$). This plays an important role in the estimation of the cost of inspection.

The search of the optimal parameters can be summarized in three main steps given as follows:

1. Fix m and for some specific sample sizes (i.e., n_1 and n_2) and mean shift ($\delta = 0$), find all possible combinations of the design parameters that yield an attained ARL_0 value of 370.4 for a prespecified value of the ASS_0 . These combination of parameters (L_1, L, L_2) are called local design parameters;

2. For each combination of the local design parameters, compute the ARL_δ (where $\delta = 0.1$ to 2.5 with a step shift of 0.1) and then calculate the corresponding $AEQL$ value;
3. Select the combination that yields the minimum $AEQL$ value to be the combination of the optimal design parameters (L_1^*, L^*, L_2^*) .

5. Performance study

5.1 Performance analysis of the SSDS \bar{X} monitoring scheme

In this section, the performance of the SSDS \bar{X} monitoring scheme is investigated in Case U by setting the nominal ARL_0 value to 370.4 with a maximum mean shift of 2.5 (i.e. $\delta_{max} = 2.5$) and ASS_0 values of 5 and 8, see Tables 1 to 4; where, for illustration purpose, m is set at 50 and 100 for Case U SSDS \bar{X} scheme, and m is assumed to approach infinity (∞) for Case K. The first row of each cell gives the ARL , $SDRL$, ASS and $ANOS$ values and the second row gives the 5th, 25th, 50th, 75th and 95th percentiles (denoted by (P5, P25, P50, P75, P95)) of the RL distribution of the Case U SSDS \bar{X} scheme. Note that the Case K properties discussed in Section 2.1 are given in the last column. Equations (21) to (23) are used to compute the IC and OOC characteristics of the RL distribution. Moreover, the ASS , $ANOS$ and $AEQL$ values are computed using Equations (24), (26) and (27), respectively.

For instance (see the second column of Table 1), for a Phase I sample of size 50 (i.e. $m = 50$), when $(n_1, n_2) = (5, 5)$, $(\delta_{min}, \delta_{max}) = (0, 2.5)$ and $ASS_0 = 5$, it is found (using the optimization model in Equations (30) to (32)) that $(L_1^*, L^*, L_2^*) = (2.9093, 3.0111, 2.9309)$ so that the proposed SSDS \bar{X} scheme satisfies $ARL_0 = 370.4$ with a minimum $AEQL = 70.72$. However, when n_2 is increased to 8, for the same values of m , n_1 , δ_{min} , δ_{max} , and ASS_0 , it is found that $(L_1^*, L^*, L_2^*) = (2.9101, 3.0108, 2.6310)$ so that the proposed scheme also satisfies $ARL_0 = 370.4$ with a minimum $AEQL = 69.06$ (see the third column of Table 1).

From Table 1, it can be seen that when $m = 50$ and $ASS_0 = 5$, if $(n_1, n_2) = (5, 5)$ there is 5% chance that the Case U SSDS \bar{X} scheme gives a signal for the first time on the 18th subgroup and a 95% chance that it signals on the 1102 subgroup in Phase II when the process is IC (shift = 0). For a small shift of size 0.3, there are 5% and 95% chances that the proposed scheme gives a signal on the 11th and 563th subgroups, respectively. For $m = 100$ with an ASS_0 of 5, when $(n_1, n_2) = (5, 5)$ (i.e. fourth column of Table 1) and with a mean shift of 0.3, there are 5% and 95% chances that the Case U SSDS \bar{X} scheme gives a signal on the 7th and 388th subgroups, respectively. For a moderate mean shift of 0.9, there are 5% and 95% chances that the Case U SSDS \bar{X} scheme signals on the first and eighteenth subgroups, respectively. These findings

confirm that the larger the Phase I sample size, the more sensitive the proposed scheme. As the mean shift increases, the proposed scheme becomes more sensitive. When we keep $n_1 = 5$ and increase n_2 (say $n_2 = 8$) for $m = 50$, for a mean shift of 0.3, there is 95% chance that the proposed scheme gives a signal on the 546th subgroup in the prospective phase. This reveals an improvement in the sensitivity of the proposed scheme when the Stage 2 sample size increases. In Case K, when $\delta = 0.3$, $(n_1, n_2) = (5, 8)$ and $(3, 10)$ with $ASS_0 = 5$, there is 5% chance that the SSDS \bar{X} scheme gives a signal on the fourth sample, see the last column of Tables 1 and 2, respectively. However, there is 95% chance that the SSDS \bar{X} scheme gives a signal on the 260th and 183rd sample, respectively. This shows that the proposed scheme performs better in Case K. From Tables 3 and 4, it can be seen that when the expected sample size (i.e. ASS_0) increases, the sensitivity of the proposed scheme increases as well. For small Phase I sample sizes (i.e. $m = 25$, on the second column of Tables 3 and 4), the detection ability of the proposed scheme is poor as compared to $m = 50$ and 100 on columns 3 to 6, respectively.

In terms of the ARL values, for small and moderate shifts in the process mean, the larger the Phase I sample (i.e. m), the more sensitive the Case U SSDS \bar{X} scheme. However, for large shifts in the process mean, the proposed scheme performs uniformly better regardless of the Phase I sample size. For small and moderate shifts, the proposed scheme is less sensitive in Case U than in Case K. This under-performance is due to the effect of estimation that deteriorates the performance of a monitoring scheme.

In terms of the $SDRL$ values, it can be seen that the practitioner-to-practitioner variability in the performance of the proposed scheme decreases as the Phase I sample size increases. The OOC $SDRL$ ($SDRL_\delta$) drop rapidly as the Phase I sample size increases. Therefore, the larger the Phase I sample size, the more reliable the results. The larger the ASS_0 , the smaller the variability in the performance outputs. In terms of the $ANOS$ values, the larger the Phase I sample, the smaller the OOC $ANOS$. For very small shifts (i.e. $0 < \delta < 0.2$), the smaller the ASS_0 value, the smaller the $ANOS$ value. When $\delta \geq 0.2$, the larger the ASS_0 value, the smaller the $ANOS$ value, which means that when the process is OOC, the proposed scheme is more efficient and cost effective for larger values of ASS_0 .

In terms of the overall performance, i.e. $AEQL$, the proposed SSDS \bar{X} scheme performs better for large Phase I sample sizes and/or large expected number of samples. As the Phase I sample size increases, the Case U properties converge towards to the Case K properties. Therefore, it is very important to study the effect of the Phase I sample size on the performance of the

proposed scheme in order to know the amount of Phase I observations required to reach the Case K performance. This topic is under investigation and will be reported in a separate article.

Table 1. The exact *ARL*, *SDRL*, *ASS*, *ANOS* (first row), Percentiles (second row), *AEQL* and optimal design parameters of the proposed scheme when $m \in \{50, 100\}$ and $m = \infty$ (i.e. Case K), $(n_1, n_2) \in \{(5, 5); (5, 8)\}$, $ASS_0 = 5$ and $\delta_{max} = 2.5$ for a nominal ARL_0 value of 370.4

<i>(ARL, SDRL, ASS, ANOS)</i> <i>(P5, P25, P50, P75, P95)</i>					
Type of design	Case U				Case K
Shift(δ)	$m = 50$		$m = 100$		$m = \infty$
0.00	(370.4, 369.98, 5.00, 1847.56) (18, 107, 254, 510, 1102)	(370.4, 371.36, 5.00, 1853.22)	(370.4, 375.44, 5.01, 1862.09) (19, 103, 256, 518, 1122)	(370.4, 373.22, 5.02, 1858.68) (19, 107, 254, 513, 1121)	(370.4, 367.15, 5.02, 1859.37) (20, 108, 257, 515, 1121)
0.10	(347.63, 340.25, 5.00, 1740.89) (24, 132, 314, 620, 1337)	(340.83, 345.39, 5.01, 1608.47)	(279.97, 278.84, 5.01, 1504.20) (20, 109, 257, 522, 1139)	(273.23, 272.54, 5.02, 1472.99) (20, 109, 258, 516, 1124)	(292.46, 288.87, 5.02, 1467.67) (16, 89, 205, 405, 870)
0.20	(333.23, 329.98, 5.00, 1669.22) (17, 96, 233, 457, 998)	(321.71, 322.17, 5.01, 1613.34)	(243.47, 242.17, 5.08, 1221.64) (12, 73, 170, 334, 723)	(241.81, 241.39, 5.03, 1215.90) (13, 71, 169, 333, 714)	(159.38, 159.45, 5.03, 801.43) (8, 46, 110, 223, 474)
0.30	(187.70, 189.80, 5.02, 941.31) (11, 55, 129, 260, 563)	(182.85, 182.73, 5.02, 918.68)	(131.10, 131.65, 5.03, 659.31) (7, 38, 91, 182, 388)	(123.83, 124.38, 5.05, 624.92) (7, 35, 84, 172, 375)	(86.17, 86.73, 5.05, 434.89) (4, 25, 59, 121, 260)
0.40	(102.65, 100.97, 5.02, 515) (6, 30, 72, 143, 307)	(97.23, 97.40, 5.03, 489.93) (5, 28, 68, 133, 292)	(69.75, 69.28, 5.05, 352.05) (4, 20, 48, 97, 208)	(66.91, 66.79, 5.08, 339.64) (4, 20, 46, 93, 202)	(47.67, 47.36, 5.08, 241.97) (3, 14, 33, 66, 143)
0.50	(56.26, 56.21, 5.03, 283.38) (3, 16, 39, 77, 170)	(54.73, 54.16, 5.06, 276.92) (3, 16, 38, 75, 165)	(39.00, 38.92, 5.07, 197.91) (3, 12, 27, 52, 117)	(36.99, 36.08, 5.11, 189.36) (2, 11, 26, 51, 109)	(28.02, 27.18, 5.12, 143.40) (2, 9, 20, 39, 82)
0.60	(32.50, 31.65, 5.05, 164.27) (2, 10, 23, 45, 97)	(30.88, 30.61, 5.09, 157.11) (2, 9, 21, 43, 93)	(22.99, 22.69, 5.11, 117.47) (2, 7, 16, 31, 69)	(21.98, 21.46, 5.18, 113.80) (2, 7, 15, 30, 64)	(17.30, 16.94, 5.18, 89.54) (1, 5, 12, 24, 52)
0.70	(19.63, 19.02, 5.07, 99.65) (1, 6, 14, 27, 58)	(19.16, 18.80, 5.12, 98.15) (1, 6, 13, 27, 55)	(14.20, 13.41, 5.16, 73.21) (1, 5, 10, 19, 41)	(13.46, 13.03, 5.25, 70.66) (1, 4, 9, 18, 40)	(11.26, 10.62, 5.25, 59.13) (1, 4, 8, 15, 33)
0.80	(12.62, 12.11, 5.10, 64.38) (1, 4, 9, 17, 36)	(12.18, 11.60, 5.16, 62.91) (1, 4, 9, 17, 35)	(9.15, 8.52, 5.21, 47.68) (1, 3, 7, 12, 26)	(8.59, 7.96, 5.34, 45.85) (1, 3, 6, 12, 24)	(7.52, 7.05, 5.34, 40.14) (1, 3, 5, 10, 22)
0.90	(8.34, 7.76, 5.13, 42.83) (1, 3, 6, 11, 24)	(7.94, 7.42, 5.21, 41.34) (1, 3, 6, 11, 23)	(6.31, 5.78, 5.27, 33.28) (1, 2, 4, 9, 18)	(6.03, 5.55, 5.43, 32.76) (1, 2, 4, 8, 17)	(5.31, 4.85, 5.43, 28.84) (1, 2, 4, 7, 15)
1.00	(5.83, 5.38, 5.15, 30.08) (1, 2, 4, 8, 16)	(5.66, 5.13, 5.25, 29.73) (1, 2, 4, 8, 16)	(4.46, 3.93, 5.42, 14.24) (1, 2, 3, 6, 13)	(4.28, 3.81, 5.53, 23.69) (1, 2, 3, 6, 12)	(4.03, 3.48, 5.53, 22.30) (1, 2, 3, 5, 11)
1.20	(3.22, 2.72, 5.20, 16.72) (1, 1, 2, 4, 9)	(3.21, 2.62, 5.31, 17.03) (1, 1, 2, 4, 8)	(2.62, 2.07, 5.45, 9.64) (1, 1, 2, 3, 7)	(2.56, 1.98, 5.68, 14.56) (1, 1, 2, 3, 6)	(2.40, 1.81, 5.68, 13.66) (1, 1, 2, 3, 6)
1.40	(2.04, 1.46, 5.20, 10.61) (1, 1, 2, 3, 5)	(2.04, 1.44, 5.32, 10.87) (1, 1, 2, 3, 5)	(1.77, 1.17, 5.39, 9.64) (1, 1, 1, 2, 4)	(1.78, 1.18, 5.72, 10.19) (1, 1, 1, 2, 4)	(1.71, 1.08, 5.71, 9.77) (1, 1, 1, 2, 4)
1.60	(1.50, 0.88, 5.17, 7.77) (1, 1, 1, 2, 3)	(1.51, 0.88, 5.27, 7.95) (1, 1, 1, 2, 3)	(1.39, 0.73, 5.39, 7.47) (1, 1, 1, 2, 3)	(1.37, 0.71, 5.62, 7.68) (1, 1, 1, 2, 3)	(1.34, 0.68, 5.62, 7.51) (1, 1, 1, 1, 3)
1.80	(1.25, 0.55, 5.12, 6.37) (1, 1, 1, 1, 2)	(1.24, 0.54, 5.18, 6.42) (1, 1, 1, 1, 2)	(1.17, 0.45, 5.27, 6.18) (1, 1, 1, 1, 2)	(1.17, 0.45, 5.44, 6.39) (1, 1, 1, 1, 2)	(1.16, 0.43, 5.44, 6.32) (1, 1, 1, 1, 2)
2.00	(1.11, 0.34, 5.06, 5.60) (1, 1, 1, 1, 2)	(1.11, 0.35, 5.10, 5.65) (1, 1, 1, 1, 2)	(1.07, 0.28, 5.16, 5.53) (1, 1, 1, 1, 2)	(1.07, 0.27, 5.61, 5.25) (1, 1, 1, 1, 2)	(1.06, 0.26, 5.25, 5.58) (1, 1, 1, 1, 2)
2.50	(1.01, 0.08, 5.01, 5.04) (1, 1, 1, 1, 1)	(1.01, 0.08, 5.01, 5.05) (1, 1, 1, 1, 1)	(1.00, 0.07, 5.02, 5.04) (1, 1, 1, 1, 1)	(1.00, 0.06, 5.03, 5.04) (1, 1, 1, 1, 1)	(1.00, 0.05, 5.03, 5.04) (1, 1, 1, 1, 1)
<i>AEQL</i>	70.72	69.06	59.59	55.28	49.91
(n_1, n_2)	(5, 5)	(5, 8)	(5, 5)	(5, 8)	(5, 8)
(L_1, L_2)	(2.9093, 3.0111) 2.9309	(2.9101, 3.0108) 2.6310	(2.9096, 3.1354) 2.9103	(2.9098, 3.1361) 2.7101	(2.9099, 3.1354) 1.9002

Table 2. The exact *ARL*, *SDRL*, *ASS*, *ANOS* (first row), Percentiles (second row), *AEQL* and optimal design parameters of the proposed chart when $m \in \{50, 100\}$ and $m = \infty$ (i.e. Case K), $(n_1, n_2) \in \{(3, 5); (3, 10)\}$, $ASS_0 = 5$ and $\delta_{max} = 2.5$ for a nominal ARL_0 value of 370.4

(ARL, SDRL, ASS, ANOS) (P5, P25, P50, P75, P95)					
Type of design	Case U				Case K
Shift(δ)	$m = 50$		$m = 100$		$m = \infty$
0.00	(370.4, 369.67, 5.11, 1898.25) (20, 108, 255, 513, 1103)	(370.4, 372.08, 4.95, 1841.79) (20, 105, 258, 517, 1115)	(370.4, 370.85, 5.08, 1874.94) (20, 108, 258, 500, 1112)	(370.4, 373.06, 5.03, 1850.61) (18, 104, 253, 507, 1115)	(370.4, 369.43, 5.03, 1873.42) (19, 107, 259, 518, 1107)
0.10	(349.72, 344.22, 5.14, 1512.51) (23, 133, 312, 618, 1342)	(357.29, 354.83, 5.06, 1693) (26, 136, 320, 628, 1362)	(280.74, 279.53, 5.12, 1450.46) (19, 109, 263, 533, 1128)	(267.35, 268.21, 5.09, 1370.19) (20, 106, 255, 515, 1102)	(267.48, 262.02, 5.09, 1361.74) (14, 75, 190, 374, 791)
0.20	(329.95, 328.82, 5.23, 1728.44) (17, 93, 226, 460, 1000)	(317.14, 318.04, 5.20, 1650.42) (16, 91, 219, 438, 943)	(222.82, 222.43, 5.22, 1163.72) (12, 65, 154, 309, 662)	(235.37, 234.30, 5.28, 1243.25) (13, 70, 164, 325, 699)	(132.85, 130.81, 5.28, 701.74) (7, 39, 93, 184, 397)
0.30	(184.41, 188.05, 5.39, 994.05) (11, 52, 125, 252, 566)	(154.15, 153.63, 5.51, 848.96) (9, 44, 107, 213, 472)	(110.50, 109.24, 5.38, 594.59) (6, 33, 79, 153, 325)	(122.11, 120.98, 5.59, 682.53) (7, 35, 85, 170, 370)	(61.16, 60.69, 5.59, 341.88) (4, 18, 42, 86, 183)
0.40	(96.62, 95.25, 5.59, 539.68) (6, 29, 68, 133, 287)	(71.17, 70.62, 5.91, 420.53) (4, 21, 49, 100, 212)	(55.06, 54.33, 5.59, 307.52) (3, 16, 39, 76, 164)	(60.09, 59.23, 5.99, 360.35) (4, 17, 42, 84, 178)	(29.18, 27.98, 5.99, 175.01) (2, 9, 21, 41, 84)
0.50	(49.86, 48.63, 5.81, 289.59) (3, 15, 35, 69, 145)	(32.75, 32.34, 6.39, 209.19) (2, 10, 23, 45, 99)	(29.09, 29.01, 5.82, 169.38) (2, 9, 20, 40, 87)	(28.90, 28.79, 6.48, 187.32) (2, 9, 20, 40, 86)	(14.68, 14.39, 6.48, 95.14) (1, 5, 10, 20, 44)
0.60	(27.34, 27.00, 6.04, 165.15) (2, 8, 19, 38, 81)	(16.65, 16.14, 6.92, 115.15) (1, 5, 12, 23, 49)	(16.68, 16.30, 6.07, 101.34) (1, 5, 12, 23, 49)	(15.47, 14.96, 7.01, 108.49) (1, 5, 11, 21, 45)	(8.34, 7.76, 7.01, 58.49) (1, 3, 6, 11, 24)
0.70	(15.95, 15.41, 6.27, 99.91) (1, 5, 11, 22, 47)	(9.15, 8.57, 7.46, 68.30) (1, 3, 7, 13, 27)	(9.92, 9.48, 6.32, 62.74) (1, 3, 7, 14, 28)	(8.53, 8.09, 7.56, 64.56) (1, 3, 6, 12, 25)	(5.03, 4.45, 7.57, 38.08) (1, 2, 4, 7, 14)
0.80	(9.62, 9.07, 6.46, 62.19) (1, 3, 7, 13, 27)	(5.51, 5.03, 7.99, 44.03) (1, 2, 4, 7, 16)	(6.27, 5.75, 6.56, 41.11) (1, 2, 4, 9, 18)	(5.07, 4.56, 8.10, 41.09) (1, 2, 4, 7, 14)	(3.38, 2.81, 8.10, 27.40) (1, 1, 2, 4, 9)
0.90	(6.29, 5.79, 6.62, 41.62) (1, 2, 4, 8, 18)	(3.57, 2.99, 8.47, 30.23) (1, 1, 3, 5, 9)	(4.99, 2.42, 6.76, 28.58) (1, 1, 2, 4, 8)	(3.40, 2.81, 8.59, 29.18) (1, 1, 3, 4, 9)	(2.41, 1.84, 8.59, 20.70) (1, 1, 2, 3, 6)
1.00	(4.18, 3.60, 6.72, 28.09) (1, 2, 3, 6, 11)	(2.57, 2.01, 8.87, 22.82) (1, 1, 2, 3, 7)	(4.23, 3.62, 6.92, 20.66) (1, 2, 3, 6, 11)	(2.46, 1.90, 8.99, 22.08) (1, 1, 2, 3, 6)	(1.91, 1.31, 8.99, 17.18) (1, 1, 1, 2, 5)
1.20	(2.28, 1.72, 6.75, 15.41) (1, 1, 2, 3, 6)	(1.62, 1.01, 9.33, 15.09) (1, 1, 1, 2, 4)	(1.83, 1.24, 7.06, 12.89) (1, 1, 1, 2, 4)	(1.54, 0.91, 9.45, 14.57) (1, 1, 1, 2, 3)	(1.36, 0.69, 9.46, 12.83) (1, 1, 1, 2, 3)
1.40	(1.54, 0.90, 6.50, 9.99) (1, 1, 1, 2, 3)	(1.25, 0.55, 9.24, 11.55) (1, 1, 1, 1, 2)	(1.34, 0.68, 6.96, 9.32) (1, 1, 1, 2, 3)	(1.22, 0.51, 9.37, 11.41) (1, 1, 1, 1, 2)	(1.17, 0.44, 9.37, 10.93) (1, 1, 1, 1, 2)
1.60	(1.20, 0.49, 6.04, 7.25) (1, 1, 1, 1, 2)	(1.11, 0.36, 8.62, 9.61) (1, 1, 1, 1, 2)	(1.12, 0.37, 6.61, 7.42) (1, 1, 1, 1, 2)	(1.09, 0.31, 8.75, 9.56) (1, 1, 1, 1, 2)	(1.07, 0.27, 8.76, 9.39) (1, 1, 1, 1, 2)
1.80	(1.08, 0.29, 5.43, 5.85) (1, 1, 1, 1, 2)	(1.05, 0.23, 7.62, 8.02) (1, 1, 1, 1, 1)	(1.04, 0.21, 6.07, 6.32) (1, 1, 1, 1, 1)	(1.04, 0.20, 7.75, 8.04) (1, 1, 1, 1, 1)	(1.03, 0.19, 7.75, 8.01) (1, 1, 1, 1, 1)
2.00	(1.02, 0.15, 4.78, 4.89) (1, 1, 1, 1, 1)	(1.02, 0.15, 6.46, 6.60) (1, 1, 1, 1, 1)	(1.02, 0.16, 5.42, 5.48) (1, 1, 1, 1, 1)	(1.02, 0.13, 6.57, 6.69) (1, 1, 1, 1, 1)	(1.01, 0.12, 6.57, 6.66) (1, 1, 1, 1, 1)
2.50	(1.00, 0.02, 3.55, 3.55) (1, 1, 1, 1, 1)	(1.00, 0.04, 4.09, 4.10) (1, 1, 1, 1, 1)	(1.00, 0.01, 3.93, 3.93) (1, 1, 1, 1, 1)	(1.00, 0.04, 4.15, 4.16) (1, 1, 1, 1, 1)	(1.00, 0.04, 4.15, 4.16) (1, 1, 1, 1, 1)
AEQL	62.32	50.50	46.82	45.79	35.03
(n_1, n_2)	(3, 5)	(3, 10)	(3, 5)	(3, 10)	(3, 10)
(L_1, L) L_2	(0.8018, 3.1071) 3.2650	(1.2903, 3.1079) 3.2210	(0.8093, 3.4354) 3.1721	(1.2693, 3.1354) 3.5001	(1.2693, 3.1354) 3.1600

Table 3. The exact *ARL*, *SDRL*, *ASS*, *ANOS* (first row), Percentiles (second row), *AEQL* and optimal design parameters of the proposed chart when $m \in \{25, 50, 100\}$ and $m = \infty$ (i.e. Case K), $(n_1, n_2) \in \{(5, 5), (5, 8)\}$, $ASS_0 = 8$ and $\delta_{max} = 2.5$ for a nominal ARL_0 value of 370.4

(ARL, SDRL, ASS, ANOS) (P5, P25, P50, P75, P95)						
Type of design	Case U					Case K
Shift(δ)	$m = 25$	$m = 50$		$m = 100$		$m = \infty$
0.00	(370.4, 373.18, 8.41, 3126.71) (20, 106, 256, 517, 1115)	(370.4, 370.84, 7.99, 2961.50) (20, 108, 257, 508, 1116)	(370.4, 365.99, 8.00, 2933.87) (21, 109, 256, 505, 1086)	(370.4, 378.83, 7.99, 3014.07) (18, 107, 256, 522, 1134)	(370.4, 361.99, 8.00, 2914.36) (19, 106, 251, 512, 1077)	(370.4, 368.56, 8.00, 2956.70) (20, 110, 258, 509, 1085)
0.10	(266.04, 266.36, 8.44, 2246.64) (13, 77, 181, 367, 806)	(168.63, 165.29, 8.04, 1356.19) (9, 50, 119, 236, 502)	(138.89, 137.37, 8.10, 1124.34) (8, 41, 97, 192, 417)	(168.88, 180.20, 8.04, 1438.60) (10, 51, 125, 249, 531)	(132.11, 157.06, 8.10, 1662.76) (35, 187, 444, 901, 671)	(133.41, 119.10, 8.10, 1608.56)
0.20	(128.55, 128.04, 8.55, 1098.77) (7, 39, 89, 179, 383)	(73.81, 74.87, 8.17, 602.87) (4, 21, 50, 102, 219)	(59.36, 57.63, 8.37, 454.94) (3, 16, 38, 75, 163)	(49.41, 79.13, 8.17, 648.65) (5, 23, 55, 110, 238)	(56.80, 88.21, 8.37, 480.60) (21, 115, 75, 143, 230)	(56.76, 55.42, 8.37, 734.41) (5, 25, 61, 120, 227)
0.30	(58.67, 57.42, 8.70, 510.52) (4, 18, 41, 81, 174)	(35.23, 34.95, 8.36, 294.47) (2, 11, 25, 49, 103)	(26.06, 25.35, 8.79, 211.56) (2, 7, 17, 33, 70)	(27.00, 36.72, 8.36, 309.25) (2, 11, 26, 51, 109)	(24.63, 24.04, 8.79, 389.23) (8, 43, 103, 203, 437)	(24.35, 23.01, 8.79, 319.64) (2, 11, 25, 51, 108)
0.40	(28.64, 28.41, 8.88, 254.43) (2, 8, 20, 39, 86)	(18.06, 17.63, 8.58, 155.04) (1, 6, 12, 25, 54)	(13.73, 12.26, 9.32, 109.35) (1, 4, 8, 16, 34)	(18.87, 18.22, 8.58, 161.96) (1, 6, 13, 26, 56)	(12.32, 56.60, 9.32, 134.38) (3, 17, 40, 79, 171)	(12.02, 11.36, 9.32, 158.68) (1, 5, 12, 24, 49)
0.50	(15.01, 14.43, 9.07, 136.12) (1, 5, 11, 21, 43)	(9.96, 9.31, 8.82, 87.86) (1, 3, 7, 14, 28)	(7.41, 5.86, 9.91, 63.54) (1, 2, 5, 9, 18)	(10.42, 10.08, 8.82, 91.89) (6.25, 5.81, 9.03, 56.44)	(6.42, 23.99, 9.91, 101.98) (4.18, 11.65, 10.50, 67.92)	(6.80, 5.33, 9.91, 87.20) (1, 3, 6, 12, 25)
0.60	(9.60, 8.16, 9.23, 79.42) (1, 3, 6, 12, 25)	(5.92, 5.41, 9.03, 53.43) (1, 2, 4, 8, 17)	(4.95, 4.44, 10.51, 41.49) (1, 1, 3, 5, 11)	(1, 2, 4, 8, 18) (4.04, 3.57, 9.19, 37.16)	(1, 4, 9, 17, 36) (2.61, 6.04, 11.06, 43.13)	(4.06, 3.65, 10.50, 53.19) (1, 2, 4, 7, 14)
0.70	(5.31, 4.89, 9.35, 49.63) (1, 2, 4, 7, 15)	(3.93, 3.42, 9.19, 36.09) (1, 1, 3, 5, 11)	(2.94, 2.03, 11.07, 28.73) (1, 1, 2, 3, 7)	(1, 1, 3, 5, 11) (2.77, 2.22, 9.29, 25.71)	(1, 2, 5, 9, 19) (2.00, 3.43, 11.55, 26.20)	(2.25, 1.74, 11.06, 36.01) (1, 1, 2, 3, 6)
0.80	(3.53, 3.04, 9.40, 33.17) (1, 1, 3, 5, 10)	(2.94, 2.22, 9.29, 25.41) (1, 1, 2, 4, 7)	(1.99, 1.43, 11.56, 22.22) (1, 1, 1, 2, 5)	(2.10, 1.56, 9.30, 19.57) (1, 1, 2, 3, 5)	(1.70, 2.15, 11.95, 22.30) (1, 1, 2, 4, 7)	(1.98, 1.69, 11.55, 26.32) (1, 1, 1, 2, 4)
0.90	(2.91, 1.97, 9.38, 23.57) (1, 1, 2, 3, 6)	(2.25, 1.44, 9.29, 19.07) (1, 1, 2, 3, 5)	(1.80, 0.87, 11.97, 17.92) (1, 1, 1, 2, 3)	(1.65, 1.05, 9.22, 15.24) (1, 1, 1, 2, 4)	(1.33, 1.31, 12.26, 23.61) (1, 1, 1, 2, 5)	(1.74, 1.12, 11.96, 20.82) (1, 1, 1, 2, 4)
1.00	(2.09, 1.31, 9.28, 17.57) (1, 1, 1, 2, 4)	(1.95, 1.03, 9.22, 15.17) (1, 1, 1, 2, 4)	(1.59, 0.61, 12.30, 15.85) (1, 1, 1, 1, 2)	(1.22, 0.52, 8.80, 10.73) (1, 1, 1, 1, 2)	(1.10, 0.62, 12.60, 16.36) (1, 1, 1, 1, 3)	(1.30, 0.74, 12.26, 17.16) (1, 1, 1, 2, 3)
1.20	(1.33, 0.67, 8.82, 11.69) (1, 1, 1, 1, 3)	(1.30, 0.52, 8.80, 10.74) (1, 1, 1, 1, 2)	(1.28, 0.29, 12.71, 13.71) (1, 1, 1, 1, 2)	(1.07, 0.27, 8.08, 8.61) (1, 1, 1, 1, 2)	(1.08, 0.29, 12.57, 13.56) (1, 1, 1, 1, 1)	(1.12, 0.37, 12.60, 14.10) (1, 1, 1, 1, 2)
1.40	(1.19, 0.33, 8.08, 8.91) (1, 1, 1, 1, 2)	(1.09, 0.27, 8.08, 8.61) (1, 1, 1, 1, 2)	(1.09, 0.14, 12.90, 13.14) (1, 1, 1, 1, 1)	(1.02, 0.13, 7.21, 7.32) (1, 1, 1, 1, 1)	(1.00, 0.05, 6.39, 6.40) (1, 1, 1, 1, 1)	(1.03, 0.16, 12.57, 12.89) (1, 1, 1, 1, 1)
1.60	(1.12, 0.15, 7.21, 7.38) (1, 1, 1, 1, 1)	(1.07, 0.13, 7.21, 7.33) (1, 1, 1, 1, 1)	(1.06, 0.06, 12.97, 13.02) (1, 1, 1, 1, 1)	(1.00, 0.02, 12.99, 12.99) (1, 1, 1, 1, 1)	(1.00, 0.02, 12.99, 13.00) (1, 1, 1, 1, 1)	(1.01, 0.08, 12.18, 12.26) (1, 1, 1, 1, 1)
1.80	(1.09, 0.07, 6.39, 6.42) (1, 1, 1, 1, 1)	(1.05, 0.04, 6.39, 6.40) (1, 1, 1, 1, 1)	(1.04, 0.02, 12.99, 12.99) (1, 1, 1, 1, 1)	(1.00, 0.00, 5.08, 5.08) (1, 1, 1, 1, 1)	(1.00, 0.00, 5.08, 5.08) (1, 1, 1, 1, 1)	(1.00, 0.03, 11.39, 11.40) (1, 1, 1, 1, 1)
2.00	(1.07, 0.02, 5.75, 5.75) (1, 1, 1, 1, 1)	(1.02, 0.01, 5.75, 5.75) (1, 1, 1, 1, 1)	(1.01, 0.02, 12.99, 13.00) (1, 1, 1, 1, 1)	(1.00, 0.00, 5.08, 5.08) (1, 1, 1, 1, 1)	(1.00, 0.01, 10.22, 10.23) (1, 1, 1, 1, 1)	(1.00, 0.02, 10.22, 10.23) (1, 1, 1, 1, 1)
2.50	(1.04, 0.00, 5.08, 5.08) (1, 1, 1, 1, 1)	(1.01, 0.00, 5.08, 5.08) (1, 1, 1, 1, 1)	(1.00, 0.00, 5.08, 12.99) (1, 1, 1, 1, 1)	(1.00, 0.00, 5.08, 5.08) (1, 1, 1, 1, 1)	(1.00, 0.00, 6.88, 6.88) (1, 1, 1, 1, 1)	(1.00, 0.00, 6.88, 6.88) (1, 1, 1, 1, 1)
AEQL	39.63	36.96	33.59	32.84	30.19	29.07
(n_1, n_2)	(5, 5)	(5, 5)	(5, 8)	(5, 5)	(5, 8)	(5, 8)
(L_1, L_2)	(0.4093, 3.4354) 3.2221	(0.524, 3.435) 3.243	(0.8870, 10.5920) 3.1850	(2.9835, 3.0083) 2.0027	(0.887, 4.866) 3.375	(0.887, 4.866) 2.975

Table 4. Exact *ARL*, *SDRL*, *ASS*, *ANOS* (first row), the Percentiles (second row), *AEQL* and optimal design parameters of the proposed chart when $m \in \{25, 50, 100\}$ and $m = \infty$ (i.e. Case K), $(n_1, n_2) \in \{(3, 5); (3, 10)\}$, $ASS_0 = 8$ and $\delta_{max} = 2.5$ for a nominal ARL_0 value of 370.4

(ARL, SDRL, ASS, ANOS) (P5, P25, P50, P75, P95)						
Type of design	Case U					Case K
Shift(δ)	$m = 25$	$m = 50$		$m = 100$		$m = \infty$
0.00	(370.4, 377.30, 7.98, 2995.65) (20, 107, 262, 516, 1131)	(370.4, 373.36, 7.98, 2979.38) (19, 108, 257, 516, 1126)	(370.4, 371.12, 8.00, 2947.74) (20, 105, 254, 513, 1111)	(370.4, 367.01, 7.98, 2968.61) (20, 108, 263, 515, 1124)	(370.4, 364.11, 8.00, 2938.58) (19, 106, 257, 507, 1090)	(370.4, 374.68, 8.00, 2982.43) (20, 110, 253, 515, 1126)
0.10	(273.72, 274.99, 7.98, 2183.83) (15, 79, 191, 378, 808)	(270.11, 271.52, 7.98, 2155.02) (15, 80, 187, 376, 800)	(134.70, 135.11, 8.07, 1086.59) (7, 39, 94, 188, 405)	(195.76, 196.16, 7.98, 1561.82) (10, 55, 135, 271, 592)	(131.27, 131.95, 8.07, 1020.31) (8, 43, 105, 209, 446)	(129.99, 121.92, 8.07, 1074.63) (11, 64, 151, 303, 466)
0.20	(150.52, 150.47, 7.97, 1200.79) (8, 43, 105, 207, 455)	(150.22, 148.41, 7.97, 1197.84) (8, 44, 105, 208, 448)	(52.96, 52.31, 8.25, 473.12) (3, 16, 37, 74, 158)	(99.86, 99.42, 7.97, 796.25) (6, 30, 69, 137, 302)	(50.03, 49.70, 8.25, 447.20) (4, 18, 41, 81, 173)	(49.29, 49.09, 8.25, 437.00) (5, 26, 61, 122, 165)
0.30	(79.85, 79.34, 7.96, 635.99) (5, 24, 55, 111, 237)	(77.18, 75.54, 7.96, 614.69) (4, 22, 54, 107, 228)	(23.82, 23.60, 8.55, 203.66) (2, 7, 17, 33, 70)	(50.80, 49.60, 7.95, 404.60) (3, 15, 36, 71, 151)	(21.47, 20.86, 8.55, 226.32) (2, 8, 19, 36, 78)	(20.87, 19.06, 8.55, 323.77) (2, 11, 27, 53, 111)
0.40	(42.89, 43.41, 7.95, 340.97) (3, 12, 30, 59, 127)	(41.25, 39.51, 7.95, 327.96) (3, 12, 29, 58, 122)	(11.71, 11.36, 8.93, 104.60) (1, 4, 8, 16, 35)	(27.53, 26.61, 7.95, 218.88) (2, 8, 19, 38, 80)	(11.96, 11.43, 8.93, 115.75) (1, 4, 9, 18, 38)	(11.84, 10.28, 8.93, 119.39) (1, 5, 13, 24, 52)
0.50	(23.54, 23.11, 7.93, 186.62) (2, 7, 16, 32, 69)	(23.29, 22.92, 7.93, 184.63) (2, 7, 16, 32, 68)	(6.54, 6.14, 9.38, 61.30) (1, 2, 5, 9, 19)	(15.74, 15.24, 7.93, 124.78) (1, 5, 11, 21, 46)	(6.07, 5.51, 9.38, 66.35) (1, 2, 5, 9, 20)	(6.20, 5.87, 9.38, 86.24) (1, 3, 6, 12, 27)
0.60	(13.47, 13.07, 7.89, 106.35) (1, 4, 10, 18, 40)	(13.23, 12.71, 7.89, 104.40) (1, 4, 9, 18, 38)	(3.95, 3.37, 9.86, 38.93) (1, 1, 3, 5, 11)	(9.48, 9.03, 7.89, 74.86) (1, 3, 7, 13, 28)	(3.84, 2.70, 9.86, 37.79) (1, 2, 3, 6, 12)	(3.80, 2.86, 9.86, 36.21) (1, 2, 4, 7, 15)
0.70	(8.32, 7.79, 7.84, 65.22) (1, 3, 6, 11, 24)	(8.25, 7.76, 7.84, 64.67) (1, 3, 6, 11, 24)	(2.69, 2.16, 10.34, 27.85) (1, 1, 2, 3, 7)	(6.02, 5.41, 7.84, 47.22) (1, 2, 4, 8, 17)	(2.62, 1.89, 10.34, 27.20) (1, 1, 2, 4, 7)	(2.61, 1.90, 10.34, 35.28) (1, 1, 2, 4, 9)
0.80	(5.45, 4.89, 7.77, 42.40) (1, 2, 4, 7, 15)	(5.42, 4.94, 7.77, 42.16) (1, 2, 4, 7, 15)	(1.97, 1.40, 10.81, 21.29) (1, 1, 1, 2, 5)	(4.10, 3.63, 7.77, 31.85) (1, 1, 3, 5, 11)	(2.06, 1.48, 10.81, 22.32) (1, 1, 2, 3, 5)	(2.32, 1.77, 10.81, 25.12) (1, 1, 2, 3, 6)
0.90	(3.77, 3.24, 7.68, 28.95) (1, 1, 3, 5, 10)	(3.73, 3.19, 7.68, 28.64) (1, 1, 3, 5, 10)	(1.65, 0.92, 11.25, 17.42) (1, 1, 1, 2, 3)	(2.93, 2.38, 7.68, 22.48) (1, 1, 2, 4, 8)	(1.62, 1.00, 11.25, 18.24) (1, 1, 1, 2, 4)	(1.78, 1.17, 11.25, 19.97) (1, 1, 1, 2, 4)
1.00	(2.27, 2.15, 7.56, 20.58) (1, 1, 2, 4, 7)	(2.74, 2.19, 7.56, 20.73) (1, 1, 2, 4, 7)	(1.42, 0.65, 11.63, 15.40) (1, 1, 1, 1, 3)	(2.22, 1.65, 7.56, 16.81) (1, 1, 2, 3, 6)	(1.36, 0.71, 11.63, 15.83) (1, 1, 1, 2, 3)	(1.47, 0.83, 11.63, 17.13) (1, 1, 1, 2, 3)
1.20	(1.74, 1.14, 7.22, 12.55) (1, 1, 1, 2, 4)	(1.71, 1.10, 7.22, 12.36) (1, 1, 1, 2, 4)	(1.21, 0.34, 12.23, 13.49) (1, 1, 1, 1, 2)	(1.51, 0.88, 7.22, 10.87) (1, 1, 1, 2, 3)	(1.12, 0.37, 12.23, 13.71) (1, 1, 1, 1, 2)	(1.16, 0.44, 12.23, 14.24) (1, 1, 1, 1, 2)
1.40	(1.28, 0.60, 6.73, 8.63) (1, 1, 1, 1, 2)	(1.28, 0.60, 6.73, 8.58) (1, 1, 1, 1, 2)	(1.05, 0.18, 12.60, 13.01) (1, 1, 1, 1, 1)	(1.19, 0.48, 6.73, 8.03) (1, 1, 1, 1, 2)	(1.04, 0.21, 12.60, 13.12) (1, 1, 1, 1, 1)	(1.06, 0.25, 12.60, 13.31) (1, 1, 1, 1, 2)
1.60	(1.10, 0.32, 6.12, 6.71) (1, 1, 1, 1, 2)	(1.10, 0.33, 6.12, 6.74) (1, 1, 1, 1, 2)	(1.03, 0.10, 12.81, 12.94) (1, 1, 1, 1, 1)	(1.07, 0.26, 6.12, 6.53) (1, 1, 1, 1, 2)	(1.01, 0.12, 12.81, 12.99) (1, 1, 1, 1, 1)	(1.02, 0.14, 12.81, 13.07) (1, 1, 1, 1, 1)
1.80	(1.03, 0.17, 5.44, 5.59) (1, 1, 1, 1, 1)	(1.03, 0.18, 5.44, 5.60) (1, 1, 1, 1, 1)	(1.01, 0.07, 12.88, 12.94) (1, 1, 1, 1, 1)	(1.02, 0.13, 5.44, 5.54) (1, 1, 1, 1, 1)	(1.01, 0.07, 12.88, 12.95) (1, 1, 1, 1, 1)	(1.01, 0.08, 12.88, 12.97) (1, 1, 1, 1, 1)
2.00	(1.01, 0.08, 4.77, 4.78) (1, 1, 1, 1, 1)	(1.01, 0.09, 5.10, 4.80) (1, 1, 1, 1, 1)	(1.00, 0.03, 12.85, 12.86) (1, 1, 1, 1, 1)	(1.00, 0.07, 4.77, 4.79) (1, 1, 1, 1, 1)	(1.00, 0.05, 12.85, 12.88) (1, 1, 1, 1, 1)	(1.00, 0.04, 12.85, 12.87) (1, 1, 1, 1, 1)
2.50	(1.00, 0.01, 3.54, 3.54) (1, 1, 1, 1, 1)	(1.00, 0.01, 3.54, 3.54) (1, 1, 1, 1, 1)	(1.00, 0.01, 12.17, 12.17) (1, 1, 1, 1, 1)	(1.00, 0.00, 3.53, 3.54) (1, 1, 1, 1, 1)	(1.00, 0.01, 12.17, 12.17) (1, 1, 1, 1, 1)	(1.00, 0.01, 12.17, 12.17) (1, 1, 1, 1, 1)
<i>AEQL</i>	41.22	40.88	28.10	34.83	27.68	27.21
(n_1, n_2)	(3, 5)	(3, 5)	(3, 10)	(3, 5)	(3, 10)	(3, 10)
(L_1, L) L_2	(0.02527, 3.088) 3.2100	(0.02549, 3.091) 3.2010	(0.6740, 5.7150) 3.3102	(0.02544, 3.088) 3.1503	(0.6736, 5.7150) 2.999	(0.6742, 5.7155) 2.945

5.2 ARL profiles of the Case U SSDS \bar{X} scheme using Case K optimal design parameters

In this sub-section, we investigate the ARL profile behaviour of the proposed SSDS \bar{X} scheme when the Case U performance is obtained using the Case K optimal design parameters (as discussed in Malela-Majika et al. (2019)) instead of the Case U optimal design parameters. To evaluate the impact of using the Case K optimal design parameters in Case U, the percentage difference ($\%Diff$) between the Case U OOC ARL (denoted as ARL_{δ_U}) and Case K OOC ARL (denoted as ARL_{δ_K}) is calculated as follows:

$$\%Diff = \left(\frac{ARL_{\delta_U} - ARL_{\delta_K}}{ARL_{\delta_K}} \right) \times 100 \quad (33)$$

Table 5 displays the ARL_{δ_U} and ARL_{δ_K} (last column) values using the Case K optimal design parameters when $n \in \{2, 5\}$, $n_1 \in \{2, 5\}$, $n_2 \in \{2, 5, 8, 11\}$ and nominal ARL_0 of 370.4. In Table 5, $m = \infty$ denotes the parameters known case (i.e. Case K). From Table 5, it can be noticed that the proposed SSDS \bar{X} scheme yields very large ARL_{δ_U} for small Phase I sample size. For instance, when $\delta = 0.5$ and $(n, n_1, n_2) = (2, 2, 2)$ for a nominal ARL_0 value of 370.4, the SSDS \bar{X} scheme yields ARL values of 160.2 and 79.41 when $m = 25$ and $m = \infty$, respectively; revealing a 101.7% percentage difference as compared to the Case K ARL value. Moreover, the results in Table 5 show that, as the Phase I sample size increases, the $\%Diff$ decreases considerably. For a large Phase I sample size (e.g. $m = 400$), the $\%Diff$ is less than 1%, meaning that the Case U SSDS \bar{X} scheme performs as if the optimal design parameters were known. Therefore, it is very important to know the number of Phase I observations for which the proposed scheme performs as if it was in Case K. As we can see from Table 5, this will depend on the average sample size as well as the Stage 1 and Stage 2 sample sizes. The finding from Table 5 also shows that the larger the average sample size, the higher the $\%Diff$.

Table 5. Case U and Case K OOC ARL (first row) and $\%Diff$ (second row) of the SSDS scheme using the Case K optimal design parameters when $n \in \{2, 5\}$, $n_1 \in \{2, 5\}$ and $n_2 \in \{2, 5, 8, 11\}$ when $NARL_0 = 370.4$

n	(n_1, n_2)	Case K Optimal parameters (L_1, L, L_2)	δ	m						∞
				25	50	100	200	300	400	
2	(2, 2)	(2.910, 3.057, 2.405)	0.5	160.20	112.64	101.41	82.01	81.11	80.01	79.41
				101.7%	41.8%	27.7%	3.3%	2.1%	0.8%	
			1.0	30.14	30.19	24.58	18.71	17.14	15.48	15.43
				95.3%	95.7%	59.3%	21.3%	11.1%	0.3%	
			1.5	11.16	8.19	6.37	5.54	5.00	4.74	4.73
				135.9%	73.2%	34.7%	17.1%	5.7%	0.2%	
	(2, 8)	(2.975, 3.005, 2.931)	2.0	5.13	3.31	3.00	2.64	2.39	2.20	2.18
				135.3%	51.8%	37.6%	21.1%	9.6%	0.9%	
			2.5	3.36	2.24	2.24	2.03	1.88	1.41	1.40
				140.0%	60.0%	60.0%	45.0%	34.3%	0.7%	
			0.5	164.20	95.02	87.64	83.63	82.23	81.09	80.34
				104.4%	18.3%	9.1%	4.1%	2.4%	0.9%	
			1.0	32.11	26.62	23.13	19.24	17.44	16.03	15.97

			101.1%	66.7%	44.8%	20.5%	9.2%	0.4%	
		1.5	12.94	9.04	7.18	5.70	5.02	4.82	4.77
			171.3%	89.5%	50.5%	19.5%	5.2%	1.0%	
		2.0	6.07	3.56	2.98	2.66	2.37	2.29	2.27
			167.4%	56.8%	31.3%	17.2%	4.4%	0.9%	
		2.5	4.01	2.62	2.21	2.05	1.76	1.44	1.43
			180.4%	83.2%	54.5%	43.4%	23.1%	0.7%	
		0.5	165.18	96.21	88.07	83.61	82.00	81.18	80.74
			104.6%	19.2%	9.1%	3.6%	1.6%	0.5%	
		1.0	34.05	28.19	24.13	19.31	18.07	16.73	16.66
			104.4%	69.2%	44.8%	15.9%	8.5%	0.4%	
(2, 11)	(2.991, 3.00, 2.998)	1.5	13.71	12.63	7.25	5.82	5.38	5.00	4.96
			176.4%	154.6%	46.2%	17.3%	8.5%	0.8%	
		2.0	6.60	6.11	2.99	2.67	2.49	2.31	2.29
			188.2%	166.8%	30.6%	16.6%	8.7%	0.9%	
		2.5	4.26	4.38	2.29	2.24	2.03	1.76	1.75
			143.4%	150.3%	30.9%	28.0%	16.0%	0.6%	

Table 5. (continues)

n	(n_1, n_2)	Case K Optimal parameters (L_1, L, L_2)	δ	m						∞
				25	50	100	200	300	400	
5	(2, 11)	(1.094, 3.234, 3.010)	0.5	116.33	30.21	22.07	22.51	21.02	19.40	19.35
				501.2%	56.1%	14.1%	16.3%	8.6%	0.3%	
			1.0	24.42	4.23	3.72	3.31	2.89	2.44	2.39
				921.8%	77.0%	55.6%	38.5%	20.9%	2.1%	
			1.5	9.06	2.51	2.04	1.69	1.61	1.49	1.48
				512.2%	69.6%	37.8%	14.2%	8.8%	0.7%	
			2.0	3.10	2.08	1.63	1.39	1.26	1.13	1.12
				176.8%	85.7%	45.5%	24.1%	12.5%	0.9%	
			2.5	2.00	1.61	1.39	1.22	1.12	1.04	1.02
				96.1%	57.8%	36.3%	19.6%	9.8%	2.0%	
	(5, 5)	(2.993, 3.001, 3.000)	0.5	120.76	41.47	32.04	29.11	27.77	26.43	26.29
				359.3%	57.7%	21.9%	10.7%	5.6%	0.5%	
			1.0	22.01	9.21	6.36	4.71	4.04	3.87	3.86
				470.2%	138.6%	64.8%	22.0%	4.7%	0.3%	
			1.5	8.86	4.92	2.28	1.61	1.50	1.42	1.42
				523.9%	246.5%	60.6%	13.4%	5.6%	0.0%	
			2.0	3.97	2.73	1.94	1.40	1.26	1.10	1.09
				264.2%	150.5%	78.0%	28.4%	15.6%	0.9%	
			2.5	3.45	1.87	1.47	1.19	1.10	1.03	1.02
				235.0%	81.6%	42.7%	15.5%	6.8%	0.0%	
	(5, 8)	(2.993, 3.001, 2.998)	0.5	104.31	26.42	21.72	20.17	19.08	18.71	18.61
				460.5%	42.0%	16.7%	8.4%	2.5%	0.5%	
			1.0	21.79	7.26	5.48	5.01	3.04	2.09	2.09
				942.6%	247.4%	162.2%	139.7%	45.5%	0.0%	
			1.5	7.08	4.54	2.19	2.01	1.63	1.24	1.22
				475.6%	269.1%	78.0%	63.4%	32.5%	0.8%	
			2.0	3.31	2.80	1.79	1.53	1.36	1.07	1.06
				212.3%	164.2%	68.9%	44.3%	28.3%	0.9%	
			2.5	2.48	1.74	1.41	1.34	1.20	1.02	1.01
				145.5%	72.3%	39.6%	32.7%	18.8%	1.0%	
	(5, 11)	(2.996, 3.000, 2.999)	0.5	99.17	25.45	21.39	19.15	17.82	17.23	17.21
				476.2%	47.9%	24.3%	11.3%	3.5%	0.1%	
			1.0	22.84	7.42	5.47	4.93	3.00	2.11	2.11
				982.5%	251.7%	159.2%	133.6%	42.2%	0.0%	
			1.5	6.59	4.50	2.16	1.97	1.61	1.15	1.14
				478.1%	294.7%	89.5%	72.8%	41.2%	0.9%	
			2.0	3.22	2.77	1.77	1.49	1.33	1.09	1.07
				198.1%	156.5%	63.9%	38.0%	23.1%	0.9%	
			2.5	2.29	1.70	1.39	1.33	1.21	1.02	1.00
				126.7%	68.3%	37.6%	31.7%	19.8%	1.0%	

Therefore, to secure stability and better OOC performance in Phase II for the proposed SSDS \bar{X} scheme, the operator must either use a high desired Phase I sample size or choose the appropriate design parameters as suggested in Tables 1 to 4.

5.3 Performance comparison

In this section, the proposed Case U SSDS \bar{X} scheme is compared to a number of well-known Case U monitoring schemes including the existing NSSDS \bar{X} , NSS and side-sensitive synthetic Shewhart \bar{X} , exponentially weighted moving average \bar{X} (denoted as \bar{X} -EWMA (λ) where λ represents the smoothing parameter) with $\lambda = 0.1$ and 0.5 , cumulative sum \bar{X} (denoted as \bar{X} -CUSUM) monitoring schemes with estimated process parameters. The competing schemes are compared in terms of the $AEQL$, the $ARARL$ and PCI values. Note that the monitoring scheme with a small $AEQL$ value is considered to be superior in performance for the range of shifts under consideration. In this example, the proposed scheme is considered to be the benchmark scheme. Therefore, for the chosen competing schemes, if its PCI and $ARARL$ values are less than one, then that particular competing scheme is declared as more efficient than the proposed Case U SSDS \bar{X} scheme. However, if the PCI and $ARARL$ values are greater than one, then the competing scheme is declared as less efficient than the proposed SSDS \bar{X} scheme. When the PCI and $ARARL$ values are equal to one, then the competing scheme and the proposed SSDS \bar{X} scheme are equivalent. For a fair comparison, the performance of the competing schemes are investigated when $(\delta_{min}, \delta_{max}) = (0, 2.5)$, $m \in \{50, 100\}$, $ASS_0 \in \{5, 8\}$ corresponding to $n \in \{5, 8\}$, $n_1 \in \{3, 5\}$, $n_2 \in \{5, 8\}$ and a nominal $ARL_0 = 370.4$. The shifts sizes are divided into three groups which are “small” ($0 < \delta \leq 0.7$), “small to moderate” ($0 < \delta \leq 1.6$), and “small to large” ($0 < \delta \leq 2.5$). In Table 6, the proposed scheme is compared to the foregoing monitoring schemes in terms of the overall performance. The results corresponding to the best monitoring scheme are highlighted in bold.

Table 6. Case U monitoring schemes performance comparison when $n = ASS_0 \in \{5, 8\}$, $n_1 \in \{3, 5\}$, $n_2 \in \{5, 8\}$, $m \in \{50, 100\}$, $\delta_{min} = 0$ and $\delta_{max} = 2.5$ with a nominal ARL_0 of 370.4

*Shift	Performance measures	Control charts							(n ₁ , n ₂)	ASS ₀ = n	m						
		NSS Synthetic \bar{X}	SS Synthetic \bar{X}	\bar{X} - EWMA(0.1)	\bar{X} - EWMA(0.5)	\bar{X} -CUSUM	NSSDS \bar{X}	SSDS \bar{X}									
Small	AEOL ARARL PCI	98.21 1.24 1.29	82.33 1.10 1.08	70.56 0.91 0.93	119.12 1.48 1.56	96.01 1.23 1.26	84.13 1.13 1.10	76.23 1.00 1.00	(3, 5)	5	50						
Small to moderate	AEOL ARARL PCI	104.24 1.49 1.47	80.32 1.17 1.14	76.79 1.11 1.09	120.24 1.64 1.70	103.72 1.50 1.47	74.22 1.08 1.05	70.76 1.00 1.00									
Small to large	AEOL ARARL PCI	86.04 1.41 1.38	70.43 1.11 1.13	94.18 1.47 1.51	110.44 1.68 1.77	100.37 1.56 1.61	67.99 1.13 1.09	62.32 1.00 1.00									
Small	AEOL ARARL PCI	72.89 1.20 1.17	68.16 1.14 1.09	60.30 0.95 0.97	73.05 1.21 1.17	71.18 1.14 1.14	68.29 1.12 1.09	62.46 1.00 1.00				(5, 8)	8				
Small to moderate	AEOL ARARL PCI	71.51 1.34 1.32	60.16 1.15 1.11	61.47 1.14 1.13	79.09 1.52 1.46	66.34 1.20 1.22	59.35 1.12 1.09	54.24 1.00 1.00									
Small to large	AEOL ARARL PCI	45.04 1.36 1.34	40.37 1.31 1.20	50.56 1.40 1.45	61.40 1.72 1.71	59.48 1.63 1.65	38.55 1.17 1.15	33.59 1.00 1.00									
Small	AEOL ARARL PCI	75.43 1.17 1.19	67.68 1.03 1.07	54.44 0.82 0.86	103.57 1.57 1.63	86.12 1.32 1.36	71.07 1.14 1.12	63.40 1.00 1.00							(3, 5)	5	
Small to moderate	AEOL ARARL PCI	79.69 1.49 1.53	61.21 1.23 1.18	59.85 1.21 1.15	85.59 1.43 1.65	75.05 1.36 1.44	56.10 1.13 1.08	51.94 1.00 1.00									
Small to large	AEOL ARARL PCI	67.65 1.48 1.44	51.18 1.12 1.09	71.02 1.41 1.52	83.51 1.74 1.78	68.71 1.49 1.47	53.95 1.19 1.15	46.82 1.00 1.00									
Small	AEOL ARARL PCI	72.43 1.21 1.18	65.68 1.10 1.07	53.51 0.84 0.87	99.43 1.58 1.62	88.09 1.41 1.44	70.12 1.17 1.14	61.25 1.00 1.00									
Small to moderate	AEOL ARARL PCI	75.45 1.46 1.52	59.06 1.21 1.19	60.76 1.26 1.23	83.28 1.56 1.68	71.26 1.47 1.44	53.23 1.10 1.07	49.59 1.00 1.00									
Small to large	AEOL ARARL PCI	65.69 1.42 1.39	59.34 1.19 1.26	68.13 1.45 1.44	74.79 1.47 1.58	66.52 1.36 1.41	51.48 1.06 1.09	47.19 1.00 1.00									

* Small: ($0 < \delta \leq 0.7$), Small to Moderate: ($0 < \delta \leq 1.6$) and Small to Large: ($0 < \delta \leq 2.5$).

From Table 6, it can be seen that regardless of the sample sizes, the EWMA (0.1) monitoring scheme outperforms the proposed scheme for “small” shift ($\delta_{max} = 0.7$) in the process mean. However, for “small to moderate” shifts (i.e. $\delta_{max} = 1.6$) as well as for the “small to large” shifts (i.e. $\delta_{max} = 2.5$), the proposed monitoring scheme outperforms all the competing schemes considered in this paper. These findings are also valid in Case K, i.e. the proposed SSDS \bar{X} monitoring scheme is superior to all the considered competing schemes in Case K. Due to restriction on the number of pages, the table for the Case K performance comparison is not provided in this paper; but for more details on this, readers are referred to Malela-Majika et al. (2019).

The findings in Table 6 are also confirmed in terms of the *ARARL* and *PCI* values. When comparing the existing DS \bar{X} scheme to the proposed scheme, we can observe the following: for “small” shifts, the SSDS \bar{X} monitoring scheme improves the existing DS \bar{X} scheme between 10% and 17%. From “small to moderate” shifts, the overall improvement is between 5% and 9%. From “small to large” shifts, the overall improvement is between 7% and 15%.

6. Illustrative example

In this section, the implementation and application of the proposed SSDS \bar{X} scheme is illustrated using the data set from Zaman et al. (2017). The data gives the information on the inside diameter of cylinder bores in an engine block and contain thirty-five samples, each of size $n = 5$. In this implementation example, each sample is considered to be a master sample which is divided into two subgroups of sizes 2 and 3 (i.e. $n_1 = 2$ and $n_2 = 3$), in Stages 1 and 2, respectively, such that $n = n_1 + n_2 = 5$. The estimated IC process mean and standard deviation (using Equations (11) and (12)) for the inside diameter of cylinder bores are $\hat{\mu}_0 = 200.15$ and $\hat{\sigma}_0 = 3.47$ millimeters (*mm*), respectively. The shift detection ability of the proposed Case U SSDS \bar{X} scheme is also compared to the one of the existing Case U NSSDS \bar{X} scheme. For $(n_1, n_2) = (2, 3)$ and $ASS_0 = 3$, the optimal combinations (L_1^*, L^*, L_2^*) of the Case U SSDS \bar{X} scheme and the Case U NSSDS \bar{X} scheme are found to be equal to (2.212, 2.576, 2.305) and (2.306, 2.614, 2.418), respectively, so that these schemes both satisfied $ARL_0 = 370.4$. A plot of the charting statistics Z_{1i} and Z_i (i.e. for Stages 1 and 2) of the two monitoring schemes are shown in Figure 3. Table 7 illustrates the operation of the Case U’s Phase II NSSDS and SSDS \bar{X} schemes using the data set on the inside diameter of cylinder bores.

It is seen that the NSSDS \bar{X} scheme does not give a signal at Stage 1. However, at the 16th, 19th and 26th sampling time, there was a need for a second sample and the process moved to Stage 2. The plotting statistics of the NSSDS \bar{X} scheme at Stage 2, Z_i , at the 16th, 19th and 26th sampling time are equal to -0.425 , 1.015 and 3.176 , respectively. It can be seen that Z_{16} and Z_{19} plot between $-L_2 = -2.418$ and $L_2 = 2.418$, which means that the DS \bar{X} scheme does not signal on the 16th and 19th sampling time. Since Z_{26} plots above L_2 , the NSSDS \bar{X} scheme gives a signal at the 26th sampling time (see Figure 3(a) and Table 7) for the first time in Stage 2.

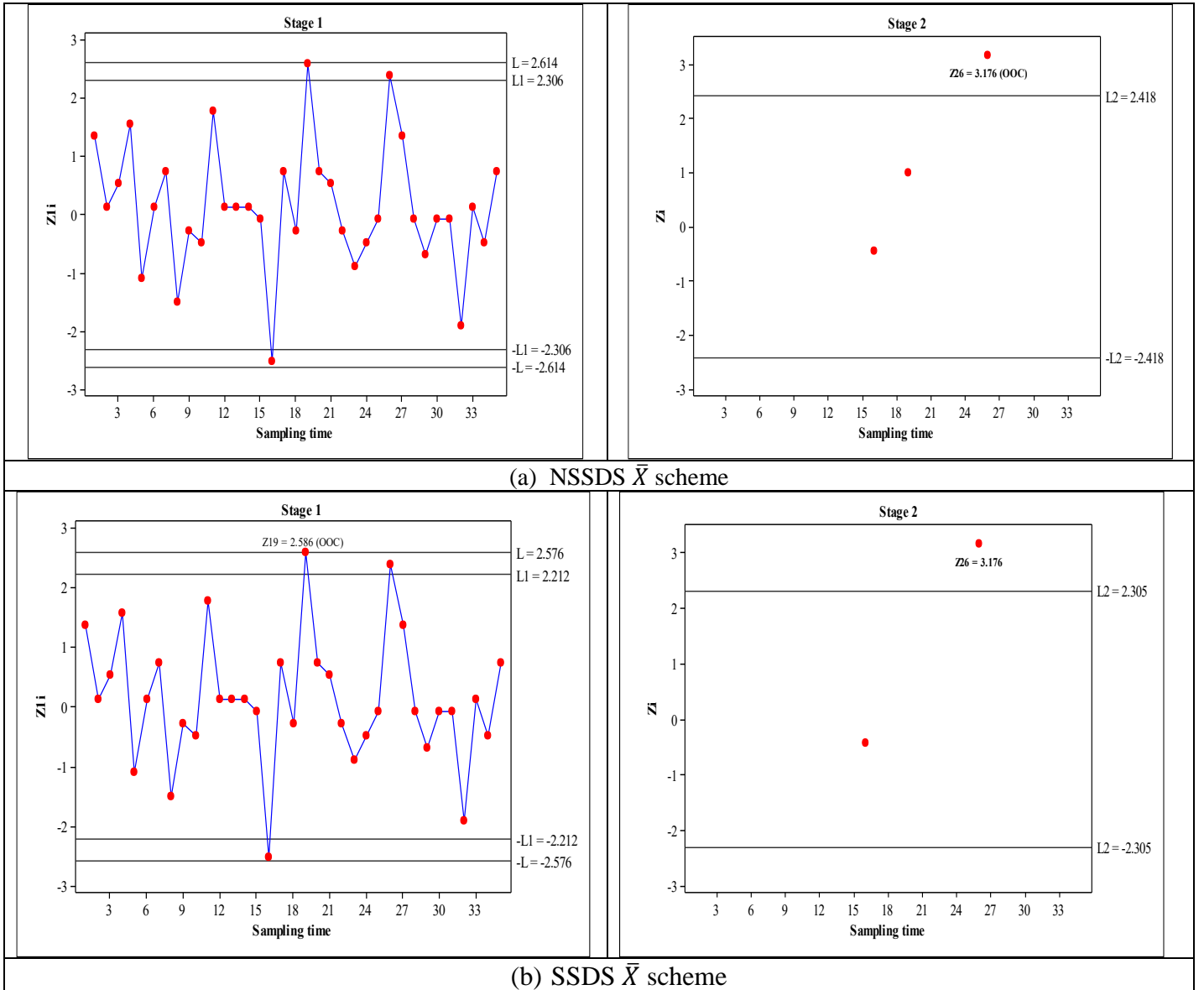


Figure 3. The Case U NSSDS and SSDS \bar{X} schemes using the data set on the inside diameter of cylinder bores in an engine block

Table 7. Illustration of the operation of the NSSDS and SSDS \bar{X} schemes using the data set on the inside diameter of cylinder bores in an engine block

	NSSDS \bar{X} chart								SSDS \bar{X} chart							
Sample No	\bar{X}_{1i}	Z_{1i}	Take a 2 nd Sample	\bar{X}_{2i}	\bar{X}_i	Z_i	Stage 1: NSSDS \bar{X} signal	Stage 2: NSSDS \bar{X} signal	\bar{X}_{1i}	Z_{1i}	Take a 2 nd Sample	\bar{X}_{2i}	\bar{X}_i	Z_i	Stage 1: SSDS \bar{X} signal	Stage 2: SSDS \bar{X} signal
1	203.5	1.3637	N				N		203.5	1.3637	N				N	
2	200.5	0.1409	N				N		200.5	0.1409	N				N	
3	201.5	0.5485	N				N		201.5	0.5485	N				N	
4	204	1.5674	N				N		204	1.5674	N				N	
5	197.5	-1.0818	N				N		197.5	-1.0818	N				N	
6	200.5	0.1409	N				N		200.5	0.1409	N				N	
7	202	0.7523	N				N		202	0.7523	N				N	
8	196.5	-1.4894	N				N		196.5	-1.4894	N				N	
9	199.5	-0.2667	N				N		199.5	-0.2667	N				N	
10	199	-0.4705	N				N		199	-0.4705	N				N	
11	204.5	1.7712	N				N		204.5	1.7712	N				N	
12	200.5	0.1409	N				N		200.5	0.1409	N				N	
13	200.5	0.1409	N				N		200.5	0.1409	N				N	
14	200.5	0.1409	N				N		200.5	0.1409	N				N	
15	200	-0.0629	N				N		200	-0.0629	N				N	
16	194	-2.5084	Y	203.67	199.8	-0.4253	N	N	194	-2.5084	Y	203.67	199.8	-0.4253	N	N
17	202	0.7523	N				N		202	0.7523	N				N	
18	199.5	-0.2667	N				N		199.5	-0.2667	N				N	
19	206.5	2.5864	Y	197.33	201	1.0152	N	N	206.5	2.5864	N			Y		
20	202	0.7523	N				N		202	0.7523	N				N	
21	201.5	0.5485	N				N		201.5	0.5485	N				N	
22	199.5	-0.2667	N				N		199.5	-0.2667	N				N	
23	198	-0.8780	N				N		198	-0.8780	N				N	
24	199	-0.4705	N				N		199	-0.4705	N				N	
25	200	-0.0629	N				N		200	-0.0629	N				N	
26	206	2.3826	Y	200.67	202.8	3.1760	N	Y	206	2.3826	Y	200.67	202.8	3.1760	N	Y
27	203.5	1.3637	N				N		203.5	1.3637	N				N	
28	200	-0.0629	N				N		200	-0.0629	N				N	
29	198.5	-0.6743	N				N		198.5	-0.6743	N				N	
30	200	-0.0629	N				N		200	-0.0629	N				N	
31	200	-0.0629	N				N		200	-0.0629	N				N	
32	195.5	-1.8970	N				N		195.5	-1.8970	N				N	
33	200.5	0.1409	N				N		200.5	0.1409	N				N	
34	199	-0.4705	N				N		199	-0.4705	N				N	
35	202	0.7523	N				N		202	0.7523	N				N	

Note: N = No and Y = Yes.

The proposed SSDS \bar{X} scheme moves for the first time to Stage 2 at the 16th sampling time. At this sampling time, Z_{16} is equal to -0.425. Since $Z_{16} \in (-L_2, L_2) = (-2.305, 2.305)$, the proposed SSDS \bar{X} scheme does not give a signal on the 16th sampling time. However, on the 19th sampling time, Z_{1i} (with, $i = 19$) equal to 2.5864 plots above $L_2 = 2.305$. Therefore, the SSDS \bar{X} scheme give a signal for the first time at the 19th sampling time in Stage 1 (see Figure 3(b) and Table 7).

This example shows that the proposed SSDS \bar{X} scheme is more sensitive than the existing NSSDS \bar{X} scheme in monitoring Phase II samples when the unknown design parameters are estimated from an IC Phase I sample. Therefore, practitioners in the industrial and non-industrial environments are advised to use the proposed SSDS \bar{X} monitoring scheme instead of the existing NSSDS \bar{X} scheme regardless of the size of the shift to be detected.

7. Synthetic SSDS scheme when parameters are unknown

The SSDS \bar{X} control chart can be further extended by adding a synthetic scheme, i.e. by adding a CRL (conforming run-length) scheme with a single integer control limit $H \geq 1$. These schemes were originally proposed in Wu and Spedding (2000) and more recently reviewed in Rakitzis et al. (2019). Khoo et al. (2011) were the first to integrate the operation of a NSSDS scheme with the CRL scheme using the regions given in Figure 4(a) to formulate the non-side-sensitive (NSS) synthetic double sampling (DS) scheme. The latter scheme was also discussed in You (2017) from a different perspective. Costa and Machado (2015) and Malela-Majika and Rapoo (2019) proposed the side-sensitive versions of Khoo et al. (2011) scheme that had an improved performance. These side-sensitive versions are called the standard, revised and modified side-sensitive (denoted by SSS, RSS and MSS) synthetic DS schemes – these are shown in Figures 4(b) and (c), respectively. For Case U, the NSS synthetic DS scheme is studied in You et al. (2015) and You (2018). In this section, we briefly focus on the MSS synthetic DS scheme.

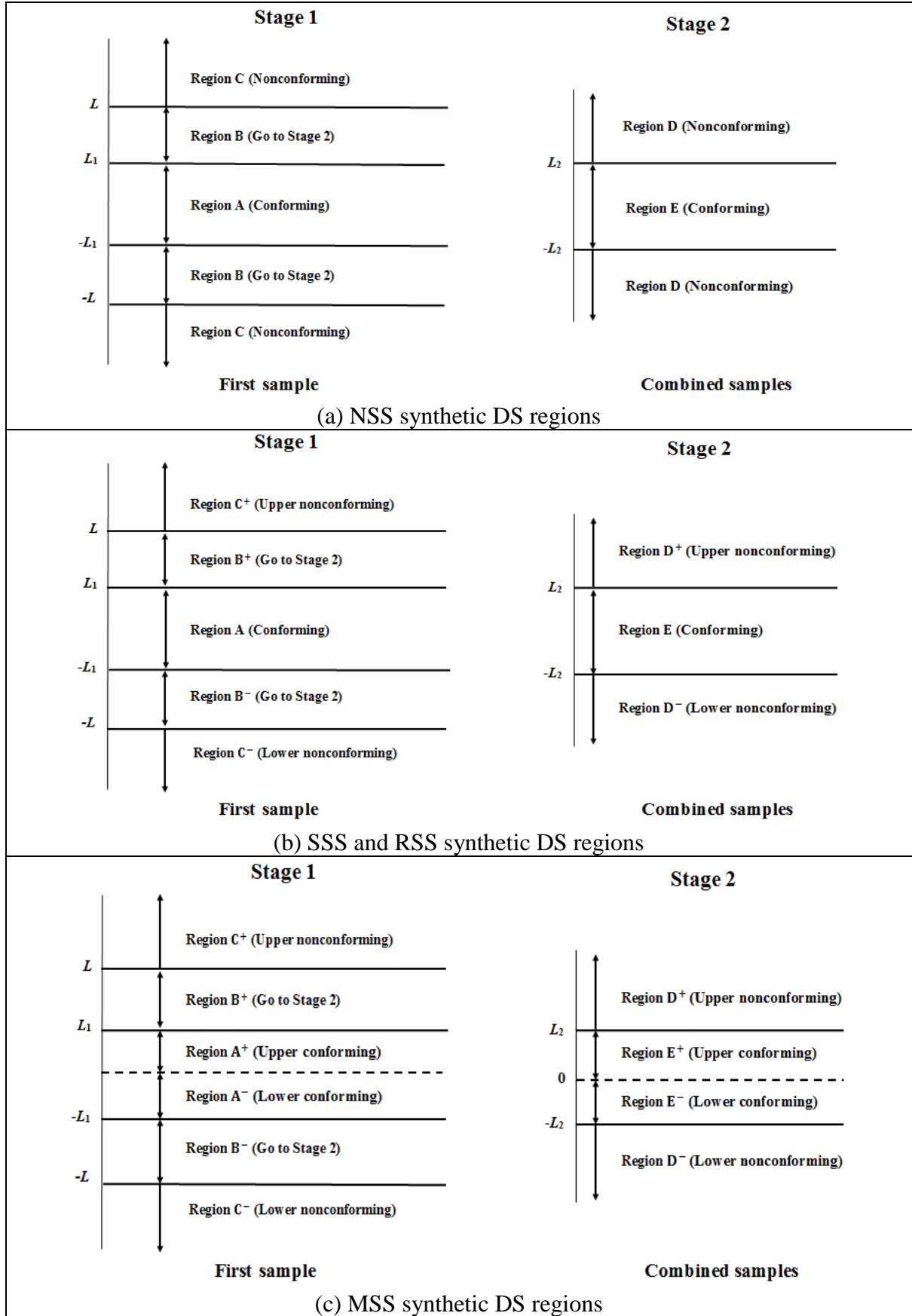


Figure 4: The charting regions in stages 1 and 2 of the synthetic DS scheme

While the basic double sampling schemes have two main designs for the charting regions (i.e. NSSDS and SSDS) and three design parameters (i.e. L, L_1, L_2); however, the synthetic DS schemes have four main designs for the charting regions (i.e. NSS, SSS, RSS and MSS) and four design parameters (i.e. L, L_1, L_2, H). Moreover, the synthetic DS schemes do not issue an OOC signal at the first sample point that falls on the nonconforming regions (i.e., the ‘OOC regions’ in Figure 1). That is, the process waits until a second sample point falls on the nonconforming region and, if these two nonconforming samples are relatively close to each other (say, $CRL \leq H$), then an OOC signal is triggered. The charting regions are divided into ten intervals – see Figure 4(c), i.e. $C^+ = [L, +\infty)$, $B^+ = [L_1, L)$, $A^+ = [0, L_1)$, $A^- = (-L_1, 0]$, $B^- = (-L_2, -L_1]$, $C^- = (-\infty, -L]$, $D^+ = [L_2, +\infty)$, $E^+ = [0, L_2)$, $E^- = (-L_2, 0]$ and $D^- = (-\infty, -L_2]$.

7.1 Operation of the MSS synthetic DS scheme

The Phase I implementation of the MSS synthetic DS scheme is the same that discussed above for the SSDS scheme; however, the Phase II operation is as follows:

1. Set the optimal design parameters ($m, n_1, n_2, L_1, L_2, L, H$).
2. Take a sample of size n_1 and calculate the standardised statistic using Equation (13) at the i^{th} sampling time of the first sample.
3. (a) If $\hat{Z}_{1i} \in A^+$, the i^{th} sample is an upper conforming, hence return to Step 2.
(b) If $\hat{Z}_{1i} \in A^-$, the i^{th} sample is a lower conforming, hence return to Step 2.
4. (a) If $\hat{Z}_{1i} \in C^+$, the i^{th} sample is an upper nonconforming; then go to Step 7(a).
(b) If $\hat{Z}_{1i} \in C^-$, the i^{th} sample is a lower nonconforming; then go to Step 7(b).
5. If $\hat{Z}_{1i} \in B^+$ or if $\hat{Z}_{1i} \in B^-$, take a second sample of size n_2 ($n_2 > n_1$) and calculate the standardised statistic using Equation (14) at the i^{th} sampling time at Stage 2.
6. (a) If $\hat{Z}_{1i} \in B^+$ and $\hat{Z}_i \in D^+$, the i^{th} sample is an upper nonconforming; then go to Step 7(a).
(b) If $\hat{Z}_{1i} \in B^-$ and $\hat{Z}_i \in D^-$, the i^{th} sample is a lower nonconforming; then go to Step 7(b). Otherwise, return to Step 2.
7. (a) Count the number of upper conforming samples between two consecutive upper nonconforming samples including the second upper nonconforming sample (i.e., denote this as CRL_U^{MSS}).

- (b) Count the number of lower conforming samples between two consecutive lower nonconforming samples including the present lower nonconforming sample (i.e., denote this as CRL_L^{MSS}).
8. If $CRL_U^{MSS} \leq H$ (or $CRL_L^{MSS} \leq H$), the process is OOC; then go to Step 9. Otherwise, the process is IC, hence return to Step 2.
9. Identify and remove the assignable cause(s); then go to Step 2.

7.2 Run-length properties of the MSS synthetic DS scheme

In order to obtain the run-length properties of the MSS synthetic DS \bar{X} scheme, the Markov chain matrix similar in principle to that proposed in Malela-Majika and Rapoo (2019) can be used. This method consist of defining a transition probability matrix (TPM) which is denoted as \mathbf{P} , with $2H + 1$ transient states and one absorbing state. Therefore, the TPM of the MSS synthetic DS \bar{X} scheme has the following structure:

$$\mathbf{P} = \begin{bmatrix} \mathbf{Q} & \mathbf{r} \\ \mathbf{0}^T & 1 \end{bmatrix} = \left[\begin{array}{cccccccccccc|c} 0 & 0 & \cdots & 0 & 0 & p_0 & p_{1L} & 0 & \cdots & 0 & 0 & p_{1U} \\ p_{0U} & 0 & \cdots & 0 & 0 & p_{0L} & p_{1L} & 0 & \cdots & 0 & 0 & p_{1U} \\ 0 & p_{0U} & \cdots & 0 & 0 & p_{0L} & p_{1L} & 0 & \cdots & 0 & 0 & p_{1U} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & p_{0U} & 0 & p_{0L} & p_{1L} & 0 & \cdots & 0 & 0 & p_{1U} \\ 0 & 0 & \cdots & 0 & p_{1U} & p_0 & p_{1L} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & p_{1U} & p_{0U} & 0 & p_{0L} & \cdots & 0 & 0 & p_{1L} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & p_{1U} & p_{0U} & 0 & 0 & \cdots & p_{0L} & 0 & p_{1L} \\ 0 & 0 & \cdots & 0 & p_{1U} & p_{0U} & 0 & 0 & \cdots & 0 & p_{0L} & p_{1L} \\ 0 & 0 & \cdots & 0 & p_{1U} & p_0 & 0 & 0 & \cdots & 0 & 0 & p_{1L} \\ \hline 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{array} \right] \quad (34)$$

where $\mathbf{0} = (0 \ 0 \ \dots 0)^T$, \mathbf{Q} is the $(2H + 1) \times (2H + 1)$ matrix of transient probabilities, \mathbf{r} is a $(2H + 1) \times 1$ vector that satisfies $\mathbf{r} = \mathbf{1} - \mathbf{Q}\mathbf{1}$ with $\mathbf{1} = (1 \ 1 \ \dots 1)^T$ and the elements of the TPM are defined as:

$$\begin{aligned} p_{0L} &= \Phi(\delta\sqrt{n_1}) - \Phi(-L_1 + \delta\sqrt{n_1}) + \int_{A^{*-}} \Phi\left(rc\delta - z\sqrt{\frac{n_1}{n_2}}\right) \phi(z)dz, \\ p_{0U} &= \Phi(L_1 + \delta\sqrt{n_1}) - \Phi(\delta\sqrt{n_1}) + \int_{A^{**}} \left[\Phi\left(cL_2 + rc\delta - z\sqrt{\frac{n_1}{n_2}}\right) - \Phi\left(rc\delta - z\sqrt{\frac{n_1}{n_2}}\right) \right] \phi(z)dz, \\ p_{1L} &= \Phi(-L + \delta\sqrt{n_1}) + \int_{B^{*-}} \Phi\left(-cL_2 + rc\delta - z\sqrt{\frac{n_1}{n_2}}\right) \phi(z)dz, \end{aligned}$$

$$p_{1U} = 1 - \Phi(L + \delta\sqrt{n_1}) + \int_{B^{**}} \left[1 - \Phi\left(cL_2 + rc\delta - z\sqrt{\frac{n_1}{n_2}}\right) \right] \phi(z) dz,$$

$$p_0 = p_{0L} + p_{0U},$$

where $A^{*-} = (-L_1 + \delta\sqrt{n_1}, 0]$, $A^{*+} = [\delta\sqrt{n_1}, L_1 + \delta\sqrt{n_1})$, $B^{*-} = [-L + \delta\sqrt{n_1}, -L_1 + \delta\sqrt{n_1})$ and $B^{*+} = (L_1 + \delta\sqrt{n_1}, L + \delta\sqrt{n_1}]$.

Thus, the zero-state c.d.f. for calculating the percentile run-length is given by

$$F_{RL}(\ell) = 1 - \int_{-\infty}^{+\infty} \int_0^{+\infty} (\mathbf{q}^T \mathbf{Q}^\ell \mathbf{1}) f_U(u) f_V(v) dv du, \quad (35)$$

where $\mathbf{q}^T = (0, \dots, 0, 1, 0, \dots, 0)$ with the unique “1” located at the $(H + 1)^{\text{th}}$ position of the TPM, $\ell \in \{1, 2, 3, \dots\}$, $f_U(u)$ and $f_V(v)$ are defined in Equations (17) and (18). Consequently, the unconditional *ARL* and *SDRL* of the MSS synthetic DS scheme are given by

$$ARL = \int_{-\infty}^{+\infty} \int_0^{+\infty} \mathbf{q}^T (\mathbf{I} - \mathbf{Q})^{-1} \mathbf{1} f_U(u) f_V(v) dv du, \quad (36)$$

and

$$SDRL = \int_{-\infty}^{+\infty} \int_0^{+\infty} [2\mathbf{q}^T (\mathbf{I} - \mathbf{Q})^{-2} \mathbf{Q} \mathbf{1} - ARL^2 + ARL]^{\frac{1}{2}} f_U(u) f_V(v) dv du,$$

respectively, where \mathbf{I} is a $(2H + 1) \times (2H + 1)$ identity matrix. For more details on the above discussion, readers are referred to Davis and Woodall (2002), Zhang et al. (2011), You et al. (2015), You (2018), Shongwe and Graham (2018), Rakitzis et al. (2019) and Malela-Majika and Rapoo (2019).

7.3 Performance comparison

In this section, the performance of the MSS synthetic DS \bar{X} scheme is briefly investigated and compared to one of the proposed SSDS \bar{X} scheme. Table 8 shows that the sensitivity of the MSS synthetic DS \bar{X} scheme depends on the Phase I sample size (i.e., m) and the CRL scheme control limit (i.e., H). The larger the value of m , the more sensitive the MSS synthetic DS \bar{X} scheme is. Moreover, the larger the design parameter H , the more sensitive the MSS synthetic DS \bar{X} scheme becomes.

Table 8. Performance comparison between the SSDS \bar{X} and MSS synthetic \bar{X} schemes when $n = 5$, $m \in \{50, 100\}$, $H \in \{1, 2\}$ for a nominal $ARL_0 = 370.4$

(ARL, SDRL) (P5, P25, P50, P75, P95)						
Shift(δ)	$m = 50$			$m = 100$		
	SSDS	MSS Synthetic DS: ($H = 1$)	MSS Synthetic DS: ($H = 2$)	SSDS	MSS Synthetic DS: ($H = 1$)	MSS Synthetic DS: ($H = 2$)
0.00	(370.4, 369.67)	(370.76, 430.96)	(369.72, 452.90)	(370.4, 370.85)	(369.09, 412.76)	(370.94, 427.14)
	(20, 108, 255, 513, 1103)	(2, 62, 227, 530, 1230)	(2, 51, 213, 523, 1270)	(20, 108, 258, 500, 1112)	(2, 69, 237, 527, 1195)	(2, 61, 230, 532, 1227)
0.10	(349.72, 344.22)	(299.01, 357.95)	(298.03, 372.74)	(280.74, 279.53)	(284.66, 325.68)	(278.88, 317.70)
	(23, 133, 312, 618, 1342)	(2, 45, 176, 422, 1013)	(2, 35, 168, 422, 1055)	(19, 109, 263, 533, 1128)	(2, 52, 183, 416, 965)	(2, 44, 175, 406, 941)
0.20	(329.95, 328.82)	(189.20, 233.80)	(187.46, 253.50)	(222.82, 222.43)	(177.47, 207.22)	(168.32, 205.79)
	(17, 93, 226, 460, 1000)	(2, 24, 108, 269, 654)	(2, 18, 97, 258, 680)	(12, 65, 154, 309, 662)	(2, 27, 107, 254.600)	(2, 19, 97, 238, 587)
0.30	(184.41, 188.05)	(101.92, 134.13)	(99.44, 142.18)	(110.50, 109.24)	(88.89, 109.26)	(85.69, 109.62)
	(11, 52, 125, 252, 566)	(1, 10, 52, 141, 370)	(1, 8, 46, 134, 372)	(6, 33, 79, 153, 325)	(1, 10, 51, 126, 311)	(1, 8, 45, 121, 307)
0.40	(96.62, 95.25)	(53.60, 75.29)	(51.13, 78.08)	(55.06, 54.33)	(44.16, 56.79)	(41.85, 56.13)
	(6, 29, 68, 133, 287)	(1, 5, 25, 72, 200)	(1, 5, 20, 66, 199)	(3, 16, 39, 76, 164)	(1, 5, 23, 62, 159)	(1, 5, 19, 58, 157)
0.50	(49.86, 48.63)	(28.40, 40.23)	(27.17, 43.52)	(29.09, 29.01)	(23.15, 30.03)	(21.89, 30.05)
	(3, 15, 35, 69, 145)	(1, 3, 13, 37, 108)	(1, 3, 11, 34, 108)	(2, 9, 20, 40, 87)	(1, 3, 11, 32, 84)	(1, 3, 10, 30, 84)
0.60	(27.34, 27.00)	(16.04, 23.03)	(15.16, 24.05)	(16.68, 16.30)	(13.31, 16.78)	(12.37, 16.13)
	(2, 8, 19, 38, 81)	(1, 2, 7, 21, 59)	(1, 2, 7, 19, 59)	(1, 5, 12, 23, 49)	(1, 2, 7, 18, 48)	(1, 2, 6, 16, 47)
0.70	(15.95, 15.41)	(9.91, 13.83)	(9.22, 13.21)	(9.92, 9.48)	(8.46, 10.28)	(8.08, 10.06)
	(1, 5, 11, 22, 47)	(1, 2, 5, 12, 37)	(1, 2, 4, 11, 35)	(1, 3, 7, 14, 28)	(1, 2, 5, 11, 29)	(1, 2, 4, 11, 28)
0.80	(9.62, 9.07)	(6.55, 8.43)	(6.10, 8.23)	(6.27, 5.75)	(5.66, 6.47)	(5.65, 6.43)
	(1, 3, 7, 13, 27)	(1, 1, 3, 8, 23)	(1, 1, 3, 8, 23)	(1, 2, 4, 9, 18)	(1, 1, 3, 7, 18)	(1, 1, 3, 7, 19)
0.90	(6.29, 5.79)	(4.68, 5.61)	(4.30, 5.64)	(4.99, 2.42)	(4.19, 4.51)	(4.20, 4.64)
	(1, 2, 4, 8, 18)	(1, 1, 3, 6, 15)	(1, 1, 2, 6, 15)	(1, 1, 2, 4, 8)	(1, 1, 3, 5, 13)	(1, 1, 2, 5, 13)
1.00	(4.18, 3.60)	(3.56, 4.04)	(3.24, 3.65)	(4.23, 3.62)	(3.20, 3.27)	(3.21, 3.03)
	(1, 2, 3, 6, 11)	(1, 1, 2, 4, 11)	(1, 1, 2, 4, 11)	(1, 2, 3, 6, 11)	(1, 1, 2, 4, 10)	(1, 1, 2, 4, 10)
1.20	(2.28, 1.72)	(2.25, 2.16)	(2.05, 2.01)	(1.83, 1.24)	(2.19, 1.63)	(2.13, 1.66)
	(1, 1, 2, 3, 6)	(1, 1, 1, 3, 7)	(1, 1, 1, 2, 6)	(1, 1, 1, 2, 4)	(1, 1, 1, 3, 6)	(1, 1, 1, 2, 6)
1.40	(1.54, 0.90)	(1.55, 1.14)	(1.50, 1.09)	(1.34, 0.68)	(1.60, 1.15)	(1.61, 1.14)
	(1, 1, 1, 2, 3)	(1, 1, 1, 2, 4)	(1, 1, 1, 2, 4)	(1, 1, 1, 2, 3)	(1, 1, 1, 2, 4)	(1, 1, 1, 2, 4)
1.60	(1.20, 0.49)	(1.27, 0.72)	(1.21, 0.65)	(1.12, 0.37)	(1.31, 0.73)	(1.31, 0.71)
	(1, 1, 1, 1, 2)	(1, 1, 1, 1, 3)	(1, 1, 1, 1, 3)	(1, 1, 1, 1, 2)	(1, 1, 1, 1, 3)	(1, 1, 1, 1, 3)
1.80	(1.08, 0.29)	(1.10, 0.42)	(1.07, 0.38)	(1.04, 0.21)	(1.16, 0.48)	(1.16, 0.46)
	(1, 1, 1, 1, 2)	(1, 1, 1, 1, 2)	(1, 1, 1, 1, 2)	(1, 1, 1, 1, 1)	(1, 1, 1, 1, 2)	(1, 1, 1, 1, 2)
2.00	(1.02, 0.15)	(1.04, 0.26)	(1.04, 0.22)	(1.02, 0.16)	(1.08, 0.31)	(1.09, 0.32)
	(1, 1, 1, 1, 1)	(1, 1, 1, 1, 2)	(1, 1, 1, 1, 2)	(1, 1, 1, 1, 1)	(1, 1, 1, 1, 2)	(1, 1, 1, 1, 2)
2.50	(1.00, 0.02)	(1.01, 0.11)	(1.00, 0.04)	(1.00, 0.01)	(1.01, 0.10)	(1.01, 0.11)
	(1, 1, 1, 1, 1)	(1, 1, 1, 1, 1)	(1, 1, 1, 1, 1)	(1, 1, 1, 1, 1)	(1, 1, 1, 1, 1)	(1, 1, 1, 1, 1)
AEQL	62.32	46.81	44.9	46.82	42.52	41.43
(L_1, L)	(0.8018, 3.1071)	(1.9326, 1.9599)	(1.9534, 2.0948)	(0.8093, 3.4354)	(1.9173, 1.9639)	(2.0112, 2.0583)
L_2	3.2650	2.4193	2.4987	3.1721	2.3714	2.4304

From Table 8, it can also be seen that the MSS synthetic DS \bar{X} scheme outperforms the proposed SSDS \bar{X} scheme for small shifts (i.e. $0 < \delta \leq 0.75$) and moderate shifts (i.e. $0.75 < \delta < 1.2$) in the process mean for the given H values. However, when $H = 1$, for moderately large shifts (i.e. $1.2 \leq \delta < 1.5$), the proposed SSDS \bar{X} monitoring scheme is slightly better than the MSS synthetic DS \bar{X} scheme; whereas, when $H = 2$, the two monitoring schemes are similar in performance. For large shifts in the process mean, i.e. $\delta > 1.5$, the two schemes are also similar in performance when $H \in \{1, 2\}$. It can also be observed that in terms of the *AEQL* values (i.e. overall performance), the MSS synthetic DS \bar{X} scheme performs better than the SSDS \bar{X} scheme regardless of the magnitude of H . In terms of the *PRL* values, the MSS synthetic DS \bar{X} scheme performs better than the proposed SSDS \bar{X} scheme for both small and moderate shifts. For large shifts, the two monitoring schemes perform almost similarly in detecting OOC signals in the process mean.

8. Conclusion and recommendations

In this paper, a SSDS \bar{X} monitoring scheme is proposed under the assumption of unknown process parameters. The performance of the proposed scheme is investigated in terms of the different characteristics of the run-length distribution as well as the *ANOS*, *ASS* and *AEQL* values. In terms of the *AEQL*, *PCI* and *ARARL* values, the proposed Case U SSDS \bar{X} scheme outperforms the competing schemes considered in this paper in many cases. Moreover, in terms of the *ASS* and *ANOS* values, the proposed monitoring scheme is found to be cost effective and sensitive compared to the competing schemes considered in this paper. Practitioners in the industrial and non-industrial environments, are recommended that when the parameters are unknown and need to be estimated, the reference sample size m must be sufficiently large (at least $m = 100$) in order to get more accurate estimates and stability in the performance of the proposed monitoring scheme as for such large Phase I sample sizes, the resulting performance is closer to the parameters known scenario. Finally, it is briefly shown that the synthetic version of the proposed scheme (i.e. MSS synthetic DS scheme) has a relatively better performance than the SSDS scheme; hence this indicate that the double sampling procedure combined with other procedures like synthetic schemes (as well as, say, variable sampling size and interval (VSSI), group-runs, etc.,) need to be thoroughly investigated to assess the extent that these enhancement techniques can improve the performance of double sampling schemes in Case U. Since the assumption of normality is often violated in practice, researchers are also recommended to design the proposed monitoring scheme for non-normal data. Moreover, this

study assumes that observations are from i.i.d. subgroups, it is equally important to investigate the scenario when the subgroup samples are autocorrelated.

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References

- Carot, V., J.M. Jabaloyes, and T. Carot. 2002. Combined double sampling and variable sampling interval chart. *International Journal of Production Research*, 40(9):2175-2186.
- Castagliola, P., P.C. Oprime, and M.B.C. Khoo. 2017. The double sampling S^2 chart with estimated process variance. *Communications in Statistics - Theory and Methods*; 46(7):3556-3573.
- Chong, N.L., M.B.C. Khoo, Z.L. Chong, and W.L. Teoh. 2018. A study on the run length properties of the side sensitive group runs double sampling (SSGRDS) control chart. *MATEC Web of Conferences*. DOI: 10.1051/mateconf/201819201005.
- Costa, A.F.B., and M.A.G. Machado. 2015. The steady-state behavior of the synthetic and side-sensitive double sampling \bar{X} charts. *Quality and Reliability Engineering International*, 31(2):297-303.
- Costa, A.F.B. 1994. \bar{X} charts with variable sampling size. *Journal of Quality Technology*, 26(3):155-163.
- Costa, A.F.B. 2017. The double sampling range chart. *Quality and Reliability Engineering International*, 33(8):2739-2745.
- Daudin, J.J. 1992. Double Sampling \bar{X} charts. *Journal of Quality Technology*, 24(2):78-87.
- Davis, R.B., and W.H. Woodall. 2002. Evaluating and improving the synthetic control chart. *Journal of Quality Technology* 34(2):200-208.
- Haq, A., and M.B.C. Khoo. 2018. A new double sampling control chart for monitoring process mean using auxiliary information. *Journal of Statistical Computation and Simulation*, 88(5):869-899.
- Haq, A., and M.B.C. Khoo. 2019. A synthetic double sampling control chart for process mean using auxiliary information. *Quality and Reliability Engineering International*. 35(6): 1803-1825.

- He, D., A. Grigoryan, and M. Sigh. 2002. Design of double-and triple-sampling \bar{X} control charts using genetic algorithms. *International Journal of Production Research*, 40(6):1387-1404.
- Irianto, D., and N. Shinozaki. 1998. An optimal double sampling \bar{X} control chart. *International Journal of Industrial Engineering— Theory, Applications and Practice*; 5(3), 226-234.
- Jensen, W.A., L.A. Jones-Farmer, C.W. Champ, and W.H. Woodall. 2006. Effects of parameter estimation on control chart properties: A literature review. *Journal of Quality Technology*, 38(4):349-364.
- Jones L.A., C.W. Champ, and S.E. Rigdon. 2004. The run length distribution of the CUSUM with estimated parameters. *Journal of Quality Technology*, 36(1):95-108.
- Khoo, M.B.C., H.C. Lee, Z. Wu, C.H. Chen and P. Castagliola. 2011. A synthetic double sampling control chart for the process mean. *IIE Transactions*, 43(1):23-38.
- Khoo, M.B.C., W.L. Teoh, P. Castagliola, and M.H. Lee. 2013b. Optimal designs of the double sampling chart with estimated parameters. *International Journal of Production Economics*, 144(1):345-357.
- Khoo, M.B.C., Z. Wu, P. Castagliola, and H.C. Lee. 2013a. A multivariate synthetic double sampling T^2 control chart. *Computers and Industrial Engineering*, 64(1):179-189.
- Lee, P.H., C.C. Torng, and L.F. Liao. 2012. An economic design of combined double sampling and variable sampling interval \bar{X} control chart. *International Journal of Production Economics*, 138(1):102-106.
- Lee, M.H., and M.B.C. Khoo. 2019. Double sampling np chart with estimated process parameter. *Communications in Statistics - Simulation and Computation*. DOI: 10.1080/03610918.2019. 1599017.
- Machado, M.A.G., and A.F.B. Costa. 2014. A side-sensitive synthetic chart combined with an \bar{X} chart. *International Journal of Production Research*, 52(11):3404-3416.
- Malela-Majika, J.C. 2019. Modified side-sensitive synthetic double sampling monitoring scheme for simultaneously monitoring the process mean and variability. *Computers and Industrial Engineering*, 130:798-814.
- Malela-Majika, J.C., and E.M. Rapoo. 2019. Side-sensitive synthetic double sampling \bar{X} control charts. *European Journal of Industrial Engineering*, 13(1):117-148.
- Malela-Majika, J.C., C.M. Motsepa, and M.A. Graham. 2019. A new double sampling \bar{X} control chart for monitoring an abrupt change in the process location; *Communications in Statistics-Simulation and Computation*. DOI: 10.1080/03610918.2019.1577970.

- Psarakis, S., A. Vyniou, and P. Castagliola. 2013. Some recent developments on the effects of parameter estimation on control charts. *Quality and Reliability Engineering International*, 30(8):1113-1129.
- Rakitzis, A.C., S. Chakraborti, S.C. Shongwe, M.A. Graham, and M.B.C. Khoo. 2019. An overview of synthetic-type control charts: Techniques and Methodology. *Quality and Reliability Engineering International*, DOI: 10.1002/qre.2491.
- Reynolds, M.R. Jr, and J. Lou. 2010. An evaluation of GLR control chart combined with \bar{X} chart. *Journal of Quality Technology*, 42(3):287-310.
- Ryu, J.H., H. Wan, and S. Kim. 2010. Optimal design of a CUSUM chart for a mean shift of unknown size. *Journal of Quality Technology*, 42(3):311-326.
- Shongwe, S.C., and M.A. Graham. 2018. A modified side-sensitive synthetic chart to monitor the process mean. *Quality Technology and Quantitative Management* 15(3):328-353.
- Shongwe, S.C., J.-C. Malela-Majika, and E.M. Rapoo. 2019. One-sided and two-sided w -of- w runs-rules schemes: An overall performance perspective and the unified run-length derivations. *Journal of Probability and Statistics*, Article ID: 6187060, 20pages. DOI: 10.1155/2019/6187060.
- Teoh, W.L., M.S. Fun, S.Y. Teh, M.B.C. Khoo, and W.C. Yeong. 2016a. Exact run length distribution of the double sampling \bar{X} chart with estimated process parameters, *South African Journal of Industrial Engineering*, 27(1):20-31.
- Teoh, W.L., M.B.C. Khoo, P. Castagliola, and S. Chakraborti. 2015. A median run length-based double-sampling \bar{X} chart with estimated parameters for minimizing the average sample size. *International Journal of Advanced Manufacturing Technology*, 80(1-4):411-426.
- Teoh, W.L., M.B.C. Khoo, P. Castagliola, and S. Chakraborti. 2014. Optimal design of the double sampling \bar{X} chart with estimated parameters based on median run length. *Computers and Industrial Engineering*; 67(1):104-115.
- Teoh, W.L., M.B.C. Khoo, S.Y. Teh. 2013. Optimal designs of the median run length based double sampling \bar{X} chart for minimizing the average sample size. *PLoS ONE*, 8(7):e68580. doi:10.1371/journal.pone.0068580.
- Teoh, W.L., W.C. Yeong, M.B.C. Khoo, and S.Y. Teh. 2016b. The performance of the double sampling \bar{X} chart with estimated parameters for skewed distributions. *Academic Journal of Science*, 5(1):237-252.

- Wu, Z., W. Yang, W. Jiang, and M.B.C. Khoo. 2008. Optimisation designs of the combined Shewhart-CUSUM control charts, *Computational Statistics & Data Analysis*, 53(2):496-506.
- Wu, Z., and T.A. Spedding. 2000. A synthetic control chart for detecting small shifts in the process mean. *Journal of Quality Technology* 32(1):32-38.
- You, H.W. 2017. Run length distribution of synthetic double sampling chart. *International Journal of Applied Engineering Research*, 12(24):14268-14271.
- You, H.W. 2018. Performance of synthetic double sampling chart with estimated parameters based on expected average run length. *Journal of Probability and Statistic*, Article ID: 7583610, 6pages. DOI: 10.1155/2018/7583610.
- You, H.W., M.B.C. Khoo, M.H. Lee, and P. Castagliola. 2016. Synthetic double sampling \bar{X} chart with estimated process parameters. *Quality Technology and Quantitative Management*, 12(4):579–604.
- Zaman, B., M.H. Lee, M. Riaz, and M.R. Abujiya. 2017. An adaptive EWMA scheme-based CUSUM accumulation error for efficient monitoring of process location. *Quality and Reliability Engineering International*, 33(8):2463-2482.
- Zhang, Y., P. Castagliola, Z. Wu, and M.B.C. Khoo. 2011. The synthetic \bar{X} chart with estimated parameters. *IIE Transactions*, 43(9):676-687.