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Methods of free probability

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ABSTRACT. This is a joint introduction to classical and free probability, which are twin sisters. We first review the foundations of classical probability, notably with the main limiting theorems (CLT, CCLT, PLT, CPLT), and with a look into examples coming from Lie groups and random matrices. Then we present the foundations and main results of free probability, notably with free limiting theorems, and with a look into examples coming from quantum groups and random matrices. We discuss then a number of more advanced aspects, in relation with free geometry and with subfactor theory.

Preface

Probability theory, and the probabilistic way of thinking, have seen a considerable surge in the last years, with virtually every single branch of mathematics being affected. It goes without saying that everything mathematics coming from quantum mechanics, which actually accounts for a big part of pure mathematics as we know it, has some probability behind, and this has become more and more visible during recent years. The same goes of course for statistical mechanics, once again somehow by definition. As in what regards classical mechanics, randomness of the initial data is certainly a very fruitful idea too. Finally, old branches of pure mathematics, such as number theory, are increasingly becoming more analytic, and more probabilistic too.

At the technical level, probability theory comes in many flavors. However, if there is one thing to be known, having interesting mathematics and physics behind, this is the fact that classical probability theory has a “twin sister”, namely free probability.

Free probability was introduced by Voiculescu in the mid 1980s, with motivation coming from general quantum mechanics, and more specifically with a number of operator algebra questions in mind. Among the main discoveries of Voiculescu was the fact that Wigner’s semicircle law, coming from advanced quantum physics and random matrices, appears as the “free analogue” of the normal law. This has led to a lot of interest in free probability, with the subject having now deep ties to operator algebras, random matrices, quantum groups, noncommutative geometry, and virtually any other branch of mathematics coming from quantum mechanics, or statistical mechanics.

This book is an introduction to free probability, with the aim of keeping things as simple and concrete as possible, while still being relatively complete. Our goals will be on one hand that of explaining the definition and main properties of free probability, in analogy with the definition and main properties of classical probability, and by keeping the presentation as elementary as possible, and on the other hand to go, at least a little bit, into each of the above-mentioned classes of examples and applications, namely operator algebras, random matrices, quantum groups and noncommutative geometry.

The first half of the book contains basic material, all beautiful and useful things, leading to free probability. Part I is concerned with classical probability, or rather with

selected topics from classical probability, which extend well to the free case. These include the standard classical limiting theorems (CLT, CCLT, PLT, CPLT), all done via the moment method and combinatorics, and then a discussion regarding Lie groups, and Weingarten calculus. Part II is an introduction to the random matrices, benefiting from the probability theory learned in Part I, and making a transition towards the free probability theory from Parts III-IV. The main results here are the classical limiting theorems of Wigner and Marchenko-Pastur, both done via the moment method and combinatorics, and with a look into the block-modified random matrices too.

The second half of the book is concerned with free probability itself, and applications. Part III deals with the definition and main properties of free probability, central here being, besides the foundations, the free analogues of the classical limiting theorems (CLT, CCLT, PLT, CPLT), following Voiculescu. Our approach is based on standard calculus and basic operator algebra theory, a bit in the spirit of the original book by Voiculescu, Dykema and Nica [90], but by attempting to make things a bit simpler, with the whole presentation meant to be as accessible to everyone as possible. Also, we will explain here the Bercovici-Pata bijection, and the block-modified random matrix models for the corresponding main free laws. As for Part IV, this deals with applications to quantum groups, noncommutative geometry, operator algebras and subfactors.

All in all, many things to be discussed. As a complement to what we will be doing here, for advanced combinatorics and operator algebra aspects you have [54], [71], [90], and for advanced random matrix theory you have [2], [29], [69]. So, in the hope that you will like free probability, and end up learning everything, from here and from [2], [29], [54], [69], [71], [90], with the precise order being more a matter of taste.

I learned myself free probability long ago, as a graduate student, from [90], with my first research paper being a 1996 note on the circular variables [6]. Later I started doing quantum groups, and some random matrices too, with free probability always in mind. I am grateful to Mireille Capitaine, Benoît Collins, Steve Curran, Ion Nechita, Roland Speicher and the others, for substantial joint work on the subject. Many thanks go as well to my cats. No serious science can be done without advice from a cat or two.

Cergy, August 2024

Teo Banica

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Part I

Classical probability

*The Magical Mystery Tour
Is coming to take you away
Coming to take you away
Take you today*

CHAPTER 1

Normal laws

1a. Probability theory

Generally speaking, probability theory is best learned by flipping coins and throwing dice. At a more advanced level, which is playing cards, we have:

THEOREM 1.1. *The probabilities at poker are as follows:*

- (1) *One pair:* 0.533.
- (2) *Two pairs:* 0.120.
- (3) *Three of a kind:* 0.053.
- (4) *Full house:* 0.006.
- (5) *Straight:* 0.005.
- (6) *Four of a kind:* 0.001.
- (7) *Flush:* 0.000.
- (8) *Straight flush:* 0.000.

PROOF. Let us consider indeed our deck of 32 cards, 7, 8, 9, 10, J , Q , K , A . The total number of possibilities for a poker hand is:

$$\binom{32}{5} = \frac{32 \cdot 31 \cdot 30 \cdot 29 \cdot 28}{2 \cdot 3 \cdot 4 \cdot 5} = 32 \cdot 31 \cdot 29 \cdot 7$$

(1) For having a pair, the number of possibilities is:

$$N = \binom{8}{1} \binom{4}{2} \times \binom{7}{3} \binom{4}{1}^3 = 8 \cdot 6 \cdot 35 \cdot 64$$

Thus, the probability of having a pair is:

$$P = \frac{8 \cdot 6 \cdot 35 \cdot 64}{32 \cdot 31 \cdot 29 \cdot 7} = \frac{6 \cdot 5 \cdot 16}{31 \cdot 29} = \frac{480}{899} = 0.533$$

(2) For having two pairs, the number of possibilities is:

$$N = \binom{8}{2} \binom{4}{2}^2 \times \binom{24}{1} = 28 \cdot 36 \cdot 24$$

Thus, the probability of having two pairs is:

$$P = \frac{28 \cdot 36 \cdot 24}{32 \cdot 31 \cdot 29 \cdot 7} = \frac{36 \cdot 3}{31 \cdot 29} = \frac{108}{899} = 0.120$$

(3) For having three of a kind, the number of possibilities is:

$$N = \binom{8}{1} \binom{4}{3} \times \binom{7}{2} \binom{4}{1}^2 = 8 \cdot 4 \cdot 21 \cdot 16$$

Thus, the probability of having three of a kind is:

$$P = \frac{8 \cdot 4 \cdot 21 \cdot 16}{32 \cdot 31 \cdot 29 \cdot 7} = \frac{3 \cdot 16}{31 \cdot 29} = \frac{48}{899} = 0.053$$

(4) For having full house, the number of possibilities is:

$$N = \binom{8}{1} \binom{4}{3} \times \binom{7}{1} \binom{4}{2} = 8 \cdot 4 \cdot 7 \cdot 6$$

Thus, the probability of having full house is:

$$P = \frac{8 \cdot 4 \cdot 7 \cdot 6}{32 \cdot 31 \cdot 29 \cdot 7} = \frac{6}{31 \cdot 29} = \frac{6}{899} = 0.006$$

(5) For having a straight, the number of possibilities is:

$$N = 4 \left[\binom{4}{1}^4 - 4 \right] = 16 \cdot 63$$

Thus, the probability of having a straight is:

$$P = \frac{16 \cdot 63}{32 \cdot 31 \cdot 29 \cdot 7} = \frac{9}{2 \cdot 31 \cdot 29} = \frac{9}{1798} = 0.005$$

(6) For having four of a kind, the number of possibilities is:

$$N = \binom{8}{1} \binom{4}{4} \times \binom{7}{1} \binom{4}{1} = 8 \cdot 7 \cdot 4$$

Thus, the probability of having four of a kind is:

$$P = \frac{8 \cdot 7 \cdot 4}{32 \cdot 31 \cdot 29 \cdot 7} = \frac{1}{31 \cdot 29} = \frac{1}{899} = 0.001$$

(7) For having a flush, the number of possibilities is:

$$N = 4 \left[\binom{8}{4} - 4 \right] = 4 \cdot 66$$

Thus, the probability of having a flush is:

$$P = \frac{4 \cdot 66}{32 \cdot 31 \cdot 29 \cdot 7} = \frac{33}{4 \cdot 31 \cdot 29 \cdot 7} = \frac{9}{25172} = 0.000$$

(8) For having a straight flush, the number of possibilities is:

$$N = 4 \cdot 4$$

Thus, the probability of having a straight flush is:

$$P = \frac{4 \cdot 4}{32 \cdot 31 \cdot 29 \cdot 7} = \frac{1}{2 \cdot 31 \cdot 29 \cdot 7} = \frac{1}{12586} = 0.000$$

Thus, we have obtained the numbers in the statement. \square

Summarizing, probability is basically about binomials and factorials, and ultimately about numbers. We will see later that, in connection with more advanced questions, of continuous nature, some standard calculus comes into play as well.

Let us discuss now the general theory. The fundamental result in probability is the Central Limit Theorem (CLT), and our first task will be that of explaining this. With the idea in mind of doing things a bit abstractly, our starting point will be:

DEFINITION 1.2. *Let X be a probability space, that is, a space with a probability measure, and with the corresponding integration denoted E , and called expectation.*

- (1) *The random variables are the real functions $f \in L^\infty(X)$.*
- (2) *The moments of such a variable are the numbers $M_k(f) = E(f^k)$.*
- (3) *The law of such a variable is the measure given by $M_k(f) = \int_{\mathbb{R}} x^k d\mu_f(x)$.*

Here the fact that μ_f exists indeed is well-known. By linearity, we would like to have a real probability measure making hold the following formula, for any $P \in \mathbb{R}[X]$:

$$E(P(f)) = \int_{\mathbb{R}} P(x) d\mu_f(x)$$

By using a standard continuity argument, it is enough to have this formula for the characteristic functions χ_I of the measurable sets of real numbers $I \subset \mathbb{R}$:

$$E(\chi_I(f)) = \int_{\mathbb{R}} \chi_I(x) d\mu_f(x)$$

But this latter formula, which reads $P(f \in I) = \mu_f(I)$, can serve as a definition for μ_f , and we are done. Alternatively, assuming some familiarity with measure theory, μ_f is the push-forward of the probability measure on X , via the function $f : X \rightarrow \mathbb{R}$.

Next in line, we need to talk about independence. We can do this as follows:

DEFINITION 1.3. *Two variables $f, g \in L^\infty(X)$ are called independent when*

$$E(f^k g^l) = E(f^k) E(g^l)$$

happens, for any $k, l \in \mathbb{N}$.

Again, this definition hides some non-trivial things. Indeed, by linearity, we would like to have a formula as follows, valid for any polynomials $P, Q \in \mathbb{R}[X]$:

$$E[P(f)Q(g)] = E[P(f)] E[Q(g)]$$

By using a continuity argument, it is enough to have this formula for characteristic functions χ_I, χ_J of the measurable sets of real numbers $I, J \subset \mathbb{R}$:

$$E[\chi_I(f)\chi_J(g)] = E[\chi_I(f)] E[\chi_J(g)]$$

Thus, we are led to the usual definition of independence, namely:

$$P(f \in I, g \in J) = P(f \in I) P(g \in J)$$

All this might seem a bit abstract, but in practice, the idea is of course that f, g must be independent, in an intuitive, real-life sense. As a first result now, we have:

PROPOSITION 1.4. *Assuming that $f, g \in L^\infty(X)$ are independent, we have*

$$\mu_{f+g} = \mu_f * \mu_g$$

where $*$ is the convolution of real probability measures.

PROOF. We have the following computation, using the independence of f, g :

$$\begin{aligned} M_k(f+g) &= E((f+g)^k) \\ &= \sum_r \binom{k}{r} E(f^r g^{k-r}) \\ &= \sum_r \binom{k}{r} M_r(f) M_{k-r}(g) \end{aligned}$$

On the other hand, by using the Fubini theorem, we have as well:

$$\begin{aligned} \int_{\mathbb{R}} x^k d(\mu_f * \mu_g)(x) &= \int_{\mathbb{R} \times \mathbb{R}} (x+y)^k d\mu_f(x) d\mu_g(y) \\ &= \sum_r \binom{k}{r} \int_{\mathbb{R}} x^r d\mu_f(x) \int_{\mathbb{R}} y^{k-r} d\mu_g(y) \\ &= \sum_r \binom{k}{r} M_r(f) M_{k-r}(g) \end{aligned}$$

Thus μ_{f+g} and $\mu_f * \mu_g$ have the same moments, so they coincide, as desired. \square

Here is now a second result on independence, which is something more advanced:

THEOREM 1.5. *Assuming that $f, g \in L^\infty(X)$ are independent, we have*

$$F_{f+g} = F_f F_g$$

where $F_f(x) = E(e^{ixf})$ is the Fourier transform.

PROOF. We have the following computation, using Proposition 1.4 and Fubini:

$$\begin{aligned}
 F_{f+g}(x) &= \int_{\mathbb{R}} e^{ixz} d\mu_{f+g}(z) \\
 &= \int_{\mathbb{R}} e^{ixz} d(\mu_f * \mu_g)(z) \\
 &= \int_{\mathbb{R} \times \mathbb{R}} e^{ix(z+t)} d\mu_f(z) d\mu_g(t) \\
 &= \int_{\mathbb{R}} e^{ixz} d\mu_f(z) \int_{\mathbb{R}} e^{ixt} d\mu_g(t) \\
 &= F_f(x) F_g(x)
 \end{aligned}$$

Thus, we are led to the conclusion in the statement. \square

This was for the foundations of probability theory, quickly explained. For further reading, a classical book is Feller [46]. A nice, more modern book is Durrett [43].

1b. Central limits

The main result in classical probability is the Central Limit Theorem (CLT), that we will explain now. Let us first discuss the normal distributions, that we will see later to appear as limiting laws in the CLT. We will need the following standard result:

THEOREM 1.6. *We have the following formula,*

$$\int_{\mathbb{R}} e^{-x^2} dx = \sqrt{\pi}$$

called Gauss integral formula.

PROOF. Let I be the integral in the statement. By using polar coordinates, namely $x = r \cos t$, $y = r \sin t$, with the corresponding Jacobian being r , we have:

$$\begin{aligned}
 I^2 &= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-x^2-y^2} dx dy \\
 &= \int_0^{2\pi} \int_0^{\infty} e^{-r^2} r dr dt \\
 &= 2\pi \int_0^{\infty} \left(-\frac{e^{-r^2}}{2} \right)' dr \\
 &= \pi
 \end{aligned}$$

Thus, we are led to the formula in the statement. \square

We can now introduce the normal distributions, as follows:

DEFINITION 1.7. *The normal law of parameter 1 is the following measure:*

$$g_1 = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

More generally, the normal law of parameter $t > 0$ is the following measure:

$$g_t = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} dx$$

These are also called Gaussian distributions, with “g” standing for Gauss.

The above laws are usually denoted $\mathcal{N}(0, 1)$ and $\mathcal{N}(0, t)$, but since we will be doing in this book all kinds of probability, we will use simplified notations for all our measures. Let us mention as well that the normal laws traditionally have 2 parameters, the mean and the variance, but here we will not need the mean, all our theory using centered laws. Finally, observe that the above laws have indeed mass 1, as they should, due to:

$$\int_{\mathbb{R}} e^{-x^2/2t} dx = \int_{\mathbb{R}} e^{-y^2} \sqrt{2t} dy = \sqrt{2\pi t}$$

Generally speaking, the normal laws appear as bit everywhere, in real life. The reasons for this come from the Central Limit Theorem (CLT), that we will explain in a moment, after developing some more general theory. As a first result, we have:

PROPOSITION 1.8. *We have the variance formula*

$$V(g_t) = t$$

valid for any $t > 0$.

PROOF. The first moment is 0, because our normal law g_t is centered. As for the second moment, this can be computed as follows:

$$\begin{aligned} M_2 &= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} x^2 e^{-x^2/2t} dx \\ &= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} (tx) \left(-e^{-x^2/2t} \right)' dx \\ &= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} t e^{-x^2/2t} dx \\ &= t \end{aligned}$$

We conclude from this that the variance is $V = M_2 = t$. □

Here is another result, which is widely useful in practice:

THEOREM 1.9. *We have the following formula, valid for any $t > 0$:*

$$F_{g_t}(x) = e^{-tx^2/2}$$

*In particular, the normal laws satisfy $g_s * g_t = g_{s+t}$, for any $s, t > 0$.*

PROOF. The Fourier transform formula can be established as follows:

$$\begin{aligned}
F_{g_t}(x) &= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} e^{-z^2/2t+ixz} dz \\
&= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} e^{-(z/\sqrt{2t}-\sqrt{t/2}iz)^2-tx^2/2} dz \\
&= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} e^{-y^2-tx^2/2} \sqrt{2t} dy \\
&= \frac{1}{\sqrt{\pi}} e^{-tx^2/2} \int_{\mathbb{R}} e^{-y^2} dy \\
&= e^{-tx^2/2}
\end{aligned}$$

As for $g_s * g_t = g_{s+t}$, this follows via Theorem 1.5, $\log F_{g_t}$ being linear in t . \square

We are now ready to state and prove the CLT, as follows:

THEOREM 1.10 (CLT). *Given real variables $f_1, f_2, f_3, \dots \in L^\infty(X)$ which are i.i.d., centered, and with common variance $t > 0$, we have*

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n f_i \sim g_t$$

with $n \rightarrow \infty$, in moments.

PROOF. In terms of moments, the Fourier transform is given by:

$$\begin{aligned}
F_f(x) &= E \left(\sum_{r=0}^{\infty} \frac{(ixf)^r}{r!} \right) \\
&= \sum_{r=0}^{\infty} \frac{(ix)^r E(f^r)}{r!} \\
&= \sum_{r=0}^{\infty} \frac{i^r M_r(f)}{r!} x^r
\end{aligned}$$

Thus, the Fourier transform of the variable in the statement is:

$$\begin{aligned}
F(x) &= \left[F_f \left(\frac{x}{\sqrt{n}} \right) \right]^n \\
&= \left[1 - \frac{tx^2}{2n} + O(n^{-2}) \right]^n \\
&\simeq e^{-tx^2/2}
\end{aligned}$$

But this function being the Fourier transform of g_t , we obtain the result. \square

Let us discuss now some further properties of the normal law. We first have:

PROPOSITION 1.11. *The even moments of the normal law are the numbers*

$$M_k(g_t) = t^{k/2} \times k!!$$

where $k!! = (k-1)(k-3)(k-5)\dots$, and the odd moments vanish.

PROOF. We have the following computation, valid for any integer $k \in \mathbb{N}$:

$$\begin{aligned} M_k &= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} y^k e^{-y^2/2t} dy \\ &= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} (ty^{k-1}) \left(-e^{-y^2/2t}\right)' dy \\ &= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} t(k-1)y^{k-2} e^{-y^2/2t} dy \\ &= t(k-1) \times \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} y^{k-2} e^{-y^2/2t} dy \\ &= t(k-1)M_{k-2} \end{aligned}$$

Now recall from the proof of Proposition 1.8 that we have $M_0 = 1$, $M_1 = 0$. Thus by recurrence, we are led to the formula in the statement. \square

We have the following alternative formulation of the above result:

PROPOSITION 1.12. *The moments of the normal law are the numbers*

$$M_k(g_t) = t^{k/2} |P_2(k)|$$

where $P_2(k)$ is the set of pairings of $\{1, \dots, k\}$.

PROOF. Let us count the pairings of $\{1, \dots, k\}$. In order to have such a pairing, we must pair 1 with one of the numbers $2, \dots, k$, and then use a pairing of the remaining $k-2$ numbers. Thus, we have the following recurrence formula:

$$|P_2(k)| = (k-1)|P_2(k-2)|$$

As for the initial data, this is $P_1 = 0$, $P_2 = 1$. Thus, we are led to the result. \square

We are not done yet, and here is one more improvement of the above:

THEOREM 1.13. *The moments of the normal law are the numbers*

$$M_k(g_t) = \sum_{\pi \in P_2(k)} t^{|\pi|}$$

where $P_2(k)$ is the set of pairings of $\{1, \dots, k\}$, and $|\cdot|$ is the number of blocks.

PROOF. This follows indeed from Proposition 1.12, because the number of blocks of a pairing of $\{1, \dots, k\}$ is trivially $k/2$, independently of the pairing. \square

We will see later in this book that many other interesting probability distributions are subject to similar formulae regarding their moments, involving partitions.

1c. Spherical integrals

In a purely mathematical context, the simplest way of recovering the normal laws is by looking at the coordinates over the real spheres $S_{\mathbb{R}}^{N-1}$, in the $N \rightarrow \infty$ limit. To start with, at $N = 2$ the sphere is the unit circle \mathbb{T} , and with $z = e^{it}$ the coordinates are $\cos t, \sin t$. Let us first integrate powers of these coordinates. We have here:

PROPOSITION 1.14. *We have the following formulae,*

$$\int_0^{\pi/2} \cos^k t dt = \int_0^{\pi/2} \sin^k t dt = \left(\frac{\pi}{2}\right)^{\varepsilon(k)} \frac{k!!}{(k+1)!!}$$

where $\varepsilon(k) = 1$ if k is even, and $\varepsilon(k) = 0$ if k is odd.

PROOF. Let us call I_k the integral on the left in the statement. In order to compute it, we use partial integration. We have the following formula:

$$\begin{aligned} (\cos^k t \sin t)' &= k \cos^{k-1} t (-\sin t) \sin t + \cos^k t \cos t \\ &= (k+1) \cos^{k+1} t - k \cos^{k-1} t \end{aligned}$$

By integrating between 0 and $\pi/2$, we obtain the following formula:

$$(k+1)I_{k+1} = kI_{k-1}$$

Thus we can compute I_k by recurrence, and we obtain in this way:

$$\begin{aligned} I_k &= \frac{k-1}{k} I_{k-2} \\ &= \frac{k-1}{k} \cdot \frac{k-3}{k-2} I_{k-4} \\ &= \frac{k-1}{k} \cdot \frac{k-3}{k-2} \cdot \frac{k-5}{k-4} I_{k-6} \\ &\vdots \\ &= \frac{k!!}{(k+1)!!} I_{1-\varepsilon(k)} \end{aligned}$$

The initial data being $I_0 = \pi/2$ and $I_1 = 1$, we obtain the result. As for the second formula, this follows from the first one, with the change of variables $t = \pi/2 - s$. \square

More generally now, we have the following result:

THEOREM 1.15. *We have the following formula,*

$$\int_0^{\pi/2} \cos^r t \sin^s t dt = \left(\frac{\pi}{2}\right)^{\varepsilon(r)\varepsilon(s)} \frac{r!!s!!}{(r+s+1)!!}$$

where $\varepsilon(r) = 1$ if r is even, and $\varepsilon(r) = 0$ if r is odd.

PROOF. Let us call I_{rs} the integral in the statement. In order to do the partial integration, observe that we have the following formula:

$$\begin{aligned} (\cos^r t \sin^s t)' &= r \cos^{r-1} t (-\sin t) \sin^s t + \cos^r t \cdot s \sin^{s-1} t \cos t \\ &= -r \cos^{r-1} t \sin^{s+1} t + s \cos^{r+1} t \sin^{s-1} t \end{aligned}$$

By integrating between 0 and $\pi/2$, we obtain, for $r, s > 0$:

$$rI_{r-1,s+1} = sI_{r+1,s-1}$$

Thus, we can compute I_{rs} by recurrence. When s is even we have:

$$\begin{aligned} I_{rs} &= \frac{s-1}{r+1} I_{r+2,s-2} \\ &= \frac{s-1}{r+1} \cdot \frac{s-3}{r+3} I_{r+4,s-4} \\ &= \frac{s-1}{r+1} \cdot \frac{s-3}{r+3} \cdot \frac{s-5}{r+5} I_{r+6,s-6} \\ &\vdots \\ &= \frac{r!!s!!}{(r+s)!!} I_{r+s} \end{aligned}$$

But the last term comes from Proposition 1.14, and we obtain the result:

$$\begin{aligned} I_{rs} &= \frac{r!!s!!}{(r+s)!!} I_{r+s} \\ &= \frac{r!!s!!}{(r+s)!!} \left(\frac{\pi}{2}\right)^{\varepsilon(r+s)} \frac{(r+s)!!}{(r+s+1)!!} \\ &= \left(\frac{\pi}{2}\right)^{\varepsilon(r)\varepsilon(s)} \frac{r!!s!!}{(r+s+1)!!} \end{aligned}$$

Observe that this gives the result for r even as well, by symmetry. In the remaining case now, where both the exponents r, s are odd, we can use once again the formula $rI_{r-1,s+1} = sI_{r+1,s-1}$ found above, and the recurrence goes as follows:

$$\begin{aligned} I_{rs} &= \frac{s-1}{r+1} I_{r+2,s-2} \\ &= \frac{s-1}{r+1} \cdot \frac{s-3}{r+3} I_{r+4,s-4} \\ &= \frac{s-1}{r+1} \cdot \frac{s-3}{r+3} \cdot \frac{s-5}{r+5} I_{r+6,s-6} \\ &\vdots \\ &= \frac{r!!s!!}{(r+s-1)!!} I_{r+s-1,1} \end{aligned}$$

In order to compute the last term, observe that we have:

$$\begin{aligned} I_{r1} &= \int_0^{\pi/2} \cos^r t \sin t \, dt \\ &= -\frac{1}{r+1} \int_0^{\pi/2} (\cos^{r+1} t)' \, dt \\ &= \frac{1}{r+1} \end{aligned}$$

Thus, we obtain the formula in the statement, the exponent of $\pi/2$ appearing there being $\varepsilon(r)\varepsilon(s) = 0 \cdot 0 = 0$ in the present case, and this finishes the proof. \square

In order to deal now with the higher spheres, we will use spherical coordinates:

THEOREM 1.16. *We have spherical coordinates in N dimensions,*

$$\begin{cases} x_1 &= r \cos t_1 \\ x_2 &= r \sin t_1 \cos t_2 \\ \vdots & \\ x_{N-1} &= r \sin t_1 \sin t_2 \dots \sin t_{N-2} \cos t_{N-1} \\ x_N &= r \sin t_1 \sin t_2 \dots \sin t_{N-2} \sin t_{N-1} \end{cases}$$

the corresponding Jacobian being given by the following formula:

$$J(r, t) = r^{N-1} \sin^{N-2} t_1 \sin^{N-3} t_2 \dots \sin^2 t_{N-3} \sin t_{N-2}$$

PROOF. The fact that we have indeed spherical coordinates is clear. Regarding now the Jacobian, by developing over the last column, we have:

$$\begin{aligned} J_N &= r \sin t_1 \dots \sin t_{N-2} \sin t_{N-1} \times \sin t_{N-1} J_{N-1} \\ &+ r \sin t_1 \dots \sin t_{N-2} \cos t_{N-1} \times \cos t_{N-1} J_{N-1} \\ &= r \sin t_1 \dots \sin t_{N-2} (\sin^2 t_{N-1} + \cos^2 t_{N-1}) J_{N-1} \\ &= r \sin t_1 \dots \sin t_{N-2} J_{N-1} \end{aligned}$$

Thus, we obtain the formula in the statement, by recurrence. \square

As a first application, we can compute the volume of the sphere:

THEOREM 1.17. *The volume of the unit sphere in \mathbb{R}^N is given by*

$$V = \left(\frac{\pi}{2}\right)^{[N/2]} \frac{2^N}{(N+1)!!}$$

with our usual convention $m!! = (m-1)(m-3)(m-5)\dots$ for double factorials.

PROOF. If we denote by Q the positive part of the sphere, obtained by cutting the sphere in 2^N parts, we have, by using Theorems 1.15 and 1.16 and Fubini:

$$\begin{aligned}
\frac{V}{2^N} &= \int_0^1 \int_0^{\pi/2} \dots \int_0^{\pi/2} r^{N-1} \sin^{N-2} t_1 \dots \sin t_{N-2} dr dt_1 \dots dt_{N-1} \\
&= \int_0^1 r^{N-1} dr \int_0^{\pi/2} \sin^{N-2} t_1 dt_1 \dots \int_0^{\pi/2} \sin t_{N-2} dt_{N-2} \int_0^{\pi/2} 1 dt_{N-1} \\
&= \frac{1}{N} \times \left(\frac{\pi}{2}\right)^{[N/2]} \times \frac{(N-2)!!}{(N-1)!!} \cdot \frac{(N-3)!!}{(N-2)!!} \dots \frac{2!!}{3!!} \cdot \frac{1!!}{2!!} \cdot 1 \\
&= \left(\frac{\pi}{2}\right)^{[N/2]} \frac{1}{(N+1)!!}
\end{aligned}$$

Here we have used the following formula for computing the exponent of $\pi/2$, where $\varepsilon(r) = 1$ if r is even and $\varepsilon(r) = 0$ if r is odd, as in Theorem 1.15:

$$\begin{aligned}
\varepsilon(0) + \varepsilon(1) + \varepsilon(2) + \dots + \varepsilon(N-2) &= 1 + 0 + 1 + 0 + \dots + \varepsilon(N-2) \\
&= \left[\frac{N-2}{2} \right] + 1 \\
&= \left[\frac{N}{2} \right]
\end{aligned}$$

Thus, we are led to the conclusion in the statement. \square

Let us discuss now the computation of the arbitrary polynomial integrals, over the spheres of arbitrary dimension. The result here is as follows:

THEOREM 1.18. *The spherical integral of $x_{i_1} \dots x_{i_r}$ vanishes, unless each index $a \in \{1, \dots, N\}$ appears an even number of times in the sequence i_1, \dots, i_r . We have*

$$\int_{S_{\mathbb{R}}^{N-1}} x_{i_1} \dots x_{i_r} dx = \frac{(N-1)!! k_1!! \dots k_N!!}{(N + \sum k_i - 1)!!}$$

with k_a being this number of occurrences.

PROOF. In what concerns the first assertion, regarding vanishing when some multiplicity k_a is odd, this follows via the change of variables $x_a \rightarrow -x_a$. Regarding now the formula in the statement, assume that we are in the case $k_a \in 2\mathbb{N}$, for any $a \in \{1, \dots, N\}$. The integral in the statement can be written in spherical coordinates, as follows:

$$I = \frac{2^N}{V} \int_0^{\pi/2} \dots \int_0^{\pi/2} x_1^{k_1} \dots x_N^{k_N} J dt_1 \dots dt_{N-1}$$

In this formula V is the volume of the sphere, J is the Jacobian, and the 2^N factor comes from the restriction to the $1/2^N$ part of the sphere where all the coordinates are

positive. According to the formula in Theorem 1.17, the normalization constant is:

$$\frac{2^N}{V} = \left(\frac{2}{\pi}\right)^{[N/2]} (N+1)!!$$

As for the unnormalized integral, this is given by:

$$\begin{aligned} I' = & \int_0^{\pi/2} \dots \int_0^{\pi/2} (\cos t_1)^{k_1} (\sin t_1 \cos t_2)^{k_2} \\ & \vdots \\ & (\sin t_1 \sin t_2 \dots \sin t_{N-2} \cos t_{N-1})^{k_{N-1}} \\ & (\sin t_1 \sin t_2 \dots \sin t_{N-2} \sin t_{N-1})^{k_N} \\ & \sin^{N-2} t_1 \sin^{N-3} t_2 \dots \sin^2 t_{N-3} \sin t_{N-2} \\ & dt_1 \dots dt_{N-1} \end{aligned}$$

By rearranging the terms, we obtain:

$$\begin{aligned} I' = & \int_0^{\pi/2} \cos^{k_1} t_1 \sin^{k_2+\dots+k_N+N-2} t_1 dt_1 \\ & \int_0^{\pi/2} \cos^{k_2} t_2 \sin^{k_3+\dots+k_N+N-3} t_2 dt_2 \\ & \vdots \\ & \int_0^{\pi/2} \cos^{k_{N-2}} t_{N-2} \sin^{k_{N-1}+k_N+1} t_{N-2} dt_{N-2} \\ & \int_0^{\pi/2} \cos^{k_{N-1}} t_{N-1} \sin^{k_N} t_{N-1} dt_{N-1} \end{aligned}$$

Now by using the formula in Theorem 1.15, this gives:

$$\begin{aligned} I' = & \frac{k_1!!(k_2 + \dots + k_N + N - 2)!!}{(k_1 + \dots + k_N + N - 1)!!} \left(\frac{\pi}{2}\right)^{\varepsilon(N-2)} \\ & \frac{k_2!!(k_3 + \dots + k_N + N - 3)!!}{(k_2 + \dots + k_N + N - 2)!!} \left(\frac{\pi}{2}\right)^{\varepsilon(N-3)} \\ & \vdots \\ & \frac{k_{N-2}!!(k_{N-1} + k_N + 1)!!}{(k_{N-2} + k_{N-1} + k_N + 2)!!} \left(\frac{\pi}{2}\right)^{\varepsilon(1)} \\ & \frac{k_{N-1}!!k_N!!}{(k_{N-1} + k_N + 1)!!} \left(\frac{\pi}{2}\right)^{\varepsilon(0)} \end{aligned}$$

Now observe that the various double factorials multiply up to quantity in the statement, modulo a $(N - 1)!!$ factor, and that the $\pi/2$ factors multiply up to:

$$F = \left(\frac{\pi}{2}\right)^{[N/2]}$$

Thus by multiplying by the normalization constant, we obtain the result. \square

We can now recover the normal laws, geometrically, as follows:

THEOREM 1.19. *The moments of the hyperspherical variables are*

$$\int_{S_{\mathbb{R}}^{N-1}} x_i^p dx = \frac{(N-1)!!p!!}{(N+p-1)!!}$$

and the rescaled variables $y_i = \sqrt{N}x_i$ become normal and independent with $N \rightarrow \infty$.

PROOF. The moment formula in the statement follows from Theorem 1.18. As a consequence, with $N \rightarrow \infty$ we have the following estimate:

$$\int_{S_{\mathbb{R}}^{N-1}} x_i^p dx \simeq N^{-p/2} \times p!! = N^{-p/2} M_p(g_1)$$

Thus, the rescaled variables $\sqrt{N}x_i$ become normal with $N \rightarrow \infty$, as claimed. As for the proof of the asymptotic independence, this is standard too, once again by using the formula in Theorem 1.18. Indeed, the joint moments of x_1, \dots, x_N are given by:

$$\int_{S_{\mathbb{R}}^{N-1}} x_1^{k_1} \dots x_N^{k_N} dx = \frac{(N-1)!!k_1!! \dots k_N!!}{(N + \sum k_i - 1)!!} \simeq N^{-\sum k_i} \times k_1!! \dots k_N!!$$

By rescaling, the joint moments of the variables $y_i = \sqrt{N}x_i$ are given by:

$$\int_{S_{\mathbb{R}}^{N-1}} y_1^{k_1} \dots y_N^{k_N} dx \simeq k_1!! \dots k_N!!$$

Thus, we have multiplicativity, and so independence with $N \rightarrow \infty$, as claimed. \square

As a last result about the normal laws, we can recover these as well in connection with rotation groups. Indeed, we have the following reformulation of Theorem 1.19:

THEOREM 1.20. *We have the integration formula*

$$\int_{O_N} U_{ij}^p dU = \frac{(N-1)!!p!!}{(N+p-1)!!}$$

and the rescaled variables $V_{ij} = \sqrt{N}U_{ij}$ become normal and independent with $N \rightarrow \infty$.

PROOF. We use the basic fact that the rotations $U \in O_N$ act on the points of the real sphere $z \in S_{\mathbb{R}}^{N-1}$, with the stabilizer of $z = (1, 0, \dots, 0)$ being the subgroup $O_{N-1} \subset O_N$. In algebraic terms, this gives an identification as follows:

$$S_{\mathbb{R}}^{N-1} = O_N/O_{N-1}$$

In functional analytic terms, this result provides us with an embedding as follows, for any i , which makes correspond the respective integration functionals:

$$C(S_{\mathbb{R}}^{N-1}) \subset C(O_N) \quad , \quad x_i \rightarrow U_{1i}$$

With this identification made, the result follows from Theorem 1.19. □

We will see later, following [36], [95], that the relation between the orthogonal group O_N and the normal laws goes well beyond Theorem 1.20. And we will see as well, following [8], [13] and related papers, that there are also “free versions” of all this.

1d. Complex variables

We have seen so far a number of interesting results regarding the normal laws, and their geometric interpretation. As a last topic for this chapter, let us discuss now the complex analogues of all this. To start with, we have the following definition:

DEFINITION 1.21. *The complex Gaussian law of parameter $t > 0$ is*

$$G_t = \text{law} \left(\frac{1}{\sqrt{2}}(a + ib) \right)$$

where a, b are independent, each following the law g_t .

As in the real case, these measures form convolution semigroups:

THEOREM 1.22. *The complex Gaussian laws have the property*

$$G_s * G_t = G_{s+t}$$

for any $s, t > 0$, and so they form a convolution semigroup.

PROOF. This follows indeed from the real result, namely $g_s * g_t = g_{s+t}$, established in Theorem 1.9, simply by taking real and imaginary parts. □

We have as well the following complex analogue of the CLT:

THEOREM 1.23 (CCLT). *Given complex variables $f_1, f_2, f_3, \dots \in L^\infty(X)$ which are i.i.d., centered, and with common variance $t > 0$, we have*

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n f_i \sim G_t$$

with $n \rightarrow \infty$, in moments.

PROOF. This follows indeed from the real CLT, established in Theorem 1.10, simply by taking the real and imaginary parts of all variables involved. \square

Regarding now the moments, the situation is more complicated than in the real case, because in order to have good results, we have to deal with both the complex variables, and their conjugates. Let us formulate the following definition:

DEFINITION 1.24. *The moments a complex variable $f \in L^\infty(X)$ are the numbers*

$$M_k = E(f^k)$$

depending on colored integers $k = \circ \bullet \bullet \circ \dots$, with the conventions

$$f^\emptyset = 1 \quad , \quad f^\circ = f \quad , \quad f^\bullet = \bar{f}$$

and multiplicativity, in order to define the colored powers f^k .

Observe that, since f, \bar{f} commute, we can permute terms, and restrict the attention to exponents of type $k = \dots \circ \circ \circ \bullet \bullet \bullet \dots$, if we want to. However, our result about the complex Gaussian laws, and other complex laws, later on, will actually look better without doing is, so we will use Definition 1.24 as stated. We first have:

THEOREM 1.25. *The moments of the complex normal law are given by*

$$M_k(G_t) = \begin{cases} t^p p! & (k \text{ uniform, of length } 2p) \\ 0 & (k \text{ not uniform}) \end{cases}$$

where $k = \circ \bullet \bullet \circ \dots$ is called uniform when it contains the same number of \circ and \bullet .

PROOF. We must compute the moments, with respect to colored integer exponents $k = \circ \bullet \bullet \circ \dots$, of the variable from Definition 1.21, namely:

$$f = \frac{1}{\sqrt{2}}(a + ib)$$

We can assume that we are in the case $t = 1$, and the proof here goes as follows:

(1) As a first observation, in the case where our exponent $k = \circ \bullet \bullet \circ \dots$ is not uniform, a standard rotation argument shows that the corresponding moment of f vanishes. To be more precise, the variable $f' = wf$ is complex Gaussian too, for any complex number $w \in \mathbb{T}$, and from $M_k(f) = M_k(f')$ we obtain $M_k(f) = 0$, in this case.

(2) In the uniform case now, where the exponent $k = \circ \bullet \bullet \circ \dots$ consists of p copies of \circ and p copies of \bullet , the corresponding moment can be computed as follows:

$$\begin{aligned}
M_k &= \int (f\bar{f})^p \\
&= \frac{1}{2^p} \int (a^2 + b^2)^p \\
&= \frac{1}{2^p} \sum_r \binom{p}{r} \int a^{2r} \int b^{2p-2r} \\
&= \frac{1}{2^p} \sum_r \binom{p}{r} (2r)!! (2p-2r)!! \\
&= \frac{1}{2^p} \sum_r \frac{p!}{r!(p-r)!} \cdot \frac{(2r)!}{2^r r!} \cdot \frac{(2p-2r)!}{2^{p-r} (p-r)!} \\
&= \frac{p!}{4^p} \sum_r \binom{2r}{r} \binom{2p-2r}{p-r}
\end{aligned}$$

(3) In order to finish now the computation, let us recall that we have the following formula, coming from the generalized binomial formula, or from the Taylor formula:

$$\frac{1}{\sqrt{1+t}} = \sum_{q=0}^{\infty} \binom{2q}{q} \left(\frac{-t}{4}\right)^q$$

By taking the square of this series, we obtain the following formula:

$$\begin{aligned}
\frac{1}{1+t} &= \sum_{qr} \binom{2q}{q} \binom{2r}{r} \left(\frac{-t}{4}\right)^{q+r} \\
&= \sum_p \left(\frac{-t}{4}\right)^p \sum_r \binom{2r}{r} \binom{2p-2r}{p-r}
\end{aligned}$$

Now by looking at the coefficient of t^p on both sides, we conclude that the sum on the right equals 4^p . Thus, we can finish the moment computation in (2), as follows:

$$M_k = \frac{p!}{4^p} \times 4^p = p!$$

We are therefore led to the conclusion in the statement. \square

As before with the real Gaussian laws, a better-looking statement is in terms of partitions. Given a colored integer $k = \circ \bullet \bullet \circ \dots$, we say that a pairing $\pi \in P_2(k)$ is matching when it pairs $\circ - \bullet$ symbols. With this convention, we have the following result:

THEOREM 1.26. *The moments of the complex normal law are the numbers*

$$M_k(G_t) = \sum_{\pi \in \mathcal{P}_2(k)} t^{|\pi|}$$

where $\mathcal{P}_2(k)$ are the matching pairings of $\{1, \dots, k\}$, and $|\cdot|$ is the number of blocks.

PROOF. This is a reformulation of Theorem 1.25. Indeed, we can assume that we are in the case $t = 1$, and here we know from Theorem 1.25 that the moments are:

$$M_k = \begin{cases} (|k|/2)! & (k \text{ uniform}) \\ 0 & (k \text{ not uniform}) \end{cases}$$

On the other hand, the numbers $|\mathcal{P}_2(k)|$ are given by exactly the same formula. Indeed, in order to have a matching pairing of k , our exponent $k = \circ \bullet \bullet \circ \dots$ must be uniform, consisting of p copies of \circ and p copies of \bullet , with $p = |k|/2$. But then the matching pairings of k correspond to the permutations of the \bullet symbols, as to be matched with \circ symbols, and so we have $p!$ such pairings. Thus, we have the same formula as for the moments of f , and we are led to the conclusion in the statement. \square

In practice, we also need to know how to compute joint moments of independent normal variables. We have here the following result, to be heavily used later on:

THEOREM 1.27 (Wick formula). *Given independent variables f_i , each following the complex normal law G_t , with $t > 0$ being a fixed parameter, we have the formula*

$$E(f_{i_1}^{k_1} \dots f_{i_s}^{k_s}) = t^{s/2} \# \left\{ \pi \in \mathcal{P}_2(k) \mid \pi \leq \ker i \right\}$$

where $k = k_1 \dots k_s$ and $i = i_1 \dots i_s$, for the joint moments of these variables, where $\pi \leq \ker i$ means that the indices of i must fit into the blocks of π , in the obvious way.

PROOF. This is something well-known, which can be proved as follows:

(1) Let us first discuss the case where we have a single variable f , which amounts in taking $f_i = f$ for any i in the formula in the statement. What we have to compute here are the moments of f , with respect to colored integer exponents $k = \circ \bullet \bullet \circ \dots$, and the formula in the statement tells us that these moments must be:

$$E(f^k) = t^{|k|/2} |\mathcal{P}_2(k)|$$

But this is the formula in Theorem 1.26, so we are done with this case.

(2) In general now, when expanding the product $f_{i_1}^{k_1} \dots f_{i_s}^{k_s}$ and rearranging the terms, we are left with doing a number of computations as in (1), and then making the product of the expectations that we found. But this amounts in counting the partitions in the statement, with the condition $\pi \leq \ker i$ there standing for the fact that we are doing the various type (1) computations independently, and then making the product. \square

The above statement is one of the possible formulations of the Wick formula, and there are in fact many more formulations, which are all useful. We will be back to this in chapter 6 below, when discussing applications of the Wick formula. Getting back now to geometric aspects, in the spirit for what we did in the real case, we have:

THEOREM 1.28. *We have the following integration formula over the complex sphere $S_{\mathbb{C}}^{N-1} \subset \mathbb{C}^N$, with respect to the normalized uniform measure,*

$$\int_{S_{\mathbb{C}}^{N-1}} |z_1|^{2k_1} \dots |z_N|^{2k_N} dz = \frac{(N-1)!k_1! \dots k_n!}{(N + \sum k_i - 1)!}$$

valid for any exponents $k_i \in \mathbb{N}$. As for the other polynomial integrals in z_1, \dots, z_N and their conjugates $\bar{z}_1, \dots, \bar{z}_N$, these all vanish.

PROOF. Consider an arbitrary polynomial integral over $S_{\mathbb{C}}^{N-1}$, written as follows:

$$I = \int_{S_{\mathbb{C}}^{N-1}} z_{i_1} \bar{z}_{i_2} \dots z_{i_{2k-1}} \bar{z}_{i_{2k}} dz$$

By using transformations of type $p \rightarrow \lambda p$ with $|\lambda| = 1$, we see that this integral I vanishes, unless each z_a appears as many times as \bar{z}_a does, and this gives the last assertion. So, assume now that we are in the non-vanishing case. Then the k_a copies of z_a and the k_a copies of \bar{z}_a produce by multiplication a factor $|z_a|^{2k_a}$, so we have:

$$I = \int_{S_{\mathbb{C}}^{N-1}} |z_1|^{2k_1} \dots |z_N|^{2k_N} dz$$

Now by using the standard identification $S_{\mathbb{C}}^{N-1} \simeq S_{\mathbb{R}}^{2N-1}$, we obtain:

$$\begin{aligned} I &= \int_{S_{\mathbb{R}}^{2N-1}} (x_1^2 + y_1^2)^{k_1} \dots (x_N^2 + y_N^2)^{k_N} d(x, y) \\ &= \sum_{r_1 \dots r_N} \binom{k_1}{r_1} \dots \binom{k_N}{r_N} \int_{S_{\mathbb{R}}^{2N-1}} x_1^{2k_1-2r_1} y_1^{2r_1} \dots x_N^{2k_N-2r_N} y_N^{2r_N} d(x, y) \end{aligned}$$

By using the formula in Theorem 1.18, we obtain:

$$\begin{aligned} I &= \sum_{r_1 \dots r_N} \binom{k_1}{r_1} \dots \binom{k_N}{r_N} \frac{(2N-1)!!(2r_1)!! \dots (2r_N)!!(2k_1-2r_1)!! \dots (2k_N-2r_N)!!}{(2N+2\sum k_i-1)!!} \\ &= \sum_{r_1 \dots r_N} \binom{k_1}{r_1} \dots \binom{k_N}{r_N} \frac{2^{N-1}(N-1)! \prod (2r_i)! / (2^{r_i} r_i!) \prod (2k_i-2r_i)! / (2^{k_i-r_i} (k_i-r_i)!)}{2^{N+\sum k_i-1} (N+\sum k_i-1)!} \\ &= \sum_{r_1 \dots r_N} \binom{k_1}{r_1} \dots \binom{k_N}{r_N} \frac{(N-1)!(2r_1)! \dots (2r_N)!(2k_1-2r_1)! \dots (2k_N-2r_N)!}{4^{\sum k_i} (N+\sum k_i-1)! r_1! \dots r_N! (k_1-r_1)! \dots (k_N-r_N)!} \end{aligned}$$

Now observe that can rewrite this quantity in the following way:

$$\begin{aligned}
I &= \sum_{r_1 \dots r_N} \frac{k_1! \dots k_N! (N-1)! (2r_1)! \dots (2r_N)! (2k_1 - 2r_1)! \dots (2k_N - 2r_N)!}{4^{\sum k_i} (N + \sum k_i - 1)! (r_1! \dots r_N! (k_1 - r_1)! \dots (k_N - r_N)!)^2} \\
&= \sum_{r_1} \binom{2r_1}{r_1} \binom{2k_1 - 2r_1}{k_1 - r_1} \dots \sum_{r_N} \binom{2r_N}{r_N} \binom{2k_N - 2r_N}{k_N - r_N} \frac{(N-1)! k_1! \dots k_N!}{4^{\sum k_i} (N + \sum k_i - 1)!} \\
&= 4^{k_1} \times \dots \times 4^{k_N} \times \frac{(N-1)! k_1! \dots k_N!}{4^{\sum k_i} (N + \sum k_i - 1)!} \\
&= \frac{(N-1)! k_1! \dots k_N!}{(N + \sum k_i - 1)!}
\end{aligned}$$

Here we have used the following well-known identity, whose proof is standard:

$$\sum_r \binom{2r}{r} \binom{2k - 2r}{k - r} = 4^k$$

Thus, we obtain the formula in the statement. \square

Regarding now the hyperspherical variables, investigated in the above in the real case, we have similar results in the complex case, as follows:

THEOREM 1.29. *The rescalings $\sqrt{N}z_i$ of the unit complex sphere coordinates*

$$z_i : S_{\mathbb{C}}^{N-1} \rightarrow \mathbb{C}$$

as well as the rescalings $\sqrt{N}U_{ij}$ of the unitary group coordinates

$$U_{ij} : U_N \rightarrow \mathbb{C}$$

become complex Gaussian and independent with $N \rightarrow \infty$.

PROOF. We have several assertions to be proved, the idea being as follows:

(1) According to the formula in Theorem 1.28, the polynomials integrals in z_i, \bar{z}_i vanish, unless the number of z_i, \bar{z}_i is the same. In this latter case these terms can be grouped together, by using $z_i \bar{z}_i = |z_i|^2$, and the relevant integration formula is:

$$\int_{S_{\mathbb{C}}^{N-1}} |z_i|^{2k} dz = \frac{(N-1)! k!}{(N+k-1)!}$$

Now with $N \rightarrow \infty$, we obtain from this the following estimate:

$$\int_{S_{\mathbb{C}}^{N-1}} |z_i|^{2k} dx \simeq N^{-k} \times k!$$

Thus, the rescaled variables $\sqrt{N}z_i$ become normal with $N \rightarrow \infty$, as claimed.

(2) As for the proof of the asymptotic independence, this is standard too, again by using Theorem 1.28. Indeed, the joint moments of z_1, \dots, z_N are given by:

$$\begin{aligned} \int_{S_{\mathbb{R}}^{N-1}} |z_1|^{2k_1} \dots |z_N|^{2k_N} dx &= \frac{(N-1)!k_1! \dots k_n!}{(N + \sum k_i - 1)!} \\ &\simeq N^{-\sum k_i} \times k_1! \dots k_N! \end{aligned}$$

By rescaling, the joint moments of the variables $y_i = \sqrt{N}z_i$ are given by:

$$\int_{S_{\mathbb{R}}^{N-1}} |y_1|^{2k_1} \dots |y_N|^{2k_N} dx \simeq k_1! \dots k_N!$$

Thus, we have multiplicativity, and so independence with $N \rightarrow \infty$, as claimed.

(3) Regarding the last assertion, we can use the basic fact that the rotations $U \in U_N$ act on the points of the sphere $z \in S_{\mathbb{C}}^{N-1}$, with the stabilizer of $z = (1, 0, \dots, 0)$ being the subgroup $U_{N-1} \subset U_N$. In algebraic terms, this gives an identification as follows:

$$S_{\mathbb{C}}^{N-1} = U_N/U_{N-1}$$

In functional analytic terms, this result provides us with an embedding as follows, for any i , which makes correspond the respective integration functionals:

$$C(S_{\mathbb{C}}^{N-1}) \subset C(U_N) \quad , \quad x_i \rightarrow U_{1i}$$

With this identification made, the result follows from (1,2). □

As already mentioned in the real context, it is possible to get beyond such results, by using advanced group theory. We will be back to this, in chapter 4 below. It is also possible to formulate “free versions” of all the above, and we will do this later.

So long for the basics of probability theory, quickly explained. For further theory, the best is to go to a dedicated probability book, such as the one of Feller [46], or Durrett [43]. Alternatively, you can learn good probability theory from the preliminary chapters of more specialized probability books, and with the comment here that, among probabilists, the random matrix people know well their job, and are very close to what we will be doing in this book. Well-known introductions to random matrices include the classical and delightful book by Mehta [66], the more modern book by Anderson, Guionnet and Zeitouni [2], the books by Bose [29], Mingo and Speicher [69], and many more.

Needless to say, you can also learn reliable probability theory from physicists, or other scientists. In fact, probability theory was fully accepted only recently, in the late 20th century, as a respectable branch of mathematics, and if there are some scientists who have taken probability seriously, and this since ever, these are the physicists.

1e. Exercises

Things have been quite classical in this opening chapter, and there are just a few further things that need to be learned. First, in connection with the CLT, we have:

EXERCISE 1.30. *Look up the CLT, which was done here in moments, learn how the convergence can be improved, and write a brief account of that.*

This is a bit vague, but at this stage, learning more theory would be a good thing.

EXERCISE 1.31. *Prove that the area of the unit sphere in \mathbb{R}^N is given by*

$$A = \left(\frac{\pi}{2}\right)^{[N/2]} \frac{2^N}{(N-1)!!}$$

with our usual convention $N!! = (N-1)(N-3)(N-5)\dots$ for double factorials.

Here you can either recycle our proof for V , by making changes where needed, or deduce the result from our result for V . In any case, think first at $N = 2$.

EXERCISE 1.32. *Establish the following integration formula over $S_{\mathbb{R}}^{N-1} \subset \mathbb{R}^N$, with respect to the normalized measure, valid for any exponents $p_i \in \mathbb{N}$,*

$$\int_{S_{\mathbb{R}}^{N-1}} |x_1^{p_1} \dots x_N^{p_N}| dx = \left(\frac{2}{\pi}\right)^{\Sigma(p_1, \dots, p_N)} \frac{(N-1)!! p_1!! \dots p_N!!}{(N + \Sigma p_i - 1)!!}$$

where $\Sigma = [\text{odds}/2]$ if N is odd and $\Sigma = [(\text{odds} + 1)/2]$ if N is even, where “odds” denotes the number of odd numbers in the sequence p_1, \dots, p_N .

Observe that this generalizes the integration formula for monomials that we established in the above, because odd powers lead to 0 integrals. The proof can only be similar.

EXERCISE 1.33. *Compute the density of the hyperspherical law at $N = 4$, that is, the law of one of the coordinates over the unit sphere $S_{\mathbb{R}}^3 \subset \mathbb{R}^4$.*

If you find something very interesting, as an answer here, do not be surprised. After all, $S_{\mathbb{R}}^3$ is the sphere of space-time, having its own magic. We will be back to this.

CHAPTER 2

Poisson laws

2a. Poisson limits

We have seen so far that the centered normal laws g_t and their complex analogues G_t , which appear from the Central Limit Theorem (CLT), have interesting combinatorial properties, and appear in several group-theoretical and geometric contexts.

We discuss here the discrete counterpart of these results. The mathematics will involve the Poisson laws p_t , which appear via the Poisson Limit Theorem (PLT), and their generalized versions p_ν , called compound Poisson laws, which appear via the Compound Poisson Limit Theorem (CPLT). Let us start with the following definition:

DEFINITION 2.1. *The Poisson law of parameter 1 is the following measure,*

$$p_1 = \frac{1}{e} \sum_{k \in \mathbb{N}} \frac{\delta_k}{k!}$$

and the Poisson law of parameter $t > 0$ is the following measure,

$$p_t = e^{-t} \sum_{k \in \mathbb{N}} \frac{t^k}{k!} \delta_k$$

with the letter “p” standing for Poisson.

We are using here, as before, some simplified notations for these laws, which are in tune with the notations g_t, G_t that we used for the centered Gaussian laws. Observe that our laws have indeed mass 1, as they should, due to the following key formula:

$$e^t = \sum_{k \in \mathbb{N}} \frac{t^k}{k!}$$

We will see in the moment why these measures appear a bit everywhere, in discrete contexts, the reasons for this coming from the Poisson Limit Theorem (PLT). Let us first develop some general theory. We first have the following result:

THEOREM 2.2. *We have the following formula, for any $s, t > 0$,*

$$p_s * p_t = p_{s+t}$$

so the Poisson laws form a convolution semigroup.

PROOF. By using $\delta_k * \delta_l = \delta_{k+l}$ and the binomial formula, we obtain:

$$\begin{aligned}
p_s * p_t &= e^{-s} \sum_k \frac{s^k}{k!} \delta_k * e^{-t} \sum_l \frac{t^l}{l!} \delta_l \\
&= e^{-s-t} \sum_n \delta_n \sum_{k+l=n} \frac{s^k t^l}{k! l!} \\
&= e^{-s-t} \sum_n \frac{\delta_n}{n!} \sum_{k+l=n} \frac{n!}{k! l!} s^k t^l \\
&= e^{-s-t} \sum_n \frac{(s+t)^n}{n!} \delta_n \\
&= p_{s+t}
\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Next in line, we have the following result, which is fundamental as well:

THEOREM 2.3. *The Poisson laws appear as formal exponentials*

$$p_t = \sum_k \frac{t^k (\delta_1 - \delta_0)^{*k}}{k!}$$

with respect to the convolution of measures $*$.

PROOF. By using the binomial formula, the measure on the right is:

$$\begin{aligned}
\mu &= \sum_k \frac{t^k}{k!} \sum_{r+s=k} (-1)^s \frac{k!}{r! s!} \delta_r \\
&= \sum_k t^k \sum_{r+s=k} (-1)^s \frac{\delta_r}{r! s!} \\
&= \sum_r \frac{t^r \delta_r}{r!} \sum_s \frac{(-1)^s}{s!} \\
&= \frac{1}{e} \sum_r \frac{t^r \delta_r}{r!} \\
&= p_t
\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Regarding now the Fourier transform computation, this is as follows:

THEOREM 2.4. *The Fourier transform of p_t is given by*

$$F_{p_t}(y) = \exp((e^{iy} - 1)t)$$

for any $t > 0$.

PROOF. We have indeed the following computation:

$$\begin{aligned}
 F_{p_t}(y) &= e^{-t} \sum_k \frac{t^k}{k!} F_{\delta_k}(y) \\
 &= e^{-t} \sum_k \frac{t^k}{k!} e^{iky} \\
 &= e^{-t} \sum_k \frac{(e^{iy}t)^k}{k!} \\
 &= \exp(-t) \exp(e^{iy}t) \\
 &= \exp((e^{iy} - 1)t)
 \end{aligned}$$

Thus, we obtain the formula in the statement. \square

Observe that the above formula gives an alternative proof for Theorem 2.2, by using the fact that the logarithm of the Fourier transform linearizes the convolution. As another application, we can now establish the Poisson Limit Theorem, as follows:

THEOREM 2.5 (PLT). *We have the following convergence, in moments,*

$$\left(\left(1 - \frac{t}{n} \right) \delta_0 + \frac{t}{n} \delta_1 \right)^{*n} \rightarrow p_t$$

for any $t > 0$.

PROOF. Let us denote by ν_n the measure under the convolution sign, namely:

$$\nu_n = \left(1 - \frac{t}{n} \right) \delta_0 + \frac{t}{n} \delta_1$$

We have the following computation, for the Fourier transform of the limit:

$$\begin{aligned}
 F_{\delta_r}(y) = e^{iry} &\implies F_{\nu_n}(y) = \left(1 - \frac{t}{n} \right) + \frac{t}{n} e^{iy} \\
 &\implies F_{\nu_n^{*n}}(y) = \left(\left(1 - \frac{t}{n} \right) + \frac{t}{n} e^{iy} \right)^n \\
 &\implies F_{\nu_n^{*n}}(y) = \left(1 + \frac{(e^{iy} - 1)t}{n} \right)^n \\
 &\implies F(y) = \exp((e^{iy} - 1)t)
 \end{aligned}$$

Thus, we obtain indeed the Fourier transform of p_t , as desired. \square

At the level of moments now, things are quite subtle for the Poisson laws, combinatorially speaking, and more complicated than for the normal laws. We first have the following result, dealing with the simplest case, where the parameter is $t = 1$:

THEOREM 2.6. *The moments of p_1 are the Bell numbers,*

$$M_k(p_1) = |P(k)|$$

where $P(k)$ is the set of partitions of $\{1, \dots, k\}$.

PROOF. The moments of p_1 are given by the following formula:

$$M_k = \frac{1}{e} \sum_r \frac{r^k}{r!}$$

We therefore have the following recurrence formula for these moments:

$$\begin{aligned} M_{k+1} &= \frac{1}{e} \sum_r \frac{(r+1)^{k+1}}{(r+1)!} \\ &= \frac{1}{e} \sum_r \frac{r^k}{r!} \left(1 + \frac{1}{r}\right)^k \\ &= \frac{1}{e} \sum_r \frac{r^k}{r!} \sum_s \binom{k}{s} r^{-s} \\ &= \sum_s \binom{k}{s} \cdot \frac{1}{e} \sum_r \frac{r^{k-s}}{r!} \\ &= \sum_s \binom{k}{s} M_{k-s} \end{aligned}$$

With this done, let us try now to find a recurrence for the Bell numbers:

$$B_k = |P(k)|$$

A partition of $\{1, \dots, k+1\}$ appears by choosing s neighbors for 1, among the k numbers available, and then partitioning the $k-s$ elements left. Thus, we have:

$$B_{k+1} = \sum_s \binom{k}{s} B_{k-s}$$

Thus, our moments M_k satisfy the same recurrence as the numbers B_k . Regarding now the initial values, in what concerns the first moment of p_1 , we have:

$$M_1 = \frac{1}{e} \sum_r \frac{r}{r!} = 1$$

Also, by using the above recurrence for the numbers M_k , we obtain from this:

$$M_2 = \sum_s \binom{1}{s} M_{k-s} = 1 + 1 = 2$$

On the other hand, $B_1 = 1$ and $B_2 = 2$. Thus we obtain $M_k = B_k$, as claimed. \square

More generally now, we have the following result, dealing with the case $t > 0$:

THEOREM 2.7. *The moments of p_t with $t > 0$ are given by*

$$M_k(p_t) = \sum_{\pi \in P(k)} t^{|\pi|}$$

where $|\cdot|$ is the number of blocks.

PROOF. The moments of the Poisson law p_t with $t > 0$ are given by:

$$M_k = e^{-t} \sum_r \frac{t^r r^k}{r!}$$

We have the following recurrence formula for these moments:

$$\begin{aligned} M_{k+1} &= e^{-t} \sum_r \frac{t^{r+1} (r+1)^{k+1}}{(r+1)!} \\ &= e^{-t} \sum_r \frac{t^{r+1} r^k}{r!} \left(1 + \frac{1}{r}\right)^k \\ &= e^{-t} \sum_r \frac{t^{r+1} r^k}{r!} \sum_s \binom{k}{s} r^{-s} \\ &= \sum_s \binom{k}{s} \cdot e^{-t} \sum_r \frac{t^{r+1} r^{k-s}}{r!} \\ &= t \sum_s \binom{k}{s} M_{k-s} \end{aligned}$$

Regarding now the initial values, the first moment of p_t is given by:

$$M_1 = e^{-t} \sum_r \frac{t^r r}{r!} = e^{-t} \sum_r \frac{t^r}{(r-1)!} = t$$

Now by using the above recurrence we obtain from this:

$$M_2 = t \sum_s \binom{1}{s} M_{k-s} = t(1+t) = t + t^2$$

On the other hand, consider the numbers in the statement, namely:

$$S_k = \sum_{\pi \in P(k)} t^{|\pi|}$$

Since a partition of $\{1, \dots, k+1\}$ appears by choosing s neighbors for 1, among the k numbers available, and then partitioning the $k-s$ elements left, we have:

$$S_{k+1} = t \sum_s \binom{k}{s} S_{k-s}$$

As for the initial values of these numbers, these are $S_1 = t$, $S_2 = t + t^2$. Thus the initial values coincide, and so these numbers are the moments of p_t , as stated. \square

As a conclusion to all this, the Poisson laws p_t are now entitled to join the collection of “interesting” probability measures that we have, formed by the real and complex Gaussian laws g_t and G_t . Indeed, not only all these measures appear via key limiting theorems, and form convolution semigroups, but at the level of moments, we have:

THEOREM 2.8. *The moments of $\mu_t = p_t, g_t, G_t$ are given by the formula*

$$M_k(\mu_t) = \sum_{\pi \in D(k)} t^{|\pi|}$$

where $D = P, P_2, \mathcal{P}_2$ respectively, and $|\cdot|$ is the number of blocks.

PROOF. This follows indeed from Theorem 2.7, and from the results in chapter 1. \square

The above result raises a whole string of interesting questions. Are there more measures of this type? Is a classification of such measures possible? Are the convolution semigroup results consequences of the moment formula? What about the limiting theorems? And so on. All these questions will be answered, in due time.

2b. Derangements

In relation now with groups, and with pure mathematics in general, let us start with the following well-known, beautiful and fundamental result:

THEOREM 2.9. *The probability for a random $\sigma \in S_N$ to have no fixed points is*

$$P \simeq \frac{1}{e}$$

in the $N \rightarrow \infty$ limit, where $e = 2.718\dots$ is the usual constant from analysis.

PROOF. This is best viewed by using the inclusion-exclusion principle. Let us set:

$$S_N^k = \left\{ \sigma \in S_N \mid \sigma(k) = k \right\}$$

The set of permutations having no fixed points, called derangements, is then:

$$X_N = \left(\bigcup_k S_N^k \right)^c$$

Now the inclusion-exclusion principle tells us that we have:

$$\begin{aligned}
|X_N| &= \left| \left(\bigcup_k S_N^k \right)^c \right| \\
&= |S_N| - \sum_k |S_N^k| + \sum_{k < l} |S_N^k \cap S_N^l| - \dots + (-1)^N \sum_{k_1 < \dots < k_N} |S_N^{k_1} \cup \dots \cup S_N^{k_N}| \\
&= N! - N(N-1)! + \binom{N}{2}(N-2)! - \dots + (-1)^N \binom{N}{N}(N-N)! \\
&= \sum_{r=0}^N (-1)^r \binom{N}{r} (N-r)!
\end{aligned}$$

Thus, the probability that we are interested in, for a random permutation $\sigma \in S_N$ to have no fixed points, is given by the following formula:

$$P = \frac{|X_N|}{N!} = \sum_{r=0}^N \frac{(-1)^r}{r!}$$

Since on the right we have the expansion of $1/e$, this gives the result. \square

In order to refine now the above result, as to reach to Poisson laws, we will need some basic notions from group theory. Let us start with the following standard definition:

DEFINITION 2.10. *Given a closed subgroup $G \subset U_N$, the function*

$$\chi : G \rightarrow \mathbb{C} \quad , \quad \chi(g) = \sum_{i=1}^N g_{ii}$$

is called main character of G .

We will see later a number for motivations for the study of characters, the idea being that a compact group G can have several representations $\pi : G \subset U_N$, which can be studied via their characters $\chi_\pi : G \rightarrow \mathbb{C}$. For the moment, we will not need any kind of abstract motivations, and this because for S_N , we have the following beautiful result:

THEOREM 2.11. *Consider the symmetric group S_N , regarded as the permutation group, $S_N \subset O_N$, of the N coordinate axes of \mathbb{R}^N .*

- (1) *The main character $\chi \in C(S_N)$ counts the number of fixed points.*
- (2) *The law of $\chi \in C(S_N)$ becomes Poisson (1), in the $N \rightarrow \infty$ limit.*

PROOF. We have two things to be proved here, the idea being as follows:

(1) The permutation matrices $\sigma \in O_N$, which give the embedding $S_N \subset O_N$ in the statement, being given by $\sigma_{ij} = \delta_{i\sigma(j)}$, we have the following computation:

$$\chi(\sigma) = \sum_i \delta_{\sigma(i)i} = \# \left\{ i \in \{1, \dots, N\} \mid \sigma(i) = i \right\}$$

(2) In order to establish now the asymptotic result in the statement, we must prove the following formula, for any $r \in \mathbb{N}$, in the $N \rightarrow \infty$ limit:

$$P(\chi = r) \simeq \frac{1}{r!e}$$

We already know, from Theorem 2.9, that this formula holds at $r = 0$. In the general case now, we have to count the permutations $\sigma \in S_N$ having exactly r points. Now since having such a permutation amounts in choosing r points among $1, \dots, N$, and then permuting the $N - r$ points left, without fixed points allowed, we have:

$$\begin{aligned} \# \left\{ \sigma \in S_N \mid \chi(\sigma) = r \right\} &= \binom{N}{r} \# \left\{ \sigma \in S_{N-r} \mid \chi(\sigma) = 0 \right\} \\ &= \frac{N!}{r!(N-r)!} \# \left\{ \sigma \in S_{N-r} \mid \chi(\sigma) = 0 \right\} \\ &= N! \times \frac{1}{r!} \times \frac{\# \left\{ \sigma \in S_{N-r} \mid \chi(\sigma) = 0 \right\}}{(N-r)!} \end{aligned}$$

By dividing everything by $N!$, we obtain from this the following formula:

$$\frac{\# \left\{ \sigma \in S_N \mid \chi(\sigma) = r \right\}}{N!} = \frac{1}{r!} \times \frac{\# \left\{ \sigma \in S_{N-r} \mid \chi(\sigma) = 0 \right\}}{(N-r)!}$$

Now by using the computation at $r = 0$, that we already have, from Theorem 2.9, it follows that with $N \rightarrow \infty$ we have the following estimate:

$$P(\chi = r) \simeq \frac{1}{r!} \cdot P(\chi = 0) \simeq \frac{1}{r!} \cdot \frac{1}{e}$$

Thus, we obtain as limiting measure the Poisson law of parameter 1, as stated. \square

As a next step, let us try now to generalize what we have, namely Theorem 2.11, as to reach to the Poisson laws of arbitrary parameter $t > 0$. We will need:

DEFINITION 2.12. *Given a closed subgroup $G \subset U_N$, the function*

$$\chi_t : G \rightarrow \mathbb{C} \quad , \quad \chi_t(g) = \sum_{i=1}^{[tN]} g_{ii}$$

is called main truncated character of G , of parameter $t \in (0, 1]$.

As before with plain characters, there is some theory behind this definition, and we will discuss this later on, in chapter 4 below. In relation with the present considerations, we actually already met such truncated characters, but in a disguised form, in chapter 1, when talking about O_N, U_N . Indeed, the results there show that we have:

PROPOSITION 2.13. *For the orthogonal and unitary groups O_N, U_N , the rescalings*

$$\rho_{1/N} = \sqrt{N} \cdot \chi_{1/N}$$

become respectively real and complex Gaussian, in the $N \rightarrow \infty$ limit.

PROOF. According to our conventions, given a closed subgroup $G \subset U_N$, the main character truncated at $t = 1/N$ is simply the first coordinate:

$$\chi_{1/N}(g) = g_{11}$$

With this remark made, the conclusions from the statement follow from the computations from chapter 1, for the laws of coordinates on O_N, U_N . \square

Getting back now to the symmetric groups, we have the following result, generalizing Theorem 2.11, and which will be our final saying on the subject:

THEOREM 2.14. *Consider the symmetric group S_N , regarded as the permutation group, $S_N \subset O_N$, of the N coordinate axes of \mathbb{R}^N .*

- (1) *The variable χ_t counts the number of fixed points among $1, \dots, [tN]$.*
- (2) *The law of this variable χ_t becomes Poisson (t), in the $N \rightarrow \infty$ limit.*

PROOF. We already know from Theorem 2.11 that the results hold at $t = 1$. In general, the proof is similar, the idea being as follows:

- (1) We have indeed the following computation, coming from definitions:

$$\chi_t(\sigma) = \sum_{i=1}^{[tN]} \delta_{\sigma(i)i} = \# \left\{ i \in \{1, \dots, [tN]\} \mid \sigma(i) = i \right\}$$

- (2) Consider indeed the following sets, as in the proof of Theorem 2.9:

$$S_N^k = \left\{ \sigma \in S_N \mid \sigma(k) = k \right\}$$

The set of permutations having no fixed points among $1, \dots, [tN]$ is then:

$$X_N = \left(\bigcup_{k \leq [tN]} S_N^k \right)^c$$

As before in the proof of Theorem 2.9, we obtain by inclusion-exclusion that:

$$\begin{aligned}
P(\chi_t = 0) &= \frac{1}{N!} \sum_{r=0}^{[tN]} (-1)^r \sum_{k_1 < \dots < k_r < [tN]} |S_N^{k_1} \cap \dots \cap S_N^{k_r}| \\
&= \frac{1}{N!} \sum_{r=0}^{[tN]} (-1)^r \binom{[tN]}{r} (N-r)! \\
&= \sum_{r=0}^{[tN]} \frac{(-1)^r}{r!} \cdot \frac{[tN]!(N-r)!}{N!([tN]-r)!}
\end{aligned}$$

Now with $N \rightarrow \infty$, we obtain from this the following estimate:

$$\begin{aligned}
P(\chi_t = 0) &\simeq \sum_{r=0}^{[tN]} \frac{(-1)^r}{r!} \cdot t^r \\
&= \sum_{r=0}^{[tN]} \frac{(-t)^r}{r!} \\
&\simeq e^{-t}
\end{aligned}$$

More generally, by counting the permutations $\sigma \in S_N$ having exactly r fixed points among $1, \dots, [tN]$, as in the proof of Theorem 2.11, we obtain:

$$P(\chi_t = r) \simeq \frac{t^r}{r!e^t}$$

Thus, we obtain in the limit a Poisson law of parameter t , as stated. \square

The above result is quite fundamental, and worth proving a second time, by using an alternative method. We can indeed use the following formula:

THEOREM 2.15. *Consider the symmetric group S_N , with its standard coordinates:*

$$u_{ij} = \chi \left(\sigma \in S_N \mid \sigma(j) = i \right)$$

We have then the following integration formula,

$$\int_{S_N} u_{i_1 j_1} \dots u_{i_r j_r} = \begin{cases} \frac{(N - |\ker i|)!}{N!} & \text{if } \ker i = \ker j \\ 0 & \text{otherwise} \end{cases}$$

where $\ker i$ denotes the partition of $\{1, \dots, r\}$ whose blocks collect the equal indices of i , and where $|\cdot|$ denotes the number of blocks.

PROOF. Observe first that the above formula computes all the integrals over S_N , and this because the coordinates u_{ij} separate the points of S_N . In what regards the proof, according to the definition of u_{ij} , the integrals in the statement are given by:

$$\int_{S_N} u_{i_1 j_1} \dots u_{i_r j_r} = \frac{1}{N!} \# \left\{ \sigma \in S_N \mid \sigma(j_1) = i_1, \dots, \sigma(j_r) = i_r \right\}$$

Now observe that the existence of $\sigma \in S_N$ as above requires:

$$i_k = i_l \iff j_k = j_l$$

Thus, the integral in the statement vanishes if $\ker i \neq \ker j$. As for the case left, namely $\ker i = \ker j$, if we denote by $b \in \{1, \dots, r\}$ the number of blocks of this partition $\ker i = \ker j$, then we have $N - b$ points to be sent bijectively to $N - b$ points, and so $(N - b)!$ solutions, and our integral follows to be $(N - b)!/N!$, as claimed. \square

As an illustration for the above formula, we can now formulate, as promised:

THEOREM 2.16. *For the symmetric group $S_N \subset O_N$, regarded as group of matrices, $S_N \subset O_N$, via the standard permutation matrices, the truncated character*

$$\chi_t = \sum_{i=1}^{[tN]} u_{ii}$$

counts the number of fixed points among $\{1, \dots, [tN]\}$, and its law with respect to the counting measure becomes, with $N \rightarrow \infty$, a Poisson law of parameter t .

PROOF. The first assertion is something trivial, that we already know. Regarding now the second assertion, we can use here Theorem 2.15. With S_{rb} being the Stirling numbers, counting the partitions of $\{1, \dots, r\}$ having b blocks, we have:

$$\begin{aligned} \int_{S_N} \chi_t^r &= \sum_{i_1=1}^{[tN]} \dots \sum_{i_r=1}^{[tN]} \int_{S_N} u_{i_1 i_1} \dots u_{i_r i_r} \\ &= \sum_{\pi \in P(r)} \frac{[tN]!}{([tN] - |\pi|)!} \cdot \frac{(N - |\pi|)!}{N!} \\ &= \sum_{b=1}^{[tN]} \frac{[tN]!}{([tN] - b)!} \cdot \frac{(N - b)!}{N!} \cdot S_{rb} \end{aligned}$$

In particular with $N \rightarrow \infty$ we obtain the following formula:

$$\lim_{N \rightarrow \infty} \int_{S_N} \chi_t^r = \sum_{b=1}^r S_{rb} t^b$$

But this is a Poisson (t) moment, according to our formula for the moments of p_t , which in terms of Stirling numbers is the above one, and so we are done. \square

As a conclusion to all this, the Poisson laws p_t appear to be quite similar to the real and complex Gaussian laws g_t and G_t , in the sense that:

- (1) All these laws appear via basic limiting theorems.
- (2) They form semigroups with respect to convolution.
- (3) Their moments can be computed by counting certain partitions.
- (4) There is a relation with pure mathematics as well, involving S_N, O_N, U_N .

All this remains of course to be further discussed. We will be back to this right next, then in chapters 3-4 below, following [13], [36], [95], and then later on as well.

2c. Compound Poisson

We have so far many interesting results regarding p_t, g_t, G_t , on one hand regarding moments, with the sets of partitions P, P_2, \mathcal{P}_2 being involved, and on the other hand regarding characters, with the groups S_N, O_N, U_N involved. All this is quite nice, but looks a bit incomplete, and we are led to the following question:

QUESTION 2.17. *What is the complex analogue of the Poisson law?*

To be more precise, for obvious reasons, we would like to have a complex analogue P_t of the Poisson law p_t , as to be able to draw a nice square diagram, as follows:

$$\begin{array}{ccc} P_t & \text{---} & G_t \\ | & & | \\ p_t & \text{---} & g_t \end{array}$$

All this is quite philosophical, of course. In view of what we have, the first thought goes to moments, and we are led here to the following question, philosophical as well:

QUESTION 2.18. *Is it possible to talk about the set \mathcal{P} of matching partitions?*

To be more precise, we would like our moment formula $M_k(\mu_t) = \sum_{\pi \in D(k)} t^{|\pi|}$ to hold as well for the mysterious law P_t that we are looking for, with $D = \mathcal{P}$. So, we would like to have a set \mathcal{P} of “matching partitions”, as to be able to draw the following diagram:

$$\begin{array}{ccc} \mathcal{P} & \text{---} & \mathcal{P}_2 \\ | & & | \\ P & \text{---} & P_2 \end{array}$$

However, things are quite unclear with \mathcal{P} , because when $\pi \in P$ has all blocks having even size, $\pi \in P_{\text{even}}$, we can probably declare that we have $\pi \in \mathcal{P}$ when the equality $\# \circ = \# \bullet$ holds in each block. But when $\pi \notin P_{\text{even}}$, it is not clear at all what to do.

Let us record our conclusions in the form of a vague thought, as follows:

THOUGHT 2.19. *There are probably no complex Poisson law P_t , and no set of matching partitions \mathcal{P} . However, we should have diagrams of type*

$$\begin{array}{ccc} B_t & \text{-----} & G_t \\ | & & | \\ b_t & \text{-----} & g_t \end{array} \quad \begin{array}{ccc} \mathcal{P}_{\text{even}} & \text{-----} & \mathcal{P}_2 \\ | & & | \\ P_{\text{even}} & \text{-----} & P_2 \end{array}$$

with the laws b_t, B_t , coming from $P_{\text{even}}, \mathcal{P}_{\text{even}}$ via the formula

$$M_k(\mu_t) = \sum_{\pi \in D(k)} t^{|\pi|}$$

to be computed, and being the “true” discrete analogues of g_t, G_t .

So, this is what we have. Of course, all this looks a bit like science fiction, and shall we follow this luminous new way or not, and I would agree with you that rather not.

This being said, there is still a chance for some reasonable theory coming from groups, and we have here, as a complement to Question 2.17 and Question 2.18:

QUESTION 2.20. *What is the complex analogue of the symmetric group S_N ?*

As before with Question 2.17 and Question 2.18, this is something quite philosophical. But the subject is now quite fruitful, because there are plenty of interesting reflection groups $G_N \subset U_N$, and with a bit of luck, by studying such groups, we can reach to an answer to our questions. Let us record this in the form of a second thought, as follows:

THOUGHT 2.21. *The answers to our questions should come from a diagram of type*

$$\begin{array}{ccc} K_N & \text{-----} & U_N \\ | & & | \\ H_N & \text{-----} & O_N \end{array}$$

with $H_N \subset K_N \subset U_N$ being certain reflection groups, and with $H_N = S_N$, ideally.

Here we have used an unknown H_N instead of $H_N = S_N$ that we are interested in, and this because of Thought 2.19, which is still there, suggesting potential troubles.

Anyway, what to do? Get to work, of course, and in the lack of any clear idea, let us do some character computations for reflection groups $G_N \subset U_N$. An obvious choice here is the hyperoctahedral group, whose definition and basic properties are as follows:

THEOREM 2.22. *Consider the hyperoctahedral group $H_N \subset O_N$, consisting of the various symmetries of the hypercube in \mathbb{R}^N .*

- (1) H_N is the symmetry group of the N coordinate axes of \mathbb{R}^N .
- (2) H_N consists of the permutation-like matrices over $\{-1, 0, 1\}$.
- (3) We have the cardinality formula $|H_N| = 2^N N!$.
- (4) We have a crossed product decomposition $H_N = S_N \rtimes \mathbb{Z}_2^N$.
- (5) We have a wreath product decomposition $H_N = \mathbb{Z}_2 \wr S_N$.

PROOF. Consider indeed the standard cube in \mathbb{R}^N , which is by definition centered at 0, and having as vertices the points having coordinates ± 1 .

(1) With the above picture of the cube in hand, it is clear that the symmetries of the cube coincide with the symmetries of the N coordinate axes of \mathbb{R}^N .

(2) Each of the permutations $\sigma \in S_N$ of the N coordinate axes of \mathbb{R}^N can be further “decorated” by a sign vector $\varepsilon \in \{\pm 1\}^N$, consisting of the possible ± 1 flips which can be applied to each coordinate axis, at the arrival. In matrix terms, this gives the result.

(3) By using the above interpretation of H_N , we have the following formula:

$$|H_N| = |S_N| \cdot |\mathbb{Z}_2^N| = N! \cdot 2^N$$

(4) We know from (3) that at the level of cardinalities we have $|H_N| = |S_N \times \mathbb{Z}_2^N|$, and with a bit more work, we obtain that we have $H_N = S_N \rtimes \mathbb{Z}_2^N$, as claimed.

(5) This is simply a reformulation of (4), in terms of wreath products. \square

Getting back now to our character computations, following [10], we have:

THEOREM 2.23. *For the hyperoctahedral group $H_N \subset O_N$, the law of the variable $\chi = g_{11} + \dots + g_{ss}$ with $s = [tN]$ is, in the $N \rightarrow \infty$ limit, the following measure,*

$$b_t = e^{-t} \sum_{r=-\infty}^{\infty} \delta_r \sum_{p=0}^{\infty} \frac{(t/2)^{|r|+2p}}{(|r|+p)!p!}$$

called *Bessel law of parameter $t > 0$* .

PROOF. We regard H_N as being the symmetry group of the graph $I_N = \{I^1, \dots, I^N\}$ formed by n segments. The diagonal coefficients are then given by:

$$u_{ii}(g) = \begin{cases} 0 & \text{if } g \text{ moves } I^i \\ 1 & \text{if } g \text{ fixes } I^i \\ -1 & \text{if } g \text{ returns } I^i \end{cases}$$

Let us denote by F_g, R_g the number of segments among $\{I^1, \dots, I^s\}$ which are fixed, respectively returned by an element $g \in H_N$. With this notation, we have:

$$u_{11} + \dots + u_{ss} = F_g - R_g$$

We denote by P_N probabilities computed over H_N . The density of the law of the variable $u_{11} + \dots + u_{ss}$ at a point $r \geq 0$ is then given by the following formula:

$$D(r) = P_N(F_g - R_g = r) = \sum_{p=0}^{\infty} P_N(F_g = r + p, R_g = p)$$

Assume first that we are in the case $t = 1$. We have the following computation:

$$\begin{aligned} \lim_{N \rightarrow \infty} D(r) &= \lim_{N \rightarrow \infty} \sum_{p=0}^{\infty} (1/2)^{r+2p} \binom{r+2p}{r+p} P_N(F_g + R_g = r + 2p) \\ &= \sum_{p=0}^{\infty} (1/2)^{r+2p} \binom{r+2p}{r+p} \frac{1}{e^{(r+2p)!}} \\ &= \frac{1}{e} \sum_{p=0}^{\infty} \frac{(1/2)^{r+2p}}{(r+p)!p!} \end{aligned}$$

The general case $0 < t \leq 1$ follows by performing some modifications in the above computation. Indeed, the asymptotic density can be computed as follows:

$$\begin{aligned} \lim_{N \rightarrow \infty} D(r) &= \lim_{N \rightarrow \infty} \sum_{p=0}^{\infty} (1/2)^{r+2p} \binom{r+2p}{r+p} P_N(F_g + R_g = r + 2p) \\ &= \sum_{p=0}^{\infty} (1/2)^{r+2p} \binom{r+2p}{r+p} \frac{t^{r+2p}}{e^t (r+2p)!} \\ &= e^{-t} \sum_{p=0}^{\infty} \frac{(t/2)^{r+2p}}{(r+p)!p!} \end{aligned}$$

Together with $D(-r) = D(r)$, this gives the formula in the statement. \square

The above result is quite interesting, because the densities that we found there are the following functions, called Bessel functions of the first kind:

$$f_r(t) = \sum_{p=0}^{\infty} \frac{t^{|r|+2p}}{(|r|+p)!p!}$$

Due to this fact, the limiting measures are called Bessel laws, as mentioned in Theorem 2.23. Let us study now these Bessel laws. We first have the following result, from [10]:

THEOREM 2.24. *The Bessel laws b_t have the property*

$$b_s * b_t = b_{s+t}$$

so they form a truncated one-parameter semigroup with respect to convolution.

PROOF. With $f_r(t)$ being the Bessel functions of the first kind, we have:

$$b_t = e^{-t} \sum_{r=-\infty}^{\infty} \delta_r f_r(t/2)$$

The Fourier transform of this measure b_t is given by:

$$F_{b_t}(y) = e^{-t} \sum_{r=-\infty}^{\infty} e^{iry} f_r(t/2)$$

We compute now the derivative with respect to the variable t :

$$F_{b_t}(y)' = -F_{b_t}(y) + \frac{e^{-t}}{2} \sum_{r=-\infty}^{\infty} e^{iry} f_r'(t/2)$$

On the other hand, the derivative of f_r with $r \geq 1$ is given by:

$$\begin{aligned} f_r'(t) &= \sum_{p=0}^{\infty} \frac{(r+2p)t^{r+2p-1}}{(r+p)!p!} \\ &= \sum_{p=0}^{\infty} \frac{(r+p)t^{r+2p-1}}{(r+p)!p!} + \sum_{p=0}^{\infty} \frac{pt^{r+2p-1}}{(r+p)!p!} \\ &= \sum_{p=0}^{\infty} \frac{t^{r+2p-1}}{(r+p-1)!p!} + \sum_{p=1}^{\infty} \frac{t^{r+2p-1}}{(r+p)!(p-1)!} \\ &= \sum_{p=0}^{\infty} \frac{t^{(r-1)+2p}}{((r-1)+p)!p!} + \sum_{p=1}^{\infty} \frac{t^{(r+1)+2(p-1)}}{((r+1)+(p-1))!(p-1)!} \\ &= f_{r-1}(t) + f_{r+1}(t) \end{aligned}$$

This computation works in fact for any r , and we obtain in this way:

$$\begin{aligned} F_{b_t}(y)' &= -F_{b_t}(y) + \frac{e^{-t}}{2} \sum_{r=-\infty}^{\infty} e^{iry} (f_{r-1}(t/2) + f_{r+1}(t/2)) \\ &= -F_{b_t}(y) + \frac{e^{-t}}{2} \sum_{r=-\infty}^{\infty} e^{i(r+1)y} f_r(t/2) + e^{i(r-1)y} f_r(t/2) \\ &= -F_{b_t}(y) + \frac{e^{iy} + e^{-iy}}{2} F_{b_t}(y) \\ &= \left(\frac{e^{iy} + e^{-iy}}{2} - 1 \right) F_{b_t}(y) \end{aligned}$$

By integrating, we obtain from this the following formula:

$$F_{b_t}(y) = \exp \left(\left(\frac{e^{iy} + e^{-iy}}{2} - 1 \right) t \right)$$

Thus the log of the Fourier transform is linear in t , and we get the assertion. \square

In order to further discuss all this, and extend the above results, we will need a number of standard probabilistic preliminaries. We have the following notion, extending the Poisson limit theory developed in the beginning of the present chapter:

DEFINITION 2.25. *Associated to any compactly supported positive measure ν on \mathbb{C} , not necessarily of mass 1, is the probability measure*

$$p_\nu = \lim_{n \rightarrow \infty} \left(\left(1 - \frac{t}{n} \right) \delta_0 + \frac{1}{n} \nu \right)^{*n}$$

where $t = \text{mass}(\nu)$, called *compound Poisson law*.

In what follows we will be mainly interested in the case where the measure ν is discrete, as is for instance the case for $\nu = t\delta_1$ with $t > 0$, which produces the Poisson laws. The following standard result allows one to detect compound Poisson laws:

PROPOSITION 2.26. *For $\nu = \sum_{i=1}^s t_i \delta_{z_i}$ with $t_i > 0$ and $z_i \in \mathbb{C}$, we have*

$$F_{p_\nu}(y) = \exp \left(\sum_{i=1}^s t_i (e^{iyz_i} - 1) \right)$$

where F denotes the Fourier transform.

PROOF. Let η_n be the measure in Definition 2.25, under the convolution sign:

$$\eta_n = \left(1 - \frac{t}{n} \right) \delta_0 + \frac{1}{n} \nu$$

We have then the following computation:

$$\begin{aligned} F_{\eta_n}(y) = \left(1 - \frac{t}{n} \right) + \frac{1}{n} \sum_{i=1}^s t_i e^{iyz_i} &\implies F_{\eta_n^{*n}}(y) = \left(\left(1 - \frac{t}{n} \right) + \frac{1}{n} \sum_{i=1}^s t_i e^{iyz_i} \right)^n \\ &\implies F_{p_\nu}(y) = \exp \left(\sum_{i=1}^s t_i (e^{iyz_i} - 1) \right) \end{aligned}$$

Thus, we have obtained the formula in the statement. \square

We have as well the following result, providing an alternative to Definition 2.25, and which will be our formulation here of the Compound Poisson Limit Theorem:

THEOREM 2.27 (CPLT). For $\nu = \sum_{i=1}^s t_i \delta_{z_i}$ with $t_i > 0$ and $z_i \in \mathbb{C}$, we have

$$p_\nu = \text{law} \left(\sum_{i=1}^s z_i \alpha_i \right)$$

where the variables α_i are Poisson (t_i), independent.

PROOF. Let α be the sum of Poisson variables in the statement, namely:

$$\alpha = \sum_{i=1}^s z_i \alpha_i$$

By using some standard Fourier transform formulae, we have:

$$\begin{aligned} F_{\alpha_i}(y) = \exp(t_i(e^{iy} - 1)) &\implies F_{z_i \alpha_i}(y) = \exp(t_i(e^{iy z_i} - 1)) \\ &\implies F_\alpha(y) = \exp \left(\sum_{i=1}^s t_i (e^{iy z_i} - 1) \right) \end{aligned}$$

Thus we have indeed the same formula as in Proposition 2.26, as desired. \square

Summarizing, we have now a full generalization of the PLT. Getting back now to the Poisson and Bessel laws, with the above formalism in hand, we have:

THEOREM 2.28. *The Poisson and Bessel laws are compound Poisson laws,*

$$p_t = p_{t\delta_1} \quad , \quad b_t = p_{t\varepsilon}$$

where δ_1 is the Dirac mass at 1, and ε is the centered Bernoulli law, $\varepsilon = (\delta_1 + \delta_{-1})/2$.

PROOF. We have two assertions here, the idea being as follows:

(1) The first assertion, regarding the Poisson law p_t , is clear from Definition 2.25, which for $\nu = t\delta_1$ takes the following form:

$$p_\nu = \lim_{n \rightarrow \infty} \left(\left(1 - \frac{t}{n} \right) \delta_0 + \frac{t}{n} \delta_1 \right)^{*n}$$

Indeed, according to the PLT, the limit on the right produces the Poisson law p_t , as desired. Alternatively, the result follows as well from Proposition 2.26, which gives:

$$F_{p_\nu}(y) = \exp(t(e^{iy} - 1))$$

But the simplest way of proving the result is by invoking Theorem 2.28, which tells us that for $\nu = t\delta_1$ we have $p_\nu = \text{law}(\alpha)$, with α being Poisson (t).

(2) Regarding the second assertion, concerning b_t , the most convenient here is to use the formula of the Fourier transform found in the proof of Theorem 2.24, namely:

$$F_{b_t}(y) = \exp \left(t \left(\frac{e^{iy} + e^{-iy}}{2} - 1 \right) \right)$$

On the other hand, the formula in Proposition 2.26 gives, for $\nu = t\varepsilon$:

$$F_{p_\nu}(y) = \exp\left(\frac{t}{2}(e^{iy} - 1) + \frac{t}{2}(e^{-iy} - 1)\right)$$

Thus, with $\nu = t\varepsilon$ we have $p_\nu = b_t$, as claimed. \square

As a conclusion to all this, we can add the Bessel laws b_t to the family of “interesting” probability measures that we have, consisting so far of the real and complex Gaussian laws g_t and G_t , and the Poisson laws p_t . Indeed, the measures p_t, b_t, g_t, G_t all appear via basic limiting theorems, they form convolution semigroups, and they are related to group theory as well, and more specifically to the groups S_N, H_N, O_N, U_N .

Still missing, however, for b_t is a combinatorial formula for the moments, in the spirit of the formulae that we have for p_t, g_t, G_t . This is something quite tricky, and the formula is as follows, with P_{even} standing for the partitions all whose blocks have even size:

$$M_k(b_t) = \sum_{\pi \in P_{\text{even}}(k)} t^{|\pi|}$$

It is possible to prove this out of what we have, for instance by taking the generating function of the above numbers, then converting this series into a Fourier one, with the conclusion that we obtain indeed the Fourier transform F_{b_t} computed above. However, the computations are quite complex, and this even in the simplest case, $t = 1$, and instead of embarking into this, we will leave it for later, when we will have better tools.

2d. Bessel laws

Moving ahead, Theorem 2.28 suggests formulating the following definition, which unifies the Poisson laws with the real Bessel laws that we found above:

DEFINITION 2.29. *The Bessel law of level $s \in \mathbb{N} \cup \{\infty\}$ and parameter $t > 0$ is*

$$b_t^s = p_{t\varepsilon_s}$$

with ε_s being the uniform measure on the s -th roots of unity. The measures

$$b_t = b_t^2 \quad , \quad B_t = b_t^\infty$$

are called *real Bessel law*, and *complex Bessel law*.

Here we use the same convention as in the continuous case, namely that capital letters stand for complexifications. We will see in a moment that B_t is indeed a complexification of b_t , in a suitable sense, so that the couple b_t/B_t stands as the correct “discrete analogue” of the couple g_t/G_t . Which is something quite interesting, philosophically speaking.

In practice now, we first have to study the measures b_t^s in our standard way, meaning density, moments, Fourier, semigroup property, limiting theorems, and relation with group

theory. In what regards limiting theorems, the measures b_t^s appear by definition via the CPLT, so done with that. As a consequence of this, however, let us record:

PROPOSITION 2.30. *The Bessel laws are given by*

$$b_t^s = \text{law} \left(\sum_{k=1}^s w^k a_k \right)$$

where a_1, \dots, a_s are Poisson (t) independent, and $w = e^{2\pi i/s}$.

PROOF. At $s = 1, 2$ this is something that we already know, coming from Theorem 2.28 and its proof. In general, this follows from Theorem 2.27. \square

Following [9], where the laws b_t^s were introduced and studied, let us discuss now Fourier transforms and the semigroup property. Consider the level s exponential function:

$$\exp_s z = \sum_{k=0}^{\infty} \frac{z^{sk}}{(sk)!}$$

We have then the following formula, in terms of $w = e^{2\pi i/s}$:

$$\exp_s z = \frac{1}{s} \sum_{k=1}^s \exp(w^k z)$$

Observe that $\exp_1 = \exp$ and $\exp_2 = \cosh$. We have the following result:

THEOREM 2.31. *The Fourier transform of b_t^s is given by*

$$\log F_t^s(z) = t (\exp_s z - 1)$$

where $\exp_s z$ is as above. In particular we have the formula

$$b_t^s * b_{t'}^s = b_{t+t'}^s$$

so the measures b_t^s form a one-parameter convolution semigroup.

PROOF. Consider, as in Proposition 2.30, the variable $a = \sum_{k=1}^s w^k a_k$. We have then the following Fourier transform computation:

$$\log F_a(z) = \sum_{k=1}^s \log F_{a_k}(w^k z) = \sum_{k=1}^s \frac{t}{s} (\exp(w^k z) - 1)$$

But this gives the following formula:

$$\log F_a(z) = t \left(\left(\frac{1}{s} \sum_{k=1}^s \exp(w^k z) \right) - 1 \right) = t (\exp_s z - 1)$$

Now since b_t^s is the law of a , this gives the formula in the statement. As for the last assertion, this comes from the fact that the log of the Fourier transform is linear in t . \square

Still following [9], we can compute the density of b_t^s , as follows:

THEOREM 2.32. *We have the formula*

$$b_t^s = e^{-t} \sum_{c_1=0}^{\infty} \cdots \sum_{c_s=0}^{\infty} \frac{1}{c_1! \cdots c_s!} \left(\frac{t}{s}\right)^{c_1+\dots+c_s} \delta \left(\sum_{k=1}^s w^k c_k \right)$$

where $w = e^{2\pi i/s}$, and the δ symbol is a Dirac mass.

PROOF. The Fourier transform of the measure on the right is given by:

$$\begin{aligned} F(z) &= e^{-t} \sum_{c_1=0}^{\infty} \cdots \sum_{c_s=0}^{\infty} \frac{1}{c_1! \cdots c_s!} \left(\frac{t}{s}\right)^{c_1+\dots+c_s} F \delta \left(\sum_{k=1}^s w^k c_k \right) (z) \\ &= e^{-t} \sum_{c_1=0}^{\infty} \cdots \sum_{c_s=0}^{\infty} \frac{1}{c_1! \cdots c_s!} \left(\frac{t}{s}\right)^{c_1+\dots+c_s} \exp \left(\sum_{k=1}^s w^k c_k z \right) \\ &= e^{-t} \sum_{r=0}^{\infty} \left(\frac{t}{s}\right)^r \sum_{\Sigma c_i=r} \frac{\exp \left(\sum_{k=1}^s w^k c_k z \right)}{c_1! \cdots c_s!} \end{aligned}$$

We multiply now by e^t , and we compute the derivative with respect to t :

$$\begin{aligned} (e^t F(z))' &= \sum_{r=1}^{\infty} \frac{r}{s} \left(\frac{t}{s}\right)^{r-1} \sum_{\Sigma c_i=r} \frac{\exp \left(\sum_{k=1}^s w^k c_k z \right)}{c_1! \cdots c_s!} \\ &= \frac{1}{s} \sum_{r=1}^{\infty} \left(\frac{t}{s}\right)^{r-1} \sum_{\Sigma c_i=r} \left(\sum_{l=1}^s c_l \right) \frac{\exp \left(\sum_{k=1}^s w^k c_k z \right)}{c_1! \cdots c_s!} \\ &= \frac{1}{s} \sum_{r=1}^{\infty} \left(\frac{t}{s}\right)^{r-1} \sum_{\Sigma c_i=r} \sum_{l=1}^s \frac{\exp \left(\sum_{k=1}^s w^k c_k z \right)}{c_1! \cdots c_{l-1}! (c_l - 1)! c_{l+1}! \cdots c_s!} \end{aligned}$$

By using the variable $u = r - 1$, we obtain in this way:

$$\begin{aligned} (e^t F(z))' &= \frac{1}{s} \sum_{u=0}^{\infty} \left(\frac{t}{s}\right)^u \sum_{\Sigma d_i=u} \sum_{l=1}^s \frac{\exp \left(w^l z + \sum_{k=1}^s w^k d_k z \right)}{d_1! \cdots d_s!} \\ &= \left(\frac{1}{s} \sum_{l=1}^s \exp(w^l z) \right) \left(\sum_{u=0}^{\infty} \left(\frac{t}{s}\right)^u \sum_{\Sigma d_i=u} \frac{\exp \left(\sum_{k=1}^s w^k d_k z \right)}{d_1! \cdots d_s!} \right) \\ &= (\exp_s z)(e^t F(z)) \end{aligned}$$

On the other hand, $\Phi(t) = \exp(t \exp_s z)$ satisfies the same equation, namely:

$$\Phi'(t) = (\exp_s z) \Phi(t)$$

Thus, we have the $e^t F(z) = \Phi(t)$, which gives the following formula:

$$\begin{aligned} \log F &= \log(e^{-t} \exp(t \exp_s z)) \\ &= \log(\exp(t(\exp_s z - 1))) \\ &= t(\exp_s z - 1) \end{aligned}$$

Thus, we obtain the formulae in the statement. \square

Regarding now the questions which are left, namely moments and relation with groups, these are quite technical, and related. Let us start by discussing the relation with groups. Obviously we need here a generalization of the groups S_N, H_N , involving a parameter $s \in \mathbb{N} \cup \{\infty\}$, and the answer to this question is straightforward, as follows:

DEFINITION 2.33. *The complex reflection group $H_N^s \subset U_N$ is the group of permutations of N copies of the s -simplex. Equivalently, we have*

$$H_N^s = M_N(\mathbb{Z}_s \cup \{0\}) \cap U_N$$

telling us that H_N^s consists of the permutation-type matrices with s -th roots of unity as entries. Also equivalently, we have the formula $H_N^s = \mathbb{Z}_s \wr S_N$.

Here the equivalence between the various viewpoints on H_N^s comes as in Theorem 2.22, which corresponds to the case $s = 2$. In fact, the basic examples are as follows:

- (1) $s = 1$. Here $H_N^1 = S_N$, trivially, no matter which viewpoint we take.
- (2) $s = 2$. Here $H_N^2 = H_2$, with this coming from Theorem 2.22.
- (3) $s = \infty$. Here $H_N^\infty = K_N$ is an interesting group, and more on it later.

In general, H_N^s are well-known in group theory, the idea being that, up to a number of exceptional examples, the complex reflection groups are exactly these groups H_N^s , and their versions H_N^{sd} obtained by adding the supplementary condition $(\det U)^d = 1$.

In relation with the Bessel laws, we have the following result, from [9]:

THEOREM 2.34. *For the complex reflection group H_N^s we have, with $N \rightarrow \infty$:*

$$\chi_t \sim b_t^s$$

Moreover, the asymptotic moments of this variable are the numbers

$$M_k(b_t^s) = \sum_{\pi \in P^s(k)} t^{|\pi|}$$

where $P^s(k)$ are the partitions of $\{1, \dots, k\}$ satisfying $\#\circ = \#\bullet(s)$, in each block.

PROOF. This is something quite long, that we will discuss in detail in chapters 3-4 below, when systematically doing representation theory, the idea being as follows:

(1) At $s = 1$ the reflection group is $H_N^1 = S_N$, the Bessel law is the Poisson law, $b_t^1 = p_t$, and the formula $\chi_t \sim p_t$ with $N \rightarrow \infty$ is something that we know. As for the moment formula, where $P^1 = P$, this is something that we know too.

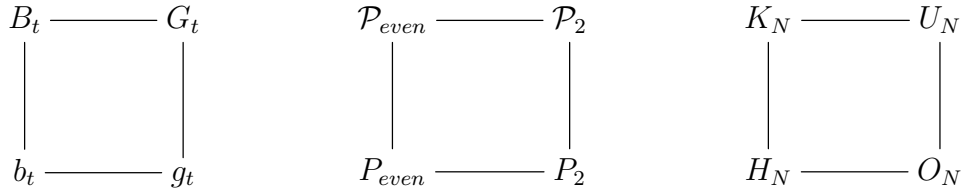
(2) At $s = 2$ the reflection group is $H_N^2 = H_N$, the Bessel law is $b_t^2 = b_t$, and the formula $\chi_t \sim b_t$ with $N \rightarrow \infty$ is something that we know. As for the moment formula, where $P^2 = P_{even}$, this is something more technical, which remains to be discussed.

(3) At $s = \infty$ the reflection group is $H_N^\infty = K_N$, the Bessel law is $b_t^\infty = B_t$, and the formula $\chi_t \sim B_t$ with $N \rightarrow \infty$ is something that can be proved as for S_N, H_N . As for the moment formula, where $P^\infty = \mathcal{P}_{even}$, this remains to be discussed.

(4) In the general case, where $s \in \mathbb{N} \cup \{\infty\}$, the formula $\chi_t \sim b_t^s$ with $N \rightarrow \infty$ can be established a bit like for S_N, H_N , and the moment formula is something quite technical. We will discuss both questions in chapter 4 below, using more advanced tools. \square

All the above is very nice, theoretically speaking, and we can now answer the various philosophical questions raised in the beginning of this section, as follows:

CONCLUSION 2.35. *There is no complex Poisson law P_t , no set of matching partitions \mathcal{P} , and no complex analogue of S_N . However, we have diagrams*



with the laws b_t, B_t being related to $P_{even}, \mathcal{P}_{even}$ and to the groups H_N, K_N in the standard way, and these laws b_t, B_t are the true discrete analogues of g_t, G_t .

Summarizing, what we did so far in this book, namely the Gaussian and Poisson laws, and their various versions, have interesting combinatorics. All the above was an introduction to this combinatorics, following the classical theory, and [9], [10], [36] and related papers. We will be back to these laws and results on numerous occasions.

Let us also mention that all the above is in fact just half of the story, because all 4 measures in Conclusion 2.35 are of “classical” nature, and there will be 4 more measures, appearing as “free versions” of these. So, expect our final result on the subject to be a cube formed by 8 measures, coming with accompanying cubes of partitions and groups. But more on this later, after some substantial work, towards the end of this book.

2e. Exercises

There has been a lot of non-trivial material in this chapter, sometimes only briefly explained, and this is because we will come back to this later, with more powerful tools. However, before that, let us start with a standard and beautiful exercise:

EXERCISE 2.36. *Prove that the Bell numbers $B_k = |P(k)|$, which are the moments of the Poisson law p_1 , have the following properties:*

$$B_{k+1} = \sum_{r=0}^k \binom{k}{r} B_r \quad , \quad B_k = \frac{1}{e} \sum_{r=0}^{\infty} \frac{r^k}{r!}$$

$$\sum_{k=0}^{\infty} \frac{B_k}{k!} z^k = e^{e^z - 1} \quad , \quad B_k = \frac{k!}{2\pi i e} \int_{|z|=1} \frac{e^{e^z}}{z^{k+1}} dz$$

Also, prove as well that we have $\log B_k/k \simeq \log k - \log \log k - 1$.

Here some of the formulae are things that we already know, from the above, some other formulae are fairly easy, and some other are more difficult.

EXERCISE 2.37. *Prove that for the cyclic group $\mathbb{Z}_N \subset O_N$ we have*

$$\text{law}(\chi) = \left(1 - \frac{1}{N}\right) \delta_0 + \frac{1}{N} \delta_N$$

and look as well at truncated characters.

This looks quite elementary, and indeed it is, matter of having things started, in relation with character computations, beyond what has been said in the above.

EXERCISE 2.38. *Prove that for the dihedral group $D_N \subset S_N$ we have*

$$\text{law}(\chi) = \begin{cases} \left(\frac{3}{4} - \frac{1}{2N}\right) \delta_0 + \frac{1}{4} \delta_2 + \frac{1}{2N} \delta_N & (N \text{ even}) \\ \left(\frac{1}{2} - \frac{1}{2N}\right) \delta_0 + \frac{1}{2} \delta_1 + \frac{1}{2N} \delta_N & (N \text{ odd}) \end{cases}$$

and look as well at truncated characters.

Again, this is something which can only be quite elementary. As a conclusion to this, the character laws for \mathbb{Z}_N, D_N have no interesting asymptotics.

EXERCISE 2.39. *Prove that, if g_{ij} are the standard coordinates of $S_N \subset O_N$,*

$$\text{law}(g_{11} + \dots + g_{ss}) = \frac{s!}{N!} \sum_{p=0}^s \frac{(N-p)!}{(s-p)!} \cdot \frac{(\delta_1 - \delta_0)^{*p}}{p!}$$

and deduce from this that such variables become Poisson, with $N \rightarrow \infty$.

As a bonus exercise, you can try to work out all the missing details for the various computations involving the complex reflection groups H_N^s , and the Bessel laws b_t^s .

CHAPTER 3

Random walks

3a. Random walks

We have learned so far the basics of theoretical probability, and time now to see if this knowledge can be of any help, in relation with concrete questions. The question that we would like to discuss, which is something very basic, is as follows:

QUESTION 3.1. *Given a graph X , with a distinguished vertex $*$:*

- (1) *What is the number L_k of length k loops on X , based at $*$?*
- (2) *Equivalently, what is the measure μ having L_k as moments?*

To be more precise, we are mainly interested in the first question, counting loops on graphs, with this being notoriously related to many applied mathematics questions, of discrete type. As for the second question, this is a technical, useful probabilistic reformulation of the first question, that we will usually prefer, in what follows.

Actually, in relation with this, the fact that a measure μ as above exists indeed is not exactly obvious. But comes from the following result, which is something rather elementary, and which can be very helpful for explicit computations:

THEOREM 3.2. *Given a graph X , with adjacency matrix $d \in M_N(0, 1)$, we have:*

$$L_k = (d^k)_{**}$$

When writing $d = UDU^t$ with $U \in O_N$ and $D = \text{diag}(\lambda_1, \dots, \lambda_N)$ with $\lambda_i \in \mathbb{R}$, we have

$$L_k = \sum_i U_{*i}^2 \lambda_i^k$$

and the real probability measure μ having these numbers as moments is given by

$$\mu = \sum_i U_{*i}^2 \delta_{\lambda_i}$$

with the delta symbols standing as usual for Dirac masses.

PROOF. There are several things going on here, the idea being as follows:

(1) According to the usual rule of matrix multiplication, the formula for the powers of the adjacency matrix $d \in M_N(0, 1)$ is as follows:

$$\begin{aligned}
(d^k)_{i_0 i_k} &= \sum_{i_1, \dots, i_{k-1}} d_{i_0 i_1} d_{i_1 i_2} \cdots d_{i_{k-1} i_k} \\
&= \sum_{i_1, \dots, i_{k-1}} \delta_{i_0 - i_1} \delta_{i_1 - i_2} \cdots \delta_{i_{k-1} - i_k} \\
&= \sum_{i_1, \dots, i_{k-1}} \delta_{i_0 - i_1 - \dots - i_{k-1} - i_k} \\
&= \#\{i_0 - i_1 - \dots - i_{k-1} - i_k\}
\end{aligned}$$

In particular, with $i_0 = i_k = *$, we obtain the following formula, as claimed:

$$(d^k)_{**} = \#\{* - i_1 - \dots - i_{k-1} - *\} = L_k$$

(2) Now since the adjacency matrix $d \in M_N(0, 1)$ is symmetric, by basic linear algebra, that we will recall in chapter 5 below, this matrix is diagonalizable, with the diagonalization being as follows, with $U \in O_N$, and $D = \text{diag}(\lambda_1, \dots, \lambda_N)$ with $\lambda_i \in \mathbb{R}$:

$$d = UDU^t$$

By using this formula, we obtain the second formula in the statement:

$$\begin{aligned}
L_k &= (d^k)_{**} \\
&= (UD^k U^t)_{**} \\
&= \sum_i U_{*i} \lambda_i^k (U^t)_{i*} \\
&= \sum_i U_{*i}^2 \lambda_i^k
\end{aligned}$$

(3) Finally, the last assertion is clear from this, because the moments of the measure in the statement, $\mu = \sum_i U_{*i}^2 \delta_{\lambda_i}$, are the following numbers:

$$\begin{aligned}
M_k &= \int_{\mathbb{R}} x^k d\mu(x) \\
&= \sum_i U_{*i}^2 \lambda_i^k \\
&= L_k
\end{aligned}$$

Observe also that μ is indeed of mass 1, because all rows of $U \in O_N$ must be of norm 1, and so $\sum_i U_{*i}^2 = 1$. Thus, we are led to the conclusions in the statement. \square

At the level of examples now, what are the simplest graphs X , that we can try to do some loop computations for? And here, we have 3 possible answers, as follows:

FACT 3.3. *The following are graphs X , with a distinguished vertex $0 \in X$:*

- (1) *The circle graph, having N vertices, with 0 being one of the vertices.*
- (2) *The segment graph, having N vertices, with 0 being the vertex at left.*
- (3) *The segment graph, having $2N + 1$ vertices, with 0 being in the middle.*

So, let us start with these. However, the computations are quite non-trivial, and you can try doing some, in order to understand what I am talking about. So, let us pull instead an analysis trick, and formulate the following modest, informal result:

THEOREM 3.4. *For the circle graph, having N vertices, the number of length k loops based at one of the vertices is approximately*

$$L_k \simeq \frac{2^k}{N}$$

in the $k \rightarrow \infty$ limit, when N is odd, and is approximately

$$L_k \simeq \begin{cases} \frac{2^{k+1}}{N} & (k \text{ even}) \\ 0 & (k \text{ odd}) \end{cases}$$

also with $k \rightarrow \infty$, when N is even. However, in what regards the two segment graphs, we can expect here things to be more complicated.

PROOF. This is something not exactly trivial, and with the way the statement is written, which is clearly informal, witnessing for that. The idea is as follows:

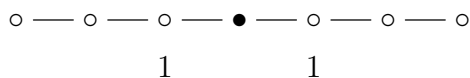
(1) Consider the circle graph X , with vertices denoted $0, 1, \dots, N - 1$. Since each vertex has valence 2, any length k path based at 0 will consist of a binary choice at the beginning, then another binary choice afterwards, and so on up to a k -th binary choice at the end. Thus, there is a total of 2^k such paths, based at 0 , and having length k .

(2) But now, based on the obvious “uniformity” of the circle, we can argue that, in the $k \rightarrow \infty$ limit, the endpoint of such a path will become random among the vertices $0, 1, \dots, N - 1$. Thus, if we want this endpoint to be 0 , as to have a loop, we have $1/N$ chances for this to happen, so the total number of loops is $L_k \simeq 2^k/N$, as stated.

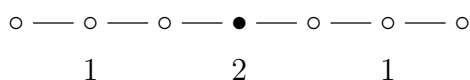
(3) With the remark, however, that the above argument works fine only when N is odd. Indeed, when N is even, the endpoint of a length k path will be random among $0, 2, \dots, 2N - 2$ when k is even, and random among $1, 3, \dots, 2N - 1$ when k is odd. Thus for getting a loop we must assume that k is even, and in this case the number of such loops is the total number of length k paths, namely 2^k , approximately divided by $N/2$, the number of points in $\{0, 2, \dots, 2N - 2\}$, which gives $L_k = 2^k/(N/2)$, as stated.

(4) Moving ahead now to the segment graphs, it is pretty much clear that for both, we lack the “uniformity” needed in (2), and this due to the 2 endpoints of the segment. In fact, thinking well, these graphs are no longer 2-valent, again due to the 2 endpoints, each having valence 1, and so even (1) must be fixed. And so, we will stop here. \square

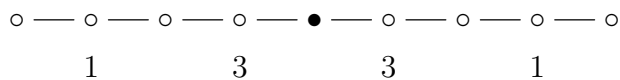
So, what to do? As an idea, let us look instead at the infinite graphs, and try to count the length k paths on \mathbb{Z} , based at 0. At $k = 1$ we have 2 such paths, ending at -1 and 1 , and the count results can be pictured as follows, in a self-explanatory way:



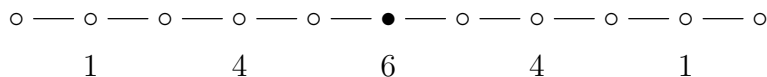
At $k = 2$ now, we have 4 paths, one of which ends at -2 , two of which end at 0 , and one of which ends at 2 . The results can be pictured as follows:



At $k = 3$ now, we have 8 paths, the distribution of the endpoints being as follows:



As for $k = 4$, here we have 16 paths, the distribution of the endpoints being as follows:



And good news, we can see in the above the Pascal triangle. Thus, getting back now to Question 3.1, we can answer it for the graph \mathbb{Z} , the result being as follows:

THEOREM 3.5. *The paths on \mathbb{Z} are counted by the binomial coefficients. In particular, the $2k$ -paths based at 0 are counted by the central binomial coefficients,*

$$L_{2k} = \binom{2k}{k}$$

and μ is the centered measure having these numbers as even moments.

PROOF. This basically follows from the above discussion, as follows:

(1) In what regards the count, we certainly have the Pascal triangle, as discovered above, and the rest is just a matter of finishing. There are many possible ways here, a straightforward one being that of arguing that the number C_k^l of length k loops $0 \rightarrow l$ is subject, due to the binary choice at the end, to the following recurrence relation:

$$C_k^l = C_{k-1}^{l-1} + C_{k-1}^{l+1}$$

But this is exactly the recurrence for the Pascal triangle, so done with the count.

(2) As for the second assertion, the first part, regarding L_{2k} , is clear from this, and the second part is more of an empty statement, with μ remaining to be computed. \square

3b. Catalan numbers

As a second illustration, let us try to count the loops of \mathbb{N} , based at 0. This is something less obvious, and at the experimental level, the result is as follows:

PROPOSITION 3.6. *The Catalan numbers C_k , counting the loops on \mathbb{N} based at 0,*

$$C_k = \#\{0 - i_1 - \dots - i_{2k-1} - 0\}$$

are numerically 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, ...

PROOF. To start with, we have indeed $C_1 = 1$, the only loop here being $0 - 1 - 0$. Then we have $C_2 = 2$, due to two possible loops, namely:

$$0 - 1 - 0 - 1 - 0$$

$$0 - 1 - 2 - 1 - 0$$

Then we have $C_3 = 5$, the possible loops here being as follows:

$$0 - 1 - 0 - 1 - 0 - 1 - 0$$

$$0 - 1 - 0 - 1 - 2 - 1 - 0$$

$$0 - 1 - 2 - 1 - 0 - 1 - 0$$

$$0 - 1 - 2 - 1 - 2 - 1 - 0$$

$$0 - 1 - 2 - 3 - 2 - 1 - 0$$

In general, the same method works, with $C_4 = 14$ being left to you, as an exercise, and with C_5 and higher to me, and I will be back with the solution, in due time. \square

Obviously, computing the numbers C_k is no easy task, and finding the formula of C_k , out of the data that we have, does not look as an easy task either. So, we will do what combinatorists do, let me teach you. The first step is to relax, then to look around, not with the aim of computing your numbers C_k , but rather with the aim of finding other objects counted by the same numbers C_k . With a bit of luck, among these objects some will be easier to count than the others, and this will eventually compute C_k .

This was for the strategy. In practice now, we first have the following result:

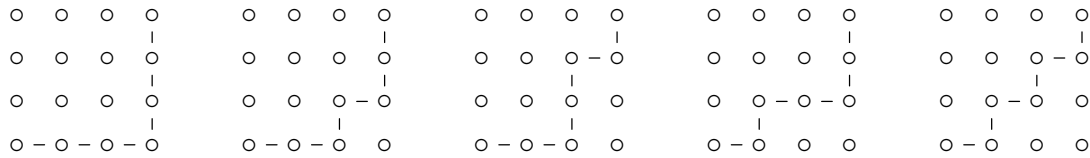
THEOREM 3.7. *The Catalan numbers C_k count:*

- (1) *The length $2k$ loops on \mathbb{N} , based at 0.*
- (2) *The noncrossing pairings of $1, \dots, 2k$.*
- (3) *The noncrossing partitions of $1, \dots, k$.*
- (4) *The length $2k$ Dyck paths in the plane.*

PROOF. All this is standard combinatorics, the idea being as follows:

(1) To start with, in what regards the various objects involved, the length $2k$ loops on \mathbb{N} are the length $2k$ loops on \mathbb{N} that we know, and the same goes for the noncrossing pairings of $1, \dots, 2k$, and for the noncrossing partitions of $1, \dots, k$, the idea here being that you must be able to draw the pairing or partition in a noncrossing way.

(2) Regarding now the length $2k$ Dyck paths in the plane, these are by definition the paths from $(0, 0)$ to (k, k) , marching North-East over the integer lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$, by staying inside the square $[0, k] \times [0, k]$, and staying as well under the diagonal of this square. As an example, here are the 5 possible Dyck paths at $n = 3$:



(3) Thus, we have definitions for all the objects involved, and in each case, if you start counting them, as we did in Proposition 3.6 with the loops on \mathbb{N} , you always end up with the same sequence of numbers, namely those found in Proposition 3.6:

$$1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, \dots$$

(4) In order to prove now that (1-4) produce indeed the same numbers, many things can be said. The idea is that, leaving aside mathematical brevity, and more specifically abstract reasonings of type $a = b, b = c \implies a = c$, what we have to do, in order to fully understand what is going on, is to establish $\binom{4}{2} = 6$ equalities, via bijective proofs.

(5) But this can be done, indeed. As an example here, the noncrossing pairings of $1, \dots, 2k$ from (2) are in bijection with the noncrossing partitions of $1, \dots, k$ from (3), via fattening the pairings and shrinking the partitions. We will leave the details here as an instructive exercise, and exercise as well, to add (1) and (4) to the picture.

(6) However, matter of having our theorem formally proved, I mean by me professor and not by you student, here is a less elegant argument, which is however very quick, and does the job. The point is that, in each of the cases (1-4) under consideration, the numbers C_k that we get are easily seen to be subject to the following recurrence:

$$C_{k+1} = \sum_{a+b=k} C_a C_b$$

The initial data being the same, namely $C_1 = 1$ and $C_2 = 2$, in each of the cases (1-4) under consideration, we get indeed the same numbers. \square

Now we can pass to the second step, namely selecting in the above list the objects that we find the most convenient to count, and count them. This leads to:

THEOREM 3.8. *The Catalan numbers are given by the formula*

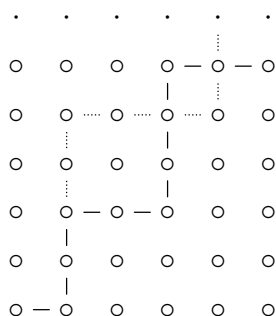
$$C_k = \frac{1}{k+1} \binom{2k}{k}$$

with this being best seen by counting the length $2k$ Dyck paths in the plane.

PROOF. This is something quite tricky, the idea being as follows:

(1) Let us count indeed the Dyck paths in the plane. For this purpose, we use a trick. Indeed, if we ignore the assumption that our path must stay under the diagonal of the square, we have $\binom{2k}{k}$ such paths. And among these, we have the “good” ones, those that we want to count, and then the “bad” ones, those that we want to ignore.

(2) So, let us count the bad paths, those crossing the diagonal of the square, and reaching the higher diagonal next to it, the one joining $(0, 1)$ and $(k, k + 1)$. In order to count these, the trick is to “flip” their bad part over that higher diagonal, as follows:



(3) Now observe that, as it is obvious on the above picture, due to the flipping, the flipped bad path will no longer end in (k, k) , but rather in $(k - 1, k + 1)$. Moreover, more is true, in the sense that, by thinking a bit, we see that the flipped bad paths are precisely those ending in $(k - 1, k + 1)$. Thus, we can count these flipped bad paths, and so the bad paths, and so the good paths too, and so good news, we are done.

(4) To finish now, by putting everything together, we have:

$$\begin{aligned} C_k &= \binom{2k}{k} - \binom{2k}{k-1} \\ &= \binom{2k}{k} - \frac{k}{k+1} \binom{2k}{k} \\ &= \frac{1}{k+1} \binom{2k}{k} \end{aligned}$$

Thus, we are led to the formula in the statement. □

We have as well another approach to all this, computation of the Catalan numbers, this time based on rock-solid standard calculus, as follows:

THEOREM 3.9. *The Catalan numbers have the following properties:*

- (1) *They satisfy $C_{k+1} = \sum_{a+b=k} C_a C_b$.*
- (2) *The series $f(z) = \sum_{k \geq 0} C_k z^k$ satisfies $z f^2 - f + 1 = 0$.*
- (3) *This series is given by $f(z) = \frac{1 - \sqrt{1 - 4z}}{2z}$.*
- (4) *We have the formula $C_k = \frac{1}{k+1} \binom{2k}{k}$.*

PROOF. This is best viewed by using noncrossing pairings, as follows:

(1) Let us count the noncrossing pairings of $\{1, \dots, 2k+2\}$. Such a pairing appears by pairing 1 to an odd number, $2a+1$, and then inserting a noncrossing pairing of $\{2, \dots, 2a\}$, and a noncrossing pairing of $\{2a+2, \dots, 2k+2\}$. Thus we have, as claimed:

$$C_{k+1} = \sum_{a+b=k} C_a C_b$$

(2) Consider now the generating series of the Catalan numbers, $f(z) = \sum_{k \geq 0} C_k z^k$. In terms of this generating series, the above recurrence gives, as desired:

$$\begin{aligned} z f^2 &= \sum_{a,b \geq 0} C_a C_b z^{a+b+1} \\ &= \sum_{k \geq 1} \sum_{a+b=k-1} C_a C_b z^k \\ &= \sum_{k \geq 1} C_k z^k \\ &= f - 1 \end{aligned}$$

(3) By solving the equation $z f^2 - f + 1 = 0$ found above, and choosing the solution which is bounded at $z = 0$, we obtain the following formula, as claimed:

$$f(z) = \frac{1 - \sqrt{1 - 4z}}{2z}$$

(4) In order to compute this function, we use the generalized binomial formula, which is as follows, with $p \in \mathbb{R}$ being an arbitrary exponent, and with $|t| < 1$:

$$(1+t)^p = \sum_{k=0}^{\infty} \binom{p}{k} t^k$$

To be more precise, this formula, which generalizes the usual binomial formula, holds indeed due to the Taylor formula, with the binomial coefficients being given by:

$$\binom{p}{k} = \frac{p(p-1)\dots(p-k+1)}{k!}$$

(5) For the exponent $p = 1/2$, the generalized binomial coefficients are:

$$\begin{aligned}
 \binom{1/2}{k} &= \frac{1/2(-1/2)(-3/2)\dots(3/2-k)}{k!} \\
 &= (-1)^{k-1} \frac{1 \cdot 3 \cdot 5 \dots (2k-3)}{2^k k!} \\
 &= (-1)^{k-1} \frac{(2k-2)!}{2^{k-1}(k-1)!2^k k!} \\
 &= \frac{(-1)^{k-1}}{2^{2k-1}} \cdot \frac{1}{k} \binom{2k-2}{k-1} \\
 &= -2 \left(\frac{-1}{4}\right)^k \cdot \frac{1}{k} \binom{2k-2}{k-1}
 \end{aligned}$$

(6) Thus the generalized binomial formula at exponent $p = 1/2$ reads:

$$\sqrt{1+t} = 1 - 2 \sum_{k=1}^{\infty} \frac{1}{k} \binom{2k-2}{k-1} \left(\frac{-t}{4}\right)^k$$

With $t = -4z$ we obtain from this the following formula:

$$\sqrt{1-4z} = 1 - 2 \sum_{k=1}^{\infty} \frac{1}{k} \binom{2k-2}{k-1} z^k$$

(7) Now back to our series f , we obtain the following formula for it:

$$\begin{aligned}
 f(z) &= \frac{1 - \sqrt{1-4z}}{2z} \\
 &= \sum_{k=1}^{\infty} \frac{1}{k} \binom{2k-2}{k-1} z^{k-1} \\
 &= \sum_{k=0}^{\infty} \frac{1}{k+1} \binom{2k}{k} z^k
 \end{aligned}$$

(8) Thus the Catalan numbers are given by the formula the statement, namely:

$$C_k = \frac{1}{k+1} \binom{2k}{k}$$

So done, and note in passing that I kept my promise, from the proof of Proposition 3.6. Indeed, with the above final formula, the numerics are easily worked out. \square

Many other things can be said about the Catalan numbers, as a continuation of the above, and about the central binomial coefficients too. We will be back to this.

In relation now with Question 3.1, we are led to the following questions:

QUESTION 3.10. *What are the following centered measures?*

- (1) *The measure having the central binomial coefficients as even moments.*
- (2) *The measure having the Catalan numbers as even moments.*

We will solve in what follows this question, among others with the aim of enlarging our menagery of interesting probability measures, consisting so far of the real and complex normal laws g_t, G_t , and of the Poisson laws p_t , and their compound versions.

3c. Stieltjes inversion

As explained above, the problem is now, how to recover a probability measure out of its moments. And the answer here, which is something non-trivial, is as follows:

THEOREM 3.11. *The density of a real probability measure μ can be recaptured from the sequence of moments $\{M_k\}_{k \geq 0}$ via the Stieltjes inversion formula*

$$d\mu(x) = \lim_{t \searrow 0} -\frac{1}{\pi} \operatorname{Im} (G(x + it)) \cdot dx$$

where the function on the right, given in terms of moments by

$$G(\xi) = \xi^{-1} + M_1 \xi^{-2} + M_2 \xi^{-3} + \dots$$

is the Cauchy transform of the measure μ .

PROOF. The Cauchy transform of our measure μ is given by:

$$\begin{aligned} G(\xi) &= \xi^{-1} \sum_{k=0}^{\infty} M_k \xi^{-k} \\ &= \int_{\mathbb{R}} \frac{\xi^{-1}}{1 - \xi^{-1}y} d\mu(y) \\ &= \int_{\mathbb{R}} \frac{1}{\xi - y} d\mu(y) \end{aligned}$$

Now with $\xi = x + it$, we obtain the following formula:

$$\begin{aligned} \operatorname{Im}(G(x + it)) &= \int_{\mathbb{R}} \operatorname{Im} \left(\frac{1}{x - y + it} \right) d\mu(y) \\ &= \int_{\mathbb{R}} \frac{1}{2i} \left(\frac{1}{x - y + it} - \frac{1}{x - y - it} \right) d\mu(y) \\ &= - \int_{\mathbb{R}} \frac{t}{(x - y)^2 + t^2} d\mu(y) \end{aligned}$$

By integrating over $[a, b]$ we obtain, with the change of variables $x = y + tz$:

$$\begin{aligned}
\int_a^b \operatorname{Im}(G(x + it)) dx &= - \int_{\mathbb{R}} \int_a^b \frac{t}{(x - y)^2 + t^2} dx d\mu(y) \\
&= - \int_{\mathbb{R}} \int_{(a-y)/t}^{(b-y)/t} \frac{t}{(tz)^2 + t^2} t dz d\mu(y) \\
&= - \int_{\mathbb{R}} \int_{(a-y)/t}^{(b-y)/t} \frac{1}{1 + z^2} dz d\mu(y) \\
&= - \int_{\mathbb{R}} \left(\arctan \frac{b - y}{t} - \arctan \frac{a - y}{t} \right) d\mu(y)
\end{aligned}$$

Now observe that with $t \searrow 0$ we have:

$$\lim_{t \searrow 0} \left(\arctan \frac{b - y}{t} - \arctan \frac{a - y}{t} \right) = \begin{cases} \frac{\pi}{2} - \frac{\pi}{2} = 0 & (y < a) \\ \frac{\pi}{2} - 0 = \frac{\pi}{2} & (y = a) \\ \frac{\pi}{2} - (-\frac{\pi}{2}) = \pi & (a < y < b) \\ 0 - (-\frac{\pi}{2}) = \frac{\pi}{2} & (y = b) \\ -\frac{\pi}{2} - (-\frac{\pi}{2}) = 0 & (y > b) \end{cases}$$

We therefore obtain the following formula:

$$\lim_{t \searrow 0} \int_a^b \operatorname{Im}(G(x + it)) dx = -\pi \left(\mu(a, b) + \frac{\mu(a) + \mu(b)}{2} \right)$$

Thus, we are led to the conclusion in the statement. \square

Before getting further, let us mention that the above result does not fully solve the moment problem, because we still have the question of understanding when a sequence of numbers M_1, M_2, M_3, \dots can be the moments of a measure μ . We have here:

THEOREM 3.12. *A sequence of numbers $M_0, M_1, M_2, M_3, \dots \in \mathbb{R}$, with $M_0 = 1$, is the series of moments of a real probability measure μ precisely when:*

$$|M_0| \geq 0 \quad , \quad \begin{vmatrix} M_0 & M_1 \\ M_1 & M_2 \end{vmatrix} \geq 0 \quad , \quad \begin{vmatrix} M_0 & M_1 & M_2 \\ M_1 & M_2 & M_3 \\ M_2 & M_3 & M_4 \end{vmatrix} \geq 0 \quad , \quad \dots$$

That is, the associated Hankel determinants must be all positive.

PROOF. This is something a bit more advanced, the idea being as follows:

(1) As a first observation, the positivity conditions in the statement tell us that the following associated linear forms must be positive:

$$\sum_{i,j=1}^n c_i \bar{c}_j M_{i+j} \geq 0$$

(2) But this is something very classical, in one sense the result being elementary, coming from the following computation, which shows that we have positivity indeed:

$$\begin{aligned} \int_{\mathbb{R}} \left| \sum_{i=1}^n c_i x^i \right|^2 d\mu(x) &= \int_{\mathbb{R}} \sum_{i,j=1}^n c_i \bar{c}_j x^{i+j} d\mu(x) \\ &= \sum_{i,j=1}^n c_i \bar{c}_j M_{i+j} \end{aligned}$$

(3) As for the other sense, here the result comes once again from the above formula, this time via some standard functional analysis. \square

As a basic application of the Stieltjes formula, let us solve the moment problem for the Catalan numbers C_k , and for the central binomial coefficients D_k . We first have:

THEOREM 3.13. *The real measure having as even moments the Catalan numbers, $C_k = \frac{1}{k+1} \binom{2k}{k}$, and having all odd moments 0 is the measure*

$$\gamma_1 = \frac{1}{2\pi} \sqrt{4 - x^2} dx$$

called *Wigner semicircle law* on $[-2, 2]$.

PROOF. In order to apply the inversion formula, our starting point will be the formula from Theorem 3.9 for the generating series of the Catalan numbers, namely:

$$\sum_{k=0}^{\infty} C_k z^k = \frac{1 - \sqrt{1 - 4z}}{2z}$$

By using this formula with $z = \xi^{-2}$, we obtain the following formula:

$$\begin{aligned} G(\xi) &= \xi^{-1} \sum_{k=0}^{\infty} C_k \xi^{-2k} \\ &= \xi^{-1} \cdot \frac{1 - \sqrt{1 - 4\xi^{-2}}}{2\xi^{-2}} \\ &= \frac{\xi}{2} \left(1 - \sqrt{1 - 4\xi^{-2}} \right) \\ &= \frac{\xi}{2} - \frac{1}{2} \sqrt{\xi^2 - 4} \end{aligned}$$

Now let us apply Theorem 3.11. The study here goes as follows:

(1) According to the general philosophy of the Stieltjes formula, the first term, namely $\xi/2$, which is “trivial”, will not contribute to the density.

(2) As for the second term, which is something non-trivial, this will contribute to the density, the rule here being that the square root $\sqrt{\xi^2 - 4}$ will be replaced by the “dual” square root $\sqrt{4 - x^2} dx$, and that we have to multiply everything by $-1/\pi$.

(3) As a conclusion, by Stieltjes inversion we obtain the following density:

$$d\mu(x) = -\frac{1}{\pi} \cdot -\frac{1}{2}\sqrt{4 - x^2} dx = \frac{1}{2\pi}\sqrt{4 - x^2} dx$$

Thus, we have obtained the measure in the statement, and we are done. \square

We have the following version of the above result:

THEOREM 3.14. *The real measure having as sequence of moments the Catalan numbers, $C_k = \frac{1}{k+1} \binom{2k}{k}$, is the measure*

$$\pi_1 = \frac{1}{2\pi} \sqrt{4x^{-1} - 1} dx$$

called *Marchenko-Pastur law* on $[0, 4]$.

PROOF. As before, we use the standard formula for the generating series of the Catalan numbers. With $z = \xi^{-1}$ in that formula, we obtain the following formula:

$$\begin{aligned} G(\xi) &= \xi^{-1} \sum_{k=0}^{\infty} C_k \xi^{-k} \\ &= \xi^{-1} \cdot \frac{1 - \sqrt{1 - 4\xi^{-1}}}{2\xi^{-1}} \\ &= \frac{1}{2} \left(1 - \sqrt{1 - 4\xi^{-1}} \right) \\ &= \frac{1}{2} - \frac{1}{2} \sqrt{1 - 4\xi^{-1}} \end{aligned}$$

With this in hand, let us apply now the Stieltjes inversion formula, from Theorem 3.11. We obtain, a bit as before in Theorem 3.13, the following density:

$$d\mu(x) = -\frac{1}{\pi} \cdot -\frac{1}{2}\sqrt{4x^{-1} - 1} dx = \frac{1}{2\pi}\sqrt{4x^{-1} - 1} dx$$

Thus, we are led to the conclusion in the statement. \square

Regarding now the central binomial coefficients, we have here:

THEOREM 3.15. *The real probability measure having as moments the central binomial coefficients, $D_k = \binom{2k}{k}$, is the measure*

$$\alpha_1 = \frac{1}{\pi\sqrt{x(4-x)}} dx$$

called *arcsine law* on $[0, 4]$.

PROOF. We have the following computation, using some standard formulae:

$$\begin{aligned}
 G(\xi) &= \xi^{-1} \sum_{k=0}^{\infty} D_k \xi^{-k} \\
 &= \frac{1}{\xi} \sum_{k=0}^{\infty} D_k \left(-\frac{t}{4}\right)^k \\
 &= \frac{1}{\xi} \cdot \frac{1}{\sqrt{1-4/\xi}} \\
 &= \frac{1}{\sqrt{\xi(\xi-4)}}
 \end{aligned}$$

But this gives the density in the statement, via Theorem 3.11. □

Finally, we have the following version of the above result:

THEOREM 3.16. *The real probability measure having as moments the middle binomial coefficients, $E_k = \binom{k}{[k/2]}$, is the following law on $[-2, 2]$,*

$$\sigma_1 = \frac{1}{2\pi} \sqrt{\frac{2+x}{2-x}} dx$$

called modified arcsine law on $[-2, 2]$.

PROOF. In terms of the central binomial coefficients D_k , we have:

$$E_{2k} = \binom{2k}{k} = \frac{(2k)!}{k!k!} = D_k$$

$$E_{2k-1} = \binom{2k-1}{k} = \frac{(2k-1)!}{k!(k-1)!} = \frac{D_k}{2}$$

Standard calculus based on the Taylor formula for $(1+t)^{-1/2}$ gives:

$$\frac{1}{2x} \left(\sqrt{\frac{1+2x}{1-2x}} - 1 \right) = \sum_{k=0}^{\infty} E_k x^k$$

With $x = \xi^{-1}$ we obtain the following formula for the Cauchy transform:

$$\begin{aligned} G(\xi) &= \xi^{-1} \sum_{k=0}^{\infty} E_k \xi^{-k} \\ &= \frac{1}{\xi} \left(\sqrt{\frac{1 + 2/\xi}{1 - 2/\xi}} - 1 \right) \\ &= \frac{1}{\xi} \left(\sqrt{\frac{\xi + 2}{\xi - 2}} - 1 \right) \end{aligned}$$

By Stieltjes inversion we obtain the density in the statement. \square

All this is very nice, and we are obviously building here, as this book goes by, some solid knowledge in classical probability. We will be back to all this later.

3d. Finite graphs

With the above done, we can come back now to walks on finite graphs, that we know from the above to be related to the eigenvalues of the adjacency matrix $d \in M_N(0, 1)$. But here, we are led to the following philosophical question, to start with:

QUESTION 3.17. *What are the most important finite graphs, that we should do our computations for?*

Not an easy question, you have to agree with me, with the answer to this obviously depending on your previous experience with mathematics, or physics, or chemistry, or computer science, or other branch of science that you are interested in, and also, on the specific problems that you are the most in love with, in that part of science.

So, we have to be subjective here. And with me writing this book, and doing some sort of complicated quantum physics, as daytime job, I will choose the ADE graphs. It is beyond our scope here to explain where these ADE graphs exactly come from, and what they are good for, but as a piece of advertisement for them, we have:

ADVERTISEMENT 3.18. *The ADE graphs classify the following:*

- (1) *Basic Lie groups and algebras.*
- (2) *Subgroups of SU_2 and of SO_3 .*
- (3) *Singularities of algebraic manifolds.*
- (4) *Basic invariants of knots and links.*
- (5) *Subfactors and planar algebras of small index.*
- (6) *Subgroups of the quantum permutation group S_4^+ .*
- (7) *Basic quantum field theories, and other physics beasts.*

Which sounds exciting, doesn't it. So, have a look at this, and with the comment that some heavy learning work is needed, in order to understand how all this works. And with the extra comment that, in view of (7), tough physics, no one really understands how all this works. A nice introduction to all this is the paper of Jones [58].

Getting to work now, we first need to know what the ADE graphs are. The A graphs, which are the simplest, are as follows, with the distinguished vertex being denoted \bullet , and with A_n having $n \geq 2$ vertices, and \tilde{A}_{2n} having $2n \geq 2$ vertices:

$$\begin{array}{cc}
 A_n = \bullet - \circ - \circ - \circ \cdots \circ - \circ - \circ & A_\infty = \bullet - \circ - \circ - \circ - \circ \cdots \\
 \\
 \begin{array}{c} \circ - \circ - \circ - \circ \cdots \circ - \circ - \circ \\ | \qquad \qquad \qquad | \\ \tilde{A}_{2n} = \bullet - \circ - \circ - \circ - \circ - \circ - \circ \end{array} & \begin{array}{c} \circ - \circ - \circ - \circ - \circ \cdots \\ | \\ \tilde{A}_\infty = \bullet - \circ - \circ - \circ - \circ \cdots \end{array}
 \end{array}$$

These A graphs do not actually look that scary, because we already met all of them in the above, and as a comment on them, summarizing the situation, we have:

COMMENT 3.19. *With the A graphs we are not really lost into quantum physics, because all these graphs are quite familiar to us, as follows:*

- (1) A_n is the segment.
- (2) A_∞ is the \mathbb{N} graph.
- (3) \tilde{A}_{2n} is the circle.
- (4) \tilde{A}_∞ is the \mathbb{Z} graph.

You might probably say, why not stopping here, and doing our unfinished business for the segment and the circle, with whatever new ideas that we might have. Good point, but in answer, these ideas will apply as well, with minimal changes, to the D graphs, which are as follows, with D_n having $n \geq 3$ vertices, and \tilde{D}_n having $n + 1 \geq 5$ vertices:

$$\begin{array}{c}
 \circ \\
 | \\
 D_n = \bullet - \circ - \circ - \circ \cdots \circ - \circ - \circ
 \end{array}$$

$$\begin{array}{c}
 \circ \qquad \qquad \qquad \circ \\
 | \qquad \qquad \qquad | \\
 \tilde{D}_n = \bullet - \circ - \circ - \circ \cdots \circ - \circ - \circ
 \end{array}$$

$$\begin{array}{c}
 \circ \\
 | \\
 D_\infty = \bullet - \circ - \circ - \circ - \circ \cdots
 \end{array}$$

As mentioned above, it is beyond our scope here to explain what the ADE graphs really stand for, but as an informal comment on these latter D graphs, we have:

COMMENT 3.20. *The D graphs are not that scary either, and they can be thought of as being certain technical versions of the A graphs.*

So, this is the situation, you have to trust me here, and for more on all this, check for instance the pair of Jones [58]. In what concerns us, we will just take the above D graphs as they come, and do our loop count work for them, without questions asked.

As another comment, the labeling conventions for the AD graphs, while very standard, can be a bit confusing. The first graph in each series is by definition as follows:

$$A_2 = \bullet - \circ \qquad \tilde{A}_2 = \begin{array}{c} \circ \\ || \\ \bullet \end{array} \qquad D_3 = \begin{array}{c} \circ \\ | \\ \bullet - \circ \end{array} \qquad \tilde{D}_4 = \begin{array}{c} \circ \quad \circ \\ \backslash \quad / \\ \bullet - \circ - \circ \end{array}$$

Finally, there are also a number of exceptional ADE graphs. First we have:

$$E_6 = \bullet - \circ - \circ - \begin{array}{c} \circ \\ | \\ \circ \end{array} - \circ - \circ$$

$$E_7 = \bullet - \circ - \circ - \circ - \begin{array}{c} \circ \\ | \\ \circ \end{array} - \circ - \circ$$

$$E_8 = \bullet - \circ - \circ - \circ - \circ - \begin{array}{c} \circ \\ | \\ \circ \end{array} - \circ - \circ$$

Then, we have extended versions of the above exceptional graphs, as follows:

$$\tilde{E}_6 = \bullet - \circ - \circ - \begin{array}{c} \circ \\ | \\ \circ \\ | \\ \circ \end{array} - \circ - \circ$$

$$\tilde{E}_7 = \bullet - \circ - \circ - \circ - \begin{array}{c} \circ \\ | \\ \circ \end{array} - \circ - \circ - \circ - \circ$$

$$\tilde{E}_8 = \bullet - \circ - \circ - \circ - \circ - \circ - \begin{array}{c} \circ \\ | \\ \circ \end{array} - \circ - \circ - \circ$$

And good news, that is all. Hard job for me to come now with a comment on these latter E graphs, along the lines of Comments 3.19 and 3.20, and here is what I have:

COMMENT 3.21. *The E graphs naturally complement the AD series, by capturing the combinatorics of certain “exceptional” phenomena in mathematics and physics.*

So long for difficult definitions and related informal talk, and as already mentioned in the above, for more on all this, have a look at the paper of Jones [58]. Getting now to work, we have some new graphs, and here is the problem that we would like to solve:

PROBLEM 3.22. *How to count loops on the ADE graphs?*

In answer, as mentioned in Comment 3.19, we are already familiar with two of the ADE graphs, namely A_∞ and \tilde{A}_∞ , which are respectively the graphs that we previously called \mathbb{N} and \mathbb{Z} . So, based on our work for these graphs, where the combinatorics naturally led us into generating series, let us formulate the following definition:

DEFINITION 3.23. *The Poincaré series of a rooted bipartite graph X is*

$$f(z) = \sum_{k=0}^{\infty} L_{2k} z^k$$

where L_{2k} is the number of $2k$ -loops based at the root.

To be more precise, observe that all the above ADE graphs are indeed bipartite. Now the point is that, for a bipartite graph, the loops based at any point must have even length. Thus, in order to study the loops on the ADE graphs, based at the root, we just have to count the above numbers L_{2k} . And then, considering the generating series $f(z)$ of these numbers, and calling this Poincaré series, is something very standard.

Before getting into computations, let us introduce as well:

DEFINITION 3.24. *The positive spectral measure μ of a rooted bipartite graph X is the real probability measure having the numbers L_{2k} as moments:*

$$\int_{\mathbb{R}} x^k d\mu(x) = L_{2k}$$

Equivalently, we must have the Stieltjes transform formula

$$f(z) = \int_{\mathbb{R}} \frac{1}{1 - xz} d\mu(x)$$

where f is the Poincaré series of X .

Here the existence of μ , and the fact that this is indeed a positive measure, meaning a measure supported on $[0, \infty)$, comes from the following simple fact:

THEOREM 3.25. *The positive spectral measure of a rooted bipartite graph X is given by the following formula, with d being the adjacency matrix of the graph,*

$$\mu = \text{law}(d^2)$$

and with the probabilistic computation being with respect to the expectation

$$A \rightarrow \langle A \rangle$$

with $\langle A \rangle$ being the $(*, *)$ -entry of a matrix A , where $*$ is the root.

PROOF. With the above conventions, we have the following computation:

$$\begin{aligned} f(z) &= \sum_{k=0}^{\infty} L_{2k} z^k \\ &= \sum_{k=0}^{\infty} \langle d^{2k} \rangle z^k \\ &= \left\langle \frac{1}{1 - d^2 z} \right\rangle \end{aligned}$$

But this shows that we have $\mu = \text{law}(d^2)$, as desired. \square

The above result shows that computing μ might be actually a simpler problem than computing f , and in practice, this is indeed the case. So, in what follows we will rather forget about loops and Definition 3.23, and use Definition 3.24 instead, with our computations to follow being based on the concrete interpretation from Theorem 3.25.

However, even with this probabilistic trick in our bag, things are not exactly trivial. So, following now [12], let us introduce as well the following notion:

DEFINITION 3.26. *The circular measure ε of a rooted bipartite graph X is given by*

$$d\varepsilon(q) = d\mu((q + q^{-1})^2)$$

where μ is the associated positive spectral measure.

To be more precise, we know from Theorem 3.25 that the positive measure μ is the spectral measure of a certain positive matrix, $d^2 \geq 0$, and it follows from this, and from basic spectral theory, that this measure is supported by the positive reals:

$$\text{supp}(\mu) \subset \mathbb{R}_+$$

But then, with this observation in hand, we can define indeed the circular measure ε as above, as being the pullback of μ via the following map:

$$\mathbb{R} \cup \mathbb{T} \rightarrow \mathbb{R}_+ \quad , \quad q \rightarrow (q + q^{-1})^2$$

As a basic example for this, to start with, assume that μ is a discrete measure, supported by n positive numbers $x_1 < \dots < x_n$, with corresponding densities p_1, \dots, p_n :

$$\mu = \sum_{i=1}^n p_i \delta_{x_i}$$

For each $i \in \{1, \dots, n\}$ the equation $(q + q^{-1})^2 = x_i$ has then four solutions, that we can denote $q_i, q_i^{-1}, -q_i, -q_i^{-1}$. And with this notation, we have:

$$\varepsilon = \frac{1}{4} \sum_{i=1}^n p_i \left(\delta_{q_i} + \delta_{q_i^{-1}} + \delta_{-q_i} + \delta_{-q_i^{-1}} \right)$$

In general, the basic properties of ε can be summarized as follows:

THEOREM 3.27. *The circular measure has the following properties:*

- (1) ε has equal density at $q, q^{-1}, -q, -q^{-1}$.
- (2) The odd moments of ε are 0.
- (3) The even moments of ε are half-integers.
- (4) When X has norm ≤ 2 , ε is supported by the unit circle.
- (5) When X is finite, ε is discrete.
- (6) If K is a solution of $d = K + K^{-1}$, then $\varepsilon = \text{law}(K)$.

PROOF. These results can be deduced from definitions, the idea being that (1-5) are trivial, and that (6) follows from the formula of μ from Theorem 3.25. \square

Getting now to computations, remember our struggle from the above, with the circle graph? We can now solve this question, majestically, as follows:

THEOREM 3.28. *The circular measure of the basic index 4 graph, namely*

$$\tilde{A}_{2n} = \begin{array}{cccccccc} \circ & - & \circ & - & \circ & \cdots & \circ & - & \circ & - & \circ \\ | & & & & & & & & & & | \\ \bullet & - & \circ & - & \circ & - & \circ & - & \circ & - & \circ \end{array}$$

is the uniform measure on the $2n$ -roots of unity.

PROOF. Let us identify the vertices of $X = \tilde{A}_{2n}$ with the group $\{w^k\}$ formed by the $2n$ -th roots of unity in the complex plane, where $w = e^{\pi i/n}$. The adjacency matrix of X acts then on the functions $f \in C(X)$ in the following way:

$$df(w^s) = f(w^{s-1}) + f(w^{s+1})$$

But this shows that we have $d = K + K^{-1}$, where K is given by:

$$Kf(w^s) = f(w^{s+1})$$

Thus we can use Theorem 3.25 and Theorem 3.27 (6), and we get:

$$\varepsilon = \text{law}(K)$$

But this is the uniform measure on the $2n$ -roots of unity, as claimed. \square

All this is very nice, so, before going ahead with more computations, let us have an excursion into subfactor theory, and explain what is behind this trick. Following Jones [60], we can introduce the theta series of a graph X , as a version of the Poincaré series, via the change of variables $z^{-1/2} = q^{1/2} + q^{-1/2}$, as follows:

DEFINITION 3.29. *The theta series of a rooted bipartite graph X is*

$$\Theta(q) = q + \frac{1-q}{1+q} f\left(\frac{q}{(1+q)^2}\right)$$

where f is the Poincaré series.

The theta series can be written as $\Theta(q) = \sum a_r q^r$, and it follows from the above formula, via some simple manipulations, that its coefficients are integers:

$$a_r \in \mathbb{Z}$$

In fact, we have the following explicit formula from Jones' paper [60], relating the coefficients of $\Theta(q) = \sum a_r q^r$ to those of the Poincaré series $f(z) = \sum c_k z^k$:

$$a_r = \sum_{k=0}^r (-1)^{r-k} \frac{2r}{r+k} \binom{r+k}{r-k} c_k$$

As an important comment now, in the case where X is the principal graph of a subfactor $A_0 \subset A_1$ of index $N > 4$, it is known from [60] that the numbers a_r are certain multiplicities associated to the planar algebra inclusion $TL_N \subset P$, as explained there. In particular, the coefficients of the theta series are in this case positive integers:

$$a_r \in \mathbb{N}$$

In relation now with the circular measure, the result here, which is quite similar to the Stieltjes transform formula from Definition 3.24, is as follows:

THEOREM 3.30. *We have the Stieltjes transform type formula*

$$2 \int \frac{1}{1-qu^2} d\varepsilon(u) = 1 + T(q)(1-q)$$

where the T series of a rooted bipartite graph X is by definition given by

$$T(q) = \frac{\Theta(q) - q}{1 - q}$$

with Θ being the associated theta series.

PROOF. This follows by applying the change of variables $q \rightarrow (q + q^{-1})^2$ to the fact that f is the Stieltjes transform of μ . Indeed, we obtain in this way:

$$\begin{aligned} 2 \int \frac{1}{1-qu^2} d\varepsilon(u) &= 1 + \frac{1-q}{1+q} f\left(\frac{q}{(1+q)^2}\right) \\ &= 1 + \Theta(q) - q \\ &= 1 + T(q)(1-q) \end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Summarizing, we have a whole menagerie of subfactor, planar algebra and bipartite graph invariants, which come in several flavors, namely series and measures, and which can be linear or circular, and which all appear as versions of the Poincaré series.

In order to discuss all this more systematically, let us introduce as well:

DEFINITION 3.31. *The series of the form*

$$\xi(n_1, \dots, n_s : m_1, \dots, m_t) = \frac{(1 - q^{n_1}) \dots (1 - q^{n_s})}{(1 - q^{m_1}) \dots (1 - q^{m_t})}$$

with $n_i, m_i \in \mathbb{N}$ are called *cyclotomic*.

It is technically convenient to allow as well $1 + q^n$ factors, to be designated by n^+ symbols in the above writing. For instance we have, by definition:

$$\xi(2^+ : 3) = \xi(4 : 2, 3)$$

Also, it is convenient in what follows to use the following notations:

$$\xi' = \frac{\xi}{1 - q} \quad , \quad \xi'' = \frac{\xi}{1 - q^2}$$

The Poincaré series of the ADE graphs are given by quite complicated formulae. However, the corresponding T series are all cyclotomic, as follows:

THEOREM 3.32. *The T series of the ADE graphs are as follows:*

- (1) For A_{n-1} we have $T = \xi(n - 1 : n)$.
- (2) For D_{n+1} we have $T = \xi(n - 1^+ : n^+)$.
- (3) For \tilde{A}_{2n} we have $T = \xi'(n^+ : n)$.
- (4) For \tilde{D}_{n+2} we have $T = \xi''(n + 1^+ : n)$.
- (5) For E_6 we have $T = \xi(8 : 3, 6^+)$.
- (6) For E_7 we have $T = \xi(12 : 4, 9^+)$.
- (7) For E_8 we have $T = \xi(5^+, 9^+ : 15^+)$.
- (8) For \tilde{E}_6 we have $T = \xi(6^+ : 3, 4)$.
- (9) For \tilde{E}_7 we have $T = \xi(9^+ : 4, 6)$.
- (10) For \tilde{E}_8 we have $T = \xi(15^+ : 6, 10)$.

PROOF. These formulae were obtained in [12], by counting loops, and then by making the following change of variables, and factorizing the resulting series:

$$z^{-1/2} = q^{1/2} + q^{-1/2}$$

An alternative proof for these formulae can be obtained by using planar algebra methods, along the lines of the paper of Jones [60]. For details here, see [12]. \square

Our purpose now will be that of converting the above technical results, regarding the T series, into some final results, regarding the corresponding circular measures ε . In order to formulate our results, we will need some more theory. First, we have:

DEFINITION 3.33. *A cyclotomic measure is a probability measure ε on the unit circle, having the following properties:*

- (1) ε is supported by the $2n$ -roots of unity, for some $n \in \mathbb{N}$.
- (2) ε has equal density at $q, q^{-1}, -q, -q^{-1}$.

As a first observation, it follows from Theorem 3.27 and from Theorem 3.32 that the circular measures of the finite ADE graphs are supported by certain roots of unity, hence are cyclotomic. We will be back to this in a moment, with details, and computations.

At the general level now, let us introduce as well the following notion:

DEFINITION 3.34. *The T series of a cyclotomic measure ε is given by*

$$1 + T(q)(1 - q) = 2 \int \frac{1}{1 - qu^2} d\varepsilon(u)$$

with ε being as usual the circular spectral measure.

Observe that this formula is nothing but the one in Theorem 3.30, written now in the other sense. In other words, if the cyclotomic measure ε happens to be the circular measure of a rooted bipartite graph, then the T series as defined above coincides with the T series as defined before. This is useful for explicit computations.

Good news, with this technology in hand, and with a computation already done, in Theorem 3.28, we are now ready to discuss the circular measures of all ADE graphs.

The idea will be that these measures are all cyclotomic, of level ≤ 3 , and can be expressed in terms of the basic polynomial densities of degree ≤ 6 , namely:

$$\alpha = \operatorname{Re}(1 - q^2)$$

$$\beta = \operatorname{Re}(1 - q^4)$$

$$\gamma = \operatorname{Re}(1 - q^6)$$

To be more precise, we have the following final result on the subject, with α, β, γ being as above, with d_n being the uniform measure on the $2n$ -th roots of unity, and with $d'_n = 2d_{2n} - d_n$ being the uniform measure on the odd $4n$ -roots of unity:

THEOREM 3.35. *The circular measures of the ADE graphs are given by:*

- (1) $A_{n-1} \rightarrow \alpha_n$.
- (2) $\tilde{A}_{2n} \rightarrow d_n$.
- (3) $D_{n+1} \rightarrow \alpha'_n$.
- (4) $\tilde{D}_{n+2} \rightarrow (d_n + d'_1)/2$.
- (5) $E_6 \rightarrow \alpha_{12} + (d_{12} - d_6 - d_4 + d_3)/2$.
- (6) $E_7 \rightarrow \beta'_9 + (d'_1 - d'_3)/2$.
- (7) $E_8 \rightarrow \alpha'_{15} + \gamma'_{15} - (d'_5 + d'_3)/2$.
- (8) $\tilde{E}_{n+3} \rightarrow (d_n + d_3 + d_2 - d_1)/2$.

PROOF. This is something which can be proved in three steps, as follows:

(1) For the simplest graph, namely the circle \tilde{A}_{2n} , we already have the result, from Theorem 3.28, with the proof there being something elementary.

(2) For the other non-exceptional graphs, that is, of type A and D, the same method works, namely direct loop counting, with some matrix tricks. See [12].

(3) In general, this follows from the T series formulae in Theorem 3.32, via some manipulations based on the general conversion formulae given above. See [12]. \square

We refer to [12] and the subsequent literature for more on all this. Also, let us point out that all this leads to a more conceptual understanding of what we did before, for the graphs \mathbb{N} and \mathbb{Z} . Indeed, even for these very basic graphs, using the unit circle and circular measures as above leads to a better understanding of the combinatorics.

3e. Exercises

We had a lot of exciting combinatorics and calculus in this chapter, and as an exercise on all this, which is quite instructive, we have:

EXERCISE 3.36. *Compute the Fourier transform of the arcsine, modified arcsine, Wigner and Marchenko-Pastur laws.*

And with the comment here that, although this exercise looks quite conceptual, of must-do type, I don't know myself the answer to it. More on this later in this book.

CHAPTER 4

Lie groups

4a. Representations

We have seen so far the foundations and basic results of classical probability. Before stepping into more complicated things, such as random matrices and free probability, we would like to clarify one important question which appeared several times, namely the computation of integrals over the compact groups of unitary matrices $G \subset U_N$, and its probabilistic consequences. The precise question that we have in mind is:

QUESTION 4.1. *Given a compact group $G \subset U_N$, how to compute the integrals*

$$I_{ij}^e = \int_G g_{i_1 j_1}^{e_1} \cdots g_{i_k j_k}^{e_k} dg$$

depending on multi-indices i, j , and of a colored integer exponent $e = \circ \bullet \bullet \circ \dots$? Then, how to use this formula in order to compute the laws of variables of type

$$f_P = P\left(\{g_{ij}\}_{i,j=1,\dots,N}\right)$$

depending on a polynomial P ? What about the $N \rightarrow \infty$ asymptotics of such laws?

All this is quite subtle, and as a basic illustration for this, we have a fundamental result from chapter 2, stating that for $G = S_N$ the law of the variable $\chi = \sum_i g_{ii}$ can be explicitly computed, and becomes Poisson (1) with $N \rightarrow \infty$. This is something truly remarkable, and it is this kind of result that we would like to systematically have.

We will discuss this in this whole chapter, and later on too. This might seem of course quite long, but believe me, it is worth the effort, because it is quite hard to do any type of advanced probability theory without knowing the answer to Question 4.1. But probably enough advertisement, let us get to work. Following Weyl [96], we first have:

DEFINITION 4.2. *A unitary representation of a compact group G is a continuous group morphism into a unitary group*

$$v : G \rightarrow U_N \quad , \quad g \rightarrow v_g$$

which can be faithful or not. The character of such a representation is the function

$$\chi : G \rightarrow \mathbb{C} \quad , \quad g \rightarrow \text{Tr}(v_g)$$

where Tr is the usual, unnormalized trace of the $N \times N$ matrices.

At the level of examples, most of the compact groups that we met so far, finite or continuous, naturally appear as closed subgroups $G \subset U_N$. In this case, the embedding $G \subset U_N$ is of course a representation, called fundamental representation. In general now, let us first discuss the various operations on the representations. We have here:

PROPOSITION 4.3. *The representations of a compact group G are subject to:*

- (1) *Making sums. Given representations v, w , of dimensions N, M , their sum is the $N + M$ -dimensional representation $v + w = \text{diag}(v, w)$.*
- (2) *Making products. Given representations v, w , of dimensions N, M , their product is the NM -dimensional representation $(v \otimes w)_{ia, jb} = v_{ij}w_{ab}$.*
- (3) *Taking conjugates. Given a N -dimensional representation v , its conjugate is the N -dimensional representation $(\bar{v})_{ij} = \bar{v}_{ij}$.*
- (4) *Spinning by unitaries. Given a N -dimensional representation v , and a unitary $U \in U_N$, we can spin v by this unitary, $v \rightarrow UvU^*$.*

PROOF. The fact that the operations in the statement are indeed well-defined, among morphisms from G to unitary groups, is indeed clear from definitions. \square

In relation now with characters, we have the following result:

PROPOSITION 4.4. *We have the following formulae, regarding characters*

$$\chi_{v+w} = \chi_v + \chi_w \quad , \quad \chi_{v \otimes w} = \chi_v \chi_w \quad , \quad \chi_{\bar{v}} = \bar{\chi}_v \quad , \quad \chi_{UvU^*} = \chi_v$$

in relation with the basic operations for the representations.

PROOF. All these assertions are elementary, by using the following well-known trace formulae, valid for any square matrices V, W , and any unitary U :

$$\begin{aligned} \text{Tr}(\text{diag}(V, W)) &= \text{Tr}(V) + \text{Tr}(W) \quad , \quad \text{Tr}(V \otimes W) = \text{Tr}(V)\text{Tr}(W) \\ \text{Tr}(\bar{V}) &= \overline{\text{Tr}(V)} \quad , \quad \text{Tr}(UVU^*) = \text{Tr}(V) \end{aligned}$$

Thus, we are led to the formulae in the statement. \square

Assume now that we are given a closed subgroup $G \subset U_N$. By using the above operations, we can construct a whole family of representations of G , as follows:

DEFINITION 4.5. *Given a closed subgroup $G \subset U_N$, its Peter-Weyl representations are the various tensor products between the fundamental representation and its conjugate:*

$$v : G \subset U_N \quad , \quad \bar{v} : G \subset U_N$$

We denote these tensor products $v^{\otimes k}$, with $k = \circ \bullet \bullet \circ \dots$ being a colored integer, with the colored tensor powers being defined according to the rules

$$v^{\otimes \circ} = v \quad , \quad v^{\otimes \bullet} = \bar{v} \quad , \quad v^{\otimes kl} = v^{\otimes k} \otimes v^{\otimes l}$$

and with the convention that $v^{\otimes \emptyset}$ is the trivial representation $1 : G \rightarrow U_1$.

Here are a few examples of such representations, namely those coming from the colored integers of length 2, which will often appear in what follows:

$$\begin{aligned} v^{\otimes\circ\circ} &= v \otimes v \quad , \quad v^{\otimes\circ\bullet} = v \otimes \bar{v} \\ v^{\otimes\bullet\circ} &= \bar{v} \otimes v \quad , \quad v^{\otimes\bullet\bullet} = \bar{v} \otimes \bar{v} \end{aligned}$$

In relation now with characters, we have the following result:

PROPOSITION 4.6. *The characters of the Peter-Weyl representations are given by*

$$\chi_{v^{\otimes k}} = (\chi_v)^k$$

with the colored powers being given by $\chi^\circ = \chi$, $\chi^\bullet = \bar{\chi}$ and multiplicativity.

PROOF. This follows indeed from the additivity, multiplicativity and conjugation formulae from Proposition 4.4, via the conventions in Definition 4.5. \square

Getting back now to our motivations, we can see the interest in the above constructions. Indeed, the joint moments of the main character $\chi = \chi_v$ and its adjoint $\bar{\chi} = \chi_{\bar{v}}$ are the expectations of the characters of various Peter-Weyl representations:

$$\int_G \chi^k = \int_G \chi_{v^{\otimes k}}$$

In order to advance, we must develop some general theory. Let us start with:

DEFINITION 4.7. *Given a compact group G , and two of its representations,*

$$v : G \rightarrow U_N \quad , \quad w : G \rightarrow U_M$$

we define the space of intertwiners between these representations as being

$$\text{Hom}(v, w) = \left\{ T \in M_{M \times N}(\mathbb{C}) \mid T v_g = w_g T, \forall g \in G \right\}$$

and we use the following conventions:

- (1) We use the notations $\text{Fix}(v) = \text{Hom}(1, v)$, and $\text{End}(v) = \text{Hom}(v, v)$.
- (2) We write $v \sim w$ when $\text{Hom}(v, w)$ contains an invertible element.
- (3) We say that v is irreducible, and write $v \in \text{Irr}(G)$, when $\text{End}(v) = \mathbb{C}1$.

The terminology here is standard, with Fix , Hom , End standing for fixed points, homomorphisms and endomorphisms. We will see later that irreducible means indecomposable, in a suitable sense. Here are now a few basic results, regarding these spaces:

PROPOSITION 4.8. *The spaces of intertwiners have the following properties:*

- (1) $T \in \text{Hom}(v, w), S \in \text{Hom}(w, z) \implies ST \in \text{Hom}(v, z)$.
- (2) $S \in \text{Hom}(v, w), T \in \text{Hom}(z, t) \implies S \otimes T \in \text{Hom}(v \otimes z, w \otimes t)$.
- (3) $T \in \text{Hom}(v, w) \implies T^* \in \text{Hom}(w, v)$.

In abstract terms, we say that the Hom spaces form a tensor $*$ -category.

PROOF. All the formulae in the statement are indeed clear from definitions, via elementary computations. As for the last assertion, this is something coming from (1,2,3). We will be back to tensor categories later on, with more details on this latter fact. \square

As a main consequence of the above result, we have:

PROPOSITION 4.9. *Given a representation $v : G \rightarrow U_N$, the linear space*

$$\text{End}(v) \subset M_N(\mathbb{C})$$

is a $$ -algebra, with respect to the usual involution of the matrices.*

PROOF. By definition, $\text{End}(v)$ is a linear subspace of $M_N(\mathbb{C})$. We know from Proposition 4.8 (1) that this subspace $\text{End}(v)$ is a subalgebra of $M_N(\mathbb{C})$, and then we know as well from Proposition 4.8 (3) that this subalgebra is stable under the involution $*$. Thus, what we have here is a $*$ -subalgebra of $M_N(\mathbb{C})$, as claimed. \square

In order to exploit the above fact, we will need a basic result from linear algebra, stating that any $*$ -algebra $A \subset M_N(\mathbb{C})$ decomposes as a direct sum, as follows:

$$A \simeq M_{N_1}(\mathbb{C}) \oplus \dots \oplus M_{N_k}(\mathbb{C})$$

Indeed, let us write the unit $1 \in A$ as $1 = p_1 + \dots + p_k$, with $p_i \in A$ being central minimal projections. Then each of the spaces $A_i = p_i A p_i$ is a subalgebra of A , and we have a decomposition $A = A_1 \oplus \dots \oplus A_k$. But since each central projection $p_i \in A$ was chosen minimal, we have $A_i \simeq M_{N_i}(\mathbb{C})$, with $N_i = \text{rank}(p_i)$, as desired.

We can now formulate our first Peter-Weyl type theorem, as follows:

THEOREM 4.10 (Peter-Weyl 1). *Let $v : G \rightarrow U_N$ be a representation, consider the algebra $A = \text{End}(v)$, and write its unit $1 = p_1 + \dots + p_k$ as above. We have then*

$$v = v_1 + \dots + v_k$$

with each v_i being an irreducible representation, obtained by restricting v to $\text{Im}(p_i)$.

PROOF. This basically follows from Proposition 4.9, as follows:

(1) We first associate to our representation $v : G \rightarrow U_N$ the corresponding action map on \mathbb{C}^N . If a linear subspace $W \subset \mathbb{C}^N$ is invariant, the restriction of the action map to W is an action map too, which must come from a subrepresentation $w \subset v$.

(2) Consider now a projection $p \in \text{End}(v)$. From $pv = vp$ we obtain that the linear space $W = \text{Im}(p)$ is invariant under v , and so this space must come from a subrepresentation $w \subset v$. It is routine to check that the operation $p \rightarrow w$ maps subprojections to subrepresentations, and minimal projections to irreducible representations.

(3) With these preliminaries in hand, let us decompose the algebra $\text{End}(v)$ as above, by using the decomposition $1 = p_1 + \dots + p_k$ into central minimal projections. If we

denote by $v_i \subset v$ the subrepresentation coming from the vector space $V_i = \text{Im}(p_i)$, then we obtain in this way a decomposition $v = v_1 + \dots + v_k$, as in the statement. \square

Here is now our second Peter-Weyl theorem, complementing Theorem 4.10:

THEOREM 4.11 (Peter-Weyl 2). *Given a closed subgroup $G \subset_v U_N$, any of its irreducible smooth representations*

$$w : G \rightarrow U_M$$

appears inside a tensor product of the fundamental representation v and its adjoint \bar{v} .

PROOF. Given a representation $w : G \rightarrow U_M$, we define the space of coefficients $C_w \subset C(G)$ of this representation as being the following linear space:

$$C_w = \text{span} \left[g \rightarrow w(g)_{ij} \right]$$

With this notion in hand, the result can be deduced as follows:

(1) The construction $w \rightarrow C_w$ is functorial, in the sense that it maps subrepresentations into linear subspaces. This is indeed something which is routine to check.

(2) A closed subgroup $G \subset_v U_N$ is a Lie group, and a representation $w : G \rightarrow U_M$ is smooth when we have an inclusion $C_w \subset \langle C_v \rangle$. This is indeed well-known.

(3) By definition of the Peter-Weyl representations, as arbitrary tensor products between the fundamental representation v and its conjugate \bar{v} , we have:

$$\langle C_v \rangle = \sum_k C_{v^{\otimes k}}$$

(4) Now by putting together the above observations (2,3) we conclude that we must have an inclusion as follows, for certain exponents k_1, \dots, k_p :

$$C_w \subset C_{v^{\otimes k_1} \oplus \dots \oplus v^{\otimes k_p}}$$

(5) By using now (1), we deduce that we have an inclusion $w \subset v^{\otimes k_1} \oplus \dots \oplus v^{\otimes k_p}$, and by applying Theorem 4.10, this leads to the conclusion in the statement. \square

4b. Haar integration

In order to further advance with Peter-Weyl theory, we need to talk about integration over G . In the finite group case the situation is trivial, as follows:

PROPOSITION 4.12. *Any finite group G has a unique probability measure which is invariant under left and right translations,*

$$\mu(E) = \mu(gE) = \mu(Eg)$$

and this is the normalized counting measure on G , given by $\mu(E) = |E|/|G|$.

PROOF. This is indeed something trivial, which follows from definitions. \square

In the general, continuous case, let us begin with the following key result:

PROPOSITION 4.13. *Given a unital positive linear form $\psi : C(G) \rightarrow \mathbb{C}$, the limit*

$$\int_{\varphi} f = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \psi^{*k}(f)$$

exists, and for a coefficient of a representation $f = (\tau \otimes id)w$ we have

$$\int_{\varphi} f = \tau(P)$$

where P is the orthogonal projection onto the 1-eigenspace of $(id \otimes \psi)w$.

PROOF. By linearity it is enough to prove the first assertion for functions of the following type, where w is a Peter-Weyl representation, and τ is a linear form:

$$f = (\tau \otimes id)w$$

Thus we are led into the second assertion, and more precisely we can have the whole result proved if we can establish the following formula, with $f = (\tau \otimes id)w$:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \psi^{*k}(f) = \tau(P)$$

In order to prove this latter formula, observe that we have:

$$\psi^{*k}(f) = (\tau \otimes \psi^{*k})w = \tau((id \otimes \psi^{*k})w)$$

Let us set $M = (id \otimes \psi)w$. In terms of this matrix, we have:

$$((id \otimes \psi^{*k})w)_{i_0 i_{k+1}} = \sum_{i_1 \dots i_k} M_{i_0 i_1} \dots M_{i_k i_{k+1}} = (M^k)_{i_0 i_{k+1}}$$

Thus we have the following formula, valid for any $k \in \mathbb{N}$:

$$(id \otimes \psi^{*k})w = M^k$$

It follows that our Cesàro limit is given by the following formula:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \psi^{*k}(f) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \tau(M^k) = \tau \left(\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n M^k \right)$$

Now since w is unitary we have $\|w\| = 1$, and so $\|M\| \leq 1$. Thus the last Cesàro limit converges, and equals the orthogonal projection onto the 1-eigenspace of M :

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n M^k = P$$

Thus our initial Cesàro limit converges as well, to $\tau(P)$, as desired. \square

When the linear form $\psi \in C(G)^*$ is faithful, we have the following finer result:

PROPOSITION 4.14. *Given a faithful unital linear form $\psi \in C(G)^*$, the limit*

$$\int_{\psi} f = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \psi^{*k}(f)$$

exists, and is independent of ψ , given on coefficients of representations by

$$\left(id \otimes \int_{\psi} \right) w = P$$

where P is the orthogonal projection onto the space $Fix(w) = \{\xi \in \mathbb{C}^n \mid w\xi = \xi\}$.

PROOF. In view of Proposition 4.13, it remains to prove that when ψ is faithful, the 1-eigenspace of the matrix $M = (id \otimes \psi)w$ equals the space $Fix(w)$.

“ \supset ” This is clear, and for any ψ , because we have the following implication:

$$w\xi = \xi \implies M\xi = \xi$$

“ \subset ” Here we must prove that, when ψ is faithful, we have:

$$M\xi = \xi \implies w\xi = \xi$$

For this purpose, assume that we have $M\xi = \xi$, and consider the following function:

$$f = \sum_i \left(\sum_j w_{ij} \xi_j - \xi_i \right) \left(\sum_k w_{ik} \xi_k - \xi_i \right)^*$$

We must prove that we have $f = 0$. Since v is unitary, we have:

$$\begin{aligned} f &= \sum_{ijk} w_{ij} w_{ik}^* \xi_j \bar{\xi}_k - \frac{1}{N} w_{ij} \xi_j \bar{\xi}_i - \frac{1}{N} w_{ik}^* \xi_i \bar{\xi}_k + \frac{1}{N^2} \xi_i \bar{\xi}_i \\ &= \sum_j |\xi_j|^2 - \sum_{ij} w_{ij} \xi_j \bar{\xi}_i - \sum_{ik} w_{ik}^* \xi_i \bar{\xi}_k + \sum_i |\xi_i|^2 \\ &= \|\xi\|^2 - \langle w\xi, \xi \rangle - \overline{\langle w\xi, \xi \rangle} + \|\xi\|^2 \\ &= 2(\|\xi\|^2 - \operatorname{Re}(\langle w\xi, \xi \rangle)) \end{aligned}$$

By using now our assumption $M\xi = \xi$, we obtain from this:

$$\begin{aligned} \psi(f) &= 2\psi(\|\xi\|^2 - \operatorname{Re}(\langle w\xi, \xi \rangle)) \\ &= 2(\|\xi\|^2 - \operatorname{Re}(\langle M\xi, \xi \rangle)) \\ &= 2(\|\xi\|^2 - \|\xi\|^2) \\ &= 0 \end{aligned}$$

Now since ψ is faithful, this gives $f = 0$, and so $w\xi = \xi$, as claimed. \square

We can now formulate a main result, as follows:

THEOREM 4.15. *Any compact group G has a unique Haar integration, which can be constructed by starting with any faithful positive unital form $\psi \in C(G)^*$, and setting:*

$$\int_G = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \psi^{*k}$$

Moreover, for any representation w we have the formula

$$\left(id \otimes \int_G \right) w = P$$

where P is the orthogonal projection onto $Fix(w) = \{\xi \in \mathbb{C}^n \mid w\xi = \xi\}$.

PROOF. Let us first go back to the general context of Proposition 4.13. Since convolving one more time with ψ will not change the Cesàro limit appearing there, the functional $\int_\psi \in C(G)^*$ constructed there has the following invariance property:

$$\int_\psi * \psi = \psi * \int_\psi = \int_\psi$$

In the case where ψ is assumed to be faithful, as in Proposition 4.14, our claim is that we have the following formula, valid this time for any $\varphi \in C(G)^*$:

$$\int_\psi * \varphi = \varphi * \int_\psi = \varphi(1) \int_\psi$$

Indeed, it is enough to prove this formula on a coefficient of a corepresentation:

$$f = (\tau \otimes id)w$$

In order to do so, consider the following two matrices:

$$P = \left(id \otimes \int_\psi \right) w \quad , \quad Q = (id \otimes \varphi)w$$

We have then the following formulae, which all follow from definitions:

$$\left(\int_\psi * \varphi \right) f = \tau(PQ) \quad , \quad \left(\varphi * \int_\psi \right) f = \tau(QP) \quad , \quad \varphi(1) \int_\psi f = \varphi(1)\tau(P)$$

Thus, in order to prove our claim, it is enough to establish the following formula:

$$PQ = QP = \psi(1)P$$

But this follows from the fact, that we know from Proposition 4.14, that $P = (id \otimes \int_\psi)w$ is the orthogonal projection onto $Fix(w)$. Thus, we proved our claim. Now observe that, with $\Delta f(g \otimes h) = f(gh)$, this formula that we proved can be written as follows:

$$\varphi \left(\int_\psi \otimes id \right) \Delta = \varphi \left(id \otimes \int_\psi \right) \Delta = \varphi \int_\psi (\cdot) 1$$

This formula being true for any $\varphi \in C(G)^*$, we can simply delete φ , and we conclude that $\int_G = \int_\psi$ has the required left and right invariance property, namely:

$$\left(\int_G \otimes id \right) \Delta = \left(id \otimes \int_G \right) \Delta = \int_G (\cdot) 1$$

Finally, the uniqueness is clear as well, because if we have two invariant integrals \int_G, \int'_G , then their convolution equals on one hand \int_G , and on the other hand, \int'_G . \square

Summarizing, we know how to integrate over G . Before getting into probabilistic applications, let us develop however more Peter-Weyl theory. We will need:

PROPOSITION 4.16. *We have a Frobenius type isomorphism*

$$Hom(v, w) \simeq Fix(v \otimes \bar{w})$$

valid for any two representations v, w .

PROOF. According to definitions, we have the following equivalences:

$$\begin{aligned} T \in Hom(v, w) &\iff Tv = wT \\ &\iff \sum_i T_{ai} v_{ij} = \sum_b w_{ab} T_{bj}, \forall a, j \end{aligned}$$

On the other hand, we have as well the following equivalences:

$$\begin{aligned} T \in Fix(v \otimes \bar{w}) &\iff (v \otimes \bar{w})T = \xi \\ &\iff \sum_{bi} v_{ji} \bar{w}_{ab} T_{bi} = T_{aj} \forall a, j \end{aligned}$$

With these formulae in hand, both inclusions follow from the unitarity of v, w . \square

We can now formulate a third Peter-Weyl theorem, as follows:

THEOREM 4.17 (Peter-Weyl 3). *The dense subalgebra $\mathcal{C}(G) \subset C(G)$ generated by the coefficients of the fundamental representation decomposes as a direct sum*

$$\mathcal{C}(G) = \bigoplus_{w \in Irr(G)} M_{\dim(w)}(\mathbb{C})$$

with the summands being pairwise orthogonal with respect to the scalar product

$$\langle f, g \rangle = \int_G f \bar{g}$$

where \int_G is the Haar integration over G .

PROOF. By combining the previous two Peter-Weyl results, Theorems 4.10 and 4.11, we deduce that we have a linear space decomposition as follows:

$$\mathcal{C}(G) = \sum_{w \in Irr(G)} C_w = \sum_{w \in Irr(G)} M_{\dim(w)}(\mathbb{C})$$

Thus, in order to conclude, it is enough to prove that for any two irreducible representations $v, w \in \text{Irr}(G)$, the corresponding spaces of coefficients are orthogonal:

$$v \not\sim w \implies C_v \perp C_w$$

But this follows from Theorem 4.15, via Proposition 4.16. Let us set indeed:

$$P_{ia,jb} = \int_G v_{ij} \bar{w}_{ab}$$

Then P is the orthogonal projection onto the following vector space:

$$\text{Fix}(v \otimes \bar{w}) \simeq \text{Hom}(v, w) = \{0\}$$

Thus we have $P = 0$, and this gives the result. \square

Finally, we have the following result, completing the Peter-Weyl theory:

THEOREM 4.18 (Peter-Weyl 4). *The characters of irreducible representations belong to the algebra*

$$\mathcal{C}(G)_{\text{central}} = \left\{ f \in \mathcal{C}(G) \mid f(gh) = f(hg), \forall g, h \in G \right\}$$

called algebra of central functions on G , and form an orthonormal basis of it.

PROOF. Observe first that $\mathcal{C}(G)_{\text{central}}$ is indeed an algebra, which contains all the characters. Conversely, consider a function $f \in \mathcal{C}(G)$, written as follows:

$$f = \sum_{w \in \text{Irr}(G)} f_w$$

The condition $f \in \mathcal{C}(G)_{\text{central}}$ states then that for any $w \in \text{Irr}(G)$, we must have:

$$f_w \in \mathcal{C}(G)_{\text{central}}$$

But this means that f_w must be a scalar multiple of χ_w , so the characters form a basis of $\mathcal{C}(G)_{\text{central}}$, as stated. Also, the fact that we have an orthogonal basis follows from Theorem 4.17. As for the fact that the characters have norm 1, this follows from:

$$\int_G \chi_w \bar{\chi}_w = \sum_{ij} \int_G w_{ii} \bar{w}_{jj} = \sum_i \frac{1}{M} = 1$$

Here we have used the fact, coming from Theorem 4.15 and Proposition 4.16, that the integrals $\int_G w_{ij} \bar{w}_{kl}$ form the orthogonal projection onto the following vector space:

$$\text{Fix}(w \otimes \bar{w}) \simeq \text{End}(w) = \mathbb{C}1$$

Thus, the proof of our theorem is now complete. \square

4c. Diagrams, easiness

In view of the above results, no matter on what we want to do with our group, we must compute the spaces $Fix(v^{\otimes k})$. It is technically convenient to slightly enlarge the class of spaces to be computed, by talking about Tannakian categories, as follows:

DEFINITION 4.19. *The Tannakian category associated to a closed subgroup $G \subset_v U_N$ is the collection $C_G = (C_G(k, l))$ of vector spaces*

$$C_G(k, l) = Hom(v^{\otimes k}, v^{\otimes l})$$

where the representations $v^{\otimes k}$ with $k = \circ \bullet \bullet \circ \dots$ colored integer, defined by

$$v^{\otimes \emptyset} = 1 \quad , \quad v^{\otimes \circ} = v \quad , \quad v^{\otimes \bullet} = \bar{v}$$

and multiplicativity, $v^{\otimes kl} = v^{\otimes k} \otimes v^{\otimes l}$, are the Peter-Weyl representations.

Let us make a summary of what we have so far, regarding these spaces $C_G(k, l)$. In order to formulate our result, let us start with the following definition:

DEFINITION 4.20. *Let H be a finite dimensional Hilbert space. A tensor category over H is a collection $C = (C(k, l))$ of linear spaces*

$$C(k, l) \subset \mathcal{L}(H^{\otimes k}, H^{\otimes l})$$

satisfying the following conditions:

- (1) $S, T \in C$ implies $S \otimes T \in C$.
- (2) If $S, T \in C$ are composable, then $ST \in C$.
- (3) $T \in C$ implies $T^* \in C$.
- (4) $C(k, k)$ contains the identity operator.
- (5) $C(\emptyset, k)$ with $k = \circ \bullet, \bullet \circ$ contain the operator $R : 1 \rightarrow \sum_i e_i \otimes e_i$.
- (6) $C(kl, lk)$ with $k, l = \circ, \bullet$ contain the flip operator $\Sigma : a \otimes b \rightarrow b \otimes a$.

Here the tensor power Hilbert spaces $H^{\otimes k}$, with $k = \circ \bullet \bullet \circ \dots$ being a colored integer, are defined by the following formulae, and multiplicativity:

$$H^{\otimes \emptyset} = \mathbb{C} \quad , \quad H^{\otimes \circ} = H \quad , \quad H^{\otimes \bullet} = \bar{H} \simeq H$$

With these conventions, we have the following result, summarizing our knowledge on the subject, coming from the results established in the above:

THEOREM 4.21. *For a closed subgroup $G \subset_v U_N$, the associated Tannakian category*

$$C_G(k, l) = Hom(v^{\otimes k}, v^{\otimes l})$$

is a tensor category over the Hilbert space $H = \mathbb{C}^N$.

PROOF. We know that the fundamental representation v acts on the Hilbert space $H = \mathbb{C}^N$, and that its conjugate \bar{v} acts on the Hilbert space $\bar{H} = \mathbb{C}^N$. Now by multiplicativity we conclude that any Peter-Weyl representation $v^{\otimes k}$ acts on the Hilbert space $H^{\otimes k}$, and so that we have embeddings as in Definition 4.20, as follows:

$$C_G(k, l) \subset \mathcal{L}(H^{\otimes k}, H^{\otimes l})$$

Regarding now the fact that the axioms (1-6) in Definition 4.20 are indeed satisfied, this is something that we basically already know. To be more precise, (1-4) are clear, and (5) follows from the fact that each element $g \in G$ is a unitary, which gives:

$$R \in \text{Hom}(1, g \otimes \bar{g}) \quad , \quad R \in \text{Hom}(1, \bar{g} \otimes g)$$

As for (6), this is something trivial, coming from the fact that the matrix coefficients $g \rightarrow g_{ij}$ and their complex conjugates $g \rightarrow \bar{g}_{ij}$ commute with each other. \square

Our purpose now will be that of showing that any closed subgroup $G \subset U_N$ is uniquely determined by its Tannakian category $C_G = (C_G(k, l))$. This result, known as Tannakian duality, is something quite deep, and extremely useful. Indeed, the idea is that what we would have here is a “linearization” of G , allowing us to do combinatorics, and to ultimately reach to concrete and powerful results, regarding G itself. We first have:

THEOREM 4.22. *Given a tensor category $C = (C(k, l))$ over a finite dimensional Hilbert space $H \simeq \mathbb{C}^N$, the following construction,*

$$G_C = \left\{ g \in U_N \mid Tg^{\otimes k} = g^{\otimes l}T \quad , \quad \forall k, l, \forall T \in C(k, l) \right\}$$

produces a closed subgroup $G_C \subset U_N$.

PROOF. This is something elementary, with the fact that the closed subset $G_C \subset U_N$ constructed in the statement is indeed stable under the multiplication, unit and inversion operation for the unitary matrices $g \in U_N$ being clear from definitions. \square

We can now formulate the Tannakian duality result, as follows:

THEOREM 4.23. *The above Tannakian constructions*

$$G \rightarrow C_G \quad , \quad C \rightarrow G_C$$

are bijective, and inverse to each other.

PROOF. This is something quite technical, obtained by doing some abstract algebra, and for details here, we refer to the Tannakian duality literature. The whole subject is actually, in modern times, for the most part of quantum algebra, and you can consult here [64], [99], both quantum group papers, for details on the above. \square

In order to reach now to more concrete things, following Brauer’s philosophy in [30], and more specifically the more modern paper [21], based on it, we have:

DEFINITION 4.24. Let $P(k, l)$ be the set of partitions between an upper colored integer k , and a lower colored integer l . A collection of subsets

$$D = \bigsqcup_{k, l} D(k, l)$$

with $D(k, l) \subset P(k, l)$ is called a category of partitions when it has the following properties:

- (1) Stability under the horizontal concatenation, $(\pi, \sigma) \rightarrow [\pi\sigma]$.
- (2) Stability under vertical concatenation $(\pi, \sigma) \rightarrow \begin{bmatrix} \sigma \\ \pi \end{bmatrix}$, with matching middle symbols.
- (3) Stability under the upside-down turning $*$, with switching of colors, $\circ \leftrightarrow \bullet$.
- (4) Each set $P(k, k)$ contains the identity partition $|| \dots ||$.
- (5) The sets $P(\emptyset, \circ\bullet)$ and $P(\emptyset, \bullet\circ)$ both contain the semicircle \cap .
- (6) The sets $P(k, \bar{k})$ with $|k| = 2$ contain the crossing partition χ .

There are many examples of such categories, as for instance the category of all pairings \mathcal{P}_2 , or of all matching pairings \mathcal{P}_2 . We will be back to examples in a moment.

Let us formulate as well the following definition, also from [21]:

DEFINITION 4.25. Given a partition $\pi \in P(k, l)$ and an integer $N \in \mathbb{N}$, we can construct a linear map between tensor powers of \mathbb{C}^N ,

$$T_\pi : (\mathbb{C}^N)^{\otimes k} \rightarrow (\mathbb{C}^N)^{\otimes l}$$

by the following formula, with e_1, \dots, e_N being the standard basis of \mathbb{C}^N ,

$$T_\pi(e_{i_1} \otimes \dots \otimes e_{i_k}) = \sum_{j_1 \dots j_l} \delta_\pi \begin{pmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_l \end{pmatrix} e_{j_1} \otimes \dots \otimes e_{j_l}$$

and with the coefficients on the right being Kronecker type symbols,

$$\delta_\pi \begin{pmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_l \end{pmatrix} \in \{0, 1\}$$

whose values depend on whether the indices fit or not.

To be more precise, we put the indices of i, j on the legs of π , in the obvious way. In case all the blocks of π contain equal indices of i, j , we set $\delta_\pi \begin{pmatrix} i \\ j \end{pmatrix} = 1$. Otherwise, we set $\delta_\pi \begin{pmatrix} i \\ j \end{pmatrix} = 0$. The relation with the Tannakian categories comes from:

PROPOSITION 4.26. The assignment $\pi \rightarrow T_\pi$ is categorical, in the sense that

$$T_\pi \otimes T_\nu = T_{[\pi\nu]} \quad , \quad T_\pi T_\nu = N^{c(\pi, \nu)} T_{[\nu]} \quad , \quad T_\pi^* = T_{\pi^*}$$

where $c(\pi, \nu)$ are certain integers, coming from the erased components in the middle.

PROOF. This is something elementary, the computations being as follows:

(1) The concatenation axiom can be checked as follows:

$$\begin{aligned}
& (T_\pi \otimes T_\nu)(e_{i_1} \otimes \dots \otimes e_{i_p} \otimes e_{k_1} \otimes \dots \otimes e_{k_r}) \\
&= \sum_{j_1 \dots j_q} \sum_{l_1 \dots l_s} \delta_\pi \begin{pmatrix} i_1 & \dots & i_p \\ j_1 & \dots & j_q \end{pmatrix} \delta_\nu \begin{pmatrix} k_1 & \dots & k_r \\ l_1 & \dots & l_s \end{pmatrix} e_{j_1} \otimes \dots \otimes e_{j_q} \otimes e_{l_1} \otimes \dots \otimes e_{l_s} \\
&= \sum_{j_1 \dots j_q} \sum_{l_1 \dots l_s} \delta_{[\pi\nu]} \begin{pmatrix} i_1 & \dots & i_p & k_1 & \dots & k_r \\ j_1 & \dots & j_q & l_1 & \dots & l_s \end{pmatrix} e_{j_1} \otimes \dots \otimes e_{j_q} \otimes e_{l_1} \otimes \dots \otimes e_{l_s} \\
&= T_{[\pi\nu]}(e_{i_1} \otimes \dots \otimes e_{i_p} \otimes e_{k_1} \otimes \dots \otimes e_{k_r})
\end{aligned}$$

(2) The composition axiom can be checked as follows:

$$\begin{aligned}
& T_\pi T_\nu(e_{i_1} \otimes \dots \otimes e_{i_p}) \\
&= \sum_{j_1 \dots j_q} \delta_\nu \begin{pmatrix} i_1 & \dots & i_p \\ j_1 & \dots & j_q \end{pmatrix} \sum_{k_1 \dots k_r} \delta_\pi \begin{pmatrix} j_1 & \dots & j_q \\ k_1 & \dots & k_r \end{pmatrix} e_{k_1} \otimes \dots \otimes e_{k_r} \\
&= \sum_{k_1 \dots k_r} N^{c(\pi, \nu)} \delta_{[\pi]} \begin{pmatrix} i_1 & \dots & i_p \\ k_1 & \dots & k_r \end{pmatrix} e_{k_1} \otimes \dots \otimes e_{k_r} \\
&= N^{c(\pi, \nu)} T_{[\pi]}(e_{i_1} \otimes \dots \otimes e_{i_p})
\end{aligned}$$

(3) Finally, the involution axiom can be checked as follows:

$$\begin{aligned}
& T_\pi^*(e_{j_1} \otimes \dots \otimes e_{j_q}) \\
&= \sum_{i_1 \dots i_p} \langle T_\pi^*(e_{j_1} \otimes \dots \otimes e_{j_q}), e_{i_1} \otimes \dots \otimes e_{i_p} \rangle e_{i_1} \otimes \dots \otimes e_{i_p} \\
&= \sum_{i_1 \dots i_p} \delta_\pi \begin{pmatrix} i_1 & \dots & i_p \\ j_1 & \dots & j_q \end{pmatrix} e_{i_1} \otimes \dots \otimes e_{i_p} \\
&= T_{\pi^*}(e_{j_1} \otimes \dots \otimes e_{j_q})
\end{aligned}$$

Summarizing, our correspondence is indeed categorical. \square

In relation now with the groups, we have the following result, from [21]:

THEOREM 4.27. *Each category of partitions $D = (D(k, l))$ produces a family of compact groups $G = (G_N)$, with $G_N \subset_v U_N$, via the formula*

$$Hom(v^{\otimes k}, v^{\otimes l}) = span \left(T_\pi \Big|_{\pi \in D(k, l)} \right)$$

and the Tannakian duality correspondence.

PROOF. Given an integer $N \in \mathbb{N}$, consider the correspondence $\pi \rightarrow T_\pi$ constructed in Definition 4.25, and then the collection of linear spaces in the statement, namely:

$$C(k, l) = \text{span} \left(T_\pi \Big| \pi \in D(k, l) \right)$$

According to Proposition 4.26, and to our axioms for the categories of partitions, from Definition 4.24, this collection of spaces $C = (C(k, l))$ satisfies the axioms for the Tannakian categories, from Definition 4.20. Thus the Tannakian duality result, Theorem 4.23, applies, and provides us with a closed subgroup $G_N \subset_v U_N$ such that:

$$C(k, l) = \text{Hom}(v^{\otimes k}, v^{\otimes l})$$

Thus, we are led to the conclusion in the statement. \square

We can now formulate a key definition, as follows:

DEFINITION 4.28. *A closed subgroup $G \subset_v U_N$ is called easy when we have*

$$\text{Hom}(v^{\otimes k}, v^{\otimes l}) = \text{span} \left(T_\pi \Big| \pi \in D(k, l) \right)$$

for any colored integers k, l , for a certain category of partitions $D \subset P$.

The notion of easiness goes back to the results of Brauer in [30] regarding the orthogonal group O_N , and the unitary group U_N , which reformulate as follows:

THEOREM 4.29. *We have the following results:*

- (1) U_N is easy, coming from the category of matching pairings \mathcal{P}_2 .
- (2) O_N is easy too, coming from the category of all pairings P_2 .

PROOF. This is something very standard, the idea being as follows:

(1) The group U_N being defined via the relations $v^* = v^{-1}$, $v^t = \bar{v}^{-1}$, the associated Tannakian category is $C = \text{span}(T_\pi | \pi \in D)$, with:

$$D = \langle \begin{array}{c} \cap \\ \bullet \bullet \end{array}, \begin{array}{c} \cap \\ \bullet \circ \end{array} \rangle = \mathcal{P}_2$$

(2) The group $O_N \subset U_N$ being defined by imposing the relations $v_{ij} = \bar{v}_{ij}$, the associated Tannakian category is $C = \text{span}(T_\pi | \pi \in D)$, with:

$$D = \langle \mathcal{P}_2, \begin{array}{c} \updownarrow \\ \bullet \bullet \end{array}, \begin{array}{c} \updownarrow \\ \bullet \circ \end{array} \rangle = P_2$$

Thus, we are led to the conclusion in the statement. \square

Beyond this, a first natural question is that of computing the easy group associated to the category P itself, and we have here the following Brauer type theorem:

THEOREM 4.30. *The symmetric group S_N , regarded as group of unitary matrices,*

$$S_N \subset O_N \subset U_N$$

via the permutation matrices, is easy, coming from the category of all partitions P .

PROOF. Consider the easy group $G \subset O_N$ coming from the category of all partitions P . Since P is generated by the one-block partition $\mu \in P(2, 1)$, we have:

$$C(G) = C(O_N) / \left\langle T_\mu \in \text{Hom}(v^{\otimes 2}, v) \right\rangle$$

The linear map associated to μ is given by the following formula:

$$T_\mu(e_i \otimes e_j) = \delta_{ij} e_i$$

Thus, the relation defining the above group $G \subset O_N$ reformulates as follows:

$$T_\mu \in \text{Hom}(v^{\otimes 2}, v) \iff v_{ij} v_{ik} = \delta_{jk} v_{ij}, \forall i, j, k$$

In other words, the elements v_{ij} must be projections, and these projections must be pairwise orthogonal on the rows of $v = (v_{ij})$. We conclude that $G \subset O_N$ is the subgroup of matrices $g \in O_N$ having the property $g_{ij} \in \{0, 1\}$. Thus we have $G = S_N$, as claimed. \square

In fact, we have the following general easiness result, from [9], regarding the series of complex reflection groups $H_N^s \subset U_N$, that we introduced in chapter 2:

THEOREM 4.31. *The group $H_N^s = \mathbb{Z}_s \wr S_N$ is easy, the corresponding category P^s consisting of the partitions satisfying $\# \circ = \# \bullet (s)$ in each block. In particular:*

- (1) S_N is easy, coming from the category P .
- (2) H_N is easy, coming from the category P_{even} .
- (3) K_N is easy, coming from the category $\mathcal{P}_{\text{even}}$.

PROOF. This is something that we already know at $s = 1$, from Theorem 4.30. In general, the proof is similar, based on Tannakian duality. To be more precise, in what regards the main assertion, the idea here is that the one-block partition $\pi \in P(s)$, which generates the category P^s in the statement, implements the relations producing the subgroup $H_N^s \subset U_N$. As for the last assertions, these follow from the following observations:

(1) At $s = 1$ we know that we have $H_N^1 = S_N$. Regarding now the corresponding category, here the condition $\# \circ = \# \bullet (1)$ is automatic, and so $P^1 = P$.

(2) At $s = 2$ we know that we have $H_N^2 = H_N$. Regarding now the corresponding category, here the condition $\# \circ = \# \bullet (2)$ reformulates as follows:

$$\# \circ + \# \bullet = 0(2)$$

Thus each block must have even size, and we obtain, as claimed, $P^2 = P_{\text{even}}$.

(3) At $s = \infty$ we know that we have $H_N^\infty = K_N$. Regarding now the corresponding category, here the condition $\# \circ = \# \bullet (\infty)$ reads:

$$\# \circ = \# \bullet$$

But this is the condition defining $\mathcal{P}_{\text{even}}$, and so $P^\infty = \mathcal{P}_{\text{even}}$, as claimed. \square

Let us go back now to probability questions, with the aim of applying the above abstract theory, to questions regarding characters. The situation here is as follows:

(1) Given a closed subgroup $G \subset_v U_N$, we know from Peter-Weyl that the moments of the main character count the fixed points of the representations $v^{\otimes k}$.

(2) On the other hand, assuming that our group $G \subset_v U_N$ is easy, coming from a category of partitions $D = (D(k, l))$, the space formed by these fixed points is spanned by the following vectors, indexed by partitions π belonging to the set $D(k) = D(0, k)$:

$$\xi_\pi = \sum_{i_1 \dots i_k} \delta_\pi(i_1 \dots i_k) e_{i_1} \otimes \dots \otimes e_{i_k}$$

(3) Thus, we are left with investigating linear independence questions for the vectors ξ_π , and once these questions solved, to compute the moments of χ .

In order to investigate linear independence questions for the vectors ξ_π , we will use the Gram matrix of these vectors. Let us begin with some standard definitions:

DEFINITION 4.32. *Let $P(k)$ be the set of partitions of $\{1, \dots, k\}$, and let $\pi, \nu \in P(k)$.*

- (1) *We write $\pi \leq \nu$ if each block of π is contained in a block of ν .*
- (2) *We let $\pi \vee \nu \in P(k)$ be the partition obtained by superposing π, ν .*

As an illustration here, at $k = 2$ we have $P(2) = \{||, \sqcup\}$, and the order is:

$$|| \leq \sqcup$$

At $k = 3$ we have $P(3) = \{|||, \sqcup|, \sqcap, |\sqcup, \sqcap\sqcup\}$, and the order relation is as follows:

$$||| \leq \sqcup|, \sqcap, |\sqcup \leq \sqcap\sqcup$$

Observe also that we have $\pi, \nu \leq \pi \vee \nu$. In fact, $\pi \vee \nu$ is the smallest partition with this property, called supremum of π, ν . Now back to the easy groups, we have:

PROPOSITION 4.33. *The Gram matrix $G_{kN}(\pi, \nu) = \langle \xi_\pi, \xi_\nu \rangle$ is given by*

$$G_{kN}(\pi, \nu) = N^{|\pi \vee \nu|}$$

where $|\cdot|$ is the number of blocks.

PROOF. According to our formula of the vectors ξ_π , we have:

$$\begin{aligned} \langle \xi_\pi, \xi_\nu \rangle &= \sum_{i_1 \dots i_k} \delta_\pi(i_1, \dots, i_k) \delta_\nu(i_1, \dots, i_k) \\ &= \sum_{i_1 \dots i_k} \delta_{\pi \vee \nu}(i_1, \dots, i_k) \\ &= N^{|\pi \vee \nu|} \end{aligned}$$

Thus, we have obtained the formula in the statement. \square

In order to study the Gram matrix, and more specifically to compute its determinant, we will need several standard facts about the partitions. We first have:

DEFINITION 4.34. *The Möbius function of any lattice, and so of P , is given by*

$$\mu(\pi, \nu) = \begin{cases} 1 & \text{if } \pi = \nu \\ -\sum_{\pi \leq \tau < \nu} \mu(\pi, \tau) & \text{if } \pi < \nu \\ 0 & \text{if } \pi \not\leq \nu \end{cases}$$

with the construction being performed by recurrence.

As an illustration here, let us go back to the set of 2-point partitions, $P(2) = \{||, \sqcap\}$. Here we have by definition:

$$\mu(||, ||) = \mu(\sqcap, \sqcap) = 1$$

Also, we know that we have $|| < \sqcap$, with no intermediate partition in between, and so the above recurrence procedure gives the following formular:

$$\mu(||, \sqcap) = -\mu(||, ||) = -1$$

Finally, we have $\sqcap \not\leq ||$, which gives $\mu(\sqcap, ||) = 0$. Thus, as a conclusion, the Möbius matrix $M_{\pi\nu} = \mu(\pi, \nu)$ of the lattice $P(2) = \{||, \sqcap\}$ is as follows:

$$M = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

The interest in the Möbius function comes from the Möbius inversion formula:

$$f(\nu) = \sum_{\pi \leq \nu} g(\pi) \implies g(\nu) = \sum_{\pi \leq \nu} \mu(\pi, \nu) f(\pi)$$

In linear algebra terms, the statement and proof of this formula are as follows:

THEOREM 4.35. *The inverse of the adjacency matrix of P , given by*

$$A_{\pi\nu} = \begin{cases} 1 & \text{if } \pi \leq \nu \\ 0 & \text{if } \pi \not\leq \nu \end{cases}$$

is the Möbius matrix of P , given by $M_{\pi\nu} = \mu(\pi, \nu)$.

PROOF. This is well-known, coming for instance from the fact that A is upper triangular. Thus, when inverting, we are led into the recurrence from Definition 4.34. \square

As an illustration here, for $P(2)$ the formula $M = A^{-1}$ appears as follows:

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1}$$

Now back to our Gram matrix considerations, we have the following result:

PROPOSITION 4.36. *The Gram matrix is given by $G_{kN} = AL$, where*

$$L(\pi, \nu) = \begin{cases} N(N-1)\dots(N-|\pi|+1) & \text{if } \nu \leq \pi \\ 0 & \text{otherwise} \end{cases}$$

and where $A = M^{-1}$ is the adjacency matrix of $P(k)$.

PROOF. We have the following computation:

$$\begin{aligned} N^{|\pi \vee \nu|} &= \# \left\{ i_1, \dots, i_k \in \{1, \dots, N\} \mid \ker i \geq \pi \vee \nu \right\} \\ &= \sum_{\tau \geq \pi \vee \nu} \# \left\{ i_1, \dots, i_k \in \{1, \dots, N\} \mid \ker i = \tau \right\} \\ &= \sum_{\tau \geq \pi \vee \nu} N(N-1)\dots(N-|\tau|+1) \end{aligned}$$

According to Proposition 4.33 and to the definition of A, L , this formula reads:

$$(G_{kN})_{\pi\nu} = \sum_{\tau \geq \pi} L_{\tau\nu} = \sum_{\tau} A_{\pi\tau} L_{\tau\nu} = (AL)_{\pi\nu}$$

Thus, we obtain the formula in the statement. \square

With the above result in hand, we can now investigate the linear independence properties of the vectors ξ_π . To be more precise, we have the following result:

THEOREM 4.37. *The determinant of the Gram matrix G_{kN} is given by*

$$\det(G_{kN}) = \prod_{\pi \in P(k)} \frac{N!}{(N-|\pi|)!}$$

and in particular, for $N \geq k$, the vectors $\{\xi_\pi \mid \pi \in P(k)\}$ are linearly independent.

PROOF. According to the formula in Proposition 4.36, we have:

$$\det(G_{kN}) = \det(A) \det(L)$$

Now if we order $P(k)$ as usual, with respect to the number of blocks, and then lexicographically, we see that A is upper triangular, and that L is lower triangular. Thus $\det(A)$ can be computed simply by making the product on the diagonal, and we obtain 1. As for $\det(L)$, this can be computed as well by making the product on the diagonal, and we obtain the number in the statement, with the technical remark that in the case $N < k$ the convention is that we obtain a vanishing determinant. \square

We refer to [16], [41], [48] for more on all this, and we will be back to this interesting topic later on in this book. Now back to the laws of characters, we can formulate:

PROPOSITION 4.38. *For an easy group $G = (G_N)$, coming from a category of partitions $D = (D(k, l))$, the asymptotic moments of the main character are given by*

$$\lim_{N \rightarrow \infty} \int_{G_N} \chi^k = \#D(k)$$

where $D(k) = D(\emptyset, k)$, with the limiting sequence on the left consisting of certain integers, and being stationary at least starting from the k -th term.

PROOF. This follows indeed from the Peter-Weyl theory, by using the linear independence result for the vectors ξ_π coming from Theorem 4.37. \square

With these preliminaries in hand, we can now state and prove:

THEOREM 4.39. *In the $N \rightarrow \infty$ limit, the laws of the main character for the main easy groups, real and complex, and discrete and continuous, are as follows,*

$$\begin{array}{ccc} K_N & \longrightarrow & U_N \\ \uparrow & & \uparrow \\ H_N & \longrightarrow & O_N \end{array} \quad : \quad \begin{array}{ccc} B_1 & \longrightarrow & G_1 \\ \uparrow & & \uparrow \\ b_1 & \longrightarrow & g_1 \end{array}$$

with these laws, namely the real and complex Gaussian and Bessel laws, being the main limiting laws in real and complex, and discrete and continuous probability.

PROOF. This follows from the above results. To be more precise, we know that the above groups are all easy, the corresponding categories of partitions being as follows:

$$\begin{array}{ccc} \mathcal{P}_{\text{even}} & \longleftarrow & \mathcal{P}_2 \\ \downarrow & & \downarrow \\ P_{\text{even}} & \longleftarrow & P_2 \end{array}$$

Thus, we can use Proposition 4.38, and we are led into counting partitions, and then recovering the measures via their moments, and this leads to the result. \square

4d. Weingarten formula

Our aim now is to go beyond what we have, with results regarding the truncated characters. Let us start with a general formula coming from Peter-Weyl, namely:

THEOREM 4.40. *The Haar integration over a closed subgroup $G \subset_v U_N$ is given on the dense subalgebra of smooth functions by the Weingarten type formula*

$$\int_G g_{i_1 j_1}^{e_1} \dots g_{i_k j_k}^{e_k} dg = \sum_{\pi, \nu \in D(k)} \delta_\pi(i) \delta_\nu(j) W_k(\pi, \nu)$$

valid for any colored integer $k = e_1 \dots e_k$ and any multi-indices i, j , where $D(k)$ is a linear basis of $\text{Fix}(v^{\otimes k})$, the associated generalized Kronecker symbols are given by

$$\delta_\pi(i) = \langle \pi, e_{i_1} \otimes \dots \otimes e_{i_k} \rangle$$

and $W_k = G_k^{-1}$ is the inverse of the Gram matrix, $G_k(\pi, \nu) = \langle \pi, \nu \rangle$.

PROOF. This is something very standard, coming from the fact that the above integrals form altogether the orthogonal projection P^k onto the following space:

$$\text{Fix}(v^{\otimes k}) = \text{span}(D(k))$$

Consider now the following linear map, with $D(k) = \{\xi_k\}$ being as in the statement:

$$E(x) = \sum_{\pi \in D(k)} \langle x, \xi_\pi \rangle \xi_\pi$$

By a standard linear algebra computation, it follows that we have $P = WE$, where W is the inverse of the restriction of E to the following space:

$$K = \text{span} \left(T_\pi \Big|_{\pi \in D(k)} \right)$$

But this restriction is the linear map given by the matrix G_k , and so W is the linear map given by the inverse matrix $W_k = G_k^{-1}$, and this gives the result. \square

In the easy case, we have the following more concrete result:

THEOREM 4.41. *For an easy group $G \subset U_N$, coming from a category of partitions $D = (D(k, l))$, we have the Weingarten formula*

$$\int_G g_{i_1 j_1}^{e_1} \dots g_{i_k j_k}^{e_k} dg = \sum_{\pi, \nu \in D(k)} \delta_\pi(i) \delta_\nu(j) W_{kN}(\pi, \nu)$$

for any $k = e_1 \dots e_k$ and any i, j , where $D(k) = D(\emptyset, k)$, δ are usual Kronecker type symbols, checking whether the indices match, and $W_{kN} = G_{kN}^{-1}$, with

$$G_{kN}(\pi, \nu) = N^{|\pi \vee \nu|}$$

where $|\cdot|$ is the number of blocks.

PROOF. We use the abstract Weingarten formula, from Theorem 4.40. Indeed, the Kronecker type symbols there are then the usual ones, as shown by:

$$\begin{aligned} \delta_{\xi_\pi}(i) &= \langle \xi_\pi, e_{i_1} \otimes \dots \otimes e_{i_k} \rangle \\ &= \left\langle \sum_j \delta_\pi(j_1, \dots, j_k) e_{j_1} \otimes \dots \otimes e_{j_k}, e_{i_1} \otimes \dots \otimes e_{i_k} \right\rangle \\ &= \delta_\pi(i_1, \dots, i_k) \end{aligned}$$

The Gram matrix being as well the correct one, we obtain the result. \square

Let us go back now to the general easy groups $G \subset U_N$, with the idea in mind of computing the laws of truncated characters. First, we have the following formula:

PROPOSITION 4.42. *The moments of truncated characters are given by the formula*

$$\int_G (g_{11} + \dots + g_{ss})^k dg = \text{Tr}(W_{kN} G_{ks})$$

where G_{kN} and $W_{kN} = G_{kN}^{-1}$ are the associated Gram and Weingarten matrices.

PROOF. We have indeed the following computation:

$$\begin{aligned} \int_G (g_{11} + \dots + g_{ss})^k dg &= \sum_{i_1=1}^s \dots \sum_{i_k=1}^s \int_G g_{i_1 i_1} \dots g_{i_k i_k} dg \\ &= \sum_{\pi, \nu \in D(k)} W_{kN}(\pi, \nu) \sum_{i_1=1}^s \dots \sum_{i_k=1}^s \delta_\pi(i) \delta_\nu(i) \\ &= \sum_{\pi, \nu \in D(k)} W_{kN}(\pi, \nu) G_{ks}(\nu, \pi) \\ &= \text{Tr}(W_{kN} G_{ks}) \end{aligned}$$

Thus, we have reached to the formula in the statement. \square

In order to process now the above formula, and reach to concrete results, we must impose on our group a uniformity condition. Let us start with:

PROPOSITION 4.43. *For an easy group $G = (G_N)$, coming from a category of partitions $D \subset P$, the following conditions are equivalent:*

- (1) $G_{N-1} = G_N \cap U_{N-1}$, via the embedding $U_{N-1} \subset U_N$ given by $u \rightarrow \text{diag}(u, 1)$.
- (2) $G_{N-1} = G_N \cap U_{N-1}$, via the N possible diagonal embeddings $U_{N-1} \subset U_N$.
- (3) D is stable under the operation which consists in removing blocks.

If these conditions are satisfied, we say that $G = (G_N)$ is uniform.

PROOF. The equivalence (1) \iff (2) comes from the inclusion $S_N \subset G_N$, which makes everything S_N -invariant. Regarding (1) \iff (3), given a subgroup $K \subset_v U_{N-1}$, consider the matrix $u = \text{diag}(v, 1)$. Our claim is that for any $\pi \in P(k)$ we have:

$$\xi_\pi \in \text{Fix}(u^{\otimes k}) \iff \xi_{\pi'} \in \text{Fix}(u^{\otimes k'}), \forall \pi' \in P(k'), \pi' \subset \pi$$

In order to prove this claim, we must study the condition on the left. We have:

$$\begin{aligned} \xi_\pi \in \text{Fix}(u^{\otimes k}) &\iff (u^{\otimes k} \xi_\pi)_{i_1 \dots i_k} = (\xi_\pi)_{i_1 \dots i_k}, \forall i \\ &\iff \sum_j (u^{\otimes k})_{i_1 \dots i_k, j_1 \dots j_k} (\xi_\pi)_{j_1 \dots j_k} = (\xi_\pi)_{i_1 \dots i_k}, \forall i \\ &\iff \sum_j \delta_\pi(j_1, \dots, j_k) u_{i_1 j_1} \dots u_{i_k j_k} = \delta_\pi(i_1, \dots, i_k), \forall i \end{aligned}$$

Now let us recall that our representation has the special form $u = \text{diag}(v, 1)$. We conclude from this that for any index $a \in \{1, \dots, k\}$, we have:

$$i_a = N \implies j_a = N$$

With this observation in hand, if we denote by i', j' the multi-indices obtained from i, j obtained by erasing all the above $i_a = j_a = N$ values, and by $k' \leq k$ the common length of these new multi-indices, our condition becomes:

$$\sum_{j'} \delta_\pi(j_1, \dots, j_k) (u^{\otimes k'})_{i' j'} = \delta_\pi(i_1, \dots, i_k), \forall i$$

Here the index j is by definition obtained from the index j' by filling with N values. In order to finish now, we have two cases, depending on i , as follows:

Case 1. Assume that the index set $\{a | i_a = N\}$ corresponds to a certain subpartition $\pi' \subset \pi$. In this case, the N values will not matter, and our formula becomes:

$$\sum_{j'} \delta_\pi(j'_1, \dots, j'_{k'}) (u^{\otimes k'})_{i' j'} = \delta_\pi(i'_1, \dots, i'_{k'})$$

Case 2. Assume now the opposite, namely that the set $\{a | i_a = N\}$ does not correspond to a subpartition $\pi' \subset \pi$. In this case the indices mix, and our formula reads $0 = 0$. Thus we have $\xi_{\pi'} \in \text{Fix}(u^{\otimes k'})$ in both cases, for any subpartition $\pi' \subset \pi$, as desired. \square

Now back to the laws of truncated characters, we have the following result:

THEOREM 4.44. *For a uniform easy group $G = (G_N)$, we have the formula*

$$\lim_{N \rightarrow \infty} \int_{G_N} \chi_t^k = \sum_{\pi \in D(k)} t^{|\pi|}$$

with $D \subset P$ being the associated category of partitions.

PROOF. We use Proposition 4.42. With $s = [tN]$, the formula there becomes:

$$\int_{G_N} \chi_t^k = \text{Tr}(W_{kN} G_{k[tN]})$$

The point now is that in the uniform case the Gram matrix, and so the Weingarten matrix too, is asymptotically diagonal. Thus, we obtain the following estimate:

$$\begin{aligned} \int_{G_N} \chi_t^k &\simeq \sum_{\pi \in D(k)} W_{kN}(\pi, \pi) G_{k[tN]}(\pi, \pi) \\ &\simeq \sum_{\pi \in D(k)} N^{-|\pi|} (tN)^{|\pi|} \\ &= \sum_{\pi \in D(k)} t^{|\pi|} \end{aligned}$$

Thus, we are led to the formula in the statement. \square

We can now enlarge our collection of truncated character results, and we have:

THEOREM 4.45. *With $N \rightarrow \infty$, the laws of truncated characters are as follows:*

- (1) For O_N we obtain the Gaussian law g_t .
- (2) For U_N we obtain the complex Gaussian law G_t .
- (3) For S_N we obtain the Poisson law p_t .
- (4) For H_N we obtain the Bessel law b_t .
- (5) For H_N^s we obtain the generalized Bessel law b_t^s .
- (6) For K_N we obtain the complex Bessel law B_t .

PROOF. We already know these results at $t = 1$. In the general case, $t > 0$, these follow via some standard combinatorics, from the formula in Theorem 4.44. \square

4e. Exercises

We had a lot of general theory in this chapter, regarding the compact groups and their representations, and the notion of easiness. As a first exercise about this, we have:

EXERCISE 4.46. *Prove that the bistochastic groups $B_N \subset O_N$ and $C_N \subset U_N$, consisting of matrices having sum 1 on each row and column, are both easy.*

This looks quite routine, by suitably adapting the proofs for O_N and U_N .

EXERCISE 4.47. *Look up the full theory of the symplectic group $Sp_N \subset U_N$, namely Brauer theorem, super-easiness, and Weingarten formula.*

Here the main reference for the general theory is the paper by Collins-Śniady [36], but you will still have to adapt the material there, which is quite advanced, as to fit with what you learned from here, as for everything to be complete.

Part II

Random matrices

*Life is a mystery
Everyone must stand alone
I hear you call my name
And it feels like home*

CHAPTER 5

Spectral measures

5a. Linear algebra

We have seen so far some interesting probability theory, dealing with usual random variables, which are by definition functions as follows, real or complex:

$$f \in L^\infty(X)$$

We discuss in what follows more advanced aspects of probability theory, which are of rather “noncommutative” nature, in relation with the random matrices:

DEFINITION 5.1. *A random matrix is a square matrix of type*

$$Z \in M_N(L^\infty(X))$$

with X being a probability space, and $N \in \mathbb{N}$ being an integer.

As basic examples, we have the usual matrices $Z \in M_N(\mathbb{C})$, obtained by taking $X = \{.\}$. Also, we have the usual random variables $Z \in L^\infty(X)$, obtained by taking $N = 1$. In general, what we have is a joint generalization of these two situations.

As a first task, we must understand what the distribution of a random matrix is. This is something non-trivial, which will take some time. Let us begin with a discussion concerning the usual matrices $A \in M_N(\mathbb{C})$. We have here the following definition:

DEFINITION 5.2. *The moments of a complex matrix $A \in M_N(\mathbb{C})$ are the following numbers, with $tr = N^{-1} \cdot Tr$ being the normalized matrix trace:*

$$M_k = tr(A^k)$$

The distribution, or law, of our matrix A is the following abstract functional:

$$\mu_A : \mathbb{C}[X] \rightarrow \mathbb{C} \quad , \quad P \rightarrow tr(P(A))$$

In the case where we have a probability measure $\mu_A \in \mathcal{P}(\mathbb{C})$ such that

$$tr(P(A)) = \int_{\mathbb{C}} P(x) d\mu_A(x)$$

we identify this complex measure with the distribution of A .

As a basic example for this, consider the case of a diagonal matrix:

$$A = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix}$$

The powers of A , with respect to integer exponents $k \in \mathbb{N}$, are as follows:

$$A^k = \begin{pmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_N^k \end{pmatrix}$$

Thus the moments of A are given by the following formula:

$$M_k = \sum_i \lambda_i^k$$

More generally now, we have the following formula, valid for any $P \in \mathbb{C}[X]$:

$$P(A) = \begin{pmatrix} P(\lambda_1) & & \\ & \ddots & \\ & & P(\lambda_N) \end{pmatrix}$$

By applying the normalized trace, we obtain from this formula:

$$\begin{aligned} \operatorname{tr}(P(A)) &= \frac{1}{N}(P(\lambda_1) + \dots + P(\lambda_N)) \\ &= \frac{1}{N} \int_{\mathbb{C}} P(x) d(\delta_{\lambda_1} + \dots + \delta_{\lambda_N})(x) \\ &= \int_{\mathbb{C}} P(x) d\left(\frac{1}{N}(\delta_{\lambda_1} + \dots + \delta_{\lambda_N})\right)(x) \end{aligned}$$

Thus, according to Definition 5.2, the law of A is the following measure:

$$\mu_A = \frac{1}{N}(\delta_{\lambda_1} + \dots + \delta_{\lambda_N})$$

Quite remarkably, the distribution always exists as a probability measure on \mathbb{C} , and is given by the above formula, as the average of the eigenvalues:

THEOREM 5.3. *For any matrix $A \in M_N(\mathbb{C})$ we have the formula*

$$\operatorname{tr}(P(A)) = \frac{1}{N}(P(\lambda_1) + \dots + P(\lambda_N))$$

where $\lambda_1, \dots, \lambda_N \in \mathbb{C}$ are the eigenvalues of A . Thus the complex measure

$$\mu_A = \frac{1}{N}(\delta_{\lambda_1} + \dots + \delta_{\lambda_N})$$

is the distribution of A , in the abstract sense of Definition 5.2.

PROOF. According to the above discussion, the result holds for the diagonal matrices. More generally now, let us discuss the case where our matrix A is diagonalizable. Here we must have a formula as follows, with D being diagonal:

$$A = PDP^{-1}$$

Now observe that the moments of A are given by the following formula:

$$\begin{aligned} \text{tr}(A^k) &= \text{tr}(PDP^{-1} \cdot PDP^{-1} \dots PDP^{-1}) \\ &= \text{tr}(PD^kP^{-1}) \\ &= \text{tr}(D^k) \end{aligned}$$

We conclude, by linearity, that the matrices A, D have the same distribution:

$$\mu_A = \mu_D$$

On the other hand, $A = PDP^{-1}$ shows that A, D have the same eigenvalues. Thus, if we denote by $\lambda_1, \dots, \lambda_N \in \mathbb{C}$ these eigenvalues, we obtain:

$$\mu_A = \frac{1}{N}(\delta_{\lambda_1} + \dots + \delta_{\lambda_N})$$

Finally, in the general case, the result follows from what we know from the above, by using the well-known fact that the diagonalizable matrices are dense. \square

Summarizing, we have a nice theory for the matrices $A \in M_N(\mathbb{C})$, paralleling that of the random variables $f \in L^\infty(X)$. It is tempting at this point to try to go further, and unify the matrices and the random variables, by talking about random matrices:

$$Z \in M_N(L^\infty(X))$$

However, we will not do this right away, because our matrix theory has a flaw. Indeed, all what has being said above does not take into account the adjoint matrix:

$$A^* = (\bar{A}_{ji})$$

To be more precise, the idea is that the matrices $A \in M_N(\mathbb{C})$ do not come alone, but rather in pairs (A, A^*) , and this because no matter what you want to do with A , of advanced type, you will run at some point into its adjoint A^* . Thus, we must talk about the moments and distribution of the pair (A, A^*) . This can be done as follows:

DEFINITION 5.4. *The generalized moments of a complex matrix $A \in M_N(\mathbb{C})$ are the following numbers, indexed by the colored integers $k = \circ \bullet \circ \dots$*

$$M_k = \text{tr}(A^k)$$

with A^k being defined by the following formulae and multiplicativity, $A^{kl} = A^k A^l$,

$$A^\emptyset = 1 \quad , \quad A^\circ = A \quad , \quad A^\bullet = A^*$$

and with $\text{tr} = N^{-1} \cdot \text{Tr}$ being as usual the normalized matrix trace.

All this might seem a bit complicated, but this is the situation, and there is no other way of dealing with such things. Indeed, since the variables A, A^* do not commute, unless the matrix is normal, $AA^* = A^*A$, which is something special, which does not happen in general, we are led to colored exponents $k = \circ \bullet \bullet \circ \dots$ and to the above definition for the moments. Regarding now the distribution, we can use here a similar idea, as follows:

DEFINITION 5.5. *The generalized distribution, or law, of a matrix $A \in M_N(\mathbb{C})$ is the abstract functional $\mu_A : \mathbb{C} \langle X, X^* \rangle \rightarrow \mathbb{C}$ given by:*

$$P \rightarrow \text{tr}(P(A))$$

In the case where we have a probability measure $\mu_A \in \mathcal{P}(\mathbb{C})$ such that

$$\text{tr}(P(A)) = \int_{\mathbb{C}} P(x) d\mu_A(x)$$

we identify this complex measure with the distribution of A .

Observe that knowing the distribution is the same as knowing the moments, because if we write our noncommutative polynomial as $P = \sum_k c_k X^k$, then we have:

$$\text{tr}(P(A)) = \text{tr} \left(\sum_k c_k A^k \right) = \sum_k c_k M_k$$

As a first result now, coming from Theorem 5.3, we have:

THEOREM 5.6. *Given a matrix $A \in M_N(\mathbb{C})$ which is self-adjoint, $A = A^*$, we have the following formula, valid for any polynomial $P \in \mathbb{C} \langle X, X^* \rangle$,*

$$\text{tr}(P(A)) = \frac{1}{N} (P(\lambda_1) + \dots + P(\lambda_N))$$

where $\lambda_1, \dots, \lambda_N \in \mathbb{C}$ are the eigenvalues of A . Thus the complex measure

$$\mu_A = \frac{1}{N} (\delta_{\lambda_1} + \dots + \delta_{\lambda_N})$$

is the distribution of A , in the abstract sense of Definition 5.4.

PROOF. This follows indeed from Theorem 5.3, because due to our self-adjointness assumption $A = A^*$, the adjoint matrix plays no role in all this. \square

Quite remarkably, Theorem 5.6 extends to the normal case. This is something non-trivial, that we will explain now, after some linear algebra. Let us start with:

PROPOSITION 5.7. *Any matrix $A \in M_N(\mathbb{C})$ which is self-adjoint, $A = A^*$, is diagonalizable, with the diagonalization being of the following type,*

$$A = UDU^*$$

with $U \in U_N$, and with $D \in M_N(\mathbb{R})$ diagonal. The converse holds too.

PROOF. Let us first prove that the eigenvalues are real. If $Ax = \lambda x$, we have:

$$\begin{aligned}\lambda \langle x, x \rangle &= \langle Ax, x \rangle \\ &= \langle x, Ax \rangle \\ &= \bar{\lambda} \langle x, x \rangle\end{aligned}$$

Thus we obtain $\lambda \in \mathbb{R}$, as claimed. Our next claim now is that the eigenspaces corresponding to different eigenvalues are pairwise orthogonal. Assume indeed that:

$$Ax = \lambda x \quad , \quad Ay = \mu y$$

We have then the following computation, by using $\lambda, \mu \in \mathbb{R}$:

$$\begin{aligned}\lambda \langle x, y \rangle &= \langle Ax, y \rangle \\ &= \langle x, Ay \rangle \\ &= \mu \langle x, y \rangle\end{aligned}$$

Thus $\lambda \neq \mu$ implies $x \perp y$, as claimed. In order now to finish, it remains to prove that the eigenspaces span the whole \mathbb{C}^N . For this purpose, we will use a recurrence method. Let us pick an eigenvector of our matrix, $Ax = \lambda x$. Assuming $x \perp y$, we have:

$$\begin{aligned}\langle Ay, x \rangle &= \langle y, Ax \rangle \\ &= \langle y, \lambda x \rangle \\ &= \lambda \langle y, x \rangle \\ &= 0\end{aligned}$$

Thus, if x is an eigenvector of A , then the vector space x^\perp is invariant under A . On the other hand, since a square matrix A is self-adjoint precisely when $\langle Ax, x \rangle \in \mathbb{R}$, we conclude that the restriction of our matrix A to the vector space x^\perp is self-adjoint. Thus, we can proceed by recurrence, and we obtain in this way the result. \square

Let us discuss as well the case of the unitary matrices. We have here:

PROPOSITION 5.8. *Any matrix $U \in M_N(\mathbb{C})$ which is unitary, $U^* = U^{-1}$, is diagonalizable, with the eigenvalues being on \mathbb{T} . More precisely we have*

$$U = VDV^*$$

with $V \in U_N$, and with $D \in M_N(\mathbb{T})$ diagonal. The converse holds too.

PROOF. Assuming $Ux = \lambda x$, we have the following formula:

$$\begin{aligned}\langle x, x \rangle &= \langle U^*Ux, x \rangle \\ &= \langle Ux, Ux \rangle \\ &= \langle \lambda x, \lambda x \rangle \\ &= |\lambda|^2 \langle x, x \rangle\end{aligned}$$

Thus we obtain $\lambda \in \mathbb{T}$, as desired. Our next claim now is that the eigenspaces corresponding to different eigenvalues are pairwise orthogonal. Assume indeed that:

$$Ux = \lambda x \quad , \quad Uy = \mu y$$

We have then the following computation, by using $U^* = U^{-1}$ and $\lambda, \mu \in \mathbb{T}$:

$$\begin{aligned} \lambda \langle x, y \rangle &= \langle Ux, y \rangle \\ &= \langle x, U^*y \rangle \\ &= \langle x, U^{-1}y \rangle \\ &= \langle x, \mu^{-1}y \rangle \\ &= \mu \langle x, y \rangle \end{aligned}$$

Thus $\lambda \neq \mu$ implies $x \perp y$, as claimed. In order now to finish, it remains to prove that the eigenspaces span the whole \mathbb{C}^N . For this purpose, we will use a recurrence method. Let us pick an eigenvector, $Ux = \lambda x$. Assuming $x \perp y$, we have:

$$\begin{aligned} \langle Uy, x \rangle &= \langle y, U^*x \rangle \\ &= \langle y, U^{-1}x \rangle \\ &= \langle y, \lambda^{-1}x \rangle \\ &= \lambda \langle y, x \rangle \\ &= 0 \end{aligned}$$

Thus, if x is an eigenvector of U , then the vector space x^\perp is invariant under U . Now since U is an isometry, so is its restriction to this space x^\perp . Thus this restriction is a unitary, and so we can proceed by recurrence, and we obtain the result. \square

We have in fact the following general result, extending what we know so far:

THEOREM 5.9. *Any matrix $A \in M_N(\mathbb{C})$ which is normal, $AA^* = A^*A$, is diagonalizable, with the diagonalization being of the following type,*

$$A = UDU^*$$

with $U \in U_N$, and with $D \in M_N(\mathbb{C})$ diagonal. The converse holds too.

PROOF. This is something quite technical. Our first claim is that a matrix A is normal precisely when the following is satisfied, for any vector x :

$$\|Ax\| = \|A^*x\|$$

Indeed, this equality can be written in the following way, which gives $AA^* = A^*A$:

$$\langle AA^*x, x \rangle = \langle A^*Ax, x \rangle$$

Our claim now is that A, A^* have the same eigenvectors, with conjugate eigenvalues:

$$Ax = \lambda x \implies A^*x = \bar{\lambda}x$$

Indeed, this follows from the following computation, and from the trivial fact that if A is normal, then so is any matrix of type $A - \lambda 1_N$, with $\lambda \in \mathbb{C}$:

$$\begin{aligned} \|(A^* - \bar{\lambda} 1_N)x\| &= \|(A - \lambda 1_N)^*x\| \\ &= \|(A - \lambda 1_N)x\| \\ &= 0 \end{aligned}$$

Let us prove now, by using this fact, that the eigenspaces of A are pairwise orthogonal. Assuming $Ax = \lambda x$ and $Ay = \mu y$ with $\lambda \neq \mu$, we have:

$$\begin{aligned} \lambda \langle x, y \rangle &= \langle Ax, y \rangle \\ &= \langle x, A^*y \rangle \\ &= \langle x, \bar{\mu}y \rangle \\ &= \mu \langle x, y \rangle \end{aligned}$$

Thus $\lambda \neq \mu$ implies $x \perp y$, as desired. In order to finish now the proof, it remains to prove that the eigenspaces of A span the whole \mathbb{C}^N . This is something quite tricky, and our plan here will be that of proving that the eigenspaces of AA^* are eigenspaces of A . In order to do so, let us pick two eigenvectors x, y of the matrix AA^* , corresponding to different eigenvalues, $\lambda \neq \mu$. The eigenvalue equations are then as follows:

$$AA^*x = \lambda x \quad , \quad AA^*y = \mu y$$

We have the following computation, by using the normality condition $AA^* = A^*A$, and the fact that the eigenvalues of AA^* , and in particular μ , are real:

$$\begin{aligned} \lambda \langle Ax, y \rangle &= \langle A\lambda x, y \rangle \\ &= \langle AAA^*x, y \rangle \\ &= \langle AA^*Ax, y \rangle \\ &= \langle Ax, AA^*y \rangle \\ &= \langle Ax, \mu y \rangle \\ &= \mu \langle Ax, y \rangle \end{aligned}$$

We conclude that we have $\langle Ax, y \rangle = 0$. But this reformulates as follows:

$$\lambda \neq \mu \implies A(E_\lambda) \perp E_\mu$$

Now since the eigenspaces of AA^* are pairwise orthogonal, and span the whole \mathbb{C}^N , we deduce that these eigenspaces are invariant under A :

$$A(E_\lambda) \subset E_\lambda$$

But with this result in hand, we can now finish. Indeed, we can decompose the problem, and the matrix A itself, following these eigenspaces of AA^* , which in practice amounts in saying that we can assume that we only have 1 eigenspace. By rescaling, this

is the same as assuming that we have $AA^* = 1$, and so we are now into the unitary case, that we know how to solve, as explained in Proposition 5.8. \square

Getting back now to the laws of matrices, Theorem 5.6 extends to the normal case, $AA^* = A^*A$. This is something non-trivial, the result being as follows:

THEOREM 5.10. *Given a matrix $A \in M_N(\mathbb{C})$ which is normal, $AA^* = A^*A$, we have the following formula, valid for any polynomial $P \in \mathbb{C} \langle X, X^* \rangle$,*

$$\text{tr}(P(A)) = \frac{1}{N}(P(\lambda_1) + \dots + P(\lambda_N))$$

where $\lambda_1, \dots, \lambda_N \in \mathbb{C}$ are the eigenvalues of A . Thus the complex measure

$$\mu_A = \frac{1}{N}(\delta_{\lambda_1} + \dots + \delta_{\lambda_N})$$

is the distribution of A , in the abstract sense of Definition 5.5.

PROOF. There are several proofs for this fact, one of them being as follows:

(1) Let us first consider the case where the matrix is diagonal:

$$A = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix}$$

The moments of A are then given by the following formula:

$$M_k = \frac{1}{N}(\lambda_1^k + \dots + \lambda_N^k)$$

Regarding now the distribution, this by definition given by:

$$\mu_A : \mathbb{C} \langle X, X^* \rangle \rightarrow \mathbb{C} \quad , \quad P \rightarrow \text{tr}(P(A))$$

Since the matrix is normal, $AA^* = A^*A$, knowing this distribution is the same as knowing its restriction to the usual polynomials in two variables:

$$\mu_A : \mathbb{C}[X, X^*] \rightarrow \mathbb{C} \quad , \quad P \rightarrow \text{tr}(P(A))$$

By using now the fact that A is diagonal, we conclude that the distribution is:

$$\mu_A : \mathbb{C}[X, X^*] \rightarrow \mathbb{C} \quad , \quad P \rightarrow \frac{1}{N}(P(\lambda_1) + \dots + P(\lambda_N))$$

But this functional corresponds to integrating P with respect to the following complex measure, that we agree to still denote by μ_A , and call distribution of A :

$$\mu_A = \frac{1}{N}(\delta_{\lambda_1} + \dots + \delta_{\lambda_N})$$

(2) In the general case now, where $A \in M_N(\mathbb{C})$ is normal and arbitrary, we can use Theorem 5.9, which tells us that A is diagonalizable, and in fact that A, A^* are jointly diagonalizable. To be more precise, let us write, as in Theorem 5.9:

$$A = UDU^*$$

Here $U \in U_N$, and $D \in M_N(\mathbb{C})$ is diagonal. The adjoint matrix is then given by:

$$A^* = UD^*U$$

As before in the diagonal matrix case, since our matrix is normal, $AA^* = A^*A$, knowing its distribution in the abstract sense of Definition 5.5 is the same as knowing the restriction of this abstract distribution to the usual polynomials in two variables:

$$\mu_A : \mathbb{C}[X, X^*] \rightarrow \mathbb{C} \quad , \quad P \rightarrow \text{tr}(P(A))$$

In order now to compute this functional, we can change the basis via the above unitary matrix $U \in U_N$, which in practice means that we can assume $U = 1$. Thus, by using now (1), if we denote by $\lambda_1, \dots, \lambda_N$ the diagonal entries of D , which are the eigenvalues of A , the distribution that we are looking for is the following functional:

$$\mu_A : \mathbb{C}[X, X^*] \rightarrow \mathbb{C} \quad , \quad P \rightarrow \frac{1}{N}(P(\lambda_1) + \dots + P(\lambda_N))$$

As before, this functional corresponds to integrating P with respect to the following complex measure, that we agree to still denote by μ_A , and call distribution of A :

$$\mu_A = \frac{1}{N}(\delta_{\lambda_1} + \dots + \delta_{\lambda_N})$$

Thus, we are led to the conclusion in the statement. \square

We can now go ahead and discuss, eventually, the case of the random matrices, where things become truly interesting. We can extend Definition 5.5, as follows:

DEFINITION 5.11. *The colored moments of a random matrix*

$$Z \in M_N(L^\infty(X))$$

are the following numbers, indexed by the colored integers $k = \circ \bullet \bullet \circ \dots$

$$M_k = \int_X \text{tr}(Z^k)$$

with the powers Z^k being defined by $Z^\circ = Z$, $Z^\bullet = Z^*$ and multiplicativity.

Observe that this notion extends indeed the notion from Definition 5.5 for the usual matrices $Z \in M_N(\mathbb{C})$, which can be recovered with $X = \{.\}$. Also, in the case $N = 1$, where our matrix is just a random variable $Z \in L^\infty(X)$, we recover in this way the usual moments, or rather the joint moments of the random variables Z, \bar{Z} . Regarding now the distribution, we can use here a similar extension, as follows:

DEFINITION 5.12. *The distribution of a random matrix $Z \in M_N(L^\infty(X))$ is the abstract functional $\mu_Z : \mathbb{C} \langle X, X^* \rangle \rightarrow \mathbb{C}$ given by:*

$$P \rightarrow \int_X \text{tr}(P(Z))$$

In the case where we have a probability measure $\mu_Z \in \mathcal{P}(\mathbb{C})$ such that

$$\text{tr}(P(Z)) = \int_{\mathbb{C}} P(x) d\mu_Z(x)$$

we identify this measure with the distribution, or law of Z .

As basic examples, for the usual matrices $Z \in M_N(\mathbb{C})$, obtained by taking $X = \{.\}$, we obtain the previous notion of distribution of a matrix, from Definition 5.5. Also, for the usual random variables $Z \in L^\infty(X)$, obtained by taking $N = 1$, we obtain in this way the previous notion of distribution of a random variable, from chapters 1-2.

5b. Bounded operators

In order to further clarify all the above, and to discuss as well what happens in the non-normal case, we will need an extension of the theory that we have, going beyond the random matrix setting, by using some basic functional analysis and spectral theory. In order to get started, let us formulate the following definition:

DEFINITION 5.13. *A Hilbert space is a complex vector space H given with a scalar product $\langle x, y \rangle$, satisfying the following conditions:*

- (1) $\langle x, y \rangle$ is linear in x , and antilinear in y .
- (2) $\overline{\langle x, y \rangle} = \langle y, x \rangle$, for any x, y .
- (3) $\langle x, x \rangle \geq 0$, for any $x \neq 0$.
- (4) H is complete with respect to the norm $\|x\| = \sqrt{\langle x, x \rangle}$.

Here the fact that $\|\cdot\|$ is indeed a norm comes from the Cauchy-Schwarz inequality, which states that if the conditions (1,2,3) above are satisfied, then we have:

$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$$

Indeed, this inequality comes from the fact that the following degree 2 polynomial, with $t \in \mathbb{R}$ and $w \in \mathbb{T}$, being positive, its discriminant must be negative:

$$f(t) = \|x + twy\|^2$$

At the level of the examples, we first have the Hilbert space $H = \mathbb{C}^N$, with its usual scalar product, taken by definition linear at left, namely:

$$\langle x, y \rangle = \sum_i x_i \bar{y}_i$$

More generally, making the link with probability, we have the following result:

PROPOSITION 5.14. *Given a measured space X , the functions $f : X \rightarrow \mathbb{C}$, taken up to equality almost everywhere, which are square-summable,*

$$\int_X |f(x)|^2 dx < \infty$$

form a Hilbert space $L^2(X)$, with the following scalar product:

$$\langle f, g \rangle = \int_X f(x) \overline{g(x)} dx$$

In the case where $X = I$ is a set endowed with its counting measure, we obtain the space $l^2(I)$ of square-summable sequences $\{x_i\}_{i \in I} \subset \mathbb{C}$, with $\langle x, y \rangle = \sum_i x_i \bar{y}_i$.

PROOF. There are several things to be proved, as follows:

(1) Our first claim is that $L^2(X)$ is a vector space, and here we must prove that $f, g \in L^2(X)$ implies $f + g \in L^2(X)$. But this leads us into proving $\|f + g\| \leq \|f\| + \|g\|$, where $\|f\| = \sqrt{\langle f, f \rangle}$. Now since this inequality holds on each subspace $\mathbb{C}^N \subset L^2(X)$ coming from step functions, this inequality holds everywhere, as desired.

(2) Our second claim is that \langle, \rangle is well-defined on $L^2(X)$. But this follows from the Cauchy-Schwarz inequality, $|\langle f, g \rangle| \leq \|f\| \cdot \|g\|$, which can be established by truncating, a bit like we established the Minkowski inequality in (1) above.

(3) It is also clear that \langle, \rangle is a scalar product on $L^2(X)$, with the remark here that if we want to have $\langle f, f \rangle > 0$ for $f \neq 0$, we must declare that $f = 0$ when $f = 0$ almost everywhere, and so that $f = g$ when $f = g$ almost everywhere, as stated.

(4) It remains to prove that $L^2(X)$ is complete with respect to $\|f\| = \sqrt{\langle f, f \rangle}$. But this is clear, because if we pick a Cauchy sequence $\{f_n\}_{n \in \mathbb{N}} \subset L^2(X)$, then we can construct a pointwise, and hence L^2 limit, $f_n \rightarrow f$, almost everywhere.

(5) Finally, the last assertion is clear, because the integration with respect to the counting measure is by definition a sum, and so we have $L^2(I) = l^2(I)$. \square

Quite remarkably, any Hilbert space must be of the form $L^2(X)$, and even of the special form $l^2(I)$. This follows indeed from the following key result:

THEOREM 5.15. *Let H be a Hilbert space.*

- (1) *Any algebraic basis of this space $\{f_i\}_{i \in I}$ can be turned into an orthonormal basis $\{e_i\}_{i \in I}$, by using the Gram-Schmidt procedure.*
- (2) *Thus, H has an orthonormal basis, and so we have $H \simeq l^2(I)$, with I being the indexing set for this orthonormal basis.*

PROOF. All this is standard by Gram-Schmidt, the idea being as follows:

(1) First of all, in finite dimensions an orthonormal basis $\{e_i\}_{i \in I}$ is by definition a usual algebraic basis, satisfying $\langle e_i, e_j \rangle = \delta_{ij}$. But the existence of such a basis follows by applying the Gram-Schmidt procedure to any algebraic basis $\{f_i\}_{i \in I}$, as claimed.

(2) In infinite dimensions, we can say that $\{f_i\}_{i \in I}$ is a basis of H when the functions f_i are linearly independent, and when the finite linear combinations of these functions f_i form a dense subspace of H . For orthogonal bases $\{e_i\}_{i \in I}$ these definitions are equivalent, and in any case, our statement makes now sense.

(3) Regarding now the proof, in infinite dimensions, this follows again from Gram-Schmidt, exactly as in the finite dimensional case, but by using this time a tool from logic, called Zorn lemma, in order to correctly do the recurrence. \square

The above result is something quite subtle, and suggests formulating:

DEFINITION 5.16. *A Hilbert space H is called separable when the following equivalent conditions are satisfied:*

- (1) *H has a countable algebraic basis $\{f_i\}_{i \in \mathbb{N}}$.*
- (2) *H has a countable orthonormal basis $\{e_i\}_{i \in \mathbb{N}}$.*
- (3) *We have $H \simeq l^2(\mathbb{N})$, isomorphism of Hilbert spaces.*

As a first observation, according to the above, there is up to isomorphism only one separable Hilbert space, namely:

$$H = l^2(\mathbb{N})$$

This is, however, quite tricky, and can be a bit misleading. Consider for instance the space $H = L^2[0, 1]$ of square-summable functions $f : [0, 1] \rightarrow \mathbb{C}$, with:

$$\langle f, g \rangle = \int_0^1 f(x) \overline{g(x)} dx$$

This space is of course separable, because we can use the basis $f_n = x^n$ with $n \in \mathbb{N}$, orthogonalized by Gram-Schmidt. However, the orthogonalization procedure is something non-trivial, so the isomorphism $H \simeq l^2(\mathbb{N})$ that we obtain is non-trivial as well.

Let us get now into the study of linear operators. We have here:

THEOREM 5.17. *Given a Hilbert space H , the linear operators $T : H \rightarrow H$ which are bounded, in the sense that the quantity*

$$\|T\| = \sup_{\|x\| \leq 1} \|Tx\|$$

is finite, form a complex algebra $B(H)$, having the following properties:

- (1) *$B(H)$ is complete with respect to $\|\cdot\|$, and so we have a Banach algebra.*
- (2) *$B(H)$ has an involution $T \rightarrow T^*$, given by $\langle Tx, y \rangle = \langle x, T^*y \rangle$.*

In addition, the norm and the involution are related by the formula $\|TT^\| = \|T\|^2$.*

PROOF. The fact that we have indeed an algebra follows from:

$$\|S + T\| \leq \|S\| + \|T\| \quad , \quad \|\lambda T\| = |\lambda| \cdot \|T\| \quad , \quad \|ST\| \leq \|S\| \cdot \|T\|$$

(1) Assuming that $\{T_k\} \subset B(H)$ is a Cauchy sequence, the sequence $\{T_k x\}$ is Cauchy for any $x \in H$, so we can define the limit $T = \lim_{k \rightarrow \infty} T_k$ by setting:

$$Tx = \lim_{k \rightarrow \infty} T_k x$$

It is routine then to check that this formula defines indeed an operator $T \in B(H)$, and that we have $T_k \rightarrow T$ in norm, and this gives the result.

(2) The existence of T^* comes from the fact that $\psi(x) = \langle Tx, y \rangle$ being a linear map $H \rightarrow \mathbb{C}$, we must have a formula as follows, for a certain vector $T^*y \in H$:

$$\psi(x) = \langle x, T^*y \rangle$$

Moreover, since this vector T^*y is unique, T^* is unique too, and we have as well:

$$(S + T)^* = S^* + T^* \quad , \quad (\lambda T)^* = \bar{\lambda} T^*$$

$$(ST)^* = T^* S^* \quad , \quad (T^*)^* = T$$

Observe also that we have indeed $T^* \in B(H)$, due to the following equality:

$$\begin{aligned} \|T\| &= \sup_{\|x\|=1} \sup_{\|y\|=1} \langle Tx, y \rangle \\ &= \sup_{\|y\|=1} \sup_{\|x\|=1} \langle x, T^*y \rangle \\ &= \|T^*\| \end{aligned}$$

(3) Regarding now the last assertion, observe first that we have:

$$\|TT^*\| \leq \|T\| \cdot \|T^*\| = \|T\|^2$$

On the other hand, we have as well the following estimate:

$$\begin{aligned} \|T\|^2 &= \sup_{\|x\|=1} | \langle Tx, Tx \rangle | \\ &= \sup_{\|x\|=1} | \langle x, T^*Tx \rangle | \\ &\leq \|T^*T\| \end{aligned}$$

Now by replacing in this formula $T \rightarrow T^*$ we obtain $\|T\|^2 \leq \|TT^*\|$. Thus, we have proved both the needed inequalities, and we are done. \square

In the case where H comes with a basis $\{e_i\}_{i \in I}$, we can talk about the infinite matrices $M \in M_I(\mathbb{C})$, with the remark that the multiplication of such matrices is not always defined, in the case $|I| = \infty$. In this context, we have the following result:

PROPOSITION 5.18. *Let H be a Hilbert space, with orthonormal basis $\{e_i\}_{i \in I}$. The bounded operators $T \in B(H)$ can be then identified with matrices $M \in M_I(\mathbb{C})$ via*

$$Tx = Mx \quad , \quad M_{ij} = \langle Te_j, e_i \rangle$$

and we obtain in this way an embedding as follows, which is multiplicative:

$$B(H) \subset M_I(\mathbb{C})$$

In the case $H = \mathbb{C}^N$ we obtain in this way the usual isomorphism $B(H) \simeq M_N(\mathbb{C})$. In the separable case we obtain in this way a proper embedding $B(H) \subset M_\infty(\mathbb{C})$.

PROOF. We have several assertions to be proved, the idea being as follows:

(1) Regarding the first assertion, given a bounded operator $T : H \rightarrow H$, let us associate to it a matrix $M \in M_I(\mathbb{C})$ as in the statement, by the following formula:

$$M_{ij} = \langle Te_j, e_i \rangle$$

It is clear that this correspondence $T \rightarrow M$ is linear, and also that its kernel is $\{0\}$. Thus, we have an embedding of linear spaces $B(H) \subset M_I(\mathbb{C})$.

(2) Our claim now is that this embedding is multiplicative. But this is clear too, because if we denote by $T \rightarrow M_T$ our correspondence, we have:

$$\begin{aligned} (M_{ST})_{ij} &= \sum_k \langle Se_k, e_i \rangle \langle Te_j, e_k \rangle \\ &= \sum_k (M_S)_{ik} (M_T)_{kj} \\ &= (M_S M_T)_{ij} \end{aligned}$$

(3) Finally, we must prove that the original operator $T : H \rightarrow H$ can be recovered from its matrix $M \in M_I(\mathbb{C})$ via the formula in the statement, namely $Tx = Mx$. But this latter formula holds for the vectors of the basis, $x = e_j$, because we have:

$$(Te_j)_i = \langle Te_j, e_i \rangle = M_{ij} = (Me_j)_i$$

Now by linearity we obtain from this that the formula $Tx = Mx$ holds everywhere, on any vector $x \in H$, and this finishes the proof of the first assertion.

(4) In finite dimensions we obtain an isomorphism, because any matrix $M \in M_N(\mathbb{C})$ determines an operator $T : \mathbb{C}^N \rightarrow \mathbb{C}^N$, according to the formula $\langle Te_j, e_i \rangle = M_{ij}$. In infinite dimensions, however, we do not have an isomorphism. For instance on $H = l^2(\mathbb{N})$ the following matrix does not define an operator:

$$M = \begin{pmatrix} 1 & 1 & \dots \\ 1 & 1 & \dots \\ \vdots & \vdots & \end{pmatrix}$$

Indeed, $T(e_1)$ should be the all-one vector, which is not square-summable. \square

5c. Operator algebras

We will be interested here in the algebras of operators, rather than in the operators themselves. The axioms here, coming from Theorem 5.17, are as follows:

DEFINITION 5.19. *A C^* -algebra is a complex algebra with unit A , having:*

- (1) *A norm $a \rightarrow \|a\|$, making it a Banach algebra (the Cauchy sequences converge).*
- (2) *An involution $a \rightarrow a^*$, which satisfies $\|aa^*\| = \|a\|^2$, for any $a \in A$.*

As basic examples here, we have the usual matrix algebras $M_N(\mathbb{C})$, with the norm and involution being the usual matrix norm and involution, given by:

$$\|A\| = \sup_{\|x\|=1} \|Ax\| \quad , \quad (A^*)_{ij} = \overline{A_{ji}}$$

Some other basic examples are the algebras $L^\infty(X)$ of essentially bounded functions $f : X \rightarrow \mathbb{C}$ on a measured space X , with the usual norm and involution, namely:

$$\|f\| = \sup_{x \in X} |f(x)| \quad , \quad f^*(x) = \overline{f(x)}$$

We can put these two basic classes of examples together, as follows:

PROPOSITION 5.20. *The random matrix algebras $A = M_N(L^\infty(X))$ are C^* -algebras, with their usual norm and involution, given by:*

$$\|Z\| = \sup_{x \in X} \|Z_x\| \quad , \quad (Z^*)_{ij} = \overline{Z_{ij}}$$

These algebras generalize both the algebras $M_N(\mathbb{C})$, and the algebras $L^\infty(X)$.

PROOF. The fact that the C^* -algebra axioms are satisfied is clear from definitions. As for the last assertion, this follows by taking $X = \{.\}$ and $N = 1$, respectively. \square

In order to study the C^* -algebras, the key observation is that, due to Theorem 5.17, the algebra $B(H)$ of bounded linear operators $T : H \rightarrow H$ on a Hilbert space H is a C^* -algebra. More generally, any closed $*$ -subalgebra $A \subset B(H)$ is a C^* -algebra. It is possible to prove that any C^* -algebra appears in this way, $A \subset B(H)$, and we will be back to this later. For the moment, let us just record the following elementary result, dealing with the random matrix case, that we are mainly interested in here:

THEOREM 5.21. *Any algebra of type $L^\infty(X)$ is an operator algebra, as follows:*

$$L^\infty(X) \subset B(L^2(X)) \quad , \quad f \rightarrow (g \rightarrow fg)$$

More generally, any random matrix algebra is an operator algebra, as follows,

$$M_N(L^\infty(X)) \subset B(\mathbb{C}^N \otimes L^2(X))$$

with the embedding being the above one, tensored with the identity.

PROOF. We have two assertions to be proved, the idea being as follows:

(1) Given $f \in L^\infty(X)$, consider the following operator, acting on $H = L^2(X)$:

$$T_f(g) = fg$$

Observe that T_f is indeed well-defined, and bounded as well, because:

$$\|fg\|_2 = \sqrt{\int_X |f(x)|^2 |g(x)|^2 d\mu(x)} \leq \|f\|_\infty \|g\|_2$$

The application $f \rightarrow T_f$ being linear, involutive, continuous, and injective as well, we obtain in this way a C^* -algebra embedding $L^\infty(X) \subset B(H)$, as desired.

(2) Regarding the second assertion, this is best viewed in the following way:

$$\begin{aligned} M_N(L^\infty(X)) &= M_N(\mathbb{C}) \otimes L^\infty(X) \\ &\subset M_N(\mathbb{C}) \otimes B(L^2(X)) \\ &= B(\mathbb{C}^N \otimes L^2(X)) \end{aligned}$$

Here we have used (1), and some standard tensor product identifications. \square

Our purpose in what follows is to develop the spectral theory of the C^* -algebras, and in particular that of the random matrix algebras $A = M_N(L^\infty(X))$ that we are interested in, one of our objectives being that of talking about spectral measures, in the normal case, in analogy with what we know about the usual matrices. Let us start with:

DEFINITION 5.22. *The spectrum of an element $a \in A$ is the set*

$$\sigma(a) = \left\{ \lambda \in \mathbb{C} \mid a - \lambda \notin A^{-1} \right\}$$

where $A^{-1} \subset A$ is the set of invertible elements.

Given an element $a \in A$, and a rational function $f = P/Q$ having poles outside $\sigma(a)$, we can construct the element $f(a) = P(a)Q(a)^{-1}$. For simplicity, we write:

$$f(a) = \frac{P(a)}{Q(a)}$$

With this convention, we have the following result:

PROPOSITION 5.23. *We have the “rational functional calculus” formula*

$$\sigma(f(a)) = f(\sigma(a))$$

valid for any rational function $f \in \mathbb{C}(X)$ having poles outside $\sigma(a)$.

PROOF. We can prove this result in two steps, as follows:

(1) Assume first that we are in the usual polynomial case, $f \in \mathbb{C}[X]$. We pick a number $\lambda \in \mathbb{C}$, and we decompose the polynomial $f - \lambda$:

$$f(X) - \lambda = c(X - p_1) \dots (X - p_n)$$

We have then, as desired, the following computation:

$$\begin{aligned} \lambda \notin \sigma(f(a)) &\iff f(a) - \lambda \in A^{-1} \\ &\iff c(a - p_1) \dots (a - p_n) \in A^{-1} \\ &\iff a - p_1, \dots, a - p_n \in A^{-1} \\ &\iff p_1, \dots, p_n \notin \sigma(a) \\ &\iff \lambda \notin f(\sigma(a)) \end{aligned}$$

(2) In the general case now, $f \in \mathbb{C}(X)$, we pick $\lambda \in \mathbb{C}$, we write $f = P/Q$, and we set $R = P - \lambda Q$. By using (1) above, we obtain:

$$\begin{aligned} \lambda \in \sigma(f(a)) &\iff R(a) \notin A^{-1} \\ &\iff 0 \in \sigma(R(a)) \\ &\iff 0 \in R(\sigma(a)) \\ &\iff \exists \mu \in \sigma(a), R(\mu) = 0 \\ &\iff \lambda \in f(\sigma(a)) \end{aligned}$$

Thus, we have obtained the formula in the statement. \square

Given an element $a \in A$, its spectral radius $\rho(a)$ is the radius of the smallest disk centered at 0 containing $\sigma(a)$. With this convention, we have the following key result:

THEOREM 5.24. *Let A be a C^* -algebra.*

- (1) *The spectrum of a norm one element is in the unit disk.*
- (2) *The spectrum of a unitary element ($a^* = a^{-1}$) is on the unit circle.*
- (3) *The spectrum of a self-adjoint element ($a = a^*$) consists of real numbers.*
- (4) *The spectral radius of a normal element ($aa^* = a^*a$) is equal to its norm.*

PROOF. We use the various results established above, as follows:

(1) This comes from the following basic formula, valid when $\|a\| < 1$:

$$\frac{1}{1 - a} = 1 + a + a^2 + \dots$$

(2) Assuming $a^* = a^{-1}$, we have the following computations:

$$\begin{aligned} \|a\| &= \sqrt{\|aa^*\|} = \sqrt{1} = 1 \\ \|a^{-1}\| &= \|a^*\| = \|a\| = 1 \end{aligned}$$

If we denote by D the unit disk, we obtain from this, by using (1):

$$\sigma(a) \subset D \quad , \quad \sigma(a^{-1}) \subset D$$

On the other hand, by using the function $f(z) = z^{-1}$, we have:

$$\sigma(a^{-1}) \subset D \implies \sigma(a) \subset D^{-1}$$

Thus we have $\sigma(a) \subset D \cap D^{-1} = \mathbb{T}$, as desired.

(3) This follows by using the result (2), just established above, and Proposition 5.23, with the following rational function, depending on a parameter $t \in \mathbb{R}$:

$$f(z) = \frac{z + it}{z - it}$$

Indeed, for $t \gg 0$ the element $f(a)$ is well-defined, and we have:

$$\left(\frac{a + it}{a - it} \right)^* = \frac{a - it}{a + it} = \left(\frac{a + it}{a - it} \right)^{-1}$$

Thus the element $f(a)$ is a unitary, and by using (2) above its spectrum is contained in \mathbb{T} . We conclude that we have an inclusion as follows:

$$f(\sigma(a)) = \sigma(f(a)) \subset \mathbb{T}$$

Thus, we obtain an inclusion $\sigma(a) \subset f^{-1}(\mathbb{T}) = \mathbb{R}$, and we are done.

(4) We already know from (1) that we have the following inequality:

$$\rho(a) \leq \|a\|$$

For the converse, we fix an arbitrary number $\rho > \rho(a)$. We have then:

$$\int_{|z|=\rho} \frac{z^n}{z - a} dz = \sum_{k=0}^{\infty} \left(\int_{|z|=\rho} z^{n-k-1} dz \right) a^k = a^{n-1}$$

By applying the norm and taking n -th roots we obtain from this:

$$\rho \geq \lim_{n \rightarrow \infty} \|a^n\|^{1/n}$$

In the case $a = a^*$ we have $\|a^n\| = \|a\|^n$ for any exponent of the form $n = 2^k$, and by taking n -th roots we get $\rho \geq \|a\|$. But this gives the missing inequality, namely:

$$\rho(a) \geq \|a\|$$

In the general case $aa^* = a^*a$ we have $a^n(a^n)^* = (aa^*)^n$. Thus $\rho(a)^2 = \rho(aa^*)$, and since the element aa^* is self-adjoint, we obtain $\rho(aa^*) = \|a\|^2$, and we are done. \square

We are now in position of proving a key result, due to Gelfand, as follows:

THEOREM 5.25. *Any commutative C^* -algebra is the form*

$$A = C(X)$$

with its “spectrum” $X = \text{Spec}(A)$ appearing as the space of characters $\chi : A \rightarrow \mathbb{C}$.

PROOF. Given a commutative C^* -algebra A , we can define X to be the set of characters $\chi : A \rightarrow \mathbb{C}$, with topology making continuous all evaluation maps $ev_a : \chi \rightarrow \chi(a)$. Then X is a compact space, and $a \rightarrow ev_a$ is a morphism of algebras, as follows:

$$ev : A \rightarrow C(X)$$

(1) We first prove that ev is involutive. For this purpose we use the following formula, which is similar to the $z = \text{Re}(z) + i\text{Im}(z)$ formula for usual complex numbers:

$$a = \frac{a + a^*}{2} + i \cdot \frac{a - a^*}{2i}$$

Thus it is enough to prove the equality $ev_{a^*} = ev_a^*$ for self-adjoint elements a . But this is the same as proving that $a = a^*$ implies that ev_a is a real function, which is in turn true, because $ev_a(\chi) = \chi(a)$ is an element of the spectrum $\sigma(a)$, contained in \mathbb{R} .

(2) Since A is commutative, each element is normal, so ev is isometric, due to:

$$\|ev_a\| = \rho(a) = \|a\|$$

(3) It remains to prove that ev is surjective. But this follows from the Stone-Weierstrass theorem, because $ev(A)$ is a closed subalgebra of $C(X)$, which separates the points. \square

As a main consequence of the Gelfand theorem, we have:

THEOREM 5.26. *For any normal element $a \in A$ we have an identification as follows:*

$$\langle a \rangle = C(\sigma(a))$$

In addition, given a function $f \in C(\sigma(a))$, we can apply it to a , and we have

$$\sigma(f(a)) = f(\sigma(a))$$

which generalizes the previous rational calculus formula, in the normal case.

PROOF. Since a is normal, the C^* -algebra $\langle a \rangle$ that is generated is commutative, so if we denote by X the space of the characters $\chi : \langle a \rangle \rightarrow \mathbb{C}$, we have:

$$\langle a \rangle = C(X)$$

Now since the map $X \rightarrow \sigma(a)$ given by evaluation at a is bijective, we obtain:

$$\langle a \rangle = C(\sigma(a))$$

Thus, we are dealing here with usual functions, and this gives all the assertions. \square

5d. Spectral measures

In order to get now towards noncommutative probability, we have to develop the theory of positive elements, and linear forms. First, we have the following result:

PROPOSITION 5.27. *For an element $a \in A$, the following are equivalent:*

- (1) a is positive, in the sense that $\sigma(a) \subset [0, \infty)$.
- (2) $a = b^2$, for some $b \in A$ satisfying $b = b^*$.
- (3) $a = cc^*$, for some $c \in A$.

PROOF. This is something very standard, as follows:

(1) \implies (2) Observe first that $\sigma(a) \subset \mathbb{R}$ implies $a = a^*$. Thus the algebra $\langle a \rangle$ is commutative, and by using Theorem 5.26, we can set $b = \sqrt{a}$.

(2) \implies (3) This is trivial, because we can simply set $c = b$.

(3) \implies (1) This is clear too, because we have:

$$\sigma(a) = \sigma(b^2) = \sigma(b)^2 \subset \mathbb{R}^2 = [0, \infty)$$

(3) \implies (1) We proceed by contradiction. By multiplying c by a suitable element of $\langle cc^* \rangle$, we are led to the existence of an element $d \neq 0$ satisfying:

$$-dd^* \geq 0$$

By writing now $d = x + iy$ with $x = x^*, y = y^*$ we have:

$$dd^* + d^*d = 2(x^2 + y^2) \geq 0$$

Thus $d^*d \geq 0$, which is easily seen to contradict the condition $-dd^* \geq 0$. \square

We can talk as well about positive linear forms, as follows:

DEFINITION 5.28. *Consider a linear map $\varphi : A \rightarrow \mathbb{C}$.*

- (1) φ is called positive when $a \geq 0 \implies \varphi(a) \geq 0$.
- (2) φ is called faithful and positive when $a \geq 0, a \neq 0 \implies \varphi(a) > 0$.

In the commutative case, $A = C(X)$, the positive linear forms appear as follows, with μ being positive, and strictly positive if we want φ to be faithful and positive:

$$\varphi(f) = \int_X f(x) d\mu(x)$$

In general, the positive linear forms can be thought of as being integration functionals with respect to some underlying "positive measures". We have:

DEFINITION 5.29. *Let A be a C^* -algebra, given with a positive trace $tr : A \rightarrow \mathbb{C}$.*

- (1) The elements $a \in A$ are called random variables.
- (2) The moments of such a variable are the numbers $M_k(a) = tr(a^k)$.
- (3) The law of such a variable is the functional $\mu_a : P \rightarrow tr(P(a))$.

Here the exponent $k = \circ \bullet \bullet \circ \dots$ is by definition a colored integer, and the powers a^k are defined by the following formulae, and multiplicativity:

$$a^\emptyset = 1 \quad , \quad a^\circ = a \quad , \quad a^\bullet = a^*$$

As for the polynomial P , this is a noncommuting $*$ -polynomial in one variable:

$$P \in \mathbb{C} \langle X, X^* \rangle$$

Observe that the law is uniquely determined by the moments, because we have:

$$P(X) = \sum_k \lambda_k X^k \implies \mu_a(P) = \sum_k \lambda_k M_k(a)$$

At the level of the general theory, we have the following key result, extending the various results that we have, regarding the self-adjoint and normal matrices:

THEOREM 5.30. *Let A be a C^* -algebra, with a trace tr , and consider an element $a \in A$ which is normal, in the sense that $aa^* = a^*a$.*

- (1) μ_a is a complex probability measure, satisfying $\text{supp}(\mu_a) \subset \sigma(a)$.
- (2) In the self-adjoint case, $a = a^*$, this measure μ_a is real.
- (3) Assuming that tr is faithful, we have $\text{supp}(\mu_a) = \sigma(a)$.

PROOF. This is something very standard, that we already know for the usual complex matrices, and whose proof in general is quite similar, as follows:

(1) In the normal case, $aa^* = a^*a$, the Gelfand theorem, or rather the subsequent continuous functional calculus theorem, tells us that we have:

$$\langle a \rangle = C(\sigma(a))$$

Thus the functional $f(a) \rightarrow tr(f(a))$ can be regarded as an integration functional on the algebra $C(\sigma(a))$, and by the Riesz theorem, this latter functional must come from a probability measure μ on the spectrum $\sigma(a)$, in the sense that we must have:

$$tr(f(a)) = \int_{\sigma(a)} f(z) d\mu(z)$$

We are therefore led to the conclusions in the statement, with the uniqueness assertion coming from the fact that the elements a^k , taken as usual with respect to colored integer exponents, $k = \circ \bullet \bullet \circ \dots$, generate the whole C^* -algebra $C(\sigma(a))$.

(2) This is something which is clear from definitions.

(3) Once again, this is something which is clear from definitions. □

As a first concrete application now, by getting back to the random matrices, and to the various questions raised in the beginning of this chapter, we have:

THEOREM 5.31. *Given a random matrix $Z \in M_N(L^\infty(X))$ which is normal,*

$$ZZ^* = Z^*Z$$

its law, which is by definition the following abstract functional,

$$\mu : \mathbb{C} \langle X, X^* \rangle \rightarrow \mathbb{C} \quad , \quad P \rightarrow \frac{1}{N} \int_X \text{tr}(P(Z))$$

when restricted to the usual polynomials in two variables,

$$\mu : \mathbb{C}[X, X^*] \rightarrow \mathbb{C} \quad , \quad P \rightarrow \frac{1}{N} \int_X \text{tr}(P(Z))$$

must come from a probability measure on the spectrum $\sigma(Z) \subset \mathbb{C}$, as follows:

$$\mu(P) = \int_{\sigma(T)} P(x) d\mu(x)$$

We agree to use the symbol μ for all these notions.

PROOF. This follows indeed from what we know from Theorem 5.30, applied to the normal element $a = Z$, belonging to the C^* -algebra $A = M_N(L^\infty(X))$. \square

5e. Exercises

In analogy with linear algebra, operator theory is a wide area of mathematics, and there are many interesting operators, and exercises about them. We first have:

EXERCISE 5.32. *Find an explicit orthonormal basis for the Hilbert space*

$$H = L^2[0, 1]$$

by starting with the algebraic basis $f_n = x^n$ with $n \in \mathbb{N}$, and applying Gram-Schmidt.

This is actually quite non-trivial, and in case you're stuck with complicated computations, better look it up, and then write an account of what you found.

EXERCISE 5.33. *Prove that for the usual matrices $A, B \in M_N(\mathbb{C})$ we have*

$$\sigma^+(AB) = \sigma^+(BA)$$

where σ^+ denotes the set of eigenvalues, taken with multiplicities.

As a remark here, we have seen that $\sigma(AB) = \sigma(BA)$ holds outside $\{0\}$, and the equality on $\{0\}$ holds as well, because AB is invertible if and only if BA is invertible. However, in what regards the eigenvalues taken with multiplicities, things are more tricky, and the answer should be somewhere inside your linear algebra knowledge.

EXERCISE 5.34. *Clarify, with examples and counterexamples, the relation between the eigenvalues of an operator $T \in B(H)$, and its spectrum $\sigma(T) \subset \mathbb{C}$.*

Here, as usual, the counterexamples can only come from the shift operator S , on the space $H = l^2(\mathbb{N})$. As a bonus exercise here, try computing the spectrum of S .

CHAPTER 6

Wigner matrices

6a. Gaussian matrices

We have now all the needed ingredients for launching some explicit random matrix computations. Our goal will be that of computing the asymptotic moments, and then the asymptotic laws, with $N \rightarrow \infty$, for the main classes of large random matrices.

Let us begin by specifying the precise classes of matrices that we are interested in. First we have the complex Gaussian matrices, which are constructed as follows:

DEFINITION 6.1. *A complex Gaussian matrix is a random matrix of type*

$$Z \in M_N(L^\infty(X))$$

which has i.i.d. centered complex normal entries.

Here we use the notion of complex normal variable, introduced and studied in chapter 1. To be more precise, the complex Gaussian law of parameter $t > 0$ is by definition the following law, with a, b being independent, each following the normal law g_t :

$$G_t = \text{law} \left(\frac{1}{\sqrt{2}}(a + ib) \right)$$

With this notion in hand, the assumption in the above definition is that all the matrix entries Z_{ij} are independent, and follow this law G_t , for a fixed value of $t > 0$. We will see that the above matrices have an interesting, and “central” combinatorics, among all kinds of random matrices, with the study of the other random matrices being usually obtained as a modification of the study of the Gaussian matrices.

As a somewhat surprising remark, using real normal variables in Definition 6.1, instead of the complex ones appearing there, leads nowhere. The correct real versions of the Gaussian matrices are the Wigner random matrices, constructed as follows:

DEFINITION 6.2. *A Wigner matrix is a random matrix of type*

$$Z \in M_N(L^\infty(X))$$

which has i.i.d. centered complex normal entries, up to the constraint $Z = Z^$.*

This definition is something a bit compacted, and to be more precise, a Wigner matrix is by definition a random matrix as follows, with the diagonal entries being real normal variables, $a_i \sim g_t$, for some $t > 0$, the upper diagonal entries being complex normal variables, $b_{ij} \sim G_t$, the lower diagonal entries being the conjugates of the upper diagonal entries, as indicated, and with all the variables a_i, b_{ij} being independent:

$$Z = \begin{pmatrix} a_1 & b_{12} & \cdots & \cdots & b_{1N} \\ \bar{b}_{12} & a_2 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & a_{N-1} & b_{N-1,N} \\ \bar{b}_{1N} & \cdots & \cdots & \bar{b}_{N-1,N} & a_N \end{pmatrix}$$

As a comment here, for many concrete applications the Wigner matrices are in fact the central objects in random matrix theory, and in particular, they are often more important than the Gaussian matrices. In fact, these are the random matrices which were first considered and investigated, a long time ago, by Wigner himself [97].

However, as we will soon discover, the Gaussian matrices are somehow more fundamental than the Wigner matrices, at least from an abstract point of view, and this will be the point of view that we will follow here, with the Gaussian matrices coming first.

Finally, we will be interested as well in the complex Wishart matrices, which are the positive versions of the above random matrices, constructed as follows:

DEFINITION 6.3. *A complex Wishart matrix is a random matrix of type*

$$Z = YY^* \in M_N(L^\infty(X))$$

with Y being a complex Gaussian matrix.

As before with the Gaussian and Wigner matrices, there are many possible comments that can be made here, of technical or historical nature, as follows:

(1) First, using real Gaussian variables instead of complex Gaussian variables in the above definition leads to a less interesting combinatorics, and we will not do this.

(2) The complex Wishart matrices were introduced and studied by Marchenko and Pastur not long after Wigner, in [65], and so historically came second.

(3) Finally, in what regards their combinatorics and applications, the Wishart matrices quite often come first, before both the Gaussian and the Wigner ones.

So long for random matrix definitions and general talk about this, with all this being at this point quite subjective, but we will soon get to work, and prove results motivating all the above. Let us summarize this preliminary discussion in the following way:

CONCLUSION 6.4. *There are three main types of random matrices, as follows:*

- (1) *The Gaussian matrices, which can be thought of as being “complex”.*
- (2) *The Wigner matrices, which can be thought of as being “real”.*
- (3) *The Wishart matrices, which can be thought of as being “positive”.*

We will study these three types of matrices in what follows, in the above precise order, with this order being the one that, technically, best fits us here. Let us also mention that there are many other interesting classes of random matrices, which are more specialized, usually appearing as modifications of the above. More on these later.

In order to compute the asymptotic laws of the Gaussian, Wigner and Wishart matrices, we use the moment method. Given a colored integer $k = \circ \bullet \bullet \circ \dots$, we say that a pairing $\pi \in \mathcal{P}_2(k)$ is matching when it pairs $\circ - \bullet$ symbols. With this convention, we have the following result, which will be our main tool for computing moments:

THEOREM 6.5 (Wick formula). *Given independent variables X_i , each following the complex normal law G_t , with $t > 0$ being a fixed parameter, we have the formula*

$$E(X_{i_1}^{k_1} \dots X_{i_s}^{k_s}) = t^{s/2} \# \left\{ \pi \in \mathcal{P}_2(k) \mid \pi \leq \ker i \right\}$$

where $k = k_1 \dots k_s$ and $i = i_1 \dots i_s$, for the joint moments of these variables.

PROOF. This is something that we know from chapter 1, the idea being as follows:

(1) In the case where we have a single complex normal variable X , which amounts in taking $X_i = X$ for any i in the formula in the statement, what we have to compute are the moments of X , with respect to colored integer exponents $k = \circ \bullet \bullet \circ \dots$, and the formula in the statement tells us that these moments must be:

$$E(X^k) = t^{|k|/2} |\mathcal{P}_2(k)|$$

(2) But this is something that we know from chapter 1, the idea being that at $t = 1$ this follows by doing some combinatorics and calculus, in analogy with the combinatorics and calculus from the real case, where the moment formula is identical, save for the matching pairings \mathcal{P}_2 being replaced by the usual pairings P_2 , and then that the general case $t > 0$ follows from this, by rescaling. Thus, we are done with this case.

(3) In general now, with several variables as in the statement, when expanding the product $X_{i_1}^{k_1} \dots X_{i_s}^{k_s}$ and rearranging the terms, we are left with doing a number of computations as in (1), and then making the product of the expectations that we found.

(4) But this amounts in counting the partitions in the statement, with the condition $\pi \leq \ker i$ there standing for the fact that we are doing the various type (1) computations independently, and then making the product. Thus, we obtain the result. \square

The above statement is one of the possible formulations of the Wick formula, and there are in fact many more formulations, which are all useful. Here is an alternative such formulation, which is quite popular, and that we will often use in what follows:

THEOREM 6.6 (Wick formula 2). *Given independent variables f_i , each following the complex normal law G_t , with $t > 0$ being a fixed parameter, we have the formula*

$$E(f_{i_1} \dots f_{i_k} f_{j_1}^* \dots f_{j_k}^*) = t^k \# \left\{ \pi \in S_k \mid i_{\pi(r)} = j_r, \forall r \right\}$$

for the non-vanishing joint moments of these variables.

PROOF. This follows from the usual Wick formula, from Theorem 6.5. With some changes in the indices and notations, the formula there reads:

$$E(f_{I_1}^{K_1} \dots f_{I_s}^{K_s}) = t^{s/2} \# \left\{ \sigma \in \mathcal{P}_2(K) \mid \sigma \leq \ker I \right\}$$

Now observe that we have $\mathcal{P}_2(K) = \emptyset$, unless the colored integer $K = K_1 \dots K_s$ is uniform, in the sense that it contains the same number of \circ and \bullet symbols. Up to permutations, the non-trivial case, where the moment is non-vanishing, is the case where the colored integer $K = K_1 \dots K_s$ is of the following special form:

$$K = \underbrace{\circ \circ \dots \circ}_k \underbrace{\bullet \bullet \dots \bullet}_k$$

So, let us focus on this case, which is the non-trivial one. Here we have $s = 2k$, and we can write the multi-index $I = I_1 \dots I_s$ in the following way:

$$I = i_1 \dots i_k j_1 \dots j_k$$

With these changes made, the above usual Wick formula reads:

$$E(f_{i_1} \dots f_{i_k} f_{j_1}^* \dots f_{j_k}^*) = t^k \# \left\{ \sigma \in \mathcal{P}_2(K) \mid \sigma \leq \ker(ij) \right\}$$

The point now is that the matching pairings $\sigma \in \mathcal{P}_2(K)$, with $K = \circ \dots \circ \bullet \dots \bullet$, of length $2k$, as above, correspond to the permutations $\pi \in S_k$, in the obvious way. With this identification made, the above modified usual Wick formula becomes:

$$E(f_{i_1} \dots f_{i_k} f_{j_1}^* \dots f_{j_k}^*) = t^k \# \left\{ \pi \in S_k \mid i_{\pi(r)} = j_r, \forall r \right\}$$

Thus, we have reached to the formula in the statement, and we are done. \square

Finally, here is one more formulation of the Wick formula, which is useful as well:

THEOREM 6.7 (Wick formula 3). *Given independent variables f_i , each following the complex normal law G_t , with $t > 0$ being a fixed parameter, we have the formula*

$$E(f_{i_1} f_{j_1}^* \dots f_{i_k} f_{j_k}^*) = t^k \# \left\{ \pi \in S_k \mid i_{\pi(r)} = j_r, \forall r \right\}$$

for the non-vanishing joint moments of these variables.

PROOF. This follows from our second Wick formula, from Theorem 6.6, simply by permuting the terms, as to have an alternating sequence of plain and conjugate variables. Alternatively, we can start with Theorem 6.5, and then perform the same manipulations as in the proof of Theorem 6.6, but with the exponent being this time as follows:

$$K = \underbrace{\circ \bullet \bullet \dots \circ \bullet}_{2k}$$

Thus, we are led to the conclusion in the statement. \square

Now by getting back to the Gaussian matrices, we have the following result:

THEOREM 6.8. *Given a sequence of Gaussian random matrices*

$$Z_N \in M_N(L^\infty(X))$$

having independent G_t variables as entries, for some $t > 0$, we have

$$M_k \left(\frac{Z_N}{\sqrt{N}} \right) \simeq t^{|k|/2} |\mathcal{NC}_2(k)|$$

for any colored integer $k = \circ \bullet \bullet \circ \dots$, in the $N \rightarrow \infty$ limit.

PROOF. This is something standard, which can be done as follows:

(1) We fix $N \in \mathbb{N}$, and we let $Z = Z_N$. Let us first compute the trace of Z^k . With $k = k_1 \dots k_s$, and with the convention $(ij)^\circ = ij$, $(ij)^\bullet = ji$, we have:

$$\begin{aligned} \text{Tr}(Z^k) &= \text{Tr}(Z^{k_1} \dots Z^{k_s}) \\ &= \sum_{i_1=1}^N \dots \sum_{i_s=1}^N (Z^{k_1})_{i_1 i_2} (Z^{k_2})_{i_2 i_3} \dots (Z^{k_s})_{i_s i_1} \\ &= \sum_{i_1=1}^N \dots \sum_{i_s=1}^N (Z_{(i_1 i_2)^{k_1}})^{k_1} (Z_{(i_2 i_3)^{k_2}})^{k_2} \dots (Z_{(i_s i_1)^{k_s}})^{k_s} \end{aligned}$$

(2) Next, we rescale our variable Z by a \sqrt{N} factor, as in the statement, and we also replace the usual trace by its normalized version, $tr = \text{Tr}/N$. Our formula becomes:

$$tr \left(\left(\frac{Z}{\sqrt{N}} \right)^k \right) = \frac{1}{N^{s/2+1}} \sum_{i_1=1}^N \dots \sum_{i_s=1}^N (Z_{(i_1 i_2)^{k_1}})^{k_1} (Z_{(i_2 i_3)^{k_2}})^{k_2} \dots (Z_{(i_s i_1)^{k_s}})^{k_s}$$

Thus, the moment that we are interested in is given by:

$$M_k \left(\frac{Z}{\sqrt{N}} \right) = \frac{1}{N^{s/2+1}} \sum_{i_1=1}^N \dots \sum_{i_s=1}^N \int_X (Z_{(i_1 i_2)^{k_1}})^{k_1} (Z_{(i_2 i_3)^{k_2}})^{k_2} \dots (Z_{(i_s i_1)^{k_s}})^{k_s}$$

(3) Let us apply now the Wick formula, from Theorem 6.5. We conclude that the moment that we are interested in is given by:

$$\begin{aligned} & M_k \left(\frac{Z}{\sqrt{N}} \right) \\ &= \frac{t^{s/2}}{N^{s/2+1}} \sum_{i_1=1}^N \dots \sum_{i_s=1}^N \# \left\{ \pi \in \mathcal{P}_2(k) \mid \pi \leq \ker \left((i_1 i_2)^{k_1}, (i_2 i_3)^{k_2}, \dots, (i_s i_1)^{k_s} \right) \right\} \\ &= t^{s/2} \sum_{\pi \in \mathcal{P}_2(k)} \frac{1}{N^{s/2+1}} \# \left\{ i \in \{1, \dots, N\}^s \mid \pi \leq \ker \left((i_1 i_2)^{k_1}, (i_2 i_3)^{k_2}, \dots, (i_s i_1)^{k_s} \right) \right\} \end{aligned}$$

(4) Our claim now is that in the $N \rightarrow \infty$ limit the combinatorics of the above sum simplifies, with only the noncrossing partitions contributing to the sum, and with each of them contributing precisely with a 1 factor, so that we will have, as desired:

$$\begin{aligned} M_k \left(\frac{Z}{\sqrt{N}} \right) &= t^{s/2} \sum_{\pi \in \mathcal{P}_2(k)} \left(\delta_{\pi \in \mathcal{NC}_2(k)} + O(N^{-1}) \right) \\ &\simeq t^{s/2} \sum_{\pi \in \mathcal{P}_2(k)} \delta_{\pi \in \mathcal{NC}_2(k)} \\ &= t^{s/2} |\mathcal{NC}_2(k)| \end{aligned}$$

(5) In order to prove this, the first observation is that when k is not uniform, in the sense that it contains a different number of \circ , \bullet symbols, we have $\mathcal{P}_2(k) = \emptyset$, and so:

$$M_k \left(\frac{Z}{\sqrt{N}} \right) = t^{s/2} |\mathcal{NC}_2(k)| = 0$$

(6) Thus, we are left with the case where k is uniform. Let us examine first the case where k consists of an alternating sequence of \circ and \bullet symbols, as follows:

$$k = \underbrace{\circ \bullet \circ \bullet \dots \circ \bullet}_{2p}$$

In this case it is convenient to relabel our multi-index $i = (i_1, \dots, i_s)$, with $s = 2p$, in the form $(j_1, l_1, j_2, l_2, \dots, j_p, l_p)$. With this done, our moment formula becomes:

$$M_k \left(\frac{Z}{\sqrt{N}} \right) = t^p \sum_{\pi \in \mathcal{P}_2(k)} \frac{1}{N^{p+1}} \# \left\{ j, l \in \{1, \dots, N\}^p \mid \pi \leq \ker (j_1 l_1, j_2 l_1, j_2 l_2, \dots, j_1 l_p) \right\}$$

Now observe that, with k being as above, we have an identification $\mathcal{P}_2(k) \simeq S_p$, obtained in the obvious way. With this done too, our moment formula becomes:

$$M_k \left(\frac{Z}{\sqrt{N}} \right) = t^p \sum_{\pi \in S_p} \frac{1}{N^{p+1}} \# \left\{ j, l \in \{1, \dots, N\}^p \mid j_r = j_{\pi(r)+1}, l_r = l_{\pi(r)}, \forall r \right\}$$

(7) We are now ready to do our asymptotic study, and prove the claim in (4). Let indeed $\gamma \in S_p$ be the full cycle, which is by definition the following permutation:

$$\gamma = (1\ 2\ \dots\ p)$$

In terms of γ , the conditions $j_r = j_{\pi(r)+1}$ and $l_r = l_{\pi(r)}$ found above read:

$$\gamma\pi \leq \ker j \quad , \quad \pi \leq \ker l$$

Counting the number of free parameters in our moment formula, we obtain:

$$\begin{aligned} M_k \left(\frac{Z}{\sqrt{N}} \right) &= \frac{t^p}{N^{p+1}} \sum_{\pi \in S_p} N^{|\pi|+|\gamma\pi|} \\ &= t^p \sum_{\pi \in S_p} N^{|\pi|+|\gamma\pi|-p-1} \end{aligned}$$

(8) The point now is that the last exponent is well-known to be ≤ 0 , with equality precisely when the permutation $\pi \in S_p$ is geodesic, which in practice means that π must come from a noncrossing partition. Thus we obtain, in the $N \rightarrow \infty$ limit, as desired:

$$M_k \left(\frac{Z}{\sqrt{N}} \right) \simeq t^p |\mathcal{NC}_2(k)|$$

This finishes the proof in the case of the exponents k which are alternating, and the case where k is an arbitrary uniform exponent is similar, by permuting everything. \square

This was for the computation, but in what regards now the interpretation of what we found, things are more complicated. The precise question is as follows:

QUESTION 6.9. *What is the abstract asymptotic distribution that we found, having as moments the numbers*

$$M_k = t^{|k|/2} |\mathcal{NC}_2(k)|$$

for any colored integer $k = \circ \bullet \bullet \circ \dots$?

As a first observation, the above moment formula is very similar to the one for the usual complex Gaussian variables G_t , from chapter 1, which was as follows:

$$N_k = t^{|k|/2} |\mathcal{P}_2(k)|$$

It is possible to make many speculations here, for instance in relation with the combinatorics from chapters 3-4, but we will do this later, once we will know more. Let us record however our observation as a partial answer to Question 6.9, as follows:

ANSWER 6.10. *The abstract asymptotic distribution that we found appears as some sort of “free analogue” of the usual complex normal law G_t , with the underlying matching pairings being now replaced by underlying matching noncrossing pairings.*

Obviously, some interesting things are going on here. We will see in a moment, after doing some more combinatorics, this time in connection with the Wigner matrices, that there are some good reasons for calling this mysterious law “circular”.

Thus, for ending with our present study with a nice conclusion, we can say that the Gaussian matrices become “asymptotically circular”, with this meaning by definition that the $N \rightarrow \infty$ moments are those computed above. This is of course something quite vague, and we will be back to it in chapters 9-12 below, when doing free probability.

6b. Wigner matrices

Moving ahead now, let us investigate the second class of random matrices that we are interested in, namely the Wigner matrices, which are by definition self-adjoint. Here our results will be far more complete than those for the Gaussian matrices.

Let us first recall from the above that a Wigner matrix is by definition a random matrix which has i.i.d. centered complex normal entries, up to the constraint $Z = Z^*$. In practice, this means that our matrix is as follows, with the diagonal entries being real normal variables, $a_i \sim g_t$, for some $t > 0$, the upper diagonal entries being complex normal variables, $b_{ij} \sim G_t$, the lower diagonal entries being the conjugates of the upper diagonal entries, as indicated, and with all the variables a_i, b_{ij} being independent:

$$Z = \begin{pmatrix} a_1 & b_{12} & \dots & \dots & b_{1N} \\ \bar{b}_{12} & a_2 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & a_{N-1} & b_{N-1,N} \\ \bar{b}_{1N} & \dots & \dots & \bar{b}_{N-1,N} & a_N \end{pmatrix}$$

As a starting point for the study of these matrices, we have the following simple fact, making the connection with the theory of Gaussian matrices developed above:

PROPOSITION 6.11. *Given a Gaussian matrix Z , with independent entries following the centered complex normal law G_t , with $t > 0$, if we write*

$$Z = \frac{1}{\sqrt{2}}(X + iY)$$

with X, Y being self-adjoint, then both X, Y are Wigner matrices, of parameter t .

PROOF. This is something elementary, which can be done in two steps, as follows:

(1) As a first observation, the result holds at $N = 1$. Indeed, here our Gaussian matrix Z is just a random variable, subject to the condition $Z \sim G_t$. But recall that the law G_t

is by definition as follows, with X, Y being independent, each following the law g_t :

$$G_t = \text{law} \left(\frac{1}{\sqrt{2}}(X + iY) \right)$$

Thus in this case, $N = 1$, the variables X, Y that we obtain in the statement, as rescaled real and imaginary parts of Z , are subject to the condition $X, Y \sim g_t$, and so are Wigner matrices of size $N = 1$ and parameter $t > 0$, as in Definition 6.2.

(2) In the general case now, $N \in \mathbb{N}$, the proof is similar, by using the basic behavior of the real and complex normal variables with respect to sums. \square

The above result is quite interesting for us, because it shows that, in order to investigate the Wigner matrices, we are basically not in need of some new computations, starting from the Wick formula, and doing combinatorics afterwards, but just of some manipulations on the results that we already have, regarding the Gaussian matrices.

To be more precise, by using this method, we obtain the following result, coming by combining the observation in Proposition 6.11 with the formula in Theorem 6.8:

THEOREM 6.12. *Given a sequence of Wigner random matrices*

$$Z_N \in M_N(L^\infty(X))$$

having independent G_t variables as entries, with $t > 0$, up to $Z_N = Z_N^$, we have*

$$M_k \left(\frac{Z_N}{\sqrt{N}} \right) \simeq t^{k/2} |\mathcal{NC}_2(k)|$$

for any integer $k \in \mathbb{N}$, in the $N \rightarrow \infty$ limit.

PROOF. This can be deduced from a direct computation based on the Wick formula, similar to that from the proof of Theorem 6.8, but the best is to deduce this result from Theorem 6.8 itself. Indeed, we know from there that for Gaussian matrices $Y_N \in M_N(L^\infty(X))$ we have the following formula, valid for any colored integer $K = \circ \bullet \bullet \circ \dots$, in the $N \rightarrow \infty$ limit, with \mathcal{NC}_2 standing for noncrossing matching pairings:

$$M_K \left(\frac{Y_N}{\sqrt{N}} \right) \simeq t^{|K|/2} |\mathcal{NC}_2(K)|$$

By doing some combinatorics, we deduce from this that we have the following formula for the moments of the matrices $Re(Y_N)$, with respect to usual exponents, $k \in \mathbb{N}$:

$$\begin{aligned}
M_k \left(\frac{Re(Y_N)}{\sqrt{N}} \right) &= 2^{-k} \cdot M_k \left(\frac{Y_N}{\sqrt{N}} + \frac{Y_N^*}{\sqrt{N}} \right) \\
&= 2^{-k} \sum_{|K|=k} M_K \left(\frac{Y_N}{\sqrt{N}} \right) \\
&\simeq 2^{-k} \sum_{|K|=k} t^{k/2} |\mathcal{NC}_2(K)| \\
&= 2^{-k} \cdot t^{k/2} \cdot 2^{k/2} |\mathcal{NC}_2(k)| \\
&= 2^{-k/2} \cdot t^{k/2} |\mathcal{NC}_2(k)|
\end{aligned}$$

Now since the matrices $Z_N = \sqrt{2}Re(Y_N)$ are of Wigner type, this gives the result. \square

Summarizing, all this brings us into counting noncrossing pairings. But here, let us recall from chapter 3 that we have the following well-known result:

THEOREM 6.13. *The Catalan numbers $C_k = |\mathcal{NC}_2(2k)|$ are as follows:*

- (1) *They satisfy $C_{k+1} = \sum_{a+b=k} C_a C_b$.*
- (2) *The series $f(z) = \sum_{k \geq 0} C_k z^k$ satisfies $zf^2 - f + 1 = 0$.*
- (3) *This series is given by $f(z) = \frac{1 - \sqrt{1-4z}}{2z}$.*
- (4) *We have the formula $C_k = \frac{1}{k+1} \binom{2k}{k}$.*

PROOF. This is something that we know well from chapter 3, with (1) coming from the definition of C_k , and with (1) \implies (2) \implies (3) \implies (4) being routine, using standard calculus. Alternatively, and also explained in chapter 3, the formula in (4) can be established as well via a bijective proof, by counting Dyck paths in the plane. \square

Getting back now to the Wigner matrices, we can convert the main result that we have about them, Theorem 6.12, into something more concrete, as follows:

THEOREM 6.14. *Given a sequence of Wigner random matrices*

$$Z_N \in M_N(L^\infty(X))$$

having independent G_t variables as entries, with $t > 0$, up to $Z_N = Z_N^$, we have*

$$M_{2k} \left(\frac{Z_N}{\sqrt{N}} \right) \simeq t^k C_k$$

in the $N \rightarrow \infty$ limit. As for the asymptotic odd moments, these all vanish.

PROOF. This follows from Theorem 6.12 and Theorem 6.13. Indeed, according to the results there, the asymptotic even moments are given by:

$$M_{2k} \left(\frac{Z_N}{\sqrt{N}} \right) \simeq t^k |NC_2(2k)| = t^k C_k$$

As for the asymptotic odd moments, once again from Theorem 6.12, we know that these all vanish. Thus, we are led to the conclusion in the statement. \square

Summarizing, we are done with the moment computations, and with the asymptotic study, for both the Gaussian and the Wigner matrices. It remains now to interpret the results that we have, with the computation of the corresponding laws. As explained before, for the Gaussian matrices this is something quite complicated, with the technology that we presently have, and this will have to wait a bit, until we do some free probability.

Regarding the Wigner matrices, however, the problems left here are very explicit, and quite elementary, and we will solve them next, in the remainder of this chapter.

6c. Semicircle laws

In order to recapture the asymptotic measure of the Wigner matrices out of the moments, which are the Catalan numbers, there are several methods available, namely:

- (1) Stieltjes inversion.
- (2) Knowledge of SU_2 .
- (3) Cheating.

The first method, which is straightforward, without any trick, is based on the Stieltjes inversion formula, that we know from chapter 3. In fact, we have already applied in chapter 3 that formula to the Catalan numbers, with the following conclusion:

PROPOSITION 6.15. *The real measure having as even moments the Catalan numbers, $C_k = \frac{1}{k+1} \binom{2k}{k}$, and having all odd moments 0 is the measure*

$$\gamma_1 = \frac{1}{2\pi} \sqrt{4-x^2} dx$$

called Wigner semicircle law on $[-2, 2]$.

PROOF. This is something that we know, but since we will need the proof in what follows, in view of some generalizations, let us briefly recall it. The starting point is the formula in Theorem 6.13 for the generating series of the Catalan numbers, namely:

$$\sum_{k=0}^{\infty} C_k z^k = \frac{1 - \sqrt{1-4z}}{2z}$$

By using this formula with $z = \xi^{-2}$, we obtain the following formula, for the Cauchy transform of the real measure that we want to compute:

$$\begin{aligned} G(\xi) &= \xi^{-1} \sum_{k=0}^{\infty} C_k \xi^{-2k} \\ &= \xi^{-1} \cdot \frac{1 - \sqrt{1 - 4\xi^{-2}}}{2\xi^{-2}} \\ &= \frac{\xi}{2} \left(1 - \sqrt{1 - 4\xi^{-2}} \right) \\ &= \frac{\xi}{2} - \frac{1}{2} \sqrt{\xi^2 - 4} \end{aligned}$$

Now let us apply the Stieltjes inversion formula, from chapter 3, namely:

$$d\mu(x) = \lim_{t \searrow 0} -\frac{1}{\pi} \operatorname{Im} (G(x + it)) \cdot dx$$

The study of the limit on the right is then straightforward, going as follows:

(1) According to the general philosophy of the Stieltjes formula, the first term in the formula of $G(\xi)$, namely $\xi/2$, which is “trivial”, will not contribute to the density.

(2) As for the second term, which is something non-trivial, this will contribute to the density, the rule here being that the square root $\sqrt{\xi^2 - 4}$ will be replaced by the “dual” square root $\sqrt{4 - x^2} dx$, and that we have to multiply everything by $-1/\pi$.

(3) As a conclusion, by Stieltjes inversion we obtain the following density:

$$d\mu(x) = -\frac{1}{\pi} \cdot -\frac{1}{2} \sqrt{4 - x^2} dx = \frac{1}{2\pi} \sqrt{4 - x^2} dx$$

Thus, we have obtained the measure in the statement, and we are done. \square

More generally now, we have the following result:

PROPOSITION 6.16. *Given $t > 0$, the real measure having as even moments the numbers $M_{2k} = t^k C_k$ and having all odd moments 0 is the measure*

$$\gamma_t = \frac{1}{2\pi t} \sqrt{4t - x^2} dx$$

called Wigner semicircle law on $[-2\sqrt{t}, 2\sqrt{t}]$.

PROOF. This follows by redoing the above Stieltjes inversion computation, with a parameter $t > 0$ added. To be more precise, as before, the starting point is the formula from Theorem 6.13 for the generating series of the Catalan numbers, namely:

$$\sum_{k=0}^{\infty} C_k z^k = \frac{1 - \sqrt{1 - 4z}}{2z}$$

By using this formula with $z = t\xi^{-2}$, we obtain the following formula, for the Cauchy transform of the real measure that we want to compute:

$$\begin{aligned}
 G(\xi) &= \xi^{-1} \sum_{k=0}^{\infty} t^k C_k \xi^{-2k} \\
 &= \xi^{-1} \cdot \frac{1 - \sqrt{1 - 4t\xi^{-2}}}{2t\xi^{-2}} \\
 &= \frac{\xi}{2t} \left(1 - \sqrt{1 - 4t\xi^{-2}} \right) \\
 &= \frac{\xi}{2t} - \frac{1}{2t} \sqrt{\xi^2 - 4t}
 \end{aligned}$$

Thus, by Stieltjes inversion we obtain the following density, as claimed:

$$d\mu(x) = \frac{1}{2\pi t} \sqrt{4t - x^2} dx$$

But simplest is in fact, perhaps a bit by cheating, simply using the result at $t = 1$, from Proposition 6.15, along with a change of variables. Indeed, by using Proposition 6.15, the even moments of the measure in the statement are given by:

$$\begin{aligned}
 M_{2k} &= \frac{1}{2\pi t} \int_{-2\sqrt{t}}^{2\sqrt{t}} \sqrt{4t - x^2} x^{2k} dx \\
 &= \frac{1}{2\pi t} \int_{-1}^1 \sqrt{4t - ty^2} (\sqrt{ty})^{2k} \sqrt{t} dy \\
 &= \frac{t^k}{2\pi} \int_{-1}^1 \sqrt{4 - y^2} y^{2k} dy \\
 &= t^k C_k
 \end{aligned}$$

As for the odd moments, these all vanish, because the density of γ_t is an even function. Thus, one way or another, we are led to the conclusion in the statement. \square

Talking cheating, another way of recovering Proposition 6.15, this time without using the Stieltjes inversion formula, but by knowing instead the answer to the question, namely the semicircle law, in advance, which is of course cheating, is as follows:

PROPOSITION 6.17. *The Catalan numbers are the even moments of*

$$\gamma_1 = \frac{1}{2\pi} \sqrt{4 - x^2} dx$$

called Wigner semicircle law. As for the odd moments of γ_1 , these all vanish.

PROOF. The even moments of the Wigner law can be computed with the change of variable $x = 2 \cos t$, and we are led to the following formula:

$$\begin{aligned}
M_{2k} &= \frac{1}{\pi} \int_0^2 \sqrt{4-x^2} x^{2k} dx \\
&= \frac{1}{\pi} \int_0^{\pi/2} \sqrt{4-4\cos^2 t} (2\cos t)^{2k} 2\sin t dt \\
&= \frac{4^{k+1}}{\pi} \int_0^{\pi/2} \cos^{2k} t \sin^2 t dt \\
&= \frac{4^{k+1}}{\pi} \cdot \frac{\pi}{2} \cdot \frac{(2k)!!2!!}{(2k+3)!!} \\
&= 2 \cdot 4^k \cdot \frac{(2k)!/2^k k!}{2^{k+1}(k+1)!} \\
&= C_k
\end{aligned}$$

As for the odd moments, these all vanish, because the density of γ_1 is an even function. Thus, we are led to the conclusion in the statement. \square

More generally, we have the following result, involving a parameter $t > 0$:

PROPOSITION 6.18. *The numbers $t^k C_k$ are the even moments of*

$$\gamma_t = \frac{1}{2\pi t} \sqrt{4t - x^2} dx$$

called semicircle law on $[-2\sqrt{t}, 2\sqrt{t}]$. As for the odd moments of γ_t , these all vanish.

PROOF. This follows indeed from what we have in Proposition 6.17, via a quick change of variables, as explained at the end of the proof of Proposition 6.16. \square

In any case, one way or another, we have our semicircle measures, and by putting now everything together, we obtain the Wigner theorem, as follows:

THEOREM 6.19. *Given a sequence of Wigner random matrices*

$$Z_N \in M_N(L^\infty(X))$$

having independent G_t variables as entries, with $t > 0$, up to $Z_N = Z_N^$, we have*

$$\frac{Z_N}{\sqrt{N}} \sim \frac{1}{2\pi t} \sqrt{4t - x^2} dx$$

in the $N \rightarrow \infty$ limit, with the limiting measure being the Wigner semicircle law γ_t .

PROOF. This follows indeed by combining Theorem 6.14 either with Proposition 6.16, and doing here an honest job, or with Proposition 6.18. \square

There are many other things that can be said about the Wigner matrices, which appear as variations of the above, and we refer here to the standard random matrix books [2], [66], [69], [90]. We will be back to them later on in this book, in chapter 10 below.

6d. Unitary groups

We discuss here an alternative interpretation of the limiting laws γ_t that we found above, by using Lie groups, the idea being that the standard semicircle law γ_1 , and more generally all the laws γ_t , naturally appear in connection with the group SU_2 .

This is something quite natural, and good to know, and will be useful for us later on. In relation with the above, the knowledge of this fact can be used as an alternative to both Stieltjes inversion, and cheating, in order to establish the Wigner theorem.

Let us start with the following fundamental group theory result, coming as a complement to the general theory for compact groups developed in chapter 4:

THEOREM 6.20. *We have the following formula,*

$$SU_2 = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \mid |\alpha|^2 + |\beta|^2 = 1 \right\}$$

which makes SU_2 isomorphic to the unit sphere $S_{\mathbb{C}}^1 \subset \mathbb{C}^2$.

PROOF. Consider an arbitrary 2×2 matrix, written as follows:

$$U = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

Assuming that we have $\det U = 1$, the inverse of this matrix is then given by:

$$U^{-1} = \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}$$

On the other hand, assuming $U \in U_2$, the inverse must be the adjoint:

$$U^{-1} = \begin{pmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta} \end{pmatrix}$$

We conclude that our matrix must be of the following special form:

$$U = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$$

Now since the determinant is 1, we must have $|\alpha|^2 + |\beta|^2 = 1$, so we are done with one direction. As for the converse, this is clear, the matrices in the statement being unitaries, and of determinant 1, and so being elements of SU_2 . Finally, we have:

$$S_{\mathbb{C}}^1 = \left\{ (\alpha, \beta) \in \mathbb{C}^2 \mid |\alpha|^2 + |\beta|^2 = 1 \right\}$$

Thus, the final assertion in the statement holds as well. \square

Next, we have the following useful reformulation of Theorem 6.20:

THEOREM 6.21. *We have the formula*

$$SU_2 = \left\{ \begin{pmatrix} p+iq & r+is \\ -r+is & p-iq \end{pmatrix} \mid p^2 + q^2 + r^2 + s^2 = 1 \right\}$$

which makes SU_2 isomorphic to the unit real sphere $S_{\mathbb{R}}^3 \subset \mathbb{R}^3$.

PROOF. We recall from Theorem 6.20 that we have:

$$SU_2 = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \mid |\alpha|^2 + |\beta|^2 = 1 \right\}$$

Now let us write our parameters $\alpha, \beta \in \mathbb{C}$, which belong to the complex unit sphere $S_{\mathbb{C}}^1 \subset \mathbb{C}^2$, in terms of their real and imaginary parts, as follows:

$$\alpha = p + iq \quad , \quad \beta = r + is$$

In terms of $p, q, r, s \in \mathbb{R}$, our formula for a generic matrix $U \in SU_2$ reads:

$$U = \begin{pmatrix} p+iq & r+is \\ -r+is & p-iq \end{pmatrix}$$

As for the condition to be satisfied by the parameters $p, q, r, s \in \mathbb{R}$, this comes the condition $|\alpha|^2 + |\beta|^2 = 1$ to be satisfied by $\alpha, \beta \in \mathbb{C}$, which reads:

$$p^2 + q^2 + r^2 + s^2 = 1$$

Thus, we are led to the conclusion in the statement. Regarding now the last assertion, recall that the unit sphere $S_{\mathbb{R}}^3 \subset \mathbb{R}^4$ is given by:

$$S_{\mathbb{R}}^3 = \left\{ (p, q, r, s) \mid p^2 + q^2 + r^2 + s^2 = 1 \right\}$$

Thus, we have an isomorphism of compact spaces $SU_2 \simeq S_{\mathbb{R}}^3$, as claimed. \square

Here is yet another useful reformulation of our main result so far, regarding SU_2 , obtained by further building on the parametrization from Theorem 6.21:

THEOREM 6.22. *We have the following formula,*

$$SU_2 = \left\{ p\beta_1 + q\beta_2 + r\beta_3 + s\beta_4 \mid p^2 + q^2 + r^2 + s^2 = 1 \right\}$$

where $\beta_1, \beta_2, \beta_3, \beta_4$ are the following matrices,

$$\beta_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad \beta_2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad , \quad \beta_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad , \quad \beta_4 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

called *Pauli spin matrices*.

PROOF. We recall from Theorem 6.21 that the group SU_2 can be parametrized by the real sphere $S_{\mathbb{R}}^3 \subset \mathbb{R}^4$, in the following way:

$$SU_2 = \left\{ \begin{pmatrix} p+iq & r+is \\ -r+is & p-iq \end{pmatrix} \mid p^2 + q^2 + r^2 + s^2 = 1 \right\}$$

But this gives the formula in the statement, with the Pauli matrices $\beta_1, \beta_2, \beta_3, \beta_4$ being the coefficients of p, q, r, s , in this parametrization. \square

The above result is often the most convenient one, when dealing with SU_2 . This is because the Pauli matrices have a number of remarkable properties, as follows:

PROPOSITION 6.23. *The Pauli matrices multiply according to the following formulae,*

$$\beta_2^2 = \beta_3^2 = \beta_4^2 = -1$$

$$\beta_2\beta_3 = -\beta_3\beta_2 = \beta_4$$

$$\beta_3\beta_4 = -\beta_4\beta_3 = \beta_2$$

$$\beta_4\beta_2 = -\beta_2\beta_4 = \beta_3$$

they conjugate according to the following rules,

$$\beta_1^* = \beta_1, \beta_2^* = -\beta_2, \beta_3^* = -\beta_3, \beta_4^* = -\beta_4$$

and they form an orthonormal basis of $M_2(\mathbb{C})$, with respect to the scalar product

$$\langle x, y \rangle = \text{tr}(xy^*)$$

with $\text{tr} : M_2(\mathbb{C}) \rightarrow \mathbb{C}$ being the normalized trace of 2×2 matrices, $\text{tr} = \text{Tr}/2$.

PROOF. The first two assertions, regarding the multiplication and conjugation rules for the Pauli matrices, follow from some elementary computations. As for the last assertion, this follows by using these rules. Indeed, the fact that the Pauli matrices are pairwise orthogonal follows from computations of the following type, for $i \neq j$:

$$\langle \beta_i, \beta_j \rangle = \text{tr}(\beta_i\beta_j^*) = \text{tr}(\pm\beta_i\beta_j) = \text{tr}(\pm\beta_k) = 0$$

As for the fact that the Pauli matrices have norm 1, this follows from:

$$\langle \beta_i, \beta_i \rangle = \text{tr}(\beta_i\beta_i^*) = \text{tr}(\pm\beta_i^2) = \text{tr}(\beta_1) = 1$$

Thus, we are led to the conclusion in the statement. \square

Now back to probability, we can recover our semicircular measures, as follows:

THEOREM 6.24. *The main character of SU_2 follows the following law,*

$$\gamma_1 = \frac{1}{2\pi} \sqrt{4-x^2} dx$$

which is the Wigner law of parameter 1.

PROOF. This follows from Theorem 6.21, by identifying SU_2 with the sphere $S_{\mathbb{R}}^3$, the variable $\chi = 2\operatorname{Re}(p)$ being semicircular. Indeed, let us write, as in Theorem 6.21:

$$SU_2 = \left\{ \begin{pmatrix} p+iq & r+is \\ -p+iq & r-is \end{pmatrix} \mid p^2 + q^2 + r^2 + s^2 = 1 \right\}$$

In this picture, the main character is given by the following formula:

$$\chi \begin{pmatrix} p+iq & r+is \\ -r+is & p-iq \end{pmatrix} = 2p$$

We are therefore left with computing the law of the following variable:

$$p \in C(S_{\mathbb{R}}^3)$$

For this purpose, we can use the moment method. Let us recall from chapter 1 that the polynomial integrals over the real spheres are given by the following formula:

$$\int_{S_{\mathbb{R}}^{N-1}} x_1^{k_1} \dots x_N^{k_N} dx = \frac{(N-1)!! k_1!! \dots k_N!!}{(N + \sum k_i - 1)!!}$$

In our case, where $N = 4$, we obtain the following moment formula:

$$\begin{aligned} \int_{S_{\mathbb{R}}^3} p^{2k} &= \frac{3!!(2k)!!}{(2k+3)!!} \\ &= 2 \cdot \frac{3 \cdot 5 \cdot 7 \dots (2k-1)}{2 \cdot 4 \cdot 6 \dots (2k+2)} \\ &= 2 \cdot \frac{(2k)!}{2^k k! 2^{k+1} (k+1)!} \\ &= \frac{1}{4^k} \cdot \frac{1}{k+1} \binom{2k}{k} \\ &= \frac{C_k}{4^k} \end{aligned}$$

Thus the variable $2p \in C(S_{\mathbb{R}}^3)$ follows the Wigner semicircle law γ_1 , as claimed. \square

Summarizing, we have managed to recover the Wigner semicircle law γ_1 out of purely geometric considerations, involving the real sphere $S_{\mathbb{R}}^3$ and the special complex rotation group SU_2 . Moreover, with a change of variable, our results extend to γ_t with $t > 0$. And this is quite interesting, philosophically, and also makes an interesting connection with the Lie group material from chapter 4, which remains to be further investigated.

Finally, as the physicists say, there is no SU_2 without SO_3 , so let us discuss as well the computation for SO_3 , that we will certainly need later. Let us start with:

PROPOSITION 6.25. *The adjoint action $SU_2 \curvearrowright M_2(\mathbb{C})$, given by $T_U(A) = UAU^*$, leaves invariant the following real vector subspace of $M_2(\mathbb{C})$,*

$$\mathbb{R}^4 = \text{span}(\beta_1, \beta_2, \beta_3, \beta_4)$$

and we obtain in this way a group morphism $SU_2 \rightarrow GL_4(\mathbb{R})$.

PROOF. We have two assertions to be proved, as follows:

(1) We must first prove that, with $E \subset M_2(\mathbb{C})$ being the real vector space in the statement, we have the following implication:

$$U \in SU_2, A \in E \implies UAU^* \in E$$

But this is clear from the multiplication rules for the Pauli matrices, from Proposition 6.23. Indeed, let us write our matrices U, A as follows:

$$U = x\beta_1 + y\beta_2 + z\beta_3 + t\beta_4$$

$$A = a\beta_1 + b\beta_2 + c\beta_3 + d\beta_4$$

We know that the coefficients x, y, z, t and a, b, c, d are all real, due to $U \in SU_2$ and $A \in E$. The point now is that when computing UAU^* , by using the various rules from Proposition 6.23, we obtain a matrix of the same type, namely a combination of $\beta_1, \beta_2, \beta_3, \beta_4$, with real coefficients. Thus, we have $UAU^* \in E$, as desired.

(2) In order to conclude, let us identify $E \simeq \mathbb{R}^4$, by using the basis $\beta_1, \beta_2, \beta_3, \beta_4$. The result found in (1) shows that we have a correspondence as follows:

$$SU_2 \rightarrow M_4(\mathbb{R}) \quad , \quad U \rightarrow (T_U)|_E$$

Now observe that for any $U \in SU_2$ and any $A \in M_2(\mathbb{C})$ we have:

$$T_{U^*}T_U(A) = U^*UAU^*U = A$$

Thus $T_{U^*} = T_U^{-1}$, and so the correspondence that we found can be written as:

$$SU_2 \rightarrow GL_4(\mathbb{R}) \quad , \quad U \rightarrow (T_U)|_E$$

But this a group morphism, due to the following computation:

$$T_U T_V(A) = UVAV^*U^* = T_{UV}(A)$$

Thus, we are led to the conclusion in the statement. □

The point now is that Proposition 6.25 can be improved as follows:

PROPOSITION 6.26. *The adjoint action $SU_2 \curvearrowright M_2(\mathbb{C})$, given by*

$$T_U(A) = UAU^*$$

leaves invariant the following real vector subspace of $M_2(\mathbb{C})$,

$$F = \text{span}_{\mathbb{R}}(\beta_2, \beta_3, \beta_4)$$

and we obtain in this way a group morphism $SU_2 \rightarrow SO_3$.

PROOF. We can do this in several steps, as follows:

(1) Our first claim is that the group morphism $SU_2 \rightarrow GL_4(\mathbb{R})$ constructed in Proposition 6.25 is in fact a morphism $SU_2 \rightarrow O_4$. In order to prove this, recall the following formula, valid for any $U \in SU_2$, from the proof of Proposition 6.25:

$$T_{U^*} = T_U^{-1}$$

We want to prove that the matrices $T_U \in GL_4(\mathbb{R})$ are orthogonal, and in view of the above formula, it is enough to prove that we have:

$$T_U^* = (T_U)^t$$

So, let us prove this. For any two matrices $A, B \in E$, we have:

$$\begin{aligned} \langle T_{U^*}(A), B \rangle &= \langle U^*AU, B \rangle \\ &= \text{tr}(U^*AUB) \\ &= \text{tr}(AUBU^*) \end{aligned}$$

On the other hand, we have as well the following formula:

$$\begin{aligned} \langle (T_U)^t(A), B \rangle &= \langle A, T_U(B) \rangle \\ &= \langle A, UBU^* \rangle \\ &= \text{tr}(AUBU^*) \end{aligned}$$

Thus we have indeed $T_U^* = (T_U)^t$, which proves our $SU_2 \rightarrow O_4$ claim.

(2) In order now to finish, recall that we have by definition $\beta_1 = 1$, as a matrix. Thus, the action of SU_2 on the vector $\beta_1 \in E$ is given by:

$$T_U(\beta_1) = U\beta_1U^* = UU^* = 1 = \beta_1$$

We conclude that $\beta_1 \in E$ is invariant under SU_2 , and by orthogonality the following subspace of E must be invariant as well under the action of SU_2 :

$$\beta_1^\perp = \text{span}_{\mathbb{R}}(\beta_2, \beta_3, \beta_4)$$

Now if we call this subspace F , and we identify $F \simeq \mathbb{R}^3$ by using the basis $\beta_2, \beta_3, \beta_4$, we obtain by restriction to F a morphism of groups as follows:

$$SU_2 \rightarrow O_3$$

But since this morphism is continuous and SU_2 is connected, its image must be connected too. Now since the target group decomposes as $O_3 = SO_3 \sqcup (-SO_3)$, and $1 \in SU_2$ gets mapped to $1 \in SO_3$, the whole image must lie inside SO_3 , and we are done. \square

The above result is quite interesting, because we will see in a moment that the morphism $SU_2 \rightarrow SO_3$ constructed there is surjective. Thus, we will have a way of parametrizing the elements $V \in SO_3$ by elements $U \in SU_2$, and so ultimately by parameters

$(x, y, z, t) \in S_{\mathbb{R}}^3$. In order to work out all this, let us start with the following result, coming as a continuation of Proposition 6.25, independently of Proposition 6.26:

PROPOSITION 6.27. *With respect to the standard basis $\beta_1, \beta_2, \beta_3, \beta_4$ of the vector space $\mathbb{R}^4 = \text{span}(\beta_1, \beta_2, \beta_3, \beta_4)$, the morphism $T : SU_2 \rightarrow GL_4(\mathbb{R})$ is given by:*

$$T_U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p^2 + q^2 - r^2 - s^2 & 2(qr - ps) & 2(pr + qs) \\ 0 & 2(ps + qr) & p^2 + r^2 - q^2 - s^2 & 2(rs - pq) \\ 0 & 2(qs - pr) & 2(pq + rs) & p^2 + s^2 - q^2 - r^2 \end{pmatrix}$$

Thus, when looking at T as a group morphism $SU_2 \rightarrow O_4$, what we have in fact is a group morphism $SU_2 \rightarrow O_3$, and even $SU_2 \rightarrow SO_3$.

PROOF. With notations from Proposition 6.25 and its proof, let us first look at the action $L : SU_2 \curvearrowright \mathbb{R}^4$ by left multiplication, $L_U(A) = UA$. We have:

$$L_U = \begin{pmatrix} p & -q & -r & -s \\ q & p & -s & r \\ r & s & p & -q \\ s & -r & q & p \end{pmatrix}$$

Similarly, in what regards now the action $R : SU_2 \curvearrowright \mathbb{R}^4$ by right multiplication, $R_U(A) = AU^*$, the corresponding matrix is given by:

$$R_U = \begin{pmatrix} p & q & r & s \\ -q & p & -s & r \\ -r & s & p & -q \\ -s & -r & q & p \end{pmatrix}$$

Now by composing, the matrix of the adjoint matrix in the statement is:

$$\begin{aligned} T_U &= R_U L_U \\ &= \begin{pmatrix} p & q & r & s \\ -q & p & -s & r \\ -r & s & p & -q \\ -s & -r & q & p \end{pmatrix} \begin{pmatrix} p & -q & -r & -s \\ q & p & -s & r \\ r & s & p & -q \\ s & -r & q & p \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p^2 + q^2 - r^2 - s^2 & 2(qr - ps) & 2(pr + qs) \\ 0 & 2(ps + qr) & p^2 + r^2 - q^2 - s^2 & 2(rs - pq) \\ 0 & 2(qs - pr) & 2(pq + rs) & p^2 + s^2 - q^2 - r^2 \end{pmatrix} \end{aligned}$$

Thus, we have the formula in the statement, and this gives the result. \square

We can now formulate a famous result, due to Euler-Rodrigues, as follows:

THEOREM 6.28. *We have the Euler-Rodrigues formula*

$$U = \begin{pmatrix} p^2 + q^2 - r^2 - s^2 & 2(qr - ps) & 2(pr + qs) \\ 2(ps + qr) & p^2 + r^2 - q^2 - s^2 & 2(rs - pq) \\ 2(qs - pr) & 2(pq + rs) & p^2 + s^2 - q^2 - r^2 \end{pmatrix}$$

with $p^2 + q^2 + r^2 + s^2 = 1$, for the generic elements of SO_3 .

PROOF. We know from the above that we have a group morphism $SU_2 \rightarrow SO_3$, given by the formula in the statement, and the problem now is that of proving that this is a double cover map, in the sense that it is surjective, and with kernel $\{\pm 1\}$.

(1) Regarding the kernel, this is elementary to compute, as follows:

$$\begin{aligned} \ker(SU_2 \rightarrow SO_3) &= \left\{ U \in SU_2 \mid T_U(A) = A, \forall A \in E \right\} \\ &= \left\{ U \in SU_2 \mid UA = AU, \forall A \in E \right\} \\ &= \left\{ U \in SU_2 \mid U\beta_i = \beta_i U, \forall i \right\} \\ &= \{\pm 1\} \end{aligned}$$

(2) Thus, we are done with this, and as a side remark here, this result shows that our morphism $SU_2 \rightarrow SO_3$ is ultimately a morphism as follows:

$$PU_2 \subset SO_3 \quad , \quad PU_2 = SU_2/\{\pm 1\}$$

Here P stands for “projective”, and it is possible to say more about the construction $G \rightarrow PG$, which can be performed for any subgroup $G \subset U_N$. But we will not get here into this, our next goal being anyway that of proving that we have $PU_2 = SO_3$.

(3) We must prove now that the morphism $SU_2 \rightarrow SO_3$ is surjective. This is something non-trivial, and there are several proofs for this, as follows:

– A first proof is by using Lie theory. To be more precise, the tangent spaces at 1 of both SU_2 and SO_3 can be explicitly computed, by doing some linear algebra, and the morphism $SU_2 \rightarrow SO_3$ follows to be surjective around 1, and then globally.

– Another proof is via representation theory. Indeed, the representations of SU_2 and SO_3 can be explicitly computed, and follow to be subject to very similar formulae, called Clebsch-Gordan rules, and this shows that $SU_2 \rightarrow SO_3$ is surjective.

– Yet another advanced proof, which is actually quite borderline for what can be called “proof”, is by using the ADE/McKay classification of the subgroups $G \subset SO_3$, which shows that there is no room strictly inside SO_3 for something as big as PU_2 .

(4) Thus, done with this, one way or another. Alternatively, a more pedestrian proof for the surjectivity of the morphism $SU_2 \rightarrow SO_3$ is based on the fact that any rotation $U \in SO_3$ has an axis, and we will leave the computations here as an instructive exercise. \square

Now back to probability, let us formulate the following definition:

DEFINITION 6.29. *The standard Marchenko-Pastur law π_1 is given by:*

$$f \sim \gamma_1 \implies f^2 \sim \pi_1$$

That is, π_1 is the law of the square of a variable following the semicircle law γ_1 .

Here the fact that π_1 is indeed well-defined comes from the fact that a measure is uniquely determined by its moments. More explicitly now, we have:

PROPOSITION 6.30. *The density of the Marchenko-Pastur law is*

$$\pi_1 = \frac{1}{2\pi} \sqrt{4x^{-1} - 1} dx$$

and the moments of this measure are the Catalan numbers.

PROOF. There are several proofs here, the simplest being by cheating. Indeed, the moments of π_1 can be computed with the change of variable $x = 4 \cos^2 t$, as follows:

$$\begin{aligned} M_k &= \frac{1}{2\pi} \int_0^4 \sqrt{4x^{-1} - 1} x^k dx \\ &= \frac{1}{2\pi} \int_0^{\pi/2} \frac{\sin t}{\cos t} \cdot (4 \cos^2 t)^k \cdot 2 \cos t \sin t dt \\ &= \frac{4^{k+1}}{\pi} \int_0^{\pi/2} \cos^{2k} t \sin^2 t dt \\ &= \frac{4^{k+1}}{\pi} \cdot \frac{\pi}{2} \cdot \frac{(2k)!!2!!}{(2k+3)!!} \\ &= 2 \cdot 4^k \cdot \frac{(2k)!/2^k k!}{2^{k+1}(k+1)!} \\ &= C_k \end{aligned}$$

Thus, we are led to the conclusion in the statement. \square

We can do now the character computation for SO_3 , as follows:

THEOREM 6.31. *The main character of SO_3 , modified by adding 1 to it, given in standard Euler-Rodrigues coordinates by*

$$\chi = 4p^2$$

follows a squared semicircle law, or Marchenko-Pastur law π_1 .

PROOF. This follows by using the quotient map $SU_2 \rightarrow SO_3$, and the result for SU_2 . Indeed, by using the Euler-Rodrigues formula, in the context of Theorem 6.24 and its proof, the main character of SO_3 , modified by adding 1 to it, is given by:

$$\chi = (3p^2 - q^2 - r^2 - s^2) + 1 = 4p^2$$

Now recall from the proof of Theorem 6.24 that we have:

$$2p \sim \gamma_1$$

On the other hand, a quick comparison between the moment formulae for the Wigner and Marchenko-Pastur laws, which are very similar, shows that we have:

$$f \sim \gamma_1 \implies f^2 \sim \pi_1$$

Thus, with $f = 2p$, we obtain the result in the statement. \square

6e. Exercises

There has been a lot of theory in this chapter, and lots of computations as well, both calculus and combinatorics. As a first instructive exercise on all this, we have:

EXERCISE 6.32. *Find a direct proof of the Wigner theorem, without passing via the Gaussian matrices.*

This is actually how this theorem was first found, via direct computations.

EXERCISE 6.33. *Look up the various properties of the Catalan numbers,*

$$C_k = \frac{1}{k+1} \binom{2k}{k}$$

and write down an account of what you learned, ideally 2 pages or so.

Here by 2 pages we mean 2 pages of statements only, without proofs, the Catalan numbers being as famous as that.

EXERCISE 6.34. *Try to axiomatize the “circular law”, having as moments the numbers*

$$M_k = |\mathcal{NC}_2(k)|$$

which should appear as asymptotic law for the Gaussian matrices.

Obviously, this looks like something quite complicated and abstract, and some good imagination is needed. In case you don't find, don't worry, we will be back to this.

EXERCISE 6.35. *Try to find what the $t > 0$ analogue of the Marchenko-Pastur law*

$$\pi_1 = \frac{1}{2\pi} \sqrt{4x^{-1} - 1} dx$$

should be.

Again, this looks like something quite complicated and abstract, and some good imagination, and love for exploration, science in general, and mathematics in particular, is needed. And again, in case you don't find, don't worry, we will be back to this.

CHAPTER 7

Wishart matrices

7a. Positive matrices

We discuss in this chapter the complex Wishart matrices, which are the positive analogues of the Gaussian and Wigner matrices. These matrices were introduced and studied by Marchenko and Pastur in [65], not long after Wigner's paper [97], and are of interest in connection with many questions. They are constructed as follows:

DEFINITION 7.1. *A complex Wishart matrix is a random matrix of type*

$$W = YY^* \in M_N(L^\infty(X))$$

with Y being a complex Gaussian matrix, with entries following the law G_t .

Due to the formula $W = YY^*$, the Wishart matrices are positive, in the general positivity sense of chapter 5. Before getting into their study, let us first develop some more theory for the positive matrices and operators. As a starting point, we have:

THEOREM 7.2. *For an operator $T \in B(H)$, the following are equivalent:*

- (1) $\langle Tx, x \rangle \geq 0$, for any $x \in H$.
- (2) T is normal, and $\sigma(T) \subset [0, \infty)$.
- (3) $T = S^2$, for some $S \in B(H)$ satisfying $S = S^*$.
- (4) $T = R^*R$, for some $R \in B(H)$.

If these conditions are satisfied, we call T positive, and write $T \geq 0$.

PROOF. We have already seen some implications in chapter 5, but the best is to forget the few partial results that we know, and prove everything, as follows:

- (1) \implies (2) Assuming $\langle Tx, x \rangle \geq 0$, with $S = T - T^*$ we have:

$$\begin{aligned} \langle Sx, x \rangle &= \langle Tx, x \rangle - \langle T^*x, x \rangle \\ &= \langle Tx, x \rangle - \langle x, Tx \rangle \\ &= \langle Tx, x \rangle - \overline{\langle Tx, x \rangle} \\ &= 0 \end{aligned}$$

The next step is to use a polarization trick, as follows:

$$\begin{aligned}
 \langle Sx, y \rangle &= \langle S(x+y), x+y \rangle - \langle Sx, x \rangle - \langle Sy, y \rangle - \langle Sy, x \rangle \\
 &= -\langle Sy, x \rangle \\
 &= \langle y, Sx \rangle \\
 &= \overline{\langle Sx, y \rangle}
 \end{aligned}$$

Thus we must have $\langle Sx, y \rangle \in \mathbb{R}$, and with $y \rightarrow iy$ we obtain $\langle Sx, y \rangle \in i\mathbb{R}$ too, and so $\langle Sx, y \rangle = 0$. Thus $S = 0$, which gives $T = T^*$. Now since T is self-adjoint, it is normal as claimed. Moreover, by self-adjointness, we have:

$$\sigma(T) \subset \mathbb{R}$$

In order to prove now that we have indeed $\sigma(T) \subset [0, \infty)$, as claimed, we must invert $T + \lambda$, for any $\lambda > 0$. For this purpose, observe that we have:

$$\begin{aligned}
 \langle (T + \lambda)x, x \rangle &= \langle Tx, x \rangle + \langle \lambda x, x \rangle \\
 &\geq \langle \lambda x, x \rangle \\
 &= \lambda \|x\|^2
 \end{aligned}$$

But this shows that $T + \lambda$ is injective. In order to prove now the surjectivity, and the boundedness of the inverse, observe first that we have:

$$\begin{aligned}
 \text{Im}(T + \lambda)^\perp &= \ker(T + \lambda)^* \\
 &= \ker(T + \lambda) \\
 &= \{0\}
 \end{aligned}$$

Thus $\text{Im}(T + \lambda)$ is dense. On the other hand, observe that we have:

$$\begin{aligned}
 \|(T + \lambda)x\|^2 &= \langle Tx + \lambda x, Tx + \lambda x \rangle \\
 &= \|Tx\|^2 + 2\lambda \langle Tx, x \rangle + \lambda^2 \|x\|^2 \\
 &\geq \lambda^2 \|x\|^2
 \end{aligned}$$

Thus for any vector in the image $y \in \text{Im}(T + \lambda)$ we have:

$$\|y\| \geq \lambda \|(T + \lambda)^{-1}y\|$$

As a conclusion to what we have so far, $T + \lambda$ is bijective and invertible as a bounded operator from H onto its image, with the following norm bound:

$$\|(T + \lambda)^{-1}\| \leq \lambda^{-1}$$

But this shows that $\text{Im}(T + \lambda)$ is complete, hence closed, and since we already knew that $\text{Im}(T + \lambda)$ is dense, our operator $T + \lambda$ is surjective, and we are done.

(2) \implies (3) Since T is normal, and with spectrum contained in $[0, \infty)$, we can use the continuous functional calculus formula for the normal operators from chapter 5, with the function $f(x) = \sqrt{x}$, as to construct a square root $S = \sqrt{T}$.

(3) \implies (4) This is trivial, because we can set $R = S$.

(4) \implies (1) This is clear, because we have the following computation:

$$\langle R^*Rx, x \rangle = \langle Rx, Rx \rangle = \|Rx\|^2$$

Thus, we have the equivalences in the statement. \square

In analogy with what happens in finite dimensions, where among the positive matrices $A \geq 0$ we have the strictly positive ones, $A > 0$, given by the fact that the eigenvalues are strictly positive, we have as well a “strict” version of the above result, as follows:

THEOREM 7.3. *For an operator $T \in B(H)$, the following are equivalent:*

- (1) T is positive and invertible.
- (2) T is normal, and $\sigma(T) \subset (0, \infty)$.
- (3) $T = S^2$, for some $S \in B(H)$ invertible, satisfying $S = S^*$.
- (4) $T = R^*R$, for some $R \in B(H)$ invertible.

If these conditions are satisfied, we call T strictly positive, and write $T > 0$.

PROOF. Our claim is that the above conditions (1-4) are precisely the conditions (1-4) in Theorem 7.2, with the assumption “ T is invertible” added. Indeed:

(1) This is clear by definition.

(2) In the context of Theorem 7.2 (2), namely when T is normal, and $\sigma(T) \subset [0, \infty)$, the invertibility of T , which means $0 \notin \sigma(T)$, gives $\sigma(T) \subset (0, \infty)$, as desired.

(3) In the context of Theorem 7.2 (3), namely when $T = S^2$, with $S = S^*$, by using the basic properties of the functional calculus for normal operators, the invertibility of T is equivalent to the invertibility of its square root $S = \sqrt{T}$, as desired.

(4) In the context of Theorem 7.2 (4), namely when $T = RR^*$, the invertibility of T is equivalent to the invertibility of R . This can be either checked directly, or deduced via the equivalence (3) \iff (4) from Theorem 7.2, by using the above argument (3). \square

As a subtlety now, we have the following complement to the above result:

PROPOSITION 7.4. *For a strictly positive operator, $T > 0$, we have*

$$\langle Tx, x \rangle > 0 \quad , \quad \forall x \neq 0$$

but the converse of this fact is not true, unless we are in finite dimensions.

PROOF. We have several things to be proved, the idea being as follows:

(1) Regarding the main assertion, the inequality can be deduced as follows, by using the fact that the operator $S = \sqrt{T}$ is invertible, and in particular injective:

$$\begin{aligned} \langle Tx, x \rangle &= \langle S^2x, x \rangle \\ &= \langle Sx, S^*x \rangle \\ &= \langle Sx, Sx \rangle \\ &= \|Sx\|^2 \\ &> 0 \end{aligned}$$

(2) In finite dimensions, assuming $\langle Tx, x \rangle > 0$ for any $x \neq 0$, we know from Theorem 7.2 that we have $T \geq 0$. Thus we have $\sigma(T) \subset [0, \infty)$, and assuming by contradiction $0 \in \sigma(T)$, we obtain that T has $\lambda = 0$ as eigenvalue, and the corresponding eigenvector $x \neq 0$ has the property $\langle Tx, x \rangle = 0$, contradiction. Thus $T > 0$, as claimed.

(3) Finally, regarding the counterexample for the converse, we can use here:

$$T = \begin{pmatrix} 1 & & & \\ & \frac{1}{2} & & \\ & & \frac{1}{3} & \\ & & & \ddots \end{pmatrix}$$

Indeed, T is well-defined and bounded, and we have $\langle Tx, x \rangle > 0$, for any vector $x \neq 0$. However, T is not invertible, and so the converse does not hold, as stated. \square

With the above results in hand, let us discuss now some decomposition results for the bounded operators $T \in B(H)$, in analogy with what we know about the usual complex numbers $z \in \mathbb{C}$. We know that any $z \in \mathbb{C}$ can be written as follows, with $a, b \in \mathbb{R}$:

$$z = a + ib$$

Also, we know that both the real and imaginary parts $a, b \in \mathbb{R}$, and more generally any real number $c \in \mathbb{R}$, can be written as follows, with $r, s \geq 0$:

$$c = r - s$$

Here is the operator theoretic generalization of these results:

PROPOSITION 7.5. *Given an operator $T \in B(H)$, the following happen:*

- (1) *We can write $T = A + iB$, with $A, B \in B(H)$ self-adjoint.*
- (2) *When $T = T^*$, we can write $T = R - S$, with $R, S \in B(H)$ positive.*
- (3) *Thus, we can write any T as a linear combination of 4 positive elements.*

PROOF. All this follows from basic spectral theory, as follows:

(1) We can use here the same formula as for complex numbers, namely:

$$T = \frac{T + T^*}{2} + i \cdot \frac{T - T^*}{2i}$$

(2) This follows from the measurable functional calculus. Indeed, assuming $T = T^*$ we have $\sigma(T) \subset \mathbb{R}$, so we can use the following decomposition formula on \mathbb{R} :

$$z = \chi_{[0,\infty)}z - \chi_{(-\infty,0)}(-z)$$

Now by applying these measurable functions to T , we obtain as formula as follows, with both the operators $T_+, T_- \in B(H)$ being positive, as desired:

$$T = T_+ - T_-$$

(3) This follows by combining the results in (1) and (2) above. \square

Going ahead with our decomposition results, another basic thing that we know about complex numbers is that any $z \in \mathbb{C}$ appears as a real multiple of a unitary:

$$z = re^{it}$$

Finding the correct operator theoretic analogue of this is quite tricky, and this even for the usual matrices $A \in M_N(\mathbb{C})$. As a basic result here, we have:

PROPOSITION 7.6. *Given an operator $T \in B(H)$, the following happen:*

- (1) *If $T = T^*$ and $\|T\| \leq 1$, we can write $T = (U + V)/2$, with U, V unitaries.*
- (2) *If $T = T^*$, we can write $T = \lambda(U + V)$, with U, V unitaries.*
- (3) *In general, we can write T as a rescaled sum of 4 unitaries.*

PROOF. This follows from the results that we have, as follows:

(1) Assuming $T = T^*$ and $\|T\| \leq 1$ we have $1 - T^2 \geq 0$, and the decomposition that we are looking for is as follows, with both the components being unitaries:

$$T = \frac{T + i\sqrt{1 - T^2}}{2} + \frac{T - i\sqrt{1 - T^2}}{2}$$

To be more precise, the square root can be extracted as in Theorem 7.2 (3), and the check of the unitarity of the components goes as follows:

$$(T + i\sqrt{1 - T^2})(T - i\sqrt{1 - T^2}) = T^2 + (1 - T^2) = 1$$

(2) This simply follows by applying (1) to the operator $T/\|T\|$.

(3) Assuming first $\|T\| \leq 1$, we know from Proposition 7.5 (1) that we can write $T = A + iB$, with A, B being self-adjoint, and satisfying $\|A\|, \|B\| \leq 1$. Now by applying (1) to both A and B , we obtain a decomposition of T as follows:

$$T = \frac{U + V + X + Y}{2}$$

In general, we can apply this to the operator $T/\|T\|$, and we obtain the result. \square

All this gets us into the multiplicative theory of the complex numbers, that we will attempt to generalize now. As a first construction, that we would like to generalize to the bounded operator setting, we have the construction of the modulus, as follows:

$$|z| = \sqrt{z\bar{z}}$$

The point now is that we can indeed generalize this construction, as follows:

PROPOSITION 7.7. *Given an operator $T \in B(H)$, we can construct a positive operator $|T| \in B(H)$, satisfying $|T|^2 = T^*T$, as follows, by using the fact that T^*T is positive:*

$$|T| = \sqrt{T^*T}$$

In the case $H = \mathbb{C}$, this gives the usual absolute value of the complex numbers:

$$|z| = \sqrt{z\bar{z}}$$

More generally, in the case where $H = \mathbb{C}^N$ is finite dimensional, we obtain in this way the usual moduli of the complex matrices $A \in M_N(\mathbb{C})$.

PROOF. We have several things to be proved, the idea being as follows:

(1) The first assertion follows from Theorem 7.2. Indeed, according to (4) there the operator T^*T is indeed positive, and then according to (2) there we can extract the square root of this latter positive operator, by applying to it the function \sqrt{z} .

(2) By functional calculus we have then $|T|^2 = T^*T$, as desired.

(3) In the case $H = \mathbb{C}$, we obtain indeed the absolute value of complex numbers.

(4) In the case where the space H is finite dimensional, $H = \mathbb{C}^N$, we obtain indeed the usual moduli of the complex matrices $A \in M_N(\mathbb{C})$. \square

As a comment here, it is possible to talk as well about the operator $\sqrt{TT^*}$, which is in general different from $\sqrt{T^*T}$. Observe that when T is normal, we have:

$$\sqrt{TT^*} = \sqrt{T^*T}$$

Regarding now the polar decomposition formula, let us start with a weak version of this statement, regarding the invertible operators, as follows:

THEOREM 7.8. *We have the polar decomposition formula*

$$T = U\sqrt{T^*T}$$

with U being a unitary, for any $T \in B(H)$ invertible.

PROOF. According to our definition of $|T| = \sqrt{T^*T}$, we have:

$$\begin{aligned} \langle |T|x, |T|y \rangle &= \langle x, |T|^2y \rangle \\ &= \langle x, T^*Ty \rangle \\ &= \langle Tx, Ty \rangle \end{aligned}$$

Thus we can define a unitary operator $U \in B(H)$ as follows:

$$U(|T|x) = Tx$$

But this formula shows that we have $T = U|T|$, as desired. \square

Observe that we have uniqueness in the above result, in what regards the choice of the unitary $U \in B(H)$, due to the fact that we can write this unitary as follows:

$$U = T(\sqrt{T^*T})^{-1}$$

More generally now, we have the following result:

THEOREM 7.9. *We have the polar decomposition formula*

$$T = U\sqrt{T^*T}$$

with U being a partial isometry, for any $T \in B(H)$.

PROOF. As before, in the proof of Theorem 7.8, we have the following equality, valid for any two vectors $x, y \in H$:

$$\langle |T|x, |T|y \rangle = \langle Tx, Ty \rangle$$

We conclude that the following linear application is well-defined, and isometric:

$$U : \text{Im}|T| \rightarrow \text{Im}(T) \quad , \quad |T|x \rightarrow Tx$$

By continuity we can extend this map U into an isometry, as follows:

$$U : \overline{\text{Im}|T|} \rightarrow \overline{\text{Im}(T)} \quad , \quad |T|x \rightarrow Tx$$

Moreover, we can further extend U into a partial isometry $U : H \rightarrow H$, by setting $Ux = 0$, for any $x \in \overline{\text{Im}|T|}^\perp$, and with this convention, the result follows. \square

Summarizing, as a first application of our spectral theory methods, we have now a full generalization of the polar decomposition result for the usual matrices.

7b. Marchenko-Pastur

Let us discuss now the complex Wishart matrices, which are the positive analogues of the Gaussian and Wigner matrices. These matrices were introduced and studied by Marchenko-Pastur in [65], not long after Wigner's paper [97], and are of interest in connection with many questions. They are constructed as follows:

DEFINITION 7.10. *A complex Wishart matrix is a random matrix of type*

$$W = YY^* \in M_N(L^\infty(X))$$

with Y being a complex Gaussian matrix, with entries following the law G_t .

There are in fact several possible definitions for the complex Wishart matrices, with some being more clever and useful than some other. To start with, we will use the above definition, which comes naturally out of what we know about the Gaussian and Wigner matrices. Once such matrices studied, we will talk about their versions, too.

Observe that, due to the defining formula $W = YY^*$, the complex Wishart matrices are obviously positive, $W \geq 0$, in the sense of the general positivity notion discussed above. Due to this key positivity property, and to the otherwise “randomness” of W , such matrices are useful in many down-to-earth contexts. More on this later.

As usual with the random matrices, we will be interested in computing the asymptotic laws of our Wishart matrices W , suitably rescaled, in the $N \rightarrow \infty$ limit. Quite surprisingly, the computation here leads to the Catalan numbers, but not exactly in the same way as for the Wigner matrices, the precise result being as follows:

THEOREM 7.11. *Given a sequence of complex Wishart matrices*

$$W_N = Y_N Y_N^* \in M_N(L^\infty(X))$$

with Y_N being $N \times N$ complex Gaussian of parameter $t > 0$, we have

$$M_k \left(\frac{W_N}{N} \right) \simeq t^k C_k$$

for any exponent $k \in \mathbb{N}$, in the $N \rightarrow \infty$ limit.

PROOF. There are several possible proofs for this result, as follows:

(1) A first method is by using the result that we have from chapter 6, for the Gaussian matrices Y_N . Indeed, we know from there that we have the following formula, valid for any colored integer $K = \circ \bullet \bullet \circ \dots$, in the $N \rightarrow \infty$ limit:

$$M_K \left(\frac{Y_N}{\sqrt{N}} \right) \simeq t^{|K|/2} |\mathcal{NC}_2(K)|$$

With $K = \circ \bullet \bullet \circ \dots$, alternating word of length $2k$, with $k \in \mathbb{N}$, this gives:

$$M_k \left(\frac{Y_N Y_N^*}{N} \right) \simeq t^k |\mathcal{NC}_2(K)|$$

Thus, in terms of the Wishart matrix $W_N = Y_N Y_N^*$ we have, for any $k \in \mathbb{N}$:

$$M_k \left(\frac{W_N}{N} \right) \simeq t^k |\mathcal{NC}_2(K)|$$

The point now is that, by doing some combinatorics, we have:

$$|\mathcal{NC}_2(K)| = |\mathcal{NC}_2(2k)| = C_k$$

Thus, we are led to the formula in the statement.

(2) A second method, that we will explain now as well, is by proving the result directly, starting from definitions. The matrix entries of our matrix $W = W_N$ are given by:

$$W_{ij} = \sum_{r=1}^N Y_{ir} \bar{Y}_{jr}$$

Thus, the normalized traces of powers of W are given by the following formula:

$$\begin{aligned} \text{tr}(W^k) &= \frac{1}{N} \sum_{i_1=1}^N \cdots \sum_{i_k=1}^N W_{i_1 i_2} W_{i_2 i_3} \cdots W_{i_k i_1} \\ &= \frac{1}{N} \sum_{i_1=1}^N \cdots \sum_{i_k=1}^N \sum_{r_1=1}^N \cdots \sum_{r_k=1}^N Y_{i_1 r_1} \bar{Y}_{i_2 r_1} Y_{i_2 r_2} \bar{Y}_{i_3 r_2} \cdots Y_{i_k r_k} \bar{Y}_{i_1 r_k} \end{aligned}$$

By rescaling now W by a $1/N$ factor, as in the statement, we obtain:

$$\text{tr} \left(\left(\frac{W}{N} \right)^k \right) = \frac{1}{N^{k+1}} \sum_{i_1=1}^N \cdots \sum_{i_k=1}^N \sum_{r_1=1}^N \cdots \sum_{r_k=1}^N Y_{i_1 r_1} \bar{Y}_{i_2 r_1} Y_{i_2 r_2} \bar{Y}_{i_3 r_2} \cdots Y_{i_k r_k} \bar{Y}_{i_1 r_k}$$

By using now the Wick rule, we obtain the following formula for the moments, with $K = \circ \bullet \circ \bullet \dots$, alternating word of length $2k$, and with $I = (i_1 r_1, i_2 r_1, \dots, i_k r_k, i_1 r_k)$:

$$\begin{aligned} M_k \left(\frac{W}{N} \right) &= \frac{t^k}{N^{k+1}} \sum_{i_1=1}^N \cdots \sum_{i_k=1}^N \sum_{r_1=1}^N \cdots \sum_{r_k=1}^N \# \left\{ \pi \in \mathcal{P}_2(K) \mid \pi \leq \ker I \right\} \\ &= \frac{t^k}{N^{k+1}} \sum_{\pi \in \mathcal{P}_2(K)} \# \left\{ i, r \in \{1, \dots, N\}^k \mid \pi \leq \ker I \right\} \end{aligned}$$

In order to compute this quantity, we use the standard bijection $\mathcal{P}_2(K) \simeq S_k$. By identifying the pairings $\pi \in \mathcal{P}_2(K)$ with their counterparts $\pi \in S_k$, we obtain:

$$M_k \left(\frac{W}{N} \right) = \frac{t^k}{N^{k+1}} \sum_{\pi \in S_k} \# \left\{ i, r \in \{1, \dots, N\}^k \mid i_s = i_{\pi(s)+1}, r_s = r_{\pi(s)}, \forall s \right\}$$

Now let $\gamma \in S_k$ be the full cycle, which is by definition the following permutation:

$$\gamma = (12 \dots k)$$

The general factor in the product computed above is then 1 precisely when following two conditions are simultaneously satisfied:

$$\gamma\pi \leq \ker i \quad , \quad \pi \leq \ker r$$

Counting the number of free parameters in our moment formula, we obtain:

$$M_k \left(\frac{W}{N} \right) = t^k \sum_{\pi \in S_k} N^{|\pi| + |\gamma\pi| - k - 1}$$

The point now is that the last exponent is well-known to be ≤ 0 , with equality precisely when the permutation $\pi \in S_k$ is geodesic, which in practice means that π must come from a noncrossing partition. Thus we obtain, in the $N \rightarrow \infty$ limit:

$$M_k \left(\frac{W}{N} \right) \simeq t^k C_k$$

Thus, we are led to the conclusion in the statement. \square

As a consequence of the above result, we have a new look on the Catalan numbers, which is more adapted to our present Wishart matrix considerations, as follows:

PROPOSITION 7.12. *The Catalan numbers $C_k = |NC_2(2k)|$ appear as well as*

$$C_k = |NC(k)|$$

where $NC(k)$ is the set of all noncrossing partitions of $\{1, \dots, k\}$.

PROOF. This follows indeed from the proof of Theorem 7.11. \square

The direct explanation for the above formula, relating noncrossing partitions and pairings, comes from the following result, which is very useful, and good to know:

PROPOSITION 7.13. *We have a bijection between noncrossing partitions and pairings*

$$NC(k) \simeq NC_2(2k)$$

which is constructed as follows:

- (1) *The application $NC(k) \rightarrow NC_2(2k)$ is the “fattening” one, obtained by doubling all the legs, and doubling all the strings as well.*
- (2) *Its inverse $NC_2(2k) \rightarrow NC(k)$ is the “shrinking” application, obtained by collapsing pairs of consecutive neighbors.*

PROOF. The fact that the two operations in the statement are indeed inverse to each other is clear, by computing the corresponding two compositions, with the remark that the construction of the fattening operation requires the partitions to be noncrossing. \square

As a comment here, the above result is something quite remarkable, in view of the total lack of relation between $P(k)$ and $P_2(2k)$. Thus, taking for granted that “classical probability is about partitions, and free probability is about noncrossing partitions”, a general principle that emerges from our study so far, and that we will fully justify later on, we have in Proposition 7.13 an endless source of things to be done, in the free case, having no classical counterpart. We will keep this discovery in our pocket, and have it pulled out of there, for some magic, on several occasions, in what follows.

Getting back now to Wishart matrices, at $t = 1$ we are led to the question of finding the law having the Catalan numbers as moments. We already know the answer to this question from chapter 6, and more specifically from our considerations there at the end, regarding SO_3 , but here is as well an independent, pedestrian solution to this question:

PROPOSITION 7.14. *The real measure having the Catalan numbers as moments is*

$$\pi_1 = \frac{1}{2\pi} \sqrt{4x^{-1} - 1} dx$$

called Marchenko-Pastur law of parameter 1.

PROOF. As already mentioned, this is something that we already know, because we came upon this when talking about SO_3 . Here are two alternative proofs:

(1) By using the Stieltjes inversion formula. In order to apply this formula, we need a simple formula for the Cauchy transform. For this purpose, our starting point will be the formula from chapter 3 for the generating series of the Catalan numbers, namely:

$$\sum_{k=0}^{\infty} C_k z^k = \frac{1 - \sqrt{1 - 4z}}{2z}$$

By using this formula with $z = \xi^{-1}$, we obtain the following formula:

$$\begin{aligned} G(\xi) &= \xi^{-1} \sum_{k=0}^{\infty} C_k \xi^{-k} \\ &= \xi^{-1} \cdot \frac{1 - \sqrt{1 - 4\xi^{-1}}}{2\xi^{-1}} \\ &= \frac{1}{2} \left(1 - \sqrt{1 - 4\xi^{-1}} \right) \\ &= \frac{1}{2} - \frac{1}{2} \sqrt{1 - 4\xi^{-1}} \end{aligned}$$

With this formula in hand, let us apply now the Stieltjes inversion formula, from chapter 3. The first term, namely $1/2$, which is trivial, will not contribute to the density. As for the second term, which is something non-trivial, this will contribute to the density, the rule here being that the square root $\sqrt{1 - 4\xi^{-1}}$ will be replaced by the “dual” square root $\sqrt{4x^{-1} - 1} dx$, and that we have to multiply everything by $-1/\pi$. Thus, by Stieltjes inversion we obtain the density in the statement, namely:

$$\begin{aligned} d\mu(x) &= -\frac{1}{\pi} \cdot -\frac{1}{2} \sqrt{4x^{-1} - 1} dx \\ &= \frac{1}{2\pi} \sqrt{4x^{-1} - 1} dx \end{aligned}$$

(2) Alternatively, if the above was too complicated, we can simply cheat, as we actually did in chapter 6, when talking about SO_3 . Indeed, the moments of the law π_1 in the

statement can be computed with the change of variable $x = 4 \cos^2 t$, as follows:

$$\begin{aligned}
M_k &= \frac{1}{2\pi} \int_0^4 \sqrt{4x^{-1} - 1} x^k dx \\
&= \frac{1}{2\pi} \int_0^{\pi/2} \frac{\sin t}{\cos t} \cdot (4 \cos^2 t)^k \cdot 2 \cos t \sin t dt \\
&= \frac{4^{k+1}}{\pi} \int_0^{\pi/2} \cos^{2k} t \sin^2 t dt \\
&= \frac{4^{k+1}}{\pi} \cdot \frac{\pi}{2} \cdot \frac{(2k)!!2!!}{(2k+3)!!} \\
&= 2 \cdot 4^k \cdot \frac{(2k)!/2^k k!}{2^{k+1}(k+1)!} \\
&= C_k
\end{aligned}$$

Thus, we are led to the conclusion in the statement. \square

Now back to the Wishart matrices, we are led to the following result:

THEOREM 7.15. *Given a sequence of complex Wishart matrices*

$$W_N = Y_N Y_N^* \in M_N(L^\infty(X))$$

with Y_N being $N \times N$ complex Gaussian of parameter 1, we have

$$\frac{W_N}{N} \sim \frac{1}{2\pi} \sqrt{4x^{-1} - 1} dx$$

with $N \rightarrow \infty$, with the limiting measure being the Marchenko-Pastur law π_1 .

PROOF. This follows indeed from the asymptotic moment computation that we have, for these matrices, from Theorem 7.11, coupled with Proposition 7.14. \square

More generally now, we have as well a straightforward parametric version of the above result, involving a parameter $t > 0$ as in Definition 7.10, as follows:

THEOREM 7.16. *Given a sequence of complex Wishart matrices*

$$W_N = Y_N Y_N^* \in M_N(L^\infty(X))$$

with Y_N being $N \times N$ complex Gaussian of parameter $t > 0$, we have

$$\frac{W_N}{tN} \sim \frac{1}{2\pi} \sqrt{4x^{-1} - 1} dx$$

with $N \rightarrow \infty$, with the limiting measure being the Marchenko-Pastur law π_1 .

PROOF. This follows again from Theorem 7.11 and Proposition 7.14. To be more precise, recall the main formula from Theorem 7.11, for the matrices as above, namely:

$$M_k \left(\frac{W_N}{N} \right) \simeq t^k C_k$$

By dividing by t^k , this formula can be written as follows:

$$M_k \left(\frac{W_N}{tN} \right) \simeq C_k$$

Now by using Proposition 7.14, we are led to the conclusion in the statement. \square

Summarizing, we have deduced the Marchenko-Pastur theorem from the result for Gaussian matrices, via some moment combinatorics. It is possible as well to be a bit more direct here, by passing through the Wigner theorem, and then recovering the Marchenko-Pastur law directly from the Wigner semicircle law, by performing a kind of square operation. But this is more or less the same thing as we did above.

7c. Parametric version

We discuss now a generalization of the above results, motivated by a whole array of concrete questions, and bringing into the picture a “true” parameter $t > 0$, which is different from the parameter $t > 0$ used above, which is something quite trivial.

For this purpose, let us go back to the definition of the Wishart matrices. There were as follows, with Y being a $N \times N$ matrix with i.i.d. entries, each following the law G_t :

$$W = YY^*$$

The point now is that, more generally, we can use in this $W = YY^*$ construction a $N \times M$ matrix Y with i.i.d. entries, each following the law G_t , with $M \in \mathbb{N}$ being arbitrary. Thus, we have a new parameter, and by ditching the old parameter $t > 0$, which was something not very interesting, we are led to the following definition, which is the “true” definition of the Wishart matrices, from [65] and the subsequent literature:

DEFINITION 7.17. *A complex Wishart matrix is a $N \times N$ matrix of the form*

$$W = YY^*$$

where Y is a $N \times M$ matrix with i.i.d. entries, each following the law G_1 .

As before with our previous Wishart matrices, that the new ones generalize, up to setting $t = 1$, we have $W \geq 0$, by definition. Due to this property, and to the otherwise “randomness” of W , these matrices are useful in many contexts. More on this later.

In order to see what is going on, combinatorially, let us compute moments. The result here is substantially more interesting than that for the previous Wishart matrices, with the new relevant numeric parameter being now the number $t = M/N$, as follows:

THEOREM 7.18. *Given a sequence of complex Wishart matrices*

$$W_N = Y_N Y_N^* \in M_N(L^\infty(X))$$

with Y_N being $N \times M$ complex Gaussian of parameter 1, we have

$$M_k \left(\frac{W_N}{N} \right) \simeq \sum_{\pi \in NC(k)} t^{|\pi|}$$

for any exponent $k \in \mathbb{N}$, in the $M = tN \rightarrow \infty$ limit.

PROOF. This is something which is very standard, as follows:

(1) Before starting, let us clarify the relation with our previous Wishart matrix results. In the case $M = N$ we have $t = 1$, and the formula in the statement reads:

$$M_k \left(\frac{W_N}{N} \right) \simeq |NC(k)|$$

Thus, what we have here is the previous Wishart matrix formula, in full generality, at the value $t = 1$ of our old parameter $t > 0$.

(2) Observe also that by rescaling, we can obtain if we want from this the previous Wishart matrix formula, in full generality, at any value $t > 0$ of our old parameter. Thus, things fine, we are indeed generalizing what we did before.

(3) In order to prove now the formula in the statement, we proceed as usual, by using the Wick formula. The matrix entries of our Wishart matrix $W = W_N$ are given by:

$$W_{ij} = \sum_{r=1}^M Y_{ir} \bar{Y}_{jr}$$

Thus, the normalized traces of powers of W are given by the following formula:

$$\begin{aligned} tr(W^k) &= \frac{1}{N} \sum_{i_1=1}^N \cdots \sum_{i_k=1}^N W_{i_1 i_2} W_{i_2 i_3} \cdots W_{i_k i_1} \\ &= \frac{1}{N} \sum_{i_1=1}^N \cdots \sum_{i_k=1}^N \sum_{r_1=1}^M \cdots \sum_{r_k=1}^M Y_{i_1 r_1} \bar{Y}_{i_2 r_1} Y_{i_2 r_2} \bar{Y}_{i_3 r_2} \cdots Y_{i_k r_k} \bar{Y}_{i_1 r_k} \end{aligned}$$

By rescaling now W by a $1/N$ factor, as in the statement, we obtain:

$$tr \left(\left(\frac{W}{N} \right)^k \right) = \frac{1}{N^{k+1}} \sum_{i_1=1}^N \cdots \sum_{i_k=1}^N \sum_{r_1=1}^M \cdots \sum_{r_k=1}^M Y_{i_1 r_1} \bar{Y}_{i_2 r_1} Y_{i_2 r_2} \bar{Y}_{i_3 r_2} \cdots Y_{i_k r_k} \bar{Y}_{i_1 r_k}$$

(4) By using now the Wick rule, we obtain the following formula for the moments, with $K = \circ \bullet \circ \bullet \dots$, alternating word of length $2k$, and $I = (i_1 r_1, i_2 r_1, \dots, i_k r_k, i_1 r_k)$:

$$\begin{aligned} M_k \left(\frac{W}{N} \right) &= \frac{1}{N^{k+1}} \sum_{i_1=1}^N \dots \sum_{i_k=1}^N \sum_{r_1=1}^M \dots \sum_{r_k=1}^M \# \left\{ \pi \in \mathcal{P}_2(K) \mid \pi \leq \ker I \right\} \\ &= \frac{1}{N^{k+1}} \sum_{\pi \in \mathcal{P}_2(K)} \# \left\{ i \in \{1, \dots, N\}^k, r \in \{1, \dots, M\}^k \mid \pi \leq \ker I \right\} \end{aligned}$$

(5) In order to compute this quantity, we use the standard bijection $\mathcal{P}_2(K) \simeq S_k$. By identifying the pairings $\pi \in \mathcal{P}_2(K)$ with their counterparts $\pi \in S_k$, we obtain:

$$M_k \left(\frac{W}{N} \right) = \frac{1}{N^{k+1}} \sum_{\pi \in S_k} \# \left\{ i \in \{1, \dots, N\}^k, r \in \{1, \dots, M\}^k \mid i_s = i_{\pi(s)+1}, r_s = r_{\pi(s)} \right\}$$

Now let $\gamma \in S_k$ be the full cycle, which is by definition the following permutation:

$$\gamma = (12 \dots k)$$

The general factor in the product computed above is then 1 precisely when following two conditions are simultaneously satisfied:

$$\gamma\pi \leq \ker i \quad , \quad \pi \leq \ker r$$

Counting the number of free parameters in our expectation formula, we obtain:

$$M_k \left(\frac{W}{N} \right) = \frac{1}{N^{k+1}} \sum_{\pi \in S_k} N^{|\gamma\pi|} M^{|\pi|} = \sum_{\pi \in S_k} N^{|\gamma\pi|-k-1} M^{|\pi|}$$

(6) Now by using the same arguments as in the case $M = N$, from the proof of Theorem 7.11, we conclude that in the $M = tN \rightarrow \infty$ limit the permutations $\pi \in S_k$ which matter are those coming from noncrossing partitions, and so that we have:

$$M_k \left(\frac{W}{N} \right) \simeq \sum_{\pi \in NC(k)} N^{-|\pi|} M^{|\pi|} = \sum_{\pi \in NC(k)} t^{|\pi|}$$

We are therefore led to the conclusion in the statement. \square

In order to recapture now the density out of the moments, we can of course use the Stieltjes inversion formula, but the computations here are a bit opaque. So, inspired from what happens at $t = 1$, let us cheat a bit, and formulate a nice definition, as follows:

DEFINITION 7.19. *The Marchenko-Pastur law π_t of parameter $t > 0$ is given by:*

$$a \sim \gamma_t \implies a^2 \sim \pi_t$$

That is, π_t the law of the square of a variable following the law γ_t .

This is certainly very nice, and we know from chapter 6 that at $t = 1$ we obtain indeed the Marchenko-Pastur law π_1 , as constructed above. In general, we have:

PROPOSITION 7.20. *The Marchenko-Pastur law of parameter $t > 0$ is*

$$\pi_t = \max(1 - t, 0)\delta_0 + \frac{\sqrt{4t - (x - 1 - t)^2}}{2\pi x} dx$$

the support being $[0, 4t^2]$, and the moments of this measure are

$$M_k = \sum_{\pi \in NC(k)} t^{|\pi|}$$

exactly as for the asymptotic moments of the complex Wishart matrices.

PROOF. This follows as usual, by doing some computations, either combinatorics, or calculus. To be more precise, we have three formulae for π_t to be connected, namely the one in Definition 7.19, and the two ones from the present statement, and the connections between them can be established exactly as we did before, at $t = 1$. \square

Summarizing, we have now a definition for the Marchenko-Pastur law π_t , which is quite elegant, via Definition 7.19, but which still requires some computations, performed in the proof of Proposition 7.20. We will see later on, in chapters 9-12, an even more elegant definition for π_t , out of its particular case π_1 which was well understood, simply obtained by considering the corresponding 1-parameter free convolution semigroup. We will also see that π_t appears as the “free version” of the Poisson law p_t , and that this can be even taken as a definition for π_t , if we really want to. More on this later.

Now back to the complex Wishart matrices that we are interested in, in this chapter, we can now formulate a final result regarding them, as follows:

THEOREM 7.21. *Given a sequence of complex Wishart matrices*

$$W_N = Y_N Y_N^* \in M_N(L^\infty(X))$$

with Y_N being $N \times M$ complex Gaussian of parameter 1, we have

$$\frac{W_N}{N} \sim \max(1 - t, 0)\delta_0 + \frac{\sqrt{4t - (x - 1 - t)^2}}{2\pi x} dx$$

with $M = tN \rightarrow \infty$, with the limiting measure being the Marchenko-Pastur law π_t .

PROOF. This follows indeed from Theorem 7.18 and Proposition 7.20. \square

As it was the case with the Gaussian and Wigner matrices, there are many other things that can be said about the complex Wishart matrices, as variations of the above. We refer here to the standard random matrix literature [2], [66], [69], [90]. We will be back to this right below, in the remainder of this chapter, with some wizarding computations from [5], and then more systematically in chapter 11 below, when doing free probability.

7d. Shifted semicircles

Our goal now, in the remainder of this chapter, will be that of explaining a surprising result, due to Aubrun [5], stating that when suitably block-transposing the entries of a complex Wishart matrix, we obtain as asymptotic distribution a shifted version of Wigner's semicircle law. Following [5], [19], let us start with the following definition:

DEFINITION 7.22. *The partial transpose of a complex Wishart matrix W of parameters (dn, dm) is the matrix*

$$\widetilde{W} = (id \otimes t)W$$

where id is the identity of $M_d(\mathbb{C})$, and t is the transposition of $M_n(\mathbb{C})$.

In more familiar terms of bases and indices, the standard decomposition $\mathbb{C}^{dn} = \mathbb{C}^d \otimes \mathbb{C}^n$ induces an algebra decomposition $M_{dn}(\mathbb{C}) = M_d(\mathbb{C}) \otimes M_n(\mathbb{C})$, and with this convention made, the partial transpose matrix \widetilde{W} constructed above has entries as follows:

$$\widetilde{W}_{ia,jb} = W_{ib,ja}$$

Our goal in what follows will be that of computing the law of \widetilde{W} , first when d, n, m are fixed, and then in the $d \rightarrow \infty$ regime. For this purpose, we will need a number of standard facts regarding the noncrossing partitions. Let us start with:

PROPOSITION 7.23. *For a permutation $\sigma \in S_p$, we have the formula*

$$|\sigma| + \#\sigma = p$$

where $|\sigma|$ is the number of cycles of σ , and $\#\sigma$ is the minimal $k \in \mathbb{N}$ such that σ is a product of k transpositions. Also, the following formula defines a distance on S_p ,

$$(\sigma, \pi) \rightarrow \#(\sigma^{-1}\pi)$$

and the set of permutations $\sigma \in S_p$ which saturate the triangular inequality

$$\#\sigma + \#(\sigma^{-1}\gamma) = \#\gamma = p - 1$$

where $\gamma \in S_p$ is a full cycle, is in bijection with the set $NC(p)$.

PROOF. All this is standard combinatorics, that we will leave as an exercise. \square

We use the standard bijection $NC(p) \simeq NC_2(2p)$, denoted $\pi \rightarrow \widetilde{\pi}$, obtained by fattening the partitions. We have the following formula, where \vee is the join operation on $NC_2(2p)$, and $\rho_{12} = (12)(34) \dots (2p-1, 2p)$ is the fattened identity permutation:

$$|\pi| = |\widetilde{\pi} \vee \rho_{12}|$$

Similarly, we have the formula $|\pi\gamma| = |\widetilde{\pi} \vee \rho_{14}|$, where ρ_{14} is the pairing corresponding to the fattening of the inverse full cycle $\gamma^{-1}(i) = i - 1$, which pairs an element $2i$ with $2(i - 1) - 1 = 2i - 3$, or, equivalently, an element $i \in \{1, \dots, 2p\}$ with $i + (-1)^{i+1}3$.

We will need the following well-known result:

PROPOSITION 7.24. *The number $||\pi||$ of blocks having even size is given by*

$$1 + ||\pi|| = |\pi\gamma|$$

for every noncrossing partition $\pi \in NC(p)$.

PROOF. We use a recurrence over the number of blocks of π . If π has just one block, its associated geodesic permutation is γ and we have:

$$|\gamma^2| = \begin{cases} 1 & (p \text{ odd}) \\ 2 & (p \text{ even}) \end{cases}$$

For the partitions π having more than one block, we can assume without loss of generality that $\pi = \hat{1}_k \sqcup \pi'$, where $\hat{1}_k$ is a contiguous block of size k . Recall that the number of blocks of the permutation $\pi\gamma$ is given by the following formula, where $\rho_{14} \in P_2(2p)$ is the pair partition which pairs an element i with $i + (-1)^{i+1}3$:

$$|\pi\gamma| = |\tilde{\pi} \vee \rho_{14}|$$

If k is an even number, $k = 2r$, consider the following partition, which contains the block $(1\ 4\ 5\ 8\ \dots\ 4r - 3\ 4r)$, along with the blocks coming from the elements of the form $4i + 2, 4i + 3$ from $\{1, \dots, 4r\}$ and from π' :

$$\sigma = \widehat{\hat{1}_{2r} \sqcup \pi'} \vee \rho_{14}$$

We can count the blocks of the join of two partitions by drawing them one beneath the other and counting the number of connected components of the curve, without taking into account the possible crossings. We conclude that we have the following formula, where ρ'_{14} is ρ_{14} restricted to the set $\{2k + 1, 2k + 2, \dots, 2p\}$:

$$|\tilde{\pi} \vee \rho_{14}| = 1 + |\tilde{\pi}' \vee \rho'_{14}|$$

If k is odd, $k = 2r + 1$, there is no extra block appearing, so we have:

$$|\tilde{\pi} \vee \rho_{14}| = |\tilde{\pi}' \vee \rho'_{14}|$$

Thus, we are led to the conclusion in the statement. □

We can now investigate the block-transposed Wishart matrices, and we have:

THEOREM 7.25. *For any $p \geq 1$ we have the formula*

$$\lim_{d \rightarrow \infty} (E \circ tr)(m\widetilde{W})^p = \sum_{\pi \in NC(p)} m^{|\pi|} n^{||\pi||}$$

where $|\cdot|$ and $||\cdot||$ are the number of blocks, and the number of blocks of even size.

PROOF. The matrix elements of the partial transpose matrix are given by:

$$\widetilde{W}_{ia,jb} = W_{ib,ja} = (dm)^{-1} \sum_{k=1}^d \sum_{c=1}^m G_{ib,kc} \bar{G}_{ja,kc}$$

This gives the following formula:

$$\begin{aligned} tr(\widetilde{W}^p) &= (dn)^{-1} (dm)^{-p} \sum_{i_1, \dots, i_p=1}^d \sum_{a_1, \dots, a_p=1}^n \prod_{s=1}^p W_{i_s a_{s+1}, i_{s+1} a_s} \\ &= (dn)^{-1} (dm)^{-p} \sum_{i_1, \dots, i_p=1}^d \sum_{a_1, \dots, a_p=1}^n \prod_{s=1}^p \sum_{j_1, \dots, j_p=1}^d \sum_{b_1, \dots, b_p=1}^m G_{i_s a_{s+1}, j_s b_s} \bar{G}_{i_{s+1} a_s, j_{s+1} b_{s+1}} \end{aligned}$$

After interchanging the product with the last two sums, the average of the general term can be computed by the Wick rule, namely:

$$E \left(\prod_{s=1}^p G_{i_s a_{s+1}, j_s b_s} \bar{G}_{i_{s+1} a_s, j_{s+1} b_{s+1}} \right) = \sum_{\pi \in S_p} \prod_{s=1}^p \delta_{i_s, i_{\pi(s)+1}} \delta_{a_{s+1}, a_{\pi(s)}} \delta_{j_s, j_{\pi(s)}} \delta_{b_s, b_{\pi(s)}}$$

Let $\gamma \in S_p$ be the full cycle $\gamma = (1 \ 2 \ \dots \ p)^{-1}$. The general factor in the above product is 1 if and only if the following four conditions are simultaneously satisfied:

$$\gamma^{-1}\pi \leq \ker i \quad , \quad \pi\gamma \leq \ker a \quad , \quad \pi \leq \ker j \quad , \quad \pi \leq \ker b$$

Counting the number of free parameters in the above equation, we obtain:

$$\begin{aligned} (E \circ tr)(\widetilde{W}^p) &= (dn)^{-1} (dm)^{-p} \sum_{\pi \in S_p} d^{|\pi| + |\gamma^{-1}\pi|} m^{|\pi|} n^{|\pi\gamma|} \\ &= \sum_{\pi \in S_p} d^{|\pi| + |\gamma^{-1}\pi| - p - 1} m^{|\pi| - p} n^{|\pi\gamma| - 1} \end{aligned}$$

The exponent of d in the last expression on the right is:

$$\begin{aligned} N(\pi) &= |\pi| + |\gamma^{-1}\pi| - p - 1 \\ &= p - 1 - (\#\pi + \#(\gamma^{-1}\pi)) \\ &= p - 1 - (\#\pi + \#(\pi^{-1}\gamma)) \end{aligned}$$

As explained in the beginning of this section, this quantity is known to be ≤ 0 , with equality iff π is geodesic, hence associated to a noncrossing partition. Thus:

$$(E \circ tr)(\widetilde{W}^p) = (1 + O(d^{-1})) m^{-p} n^{-1} \sum_{\pi \in NC(p)} m^{|\pi|} n^{|\pi\gamma|}$$

Together with $|\pi\gamma| = \|\pi\| + 1$, this gives the result. \square

We would like now to find an equation for the moment generating function of the asymptotic law of $m\widetilde{W}$. This moment generating function is defined by:

$$F(z) = \lim_{d \rightarrow \infty} (E \circ tr) \left(\frac{1}{1 - z m \widetilde{W}} \right)$$

We have the following result, regarding this moment generating function:

THEOREM 7.26. *The moment generating function of $m\widetilde{W}$ satisfies the equation*

$$(F - 1)(1 - z^2 F^2) = mzF(1 + nzF)$$

in the $d \rightarrow \infty$ limit.

PROOF. We use the formula in Theorem 7.25. If we denote by $N(p, b, e)$ the number of partitions in $NC(p)$ having b blocks and e even blocks, we have:

$$\begin{aligned} F &= 1 + \sum_{p=1}^{\infty} \sum_{\pi \in NC(p)} z^p m^{|\pi|} n^{|\pi|} \\ &= 1 + \sum_{p=1}^{\infty} \sum_{b=0}^{\infty} \sum_{e=0}^{\infty} z^p m^b n^e N(p, b, e) \end{aligned}$$

Let us try to find a recurrence formula for the numbers $N(p, b, e)$. If we look at the block containing 1, this block must have $r \geq 0$ other legs, and we get:

$$\begin{aligned} N(p, b, e) &= \sum_{r \in 2\mathbb{N}} \sum_{p=\Sigma p_i+r+1} \sum_{b=\Sigma b_i+1} \sum_{e=\Sigma e_i} N(p_1, b_1, e_1) \dots N(p_{r+1}, b_{r+1}, e_{r+1}) \\ &+ \sum_{r \in 2\mathbb{N}+1} \sum_{p=\Sigma p_i+r+1} \sum_{b=\Sigma b_i+1} \sum_{e=\Sigma e_i+1} N(p_1, b_1, e_1) \dots N(p_{r+1}, b_{r+1}, e_{r+1}) \end{aligned}$$

Here p_1, \dots, p_{r+1} are the number of points between the legs of the block containing 1, so that we have $p = (p_1 + \dots + p_{r+1}) + r + 1$, and the whole sum is split over two cases, r even or odd, because the parity of r affects the number of even blocks of our partition. Now by multiplying everything by a $z^p m^b n^e$ factor, and by carefully distributing the various powers of z, m, b on the right, we obtain the following formula:

$$\begin{aligned} z^p m^b n^e N(p, b, e) &= m \sum_{r \in 2\mathbb{N}} z^{r+1} \sum_{p=\Sigma p_i+r+1} \sum_{b=\Sigma b_i+1} \sum_{e=\Sigma e_i} \prod_{i=1}^{r+1} z^{p_i} m^{b_i} n^{e_i} N(p_i, b_i, e_i) \\ &+ mn \sum_{r \in 2\mathbb{N}+1} z^{r+1} \sum_{p=\Sigma p_i+r+1} \sum_{b=\Sigma b_i+1} \sum_{e=\Sigma e_i+1} \prod_{i=1}^{r+1} z^{p_i} m^{b_i} n^{e_i} N(p_i, b_i, e_i) \end{aligned}$$

Let us sum now all these equalities, over all $p \geq 1$ and over all $b, e \geq 0$. According to the definition of F , at left we obtain $F - 1$. As for the two sums appearing on the right,

that is, at right of the two z^{r+1} factors, when summing them over all $p \geq 1$ and over all $b, e \geq 0$, we obtain in both cases F^{r+1} . So, we have the following formula:

$$\begin{aligned} F - 1 &= m \sum_{r \in 2\mathbb{N}} (zF)^{r+1} + mn \sum_{r \in 2\mathbb{N}+1} (zF)^{r+1} \\ &= m \frac{zF}{1 - z^2 F^2} + mn \frac{z^2 F^2}{1 - z^2 F^2} \\ &= mzF \frac{1 + nzF}{1 - z^2 F^2} \end{aligned}$$

But this gives the formula in the statement, and we are done. \square

Our goal now will be that of further processing the formula in Theorem 7.26, as to reach to a formula for the density of the corresponding law. This is something quite tricky, and as a first result here, we can reformulate Theorem 7.26 as follows:

THEOREM 7.27. *The Cauchy transform of $m\widetilde{W}$ satisfies the equation*

$$(\xi G - 1)(1 - G^2) = mG(1 + nG)$$

in the $d \rightarrow \infty$ limit. Moreover, this equation simply reads

$$R = \frac{m}{2} \left(\frac{n+1}{1-z} - \frac{n-1}{1+z} \right)$$

with the substitutions $G \rightarrow z$ and $\xi \rightarrow R + z^{-1}$.

PROOF. We have two assertions to be proved, the first one being standard, and the second one being something quite magic, the idea being as follows:

(1) Consider the equation of F , found in Theorem 7.26, namely:

$$(F - 1)(1 - z^2 F^2) = mzF(1 + nzF)$$

With $z \rightarrow \xi^{-1}$ and $F \rightarrow \xi G$, so that $zF \rightarrow G$, we obtain, as desired:

$$(\xi G - 1)(1 - G^2) = mG(1 + nG)$$

(2) Thus, we have our equation for the Cauchy transform, and with this in hand, we can try to go ahead, and use somehow the Stieltjes inversion formula, in order to reach to a formula for the density. This is certainly possible, but our claim is that we can do better, by performing first some clever manipulations on the Cauchy transform.

(3) To be more precise, let us look at the equation of the Cauchy transform that we have. With the substitutions $\xi \rightarrow K$ and $G \rightarrow z$, this equation becomes:

$$(zK - 1)(1 - z^2) = mz(1 + nz)$$

The point now is that with $K \rightarrow R + z^{-1}$ this latter equation becomes:

$$zR(1 - z^2) = mz(1 + nz)$$

But the solution of this latter equation is trivial to compute, given by:

$$R = m \frac{1 + nz}{1 - z^2} = \frac{m}{2} \left(\frac{n+1}{1-z} - \frac{n-1}{1+z} \right)$$

Thus, we are led to the conclusion in the statement. \square

All the above might look a bit mysterious, but we are into difficult mathematics now, that will take us some time to be understood. In any case, the manipulations made in Theorem 7.27 are quite interesting, and suggest the following definition:

DEFINITION 7.28. *Given a real probability measure μ , define its R -transform by:*

$$G_\mu(\xi) = \int_{\mathbb{R}} \frac{d\mu(t)}{\xi - t} \implies G_\mu \left(R_\mu(\xi) + \frac{1}{\xi} \right) = \xi$$

That is, the R -transform is the inverse of the Cauchy transform, up to a ξ^{-1} factor.

This definition is actually something very deep, due to Voiculescu [86], and we will have the whole remainder of this book for exploring its subtleties. For the moment, let us just take it as such, as something natural emerging from Theorem 7.27.

Getting back now to our questions, we would like to find the probability measure having as R -transform the function in Theorem 7.27. But here, we can only expect to find some kind of modification of the Marchenko-Pastur law, so as a first piece of work, let us just compute the R -transform of the Marchenko-Pastur law. We have here:

PROPOSITION 7.29. *The R -transform of the Marchenko-Pastur law π_t is*

$$R_{\pi_t}(\xi) = \frac{t}{1 - \xi}$$

for any $t > 0$.

PROOF. This can be done in two steps, as follows:

(1) At $t = 1$, we know that the moments of π_1 are the Catalan numbers, $M_k = C_k$, and we obtain that the Cauchy transform is given by the following formula:

$$G(\xi) = \frac{1}{2} - \frac{1}{2} \sqrt{1 - 4\xi^{-1}}$$

Now with $R(\xi) = \frac{1}{1-\xi}$ being the function in the statement, at $t = 1$, we have:

$$\begin{aligned} G\left(R(\xi) + \frac{1}{\xi}\right) &= G\left(\frac{1}{1-\xi} + \frac{1}{\xi}\right) \\ &= G\left(\frac{1}{\xi - \xi^2}\right) \\ &= \frac{1}{2} - \frac{1}{2}\sqrt{1 - 4\xi + 4\xi^2} \\ &= \frac{1}{2} - \frac{1}{2}(1 - 2\xi) \\ &= \xi \end{aligned}$$

Thus, the function $R(\xi) = \frac{1}{1-\xi}$ is indeed the R -transform of π_1 , in the above sense.

(2) In the general case, $t > 0$, the proof is similar, by using the moment formula for π_t , that we know from the above. We will be back to this with full details when really needed, and more specifically in chapters 9-12 below, when doing free probability. \square

All this is very nice, and we can now further build on Theorem 7.27, as follows:

THEOREM 7.30. *The R -transform of $m\widetilde{W}$ is given by*

$$R = R_{\pi_s} - R_{\pi_t}$$

in the $d \rightarrow \infty$ limit, where $s = m(n+1)/2$ and $t = m(n-1)/2$.

PROOF. We know from Theorem 7.27 that the R -transform of $m\widetilde{W}$ is given by:

$$R = \frac{m}{2} \left(\frac{n+1}{1-z} - \frac{n-1}{1+z} \right)$$

By using now the formula in Proposition 7.29, this gives the result. \square

We can now recover the original result of Aubrun [5], as follows:

THEOREM 7.31. *For a block-transposed Wishart matrix $\widetilde{W} = (id \otimes t)W$ we have, in the $n = \beta m \rightarrow \infty$ limit, with $\beta > 0$ fixed, the formula*

$$\frac{\widetilde{W}}{d} \sim \gamma_\beta^1$$

with γ_β^1 being the shifted version of the semicircle law γ_β , with support centered at 1.

PROOF. This follows from Theorem 7.30. Indeed, in the $n = \beta m \rightarrow \infty$ limit, with $\beta > 0$ fixed, we are led to the following formula for the Stieltjes transform:

$$f(x) = \frac{\sqrt{4\beta - (1-x)^2}}{2\beta\pi}$$

But this is the density of the shifted semicircle law having support as follows:

$$S = [1 - 2\sqrt{\beta}, 1 + 2\sqrt{\beta}]$$

Thus, we are led to the conclusion in the statement. See [5], [19]. \square

Here we have used some standard free probability results at the end, which can be proved by direct computations, and we will be back to this in chapters 9-12 below.

7e. Exercises

There has been a lot of combinatorics in this chapter, in relation with the Wishart matrices and the Marchenko-Pastur laws, and as an exercise here, we have:

EXERCISE 7.32. *Work out with full details the proof of the Aubrun result regarding the block-transposed Wishart matrices, directly, out of the Wick formula.*

To be more precise, we have seen a proof of this result, but based on rather heavy, general methods from [19]. A lighter proof is the one in [5], dealing with the block-transposed Wishart matrices, in the $n = \beta m \rightarrow \infty$ regime. So, find your favorite proof, fully read and understand it, and write down a brief account of that.

CHAPTER 8

Block modifications

8a. Block modifications

We discuss in this chapter some extensions and unifications of our results from chapter 7. As before with the usual or block-transposed Wishart matrices, there will be some non-trivial combinatorics here, that we will fully understand only later, in chapters 9-12, when doing free probability. Thus, the material below will be an introduction to this.

Let us begin with some general block modification considerations, following [5] and the more recent papers [19], [20]. We have the following construction:

DEFINITION 8.1. *Given a complex Wishart $dn \times dn$ matrix, appearing as*

$$W = YY^* \in M_{dn}(L^\infty(X))$$

with Y being a complex Gaussian $dn \times dm$ matrix, and a linear map

$$\varphi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$$

we consider the following matrix, obtained by applying φ to the $n \times n$ blocks of W ,

$$\widetilde{W} = (id \otimes \varphi)W \in M_{dn}(L^\infty(X))$$

and call it block-modified Wishart matrix.

Here we are using some standard tensor product identifications, the details being as follows. Let Y be a complex Gaussian $dn \times dm$ matrix, as above:

$$Y \in M_{dn \times dm}(L^\infty(X))$$

We can then form the corresponding complex Wishart matrix, as follows:

$$W = YY^* \in M_{dn}(L^\infty(X))$$

The size of this matrix being a composite number, $N = dn$, we can regard this matrix as being a $n \times n$ matrix, with random $d \times d$ matrices as entries. Equivalently, by using standard tensor product notations, this amounts in regarding W as follows:

$$W \in M_d(L^\infty(X)) \otimes M_n(\mathbb{C})$$

With this done, we can come up with our linear map, namely:

$$\varphi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$$

We can apply φ to the tensors on the right, and we obtain a matrix as follows:

$$\widetilde{W} = (id \otimes \varphi)W \in M_d(L^\infty(X)) \otimes M_n(\mathbb{C})$$

Finally, we can forget now about tensors, and as a conclusion to all this, we have constructed a matrix as follows, that we can call block-modified Wishart matrix:

$$\widetilde{W} \in M_{dn}(L^\infty(X))$$

In practice now, what we mostly need for fully understanding Definition 8.1 are examples. Following Aubrun [5], and the series of papers by Collins and Nechita [33], [34], [35], we have the following basic examples, for our general construction:

DEFINITION 8.2. *We have the following examples of block-modified Wishart matrices $\widetilde{W} = (id \otimes \varphi)W$, coming from various linear maps $\varphi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$:*

- (1) *Wishart matrices: $\widetilde{W} = W$, obtained via $\varphi = id$.*
- (2) *Aubrun matrices: $\widetilde{W} = (id \otimes t)W$, with t being the transposition.*
- (3) *Collins-Nechita one: $\widetilde{W} = (id \otimes \varphi)W$, with $\varphi = tr(\cdot)1$.*
- (4) *Collins-Nechita two: $\widetilde{W} = (id \otimes \varphi)W$, with φ erasing the off-diagonal part.*

These examples, whose construction is something very elementary, appear in a wide context of interesting situations, for the most in connection with various questions in quantum physics [5], [33], [34], [35], [65]. They will actually serve as a main motivation for what we will be doing, in what follows. More on this later.

Getting back now to the general case, that of Definition 8.1 as stated, the linear map $\varphi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ there is certainly useful for understanding the construction of the block-modified Wishart matrix $\widetilde{W} = (id \otimes \varphi)W$, as illustrated by the above examples. In practice, however, we would like to have as block-modification “data” something more concrete, such as a usual matrix. To be more precise, we would like to use:

PROPOSITION 8.3. *We have a correspondence between linear maps*

$$\varphi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$$

and square matrices $\Lambda \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$, given by the formula

$$\Lambda_{ab,cd} = \varphi(e_{ac})_{bd}$$

where $e_{ab} \in M_n(\mathbb{C})$ are the standard generators of the matrix algebra $M_n(\mathbb{C})$, given by the formula $e_{ab} : e_b \rightarrow e_a$, with $\{e_1, \dots, e_n\}$ being the standard basis of \mathbb{C}^n .

PROOF. This is standard linear algebra. Given a linear map $\varphi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$, we can associate to it numbers $\Lambda_{ab,cd} \in \mathbb{C}$ by the formula in the statement, namely:

$$\Lambda_{ab,cd} = \varphi(e_{ac})_{bd}$$

Now by using these n^4 numbers, we can construct a $n^2 \times n^2$ matrix, as follows:

$$\Lambda = \sum_{abcd} \Lambda_{ab,cd} e_{ac} \otimes e_{bd} \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$$

Thus, we have constructed a correspondence $\varphi \rightarrow \Lambda$, and since this correspondence is injective, and the dimensions match, this correspondence is bijective, as claimed. \square

Now by getting back to the block-modified Wishart matrices, we have:

PROPOSITION 8.4. *Given a Wishart $dn \times dn$ matrix $W = YY^*$, and a linear map*

$$\varphi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$$

the entries of the corresponding block-modified matrix $\widetilde{W} = (id \otimes \varphi)W$ are given by

$$\widetilde{W}_{ia,jb} = \sum_{cd} \Lambda_{ca,db} W_{ic,jd}$$

where $\Lambda \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ is the square matrix associated to φ , as above.

PROOF. Again, this is trivial linear algebra, coming from the following computation:

$$\widetilde{W}_{ia,jb} = \sum_{cd} W_{ic,jd} \varphi(e_{cd})_{ab} = \sum_{cd} \Lambda_{ca,db} W_{ic,jd}$$

Thus, we are led to the conclusion in the statement. \square

At the level of the main examples, from Definition 8.2, the very basic linear maps $\varphi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ used there can only correspond to some basic examples of matrices $\Lambda \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$, via the correspondence in Proposition 8.3. This is indeed the case, and in order to clarify this, and at a rather conceptual level, let us formulate, inspired by the representation theory material from chapter 4, the following definition:

DEFINITION 8.5. *Let $P(k, l)$ be the set of partitions between an upper row of k points, and a lower row of l points. Associated to any $\pi \in P(k, l)$ is the linear map*

$$T_\pi(e_{i_1} \otimes \dots \otimes e_{i_k}) = \sum_{j_1 \dots j_l} \delta_\pi \begin{pmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_l \end{pmatrix} e_{j_1} \otimes \dots \otimes e_{j_l}$$

between tensor powers of \mathbb{C}^N , called “easy”, with the Kronecker type symbol on the right being given by $\delta_\pi = 1$ when the indices fit, and $\delta_\pi = 0$ otherwise.

Observe the obvious connection with notion of easy group, from chapter 4, the point being that a closed subgroup $G \subset U_N$ is easy precisely when its Tannakian category $C_G = (C_G(k, l))$ with $C_G(k, l) \subset \mathcal{L}((\mathbb{C}^N)^k, (\mathbb{C}^N)^l)$ is spanned by easy maps.

For our purposes here, we will need a slight modification of Definition 8.5, as follows:

DEFINITION 8.6. *Associated to any partition $\pi \in P(2s, 2s)$ is the linear map*

$$\varphi_\pi(e_{a_1 \dots a_s, c_1 \dots c_s}) = \sum_{b_1 \dots b_s} \sum_{d_1 \dots d_s} \delta_\pi \begin{pmatrix} a_1 & \dots & a_s & c_1 & \dots & c_s \\ b_1 & \dots & b_s & d_1 & \dots & d_s \end{pmatrix} e_{b_1 \dots b_s, d_1 \dots d_s}$$

obtained from T_π by contracting all the tensors, via the operation

$$e_{i_1} \otimes \dots \otimes e_{i_{2s}} \rightarrow e_{i_1 \dots i_s, i_{s+1} \dots i_{2s}}$$

with $\{e_1, \dots, e_N\}$ standing as usual for the standard basis of \mathbb{C}^N .

In relation with our Wishart matrix considerations, the point is that the above linear map φ_π can be viewed as a ‘‘block-modification’’ map, as follows:

$$\varphi_\pi : M_{N^s}(\mathbb{C}) \rightarrow M_{N^s}(\mathbb{C})$$

As an illustration, let us discuss the case $s = 1$. There are 15 partitions $\pi \in P(2, 2)$, and among them, the most ‘‘basic’’ are the 4 partitions $\pi \in P_{\text{even}}(2, 2)$. We have:

THEOREM 8.7. *The partitions $\pi \in P_{\text{even}}(2, 2)$ are as follows,*

$$\pi_1 = \begin{bmatrix} \circ & \bullet \\ \circ & \bullet \end{bmatrix}, \quad \pi_2 = \begin{bmatrix} \circ & \bullet \\ \bullet & \circ \end{bmatrix}, \quad \pi_3 = \begin{bmatrix} \circ & \circ \\ \bullet & \bullet \end{bmatrix}, \quad \pi_4 = \begin{bmatrix} \circ & \circ \\ \circ & \circ \end{bmatrix}$$

with the associated linear maps $\varphi_\pi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ being as follows,

$$\varphi_1(A) = A, \quad \varphi_2(A) = A^t, \quad \varphi_3(A) = \text{Tr}(A)1, \quad \varphi_4(A) = A^\delta$$

and the associated square matrices $\Lambda_\pi \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ being as follows,

$$\Lambda_{ab,cd}^1 = \delta_{ab}\delta_{cd}, \quad \Lambda_{ab,cd}^2 = \delta_{ad}\delta_{bc}, \quad \Lambda_{ab,cd}^3 = \delta_{ac}\delta_{bd}, \quad \Lambda_{ab,cd}^4 = \delta_{abcd}$$

producing the main examples of block-modified Wishart matrices, from Definition 8.2.

PROOF. This is something elementary, coming from the formula in Definition 8.6. Indeed, in the case $s = 1$, that we are interested in here, this formula becomes:

$$\varphi_\pi(e_{ac}) = \sum_{bd} \delta_\pi \begin{pmatrix} a & c \\ b & d \end{pmatrix} e_{bd}$$

Now in the case of the 4 partitions in the statement, such maps are given by:

$$\varphi_1(e_{ac}) = e_{ac}, \quad \varphi_2(e_{ac}) = e_{ca}, \quad \varphi_3(e_{ac}) = \delta_{ac} \sum_b e_{bb}, \quad \varphi_4(e_{ac}) = \delta_{ac} e_{aa}$$

Thus, we obtain the formulae in the statement. Regarding now the associated square matrices, appearing via $\Lambda_{ab,cd} = \varphi(e_{ac})_{bd}$, these are given by:

$$\Lambda_{ab,cd}^1 = \delta_{ab}\delta_{cd}, \quad \Lambda_{ab,cd}^2 = \delta_{ad}\delta_{bc}, \quad \Lambda_{ab,cd}^3 = \delta_{ac}\delta_{bd}, \quad \Lambda_{ab,cd}^4 = \delta_{abcd}$$

Thus, we are led to the conclusions in the statement. \square

As a conclusion so far to what we did in this chapter, we have a nice definition for the block-modified Wishart matrices, and then a fine-tuning of this definition, using easy maps, which in the simplest case, that of the 4 partitions $\pi \in P_{\text{even}}(2, 2)$, produces the main 4 examples of block-modified Wishart matrices. The idea in what follows will be that of doing the combinatorics, a bit as in chapter 7, as to extend the results there.

8b. Asymptotic moments

Moving ahead now, we would first like to study the distribution of the arbitrary block-modified Wishart matrices $\widetilde{W} = (id \otimes \varphi)W$. We will use as before the moment method. However, things will be more tricky in the present setting, and we will need:

DEFINITION 8.8. *The generalized colored moments of a random matrix*

$$W \in M_N(L^\infty(X))$$

with respect to a colored integer $e = e_1 \dots e_p$, and a permutation $\sigma \in S_p$, are the numbers

$$M_e^\sigma(W) = \frac{1}{N^{|\sigma|}} E \left(\sum_{i_1, \dots, i_p} W_{i_1 i_{\sigma(1)}}^{e_1} \dots W_{i_p i_{\sigma(p)}}^{e_p} \right)$$

where $|\sigma|$ is the number of cycles of σ .

This is something quite technical, in the spirit of the free probability and free cumulant work in [71], that we will need in what follows. In order to understand how these generalized moments work, consider the standard cycle in S_p , namely:

$$\gamma = (1 \rightarrow 2 \rightarrow \dots \rightarrow p \rightarrow 1)$$

If we use this cycle $\gamma \in S_p$ as our permutation $\sigma \in S_p$ in the above definition, the corresponding generalized moment of a random matrix W is then the usual moment:

$$\begin{aligned} M_e^\gamma(W) &= \frac{1}{N} E \left(\sum_{i_1, \dots, i_p} W_{i_1 i_2}^{e_1} \dots W_{i_p i_1}^{e_p} \right) \\ &= (E \circ \text{tr})(W^{e_1} \dots W^{e_p}) \end{aligned}$$

In general, we can decompose the computation of $M_e^\sigma(W)$ over the cycles of σ , and we obtain in this way a certain product of moments of W . See [71].

As a second illustration now, in relation with the usual square matrices, and more specifically with the square matrices $\Lambda \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ as in Proposition 8.3, we have the following formula, that we will use many times in what follows:

PROPOSITION 8.9. *Given a usual square matrix, of composed size,*

$$\Lambda \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$$

we have the following generalized moment formula,

$$(M_e^\sigma \otimes M_e^\tau)(\Lambda) = \frac{1}{n^{|\sigma|+|\tau|}} \sum_{i_1, \dots, i_p} \sum_{j_1, \dots, j_p} \Lambda_{i_1 j_1, i_{\sigma(1)} j_{\tau(1)}}^{e_1} \cdots \cdots \Lambda_{i_p j_p, i_{\sigma(p)} j_{\tau(p)}}^{e_p}$$

valid for any two permutations $\sigma, \tau \in S_p$, and any colored integer $e = e_1 \dots e_p$.

PROOF. This is something obvious, applying the construction in Definition 8.8 with $N = n^2$, $X = \{.\}$, $W = \Lambda$, and then making a tensor product of the corresponding moments M_e^σ , M_e^τ , regarded as linear functionals on $M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$. \square

Consider now the embedding $NC(p) \subset S_p$ obtained by “cycling inside each block”. That is, each block $b = \{b_1, \dots, b_k\}$ with $b_1 < \dots < b_k$ of a given noncrossing partition $\sigma \in NC(p)$ produces by definition the cycle $(b_1 \dots b_k)$ of the corresponding permutation $\sigma \in S_p$. Observe that the one-block partition $\gamma \in NC(p)$ corresponds in this way to the standard cycle $\gamma \in S_p$. Also, the number of blocks $|\sigma|$ of a partition $\sigma \in NC(p)$ corresponds to the number of cycles $|\sigma|$ of the corresponding permutation $\sigma \in S_p$.

With these conventions, we have the following result, from [19], [20], generalizing our various Wishart matrix moment computations, that we did so far in this book:

THEOREM 8.10. *The asymptotic moments of a block-modified Wishart matrix*

$$\widetilde{W} = (id \otimes \varphi)W$$

with parameters $d, m, n \in \mathbb{N}$ as before, are given by the formula

$$\lim_{d \rightarrow \infty} M_e \left(\frac{\widetilde{W}}{d} \right) = \sum_{\sigma \in NC(p)} (mn)^{|\sigma|} (M_e^\sigma \otimes M_e^\sigma)(\Lambda)$$

where $\Lambda \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ is the square matrix associated to $\varphi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$.

PROOF. We use the formula for the matrix entries of \widetilde{W} , directly in terms of the matrix Λ associated to the map φ , from Proposition 8.4, namely:

$$\widetilde{W}_{ia,jb} = \sum_{cd} \Lambda_{ca,db} W_{ic,jd}$$

By conjugating this formula, we obtain the following formula for the entries of the adjoint matrix \widetilde{W}^* , that we will need as well, in what follows:

$$\widetilde{W}_{ia,jb}^* = \sum_{cd} \bar{\Lambda}_{db,ca} \bar{W}_{jd,ic} = \sum_{cd} \Lambda_{ca,db}^* W_{ic,jd}$$

Thus, we have the following global formula, valid for any exponent $e \in \{1, *\}$:

$$\widetilde{W}_{ia,jb}^e = \sum_{cd} \Lambda_{ca,db}^e W_{ic,jd}$$

In order to compute the moments of \widetilde{W} , observe first that we have:

$$\begin{aligned} \text{tr}(\widetilde{W}^{e_1} \dots \widetilde{W}^{e_p}) &= \frac{1}{dn} \sum_{i_r a_r} \prod_s \widetilde{W}_{i_s a_s, i_{s+1} a_{s+1}}^{e_s} \\ &= \frac{1}{dn} \sum_{i_r a_r c_r d_r} \prod_s \Lambda_{c_s a_s, d_s a_{s+1}}^{e_s} W_{i_s c_s, i_{s+1} d_s} \\ &= \frac{1}{dn} \sum_{i_r a_r c_r d_r j_r b_r} \prod_s \Lambda_{c_s a_s, d_s a_{s+1}}^{e_s} Y_{i_s c_s, j_s b_s} \bar{Y}_{i_{s+1} d_s, j_s b_s} \end{aligned}$$

The average of the general term can be computed by the Wick rule, which gives:

$$E \left(\prod_s Y_{i_s c_s, j_s b_s} \bar{Y}_{i_{s+1} d_s, j_s b_s} \right) = \# \left\{ \sigma \in S_p \mid i_{\sigma(s)} = i_{s+1}, c_{\sigma(s)} = d_s, j_{\sigma(s)} = j_s, b_{\sigma(s)} = b_s \right\}$$

Let us look now at the above sum. The i, j, b indices range over sets having respectively d, d, m elements, and they have to be constant under the action of $\sigma\gamma^{-1}, \sigma, \sigma$. Thus when summing over these i, j, b indices we simply obtain a factor as follows:

$$f = d^{|\sigma\gamma^{-1}|} d^{|\sigma|} m^{|\sigma|}$$

Thus, we obtain the following moment formula:

$$(E \circ \text{tr})(\widetilde{W}^{e_1} \dots \widetilde{W}^{e_p}) = \frac{1}{dn} \sum_{\sigma \in S_p} d^{|\sigma\gamma^{-1}|} (dm)^{|\sigma|} \sum_{a_r c_r} \prod_s \Lambda_{c_s a_s, c_{\sigma(s)} a_{s+1}}^{e_s}$$

On the other hand, we know from Proposition 8.9 that the generalized moments of the matrix $\Lambda \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ are given by the following formula:

$$(M_e^\sigma \otimes M_e^\tau)(\Lambda) = \frac{1}{n^{|\sigma|+|\tau|}} \sum_{i_1 \dots i_p} \sum_{j_1 \dots j_p} \Lambda_{i_1 j_1, i_{\sigma(1)} j_{\tau(1)}}^{e_1} \dots \Lambda_{i_p j_p, i_{\sigma(p)} j_{\tau(p)}}^{e_p}$$

By combining the above two formulae, we obtain the following moment formula:

$$(E \circ \text{tr})(\widetilde{W}^{e_1} \dots \widetilde{W}^{e_p}) = \sum_{\sigma \in S_p} d^{|\sigma|+|\sigma\gamma^{-1}|-1} (mn)^{|\sigma|} (M_e^\sigma \otimes M_e^\tau)(\Lambda)$$

We use now the standard fact, that we know well from before, that for $\sigma \in S_p$ we have an inequality as follows, with equality precisely when $\sigma \in NC(p)$:

$$|\sigma| + |\sigma\gamma^{-1}| \leq p + 1$$

Thus with $d \rightarrow \infty$ the sum restricts over the partitions $\sigma \in NC(p)$, and we get:

$$\lim_{d \rightarrow \infty} M_e(\widetilde{W}) = d^p \sum_{\sigma \in NC(p)} (mn)^{|\sigma|} (M_e^\sigma \otimes M_e^\gamma)(\Lambda)$$

Thus, we are led to the conclusion in the statement. \square

With the above result in hand, we are left with the question of recovering the asymptotic law of $\widetilde{W} = (id \otimes \varphi)W$, out of the asymptotic moments found there. The question here only involves the matrix $\Lambda \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$, and to be more precise, given such a matrix, we would like to find the real or complex probability measure, or abstract distribution, having as colored moments the following numbers:

$$M_e = \sum_{\sigma \in NC_p} (mn)^{|\sigma|} (M_e^\sigma \otimes M_e^\gamma)(\Lambda)$$

Although this is basically a linear algebra problem, the underlying linear algebra is of quite difficult type, and this question cannot really be solved, in general. We will see however that this question can be solved for our basic examples, coming from Theorem 8.7, and more generally, for a certain joint generalization of all these examples.

8c. Basic computations

Once again by following [19], [20], let us introduce, as a solution to the questions mentioned above, the following technical notion:

DEFINITION 8.11. *We call a square matrix $\Lambda \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ multiplicative when*

$$(M_e^\sigma \otimes M_e^\gamma)(\Lambda) = (M_e^\sigma \otimes M_e^\sigma)(\Lambda)$$

*holds for any $p \in \mathbb{N}$, any exponents $e_1, \dots, e_p \in \{1, *\}$, and any $\sigma \in NC(p)$.*

This notion is something quite technical, but we will see many examples in what follows. For instance, the square matrices Λ coming from the basic linear maps φ appearing in Definition 8.2 are all multiplicative. More on this later.

Regarding now the output measure, that we want to compute, this can only appear as some kind of modification of the Marchenko-Pastur law π_t . In order to discuss such modifications, recall from chapter 7 the following key formula:

$$R_{\pi_t}(\xi) = \frac{t}{1 - \xi}$$

To be more precise, this is something that we used in chapter 7, when dealing with the block-transposed Wishart matrices. But this suggests formulating:

DEFINITION 8.12. A measure μ having as R -transform a function of type

$$R_\mu(\xi) = \sum_{i=1}^s \frac{c_i z_i}{1 - \xi z_i}$$

with $c_i > 0$ and $z_i \in \mathbb{R}$, will be called *modified Marchenko-Pastur law*.

All this might seem a bit mysterious, but we are into difficult mathematics here, so we will use the above notion as stated, and we will understand later what is behind our computations. By anticipating a bit, however, the situation is as follows:

(1) As a first comment on the above notion, there is an obvious similarity here with the theory of the compound Poisson laws from chapter 2.

(2) The truth is that π_t is the free Poisson law of parameter t , and the modified Marchenko-Pastur laws introduced above are the general compound free Poisson laws.

(3) Also, the mysterious R -transform used above is the Voiculescu R -transform [86], which is the analogue of the log of the Fourier transform in free probability.

More on all this later, in chapters 9-12 below, when systematically doing free probability. Based on this analogy, however, we can label our modified Marchenko-Pastur laws, in the same way as we labelled in chapter 2 the compound Poisson laws, as follows:

DEFINITION 8.13. We denote by π_ρ the modified Marchenko-Pastur law satisfying

$$R_\mu(\xi) = \sum_{i=1}^s \frac{c_i z_i}{1 - \xi z_i}$$

with $c_i > 0$ and $z_i \in \mathbb{R}$, with ρ being the following measure,

$$\rho = \sum_{i=1}^s c_i \delta_{z_i}$$

which is a discrete positive measure in the complex plane, not necessarily of mass 1.

Getting back now to the block-modified Wishart matrices, and to the formula in Theorem 8.10, the above abstract notions, from Definition 8.11 and from Definition 8.12, are exactly what we need for further improving all this. Again by following [19], [20], we have the following result, substantially building on Theorem 8.10:

THEOREM 8.14. Consider a block-modified Wishart matrix

$$\widetilde{W} = (id \otimes \varphi)W$$

and assume that the matrix $\Lambda \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ associated to φ is multiplicative. Then

$$\frac{\widetilde{W}}{d} \sim \pi_{mn\rho}$$

holds, in moments, in the $d \rightarrow \infty$ limit, where $\rho = \text{law}(\Lambda)$.

PROOF. This is something quite tricky, using all the above:

(1) Our starting point is the asymptotic moment formula found in Theorem 8.10, for an arbitrary block-modified Wishart matrix, namely:

$$\lim_{d \rightarrow \infty} M_e \left(\frac{\widetilde{W}}{d} \right) = \sum_{\sigma \in NC_p} (mn)^{|\sigma|} (M_e^\sigma \otimes M_e^\sigma)(\Lambda)$$

(2) Since our modification matrix $\Lambda \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ was assumed to be multiplicative, in the sense of Definition 8.11, this formula reads:

$$\lim_{d \rightarrow \infty} M_e \left(\frac{\widetilde{W}}{d} \right) = \sum_{\sigma \in NC_p} (mn)^{|\sigma|} (M_e^\sigma \otimes M_e^\sigma)(\Lambda)$$

(3) On the other hand, a bit of calculus and combinatorics show that, in the context of Definition 8.12, given a square matrix $\Lambda \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$, having distribution $\rho = \text{law}(\Lambda)$, the moments of the modified Marchenko-Pastur law $\pi_{mn\rho}$ are given by the following formula, for any choice of the extra parameter $m \in \mathbb{N}$:

$$M_e(\pi_{mn\rho}) = \sum_{\sigma \in NC_p} (mn)^{|\sigma|} (M_e^\sigma \otimes M_e^\sigma)(\Lambda)$$

(4) The point now is that with this latter formula in hand, our previous asymptotic moment formula for the block-modified Wishart matrix \widetilde{W} simply reads:

$$\lim_{d \rightarrow \infty} M_e \left(\frac{\widetilde{W}}{d} \right) = M_e(\pi_{mn\rho})$$

Thus we have indeed $\widetilde{W}/d \sim \pi_{mn\rho}$, in the $d \rightarrow \infty$ limit, as stated. \square

All the above was of course a bit technical, but we will come back later to this, with some further details, once we will have a better understanding of the R -transform, of the free Poisson limit theorem, and of the other things which are hidden in all the above. In any case, welcome to free probability. Or perhaps to theoretical physics. The above theorem was our first free probability one, in this book, and many other to follow.

Let us we work out now some explicit consequences of Theorem 8.14, by using the modified easy linear maps from Definition 8.6. We recall from there that any modified easy linear map φ_π can be viewed as a “block-modification” map, as follows:

$$\varphi_\pi : M_{N^s}(\mathbb{C}) \rightarrow M_{N^s}(\mathbb{C})$$

In order to verify that the corresponding matrices Λ_π are multiplicative, we will need to check that all the functions $\varphi(\sigma, \tau) = (M_e^\sigma \otimes M_e^\tau)(\Lambda_\pi)$ have the following property:

$$\varphi(\sigma, \gamma) = \varphi(\sigma, \sigma)$$

For this purpose, we can use the following result, coming from [20]:

PROPOSITION 8.15. *The following functions $\varphi : NC(p) \times NC(p) \rightarrow \mathbb{R}$ are multiplicative, in the sense that they satisfy the condition $\varphi(\sigma, \gamma) = \varphi(\sigma, \sigma)$:*

- (1) $\varphi(\sigma, \tau) = |\sigma\tau^{-1}| - |\tau|$.
- (2) $\varphi(\sigma, \tau) = |\sigma\tau| - |\tau|$.
- (3) $\varphi(\sigma, \tau) = |\sigma \wedge \tau| - |\tau|$.

PROOF. All this is elementary, and can be proved as follows:

(1) This follows indeed from the following computation:

$$\varphi_1(\sigma, \gamma) = |\sigma\gamma^{-1}| - 1 = p - |\sigma| = \varphi_1(\sigma, \sigma)$$

(2) This follows indeed from the following computation:

$$\varphi_2(\sigma, \gamma) = |\sigma\gamma| - 1 = |\sigma^2| - |\sigma| = \varphi_2(\sigma, \sigma)$$

(3) This follows indeed from the following computation:

$$\varphi_3(\sigma, \gamma) = |\gamma| - |\gamma| = 0 = |\sigma| - |\sigma| = \varphi_3(\sigma, \sigma)$$

Thus, we are led to the conclusions in the statement. \square

We can get back now to the easy modification maps, and we have:

PROPOSITION 8.16. *The partitions $\pi \in P_{\text{even}}(2, 2)$ are as follows,*

$$\pi_1 = \begin{bmatrix} \circ & \bullet \\ \circ & \bullet \end{bmatrix}, \quad \pi_2 = \begin{bmatrix} \circ & \bullet \\ \bullet & \circ \end{bmatrix}, \quad \pi_3 = \begin{bmatrix} \circ & \circ \\ \bullet & \bullet \end{bmatrix}, \quad \pi_4 = \begin{bmatrix} \circ & \circ \\ \circ & \circ \end{bmatrix}$$

with the associated linear maps $\varphi_\pi : M_n(\mathbb{C}) \rightarrow M_N(\mathbb{C})$ being as follows:

$$\varphi_1(A) = A \quad , \quad \varphi_2(A) = A^t \quad , \quad \varphi_3(A) = \text{Tr}(A)1 \quad , \quad \varphi_4(A) = A^\delta$$

The corresponding matrices Λ_π are all multiplicative, in the sense of Definition 8.11.

PROOF. The first part of the statement is something that we already know, from Theorem 8.7. In order to prove the last assertion, recall from Theorem 8.7 that the associated square matrices, appearing via $\Lambda_{ab,cd} = \varphi(e_{ac})_{bd}$, are given by:

$$\Lambda_{ab,cd}^1 = \delta_{ab}\delta_{cd} \quad , \quad \Lambda_{ab,cd}^2 = \delta_{ad}\delta_{bc} \quad , \quad \Lambda_{ab,cd}^3 = \delta_{ac}\delta_{bd} \quad , \quad \Lambda_{ab,cd}^4 = \delta_{abcd}$$

Since these matrices are all self-adjoint, we can assume that all the exponents are 1 in Definition 8.11, and the multiplicativity condition there becomes:

$$(M_\sigma \otimes M_\gamma)(\Lambda) = (M_\sigma \otimes M_\sigma)(\Lambda)$$

In order to check this condition, observe that for the above 4 matrices, we have:

$$\begin{aligned}
(M^\sigma \otimes M^\tau)(\Lambda_1) &= \frac{1}{n^{|\sigma|+|\tau|}} \sum_{i_1 \dots i_p} \delta_{i_{\sigma(1)} i_{\tau(1)}} \dots \delta_{i_{\sigma(p)} i_{\tau(p)}} = n^{|\sigma\tau^{-1}|-|\sigma|-|\tau|} \\
(M^\sigma \otimes M^\tau)(\Lambda_2) &= \frac{1}{n^{|\sigma|+|\tau|}} \sum_{i_1 \dots i_p} \delta_{i_1 i_{\sigma\tau(1)}} \dots \delta_{i_p i_{\sigma\tau(p)}} = n^{|\sigma\tau|-|\sigma|-|\tau|} \\
(M^\sigma \otimes M^\tau)(\Lambda_3) &= \frac{1}{n^{|\sigma|+|\tau|}} \sum_{i_1 \dots i_p} \sum_{j_1 \dots j_p} \delta_{i_1 i_{\sigma(1)}} \delta_{j_1 j_{\tau(1)}} \dots \delta_{i_p i_{\sigma(p)}} \delta_{j_p j_{\tau(p)}} = 1 \\
(M^\sigma \otimes M^\tau)(\Lambda_4) &= \frac{1}{n^{|\sigma|+|\tau|}} \sum_{i_1 \dots i_p} \delta_{i_1 i_{\sigma(1)} i_{\tau(1)}} \dots \delta_{i_p i_{\sigma(p)} i_{\tau(p)}} = n^{|\sigma \wedge \tau|-|\sigma|-|\tau|}
\end{aligned}$$

By using now the results in Proposition 8.15, this gives the result. \square

Summarizing, the partitions $\pi \in P_{\text{even}}(2, 2)$ provide us with some concrete input for Theorem 8.14. The point now is that, when using this input, we obtain the main known computations for the block-modified Wishart matrices, from [5], [33], [34], [65]:

THEOREM 8.17. *The asymptotic distribution results for the block-modified Wishart matrices coming from the partitions $\pi_1, \pi_2, \pi_3, \pi_4 \in P_{\text{even}}(2, 2)$ are as follows:*

- (1) *Marchenko-Pastur:* $\frac{1}{d}W \sim \pi_t$, where $t = m/n$.
- (2) *Aubrun type:* $\frac{1}{d}(id \otimes t)W \sim \pi_\nu$, with $\nu = \frac{m(n-1)}{2}\delta_{-1} + \frac{m(n+1)}{2}\delta_1$.
- (3) *Collins-Nechita one:* $n(id \otimes tr(\cdot)1)W \sim \pi_t$, where $t = mn$.
- (4) *Collins-Nechita two:* $\frac{1}{d}(id \otimes (\cdot)^\delta)W \sim \pi_m$.

PROOF. All these results follow from Theorem 8.14, with the maps $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ in Proposition 8.16 producing the 4 matrices in the statement, modulo some rescalings, and with the computation of the corresponding distributions being as follows:

(1) Here $\Lambda = \sum_{ac} e_{ac} \otimes e_{ac}$, and so $\Lambda = nP$, where P is the rank one projection on $\sum_a e_a \otimes e_a \in \mathbb{C}^n \otimes \mathbb{C}^n$. Thus we have the following formula, which gives the result:

$$\rho = \frac{n^2 - 1}{n^2} \delta_0 + \frac{1}{n^2} \delta_n$$

(2) Here $\Lambda = \sum_{ac} e_{ac} \otimes e_{ca}$ is the flip operator, $\Lambda(e_c \otimes e_a) = e_a \otimes e_c$. Thus $\rho = \frac{n-1}{2n} \delta_{-1} + \frac{n+1}{2n} \delta_1$, and so we have the following formula, which gives the result:

$$mn\rho = \frac{m(n-1)}{2} \delta_{-1} + \frac{m(n+1)}{2} \delta_1$$

(3) Here $\Lambda = \sum_{ab} e_{aa} \otimes e_{bb}$ is the identity matrix, $\Lambda = 1$. Thus in this case we have the following formula, which gives $\pi_{mn\rho} = \pi_{mn}$, and so $n\widetilde{W} \sim \pi_{mn}$, as claimed:

$$\rho = \delta_1$$

(4) Here $\Lambda = \sum_a e_{aa} \otimes e_{aa}$ is the orthogonal projection on $\text{span}(e_a \otimes e_a) \subset \mathbb{C}^n \otimes \mathbb{C}^n$. Thus we have the following formula, which gives the result:

$$\rho = \frac{n-1}{n}\delta_0 + \frac{1}{n}\delta_1$$

Summarizing, we have proved all the assertions in the statement. \square

8d. Further results

We develop now some general theory, for the partitions $\pi \in P_{\text{even}}(2s, 2s)$, with $s \in \mathbb{N}$. Let us begin with a reformulation of Definition 8.6, in terms of square matrices:

PROPOSITION 8.18. *Given $\pi \in P(2s, 2s)$, the square matrix $\Lambda_\pi \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ associated to the linear map $\varphi_\pi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$, with $n = N^s$, is given by:*

$$(\Lambda_\pi)_{a_1 \dots a_s, b_1 \dots b_s, c_1 \dots c_s, d_1 \dots d_s} = \delta_\pi \begin{pmatrix} a_1 & \dots & a_s & c_1 & \dots & c_s \\ b_1 & \dots & b_s & d_1 & \dots & d_s \end{pmatrix}$$

In addition, we have $\Lambda_\pi^* = \Lambda_{\pi^\circ}$, where $\pi \rightarrow \pi^\circ$ is the blockwise middle symmetry.

PROOF. The formula for Λ_π follows from the formula of φ_π from Definition 8.6, by using our standard convention $\Lambda_{ab,cd} = \varphi(e_{ac})_{bd}$. Regarding now the second assertion, observe that with $\pi \rightarrow \pi^\circ$ being as above, for any multi-indices a, b, c, d we have:

$$\delta_\pi \begin{pmatrix} c_1 & \dots & c_s & a_1 & \dots & a_s \\ d_1 & \dots & d_s & b_1 & \dots & b_s \end{pmatrix} = \delta_{\pi^\circ} \begin{pmatrix} a_1 & \dots & a_s & c_1 & \dots & c_s \\ b_1 & \dots & b_s & d_1 & \dots & d_s \end{pmatrix}$$

Since Λ_π is real, we conclude we have the following formula:

$$(\Lambda_\pi^*)_{ab,cd} = (\Lambda_\pi)_{cd,ab} = (\Lambda_{\pi^\circ})_{ab,cd}$$

This being true for any a, b, c, d , we obtain $\Lambda_\pi^* = \Lambda_{\pi^\circ}$, as claimed. \square

In order to compute now the generalized $*$ -moments of Λ_π , we first have:

PROPOSITION 8.19. *With $\pi \in P(2s, 2s)$ and Λ_π being as above, we have*

$$(M_\sigma^e \otimes M_\tau^e)(\Lambda_\pi) = \frac{1}{n^{|\sigma|+|\tau|}} \sum_{i_1^1 \dots i_p^s} \sum_{j_1^1 \dots j_p^s} \delta_{\pi^{e_1}} \begin{pmatrix} i_1^1 & \dots & i_1^s & i_{\sigma(1)}^1 & \dots & i_{\sigma(1)}^s \\ j_1^1 & \dots & j_1^s & j_{\tau(1)}^1 & \dots & j_{\tau(1)}^s \end{pmatrix} \\ \vdots \\ \delta_{\pi^{e_p}} \begin{pmatrix} i_p^1 & \dots & i_p^s & i_{\sigma(p)}^1 & \dots & i_{\sigma(p)}^s \\ j_p^1 & \dots & j_p^s & j_{\tau(p)}^1 & \dots & j_{\tau(p)}^s \end{pmatrix}$$

with the exponents $e_1, \dots, e_p \in \{1, *\}$ at left corresponding to $e_1, \dots, e_p \in \{1, \circ\}$ at right.

PROOF. In multi-index notation, the general formula for the generalized *-moments for a tensor product square matrix $\Lambda \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$, with $n = N^s$, is:

$$(M_\sigma^e \otimes M_\tau^e)(\Lambda) = \frac{1}{n^{|\sigma|+|\tau|}} \sum_{i_1^1 \dots i_p^s} \sum_{j_1^1 \dots j_p^s} \Lambda_{i_1^1 \dots i_p^s j_1^1 \dots j_p^s}^{e_1} \dots \Lambda_{i_{\sigma(1)}^s \dots i_{\sigma(1)}^s j_{\tau(1)}^1 \dots j_{\tau(1)}^s}^{e_1} \\ \vdots \\ \Lambda_{i_p^1 \dots i_p^s j_p^1 \dots j_p^s}^{e_p} \dots \Lambda_{i_{\sigma(p)}^1 \dots i_{\sigma(p)}^s j_{\tau(p)}^1 \dots j_{\tau(p)}^s}^{e_p}$$

By using now the formulae in Proposition 8.3 for the matrix entries of Λ_π , and of its adjoint matrix $\Lambda_\pi^* = \Lambda_{\pi^\circ}$, this gives the formula in the statement. \square

As a conclusion, the quantities $(M_\sigma^e \otimes M_\tau^e)(\Lambda_\pi)$ that we are interested in can be theoretically computed in terms of π , but the combinatorics is quite non-trivial. As explained in [20], some simplifications appear in the symmetric case, $\pi = \pi^\circ$. Indeed, for such partitions we can use the following decomposition result:

PROPOSITION 8.20. *Each symmetric partition $\pi \in P_{\text{even}}(2s, 2s)$ has a finest symmetric decomposition $\pi = [\pi_1, \dots, \pi_R]$, with the components π_t being of two types, as follows:*

- (1) *Symmetric blocks of π . Such a block must have $r + r$ matching upper legs and $v + v$ matching lower legs, with $r + v > 0$.*
- (2) *Unions $\beta \sqcup \beta^\circ$ of asymmetric blocks of π . Here β must have $r + u$ unmatched upper legs and $v + w$ unmatched lower legs, with $r + u + v + w > 0$.*

PROOF. Consider indeed the block decomposition of our partition, $\pi = [\beta_1, \dots, \beta_T]$. Then $[\beta_1, \dots, \beta_T] = [\beta_1^\circ, \dots, \beta_T^\circ]$, so each block $\beta \in \pi$ is either symmetric, $\beta = \beta^\circ$, or is asymmetric, and disjoint from β° , which must be a block of π too. The result follows. \square

The idea will be that of decomposing over the components of π . First, we have:

PROPOSITION 8.21. *For the pairing $\eta \in P_{\text{even}}(2s, 2s)$ having horizontal strings,*

$$\eta = \begin{bmatrix} a & b & c & \dots & a & b & c & \dots \\ \alpha & \beta & \gamma & \dots & \alpha & \beta & \gamma & \dots \end{bmatrix}$$

we have $(M_\sigma \otimes M_\tau)(\Lambda_\eta) = 1$, for any $p \in \mathbb{N}$, and any $\sigma, \tau \in NC(p)$.

PROOF. As a first observation, the result holds at $s = 1$, due to the computations in the proof of Proposition 8.16. In general, by using Proposition 8.19, we obtain:

$$(M_\sigma \otimes M_\tau)(\Lambda_\eta) = \frac{1}{n^{|\sigma|+|\tau|}} \sum_{i_1^1 \dots i_p^s} \sum_{j_1^1 \dots j_p^s} \delta_{i_1^1 i_{\sigma(1)}^1} \dots \delta_{i_p^s i_{\sigma(p)}^s} \cdot \delta_{j_1^1 j_{\tau(1)}^1} \dots \delta_{j_p^s j_{\tau(p)}^s} \\ \vdots \\ \delta_{i_p^1 i_{\sigma(p)}^1} \dots \delta_{i_p^s i_{\sigma(p)}^s} \cdot \delta_{j_p^1 j_{\tau(p)}^1} \dots \delta_{j_p^s j_{\tau(p)}^s}$$

By transposing the two $p \times s$ matrices of Kronecker symbols, we obtain:

$$(M_\sigma \otimes M_\tau)(\Lambda_\eta) = \frac{1}{n^{|\sigma|+|\tau|}} \sum_{i_1^1 \dots i_p^1} \sum_{j_1^1 \dots j_p^1} \delta_{i_1^1 i_{\sigma(1)}^1} \dots \delta_{i_p^1 i_{\sigma(p)}^1} \cdot \delta_{j_1^1 j_{\tau(1)}^1} \dots \delta_{j_p^1 j_{\tau(p)}^1}$$

$$\vdots$$

$$\sum_{i_1^s \dots i_p^s} \sum_{j_1^s \dots j_p^s} \delta_{i_1^s i_{\sigma(1)}^s} \dots \delta_{i_p^s i_{\sigma(p)}^s} \cdot \delta_{j_1^s j_{\tau(1)}^s} \dots \delta_{j_p^s j_{\tau(p)}^s}$$

We can now perform all the sums, and we obtain in this way:

$$(M_\sigma \otimes M_\tau)(\Lambda_\eta) = \frac{1}{n^{|\sigma|+|\tau|}} (N^{|\sigma|} N^{|\tau|})^s = 1$$

Thus, the formula in the statement holds indeed. \square

We can now perform the decomposition over the components, as follows:

THEOREM 8.22. *Assuming that $\pi \in P_{\text{even}}(2s, 2s)$ is symmetric, $\pi = \pi^\circ$, we have*

$$(M_\sigma \otimes M_\tau)(\Lambda_\pi) = \prod_{t=1}^R (M_\sigma \otimes M_\tau)(\Lambda_{\pi_t})$$

whenever $\pi = [\pi_1, \dots, \pi_R]$ is a decomposition into symmetric subpartitions, which each π_t being completed with horizontal strings, coming from the standard pairing η .

PROOF. We use the general formula in Proposition 8.19. In the symmetric case the various e_x exponents disappear, and we can write the formula there as follows:

$$(M_\sigma \otimes M_\tau)(\Lambda_\pi) = \frac{1}{n^{|\sigma|+|\tau|}} \# \left\{ i, j \mid \ker \begin{pmatrix} i_x^1 & \dots & i_x^s & i_{\sigma(x)}^1 & \dots & i_{\sigma(x)}^s \\ j_x^1 & \dots & j_x^s & j_{\tau(x)}^1 & \dots & j_{\tau(x)}^s \end{pmatrix} \leq \pi, \forall x \right\}$$

The point now is that in this formula, the number of double arrays $[ij]$ that we are counting naturally decomposes over the subpartitions π_t . Thus, we have a formula of the following type, with K being a certain normalization constant:

$$(M_\sigma \otimes M_\tau)(\Lambda_\pi) = K \prod_{t=1}^R (M_\sigma \otimes M_\tau)(\Lambda_{\pi_t})$$

Regarding now the precise value of K , our claim is that this is given by:

$$K = \frac{n^{(|\sigma|+|\tau|)R}}{n^{|\sigma|+|\tau|}} \cdot \frac{1}{n^{(|\sigma|+|\tau|)(R-1)}} = 1$$

Indeed, the fraction on the left comes from the standard $\frac{1}{n^{|\sigma|+|\tau|}}$ normalizations of all the $(M_\sigma \otimes M_\tau)(\Lambda)$ quantities involved. As for the term on the right, this comes from the contribution of the horizontal strings, which altogether contribute as the strings of the standard pairing $\eta \in P_{\text{even}}(2s, 2s)$, counted $R - 1$ times. But, according to Proposition 8.21, the strings of η contribute with a $n^{|\sigma|+|\tau|}$ factor, and this gives the result. \square

Summarizing, in the easy case we are led to the study of the partitions $\pi \in P_{\text{even}}(2s, 2s)$ which are symmetric, and we have so far a decomposition formula for them.

Let us keep building on the material developed above. Our purpose will be that of converting Theorem 8.22 into an explicit formula, that we can use later on. For this, we have to compute the contributions of the components. First, we have:

PROPOSITION 8.23. *For a symmetric partition $\pi \in P_{\text{even}}(2s, 2s)$, consisting of one symmetric block, completed with horizontal strings, we have*

$$(M_\sigma \otimes M_\tau)(\Lambda_\pi) = N^{|\lambda| - r|\sigma| - v|\tau|}$$

where $\lambda \in P(p)$ is a partition constructed as follows,

$$\lambda = \begin{cases} \sigma \wedge \tau & \text{if } r, v \geq 1 \\ \sigma & \text{if } r \geq 1, v = 0 \\ \tau & \text{if } r = 0, v \geq 1 \end{cases}$$

and where r/v is half of the number of upper/lower legs of the symmetric block.

PROOF. Let us denote by a_1, \dots, a_r and b_1, \dots, b_v the upper and lower legs of the symmetric block, appearing at left, and by A_1, \dots, A_{s-r} and B_1, \dots, B_{s-v} the remaining legs, appearing at left as well. With this convention, Proposition 8.19 gives:

$$(M_\sigma \otimes M_\tau)(\Lambda_\pi) = \frac{1}{n^{|\sigma|+|\tau|}} \sum_{i_1^1 \dots i_p^s} \sum_{j_1^1 \dots j_p^s} \prod_x \delta_{i_x^{a_1} \dots i_x^{a_r} i_{\sigma(x)}^{a_1} \dots i_{\sigma(x)}^{a_r} j_x^{b_1} \dots j_x^{b_v} j_{\tau(x)}^{b_1} \dots j_{\tau(x)}^{b_v}} \delta_{i_x^{A_1} i_{\sigma(x)}^{A_1} \dots \delta_{i_x^{A_{s-r}} i_{\sigma(x)}^{A_{s-r}}} \delta_{j_x^{B_1} j_{\tau(x)}^{B_1} \dots \delta_{j_x^{B_{s-v}} j_{\tau(x)}^{B_{s-v}}}$$

If we denote by k_1, \dots, k_p the common values of the indices affected by the long Kronecker symbols, coming from the symmetric block, we have then:

$$(M_\sigma \otimes M_\tau)(\Lambda_\pi) = \frac{1}{n^{|\sigma|+|\tau|}} \sum_{k_1 \dots k_p} \sum_{i_1^1 \dots i_p^s} \prod_x \delta_{i_x^{a_1} \dots i_x^{a_r} i_{\sigma(x)}^{a_1} \dots i_{\sigma(x)}^{a_r} k_x \cdot \delta_{i_x^{A_1} i_{\sigma(x)}^{A_1} \dots \delta_{i_x^{A_{s-r}} i_{\sigma(x)}^{A_{s-r}}} \sum_{j_1^1 \dots j_p^s} \prod_x \delta_{j_x^{b_1} \dots j_x^{b_v} j_{\tau(x)}^{b_1} \dots j_{\tau(x)}^{b_v} k_x \cdot \delta_{j_x^{B_1} j_{\tau(x)}^{B_1} \dots \delta_{j_x^{B_{s-v}} j_{\tau(x)}^{B_{s-v}}}$$

Let us compute now the contributions of the various i, j indices involved. If we regard both i, j as being $p \times s$ arrays of indices, the situation is as follows:

– On the a_1, \dots, a_r columns of i , the equations are $i_x^{a_e} = i_{\sigma(x)}^{a_e} = k_x$ for any e, x . Thus when $r \neq 0$ we must have $\ker k \leq \sigma$, in order to have solutions, and if this condition is

satisfied, the solution is unique. As for the case $r = 0$, here there is no special condition to be satisfied by k , and we have once again a unique solution.

– On the A_1, \dots, A_{s-r} columns of i , the conditions on the indices are the “trivial” ones, examined in the proof of Proposition 8.21. According to the computation there, the total contribution coming from these indices is $(N^{|\sigma|})^{s-r} = N^{(s-r)|\sigma|}$.

– Regarding now j , the situation is similar, with a unique solution coming from the b_1, \dots, b_v columns, provided that the condition $\ker k \leq \tau$ is satisfied at $v \neq 0$, and with a total $N^{(s-v)|\tau|}$ contribution coming from the B_1, \dots, B_{s-v} columns.

As a conclusion, in order to have solutions i, j , we are led to the condition $\ker k \leq \lambda$, where $\lambda \in \{\sigma \wedge \tau, \sigma, \tau\}$ is the partition constructed in the statement. Now by putting everything together, we deduce that we have the following formula:

$$\begin{aligned} (M_\sigma \otimes M_\tau)(\Lambda_\pi) &= \frac{1}{n^{|\sigma|+|\tau|}} \sum_{\ker k \leq \lambda} N^{(s-r)|\sigma|+(s-v)|\tau|} \\ &= N^{-s|\sigma|-s|\tau|} N^{|\lambda|} N^{(s-r)|\sigma|+(s-v)|\tau|} \\ &= N^{|\lambda|-r|\sigma|-v|\tau|} \end{aligned}$$

Thus, we have obtained the formula in the statement, and we are done. □

In the two-block case now, we have a similar result, as follows:

PROPOSITION 8.24. *For a symmetric partition $\pi \in P_{\text{even}}(2s, 2s)$, consisting of a symmetric union $\beta \sqcup \beta^\circ$ of two asymmetric blocks, completed with horizontal strings, we have*

$$(M_\sigma \otimes M_\tau)(\Lambda_\pi) = N^{|\lambda|-(r+u)|\sigma|-(v+w)|\tau|}$$

where $r + u$ and $v + w$ represent the number of upper and lower legs of β , and where $\lambda \in P(p)$ is a partition constructed according to the following table,

$ru \setminus vw$	11	10	01	00
11	$\sigma^2 \wedge \sigma\tau \wedge \sigma\tau^{-1}$	$\sigma^2 \wedge \sigma\tau^{-1}$	$\sigma^2 \wedge \sigma\tau$	σ^2
10	$\sigma\tau \wedge \sigma\tau^{-1}$	$\sigma\tau^{-1}$	$\sigma\tau$	\emptyset
01	$\tau\sigma \wedge \tau^2$	$\tau\sigma$	$\tau^{-1}\sigma$	\emptyset
00	τ^2	\emptyset	\emptyset	–

with the 1/0 indexing symbols standing for the positivity/nullness of the corresponding variables r, u, v, w , and where \emptyset denotes a formal partition, having 0 blocks.

PROOF. Let us denote by a_1, \dots, a_r and c_1, \dots, c_u the upper legs of β , by b_1, \dots, b_v and d_1, \dots, d_w the lower legs of β , and by A_1, \dots, A_{s-r-u} and B_1, \dots, B_{s-v-w} the remaining

legs of π , not belonging to $\beta \sqcup \beta^\circ$. The formula in Proposition 8.19 gives:

$$(M_\sigma \otimes M_\tau)(\Lambda_\pi) = \frac{1}{n^{|\sigma|+|\tau|}} \sum_{i_1^1 \dots i_p^s} \sum_{j_1^1 \dots j_p^s} \prod_x \delta_{i_x^{a_1} \dots i_x^{a_r} i_{\sigma(x)}^{c_1} \dots i_{\sigma(x)}^{c_u} j_x^{b_1} \dots j_x^{b_v} j_{\tau(x)}^{d_1} \dots j_{\tau(x)}^{d_w}} \delta_{i_x^{c_1} \dots i_x^{c_u} i_{\sigma(x)}^{a_1} \dots i_{\sigma(x)}^{a_r} j_x^{d_1} \dots j_x^{d_w} j_{\tau(x)}^{b_1} \dots j_{\tau(x)}^{b_v}} \delta_{i_x^{A_1} i_{\sigma(x)}^{A_1} \dots \dots \delta_{i_x^{A_{s-r}} i_{\sigma(x)}^{A_{s-r-u}}} \delta_{j_x^{B_1} j_{\tau(x)}^{B_1} \dots \dots \delta_{j_x^{B_{s-v}} j_{\tau(x)}^{B_{s-v-w}}}}$$

We have now two long Kronecker symbols, coming from $\beta \sqcup \beta^\circ$, and if we denote by k_1, \dots, k_p and l_1, \dots, l_p the values of the indices affected by them, we obtain:

$$(M_\sigma \otimes M_\tau)(\Lambda_\pi) = \frac{1}{n^{|\sigma|+|\tau|}} \sum_{k_1 \dots k_p} \sum_{l_1 \dots l_p} \sum_{i_1^1 \dots i_p^s} \prod_x \delta_{i_x^{a_1} \dots i_x^{a_r} i_{\sigma(x)}^{c_1} \dots i_{\sigma(x)}^{c_u} k_x} \cdot \delta_{i_x^{c_1} \dots i_x^{c_u} i_{\sigma(x)}^{a_1} \dots i_{\sigma(x)}^{a_r} l_x} \cdot \delta_{i_x^{A_1} i_{\sigma(x)}^{A_1} \dots \dots \delta_{i_x^{A_{s-r-u}} i_{\sigma(x)}^{A_{s-r-u}}} \sum_{j_1^1 \dots j_p^s} \prod_x \delta_{j_x^{b_1} \dots j_x^{b_v} j_{\tau(x)}^{d_1} \dots j_{\tau(x)}^{d_w} k_x} \cdot \delta_{j_x^{d_1} \dots j_x^{d_w} j_{\tau(x)}^{b_1} \dots j_{\tau(x)}^{b_v} l_x} \cdot \delta_{j_x^{B_1} j_{\tau(x)}^{B_1} \dots \dots \delta_{j_x^{B_{s-v-w}} j_{\tau(x)}^{B_{s-v-w}}}}$$

Let us compute now the contributions of the various i, j indices. On the a_1, \dots, a_r and c_1, \dots, c_u columns of i , regarded as an $p \times s$ array, the equations are as follows:

$$i_x^{a_e} = i_{\sigma(x)}^{c_f} = k_x \quad , \quad i_x^{c_f} = i_{\sigma(x)}^{a_e} = l_x$$

If we denote by i_x the common value of the $i_x^{a_e}$ indices, when e varies, and by I_x the common value of the $i_x^{c_f}$ indices, when f varies, these equations simply become:

$$i_x = I_{\sigma(x)} = k_x \quad , \quad I_x = i_{\sigma(x)} = l_x$$

Thus we have 0 or 1 solutions. To be more precise, depending now on the positivity/nullness of the parameters r, u , we are led to 4 cases, as follows:

Case 11. Here $r, u \geq 1$, and we must have $k_x = l_{\sigma(x)}, k_{\sigma(x)} = l_x$.

Case 10. Here $r \geq 1, u = 0$, and we must have $k_{\sigma(x)} = l_x$.

Case 01. Here $r = 0, u \geq 1$, and we must have $k_x = l_{\sigma(x)}$.

Case 00. Here $r = u = 0$, and there is no condition on k, l .

In what regards now the A_1, \dots, A_{s-r} columns of i , the conditions on the indices are the “trivial” ones, examined in the proof of Proposition 8.21. According to the computation there, the total contribution coming from these indices is:

$$C_i = (N^{|\sigma|})^{s-r} = N^{(s-r)|\sigma|}$$

The study for the j indices is similar, and we will only record here the final conclusions. First, in what regards the b_1, \dots, b_v and d_1, \dots, d_w columns of j , the same discussion as above applies, and we have once again 0 or 1 solutions, as follows:

Case 11'. Here $v, w \geq 1$, and we must have $k_x = l_{\tau(x)}, k_{\tau(x)} = l_x$.

Case 10'. Here $v \geq 1, w = 0$, and we must have $k_{\tau(x)} = l_x$.

Case 01'. Here $v = 0, w \geq 1$, and we must have $k_x = l_{\tau(x)}$.

Case 00'. Here $v = w = 0$, and there is no condition on k, l .

As for the B_1, \dots, B_{s-v-w} columns of j , the conditions on the indices here are “trivial”, as in Proposition 8.21, and the total contribution coming from these indices is:

$$C_j = (N^{|\tau|})^{s-v-w} = N^{(s-v-w)|\tau|}$$

Let us put now everything together. First, we must merge the conditions on k, l found in the cases 00-11 above with those found in the cases 00'-11'. There are $4 \times 4 = 16$ computations to be performed here, and the “generic” computation, corresponding to the merger of case 11 with the case 11', is as follows:

$$\begin{aligned} & k_x = l_{\sigma(x)}, k_{\sigma(x)} = l_x, k_x = l_{\tau(x)}, k_{\tau(x)} = l_x \\ \iff & l_x = k_{\sigma(x)}, k_x = l_{\sigma(x)}, k_x = l_{\tau(x)}, k_x = l_{\tau^{-1}(x)} \\ \iff & l_x = k_{\sigma(x)}, k_x = k_{\sigma^2(x)} = k_{\sigma\tau(x)} = k_{\sigma\tau^{-1}(x)} \end{aligned}$$

Thus in this case l is uniquely determined by k , and k itself must satisfy:

$$\ker k \leq \sigma^2 \wedge \sigma\tau \wedge \sigma\tau^{-1}$$

We conclude that the total contribution of the k, l indices in this case is:

$$C_{kl}^{11,11} = N^{|\sigma^2 \wedge \sigma\tau \wedge \sigma\tau^{-1}|}$$

In the remaining 15 cases the computations are similar, with some of the above 4 conditions, that we started with, disappearing. The conclusion is that the total contribution of the k, l indices is as follows, with λ being the partition in the statement:

$$C_{kl} = N^{|\lambda|}$$

With this result in hand, we can now finish our computation, as follows:

$$\begin{aligned} (M_\sigma \otimes M_\tau)(\Lambda_\pi) &= \frac{1}{n^{|\sigma|+|\tau|}} C_{kl} C_i C_j \\ &= N^{|\lambda| - (r+u)|\sigma| - (v+w)|\tau|} \end{aligned}$$

Thus, we have obtained the formula in the statement, and we are done. \square

As a conclusion now to all this, we have the following result:

THEOREM 8.25. *For a symmetric partition $\pi \in P_{\text{even}}(2s, 2s)$, having only one component, in the sense of Proposition 8.20, completed with horizontal strings, we have*

$$(M_\sigma \otimes M_\tau)(\Lambda_\pi) = N^{|\lambda| - r|\sigma| - v|\tau|}$$

where $\lambda \in P(p)$ is the partition constructed as in Proposition 8.23 and Proposition 8.24, and where r/v is half of the total number of upper/lower legs of the component.

PROOF. This follows indeed from Proposition 8.23 and Proposition 8.24. \square

Generally speaking, the formula that we found in Theorem 8.25 does not lead to the multiplicativity condition from Definition 8.11, and this due to the fact that the various partitions $\lambda \in P_p$ constructed in Proposition 8.24 have in general a quite complicated combinatorics. To be more precise, we first have the following result:

PROPOSITION 8.26. *For a symmetric partition $\pi \in P_{\text{even}}(2s, 2s)$ we have*

$$(M_\sigma \otimes M_\tau)(\Lambda_\pi) = N^{f_1 + f_2}$$

where f_1, f_2 are respectively linear combinations of the following quantities:

- (1) $1, |\sigma|, |\tau|, |\sigma \wedge \tau|, |\sigma\tau|, |\sigma\tau^{-1}|, |\tau\sigma|, |\tau^{-1}\sigma|.$
- (2) $|\sigma^2|, |\tau^2|, |\sigma^2 \wedge \sigma\tau|, |\sigma^2 \wedge \sigma\tau^{-1}|, |\tau\sigma \wedge \tau^2|, |\sigma\tau \wedge \sigma\tau^{-1}|, |\sigma^2 \wedge \sigma\tau \wedge \sigma\tau^{-1}|.$

PROOF. This follows indeed by combining Theorem 8.22 and Theorem 8.25, with concrete input from Proposition 8.23 and Proposition 8.24. \square

In the above result, the partitions in (1) lead to the multiplicativity condition in Definition 8.11, and so to compound free Poisson laws, via Theorem 8.14. However, the partitions in (2) have a more complicated combinatorics, which does not fit with Definition 8.11, nor with the finer multiplicativity notions introduced in [20].

Summarizing, in order to extend the 4 basic computations that we have, we must fine-tune our formalism. A natural answer here comes from the following result:

PROPOSITION 8.27. *For a partition $\pi \in P(2s, 2s)$, the following are equivalent:*

- (1) φ_π is unital modulo scalars, i.e. $\varphi_\pi(1) = c1$, with $c \in \mathbb{C}$.
- (2) $\begin{bmatrix} \mu \\ \pi \end{bmatrix} = \mu$, where $\mu \in P(0, 2s)$ is the pairing connecting $\{i\} - \{i + s\}$, and where $\begin{bmatrix} \mu \\ \pi \end{bmatrix} \in P(0, 2s)$ is the partition obtained by putting μ on top of π .

In addition, these conditions are satisfied for the 4 partitions in $P_{\text{even}}(2, 2)$.

PROOF. We use the formula of φ_π from Definition 8.6, namely:

$$\varphi_\pi(e_{a_1 \dots a_s, c_1 \dots c_s}) = \sum_{b_1 \dots b_s} \sum_{d_1 \dots d_s} \delta_\pi \begin{pmatrix} a_1 & \dots & a_s & c_1 & \dots & c_s \\ b_1 & \dots & b_s & d_1 & \dots & d_s \end{pmatrix} e_{b_1 \dots b_s, d_1 \dots d_s}$$

By summing over indices $a_i = c_i$, we obtain the following formula:

$$\varphi_\pi(1) = \sum_{a_1 \dots a_s} \sum_{b_1 \dots b_s} \sum_{d_1 \dots d_s} \delta_\pi \begin{pmatrix} a_1 & \dots & a_s & a_1 & \dots & a_s \\ b_1 & \dots & b_s & d_1 & \dots & d_s \end{pmatrix} e_{b_1 \dots b_s, d_1 \dots d_s}$$

Let us first find out when $\varphi_\pi(1)$ is diagonal. In order for this condition to hold, the off-diagonal terms of $\varphi_\pi(1)$ must all vanish, and so we must have:

$$b \neq d \implies \delta_\pi \begin{pmatrix} a_1 & \dots & a_s & a_1 & \dots & a_s \\ b_1 & \dots & b_s & d_1 & \dots & d_s \end{pmatrix} = 0, \forall a$$

Our claim is that for any $\pi \in P(2s, 2s)$ we have the following formula:

$$\sup_{a_1 \dots a_s} \delta_\pi \begin{pmatrix} a_1 & \dots & a_s & a_1 & \dots & a_s \\ b_1 & \dots & b_s & d_1 & \dots & d_s \end{pmatrix} = \delta_{[\frac{\mu}{\pi}]}(b_1 \dots b_s \ d_1 \dots d_s)$$

Indeed, each of the terms of the sup on the left are smaller than the quantity on the right, so \leq holds. Also, assuming $\delta_{[\frac{\mu}{\pi}]}(bd) = 1$, we can take a_1, \dots, a_s to be the indices appearing on the strings of μ , and we obtain the following formula:

$$\delta_\pi \begin{pmatrix} a & a \\ b & d \end{pmatrix} = 1$$

Thus, we have equality. Now with this equality in hand, we conclude that we have:

$$\begin{aligned} \varphi_\pi(1) &= \varphi_\pi(1)^\delta \\ \iff \delta_{[\frac{\mu}{\pi}]}(b_1 \dots b_s \ d_1 \dots d_s) &= 0, \forall b \neq d \\ \iff \delta_{[\frac{\mu}{\pi}]}(b_1 \dots b_s \ d_1 \dots d_s) &\leq \delta_\mu(b_1 \dots b_s \ d_1 \dots d_s), \forall b, d \\ \iff \left[\frac{\mu}{\pi} \right] &\leq \mu \end{aligned}$$

Let us investigate now when (1) holds. We already know that π must satisfy $[\frac{\mu}{\pi}] \leq \mu$, and the remaining conditions, concerning the diagonal terms, are as follows:

$$\sum_{a_1 \dots a_s} \delta_\pi \begin{pmatrix} a_1 & \dots & a_s & a_1 & \dots & a_s \\ b_1 & \dots & b_s & b_1 & \dots & b_s \end{pmatrix} = c, \forall b$$

As a first observation, the quantity on the left is a decreasing function of $\lambda = \ker b$. Now in order for this decreasing function to be constant, we must have:

$$\sum_{a_1 \dots a_s} \delta_\pi \begin{pmatrix} a_1 & \dots & a_s & a_1 & \dots & a_s \\ 1 & \dots & s & 1 & \dots & s \end{pmatrix} = \sum_{a_1 \dots a_s} \delta_\pi \begin{pmatrix} a_1 & \dots & a_s & a_1 & \dots & a_s \\ 1 & \dots & 1 & 1 & \dots & 1 \end{pmatrix}$$

We conclude that the condition $[\frac{\mu}{\pi}] \leq \mu$ must be strengthened into $[\frac{\mu}{\pi}] = \mu$, as claimed. Finally, the last assertion is clear, by using either (1) or (2). \square

In the symmetric case, $\pi = \pi^\circ$, we have the following result:

PROPOSITION 8.28. *Given a partition $\pi \in P(2s, 2s)$ which is symmetric, φ_π is unital modulo scalars precisely when its symmetric components are as follows,*

- (1) *Symmetric blocks with $v \leq 1$,*
- (2) *Unions of asymmetric blocks with $r + u = 0, v + w = 1$,*
- (3) *Unions of asymmetric blocks with $r + u \geq 1, v + w \leq 1$,*

with the conventions from Proposition 8.20 for the values of r, u, v, w .

PROOF. This follows from what we have, the idea being as follows:

– We know from Proposition 8.27 that the condition in the statement is equivalent to $[\frac{\mu}{\pi}] = \mu$, and we can see from this that π satisfies the condition if and only if all the symmetric components of π satisfy the condition. Thus, we must simply check the validity of $[\frac{\mu}{\pi}] = \mu$ for the partitions in Proposition 8.20, and this gives the result.

– To be more precise, for the 1-block components the study is trivial, and we are led to (1). Regarding the 2-block components, in the case $r + u = 0$ we must have $v + w = 1$, as stated in (2). Finally, assuming $r + u \geq 1$, when constructing $[\frac{\mu}{\pi}]$ all the legs on the bottom will become connected, and so we must have $v + w \leq 1$, as stated in (3). \square

Summarizing, the condition that φ_π is unital modulo scalars is a natural generalization of what happens for the 4 basic partitions in $P_{\text{even}}(2, 2)$, and in the symmetric case, we have a good understanding of such partitions. However, the associated matrices Λ_π still fail to be multiplicative, and we must come up with a second condition, coming from:

THEOREM 8.29. *If $\pi \in P(2s, 2s)$ is symmetric, the following are equivalent:*

- (1) *The linear maps $\varphi_\pi, \varphi_{\pi^*}$ are both unital modulo scalars.*
- (2) *The symmetric components have ≤ 2 upper legs, and ≤ 2 lower legs.*
- (3) *The symmetric components appear as copies of the 4 elements of $P_{\text{even}}(2, 2)$.*

PROOF. By applying the results in Proposition 8.28 to the partitions π, π^* , and by merging these results, we conclude that the equivalence (1) \iff (2) holds indeed. As for the equivalence (2) \iff (3), this is clear from definitions. \square

Let us put now everything together. The idea will be that of using the partitions found in Theorem 8.29 as an input for Proposition 8.26, and then for the general block-modification machinery developed in the beginning of this chapter. We will need:

PROPOSITION 8.30. *The following functions $\varphi : NC(p) \times NC(p) \rightarrow \mathbb{R}$ are multiplicative, in the sense that they satisfy the condition $\varphi(\sigma, \gamma) = \varphi(\sigma, \sigma)$:*

- (1) $\varphi(\sigma, \tau) = |\tau\sigma| - |\tau|.$
- (2) $\varphi(\sigma, \tau) = |\tau^{-1}\sigma| - |\tau|.$

PROOF. This follows from some standard combinatorics, the idea being as follows:

(1) We can use here the well-known fact, explained in chapter 7, that the numbers $|\gamma\sigma| - 1$ and $|\sigma^2| - |\sigma|$ are equal, both counting the number of blocks of σ having even size. Thus we have the following computation, which gives the result:

$$\varphi_1(\sigma, \gamma) = |\gamma\sigma| - 1 = |\sigma^2| - |\sigma| = \varphi_1(\sigma, \sigma)$$

(2) Here we can use the well-known formula $|\sigma\gamma^{-1}| - 1 = p - |\sigma|$, and the fact that $\sigma\gamma^{-1}, \gamma^{-1}\sigma$ have the same cycle structure as the left and right Kreweras complements of σ , and so have the same number of blocks. Thus we have the following computation:

$$\varphi_2(\sigma, \gamma) = |\gamma^{-1}\sigma| - 1 = p - |\sigma| = \varphi_2(\sigma, \sigma)$$

But this gives the second formula in the statement, and we are done. □

We can now formulate our main multiplicativity result, as follows:

PROPOSITION 8.31. *Assuming that $\pi \in P_{\text{even}}(2s, 2s)$ is symmetric, $\pi = \pi^\circ$, and is such that $\varphi_\pi, \varphi_{\pi^*}$ are unital modulo scalars, we have a formula of the following type:*

$$(M_\sigma \otimes M_\tau)(\Lambda_\pi) = N^{a+b|\sigma|+c|\tau|+d|\sigma\wedge\tau|+e|\sigma\tau|+f|\sigma\tau^{-1}|+g|\tau\sigma|+h|\tau^{-1}\sigma|}$$

Moreover, the square matrix Λ_π is multiplicative, in the sense of Definition 8.11.

PROOF. The first assertion follows from Proposition 8.26. Indeed, according to the various results in Theorem 8.29, the list of partitions appearing in Proposition 8.26 (2) disappears in the case where both $\varphi_\pi, \varphi_{\pi^*}$ are unital modulo scalars, and this gives the result. As for the second assertion, this follows from the formula in the statement, and from the various results in Proposition 8.15 and Proposition 8.30. □

As a main consequence, Theorem 8.14 applies, and gives:

THEOREM 8.32. *Given a partition $\pi \in P_{\text{even}}(2s, 2s)$ which is symmetric, $\pi = \pi^\circ$, and which is such that $\varphi_\pi, \varphi_{\pi^*}$ are unital modulo scalars, for the corresponding block-modified Wishart matrix $\widetilde{W} = (id \otimes \varphi_\pi)W$ we have the asymptotic convergence formula*

$$m\widetilde{W} \sim \pi_{mnp}$$

in $*$ -moments, in the $d \rightarrow \infty$ limit, where $\rho = \text{law}(\Lambda_\pi)$.

PROOF. This follows by putting together the results that we have. Indeed, due to Proposition 8.31, Theorem 8.14 applies, and gives the convergence result. □

Summarizing, we have now an explicit block-modification machinery, valid for certain suitable partitions $\pi \in P_{\text{even}}(2s, 2s)$, which improves the previous theory from [20].

As a conclusion to all this, the block modification of the complex Wishart matrices leads, somehow out of nothing, to a whole new world, populated by beasts such as the R -transform, the modified Marchenko-Pastur laws, and many more. Looks like we have opened the Pandora box. We will see however later, in chapters 9-12 below, that this whole new world, called free probability, is in fact not that much different from ours.

8e. Exercises

There has been a lot of tough combinatorics in this chapter, which was rather research grade, and as an exercise here, which is research grade too, we have:

EXERCISE 8.33. *Do some more combinatorics for the block-modified Wishart matrices, as to further generalize the results discussed in this chapter.*

This is of course something quite non-trivial. In what follows we will be back to this, but rather with some conceptual interpretations, instead of new computations.

Part III

Free probability

*Winterlude, Winterlude, my little daisy
Winterlude by the telephone wire
Winterlude, it's making me lazy
Come on, sit by the logs in the fire*

CHAPTER 9

Free probability

9a. Freeness

Welcome to free probability. We have met some already, and in this chapter and in the next three ones we discuss the foundations and main results of free probability, in analogy with the foundations and main results of classical probability.

The common framework for classical and free probability is “noncommutative probability”. This is something very general, that we already met in connection with the random matrices, in chapters 5-8. We first recall this material. Let us start with:

DEFINITION 9.1. *A C^* -algebra is a complex algebra A , having a norm $\|\cdot\|$ making it a Banach algebra, and an involution $*$, related to the norm by the formula*

$$\|aa^*\| = \|a\|^2$$

which must hold for any $a \in A$.

As a basic example, the algebra $B(H)$ of the bounded linear operators $T : H \rightarrow H$ on a complex Hilbert space H is a C^* -algebra, with the usual norm and involution:

$$\|T\| = \sup_{\|x\|=1} \|Tx\| \quad , \quad \langle Tx, y \rangle = \langle x, T^*y \rangle$$

More generally, any closed $*$ -subalgebra of $B(H)$ is a C^* -algebra. It is possible to prove that any C^* -algebra appears in this way, as explained in chapter 5:

$$A \subset B(H)$$

In finite dimensions we have $H = \mathbb{C}^N$, and so the operator algebra $B(H)$ is the usual matrix algebra $M_N(\mathbb{C})$, with the usual norm and involution, namely:

$$\|M\| = \sup_{\|x\|=1} \|Mx\| \quad , \quad (M^*)_{ij} = \bar{M}_{ji}$$

As explained in chapter 4, in the context of Peter-Weyl theory, some algebra shows that the finite dimensional C^* -algebras are the direct sums of matrix algebras:

$$A = M_{n_1}(\mathbb{C}) \oplus \dots \oplus M_{n_k}(\mathbb{C})$$

Summarizing, the C^* -algebra formalism is something in between the $*$ -algebras, which are purely algebraic objects, and whose theory basically leads nowhere, and the fully advanced operator algebras, which are the von Neumann algebras. More on this later.

As yet another class of examples now, which are of particular importance for us, we have various algebras of functions $f : X \rightarrow \mathbb{C}$. The theory here is as follows:

THEOREM 9.2. *The commutative C^* -algebras are the algebras of type $C(X)$, with X being a compact space, the correspondence being as follows:*

- (1) *Given a compact space X , the algebra $C(X)$ of continuous functions $f : X \rightarrow \mathbb{C}$ is a commutative C^* -algebra, with norm and involution as follows:*

$$\|f\| = \sup_{x \in X} |f(x)| \quad , \quad f^*(x) = \overline{f(x)}$$

- (2) *Conversely, any commutative C^* -algebra can be written as $A = C(X)$, with its “spectrum” appearing as the space of Banach algebra characters of A :*

$$X = \{\chi : A \rightarrow \mathbb{C}\}$$

In view of this, given an arbitrary C^ -algebra A , not necessarily commutative, we agree to write $A = C(X)$, and call the abstract space X a compact quantum space.*

PROOF. This is something that we know from chapter 5, the idea being as follows:

(1) First of all, the fact that $C(X)$ is a Banach algebra is clear, because a uniform limit of continuous functions must be continuous. As for the formula $\|ff^*\| = \|f\|^2$, this is something trivial for functions, because on both sides we obtain $\sup_{x \in X} |f(x)|^2$.

(2) Given a commutative C^* -algebra A , the character space $X = \{\chi : A \rightarrow \mathbb{C}\}$ is indeed compact, and we have an evaluation morphism $ev : A \rightarrow C(X)$. The tricky point, which follows from basic spectral theory, is to prove that ev is indeed isometric. \square

The above result is quite interesting for us, because it allows one to formally write any C^* -algebra as $A = C(X)$, with X being a noncommutative compact space. This is certainly something very nice, and in order to do now some probability theory over such spaces X , we would need probability measures μ . But, the problem is that these measures μ are impossible to define, because our spaces X have no points in general.

However, we can trick, and do probability theory just by using expectations functionals $E : A \rightarrow \mathbb{C}$, instead of the probability measures μ themselves. These expectations are called traces, are denoted $tr : A \rightarrow \mathbb{C}$, and their axiomatization is as follows:

DEFINITION 9.3. *A trace, or expectation, or integration functional, on a C^* -algebra A is a linear form $tr : A \rightarrow \mathbb{C}$ having the following properties:*

- (1) *tr is unital, and continuous.*
- (2) *tr is positive, $a \geq 0 \implies \varphi(a) \geq 0$.*
- (3) *tr has the trace property $tr(ab) = tr(ba)$.*

We call tr faithful when $a > 0 \implies \varphi(a) > 0$.

In the commutative case, $A = C(X)$, the Riesz theorem shows that the positive traces $tr : A \rightarrow \mathbb{C}$ appear as integration functionals with respect to positive measures μ :

$$tr(f) = \int_X f(x) d\mu(x)$$

Moreover, the unitality of tr corresponds to the fact that μ has mass one, and the faithfulness of tr corresponds to the faithfulness of μ . Thus, in general, when A is no longer commutative, in order to do probability theory on the underlying noncommutative compact space X , what we need is a faithful trace $tr : A \rightarrow \mathbb{C}$ as above.

So, this will be our philosophy in what follows, a noncommutative probability space (X, μ) being something abstract, corresponding in practice to a pair (A, tr) . This is of course something a bit simplified, because associated to any space X , noncommutative or even classical, there are in fact many possible C^* -algebras of functions $f : X \rightarrow \mathbb{C}$, such as $C(X)$, $L^\infty(X)$ and so on, and for a better theory, we would have to make a choice between these various C^* -algebras associated to X . But let us not worry with this for the moment, what we have is good for starting some computations, so let us just do these computations, see what we get, and we will come back later to more about formalism.

Going ahead with definitions, everything in what follows will be based on:

DEFINITION 9.4. *Let A be a C^* -algebra, given with a trace $tr : A \rightarrow \mathbb{C}$.*

- (1) *The elements $a \in A$ are called random variables.*
- (2) *The moments of such a variable are the numbers $M_k(a) = tr(a^k)$.*
- (3) *The law of such a variable is the functional $\mu : P \rightarrow tr(P(a))$.*

Here $k = \circ \bullet \bullet \circ \dots$ is by definition a colored integer, and the corresponding powers a^k are defined by the following formulae, and multiplicativity:

$$a^\emptyset = 1 \quad , \quad a^\circ = a \quad , \quad a^\bullet = a^*$$

As for the polynomial P , this is a noncommuting $*$ -polynomial in one variable:

$$P \in \mathbb{C} \langle X, X^* \rangle$$

Observe that the law is uniquely determined by the moments, because we have:

$$P(X) = \sum_k \lambda_k X^k \implies \mu(P) = \sum_k \lambda_k M_k(a)$$

Generally speaking, the above definition is something quite abstract, but there is no other way of doing things, at least at this level of generality. However, in certain special cases, the formalism simplifies, and we recover more familiar objects, as follows:

THEOREM 9.5. *Assuming that $a \in A$ is normal, $aa^* = a^*a$, its law corresponds to a probability measure on its spectrum $\sigma(a) \subset \mathbb{C}$, according to the following formula:*

$$\text{tr}(P(a)) = \int_{\sigma(a)} P(x) d\mu(x)$$

When the trace is faithful we have $\text{supp}(\mu) = \sigma(a)$. Also, in the particular case where the variable is self-adjoint, $a = a^$, this law is a real probability measure.*

PROOF. This is something very standard, coming from the continuous functional calculus in C^* -algebras, explained in chapter 5. In fact, we can deduce from there that more is true, in the sense that the following formula holds, for any $f \in C(\sigma(a))$:

$$\text{tr}(f(a)) = \int_{\sigma(a)} f(x) d\mu(x)$$

In addition, assuming that we are in the case $A \subset B(H)$, the measurable functional calculus tells us that the above formula holds in fact for any $f \in L^\infty(\sigma(a))$. \square

We have the following independence notion, generalizing the one from chapter 1:

DEFINITION 9.6. *Two subalgebras $A, B \subset C$ are called independent when the following condition is satisfied, for any $a \in A$ and $b \in B$:*

$$\text{tr}(ab) = \text{tr}(a)\text{tr}(b)$$

Equivalently, the following condition must be satisfied, for any $a \in A$ and $b \in B$:

$$\text{tr}(a) = \text{tr}(b) = 0 \implies \text{tr}(ab) = 0$$

Also, two variables $a, b \in C$ are called independent when the algebras that they generate,

$$A = \langle a \rangle \quad , \quad B = \langle b \rangle$$

are independent inside C , in the above sense.

Observe that the above two independence conditions are indeed equivalent, with this following from the following computation, with the convention $a' = a - \text{tr}(a)$:

$$\begin{aligned} \text{tr}(ab) &= \text{tr}[(a' + \text{tr}(a))(b' + \text{tr}(b))] \\ &= \text{tr}(a'b') + \text{tr}(a')\text{tr}(b) + \text{tr}(a)\text{tr}(b') + \text{tr}(a)\text{tr}(b) \\ &= \text{tr}(a'b') + \text{tr}(a)\text{tr}(b) \\ &= \text{tr}(a)\text{tr}(b) \end{aligned}$$

The other remark is that the above notion generalizes indeed the usual notion of independence, from the classical case, the precise result here being as follows:

THEOREM 9.7. *Given two compact measured spaces X, Y , the algebras*

$$C(X) \subset C(X \times Y) \quad , \quad C(Y) \subset C(X \times Y)$$

are independent in the above sense, and a converse of this fact holds too.

PROOF. We have two assertions here, the idea being as follows:

(1) First of all, given two abstract compact spaces X, Y , we have embeddings of algebras as in the statement, defined by the following formulae:

$$f \rightarrow [(x, y) \rightarrow f(x)] \quad , \quad g \rightarrow [(x, y) \rightarrow g(y)]$$

In the measured space case now, the Fubini theorems tells us that we have:

$$\int_{X \times Y} f(x)g(y) = \int_X f(x) \int_Y g(y)$$

Thus, the algebras $C(X), C(Y)$ are independent in the sense of Definition 9.6.

(2) Conversely, assume that $A, B \subset C$ are independent, with C being commutative. Let us write our algebras as follows, with X, Y, Z being certain compact spaces:

$$A = C(X) \quad , \quad B = C(Y) \quad , \quad C = C(Z)$$

In this picture, the inclusions $A, B \subset C$ must come from quotient maps, as follows:

$$p : Z \rightarrow X \quad , \quad q : Z \rightarrow Y$$

Regarding now the independence condition from Definition 9.6, in the above picture, this tells us that the following equality must happen:

$$\int_Z f(p(z))g(q(z)) = \int_Z f(p(z)) \int_X g(q(z))$$

Thus we are in a Fubini type situation, and we obtain from this:

$$X \times Y \subset Z$$

Thus, the independence of the algebras $A, B \subset C$ appears as in (1) above. \square

It is possible to develop some theory here, but this is ultimately not very interesting. As a much more interesting notion now, we have Voiculescu's freeness [85]:

DEFINITION 9.8. *Two subalgebras $A, B \subset C$ are called free when the following condition is satisfied, for any $a_i \in A$ and $b_i \in B$:*

$$tr(a_i) = tr(b_i) = 0 \implies tr(a_1 b_1 a_2 b_2 \dots) = 0$$

Also, two variables $a, b \in C$ are called free when the algebras that they generate,

$$A = \langle a \rangle \quad , \quad B = \langle b \rangle$$

are free inside C , in the above sense.

In short, freeness appears by definition as a kind of "free analogue" of usual independence, taking into account the fact that the variables do not necessarily commute. As a first observation, of theoretical nature, there is actually a certain lack of symmetry

between Definition 9.6 and Definition 9.8, because in contrast to the former, the latter does not include an explicit formula for the quantities of the following type:

$$\text{tr}(a_1 b_1 a_2 b_2 \dots)$$

However, this is not an issue, and is simply due to the fact that the formula in the free case is something more complicated, the precise result being as follows:

PROPOSITION 9.9. *Assuming that $A, B \subset C$ are free, the restriction of tr to $\langle A, B \rangle$ can be computed in terms of the restrictions of tr to A, B . To be more precise,*

$$\text{tr}(a_1 b_1 a_2 b_2 \dots) = P\left(\{tr(a_{i_1} a_{i_2} \dots)\}_i, \{tr(b_{j_1} b_{j_2} \dots)\}_j\right)$$

where P is certain polynomial in several variables, depending on the length of the word $a_1 b_1 a_2 b_2 \dots$, and having as variables the traces of products of type

$$a_{i_1} a_{i_2} \dots, \quad b_{j_1} b_{j_2} \dots$$

with the indices being chosen increasing, $i_1 < i_2 < \dots$ and $j_1 < j_2 < \dots$

PROOF. This is something a bit theoretical, so let us begin with an example. Our claim is that if a, b are free then, exactly as in the case where we have independence:

$$\text{tr}(ab) = \text{tr}(a)\text{tr}(b)$$

Indeed, let us go back to the computation performed after Definition 9.6, which was as follows, with the convention $a' = a - \text{tr}(a)$:

$$\begin{aligned} \text{tr}(ab) &= \text{tr}[(a' + \text{tr}(a))(b' + \text{tr}(b))] \\ &= \text{tr}(a'b') + \text{tr}(a')\text{tr}(b) + \text{tr}(a)\text{tr}(b') + \text{tr}(a)\text{tr}(b) \\ &= \text{tr}(a'b') + \text{tr}(a)\text{tr}(b) \\ &= \text{tr}(a)\text{tr}(b) \end{aligned}$$

Our claim is that this computation perfectly works under the sole freeness assumption. Indeed, the only non-trivial equality is the last one, which follows from:

$$\text{tr}(a') = \text{tr}(b') = 0 \implies \text{tr}(a'b') = 0$$

In general, the situation is of course more complicated than this, but the same trick applies. To be more precise, we can start our computation as follows:

$$\begin{aligned} \text{tr}(a_1 b_1 a_2 b_2 \dots) &= \text{tr}[(a'_1 + \text{tr}(a_1))(b'_1 + \text{tr}(b_1))(a'_2 + \text{tr}(a_2))(b'_2 + \text{tr}(b_2)) \dots] \\ &= \text{tr}(a'_1 b'_1 a'_2 b'_2 \dots) + \text{other terms} \\ &= \text{other terms} \end{aligned}$$

Observe that we have used here the freeness condition, in the following form:

$$\text{tr}(a'_i) = \text{tr}(b'_i) = 0 \implies \text{tr}(a'_1 b'_1 a'_2 b'_2 \dots) = 0$$

Now regarding the “other terms”, those which are left, each of them will consist of a product of traces of type $tr(a_i)$ and $tr(b_i)$, and then a trace of a product still remaining to be computed, which is of the following form, for some elements $\alpha_i \in A$ and $\beta_i \in B$:

$$tr(\alpha_1\beta_1\alpha_2\beta_2\dots)$$

To be more precise, the variables $\alpha_i \in A$ appear as ordered products of those $a_i \in A$ not getting into individual traces $tr(a_i)$, and the variables $\beta_i \in B$ appear as ordered products of those $b_i \in B$ not getting into individual traces $tr(b_i)$. Now since the length of each such alternating product $\alpha_1\beta_1\alpha_2\beta_2\dots$ is smaller than the length of the original product $a_1b_1a_2b_2\dots$, we are led into of recurrence, and this gives the result. \square

Let us discuss now some models for independence and freeness. We have the following result, from [85], which clarifies the analogy between independence and freeness:

THEOREM 9.10. *Given two algebras (A, tr) and (B, tr) , the following hold:*

- (1) *A, B are independent inside their tensor product $A \otimes B$, endowed with its canonical tensor product trace, given by $tr(a \otimes b) = tr(a)tr(b)$.*
- (2) *A, B are free inside their free product $A * B$, endowed with its canonical free product trace, given by the formulae in Proposition 9.9.*

PROOF. Both the above assertions are clear from definitions, as follows:

(1) This is clear with either of the definitions of the independence, from Definition 9.6, because we have by construction of the product trace:

$$\begin{aligned} tr(ab) &= tr[(a \otimes 1)(1 \otimes b)] \\ &= tr(a \otimes b) \\ &= tr(a)tr(b) \end{aligned}$$

Observe that there is a relation here with Theorem 9.7 as well, due to the following formula for compact spaces, with \otimes being a topological tensor product:

$$C(X \times Y) = C(X) \otimes C(Y)$$

To be more precise, the present statement generalizes the first assertion in Theorem 9.7, and the second assertion tells us that this generalization is more or less the same thing as the original statement. All this comes of course from basic measure theory.

(2) This is clear too from definitions, the only point being that of showing that the notion of freeness, or the recurrence formulae in Proposition 9.9, can be used in order to construct a canonical free product trace, on the free product of the algebras involved:

$$tr : A * B \rightarrow \mathbb{C}$$

But this can be checked for instance by using a GNS construction. Indeed, consider the GNS constructions for the algebras (A, tr) and (B, tr) :

$$A \rightarrow B(l^2(A)) \quad , \quad B \rightarrow B(l^2(B))$$

By taking the free product of these representations, we obtain a representation as follows, with the $*$ on the right being a free product of pointed Hilbert spaces:

$$A * B \rightarrow B(l^2(A) * l^2(B))$$

Now by composing with the linear form $T \rightarrow \langle T\xi, \xi \rangle$, where $\xi = 1_A = 1_B$ is the common distinguished vector of $l^2(A)$, $l^2(B)$, we obtain a linear form, as follows:

$$tr : A * B \rightarrow \mathbb{C}$$

It is routine then to check that tr is indeed a trace, and this is the “canonical free product trace” from the statement. Then, an elementary computation shows that A, B are free inside $A * B$, with respect to this trace, and this finishes the proof. See [85]. \square

9b. Free convolution

All the above was quite theoretical, and as a concrete application of the above results, bringing us into probability, we have the following result, from [86]:

THEOREM 9.11. *We have a free convolution operation \boxplus for the distributions*

$$\mu : \mathbb{C} \langle X, X^* \rangle \rightarrow \mathbb{C}$$

which is well-defined by the following formula, with a, b taken to be free:

$$\mu_a \boxplus \mu_b = \mu_{a+b}$$

This restricts to an operation, still denoted \boxplus , on the real probability measures.

PROOF. We have several verifications to be performed here, as follows:

(1) We first have to check that given two variables a, b which live respectively in certain C^* -algebras A, B , we can recover inside some C^* -algebra C , with exactly the same distributions μ_a, μ_b , as to be able to sum them and talk about μ_{a+b} . But this comes from Theorem 9.10, because we can set $C = A * B$, as explained there.

(2) The other verification which is needed is that of the fact that if two variables a, b are free, then the distribution μ_{a+b} depends only on the distributions μ_a, μ_b . But for this purpose, we can use the general formula from Proposition 9.9, namely:

$$tr(a_1 b_1 a_2 b_2 \dots) = P\left(\{tr(a_{i_1} a_{i_2} \dots)\}_i, \{tr(b_{j_1} b_{j_2} \dots)\}_j\right)$$

Now by plugging in arbitrary powers of a, b as variables a_i, b_j , we obtain a family of formulae of the following type, with Q being certain polyomials:

$$tr(a^{k_1} b^{l_1} a^{k_2} b^{l_2} \dots) = Q\left(\{tr(a^k)\}_k, \{tr(b^l)\}_l\right)$$

Thus the moments of $a + b$ depend only on the moments of a, b , with of course colored exponents in all this, according to our moment conventions, and this gives the result.

(3) Finally, in what regards the last assertion, regarding the real measures, this is clear from the fact that if the variables a, b are self-adjoint, then so is their sum $a + b$. \square

Along the same lines, but with some technical subtleties this time, we can talk as well about multiplicative free convolution, following [87], as follows:

THEOREM 9.12. *We have a free convolution operation \boxtimes for the distributions*

$$\mu : \mathbb{C} \langle X, X^* \rangle \rightarrow \mathbb{C}$$

which is well-defined by the following formula, with a, b taken to be free:

$$\mu_a \boxtimes \mu_b = \mu_{ab}$$

In the case of the self-adjoint variables, we can equally set

$$\mu_a \boxtimes \mu_b = \mu_{\sqrt{ab}\sqrt{a}}$$

and so we have an operation, still denoted \boxtimes , on the real probability measures.

PROOF. We have two statements here, the idea being as follows:

(1) The verifications for the fact that \boxtimes as above is indeed well-defined at the general distribution level are identical to those done before for \boxplus , with the result basically coming from the formula in Proposition 9.9, and with Theorem 9.10 invoked as well, in order to say that we have a model, and so we can indeed use this formula.

(2) Regarding now the last assertion, regarding the real measures, this was something trivial for \boxplus , but is something trickier now for \boxtimes , because if we take a, b to be self-adjoint, their product ab will in general not be self-adjoint, and definitely it will be not if we want a, b to be free, and so the formula $\mu_a \boxtimes \mu_b = \mu_{ab}$ will apparently makes us exit the world of real probability measures. However, this is not exactly the case. Indeed, let us set:

$$c = \sqrt{ab}\sqrt{a}$$

This new variable is then self-adjoint, and its moments are given by:

$$\begin{aligned} tr(c^k) &= tr[(\sqrt{ab}\sqrt{a})^k] \\ &= tr[\sqrt{aba} \dots ab\sqrt{a}] \\ &= tr[\sqrt{a} \cdot \sqrt{aba} \dots ab] \\ &= tr[(ab)^k] \end{aligned}$$

Thus, we are led to the conclusion in the statement. \square

We would like now to have linearization results for \boxplus and \boxtimes , in the spirit of the known results for $*$ and \times . We will do this slowly, in several steps. As a first objective, we would like to convert our one and only modeling result so far, namely Theorem 9.10, which is a rather abstract result, into something more concrete. Let us start with:

THEOREM 9.13. *Let Γ be a discrete group, and consider the complex group algebra $\mathbb{C}[\Gamma]$, with involution given by the fact that all group elements are unitaries:*

$$g^* = g^{-1} \quad , \quad \forall g \in \Gamma$$

The maximal C^ -seminorm on $\mathbb{C}[\Gamma]$ is then a C^* -norm, and the closure of $\mathbb{C}[\Gamma]$ with respect to this norm is a C^* -algebra, denoted $C^*(\Gamma)$. Moreover,*

$$\text{tr}(g) = \delta_{g1}$$

defines a positive unital trace $\text{tr} : C^(\Gamma) \rightarrow \mathbb{C}$, which is faithful on $\mathbb{C}[\Gamma]$.*

PROOF. We have two assertions to be proved, the idea being as follows:

(1) In order to prove the first assertion, regarding the maximal seminorm which is a norm, we must find a $*$ -algebra embedding as follows, with H being a Hilbert space:

$$\mathbb{C}[\Gamma] \subset B(H)$$

For this purpose, consider the Hilbert space $H = l^2(\Gamma)$, having the family $\{h\}_{h \in \Gamma}$ as orthonormal basis. Our claim is that we have an embedding, as follows:

$$\pi : \mathbb{C}[\Gamma] \subset B(H) \quad , \quad \pi(g)(h) = gh$$

Indeed, since $\pi(g)$ maps the basis $\{h\}_{h \in \Gamma}$ into itself, this operator is well-defined and bounded, and is an isometry. It is also clear from the formula $\pi(g)(h) = gh$ that $g \rightarrow \pi(g)$ is a morphism of algebras, and since this morphism maps the unitaries $g \in \Gamma$ into isometries, this is a morphism of $*$ -algebras. Finally, the faithfulness of π is clear.

(2) Regarding the second assertion, we can use here once again the above construction. Indeed, we can define a linear form on the image of $C^*(\Gamma)$, as follows:

$$\text{tr}(T) = \langle T\delta_1, \delta_1 \rangle$$

This functional is then positive, and is easily seen to be a trace. Moreover, on the group elements $g \in \Gamma$, this functional is given by the following formula:

$$\text{tr}(g) = \delta_{g1}$$

Thus, it remains to show that tr is faithful on $\mathbb{C}[\Gamma]$. But this follows from the fact that tr is faithful on the image of $C^*(\Gamma)$, which contains $\mathbb{C}[\Gamma]$. \square

As an illustration, we have the following more precise result, in the abelian case:

PROPOSITION 9.14. *Given a discrete abelian group Γ , we have an isomorphism*

$$C^*(\Gamma) \simeq C(G)$$

where $G = \widehat{\Gamma}$ is its Pontrjagin dual, formed by the characters $\chi : \Gamma \rightarrow \mathbb{T}$. Moreover,

$$\text{tr}(g) = \delta_{g1}$$

corresponds in this way to the Haar integration over G .

PROOF. We have two assertions to be proved, the idea being as follows:

(1) Since Γ is abelian, $A = C^*(\Gamma)$ is commutative, so by the Gelfand theorem we have $A = C(X)$. The spectrum $X = \text{Spec}(A)$, consisting of the characters $\chi : C^*(\Gamma) \rightarrow \mathbb{C}$, can be then identified with the Pontrjagin dual $G = \widehat{\Gamma}$, and this gives the result.

(2) Regarding now the last assertion, we must prove here that we have:

$$\text{tr}(f) = \int_G f(x) dx$$

But this is clear via the above identifications, for instance because the linear form $\text{tr}(g) = \delta_{g1}$, when viewed as a functional on $C(G)$, is left and right invariant. \square

Getting back now to our questions, we can now formulate a general modelling result for independence and freeness, providing us with large classes of examples, as follows:

THEOREM 9.15. *We have the following results, valid for group algebras:*

- (1) $C^*(\Gamma), C^*(\Lambda)$ are independent inside $C^*(\Gamma \times \Lambda)$.
- (2) $C^*(\Gamma), C^*(\Lambda)$ are free inside $C^*(\Gamma * \Lambda)$.

PROOF. In order to prove these results, we have two possible methods:

(1) We can either use the general results in Theorem 9.10, along with the following two isomorphisms, which are both standard:

$$C^*(\Gamma \times \Lambda) = C^*(\Lambda) \otimes C^*(\Gamma) \quad , \quad C^*(\Gamma * \Lambda) = C^*(\Lambda) * C^*(\Gamma)$$

(2) Or, we can prove this directly, by using the fact that each algebra is spanned by the corresponding group elements. Indeed, this shows that it is enough to check the independence and freeness formulae on group elements, which is in turn trivial. \square

9c. Linearization

We have seen so far the foundations of free probability, in analogy with those of classical probability, taken with a functional analysis touch. The idea now is that with a bit of luck, the basic theory from the classical case, namely the Fourier transform, and then the CLT, should have free extensions. Let us begin our discussion with the following definition, from [86], coming from the theory developed in the above:

DEFINITION 9.16. *The real probability measures are subject to operations $*$ and \boxplus , called classical and free convolution, given by the formulae*

$$\mu_a * \mu_b = \mu_{a+b} \quad , \quad \mu_\alpha \boxplus \mu_\beta = \mu_{\alpha+\beta}$$

with a, b being independent, and α, β being free, and all variables being self-adjoint.

The problem now is that of linearizing these operations $*$ and \boxplus . In what regards $*$, we know from chapter 1 that this operation is linearized by the logarithm $\log F$ of the Fourier transform, which in the present setting, where $E = tr$, is given by:

$$F_a(x) = tr(e^{ixa})$$

In order to find a similar result for \boxplus , we need some efficient models for the pairs of free random variables (a, b) . This is a priori not a problem, because once we have $a \in A$ and $b \in B$, we can form the free product $A * B$, which contains a, b as free variables.

However, the initial choice, that of the variables $a \in A, b \in B$ modeling some given laws $\mu, \nu \in \mathcal{P}(\mathbb{R})$, matters a lot. Indeed, any kind of abstract choice here would lead us into an abstract algebra $A * B$, and so into the abstract combinatorics of the free convolution, that cannot be solved with bare hands, and that we want to avoid.

In short, we must be tricky, at least in what concerns the beginning of our computation. Following [86], the idea will be that of temporarily lifting the self-adjointness assumption on our variables a, b , and looking instead for random variables α, β , not necessarily self-adjoint, modelling in integer moments our given laws $\mu, \nu \in \mathcal{P}(\mathbb{R})$, as follows:

$$tr(\alpha^k) = M_k(\mu) \quad , \quad tr(\beta^k) = M_k(\nu)$$

To be more precise, assuming that α, β are indeed not self-adjoint, the above formulae are not the general formulae for α, β , simply because these latter formulae involve colored integers $k = \circ \bullet \bullet \circ \dots$ as exponents. Thus, in the context of the above formulae, μ, ν are not the distributions of α, β , but just some “parts” of these distributions.

Now with this idea in mind, due to Voiculescu and quite tricky, the solution to the law modelling problem comes in a quite straightforward way, involving the good old Hilbert space $H = l^2(\mathbb{N})$ and the good old shift operator $S \in B(H)$, as follows:

THEOREM 9.17. *Consider the shift operator on the space $H = l^2(\mathbb{N})$, given by $S(e_i) = e_{i+1}$. The variables of the following type, with $f \in \mathbb{C}[X]$ being a polynomial,*

$$S^* + f(S)$$

model then in moments, up to finite order, all the distributions $\mu : \mathbb{C}[X] \rightarrow \mathbb{C}$.

PROOF. We have already met the shift S in chapter 5, as the simplest example of an isometry which is not a unitary, $S^*S = 1, SS^* = 1$, with this coming from:

$$S^*(e_i) = \begin{cases} e_{i-1} & (i > 0) \\ 0 & (i = 0) \end{cases}$$

Consider now a variable as in the statement, namely:

$$T = S^* + a_0 + a_1S + a_2S^2 + \dots + a_nS^n$$

The computation of the moments of T is then as follows:

- We first have $\text{tr}(T) = a_0$.
- Then the computation of $\text{tr}(T^2)$ will involve a_1 .
- Then the computation of $\text{tr}(T^3)$ will involve a_2 .
- And so on.

Thus, we are led to a certain recurrence, that we will not attempt to solve now, with bare hands, but which definitely gives the conclusion in the statement. \square

Before getting further, with free products of such models, let us work out a very basic example, which is something fundamental, that we will need in what follows:

PROPOSITION 9.18. *In the context of the above correspondence, the variable*

$$T = S + S^*$$

follows the Wigner semicircle law, $\gamma_1 = \frac{1}{2\pi}\sqrt{4-x^2}dx$.

PROOF. In order to compute the law of variable T in the statement, we can use the moment method. The moments of this variable are as follows:

$$\begin{aligned} M_k &= \text{tr}(T^k) \\ &= \text{tr}((S + S^*)^k) \\ &= \#(1 \in (S + S^*)^k) \end{aligned}$$

Now since the S shifts to the right on \mathbb{N} , and S^* shifts to the left, while remaining positive, we are left with counting the length k paths on \mathbb{N} starting and ending at 0. Since there are no such paths when $k = 2r + 1$ is odd, the odd moments vanish:

$$M_{2r+1} = 0$$

In the case where $k = 2r$ is even, such paths on \mathbb{N} are best represented as paths in the upper half-plane, starting at 0, and going at each step NE or SE, depending on whether the original path on \mathbb{N} goes at right or left, and finally ending at $k \in \mathbb{N}$. With this picture we are led to the following formula for the number of such paths:

$$M_{2r+2} = \sum_s M_{2s} M_{2r-s}$$

But this is exactly the recurrence formula for the Catalan numbers, and so:

$$M_{2r} = \frac{1}{r+1} \binom{2r}{r}$$

Summarizing, the odd moments of T vanish, and the even moments are the Catalan numbers. But these numbers being the moments of the Wigner semicircle law γ_1 , as explained in chapter 3, we are led to the conclusion in the statement. \square

Getting back now to our linearization program for \boxplus , the next step is that of taking a free product of the model found in Theorem 9.17 with itself. There are two approaches here, one being a bit abstract, and the other one being more concrete. We will explain in what follows both of them. The abstract approach, which is quite nice, making a link with our main modeling result so far, involving group algebras, is as follows:

PROPOSITION 9.19. *We can talk about semigroup algebras $C^*(\Gamma) \subset B(l^2(\Gamma))$, exactly as we did for the group algebras, and at the level of examples:*

- (1) *With $\Gamma = \mathbb{N}$ we recover the shift algebra $A = \langle S \rangle$ on $H = l^2(\mathbb{N})$.*
- (2) *With $\Gamma = \mathbb{N} * \mathbb{N}$, we obtain the algebra $A = \langle S_1, S_2 \rangle$ on $H = l^2(\mathbb{N} * \mathbb{N})$.*

PROOF. We can talk indeed about semigroup algebras $C^*(\Gamma) \subset B(l^2(\Gamma))$, exactly as we did for the group algebras, the only difference coming from the fact that the semigroup elements $g \in \Gamma$ will now correspond to isometries, which are not necessarily unitaries. Now this construction in hand, both the assertions are clear, as follows:

(1) With $\Gamma = \mathbb{N}$ we recover indeed the shift algebra $A = \langle S \rangle$ on the Hilbert space $H = l^2(\mathbb{N})$, the shift S itself being the isometry associated to the element $1 \in \mathbb{N}$.

(2) With $\Gamma = \mathbb{N} * \mathbb{N}$ we recover the double shift algebra $A = \langle S_1, S_2 \rangle$ on the Hilbert space $H = l^2(\mathbb{N} * \mathbb{N})$, the two shifts S_1, S_2 themselves being the isometries associated to two copies of the element $1 \in \mathbb{N}$, one for each of the two copies of \mathbb{N} which are present. \square

In what follows we will rather use an equivalent, second approach to our problem, which is exactly the same thing, but formulated in a less abstract way, as follows:

PROPOSITION 9.20. *We can talk about the algebra of creation operators*

$$S_x : v \rightarrow x \otimes v$$

on the free Fock space associated to a real Hilbert space H , given by

$$F(H) = \mathbb{C}\Omega \oplus H \oplus H^{\otimes 2} \oplus \dots$$

and at the level of examples, we have:

- (1) *With $H = \mathbb{C}$ we recover the shift algebra $A = \langle S \rangle$ on $H = l^2(\mathbb{N})$.*
- (2) *With $H = \mathbb{C}^2$, we obtain the algebra $A = \langle S_1, S_2 \rangle$ on $H = l^2(\mathbb{N} * \mathbb{N})$.*

PROOF. We can talk indeed about the algebra $A(H)$ of creation operators on the free Fock space $F(H)$ associated to a real Hilbert space H , with the remark that, in terms of the abstract semigroup notions from Proposition 9.19, we have:

$$A(\mathbb{C}^k) = C^*(\mathbb{N}^{*k}) \quad , \quad F(\mathbb{C}^k) = l^2(\mathbb{N}^{*k})$$

As for the assertions (1,2) in the statement, these are both clear, either directly, or by passing via (1,2) from Proposition 9.19, which were both clear as well. \square

The advantage with this latter model comes from the following result, from [86], which has a very simple formulation, without linear combinations or anything:

PROPOSITION 9.21. *Given a real Hilbert space H , and two orthogonal vectors $x \perp y$, the corresponding creation operators S_x and S_y are free with respect to*

$$\text{tr}(T) = \langle T\Omega, \Omega \rangle$$

called trace associated to the vacuum vector.

PROOF. In standard tensor product notation for the elements of the free Fock space $F(H)$, the formula of a creation operator associated to a vector $x \in H$ is as follows:

$$S_x(y_1 \otimes \dots \otimes y_n) = x \otimes y_1 \otimes \dots \otimes y_n$$

As for the formula of the adjoint of this creation operator, called annihilation operator associated to the vector $x \in H$, this is as follows:

$$S_x^*(y_1 \otimes \dots \otimes y_n) = \langle x, y_1 \rangle \otimes y_2 \otimes \dots \otimes y_n$$

We obtain from this the following formula, which holds for any two vectors $x, y \in H$:

$$S_x^* S_y = \langle x, y \rangle \text{id}$$

With these formulae in hand, the result follows by doing some elementary computations, in the spirit of those done for the group algebras, in the above. \square

With this technology in hand, let us go back to our linearization program for \boxplus . We know from Theorem 9.17 how to model the individual distributions $\mu \in \mathcal{P}(\mathbb{R})$, and by combining this with Proposition 9.10 and Proposition 9.21, we therefore know how to freely model pairs of distributions $\mu, \nu \in \mathcal{P}(\mathbb{R})$, as required by the convolution problem. We are therefore left with doing the sum in the model, and then computing its distribution. And the point here is that, still following [86], we have:

THEOREM 9.22. *Given two polynomials $f, g \in \mathbb{C}[X]$, consider the variables*

$$S^* + f(S) \quad , \quad T^* + g(T)$$

where S, T are two creation operators, or shifts, associated to a pair of orthogonal norm 1 vectors. These variables are then free, and their sum has the same law as

$$R^* + (f + g)(R)$$

with R being the usual shift on $l^2(\mathbb{N})$.

PROOF. We have two assertions here, the idea being as follows:

(1) The freeness assertion comes from the general freeness result from Proposition 9.21, via the various identifications coming from the previous results.

(2) Regarding the second assertion, the idea is that this comes from a 45° rotation trick. Let us write indeed the two variables in the statement as follows:

$$\begin{aligned} X &= S^* + a_0 + a_1 S + a_2 S^2 + \dots \\ Y &= T^* + b_0 + b_1 T + a_2 T^2 + \dots \end{aligned}$$

Now let us perform the following 45° base change, on the real span of the vectors $s, t \in H$ producing our two shifts S, T , as follows:

$$r = \frac{s+t}{\sqrt{2}} \quad , \quad u = \frac{s-t}{\sqrt{2}}$$

The new shifts, associated to these vectors $r, u \in H$, are then given by:

$$R = \frac{S+T}{\sqrt{2}} \quad , \quad U = \frac{S-T}{\sqrt{2}}$$

By using now these two new shifts, which are free according to Proposition 9.21, we obtain the following equality of distributions:

$$\begin{aligned} X+Y &= S^* + T^* + \sum_k a_k S^k + b_k T^k \\ &= \sqrt{2}R^* + \sum_k a_k \left(\frac{R+U}{\sqrt{2}} \right)^k + b_k \left(\frac{R-U}{\sqrt{2}} \right)^k \\ &\sim \sqrt{2}R^* + \sum_k a_k \left(\frac{R}{\sqrt{2}} \right)^k + b_k \left(\frac{R}{\sqrt{2}} \right)^k \\ &\sim R^* + \sum_k a_k R^k + b_k R^k \end{aligned}$$

To be more precise, here at the end we have used the freeness property of R, U in order to cut U from the computation, as it cannot bring anything, and then we did a basic rescaling at the very end. Thus, we are led to the conclusion in the statement. \square

As a conclusion, the operation $\mu \rightarrow f$ from Theorem 9.17 linearizes \boxplus . In order to reach now to something concrete, we are left with a computation inside $C^*(\mathbb{N})$, which is elementary, and whose conclusion is that $R_\mu = f$ can be recaptured from μ via the Cauchy transform G_μ . The precise result here, due to Voiculescu [86], is as follows:

THEOREM 9.23. *Given a real probability measure μ , define its R -transform as follows:*

$$G_\mu(\xi) = \int_{\mathbb{R}} \frac{d\mu(t)}{\xi - t} \implies G_\mu \left(R_\mu(\xi) + \frac{1}{\xi} \right) = \xi$$

The free convolution operation is then linearized by this R -transform.

PROOF. This can be done by using the above results, in several steps, as follows:

(1) According to Theorem 9.22, the operation $\mu \rightarrow f$ from Theorem 9.17 linearizes the free convolution operation \boxplus . We are therefore left with a computation inside $C^*(\mathbb{N})$. To be more precise, consider a variable as in Theorem 9.17:

$$X = S^* + f(S)$$

In order to establish the result, we must prove that the R -transform of X , constructed according to the procedure in the statement, is the function f itself.

(2) In order to do so, we fix $|z| < 1$ in the complex plane, and we set:

$$q_z = \delta_0 + \sum_{k=1}^{\infty} z_k \delta_k$$

The shift and its adjoint act then on this vector as follows:

$$S q_z = z^{-1}(q_z - \delta_0) \quad , \quad S^* q_z = z q_z$$

It follows that the adjoint of our operator X acts on this vector as follows:

$$\begin{aligned} X^* q_z &= (S + f(S^*)) q_z \\ &= z^{-1}(q_z - \delta_0) + f(z) q_z \\ &= (z^{-1} + f(z)) q_z - z^{-1} \delta_0 \end{aligned}$$

Now observe that the above formula can be written as follows:

$$z^{-1} \delta_0 = (z^{-1} + f(z) - X^*) q_z$$

The point now is that when $|z|$ is small, the operator appearing on the right is invertible. Thus, we can rewrite the above formula as follows:

$$(z^{-1} + f(z) - X^*)^{-1} \delta_0 = z q_z$$

Now by applying the trace, we are led to the following formula:

$$\begin{aligned} \text{tr} [(z^{-1} + f(z) - X^*)^{-1}] &= \langle (z^{-1} + f(z) - X^*)^{-1} \delta_0, \delta_0 \rangle \\ &= \langle z q_z, \delta_0 \rangle \\ &= z \end{aligned}$$

(3) Let us apply now the procedure in the statement to the real probability measure μ modelled by X . The Cauchy transform G_μ is then given by:

$$\begin{aligned} G_\mu(\xi) &= \text{tr}((\xi - X)^{-1}) \\ &= \overline{\text{tr}((\bar{\xi} - X^*)^{-1})} \\ &= \text{tr}((\xi - X^*)^{-1}) \end{aligned}$$

Now observe that, with the choice $\xi = z^{-1} + f(z)$ for our complex variable, the trace formula found in (2) above tells us that we have:

$$G_\mu(z^{-1} + f(z)) = z$$

Thus, by definition of the R -transform, we have the following formula:

$$R_\mu(z) = f(z)$$

But this finishes the proof, as explained before in step (1) above. \square

Summarizing, the situation in free probability is quite similar to the one in classical probability, the product spaces needed for the basic properties of the Fourier transform being replaced by something “noncommutative”, namely the free Fock space models. This is of course something quite surprising, and the credit for this remarkable discovery, which has drastically changed operator algebras, goes to Voiculescu’s paper [86].

9d. Central limits

With the above linearization technology in hand, we can do many things. First, we have the following free analogue of the CLT, at variance 1, due to Voiculescu [86]:

THEOREM 9.24. *Given self-adjoint variables x_1, x_2, x_3, \dots which are f.i.d., centered, with variance 1, we have, with $n \rightarrow \infty$, in moments,*

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n x_i \sim \gamma_1$$

with the limiting measure being the Wigner semicircle law on $[-2, 2]$:

$$\gamma_1 = \frac{1}{2\pi} \sqrt{4 - x^2} dx$$

Due to this, we also call this Wigner law free Gaussian law.

PROOF. We follow the same idea as in the proof of the CLT, from chapter 1:

(1) The R -transform of the variable in the statement on the left can be computed by using the linearization property from Theorem 9.23, and is given by:

$$R(\xi) = nR_x \left(\frac{\xi}{\sqrt{n}} \right) \simeq \xi$$

(2) Regarding now the right term, our first claim here is that the Cauchy transform of the Wigner law γ_1 satisfies the following equation:

$$G_{\gamma_1} \left(\xi + \frac{1}{\xi} \right) = \xi$$

Indeed, we know from chapter 3 that the even moments of γ_1 are given by:

$$\frac{1}{2\pi} \int_{-2}^2 \sqrt{4 - x^2} x^{2k} dx = C_k$$

On the other hand, we also know from chapter 3 that the generating series of the Catalan numbers is given by the following formula:

$$\sum_{k=0}^{\infty} C_k z^k = \frac{1 - \sqrt{1 - 4z}}{2z}$$

By using this formula with $z = y^{-2}$, we obtain the following formula:

$$\begin{aligned}
 G_{\gamma_1}(y) &= y^{-1} \sum_{k=0}^{\infty} C_k y^{-2k} \\
 &= y^{-1} \cdot \frac{1 - \sqrt{1 - 4y^{-2}}}{2y^{-2}} \\
 &= \frac{y}{2} \left(1 - \sqrt{1 - 4y^{-2}}\right) \\
 &= \frac{y}{2} - \frac{1}{2} \sqrt{y^2 - 4}
 \end{aligned}$$

Now with $y = \xi + \xi^{-1}$, this formula becomes, as claimed in the above:

$$\begin{aligned}
 G_{\gamma_1} \left(\xi + \frac{1}{\xi} \right) &= \frac{\xi + \xi^{-1}}{2} - \frac{1}{2} \sqrt{\xi^2 + \xi^{-2} - 2} \\
 &= \frac{\xi + \xi^{-1}}{2} - \frac{\xi^{-1} - \xi}{2} \\
 &= \xi
 \end{aligned}$$

(3) We conclude from the formula found in (2) and from Theorem 9.23 that the R -transform of the Wigner semicircle law γ_1 is given by the following formula:

$$R_{\gamma_1}(\xi) = \xi$$

Observe that this follows in fact as well from the following formula, coming from Proposition 9.18, and from the technical details of the R -transform:

$$S + S^* \sim \gamma_1$$

Thus, the laws in the statement have the same R -transforms, so they are equal. \square

Summarizing, we have proved the free CLT at $t = 1$. The passage to the general case, where $t > 0$ is arbitrary, is routine, and still following Voiculescu [86], we have:

THEOREM 9.25 (Free CLT). *Given self-adjoint variables x_1, x_2, x_3, \dots which are f.i.d., centered, with variance $t > 0$, we have, with $n \rightarrow \infty$, in moments,*

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n x_i \sim \gamma_t$$

with the limiting measure being the Wigner semicircle law on $[-2\sqrt{t}, 2\sqrt{t}]$:

$$\gamma_t = \frac{1}{2\pi t} \sqrt{4t - x^2} dx$$

Due to this, we also call this Wigner law free Gaussian law.

PROOF. We follow the above proof at $t = 1$, by making changes where needed:

(1) The R -transform of the variable in the statement on the left can be computed by using the linearization property from Theorem 9.23, and is given by:

$$R(\xi) = nR_x\left(\frac{\xi}{\sqrt{n}}\right) \simeq t\xi$$

(2) Regarding now the right term, our claim here is that we have:

$$G_{\gamma_t}\left(t\xi + \frac{1}{\xi}\right) = \xi$$

Indeed, we know from chapter 5 that the even moments of γ_t are given by:

$$\frac{1}{2\pi t} \int_{-2\sqrt{t}}^{2\sqrt{t}} \sqrt{4t - x^2} x^{2k} dx = t^k C_k$$

On the other hand, we know from chapter 3 that we have the following formula:

$$\sum_{k=0}^{\infty} C_k z^k = \frac{1 - \sqrt{1 - 4z}}{2z}$$

By using this formula with $z = ty^{-2}$, we obtain the following formula:

$$\begin{aligned} G_{\gamma_t}(y) &= y^{-1} \sum_{k=0}^{\infty} t^k C_k y^{-2k} \\ &= y^{-1} \cdot \frac{1 - \sqrt{1 - 4ty^{-2}}}{2ty^{-2}} \\ &= \frac{y}{2t} \left(1 - \sqrt{1 - 4ty^{-2}}\right) \\ &= \frac{y}{2t} - \frac{1}{2t} \sqrt{y^2 - 4t} \end{aligned}$$

Now with $y = t\xi + \xi^{-1}$, this formula becomes, as claimed in the above:

$$\begin{aligned} G_{\gamma_t}\left(t\xi + \frac{1}{\xi}\right) &= \frac{t\xi + \xi^{-1}}{2t} - \frac{1}{2t} \sqrt{t^2\xi^2 + \xi^{-2} - 2t} \\ &= \frac{t\xi + \xi^{-1}}{2t} - \frac{\xi^{-1} - t\xi}{2t} \\ &= \xi \end{aligned}$$

(3) We conclude from the formula found in (2) and from Theorem 9.23 that the R -transform of the Wigner semicircle law γ_t is given by the following formula:

$$R_{\gamma_t}(\xi) = t\xi$$

Thus, the laws in the statement have the same R -transforms, so they are equal. \square

Regarding the limiting measures γ_t , that we already met in the previous chapters, in relation with the Wigner matrices, one problem that we were having was that of understanding how γ_t exactly appears, out of γ_1 . We can now solve this question:

THEOREM 9.26. *The Wigner semicircle laws have the property*

$$\gamma_s \boxplus \gamma_t = \gamma_{s+t}$$

so they form a 1-parameter semigroup with respect to free convolution.

PROOF. This follows either from Theorem 9.25, or from Theorem 9.23, by using the fact that the R -transform of γ_t , which is given by $R_{\gamma_t}(\xi) = t\xi$, is linear in t . \square

As a conclusion to what we have so far, we have:

THEOREM 9.27. *The Gaussian laws g_t and the Wigner laws γ_t , given by*

$$g_t = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} dx \quad , \quad \gamma_t = \frac{1}{2\pi t} \sqrt{4t - x^2} dx$$

have the following properties:

- (1) *They appear via the CLT, and the free CLT.*
- (2) *They form semigroups with respect to $*$ and \boxplus .*
- (3) *Their transforms are $\log F_{g_t}(x) = -tx^2/2$, $R_{\gamma_t}(x) = tx$.*
- (4) *Their moments are $M_k = \sum_{\pi \in D(k)} t^{|\pi|}$, with $D = P_2, NC_2$.*

PROOF. These are all results that we already know, the idea being as follows:

(1,2) These assertions follow from (3,4), via the general theory.

(3,4) These assertions follow by doing some combinatorics and calculus. \square

To summarize, our initial purpose for this chapter was to vaguely explore the basics of free probability, but all of a sudden, due to the power of Voiculescu's R -transform [86], we are now into stating and proving results which are on par with what we have been doing in the first part of this book, namely reasonably advanced probability theory.

This is certainly quite encouraging, and we will keep developing free probability in what follows, in the remainder of this book, with free analogues of everything, or almost, of what we have been doing in chapters 1-4, in relation with classical probability and its applications, and also with some conceptual explanations, and technical enhancements, of what we have been doing in chapters 5-8, in relation with the random matrices.

9e. Exercises

There has been a lot of exciting theory in this chapter, for the most in relation with various free product constructions, and as a first exercise on all this, we have:

EXERCISE 9.28. *Prove that given two algebras (A, tr) and (B, tr) , these algebras are free inside their free product $A * B$, endowed with its canonical free product trace.*

This is something that we already discussed in the above, but with some details missing. Time now to have this done, with all the details.

EXERCISE 9.29. *State and prove a complex analogue of the free CLT, as well as an analogue of the PLT, and study the limiting measures.*

This is something very instructive, and normally all the needed tools, namely the R -transform, and the free CLT as an illustration, are there. Of course, this is more than a regular exercise, and we will be back to this, in what follows, on several occasions.

CHAPTER 10

Circular variables

10a. Circular variables

We have seen so far that free probability theory leads to a remarkable free analogue of the CLT, with the limiting measure being the Wigner semicircle law. This is certainly something very interesting, theoretically speaking, and by reminding the fact that the Wigner laws appear in connection with many fundamental questions in mathematics, in relation with random walks on graphs, with Lie groups, and with random matrices as well, there are certainly many things to be done, as a continuation of this.

However, no hurry, and we will do this slowly. As a first objective, which is something quite straightforward, now that we have a free CLT, we would like to have as well a free analogue of the complex central limiting theorem (CCLT), adding to the classical CCLT, and providing us with free analogues Γ_t of the complex Gaussian laws G_t .

This will be something quite technical, and in order to get started, let us begin by recalling the theory of the complex Gaussian laws G_t . We first have:

DEFINITION 10.1. *The complex Gaussian law of parameter $t > 0$ is*

$$G_t = \text{law} \left(\frac{1}{\sqrt{2}}(a + ib) \right)$$

where a, b are independent, each following the law g_t .

There are many things that can be said about these laws, simply by adapting the known results from the real case, regarding the usual normal laws g_t . As a first such result, the above measures form convolution semigroups:

PROPOSITION 10.2. *The complex Gaussian laws have the property*

$$G_s * G_t = G_{s+t}$$

for any $s, t > 0$, and so they form a convolution semigroup.

PROOF. This is something that we know from chapter 1, coming from $g_s * g_t = g_{s+t}$, by taking the real and imaginary parts of all variables involved. \square

We have as well the following complex analogue of the CLT:

THEOREM 10.3 (CCLT). *Given complex variables $f_1, f_2, f_3, \dots \in L^\infty(X)$ which are i.i.d., centered, and with variance $t > 0$, we have, with $n \rightarrow \infty$, in moments,*

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n f_i \sim G_t$$

where G_t is the complex Gaussian law of parameter t .

PROOF. This is something that we know too from chapter 1, which follows from the real CLT, by taking real and imaginary parts. Indeed, let us write:

$$f_i = \frac{1}{\sqrt{2}}(x_i + iy_i)$$

The variables x_i satisfy then the assumptions of the CLT, so their rescaled averages converge to a normal law g_t , and the same happens for the variables y_i . The limiting laws that we obtain being independent, their rescaled sum is complex Gaussian, as desired. \square

Regarding now the moments, we have here the following result:

PROPOSITION 10.4. *The moments of the complex normal law are the numbers*

$$M_k(G_t) = t^{|k|/2} |\mathcal{P}_2(k)|$$

where $\mathcal{P}_2(k)$ is the set of matching pairings of $\{1, \dots, k\}$.

PROOF. This is again something that we know well too, from chapter 1, the idea being as follows, with $c = \frac{1}{\sqrt{2}}(a + ib)$ being the variable in Definition 10.1:

(1) In the case where k contains a different number of \circ and \bullet symbols, a rotation argument shows that the corresponding moment of c vanishes. But in this case we also have $\mathcal{P}_2(k) = \emptyset$, so the formula in the statement holds indeed, as $0 = 0$.

(2) In the case left, where k consists of p copies of \circ and p copies of \bullet , the corresponding moment is the p -th moment of $|c|^2$, which by some calculus is $t^p p!$. But in this case we have as well $|\mathcal{P}_2(k)| = p!$, so the formula in the statement holds indeed, as $t^p p! = t^p p!$. \square

As a final basic result regarding the laws G_t , we have the Wick formula:

THEOREM 10.5. *Given independent variables X_i , each following the complex normal law G_t , with $t > 0$ being a fixed parameter, we have the Wick formula*

$$E(X_{i_1}^{k_1} \dots X_{i_s}^{k_s}) = t^{s/2} \# \left\{ \pi \in \mathcal{P}_2(k) \mid \pi \leq \ker i \right\}$$

where $k = k_1 \dots k_s$ and $i = i_1 \dots i_s$, for the joint moments of these variables.

PROOF. This is something from chapter 1 too, the idea being as follows:

(1) In the case where we have a single complex normal variable X , we have to compute the moments of X , with respect to colored integer exponents $k = \circ\bullet\bullet\circ\dots$, and the formula in the statement coincides with the one in Theorem 10.4, namely:

$$E(X^k) = t^{|k|/2} |\mathcal{P}_2(k)|$$

(2) In general now, when expanding $X_{i_1}^{k_1} \dots X_{i_s}^{k_s}$ and rearranging the terms, we are left with doing a number of computations as in (1), then making the product of the numbers that we found. But this amounts in counting the partitions in the statement. \square

Let us discuss now the free analogues of the above results. As in the classical case, there is actually not so much work to be done here, in order to get started, because we can obtain the free convolution and central limiting results, simply by taking the real and imaginary parts of our variables. Following Voiculescu [85], [86], we first have:

DEFINITION 10.6. *The Voiculescu circular law of parameter $t > 0$ is given by*

$$\Gamma_t = \text{law} \left(\frac{1}{\sqrt{2}}(a + ib) \right)$$

where a, b are free, each following the Wigner semicircle law γ_t .

In other words, the passage $\gamma_t \rightarrow \Gamma_t$ is by definition entirely similar to the passage $g_t \rightarrow G_t$ from the classical case, by taking real and imaginary parts. As before in other similar situations, the fact that Γ_t is indeed well-defined is clear from definitions.

Let us start with a number of straightforward results, obtained by complexifying the free probability theory that we have. As a first result, we have, as announced above:

PROPOSITION 10.7. *The Voiculescu circular laws have the property*

$$\Gamma_s \boxplus \Gamma_t = \Gamma_{s+t}$$

so they form a 1-parameter semigroup with respect to free convolution.

PROOF. This follows from our result from chapter 9 stating that the Wigner laws γ_t have the free semigroup convolution property, by taking real and imaginary parts. \square

Next in line, also as announced above, and also from [86], we have the following natural free analogue of the complex central limiting theorem (CCLT):

THEOREM 10.8 (Free CCLT). *Given random variables x_1, x_2, x_3, \dots which are f.i.d., centered, with variance $t > 0$, we have, with $n \rightarrow \infty$, in moments,*

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n x_i \sim \Gamma_t$$

where Γ_t is the Voiculescu circular law of parameter t .

PROOF. This follows indeed from the free CLT, established in chapter 9, by taking real and imaginary parts. Indeed, let us write:

$$x_i = \frac{1}{\sqrt{2}}(y_i + iz_i)$$

The variables y_i satisfy then the assumptions of the free CLT, and so their rescaled averages converge to a semicircle law γ_t , and the same happens for the variables z_i :

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n y_i \sim \gamma_t \quad , \quad \frac{1}{\sqrt{n}} \sum_{i=1}^n z_i \sim \gamma_t$$

Now since the two limiting semicircle laws that we obtain in this way are free, their rescaled sum is circular, in the sense of Definition 10.6, and this gives the result. \square

Summarizing, we have so far complex analogues of both the classical and free CLT, and the basic theory of the limiting measures, including their semigroup property. As a conclusion to all this, let us formulate the following statement:

THEOREM 10.9. *We have classical and free limiting theorems, as follows,*

$$\begin{array}{ccc} FCLT & \text{————} & FCCLT \\ | & & | \\ CLT & \text{————} & CCLT \end{array}$$

the limiting laws being the following measures,

$$\begin{array}{ccc} \gamma_t & \text{————} & \Gamma_t \\ | & & | \\ g_t & \text{————} & G_t \end{array}$$

which form classical and free convolution semigroups.

PROOF. This follows indeed from the various results established above. To be more precise, the results about the left edge of the square are from the previous chapter, and the results about the right edge are those discussed in the above. \square

Going ahead with more study of the Voiculescu circular variables, less trivial now is the computation of their moments. We will do this in what follows, among others in order to expand Theorem 10.9 into something much sharper, involving as well moments.

For our computations, we will need explicit models for the circular variables. Following [86], and the material in chapter 9, let us start with the following key result:

PROPOSITION 10.10. *Let H be the complex Hilbert space having as basis the colored integers $k = \circ \bullet \bullet \circ \dots$, and consider the shift operators on this space:*

$$S : k \rightarrow \circ k \quad , \quad T : k \rightarrow \bullet k$$

We have then the following equalities of distributions,

$$S + S^* \sim \gamma_1 \quad , \quad S + T^* \sim \Gamma_1$$

with respect to the state $\varphi(T) = \langle Te, e \rangle$, where e is the empty word.

PROOF. This is standard free probability, the idea being as follows:

(1) The first formula, namely $S + S^* \sim \gamma_1$, is something that we already know, in a slightly different formulation, from chapter 9, when proving the CLT.

(2) As for the second formula, $S + T^* \sim \Gamma_1$, this follows from the first formula, by using the freeness results and the rotation tricks established in chapter 9. \square

At the combinatorial level now, we have the following result, which is in analogy with the moment theory of the Wigner semicircle law, developed above:

THEOREM 10.11. *A variable $a \in A$ follows the law Γ_1 precisely when its moments are*

$$\text{tr}(a^k) = |\mathcal{NC}_2(k)|$$

for any colored integer $k = \circ \bullet \bullet \circ \dots$

PROOF. By using Proposition 10.10, it is enough to do the computation in the model there. To be more precise, we can use the following explicit formulae for S, T :

$$S : k \rightarrow \circ k \quad , \quad T : k \rightarrow \bullet k$$

With these formulae in hand, our claim is that we have the following formula:

$$\langle (S + T^*)^k e, e \rangle = |\mathcal{NC}_2(k)|$$

In order to prove this formula, we can proceed as for the semicircle laws, in chapter 9 above. Indeed, let us expand the quantity $(S + T^*)^k$, and then apply the state φ .

With respect to the previous computation, from chapter 9, what happens is that the contributions will come this time via the following formulae, which must successively apply, as to collapse the whole product of S, S^*, T, T^* variables into a 1 quantity:

$$S^*S = 1 \quad , \quad T^*T = 1$$

As before, in the proof for the semicircle laws, from chapter 9, these applications of the rules $S^*S = 1$, $T^*T = 1$ must appear in a noncrossing manner, but what happens now, in contrast with the computation from the proof in chapter 9 where $S + S^*$ was self-adjoint, is that at each point where the exponent k has a \circ entry we must use $T^*T = 1$, and at each point where the exponent k has a \bullet entry we must use $S^*S = 1$. Thus the contributions, which are each worth 1, are parametrized by the partitions $\pi \in \mathcal{NC}_2(k)$. Thus, we obtain the above moment formula, as desired. \square

More generally now, by rescaling, we have the following result:

THEOREM 10.12. *A variable $a \in A$ is circular, $a \sim \Gamma_t$, precisely when its moments are given by the formula*

$$\operatorname{tr}(a^k) = t^{|k|/2} |\mathcal{NC}_2(k)|$$

for any colored integer $k = \circ \bullet \bullet \circ \dots$

PROOF. This follows indeed from Theorem 10.11, by rescaling. Alternatively, we can get this as well directly, by suitably modifying Proposition 10.10 first. \square

Even more generally now, we have the following free version of the Wick rule:

THEOREM 10.13. *Given free variables a_i , each following the Voiculescu circular law Γ_t , with $t > 0$ being a fixed parameter, we have the Wick type formula*

$$\operatorname{tr}(a_{i_1}^{k_1} \dots a_{i_s}^{k_s}) = t^{s/2} \# \left\{ \pi \in \mathcal{NC}_2(k) \mid \pi \leq \ker i \right\}$$

where $k = k_1 \dots k_s$ and $i = i_1 \dots i_s$, for the joint moments of these variables, with the inequality $\pi \leq \ker i$ on the right being taken in a technical, appropriate sense.

PROOF. This follows a bit as in the classical case, the idea being as follows:

(1) In the case where we have a single complex normal variable a , we have to compute the moments of a , with respect to colored integer exponents $k = \circ \bullet \bullet \circ \dots$, and the formula in the statement coincides with the one in Theorem 10.12, namely:

$$\operatorname{tr}(a^k) = t^{|k|/2} |\mathcal{NC}_2(k)|$$

(2) In general now, when expanding the product $a_{i_1}^{k_1} \dots a_{i_s}^{k_s}$ and rearranging the terms, we are left with doing a number of computations as in (1), and then making the product of the expectations that we found. But this amounts precisely in counting the partitions in the statement, with the condition $\pi \leq \ker i$ there standing precisely for the fact that we are doing the various type (1) computations independently. \square

All the above was a bit brief, based on Voiculescu's original paper [86], and on his foundational free probability book with Dykema and Nica [90]. The combinatorics of the free families of circular variables, called "circular systems", is something quite subtle, and there has been a lot of work developed in this direction. For a complement to the above material, with a systematic study using advanced tools from combinatorics, we refer to the more recent book by Nica and Speicher [71]. We will be actually back to this, in this book too, namely in chapter 12 below, when talking about free cumulants.

On the same topic, let us mention as well that various technical extensions and generalizations of the above results can be found, hidden as technical lemmas, throughout the random matrix and operator algebra literature, in connection with free probability, with the notable users of the circular systems including, besides Voiculescu himself, Dykema [44], Mingo, Nica, Speicher [67], [69], [71], [79], [80], and Shlyakhtenko [78].

Getting back now to the case of the single variables, from Theorem 10.12, the formula there has the following more conceptual interpretation:

THEOREM 10.14. *The moments of the Voiculescu laws are the numbers*

$$M_k(\Gamma_t) = \sum_{\pi \in \mathcal{NC}_2(k)} t^{|\pi|}$$

with “ \mathcal{NC}_2 ” standing for the noncrossing matching pairings.

PROOF. This follows from the formula in Theorem 10.12. Indeed, we know from there that a variable $a \in A$ is circular, of parameter $t > 0$, precisely when we have the following formula, for any colored integer $k = \circ \bullet \bullet \circ \dots$:

$$tr(a^k) = t^{|k|/2} |\mathcal{NC}_2(k)|$$

Now since the number of blocks of a pairing $\pi \in \mathcal{NC}_2(k)$ is given by $|\pi| = |k|/2$, this formula can be written in the following alternative way:

$$tr(a^k) = \sum_{\pi \in \mathcal{NC}_2(k)} t^{|\pi|}$$

Thus, we are led to the conclusion in the statement. □

All this is quite nice, when compared with the similar results from the classical case, regarding the complex Gaussian laws, that we established above, and with other results of the same type as well. As a conclusion to these considerations, we can now formulate a global result regarding the classical and free complex Gaussian laws, as follows:

THEOREM 10.15. *The complex Gaussian laws G_t and the circular Voiculescu laws Γ_t , given by the formulae*

$$G_t = law \left(\frac{1}{\sqrt{2}}(a + ib) \right) \quad , \quad \Gamma_t = law \left(\frac{1}{\sqrt{2}}(\alpha + i\beta) \right)$$

where $a, b/\alpha, \beta$ are independent/free, following g_t/γ_t , have the following properties:

- (1) They appear via the complex CLT, and the free complex CLT.
- (2) They form semigroups with respect to the operations $*$ and \boxplus .
- (3) Their moments are $M_k = \sum_{\pi \in D(k)} t^{|\pi|}$, with $D = \mathcal{P}_2, \mathcal{NC}_2$.

PROOF. This is a summary of results that we know, the idea being as follows:

(1) This is something quite straightforward, by using the linearization results provided by the logarithm of the Fourier transform, and by the R -transform.

(2) This is quite straightforward, too, once again by using the linearization results provided by the logarithm of the Fourier transform, and by the R -transform.

(3) This comes by doing some combinatorics and calculus in the classical case, and some combinatorics and operator theory in the free case, as explained above. □

More generally now, we can put everything together, with some previous results included as well, and we have the following result at the level of the moments of the asymptotic laws that we found so far, in classical and free probability:

THEOREM 10.16. *The moments of the various central limiting measures, namely*

$$\begin{array}{ccc} \gamma_t & \text{-----} & \Gamma_t \\ | & & | \\ g_t & \text{-----} & G_t \end{array}$$

are always given by the same formula, involving partitions, namely

$$M_k = \sum_{\pi \in D(k)} t^{|\pi|}$$

where the sets of partitions $D(k)$ in question are respectively

$$\begin{array}{ccc} NC_2 & \longleftarrow & \mathcal{NC}_2 \\ | & & | \\ P_2 & \longleftarrow & \mathcal{P}_2 \end{array}$$

and where $|\cdot|$ is the number of blocks.

PROOF. This follows by putting together the various moment results that we have, from the previous chapter, and from Theorem 10.15. \square

Summarizing, we are done with the combinatorial program outlined in the beginning of the present chapter. We will be back to this in the next chapter, by adding some new laws to the picture, coming from the classical and free PLT and CPLT, and then in the chapter afterwards, 12 below, with full conceptual explanations for all this.

10b. Multiplicative results

With the above basic combinatorial study done, let us discuss now a number of more advanced results regarding the Voiculescu circular laws Γ_t , which are of multiplicative nature, and quite often have no classical counterpart. Things here will be quite technical, and all that follows will be rather an introduction to the subject.

In general now, in order to deal with multiplicative questions for the free random variables, we are in need of results regarding the multiplicative free convolution operation \boxtimes . Let us recall from chapter 9 that we have the following result:

DEFINITION 10.17. We have a free convolution operation \boxtimes , constructed as follows:

- (1) For abstract distributions, via $\mu_a \boxtimes \mu_b = \mu_{ab}$, with a, b free.
- (2) For real measures, via $\mu_a \boxtimes \mu_b = \mu_{\sqrt{ab}\sqrt{a}}$, with a, b self-adjoint and free.

All this is quite tricky, explained in chapter 9, the idea being that, while (1) is straightforward, (2) is not, and comes by considering the variable $c = \sqrt{ab}\sqrt{a}$, which unlike ab is always self-adjoint, and whose moments are given by:

$$\begin{aligned} \operatorname{tr}(c^k) &= \operatorname{tr}[(\sqrt{ab}\sqrt{a})^k] \\ &= \operatorname{tr}[\sqrt{aba} \dots ab\sqrt{a}] \\ &= \operatorname{tr}[\sqrt{a} \cdot \sqrt{aba} \dots ab] \\ &= \operatorname{tr}[(ab)^k] \end{aligned}$$

As a remark here, observe that we have used in the above, and actually for the first time since talking about freeness, the trace property of the trace, namely:

$$\operatorname{tr}(ab) = \operatorname{tr}(ba)$$

This is quite interesting, philosophically speaking, because in the operator algebra world there are many interesting examples of subalgebras $A \subset B(H)$ coming with natural linear forms $\varphi : A \rightarrow \mathbb{C}$ which are continuous and positive, but which are not traces. See [28]. It is possible to do a bit of free probability on such algebras, but not much.

Quite remarkably, the free multiplicative convolution operation \boxtimes can be linearized, in analogy with what happens for the usual multiplicative convolution \times , and the additive operations $*$, \boxplus as well. We have here the following result, due to Voiculescu [87]:

THEOREM 10.18. The free multiplicative convolution operation \boxtimes for the real probability measures $\mu \in \mathcal{P}(\mathbb{R})$ can be linearized as follows:

- (1) Start with the sequence of moments M_k , then compute the moment generating function, or Stieltjes transform of the measure:

$$f(z) = 1 + M_1z + M_2z^2 + M_3z^3 + \dots$$

- (2) Perform the following operations to the Stieltjes transform:

$$\psi(z) = f(z) - 1$$

$$\psi(\chi(z)) = z$$

$$S(z) = \left(1 + \frac{1}{z}\right) \chi(z)$$

- (3) Then $\log S$ linearizes the free multiplicative convolution, $S_{\mu \boxtimes \nu} = S_\mu S_\nu$.

PROOF. There are several proofs here, with the original proof of Voiculescu [87] being quite similar to the proof of the R -transform theorem, using free Fock space models, then with a proof by Haagerup [52], obtained by further improving on this, and finally with the proof from the book of Nica and Speicher [71], using pure combinatorics. The proof of Haagerup [52], which is the most in tune with the present book, is as follows:

(1) According to our conventions from Definition 10.17, we want to prove that, given noncommutative variables a, b which are free, we have the following formula:

$$S_{\mu_{ab}}(z) = S_{\mu_a}(z)S_{\mu_b}(z)$$

(2) For this purpose, consider the orthogonal shifts S, T on the free Fock space, as in chapter 9. By using the algebraic arguments from chapter 9, from the proof of the R -transform theorem, we can assume as there that our variables have a special form, that fits our present objectives, and to be more specifically, the following form:

$$a = (1 + S)f(S^*) \quad , \quad b = (1 + T)g(T^*)$$

Our claim, which will prove the theorem, is that we have the following formulae, for the S -transforms of the various variables involved:

$$S_{\mu_a}(z) = \frac{1}{f(z)} \quad , \quad S_{\mu_b}(z) = \frac{1}{g(z)} \quad , \quad S_{\mu_{ab}}(z) = \frac{1}{f(z)g(z)}$$

(3) Let us first compute S_{μ_a} . We know that we have $a = (1 + S)f(S^*)$, with S being the shift on $l^2(\mathbb{N})$. Given $|z| < 1$, consider the following vector:

$$p = \sum_{k \geq 0} z^k e_k$$

The shift and its adjoint act on this vector in the following way:

$$Sp = \sum_{k \geq 0} z^k e_{k+1} = \frac{p - e_0}{z}$$

$$S^*p = \sum_{k \geq 1} z^k e_{k-1} = zp$$

Thus $f(S^*)p = f(z)p$, and we deduce from this that we have:

$$\begin{aligned} ap &= (1 + S)f(z)p \\ &= f(z)(p + Sp) \\ &= f(z) \left(p + \frac{p - e_0}{z} \right) \\ &= \left(1 + \frac{1}{z} \right) f(z)p - \frac{f(z)}{z} e_0 \end{aligned}$$

By dividing everything by $(1 + 1/z)f(z)$, this formula becomes:

$$\frac{z}{1+z} \cdot \frac{1}{f(z)} ap = p - \frac{e_0}{1+z}$$

We can write this latter formula in the following way:

$$\left(1 - \frac{z}{1+z} \cdot \frac{1}{f(z)} a\right) p = \frac{e_0}{1+z}$$

Now by inverting, we obtain from this the following formula:

$$\left(1 - \frac{z}{1+z} \cdot \frac{1}{f(z)} a\right)^{-1} e_0 = (1+z)p$$

(4) But this gives us the formula of S_{μ_a} . Indeed, consider the following function:

$$\rho(z) = \frac{z}{1+z} \cdot \frac{1}{f(z)}$$

With this notation, the formula that we found in (3) becomes:

$$(1 - \rho(z)a)^{-1} e_0 = (1+z)p$$

By using this, in terms of $\varphi(T) = \langle Te_0, e_0 \rangle$, we obtain:

$$\begin{aligned} \varphi((1 - \rho(z)a)^{-1}) &= \langle (1 - \rho(z)a)^{-1} e_0, e_0 \rangle \\ &= \langle (1+z)p, e_0 \rangle \\ &= 1+z \end{aligned}$$

Thus the above function ρ is the inverse of the following function:

$$\psi(z) = \varphi\left(\frac{1}{1-za}\right) - 1$$

But this latter function is the ψ function from the statement, and so ρ is the function χ from the statement, and we can finish our computation, as follows:

$$\begin{aligned} S_{\mu_a}(z) &= \frac{1+z}{z} \cdot \rho(z) \\ &= \frac{1+z}{z} \cdot \frac{z}{1+z} \cdot \frac{1}{f(z)} \\ &= \frac{1}{f(z)} \end{aligned}$$

(5) A similar computation, or just a symmetry argument, gives $S_{\mu_b}(z) = 1/g(z)$. In order to compute now $S_{\mu_{ab}}(z)$, we use a similar trick. Consider the following vector of $l^2(\mathbb{N} * \mathbb{N})$, with the primes and double primes referring to the two copies of \mathbb{N} :

$$q = e_0 + \sum_{k \geq 1} (e'_1 + e''_1 + e'_1 \otimes e''_1)^{\otimes k}$$

The adjoints of the shifts S, T act as follows on this vector:

$$S^*q = z(1 + T)q \quad , \quad T^*q = zq$$

By using these formulae, we have the following computation:

$$\begin{aligned} abq &= (1 + S)f(S^*)(1 + T)g(T^*)q \\ &= (1 + S)f(S^*)(1 + T)g(z)q \\ &= g(z)(1 + S)f(S^*)(1 + T)q \end{aligned}$$

In order to compute the last term, observe that we have:

$$\begin{aligned} S^*(1 + T)q &= (S^* + S^*T)q \\ &= S^*q \\ &= z(1 + T)q \end{aligned}$$

Thus $f(S^*)(1 + T)q = f(z)(1 + T)q$, and back to our computation, we have:

$$\begin{aligned} abq &= g(z)(1 + S)f(z)(1 + T)q \\ &= f(z)g(z)(1 + S)(1 + T)q \\ &= f(z)g(z) \left(\frac{1 + z}{z} \cdot q - \frac{e_0}{z} \right) \end{aligned}$$

Now observe that we can write this formula as follows:

$$\left(1 - \frac{z}{1 + z} \cdot \frac{1}{f(z)g(z)} \cdot ab \right) q = \frac{e_0}{1 + z}$$

By inverting, we obtain from this the following formula:

$$\left(1 - \frac{z}{1 + z} \cdot \frac{1}{f(z)g(z)} \cdot ab \right)^{-1} e_0 = (1 + z)q$$

(6) But this formula that we obtained is similar to the formula that we obtained at the end of (3) above. Thus, we can use the same argument as in (4), and we obtain:

$$S_{\mu_{ab}}(z) = \frac{1}{f(z)g(z)}$$

We are therefore done with the computations, and this finishes the proof. \square

Getting back now to the circular variables, let us look at the polar decomposition of such variables. In order to discuss this, let us start with a well-known result:

THEOREM 10.19. *We have the following results:*

- (1) Any matrix $T \in M_N(\mathbb{C})$ has a polar decomposition, $T = U|T|$.
- (2) Assuming $T \in A \subset M_N(\mathbb{C})$, we have $U, |T| \in A$.
- (3) Any operator $T \in B(H)$ has a polar decomposition, $T = U|T|$.
- (4) Assuming $T \in A \subset B(H)$, we have $U, |T| \in \bar{A}$, weak closure.

PROOF. All this is standard, the idea being as follows:

(1) In each case under consideration, the first observation is that the matrix or general operator T^*T being positive, it has a square root:

$$|T| = \sqrt{T^*T}$$

(2) With this square root extracted, in the invertible case we can compare the action of T and $|T|$, and we conclude that we have $T = U|T|$, with U being a unitary. In the general, non-invertible case, a similar analysis leads to the conclusion that we have as well $T = U|T|$, but with U being this time a partial isometry.

(3) In what regards now algebraic and topological aspects, in finite dimensions the extraction of the square root, and so the polar decomposition itself, takes place over the matrix blocks of the ambient algebra $A \subset M_N(\mathbb{C})$, and so takes place inside A itself.

(4) In infinite dimensions however, we must take the weak closure, an illustrating example here being the functions $f \in A$ belonging to the algebra $A = C(X)$, represented on $H = L^2(X)$, whose polar decomposition leads into the bigger algebra $\bar{A} = L^\infty(X)$. \square

Summarizing, we have a basic linear algebra result, regarding the polar decomposition of the usual matrices, and in infinite dimensions pretty much the same happens, with the only subtlety coming from the fact that the ambient operator algebra $A \subset B(H)$ must be taken weakly closed. We will be back to this, with more details, in chapter 15 below, when talking about such algebras $A \subset B(H)$, which are called von Neumann algebras.

In connection with our probabilistic questions, we first have the following result:

PROPOSITION 10.20. *The polar decomposition of semicircular variables is $s = eq$, with the variables e, q being as follows:*

- (1) e has moments $1, 0, 1, 0, 1, \dots$
- (2) q is quarter-circular.
- (3) e, q are independent.

PROOF. It is enough to prove the result in a model of our choice, and the best choice here is the most straightforward model for the semicircular variables, namely:

$$s = x \in L^\infty([-2, 2], \gamma_1)$$

To be more precise, we endow the interval $[-2, 2]$ with the probability measure γ_1 , and we consider here the variable $s = x = (x \rightarrow x)$, which is trivially semicircular. The polar decomposition of this variable is then $s = eq$, with e, q being as follows:

$$e = \operatorname{sgn}(x) \quad , \quad q = |x|$$

Now since e has moments $1, 0, 1, 0, 1, \dots$, and also q is quarter-circular, and finally e, q are independent, this gives the result in our model, and so in general. \square

Less trivial now is the following result, due to Voiculescu [88]:

THEOREM 10.21. *The polar decomposition of circular variables is $c = uq$, with the variables u, q being as follows:*

- (1) u is a Haar unitary.
- (2) q is quarter-circular.
- (3) u, q are free.

PROOF. This is something which looks quite similar to Proposition 10.20, but which is more difficult, and can be however proved, via various techniques:

(1) The original proof, by Voiculescu in [88], uses Gaussian random matrix models for the circular variables. We will discuss this proof at the end of the present chapter, after developing the needed Gaussian random matrix model technology.

(2) A second proof, obtained by pure combinatorics, in the spirit of Theorem 10.13, regarding the free Wick formula, and of Theorem 10.18, regarding the S -transform, or rather in the spirit of the underlying combinatorics of these results, is the one in [71].

(3) Finally, there is as well a third proof, from [6], more in the spirit of the free Fock space proofs for the R and S transform results, from [86], [87], using a suitable generalization of the free Fock spaces. We will discuss this proof right below. \square

10c. Semigroup models

We discuss here, following [6], the direct approach to Theorem 10.21, with purely algebraic techniques. We will use semigroup algebras, jointly generalizing the main models that we have, namely group algebras, and free Fock spaces. Let us start with:

DEFINITION 10.22. *We call “semigroup” a unital semigroup, embeddable into a group:*

$$M \subset G$$

For such a semigroup M , we use the notation

$$M^{-1} = \left\{ m^{-1} \mid m \in M \right\}$$

regarded as a subset of some group G containing M , as above.

As a first observation, the above embeddability assumption $M \subset G$ tells us that the usual group cancellation rules hold in M , namely:

$$ab = ac \implies b = c$$

$$ba = ca \implies b = c$$

Regarding the precise relation between M and the various groups G containing it, it is possible to talk here about the Grothendieck group G associated to such a semigroup M . However, we will not need this in what follows, and use Definition 10.22 as such.

With the above definition in hand, we have the following construction, which unifies the main models that we have, namely the group algebras, and the free Fock spaces:

PROPOSITION 10.23. *Let M be a semigroup. By using the left simplifiability of M we can define, as for the discrete groups, an embedding of semigroups, as follows:*

$$(M, \cdot) \rightarrow (B(l^2(M)), \circ)$$

$$m \rightarrow \lambda_M(m) = [\delta_n \rightarrow \delta_{mn}]$$

Via this embedding, the C^ -algebra $C^*(M) \subset B(l^2(M))$ generated by $\lambda_M(M)$, together with the following canonical state, is a noncommutative random variable algebra:*

$$\tau_M(T) = \langle T\delta_e, \delta_e \rangle$$

Also, the operators in $\lambda_M(M)$ are isometries, but not necessarily unitaries.

PROOF. Everything here is standard, as for the usual group algebras, with the only subtlety appearing at the level of the isometry property of the operators $\lambda_M(m)$. To be more precise, for every $m \in M$, the adjoint operator $\lambda_M(m)^*$ is given by:

$$\lambda_M(m)^*(\delta_n) = \sum_{x \in M} \langle \lambda_M(m)^*\delta_n, \delta_x \rangle \delta_x = \sum_{x \in M} \delta_{n, mx} \delta_x$$

Thus we have indeed the isometry property for these operators, namely:

$$\lambda_M(m)^*\lambda_M(m) = 1$$

As for the unitarity property of the such operators, this definitely holds in the usual discrete group case, $M = G$, but not in general. As a basic example here, for the semigroup $M = \mathbb{N}$, which satisfies of course the assumptions in Definition 10.22, the operator $\lambda_M(m)$ associated to the element $m = 1 \in \mathbb{N}$ is the usual shift:

$$\lambda_{\mathbb{N}}(1) = S \in B(l^2(\mathbb{N}))$$

But this shift S , that we know well from the above, is an isometry which is not a unitary. Thus, we are led to the conclusions in the statement. \square

At the level of examples now, as announced above, we have:

PROPOSITION 10.24. *The construction $M \rightarrow C^*(M)$ is as follows:*

- (1) *For the discrete groups, $M = G$, we obtain in this way the usual discrete group algebras $C^*(G)$, as previously constructed in the above.*
- (2) *For a free semigroup, $M = \mathbb{N}^I$, we obtain the algebra of creation operators over the full Fock space over \mathbb{R}^I , with the state associated to the vacuum vector.*

PROOF. All this is clear from definitions, with (1) being obvious, and (2) coming via our usual identifications for the free Fock spaces and related algebras. \square

As a key observation now, enabling us to do some probability, we have:

PROPOSITION 10.25. *If $M \subset N$ are semigroups satisfying the condition*

$$M(N - M) = N - M$$

then for every family $\{a_i\}_{i \in I}$ of elements in M , we have the formula

$$\{\lambda_N(a_i)\}_{i \in I} \sim \{\lambda_M(a_i)\}_{i \in I}$$

as an equality of joint distributions, with respect to the canonical states.

PROOF. Assuming $M \subset N$ we have $l^2(M) \subset l^2(N)$, and for $m, m' \in M$ we have:

$$\lambda_M(m)\delta_{m'} = \lambda_N(m)\delta_{m'}$$

Thus if we suppose $M(N - M) = N - M$, as in the statement, then we have:

$$\begin{aligned} \lambda_M(m)^* \delta_{m'} &= \sum_{x \in M} \delta_{m', mx} \delta_x \\ &= \sum_{x \in N} \delta_{m', mx} \delta_x \\ &= \lambda_N(m)^* \delta_{m'} \end{aligned}$$

In particular, if $m_1, \dots, m_k \in M$, and $\alpha_1, \dots, \alpha_k$ are exponents in $\{1, *\}$, then:

$$\lambda_M(m_1)^{\alpha_1} \dots \lambda_M(m_k)^{\alpha_k} \delta_e = \lambda_N(m_1)^{\alpha_1} \dots \lambda_N(m_k)^{\alpha_k} \delta_e$$

Thus, we are led to the conclusion in the statement. \square

Following [6], let us introduce the following technical notion:

DEFINITION 10.26. *Let N be a semigroup. Consider the following order on it:*

$$a \preceq_N b \iff b \in aN$$

We say that N is in the class E if it satisfies one of the following equivalent conditions:

- (1) *For \preceq_N every bounded subset is totally ordered.*
- (2) *$a \preceq c, b \preceq c \implies a \preceq b$ or $b \preceq a$.*
- (3) *$aN \cap bN \neq \emptyset \implies aN \subset bN$ or $bN \subset aN$.*
- (4) *$NN^{-1} \cap N^{-1}N = N \cup N^{-1}$.*

Also by following [6], let us introduce as well the following notion, which is something standard in the combinatorial theory of semigroups:

DEFINITION 10.27. *Let $(a_i)_{i \in I}$ be a family of elements in a semigroup N .*

- (1) *We say that $(a_i)_{i \in I}$ is a code if the semigroup $M \subset N$ generated by the a_i is isomorphic to \mathbb{N}^{*I} , via $a_i \rightarrow e_i$, and satisfies $M(N - M) = N - M$.*
- (2) *We say that $(a_i)_{i \in I}$ is a prefix if $a_i \in a_j N \implies i = j$, which means that the elements a_i are not comparable via the order relation \preceq_N .*

In our probabilistic setting, the notion of code is of interest, due to:

PROPOSITION 10.28. *Assuming that $(a_i, b_i)_{i \in I}$ is a code, the family*

$$\left(\frac{1}{2}(\lambda_N(a_i) + \lambda_N(b_i)^*) \right)_{i \in I}$$

is a circular family, in the sense of free probability theory.

PROOF. Let $(a_i, b_i)_{i \in I}$ be a code, and consider the following family:

$$\left(\lambda_N(a_i), \lambda_N(b_i) \right)_{i \in I} \in B(l^2(N))$$

By using Proposition 10.25, this family has the same distribution as a family of creation operators associated to a family of $2I$ orthonormal vectors, on the free Fock space:

$$\left(\lambda_{\mathbb{N}^*I}(e_i), \lambda_{\mathbb{N}^*I}(f_i) \right)_{i \in I} \in B(l^2(N^*I))$$

Thus, we obtain the result, via the standard facts about the circular systems on free Fock spaces, that we know from chapter 9. \square

In view of this, the following result provides us with a criterion for finding circular systems in the algebras of the semigroups in the class E , from Definition 10.26:

PROPOSITION 10.29. *For a semigroup $N \in E$, a family*

$$(a_i)_{i \in I} \subset N$$

having at least two elements is a prefix if and only if it is a code.

PROOF. We have two implications to be proved, as follows:

(1) Let first $(a_i)_{i \in I}$ be a code which is not a prefix, for instance because we have $a_i = a_j n$ with $i \neq j, n \in N$. Then n is in the semigroup M generated by the a_k and $a_i = a_j n$ with $i \neq j$, so M cannot be free, and this is a contradiction, as desired.

(2) Conversely, suppose now that $(a_i)_{i \in I}$ is a prefix and let, with $m \in N$:

$$A = a_{i_1}^{\alpha_1} \dots a_{i_n}^{\alpha_n} m = a_{j_1}^{\beta_1} \dots a_{j_s}^{\beta_s}$$

We have then $a_{i_1} \preceq A$, $a_{j_1} \preceq A$, and so $i_1 = j_1$. We can therefore simplify A to the left by a_{i_1} . A recurrence on $\sum \alpha_i$ shows then that we have $n \leq s$ and:

$$\begin{aligned} a_{i_k} &= a_{j_k} \quad , \quad \forall k \leq n \\ \alpha_k &= \beta_k \quad , \quad \forall k < n \\ \alpha_n &\leq \beta_n \\ m &= a_{j_n}^{\beta_n - \alpha_n} a_{j_{n+1}}^{\beta_{n+1}} \dots a_{j_s}^{\beta_s} \end{aligned}$$

Finally, we know that m is in the semigroup generated by the a_i , so we have a code. Moreover, for $m = e$ we obtain that we have $n = s$, $a_{j_k} = a_{i_k}$ and $\alpha_k = \beta_k$ for any $k \leq n$. Thus the variables a_i freely generate the semigroup M , and so the family $(a_i)_{i \in I}$ is a code. Thus, we are led to the conclusion in the statement. \square

Summarizing, we have some good freeness results, for our semigroups. Before getting into applications, let us discuss now the examples. We have here the following result:

PROPOSITION 10.30. *The class E has the following properties:*

- (1) *All the groups are in E .*
- (2) *The positive parts of totally ordered abelian groups are in E .*
- (3) *If G is a group and $M \in E$, then $M \times G \in E$.*
- (4) *If A_1, A_2 are in E , then the free product $A_1 * A_2$ is in E .*

PROOF. This is something elementary, whose proof goes as follows:

- (1) This is obvious, coming from definitions.
- (2) This is obvious as well, because M is here totally ordered by \preceq_M .
- (3) Let G be a group and $M \in E$. We have then, as desired:

$$\begin{aligned}
 & (M \times G)(M \times G)^{-1} \cap (M \times G)^{-1}(M \times G) \\
 = & (M \times G)(M^{-1} \times G) \cap (M^{-1} \times G)(M \times G) \\
 = & (MM^{-1} \times G) \cap (M^{-1}M \times G) \\
 = & (MM^{-1} \cap M^{-1}M) \times G \\
 = & (M \cup M^{-1}) \times G \\
 = & (M \times G) \cup (M^{-1} \times G) \\
 = & (M \times G) \cup (M \times G)^{-1}
 \end{aligned}$$

- (4) Let $a, b, c \in A_1 * A_2$ such that $ab = c$. We write, as reduced words:

$$a = x_1 \dots x_n \quad , \quad b = y_1 \dots y_m \quad , \quad c = z_1 \dots z_p$$

Now let s be such that the following equalities happen:

$$x_n y_1 = 1 \quad , \quad \dots \quad , \quad x_{n-s+1} y_s = 1 \quad , \quad x_{n-s} y_{s+1} \neq 1$$

Consider now the following element:

$$u = x_{n-s+1} \dots x_n = (y_1 \dots y_s)^{-1}$$

We have then the following computation:

$$c = ab = x_1 \dots x_{n-s} y_{s+1} \dots y_m$$

Now let $i \in \{1, 2\}$ be such that $z_{n-s} \in A_i$. There are two cases:

– If $x_{n-s} \in A_1$ and $y_{s+1} \in A_2$ or if $x_{n-s} \in A_2$ and $y_{s+1} \in A_1$, then $x_1 \dots x_{n-s} y_{s+1} \dots y_m$ is a reduced word. In particular, we have $x_1 = z_1, x_2 = z_2$, and so on up to $x_{n-s} = z_{n-s}$. Thus we have $a = z_1 \dots z_{n-s} u$, with u invertible.

– If $x_{n-s}, y_{s+1} \in A_i$ then $x_1 = z_1$ and so on, up to $x_{n-s-1} = z_{n-s-1}$ and $x_{n-s} y_{s+1} = z_{n-s}$. In this case we have $a = z_1 \dots z_{n-s-1} x_{n-s} u$, with u invertible.

Now observe that in both cases we obtained that a is of the form $z_1 \dots z_f x u$ for some f , with u invertible and such that if $z_{f+1} \in A_i$, then there exists $y \in A_i$ such that:

$$xy = z_{f+1}$$

Indeed, we can take $f = n - s - 1$ and $x = z_{n-s}, y = 1$ in the first case, and $x = x_{n-s}, y = y_{s+1}$ in the second one. Suppose now that $A_1, A_2 \in E$ and let $a, b, a', b' \in A_1 * A_2$ such that $ab = a'b'$. Let $z_1 \dots z_p$ be the decomposition of $ab = a'b'$ as a reduced word. Then we can decompose our words, as above, in the following way:

$$a = z_1 \dots z_f x u \quad , \quad a' = z_1 \dots z_{f'} x' u'$$

We have to show that $a = a'm$ or that $a' = am$ for some $m \in A_1 * A_2$. But this is clear in all three cases that can appear, namely $f < f', f' < f, f = f'$. \square

We can now formulate a main result about semigroup freeness, as follows:

THEOREM 10.31. *The following happen:*

- (1) *Given $M \subset N$, both in the class E , satisfying $M(N - M) = N - M$, any x in the $*$ -algebra generated by $\lambda(M)$ can be written as follows, with $p_i, q_i \in M$:*

$$x = \sum_i a_i \lambda_N(p_i) \lambda_N(q_i)^*$$

- (2) *Assume $A, B \in E$, and let x be an element of the $*$ -algebra generated by $\lambda_{A*B}(A)$ such that $\tau(x) = 0$. If W_A, W_B are respectively the sets of reduced words beginning by an element of A, B , then x acts as follows:*

$$l^2(W_B \cup \{e\}) \rightarrow l^2(W_A)$$

- (3) *Let $A, B \in E$. Then $\lambda_{A*B}(A)$ and $\lambda_{A*B}(B)$ are free.*

PROOF. This follows from our results so far, the idea being is as follows:

(1) It is enough to prove this for elements of the form $x = \lambda(m)^* \lambda(n)$ with $m, n \in M$, because the general case will follow easily from this. In order to do so, observe that $x = \lambda(m)^* \lambda(n)$ is different from 0 precisely when there exist $a, b \in N$ such that:

$$\langle \lambda(m)^* \lambda(n) \delta_a, \delta_b \rangle \neq 0$$

That is, the following condition must be satisfied:

$$na = mb$$

We know that there exists $c \in N$ with $n = mc$ or with $m = nc$. Moreover, as $M(N - M) = N - M$, it follows that $c \in M$. Thus $x = \lambda(m)^* \lambda(n) \neq 0$ implies that $x = \lambda(c)$ or $x = \lambda(c)^*$ with $c \in M$, and this finishes the proof.

- (2) We apply (1) with $M = A$ and $N = A * B$ for writing, with $p_i, q_i \in A$:

$$x = \sum_i a_i \lambda(p_i) \lambda(q_i)^*$$

Consider now the following element:

$$\tau(\lambda(p_i)\lambda(q_i)^*) = \sum_x \delta_{e,p_i x} \delta_{e,q_i x}$$

This element is nonzero precisely when $p_i = q_i$ is invertible, and in this case:

$$\lambda(p_i)\lambda(q_i)^* = 1$$

Now since we assumed $\tau(x) = 0$, it follows that we can write:

$$x = \sum a_i \lambda(p_i)\lambda(q_i)^* \quad , \quad \tau(\lambda(p_i)\lambda(q_i)^*) = 0$$

By linearity, it is enough to prove the result for $x = \lambda(p_i)\lambda(q_i)^*$. Let $m \in W_B \cup \{e\}$ and suppose that $x\delta_m \neq 0$. Then $\lambda(q_i)^*\delta_m \neq 0$ implies that $m = q_i c$ for some word $c \in A * B$. As $q_i \in A$ and $m \in W_B \cup \{e\}$, it follows that q_i is invertible. Now observe that:

$$p_i q_i^{-1} = 1 \implies \tau(x) = 1$$

It follows that we have, as desired:

$$x\delta_m = \delta_{p_i q_i^{-1} m} \in l^2(W_A)$$

(3) This follows from (2) above. Indeed, let $P = x_n \dots x_1$ be a product of elements in $\ker(\tau)$, such that x_{2k} is in the $*$ -algebra generated by $\lambda(B)$ and x_{2k+1} is in the $*$ -algebra generated by $\lambda(A)$. Then $x_1 \delta_e \in l^2(W_A)$. Thus $x_2 x_1 \delta_e \in l^2(W_B)$, and so on. By a recurrence, $P \delta_e$ is in $l^2(W_A)$ or in $l^2(W_B)$. But this implies that $\tau(P) = 0$, as desired. \square

As a main application of the above semigroup technology, we have:

THEOREM 10.32. *Consider a Haar unitary u , free from a semicircular s . Then*

$$c = us$$

is a circular variable.

PROOF. Denote by z the image of $1 \in \mathbb{Z}$ and by n the image of $1 \in \mathbb{N}$ by the canonical embeddings into the free product $\mathbb{Z} * \mathbb{N}$. Let $\lambda = \lambda_{\mathbb{Z} * \mathbb{N}}$. We know that $\mathbb{Z} * \mathbb{N} \in E$. Also (zn, nz^{-1}) is obviously a prefix, so it is a code. Thus, the following variable is circular:

$$c = \frac{1}{2}(\lambda(zn) + \lambda(nz^{-1})^*)$$

The point now is that we have the following formula:

$$\frac{1}{2}(\lambda(zn) + \lambda(nz^{-1})^*) = us$$

But this gives the result, in our model and so in general as well, because $u = \lambda(z)$ is a Haar-unitary, $s = 1/2(\lambda(n) + \lambda(n)^*)$ is semicircular, and u and s are free. \square

We can now recover the Voiculescu polar decomposition result for the circular variables, obtained in [88], by using random matrix techniques, as follows:

THEOREM 10.33. *Consider the polar decomposition of a circular variable, in some von Neumann algebraic probability space with faithful normal state:*

$$x = vb$$

Then v is Haar unitary, b is quarter-circular, and (v, b) are free.

PROOF. This follows by suitably manipulating Theorem 10.32, as to replace the semi-circular element there by a quarter-circular. Consider indeed the following group:

$$G = \mathbb{Z} * (\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$$

Let z, t, a be the images of the following elements, into this group G :

$$1 \in \mathbb{Z} \quad , \quad (1, \hat{0}) \in \mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z}) \quad , \quad (0, \hat{1}) \in \mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})$$

Let $u = \lambda_G(z)$, $d = \lambda_G(a)$ and choose a quarter-circular $q \in C^*(\lambda_G(t))$. Then (q, d) are independent, so dq is semicircular, and so $c = udq$ is circular, and:

- The module of c is q , which is a quarter-circular.
- The polar part of c is ud , which is obviously a Haar unitary.
- Consider the automorphism of G which is the identity on $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and maps $z \rightarrow za$. This extends to a trace-preserving automorphism of $C^*(G)$ which maps:

$$u \rightarrow ud \quad , \quad q \rightarrow q$$

Since u, q are free, it follows that ud, q are free too, finishing the proof. □

10d. Gaussian matrices

As an application of the semicircular and circular variable theory developed so far, and of free probability in general, let us go back now to the random matrices. Following Voiculescu’s paper [88], we will prove now a number of key freeness results for them, complementing the basic random matrix theory developed in chapters 6-7. As a first result, completing our asymptotic law study for the Gaussian matrices, we have:

THEOREM 10.34. *Given a sequence of complex Gaussian matrices*

$$Z_N \in M_N(L^\infty(X))$$

having independent G_t variables as entries, with $t > 0$, we have

$$\frac{Z_N}{\sqrt{N}} \sim \Gamma_t$$

in the $N \rightarrow \infty$ limit, with the limiting measure being Voiculescu’s circular law.

PROOF. We know from chapter 6, with this having been actually our very first moment computation for random matrices, in this book, that the asymptotic moments of the complex Gaussian matrices are given by the following formula:

$$M_k \left(\frac{Z_N}{\sqrt{N}} \right) \simeq t^{|k|/2} |\mathcal{NC}_2(k)|$$

On the other hand, we also know from the above that an abstract noncommutative variable $a \in A$ is circular, following the law Γ_t , precisely when its moments are:

$$M_k(a) = t^{|k|/2} |\mathcal{NC}_2(k)|$$

Thus, we are led to the conclusion in the statement. \square

The above result is of course something quite theoretical, and having it formulated as such is certainly something nice. However, and here comes our point, it is actually possible to use free probability theory in order to go well beyond this, with this time some truly “new” results on the random matrices. We will explain this now, following Voiculescu’s paper [88]. Let us begin with the Wigner matrices. We have here:

THEOREM 10.35. *Given a family of sequences of Wigner matrices,*

$$Z_N^i \in M_N(L^\infty(X)) \quad , \quad i \in I$$

with pairwise independent entries, each following the complex normal law G_t , with $t > 0$, up to the constraint $Z_N^i = (Z_N^i)^$, the rescaled sequences of matrices*

$$\frac{Z_N^i}{\sqrt{N}} \in M_N(L^\infty(X)) \quad , \quad i \in I$$

become with $N \rightarrow \infty$ semicircular, each following the Wigner law γ_t , and free.

PROOF. This is something quite subtle, the idea being as follows:

(1) First of all, we know from chapter 6 that for any $i \in I$ the corresponding sequence of rescaled Wigner matrices becomes semicircular in the $N \rightarrow \infty$ limit:

$$\frac{Z_N^i}{\sqrt{N}} \simeq \gamma_t$$

(2) Thus, what is new here, and that we have to prove, is the asymptotic freeness assertion. For this purpose we can assume that we are dealing with the case of 2 sequences of matrices, $|I| = 2$. So, assume that we have Wigner matrices as follows:

$$Z_N, Z'_N \in M_N(L^\infty(X))$$

We have to prove that these matrices become asymptotically free, with $N \rightarrow \infty$.

(3) But this something that can be proved directly, via various routine computations with partitions, which simplify as usual in the $N \rightarrow \infty$ limit, and bring freeness.

(4) However, we can prove this as well by using a trick, based on the result in Theorem 10.34. Consider indeed the following random matrix:

$$Y_N = \frac{1}{\sqrt{2}}(Z_N + iZ'_N)$$

This is then a complex Gaussian matrix, and so by using Theorem 10.34, we obtain that in the limit $N \rightarrow \infty$, we have:

$$\frac{Y_N}{\sqrt{N}} \simeq \Gamma_t$$

Now recall that the circular law Γ_t was by definition the law of the following variable, with a, b being semicircular, each following the law γ_t , and free:

$$c = \frac{1}{\sqrt{2}}(a + ib)$$

We are therefore in the situation where the variable $(Z_N + iZ'_N)/\sqrt{N}$, which has asymptotically semicircular real and imaginary parts, converges to the distribution of $a + ib$, equally having semicircular real and imaginary parts, but with these real and imaginary parts being free. Thus Z_N, Z'_N become asymptotically free, as desired. \square

Getting now to the complex case, we have a similar result here, as follows:

THEOREM 10.36. *Given a family of sequences of complex Gaussian matrices,*

$$Z_N^i \in M_N(L^\infty(X)) \quad , \quad i \in I$$

with pairwise independent entries, each following the complex normal law G_t , with $t > 0$, the rescaled sequences of matrices

$$\frac{Z_N^i}{\sqrt{N}} \in M_N(L^\infty(X)) \quad , \quad i \in I$$

become with $N \rightarrow \infty$ circular, each following the Voiculescu law Γ_t , and free.

PROOF. This follows from Theorem 10.35, which applies to the real and imaginary parts of our complex Gaussian matrices, and gives the result. \square

The above results are interesting for both free probability and random matrices. As an illustration here, we have the following application to free probability:

THEOREM 10.37. *Consider the polar decomposition of a circular variable in some von Neumann algebraic probability space with faithful normal state:*

$$x = vb$$

Then v is Haar-unitary, b is quarter-circular and (v, b) are free.

PROOF. This is indeed easy to see in the Gaussian matrix model provided by Theorem 10.36 above, and for details here, we refer to Voiculescu's paper [88]. \square

There are many other applications along these lines, and conversely, free probability can be used as well for the detailed study of the Wigner and Gaussian matrices.

For further results on the topics discussed above, we recommend, besides Voiculescu's papers [85], [86], [87], [88], [89], and book [90] with Dykema and Nica, [22], [25], [47], [71], [79], [80] for general free probability, [2], [42], [49], [51], [55], [67], [69], [84] for random matrix theory, and [26], [44], [53], [61], [77], [78] for applications to operator algebras. But do not worry, we will come back to some of these topics, in what follows.

10e. Exercises

There has been a lot of interesting combinatorics in this chapter, and as an instructive exercise on all this, we have:

EXERCISE 10.38. *Try finding the classical analogue of the polar decomposition result of the circular variables, that we found in the above.*

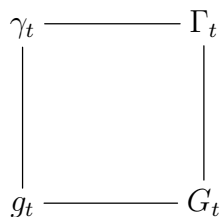
This is something a bit vague, but very instructive. In case you are stuck, try thinking at the passage $O_N \rightarrow U_N$, say at the level of the corresponding Lie algebras, and then at the corresponding laws of coordinates, in the $N \rightarrow \infty$ limit. And if you are still stuck, even with this indication, wait for it: we will be back to this, later in this book.

CHAPTER 11

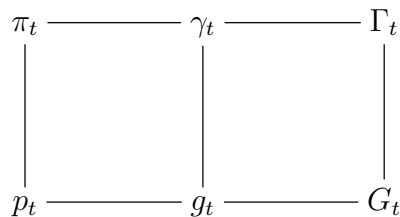
Poisson limits

11a. Poisson limits

We have seen that free probability leads to two key limiting theorems, namely the free analogues of the CLT and CCLT. The limiting measures are the Wigner semicircle laws γ_t and the Voiculescu circular laws Γ_t . Together with the Gaussian laws g_t and G_t coming from the classical CLT and CCLT, these laws form a square diagram, as follows:

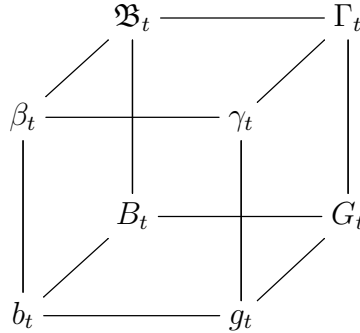


Motivated by this, in this chapter we develop more free limiting theorems. First, we will find a free analogue of the PLT, with the corresponding limiting measures, appearing as the free analogues of the Poisson laws p_t , being the Marchenko-Pastur laws π_t . This will lead to an extension to the above square diagram, into a rectangle, as follows:



More generally, we will find a free analogue of the compound Poisson limit theorem (CPLT), that we know from chapter 2. At the level of the philosophy, and of the above diagram, there are no complex analogues of p_t, π_t , but by using certain measures found via the classical and free CPLT, namely the real and purely complex Bessel laws b_t, B_t discussed in chapter 2, and their free analogues β_t, \mathfrak{B}_t to be discussed here, we will be

able to modify and then fold the diagram, as to complete it into a cube, as follows:



Which is of course quite nice, theoretically speaking, because this leads to a kind of 3D orientation inside classical and free probability, which is something very useful.

Getting started now, we would first like to have a free analogue of the Poisson Limit Theorem (PLT). Although elementary from what we have, this was something not done by Voiculescu himself, and not appearing in the foundational book [90], and only explained later, in the book of Hiai and Petz [54]. The statement is as follows:

THEOREM 11.1 (Free PLT). *The following limit converges, for any $t > 0$,*

$$\lim_{n \rightarrow \infty} \left(\left(1 - \frac{t}{n} \right) \delta_0 + \frac{t}{n} \delta_1 \right)^{\boxplus n}$$

and we obtain the Marchenko-Pastur law of parameter t ,

$$\pi_t = \max(1 - t, 0) \delta_0 + \frac{\sqrt{4t - (x - 1 - t)^2}}{2\pi x} dx$$

also called free Poisson law of parameter t .

PROOF. Consider the measure in the statement, under the convolution sign:

$$\eta = \left(1 - \frac{t}{n} \right) \delta_0 + \frac{t}{n} \delta_1$$

The Cauchy transform of this measure is easy to compute, and is given by:

$$G_\eta(\xi) = \left(1 - \frac{t}{n} \right) \frac{1}{\xi} + \frac{t}{n} \cdot \frac{1}{\xi - 1}$$

In order to prove the result, we want to compute the following R -transform:

$$R = R_{\eta^{\boxplus n}}(y) = nR_\eta(y)$$

According to the formula of G_η , the equation for this function R is as follows:

$$\left(1 - \frac{t}{n} \right) \frac{1}{1/y + R/n} + \frac{t}{n} \cdot \frac{1}{1/y + R/n - 1} = y$$

By multiplying both sides by n/y , this equation can be written as:

$$\frac{t + yR}{1 + yR/n} = \frac{t}{1 + yR/n - y}$$

With $n \rightarrow \infty$ things simplify, and we obtain the following formula:

$$t + yR = \frac{t}{1 - y}$$

Thus we have the following formula, for the R -transform that we are interested in:

$$R = \frac{t}{1 - y}$$

But this gives the result, since R_{π_t} is elementary to compute from what we have, by “doubling” the results for the Wigner law γ_t , and is given by the same formula. \square

As in the continuous case, most of the basic theory of π_t was already done before, in chapters 6-7, with all this partly coming from the theory of SO_3 , at $t = 1$. One thing which was missing there, however, was that of understanding how the law π_t , with parameter $t > 0$, exactly appears, out of π_1 . We can now solve this question:

THEOREM 11.2. *The Marchenko-Pastur laws have the property*

$$\pi_s \boxplus \pi_t = \pi_{s+t}$$

so they form a 1-parameter semigroup with respect to free convolution.

PROOF. This follows either from Theorem 11.1, or from the fact that the R -transform of π_t , computed in the proof of Theorem 11.1, is linear in t . \square

All this is very nice, conceptually speaking, and we can now summarize the various discrete probability results that we have, classical and free, as follows:

THEOREM 11.3. *The Poisson laws p_t and the Marchenko-Pastur laws π_t , given by*

$$p_t = e^{-t} \sum_k \frac{t^k}{k!} \delta_k$$

$$\pi_t = \max(1 - t, 0)\delta_0 + \frac{\sqrt{4t - (x - 1 - t)^2}}{2\pi x} dx$$

have the following properties:

- (1) *They appear via the PLT, and the free PLT.*
- (2) *They form semigroups with respect to $*$ and \boxplus .*
- (3) *Their transforms are $\log F_{p_t}(x) = t(e^{ix} - 1)$, $R_{\pi_t}(x) = t/(1 - x)$.*
- (4) *Their moments are $M_k = \sum_{\pi \in D(k)} t^{|\pi|}$, with $D = P, NC$.*

PROOF. These are all results that we already know, from here and from the previous chapters. To be more precise:

- (1) The PLT is from chapter 2, and the FPLT is from here.
- (2) The semigroup properties are from chapter 2, and from here.
- (3) The formula for F_{p_t} is from chapter 2, and the one for R_{π_t} , from here.
- (4) The moment formulae follow from the formulae of functional transforms. □

We can in fact merge this with our previous continuous results, and we obtain:

THEOREM 11.4. *The moments of the various central limiting measures, namely*

$$\begin{array}{ccccc}
 \pi_t & \text{-----} & \gamma_t & \text{-----} & \Gamma_t \\
 | & & | & & | \\
 p_t & \text{-----} & g_t & \text{-----} & G_t
 \end{array}$$

are always given by the same formula, involving partitions, namely

$$M_k = \sum_{\pi \in D(k)} t^{|\pi|}$$

where the sets of partitions $D(k)$ in question are respectively

$$\begin{array}{ccccc}
 \pi_t & \text{-----} & \gamma_t & \text{-----} & \Gamma_t \\
 | & & | & & | \\
 p_t & \text{-----} & g_t & \text{-----} & G_t
 \end{array}$$

and where $|\cdot|$ is the number of blocks.

PROOF. This follows indeed by putting together the various results that we have, from chapter 10 for the square on the right, and from here for the edge on the left. □

We will later some more conceptual explanations for all this, featuring classical and free cumulants, classical and free quantum groups, and many more.

Moving ahead now, let us try to find a free analogue of the CPLT. We will follow the CPLT material from chapter 2, by performing modifications where needed, as to replace everywhere classical probability with free probability. Let us start with the following straightforward definition, similar to the one from the classical case:

DEFINITION 11.5. Associated to any compactly supported positive measure ρ on \mathbb{C} is the probability measure

$$\pi_\rho = \lim_{n \rightarrow \infty} \left(\left(1 - \frac{c}{n}\right) \delta_0 + \frac{1}{n} \rho \right)^{\boxplus n}$$

where $c = \text{mass}(\rho)$, called compound free Poisson law.

In what follows we will be mostly interested in the case where ρ is discrete, as is for instance the case for the measure $\rho = t\delta_1$ with $t > 0$, which produces the free Poisson laws. The following result allows one to detect compound free Poisson laws:

PROPOSITION 11.6. For a discrete measure, written as

$$\rho = \sum_{i=1}^s c_i \delta_{z_i}$$

with $c_i > 0$ and $z_i \in \mathbb{C}$, we have the following formula,

$$R_{\pi_\rho}(y) = \sum_{i=1}^s \frac{c_i z_i}{1 - y z_i}$$

where R denotes as usual the Voiculescu R -transform.

PROOF. In order to prove this result, let η_n be the measure appearing in Definition 11.5, under the free convolution sign, namely:

$$\eta_n = \left(1 - \frac{c}{n}\right) \delta_0 + \frac{1}{n} \rho$$

The Cauchy transform of η_n is then given by the following formula:

$$G_{\eta_n}(\xi) = \left(1 - \frac{c}{n}\right) \frac{1}{\xi} + \frac{1}{n} \sum_{i=1}^s \frac{c_i}{\xi - z_i}$$

Consider now the R -transform of the measure $\eta_n^{\boxplus n}$, which is given by:

$$R_{\eta_n^{\boxplus n}}(y) = n R_{\eta_n}(y)$$

By using the general theory of the R -transform, from chapter 9, the above formula of G_{η_n} shows that the equation for $R = R_{\eta_n^{\boxplus n}}$ is as follows:

$$\begin{aligned} & \left(1 - \frac{c}{n}\right) \frac{1}{1/y + R/n} + \frac{1}{n} \sum_{i=1}^s \frac{c_i}{1/y + R/n - z_i} = y \\ \implies & \left(1 - \frac{c}{n}\right) \frac{1}{1 + yR/n} + \frac{1}{n} \sum_{i=1}^s \frac{c_i}{1 + yR/n - yz_i} = 1 \end{aligned}$$

Now multiplying by n , then rearranging the terms, and letting $n \rightarrow \infty$, we get:

$$\begin{aligned} \frac{c + yR}{1 + yR/n} = \sum_{i=1}^s \frac{c_i}{1 + yR/n - yz_i} &\implies c + yR_{\pi_\rho}(y) = \sum_{i=1}^s \frac{c_i}{1 - yz_i} \\ &\implies R_{\pi_\rho}(y) = \sum_{i=1}^s \frac{c_i z_i}{1 - yz_i} \end{aligned}$$

Thus, we are led to the conclusion in the statement. \square

We have as well the following result, providing an alternative to Definition 11.5, and which, together with Definition 11.5, can be thought of as being the free CPLT:

THEOREM 11.7. *For a discrete measure, written as*

$$\rho = \sum_{i=1}^s c_i \delta_{z_i}$$

with $c_i > 0$ and $z_i \in \mathbb{C}$, we have the formula

$$\pi_\rho = \text{law} \left(\sum_{i=1}^s z_i \alpha_i \right)$$

where the variables α_i are free Poisson(c_i), free.

PROOF. Let α be the sum of free Poisson variables in the statement:

$$\alpha = \sum_{i=1}^s z_i \alpha_i$$

In order to prove the result, we will show that the R -transform of α is given by the formula in Proposition 11.6. We have the following computation:

$$\begin{aligned} R_{\alpha_i}(y) = \frac{c_i}{1 - y} &\implies R_{z_i \alpha_i}(y) = \frac{c_i z_i}{1 - yz_i} \\ &\implies R_\alpha(y) = \sum_{i=1}^s \frac{c_i z_i}{1 - yz_i} \end{aligned}$$

Thus we have the same formula as in Proposition 11.6, and we are done. \square

All the above is quite general, and in practice, in order to obtain concrete results, the simplest measures that we can use as “input” for the CPLT are the same measures as those that we used in the classical case, namely the measures of type $\rho = t\varepsilon_s$, with $t > 0$, and with ε_s being the uniform measure on the s -th roots of unity. We discuss this in what follows, by following the classical material from chapter 2, and the paper [9].

Let us also mention that we already met in fact the compound free Poisson laws in chapters 7-8, when discussing the asymptotic distributions of the block-modified Wishart matrices. We will clarify this as well, at the end of the present chapter.

11b. Bessel laws

As mentioned above, for various reasons, including the construction of the “standard cube” discussed in the beginning of this chapter, we are interested in the applications of the free CPLT with the “simplest” input measures, with these simplest measures being those of type $\rho = t\varepsilon_s$, with $t > 0$, and with ε_s being the uniform measure on the s -th roots of unity. We are led in this way the following class of measures:

DEFINITION 11.8. *The Bessel and free Bessel laws, depending on parameters $s \in \mathbb{N} \cup \{\infty\}$ and $t > 0$, are the following compound Poisson and free Poisson laws,*

$$b_t^s = p_{t\varepsilon_s} \quad , \quad \beta_t^s = \pi_{t\varepsilon_s}$$

with ε_s being the uniform measure on the s -th roots of unity. In particular:

- (1) At $s = 1$ we recover the Poisson laws p_t, π_t .
- (2) At $s = 2$ we have the real Bessel laws b_t, β_t .
- (3) At $s = \infty$ we have the complex Bessel laws B_t, \mathfrak{B}_t .

The terminology here comes from the fact, that we know from chapter 2, that the density of the measure b_t , appearing at $s = 2$, is a Bessel function of the first kind. This was something first discovered in [10], and we refer to that paper, and to the subsequent literature, including [9], for more comments on this phenomenon.

Our next task will be that upgrading our results about the free Poisson law π_t in this setting, using a parameter $s \in \mathbb{N} \cup \{\infty\}$. First, we have the following result:

THEOREM 11.9. *The free Bessel laws have the property*

$$\beta_t^s \boxplus \beta_{t'}^s = \beta_{t+t'}^s$$

so they form a 1-parameter semigroup with respect to free convolution.

PROOF. This follows indeed from the fact that the R -transform of β_t^s is linear in t , which is something that we already know, from the above. □

Let us discuss now, following the paper [9], some more advanced aspects of the free Bessel laws. Given a real probability measure μ , one can ask whether the convolution powers $\mu^{\boxtimes s}$ and $\mu^{\boxplus t}$ exist, for various values of the parameters $s, t > 0$. For the free Poisson law, the answer to these questions is as follows:

PROPOSITION 11.10. *The free convolution powers of the free Poisson law*

$$\pi^{\boxtimes s} \quad , \quad \pi^{\boxplus t}$$

exist for any positive values of the parameters, $s, t > 0$.

PROOF. We have two measures to be studied, the idea being as follows:

(1) The free Poisson law π is by definition the $t = 1$ particular case of the free Poisson law of parameter t , or Marchenko-Pastur law of parameter $t > 0$, given by:

$$\pi_t = \max(1 - t, 0)\delta_0 + \frac{\sqrt{4t - (x - 1 - t)^2}}{2\pi x} dx$$

The Cauchy transform of this measure is given by:

$$G(\xi) = \frac{(\xi + 1 - t) + \sqrt{(\xi + 1 - t)^2 - 4\xi}}{2\xi}$$

We can compute now the R transform, by proceeding as follows:

$$\begin{aligned} \xi G^2 + 1 = (\xi + 1 - t)G &\implies Kz^2 + 1 = (K + 1 - t)z \\ &\implies Rz^2 + z + 1 = (R + 1 - t)z + 1 \\ &\implies Rz = R - t \\ &\implies R = t/(1 - z) \end{aligned}$$

The last expression being linear in t , the measures π_t form a semigroup with respect to free convolution. Thus we have $\pi_t = \pi^{\boxplus t}$, which proves the second assertion.

(2) Regarding now the measure $\pi^{\boxtimes s}$, there is no explicit formula for its density. However, we can prove that this measure exists, by using some abstract results. Indeed, we have the following computation for the S transform of π_t :

$$\begin{aligned} \xi G^2 + 1 = (\xi + 1 - t)G &\implies zf^2 + 1 = (1 + z - zt)f \\ &\implies z(\psi + 1)^2 + 1 = (1 + z - zt)(\psi + 1) \\ &\implies \chi(z + 1)^2 + 1 = (1 + \chi - \chi t)(z + 1) \\ &\implies \chi(z + 1)(t + z) = z \\ &\implies S = 1/(t + z) \end{aligned}$$

In particular at $t = 1$ we have the following formula:

$$S(z) = \frac{1}{1 + z}$$

Thus the Σ transform of π , which is by definition $\Sigma(z) = S(z/(1 - z))$, is given by:

$$\Sigma(z) = 1 - z$$

On the other hand, it is well-known from the general theory of the S -transform that the Σ transforms of the probability measures which are \boxtimes -infinitely divisible are the functions of the form $\Sigma(z) = e^{v(z)}$, where $v : \mathbb{C} - [0, \infty) \rightarrow \mathbb{C}$ is analytic, satisfying:

$$v(\bar{z}) = \bar{v}(z) \quad , \quad v(\mathbb{C}^+) \subset \mathbb{C}^-$$

Now in the case of the free Poisson law, the function $v(z) = \log(1 - z)$ satisfies these properties, and we are led to the conclusion in the statement. See [9]. \square

Getting now towards the free Bessel laws, we have the following remarkable identity, in relation with the above convolution powers of π , also established in [9]:

THEOREM 11.11. *We have the formula*

$$\pi^{\boxtimes s-1} \boxtimes \pi^{\boxplus t} = ((1-t)\delta_0 + t\delta_1) \boxtimes \pi^{\boxtimes s}$$

valid for any $s \geq 1$, and any $t \in (0, 1]$.

PROOF. We know from the previous proof that the S transform of the free Poisson law π is given by the following formula:

$$S_1(z) = \frac{1}{1+z}$$

We also know from there that the S transform of $\pi^{\boxplus t}$ is given by:

$$S_t(z) = \frac{1}{t+z}$$

Thus the measure on the left in the statement has the following S transform:

$$S(z) = \frac{1}{(1+z)^{s-1}} \cdot \frac{1}{t+z}$$

The S transform of $\alpha_t = (1-t)\delta_0 + t\delta_1$ can be computed as follows:

$$\begin{aligned} f = 1 + tz/(1-z) &\implies \psi = tz/(1-z) \\ &\implies z = t\chi/(1-\chi) \\ &\implies \chi = z/(t+z) \\ &\implies S = (1+z)/(t+z) \end{aligned}$$

Thus the measure on the right in the statement has the following S transform:

$$S(z) = \frac{1}{(1+z)^s} \cdot \frac{1+z}{t+z}$$

Thus the S transforms of our two measures are the same, and we are done. \square

The relation with the free Bessel laws, as previously defined, comes from:

THEOREM 11.12. *The free Bessel law is the real probability measure β_t^s , with*

$$(s, t) \in (0, \infty) \times (0, \infty) - (0, 1) \times (1, \infty)$$

defined concretely as follows:

- (1) For $s \geq 1$ we set $\beta_t^s = \pi^{\boxtimes s-1} \boxtimes \pi^{\boxplus t}$.
- (2) For $t \leq 1$ we set $\beta_t^s = ((1-t)\delta_0 + t\delta_1) \boxtimes \pi^{\boxtimes s}$.

PROOF. This follows indeed from the above results. To be more precise, these results show that the measures constructed in the statement exist indeed, and coincide with the free Bessel laws, as previously defined, as compound free Poisson laws. \square

In view of the above, we can regard the free Bessel law β_t^s as being a natural two-parameter generalization of the free Poisson law π , in connection with Voiculescu’s free convolution operations \boxtimes and \boxplus . Observe that we have the following formulae:

$$\begin{cases} \beta_1^s = \pi^{\boxtimes s} \\ \beta_t^1 = \pi^{\boxplus t} \end{cases}$$

As a comment here, concerning the precise range of the parameters (s, t) , the above results can be probably improved. The point is that the measure β_t^s still exists for certain points in the critical rectangle $(0, 1) \times (1, \infty)$, but not for all of them. To be more precise, the known numeric checks for this question, discussed in [9], show that the critical values of (s, t) tend to form an algebraic curve contained in $(0, 1) \times (1, \infty)$, having $s = 1$ as an asymptote. However, the case we are the most interested in is $t \in (0, 1]$, and here there is no problem, because β_t^s exists for any $s > 0$. Thus, we will stop this discussion here.

As before following [9], we have the following result:

PROPOSITION 11.13. *The Stieltjes transform of β_t^s satisfies:*

$$f = 1 + zf^s(f + t - 1)$$

In particular at $t = 1$ we have the formula $f = 1 + zf^{s+1}$.

PROOF. We have the following computation:

$$\begin{aligned} S = \frac{1}{(1+z)^{s-1}} \cdot \frac{1}{t+z} &\implies \chi = \frac{z}{(1+z)^s} \cdot \frac{1}{t+z} \\ &\implies z = \frac{\psi}{(1+\psi)^s} \cdot \frac{1}{t+\psi} \\ &\implies z = \frac{f-1}{f^s} \cdot \frac{1}{t+f-1} \end{aligned}$$

Thus, we obtain the equation in the statement. \square

At $t = 1$, we have in fact the following result, also from [9], which is more explicit:

THEOREM 11.14. *The Stieltjes transform of β_1^s with $s \in \mathbb{N}$ is given by*

$$f(z) = \sum_{p \in NC_s} z^{k(p)}$$

where NC_s is the set of noncrossing partitions all whose blocks have as size multiples of s , and where $k : NC_s \rightarrow \mathbb{N}$ is the normalized length.

PROOF. With the notation $C_k = \#NC_s(k)$, where $NC_s(k) \subset NC_s$ consists of the partitions of $\{1, \dots, sk\}$ belonging to NC_s , the sum on the right is:

$$f(z) = \sum_k C_k z^k$$

For a given partition $p \in NC_s(k+1)$ we can consider the last s legs of the first block, and make cuts at right of them. This gives a decomposition of p into $s+1$ partitions in NC_s , and we obtain in this way the following recurrence formula for the numbers C_k :

$$C_{k+1} = \sum_{\Sigma k_i = k} C_{k_0} \dots C_{k_s}$$

By multiplying now by z^{k+1} , and then summing over k , we obtain that the generating series of these numbers C_k satisfies the following equation:

$$f - 1 = z f^{s+1}$$

But this is the equation found in Proposition 11.13, so we obtain the result. □

Next, still following [9], we have the following result, dealing with the case $t > 0$:

THEOREM 11.15. *The Stieltjes transform of β_t^s with $s \in \mathbb{N}$ is given by:*

$$f(z) = \sum_{p \in NC_s} z^{k(p)} t^{b(p)}$$

where $k, b : NC_s \rightarrow \mathbb{N}$ are the normalized length, and the number of blocks.

PROOF. With notations from the previous proof, let F_{kb} be the number of partitions in $NC_s(k)$ having b blocks, and set $F_{kb} = 0$ for other integer values of k, b . All sums will be over integer indices ≥ 0 . The sum on the right in the statement is then:

$$f(z) = \sum_{kb} F_{kb} z^k t^b$$

The recurrence formula for the numbers C_k in the previous proof becomes:

$$\sum_b F_{k+1,b} = \sum_{\Sigma k_i = k} \sum_{b_i} F_{k_0 b_0} \dots F_{k_s b_s}$$

In this formula, each term contributes to $F_{k+1,b}$ with $b = \Sigma b_i$, except for those of the form $F_{00} F_{k_1 b_1} \dots F_{k_s b_s}$, which contribute to $F_{k+1,b+1}$. We get:

$$\begin{aligned} F_{k+1,b} &= \sum_{\Sigma k_i = k} \sum_{\Sigma b_i = b} F_{k_0 b_0} \dots F_{k_s b_s} \\ &+ \sum_{\Sigma k_i = k} \sum_{\Sigma b_i = b-1} F_{k_1 b_1} \dots F_{k_s b_s} \\ &- \sum_{\Sigma k_i = k} \sum_{\Sigma b_i = b} F_{k_1 b_1} \dots F_{k_s b_s} \end{aligned}$$

This gives the following formula for the polynomials $P_k = \sum_b F_{kb} t^b$:

$$P_{k+1} = \sum_{\Sigma k_i = k} P_{k_0} \dots P_{k_s} + (t-1) \sum_{\Sigma k_i = k} P_{k_1} \dots P_{k_s}$$

Consider now the following generating function:

$$f = \sum_k P_k z^k$$

In terms of this generating function, we get the following equation:

$$f - 1 = z f^{s+1} + (t-1) z f^s$$

But this is the same as the equation of the Stieltjes transform of β_t^s , namely:

$$f = 1 + z f^s (f + t - 1)$$

Thus, we are led to the conclusion in the statement. \square

Let us discuss now the computation of the moments of the free Bessel laws. The idea will be that of expressing these moments in terms of generalized binomial coefficients. We recall that the coefficient corresponding to $\alpha \in \mathbb{R}$, $k \in \mathbb{N}$ is:

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k!}$$

We denote by m_1, m_2, m_3, \dots the sequence of moments of a given probability measure. With this convention, we first have the following result, from [9]:

THEOREM 11.16. *The moments of β_1^s with $s > 0$ are*

$$m_k = \frac{1}{sk+1} \binom{sk+k}{k}$$

which are the Fuss-Catalan numbers.

PROOF. In the case $s \in \mathbb{N}$, we know that we have $m_k = \#NC_s(k)$. The formula in the statement follows then by counting such partitions. In the general case $s > 0$, observe first that the Fuss-Catalan number in the statement is a polynomial in s :

$$\frac{1}{sk+1} \binom{sk+k}{k} = \frac{(sk+2)(sk+3)\dots(sk+k)}{k!}$$

Thus, in order to pass from the case $s \in \mathbb{N}$ to the case $s > 0$, it is enough to check that the k -th moment of π_{s1} is analytic in s . But this is clear from the equation $f = 1 + z f^{s+1}$ of the Stieltjes transform of π_{s1} , and this gives the result. \square

We have as well the following result, which deals with the general case $t > 0$:

THEOREM 11.17. *The moments of β_t^s with $s > 0$ are*

$$m_k = \sum_{b=1}^k \frac{1}{b} \binom{k-1}{b-1} \binom{sk}{b-1} t^b$$

which are the Fuss-Narayana numbers.

PROOF. In the case $s \in \mathbb{N}$, we know from the above that we have the following formula, where F_{kb} is the number of partitions in $NC_s(k)$ having b blocks:

$$m_k = \sum_b F_{kb} t^b$$

With this observation in hand, the formula in the statement follows by counting such partitions, with this count being well-known. This result can be then extended to any parameter $s > 0$, by using a standard complex variable argument, as before. See [9]. \square

In the case $s \notin \mathbb{N}$, the moments of β_t^s can be further expressed in terms of gamma functions. In the case $s = 1/2$, the result, also from [9], is as follows:

THEOREM 11.18. *The moments of $\beta_1^{1/2}$ are given by the following formulae:*

$$m_{2p} = \frac{1}{p+1} \binom{3p}{p}$$

$$m_{2p-1} = \frac{2^{-4p+3p}}{(6p-1)(2p+1)} \cdot \frac{p!(6p)!}{(2p)!(2p)!(3p)!}$$

PROOF. According to our various results above, the even moments of the free Bessel law β_t^s with $s = n - 1/2$, $n \in \mathbb{N}$, are given by:

$$m_{2p} = \frac{1}{(n-1/2)(2p)+1} \binom{(n+1/2)2p}{2p}$$

$$= \frac{1}{(2n-1)p+1} \binom{(2n+1)p}{2p}$$

With $n = 1$ we get the formula in the statement. Now for the odd moments, we can use here the following well-known identity:

$$\binom{m-1/2}{k} = \frac{4^{-k}}{k!} \cdot \frac{(2m)!}{m!} \cdot \frac{(m-k)!}{(2m-2k)!}$$

With $m = 2np + p - n$ and $k = 2p - 1$ we get:

$$\begin{aligned} m_{2p-1} &= \frac{1}{(n-1/2)(2p-1)+1} \binom{(n+1/2)(2p-1)}{2p-1} \\ &= \frac{2}{(2n-1)(2p-1)+2} \binom{(2np+p-n)-1/2}{2p-1} \\ &= \frac{2^{-4p+3}}{(2p-1)!} \cdot \frac{(4np+2p-2n)!}{(2np+p-n)!} \cdot \frac{(2np-p-n+1)!}{(4np-2p-2n+3)!} \end{aligned}$$

In particular with $n = 1$ we obtain:

$$\begin{aligned} m_{2p-1} &= \frac{2^{-4p+3}}{(2p-1)!} \cdot \frac{(6p-2)!}{(3p-1)!} \cdot \frac{p!}{(2p+1)!} \\ &= \frac{2^{-4p+3}(2p)}{(2p)!} \cdot \frac{(6p)!(3p)}{(3p)!(6p-1)6p} \cdot \frac{p!}{(2p)!(2p+1)} \end{aligned}$$

But this gives the formula in the statement. \square

There are many other interesting things, of both combinatorial and complex analytic nature, that can be said about the free Bessel laws, their moments and their densities, and we refer here to [9]. Also, there is as well a relation with the combinatorics of the intermediate subfactors, and the Fuss-Catalan algebra of Bisch and Jones [27]. All this is a bit technical, and we will be back to it later, when taking about subfactors.

In what follows we will rather focus on the free Bessel laws that we are truly interested in, namely those appearing at $s = 1, 2, \infty$. We will be particularly interested in the cases $s = 2, \infty$, which can be thought of as being “fully real” and “purely complex”.

Also, instead of insisting on combinatorics and complex analysis, we will rather discuss the question of finding matrix models for the free Bessel laws, which is of key importance, in view of the various random matrix considerations from chapters 5-8.

11c. The standard cube

Let us get back now to the fundamental question, mentioned in the beginning of this chapter, of arranging the main probability measures that we know, classical and free, into a cube, and this as for having a kind of 3D orientation, inside probability at large. For this purpose, we will need the following result, coming from the above study:

THEOREM 11.19. *The moments of β_t^s are the numbers*

$$M_k = \sum_{\pi \in NC^s(k)} t^{|\pi|}$$

where NC^s are the noncrossing partitions satisfying $\# \circ = \# \bullet (s)$ in each block.

PROOF. At $t = 1$ the formula to be proved is as follows:

$$M_k(\beta_1^s) = |NC^s(k)|$$

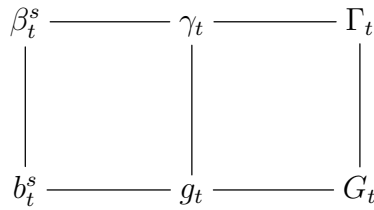
But this can be proved by using Theorem 11.14, via the bijection between the set NC_s there and the set NC^s here. At $t > 0$ now, the formula to be proved is as follows:

$$M_k(\beta_t^s) = \sum_{\pi \in NC^s(k)} t^{|\pi|}$$

But this can be proved again by doing some computations, or by using Theorem 11.15, via the bijection between the set NC_s there and the set NC^s here. \square

At the combinatorial level, this is quite interesting, and we have:

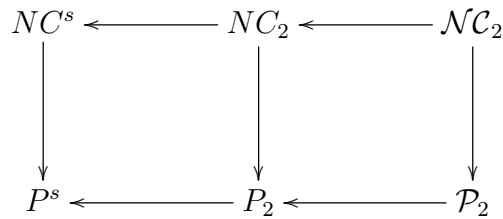
THEOREM 11.20. *The various classical and free central limiting measures,*



have moments always given by the same formula, involving partitions, namely

$$M_k = \sum_{\pi \in D(k)} t^{|\pi|}$$

where the sets of partitions $D(k)$ in question are respectively

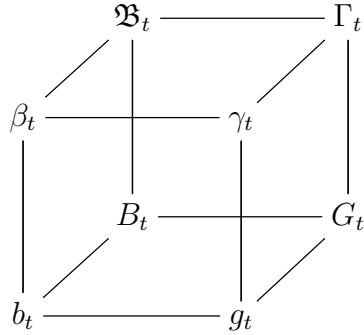


and where $|\cdot|$ is the number of blocks.

PROOF. This follows by putting together the various moment results that we have, namely those from chapter 10, and those from Theorem 11.19. \square

The above result is quite nice, and is complete as well, containing all the moment results that we have established so far, throughout this book. However, forgetting about being as general as possible, we can in fact do better. Nothing in life is better than having some 3D orientation, and as a main application of the above, we can modify a bit the above diagram, as to have a nice-looking cube, as follows:

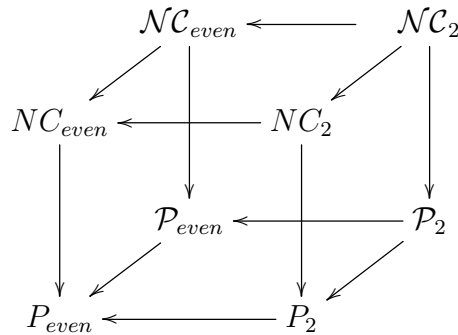
THEOREM 11.21. *The moments of the main central limiting measures,*



are always given by the same formula, involving partitions, namely

$$M_k = \sum_{\pi \in D(k)} t^{|\pi|}$$

where the sets of partitions $D(k)$ in question are respectively



and where $|\cdot|$ is the number of blocks.

PROOF. This follows by putting together the various moment results that we have. To be more precise, the result follows from Theorem 11.20, by restricting the attention on the left to the cases $s = 2, \infty$, which can be thought of as being “fully real” and “purely complex”, and then folding the 8-measure diagram into a cube, as above. \square

The above cube, which is something very nice, will basically keep us busy for the rest of this book. Among others, we will see later more conceptual explanations for it.

Importantly, we will find as well an axiomatization for all this, with the result, called “Ground Zero theorem”, stating that, when imposing a number of strong combinatorial axioms, only the above cube, which is obviously rock-solid, survives. More later.

11d. Matrix models

We discuss here the relation between the above free PLT theory and the random matrices. As a starting point, the free Poisson laws π_t that we found in the above, via the free PLT, coincide with the Marchenko-Pastur laws, shown in chapter 7 to appear as limiting laws for the complex Wishart matrices. This is certainly nice, conceptually speaking, but the point is that we can now truly improve the Marchenko-Pastur result from chapter 7, with an asymptotic freeness statement added, as follows:

THEOREM 11.22. *Given a family of sequences of complex Wishart matrices,*

$$Z_N^i = Y_N^i (Y_N^i)^* \in M_N(L^\infty(X)) \quad , \quad i \in I$$

with each Y_N^i being a $N \times M$ matrix, with entries following the normal law G_1 , and with all these entries being pairwise independent, the rescaled sequences of matrices

$$\frac{Z_N^i}{N} \in M_N(L^\infty(X)) \quad , \quad i \in I$$

become with $M = tN \rightarrow \infty$ Marchenko-Pastur, each following the law π_t , and free.

PROOF. Here the first assertion is the Marchenko-Pastur theorem, and the second assertion follows from the freeness result for the Gaussian matrices, from chapter 10. \square

At a more technical level now, we know from chapters 5-8 that the random matrices provide explicit models for most of the limiting laws appearing in free probability. This is surely an important phenomenon, and in fact, by pushing things a bit, free probability can be even regarded as a theory providing a conceptual framework for random matrix theory. Our goal now, with the standard cube from the previous section in mind, will be that of completing what we know, with matrix models for the free Bessel laws β_t^s . We have two types of models to be investigated, which are both fundamental, as follows:

(1) **Multiplicative models.** We know from chapters 5-8 that by multiplying two Gaussian matrices we obtain a Wishart matrix, and so a model for the free Poisson law π_t . Following [9], we will generalize here such constructions, by looking at more general products of Gaussian matrices, which will turn to be related to the laws β_t^s .

(2) **Block-modified models.** We also know from chapters 5-8 that by performing suitable block modifications on a complex Wishart matrix we obtain certain modifications of the free Poisson law π_t , which are compound free Poisson laws. We will further discuss here this phenomenon, with the aim of modelling in this way the laws β_t^s .

Summarizing, many things to be done, which promise to be quite technical. Let us start with the multiplicative models. We will first restrict attention to the case $t = 1$, since we have $\beta_t^s = \pi^{\boxtimes s-1} \boxtimes \pi^{\boxplus t}$, and therefore matrix models for β_t^s will follow from matrix models for $\pi^{\boxtimes s}$. Following [9], we first have the following result:

THEOREM 11.23. *Let G_1, \dots, G_s be a family of $N \times N$ independent matrices formed by independent centered Gaussian variables, of variance $1/N$. Then with*

$$M = G_1 \dots G_s$$

the moments of the spectral distribution of MM^ converge, up to a normalization, to the corresponding moments of β_1^s , as $N \rightarrow \infty$.*

PROOF. We prove this by recurrence. At $s = 1$ it is well-known that MM^* is a model for $\beta_1^1 = \pi$. So, assume that the result holds for $s - 1 \geq 1$. We have:

$$\begin{aligned} \text{tr}(MM^*)^k &= \text{tr}(G_1 \dots G_s G_s^* \dots G_1^*)^k \\ &= \text{tr}(G_1(G_2 \dots G_s G_s^* \dots G_1^* G_1)^{k-1} G_2 \dots G_s G_s^* \dots G_1^*) \end{aligned}$$

We can pass the first G_1 matrix to the right, and we get:

$$\begin{aligned} \text{tr}(MM^*)^k &= \text{tr}((G_2 \dots G_s G_s^* \dots G_1^* G_1)^{k-1} G_2 \dots G_s G_s^* \dots G_1^* G_1) \\ &= \text{tr}(G_2 \dots G_s G_s^* \dots G_1^* G_1)^k \\ &= \text{tr}((G_2 \dots G_s G_s^* \dots G_2^*)(G_1^* G_1))^k \end{aligned}$$

We know that $G_1^* G_1$ is a Wishart matrix, hence is a model for π :

$$G_1^* G_1 \sim \pi$$

Also, we know by recurrence that $G_2 \dots G_s G_s^* \dots G_2^*$ gives a matrix model for β_1^{s-1} :

$$G_2 \dots G_s G_s^* \dots G_2^* \sim \beta_1^{s-1}$$

Now since the matrices $G_1^* G_1$ and $G_2 \dots G_s G_s^* \dots G_2^*$ are asymptotically free, their product gives a matrix model for $\pi_{s-1,1} \boxtimes \pi_{11} = \beta_1^s$, and we are done. \square

We should mention that the above result, from [9], has inspired a whole string of extensions and generalizations. We refer here to [9] and the subsequent literature. Again following [9], we have as well the following result, which is of different nature:

THEOREM 11.24. *If W is a complex Wishart matrix of parameters (sN, N) and*

$$D = \begin{pmatrix} 1_N & 0 & & 0 \\ 0 & w1_N & & 0 \\ & & \ddots & \\ 0 & 0 & & w^{s-1}1_N \end{pmatrix}$$

with $w = e^{2\pi i/s}$ then the moments of the spectral distribution of $(DW)^s$ converge, up to a normalization, to the corresponding moments of β_1^s , as $N \rightarrow \infty$.

PROOF. We use the following complex Wishart matrix formula of Graczyk, Letac and Massam [49], whose proof is via standard combinatorics:

$$E(\text{Tr}(DW)^K) = \sum_{\sigma \in S_K} \frac{M^{\gamma(\sigma^{-1}\pi)}}{M^K} r_\sigma(D)$$

Here W is by definition a complex Wishart matrix of parameters (M, N) , and D is a deterministic $M \times M$ matrix. As for the right term, this is as follows:

- (1) π is the cycle $(1, \dots, K)$.
- (2) $\gamma(\sigma)$ is the number of disjoint cycles of σ .
- (3) If we denote by $C(\sigma)$ the set of such cycles and for any cycle c , by $|c|$ its length, then the function on the right is given by:

$$r_\sigma(D) = \prod_{c \in C(\sigma)} \text{Tr}(D^{|c|})$$

In our situation we have $K = sk$ and $M = sN$, and we get:

$$E(\text{Tr}(DW)^{sk}) = \sum_{\sigma \in S_{sk}} \frac{(sN)^{\gamma(\sigma^{-1}\pi)}}{(sN)^{sk}} r_\sigma(D)$$

Now since D is uniformly formed by s -roots of unity, we have:

$$\text{Tr}(D^p) = \begin{cases} sN & \text{if } s|p \\ 0 & \text{if } s \nmid p \end{cases}$$

Thus if we denote by S_{sk}^s the set of permutations $\sigma \in S_{sk}$ having the property that all the cycles of σ have length multiple of s , the above formula reads:

$$E(\text{Tr}(DW)^{sk}) = \sum_{\sigma \in S_{sk}^s} \frac{(sN)^{\gamma(\sigma^{-1}\pi)}}{(sN)^{sk}} (sN)^{\gamma(\sigma)}$$

In terms of the normalized trace tr , we obtain the following formula:

$$E(tr(DW)^{sk}) = \sum_{\sigma \in S_{sk}^s} (sN)^{\gamma(\sigma^{-1}\pi) + \gamma(\sigma) - sk - 1}$$

The exponent on the right, say L_σ , can be estimated by using the distance on the Cayley graph of S_{sk} , in the following way:

$$\begin{aligned} L_\sigma &= \gamma(\sigma^{-1}\pi) + \gamma(\sigma) - sk - 1 \\ &= (sk - d(\sigma, \pi)) + (sk - d(e, \sigma)) - sk - 1 \\ &= sk - 1 - (d(e, \sigma) + d(\sigma, \pi)) \\ &\leq sk - 1 - d(e, \pi) \\ &= 0 \end{aligned}$$

Now when taking the limit $N \rightarrow \infty$ in the above formula of $E(\text{tr}(DW)^{sk})$, the only terms that count are those coming from permutations $\sigma \in S_{sk}^s$ having the property $L_\sigma = 0$, which each contribute with a 1 value. We therefore obtain:

$$\begin{aligned} \lim_{N \rightarrow \infty} E(\text{tr}(DW)^{sk}) &= \#\{\sigma \in S_{sk}^s \mid L_\sigma = 0\} \\ &= \#\{\sigma \in S_{sk}^s \mid d(e, \sigma) + d(\sigma, \pi) = d(e, \pi)\} \\ &= \#\{\sigma \in S_{sk}^s \mid \sigma \in [e, \pi]\} \end{aligned}$$

But this number that we obtained is well-known to be the same as the number of noncrossing partitions of $\{1, \dots, sk\}$ having all blocks of size multiple of s . Thus we have reached to the sets $NC_s(k)$ from the above, and we are done. \square

As a consequence of the above random matrix formula, we have the following alternative approach to the free CPLT, in the case of the free Bessel laws, from [9]:

THEOREM 11.25. *The moments of the free Bessel law $\pi_{s,1}$ with $s \in \mathbb{N}$ coincide with those of the variable*

$$\left(\sum_{k=1}^s w^k \alpha_k \right)^s$$

where $\alpha_1, \dots, \alpha_s$ are free random variables, each of them following the free Poisson law of parameter $1/s$, and $w = e^{2\pi i/s}$.

PROOF. This is something that we already know, coming from the combinatorics of the free CPLT, but we can prove this now by using random matrices as well. For this purpose, let G_1, \dots, G_s be a family of independent $sN \times N$ matrices formed by independent, centered complex Gaussian variables, of variance $1/(sN)$. The following matrices H_1, \dots, H_s are then complex Gaussian and independent as well:

$$H_k = \frac{1}{\sqrt{s}} \sum_{p=1}^s w^{kp} G_p$$

Thus the following matrix provides a model for the variable $\sum w^k \alpha_k$:

$$\begin{aligned} M &= \sum_{k=1}^s w^k H_k H_k^* \\ &= \frac{1}{s} \sum_{k=1}^s \sum_{p=1}^s \sum_{q=1}^s w^{k+kp-kq} G_p G_q^* \\ &= \sum_{p=1}^s \sum_{q=1}^s \left(\frac{1}{s} \sum_{k=1}^s (w^{1+p-q})^k \right) G_p G_q^* \\ &= G_1 G_2^* + G_2 G_3^* + \dots + G_{s-1} G_s^* + G_s G_1^* \end{aligned}$$

Now observe that this matrix can be written as follows:

$$\begin{aligned}
M &= (G_1 \ G_2 \ \dots \ G_{s-1} \ G_s) \begin{pmatrix} G_2^* \\ G_3^* \\ \vdots \\ G_s^* \\ G_1^* \end{pmatrix} \\
&= (G_1 \ G_2 \ \dots \ G_{s-1} \ G_s) \begin{pmatrix} 0 & 1_N & 0 & \dots & 0 \\ 0 & 0 & 1_N & \dots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \dots & 1_N \\ 1_N & 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} G_1^* \\ G_2^* \\ \vdots \\ G_{s-1}^* \\ G_s^* \end{pmatrix} \\
&= GOG^*
\end{aligned}$$

In this formula $G = (G_1 \ \dots \ G_s)$ is the $sN \times sN$ Gaussian matrix obtained by concatenating G_1, \dots, G_s , and O is the matrix in the middle. But this latter matrix is of the form $O = UDU^*$ with U unitary, so and we have:

$$M = GUDU^*G^*$$

Now since GU is a Gaussian matrix, M has the same law as the following matrix:

$$M' = GDG^*$$

By using this, we obtain the following moment formula:

$$\begin{aligned}
E \left(\left(\sum_{l=1}^s w^l \alpha_l \right)^{sk} \right) &= \lim_{N \rightarrow \infty} E(\text{tr}(M^{sk})) \\
&= \lim_{N \rightarrow \infty} E(\text{tr}(GDG^*)^{sk}) \\
&= \lim_{N \rightarrow \infty} E(\text{tr}(D(G^*G))^{sk})
\end{aligned}$$

Thus with $W = G^*G$ we get the result. \square

Summarizing, we have applications to the random matrices, and random matrix models for all the 8 basic probability laws, appearing from limiting theorems. As already mentioned, the above results, from [9], have inspired a whole string of extensions and generalizations. We refer here to [9] and the subsequent literature.

As a last topic regarding the free CPLT, which is perhaps the most important, let us review now the results regarding the block-modified Wishart matrices from chapter 8, with free probability tools. We will see in particular that the laws obtained there are free combinations of free Poisson laws, or compound free Poisson laws.

Consider a complex Wishart matrix of parameters (dn, dm) . In other words, we start with a $dn \times dm$ matrix Y having independent complex G_1 entries, and we set:

$$W = YY^*$$

This matrix has size $dn \times dn$, and is best thought of as being a $d \times d$ array of $n \times n$ matrices. We will be interested here in the study of the block-modified versions of W , obtained by applying to the $n \times n$ blocks a given linear map, as follows:

$$\varphi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$$

We recall from chapter 8 that we have the following asymptotic moment formula, extending the usual moment computation for the Wishart matrices:

THEOREM 11.26. *The asymptotic moments of a block-modified Wishart matrix*

$$\widetilde{W} = (id \otimes \varphi)W$$

with parameters $d, m, n \in \mathbb{N}$, as above, are given by the formula

$$\lim_{d \rightarrow \infty} M_e \left(\frac{\widetilde{W}}{d} \right) = \sum_{\sigma \in NC_p} (mn)^{|\sigma|} (M_e^\sigma \otimes M_e^\sigma)(\Lambda)$$

where $\Lambda \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ is the square matrix associated to $\varphi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$.

PROOF. This is something that we know well from chapter 8, coming from the Wick formula, and with the correspondence between linear maps $\varphi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ and square matrices $\Lambda \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ being as well explained there. \square

As explained in chapter 8, it is possible to further build on the above result, with some concrete applications, by doing some combinatorics and calculus. That combinatorics and calculus was something a bit ad-hoc in the context of chapter 8, and congratulations of course for having survived that. With the free probability theory that we learned so far, we can now clarify all this. Following [19], [20], we first have the following result:

PROPOSITION 11.27. *Given a square matrix $\Lambda \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$, having distribution*

$$\rho = law(\Lambda)$$

the moments of the compound free Poisson law $\pi_{mn\rho}$ are given by

$$M_e(\pi_{mn\rho}) = \sum_{\sigma \in NC_p} (mn)^{|\sigma|} (M_e^\sigma \otimes M_e^\sigma)(\Lambda)$$

for any choice of the extra parameter $m \in \mathbb{N}$.

PROOF. This can be proved in several ways, as follows:

(1) A first method is by a straightforward computation, based on the general formula of the R -transform of the compound free Poisson laws, given in the above, and we will leave the computations here, which are all elementary, as an instructive exercise.

(2) Another method, originally used in [20], is by using the well-known fact, that we will discuss in a moment, in chapter 12 below, that the free cumulants of $\pi_{mn\rho}$ coincide with the moments of $mn\rho$. Thus, these free cumulants are given by:

$$\begin{aligned}\kappa_e(\pi_{mn\rho}) &= M_e(mn\rho) \\ &= mn \cdot M_e(\Lambda) \\ &= mn \cdot (M_e^\gamma \otimes M_e^\gamma)(\Lambda)\end{aligned}$$

By using now Speicher's free moment-cumulant formula, from [71], [79], to be explained in chapter 12 below as well, this gives the result. \square

We can see now an obvious similarity with the formula in Theorem 11.26. In order to exploit this similarity, once again by following [20], let us introduce:

DEFINITION 11.28. *We call a square matrix $\Lambda \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ multiplicative when*

$$(M_e^\sigma \otimes M_e^\sigma)(\Lambda) = (M_e^\sigma \otimes M_e^\sigma)(\Lambda)$$

*holds for any $p \in \mathbb{N}$, any exponents $e_1, \dots, e_p \in \{1, *\}$, and any $\sigma \in NC_p$.*

This notion is something quite technical, but we will see many examples in what follows. For instance, the square matrices Λ coming from the basic linear maps φ appearing in chapter 8 are all multiplicative. Now with the above notion in hand, we can formulate an asymptotic result regarding the block-modified Wishart matrices, as follows:

THEOREM 11.29. *Consider a block-modified Wishart matrix*

$$\widetilde{W} = (id \otimes \varphi)W$$

and assume that the matrix $\Lambda \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ associated to φ is multiplicative. Then

$$\frac{\widetilde{W}}{d} \sim \pi_{mn\rho}$$

holds, in moments, in the $d \rightarrow \infty$ limit, where $\rho = law(\Lambda)$.

PROOF. By comparing the moment formulae in Theorem 11.26 and in Proposition 11.27, we conclude that the asymptotic formula $\frac{\widetilde{W}}{d} \sim \pi_{mn\rho}$ is equivalent to the following equality, which should hold for any $p \in \mathbb{N}$, and any exponents $e_1, \dots, e_p \in \{1, *\}$:

$$\sum_{\sigma \in NC_p} (mn)^{|\sigma|} (M_e^\sigma \otimes M_e^\sigma)(\Lambda) = \sum_{\sigma \in NC_p} (mn)^{|\sigma|} (M_e^\sigma \otimes M_e^\sigma)(\Lambda)$$

Now by assuming that Λ is multiplicative, in the sense of Definition 11.28, these two sums are trivially equal, and this gives the result. \square

Summarizing, we have now a much better understanding of what is going on with the block-modified Wishart matrices, and in particular with what exactly is behind Theorem

11.29. For the continuation of all this, we refer to [5], [19], [20] and the subsequent literature on the subject, including the more recent papers [4], [48], [68].

In what concerns us, we will rather navigate in what follows towards quantum algebra, but we will be back to random matrix questions on several occasions, and notably in chapter 16 below, in the context of an all-catching final discussion, regarding the relation between Voiculescu's free probability and Jones' subfactor theory.

11e. Exercises

Things have been quite technical in this chapter, and as unique exercise here, which is unfortunately even more technical than what has been said above, we have:

EXERCISE 11.30. *Find block-modified matrix models for the free Bessel laws.*

This is something which is not very obvious, and also, needless to say, was not something solved in the above. In case you get stuck with this, of course look it up.

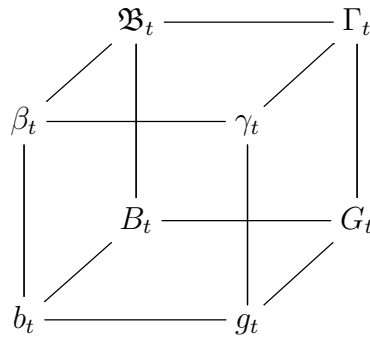
CHAPTER 12

The bijection

12a. Cumulants

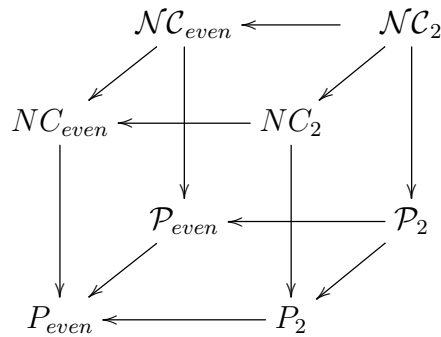
In this chapter we discuss the precise abstract relation between classical and free probability. This is something quite tricky, and as a starting point, we have the following statement, that we know from the above, and which is something very concrete:

THEOREM 12.1. *The moments of the main limiting measures in classical and free probability, real and complex, and discrete and continuous,*



are always given by the same formula, $M_k = \sum_{\pi \in D(k)} t^{|\pi|}$, where $D \subset P$ is a certain set of partitions associated to the measure, and where $|\cdot|$ is the number of blocks.

PROOF. This is something that we know well, the sets of partitions being:



For full details on all this, we refer to the previous chapters. □

What is interesting with the above cube is that it provides us with some 3D orientation in noncommutative probability, taken at large. To be more precise, the 3 “coordinate axes” that we have there, corresponding to the 3 pairs of opposing faces, are:

- (1) Real vs. complex.
- (2) Discrete vs. continuous.
- (3) Classical vs. free.

All this is very nice, and potentially fruitful. In what follows we will be mainly interested in what happens on the vertical, classical vs. free. And here, just by looking at the upper and lower faces of the cube, and how they are connected, we conclude that there should be a bijection between classical and free probability, having something to do with crossing and noncrossing partitions. Thus, we are led to:

QUESTION 12.2. *What is the exact bijection between classical and free limiting laws, which connects the upper and lower faces of the standard cube?*

This is certainly a very interesting and fundamental question, and fortunately, there is a simple answer to it, known since the paper of Bercovici-Pata [24], who first axiomatized this bijection. Explaining all this, Bercovici-Pata bijection, will be our next task.

Getting to work now, what we have in Theorem 12.1 is of rather advanced nature, regarding some special measures. In order to explain the Bercovici-Pata bijection, which basically deals with arbitrary probability measures, it is better to forget Theorem 12.1, and go back to the basics. And talking basics now, probability and combinatorics at large, of quite general type, we have here the following key definition, due to Rota:

DEFINITION 12.3. *Associated to any real probability measure $\mu = \mu_f$ is the following modification of the logarithm of the Fourier transform $F_\mu(\xi) = E(e^{i\xi f})$,*

$$K_\mu(\xi) = \log E(e^{\xi f})$$

called cumulant-generating function. The Taylor coefficients $k_n(\mu)$ of this series, given by

$$K_\mu(\xi) = \sum_{n=1}^{\infty} k_n(\mu) \frac{\xi^n}{n!}$$

are called cumulants of the measure μ . We also use the notations k_f, K_f for these cumulants and their generating series, where f is a variable following the law μ .

In other words, the cumulants are more or less the coefficients of the logarithm of the Fourier transform $\log F_\mu$, up to some normalizations. To be more precise, we have $K_\mu(\xi) = \log F_\mu(-i\xi)$, so the formula relating $\log F_\mu$ to the cumulants $k_n(\mu)$ is:

$$\log F_\mu(-i\xi) = \sum_{n=1}^{\infty} k_n(\mu) \frac{\xi^n}{n!}$$

Equivalently, the formula relating $\log F_\mu$ to the cumulants $k_n(\mu)$ is:

$$\log F_\mu(\xi) = \sum_{n=1}^{\infty} k_n(\mu) \frac{(i\xi)^n}{n!}$$

We will see in a moment the reasons for the above normalizations, namely change of variables $\xi \rightarrow -i\xi$, and Taylor coefficients instead of plain coefficients, the idea being that for simple laws like g_t, p_t , we will obtain in this way very simple quantities. Let us also mention that there is a reason for indexing the cumulants by $n = 1, 2, 3, \dots$ instead of $n = 0, 1, 2, \dots$, and more on this later, once we will have some theory and examples.

As a first observation, the sequence of cumulants k_1, k_2, k_3, \dots appears as a modification of the sequence of moments M_1, M_2, M_3, \dots , the numerics being as follows:

PROPOSITION 12.4. *The sequence of cumulants k_1, k_2, k_3, \dots appears as a modification of the sequence of moments M_1, M_2, M_3, \dots , and uniquely determines μ . We have*

$$\begin{aligned} k_1 &= M_1 \\ k_2 &= -M_1^2 + M_2 \\ k_3 &= 2M_1^3 - 3M_1M_2 + M_3 \\ k_4 &= -6M_1^4 + 12M_1^2M_2 - 3M_2^2 - 4M_1M_3 + M_4 \\ &\vdots \end{aligned}$$

in one sense, and in the other sense we have

$$\begin{aligned} M_1 &= k_1 \\ M_2 &= k_1^2 + k_2 \\ M_3 &= k_1^3 + 3k_1k_2 + k_3 \\ M_4 &= k_1^4 + 6k_1^2k_2 + 3k_2^2 + 4k_1k_3 + k_4 \\ &\vdots \end{aligned}$$

with in both cases the correspondence being polynomial, with integer coefficients.

PROOF. Here all the theoretical assertions regarding moments and cumulants are clear from definitions, and the numerics are clear from definitions too. To be more precise, we know from Definition 12.3 that the cumulants are defined by the following formula:

$$\log E(e^{\xi f}) = \sum_{s=1}^{\infty} k_s(f) \frac{\xi^s}{s!}$$

By exponentiating, we obtain from this the following formula:

$$E(e^{\xi f}) = \exp \left(\sum_{s=1}^{\infty} k_s(f) \frac{\xi^s}{s!} \right)$$

Now by looking at the terms of order 1, 2, 3, 4, this gives the above formulae. \square

Obviously, there should be some explicit formulae for the correspondences in Proposition 12.4. This is indeed the case, but things here are quite tricky, and we will discuss this later, once we will have enough motivations for the study of the cumulants.

The interest in cumulants comes from the fact that $\log F_\mu$, and so the cumulants $k_n(\mu)$ too, linearize the convolution. To be more precise, we have the following result:

THEOREM 12.5. *The cumulants have the following properties:*

- (1) $k_n(cf) = c^n k_n(f)$.
- (2) $k_1(f+d) = k_1(f) + d$, and $k_n(f+d) = k_n(f)$ for $n > 1$.
- (3) $k_n(f+g) = k_n(f) + k_n(g)$, if f, g are independent.

PROOF. Here (1) and (2) are both clear from definitions, because we have the following computation, valid for any $c, d \in \mathbb{R}$, which gives the results:

$$\begin{aligned} K_{cf+d}(\xi) &= \log E(e^{\xi(cf+d)}) \\ &= \log[e^{\xi d} \cdot E(e^{\xi cf})] \\ &= \xi d + K_f(c\xi) \end{aligned}$$

As for (3), this follows from the fact that the Fourier transform $F_f(\xi) = E(e^{i\xi f})$ satisfies the following formula, whenever f, g are independent random variables:

$$F_{f+g}(\xi) = F_f(\xi)F_g(\xi)$$

Indeed, by applying the logarithm, we obtain the following formula:

$$\log F_{f+g}(\xi) = \log F_f(\xi) + \log F_g(\xi)$$

With the change of variables $\xi \rightarrow -i\xi$, we obtain the following formula:

$$K_{f+g}(\xi) = K_f(\xi) + K_g(\xi)$$

Thus, at the level of coefficients, we obtain $k_n(f+g) = k_n(f) + k_n(g)$, as claimed. \square

At the level of examples now, we have the following result:

PROPOSITION 12.6. *The sequence of cumulants k_1, k_2, k_3, \dots is as follows:*

- (1) *For $\mu = \delta_c$ the cumulants are $c, 0, 0, \dots$*
- (2) *For $\mu = g_t$ the cumulants are $0, t, 0, 0, \dots$*
- (3) *For $\mu = p_t$ the cumulants are t, t, t, \dots*
- (4) *For $\mu = b_t$ the cumulants are $0, t, 0, t, \dots$*

PROOF. We have 4 computations to be done, the idea being as follows:

- (1) For $\mu = \delta_c$ we have the following computation:

$$\begin{aligned} K_\mu(\xi) &= \log E(e^{c\xi}) \\ &= \log(e^{c\xi}) \\ &= c\xi \end{aligned}$$

But the plain coefficients of this series are the numbers $c, 0, 0, \dots$, and so the Taylor coefficients of this series are these same numbers $c, 0, 0, \dots$, as claimed.

- (2) For $\mu = g_t$ we have the following computation:

$$\begin{aligned} K_\mu(\xi) &= \log F_\mu(-i\xi) \\ &= \log \exp[-t(-i\xi)^2/2] \\ &= t\xi^2/2 \end{aligned}$$

But the plain coefficients of this series are the numbers $0, t/2, 0, 0, \dots$, and so the Taylor coefficients of this series are the numbers $0, t, 0, 0, \dots$, as claimed.

- (3) For $\mu = p_t$ we have the following computation:

$$\begin{aligned} K_\mu(\xi) &= \log F_\mu(-i\xi) \\ &= \log \exp[(e^{i(-i\xi)} - 1)t] \\ &= (e^\xi - 1)t \end{aligned}$$

But the plain coefficients of this series are the numbers $t/n!$, and so the Taylor coefficients of this series are the numbers t, t, t, \dots , as claimed.

- (4) For $\mu = b_t$ we have the following computation:

$$\begin{aligned} K_\mu(\xi) &= \log F_\mu(-i\xi) \\ &= \log \exp \left[\left(\frac{e^\xi + e^{-\xi}}{2} - 1 \right) t \right] \\ &= \left(\frac{e^\xi + e^{-\xi}}{2} - 1 \right) t \end{aligned}$$

But the plain coefficients of this series are the numbers $(1 + (-1)^n)t/n!$, so the Taylor coefficients of this series are the numbers $0, t, 0, t, \dots$, as claimed. \square

At a more theoretical level, we have the following result, generalizing (3,4) above, and which is something very useful, when dealing with the compound Poisson laws:

THEOREM 12.7. *For a compound Poisson law p_ν we have*

$$k_n(p_\nu) = M_n(\nu)$$

valid for any integer $n \geq 1$.

PROOF. We can assume, by using a continuity argument, that our measure ν is discrete, as follows, with $t_i > 0$ and $z_i \in \mathbb{R}$, and with the sum being finite:

$$\nu = \sum_i t_i \delta_{z_i}$$

By using now the Fourier transform formula for p_ν from chapter 11, we obtain:

$$\begin{aligned} K_{p_\nu}(\xi) &= \log F_{p_\nu}(-i\xi) \\ &= \log \exp \left[\sum_i t_i (e^{\xi z_i} - 1) \right] \\ &= \sum_i t_i \sum_{n \geq 1} \frac{(\xi z_i)^n}{n!} \\ &= \sum_{n \geq 1} \frac{\xi^n}{n!} \sum_i t_i z_i^n \\ &= \sum_{n \geq 1} \frac{\xi^n}{n!} M_n(\nu) \end{aligned}$$

Thus, we are led to the conclusion in the statement. □

12b. Inversion formula

Getting back to theory now, the sequence of cumulants k_1, k_2, k_3, \dots appears as a modification of the sequence of moments M_1, M_2, M_3, \dots , and understanding the relation between moments and cumulants will be our next task. We recall from Proposition 12.4 that we have the following formulae, for the cumulants in terms of moments:

$$\begin{aligned} k_1 &= M_1 \\ k_2 &= -M_1^2 + M_2 \\ k_3 &= 2M_1^3 - 3M_1M_2 + M_3 \\ k_4 &= -6M_1^4 + 12M_1^2M_2 - 3M_2^2 - 4M_1M_3 + M_4 \\ &\vdots \end{aligned}$$

Also, we have the following formulae, for the moments in terms of cumulants:

$$\begin{aligned} M_1 &= k_1 \\ M_2 &= k_1^2 + k_2 \\ M_3 &= k_1^3 + 3k_1k_2 + k_3 \\ M_4 &= k_1^4 + 6k_1^2k_2 + 3k_2^2 + 4k_1k_3 + k_4 \\ &\vdots \end{aligned}$$

In order to understand what exactly is going on, with moments and cumulants, which reminds a bit the Möbius inversion formula, we need to do some combinatorics, in relation with partitions. So, let us go back to the material from chapter 4, where some theory for the partitions was developed. We recall that we have the following definition:

DEFINITION 12.8. *The Möbius function of any lattice, and so of P , is given by*

$$\mu(\pi, \nu) = \begin{cases} 1 & \text{if } \pi = \nu \\ -\sum_{\pi \leq \tau < \nu} \mu(\pi, \tau) & \text{if } \pi < \nu \\ 0 & \text{if } \pi \not\leq \nu \end{cases}$$

with the construction being performed by recurrence.

This is something that we already discussed in chapter 4, and as a first example here, the Möbius matrix $M_{\pi\nu} = \mu(\pi, \nu)$ of the lattice $P(2) = \{|\!, |\!, \sqcup\}$ is as follows:

$$M = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

At $k = 3$ now, we have the following formula for the Möbius matrix $M_{\pi\nu} = \mu(\pi, \nu)$, once again written with the indices picked increasing in $P(3) = \{|\!, |\!, \sqcup, \sqcap, |\sqcup, \sqcap\sqcup\}$:

$$M = \begin{pmatrix} 1 & -1 & -1 & -1 & 2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

In general, as explained in chapter 4, the Möbius matrix of $P(k)$ looks a bit like the above matrices at $k = 2, 3$, being upper triangular, with 1 on the diagonal, and so on.

Back to the general case now, the main interest in the Möbius function comes from the Möbius inversion formula, which can be formulated as follows:

THEOREM 12.9. *We have the following implication,*

$$f(\pi) = \sum_{\nu \leq \pi} g(\nu) \quad \Longrightarrow \quad g(\pi) = \sum_{\nu \leq \pi} \mu(\nu, \pi) f(\nu)$$

valid for any two functions $f, g : P(n) \rightarrow \mathbb{C}$.

PROOF. The above formula is in fact a linear algebra result, so let us start with some linear algebra. Consider the adjacency matrix of P , given by the following formula:

$$A_{\pi\nu} = \begin{cases} 1 & \text{if } \pi \leq \nu \\ 0 & \text{if } \pi \not\leq \nu \end{cases}$$

Our claim is that the inverse of this matrix is the Möbius matrix of P , given by:

$$M_{\pi\nu} = \mu(\pi, \nu)$$

Indeed, the above matrix A is upper triangular, and when trying to invert it, we are led to the recurrence in Definition 12.8, so to the Möbius matrix M . Thus we have:

$$M = A^{-1}$$

Now by applying this equality of matrices to vectors, regarded as complex functions on $P(n)$, we are led to the inversion formula in the statement. \square

As a first illustration, for $P(2)$ the formula $M = A^{-1}$ appears as follows:

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1}$$

At $k = 3$ now, the formula $M = A^{-1}$ for $P(3)$ reads:

$$\begin{pmatrix} 1 & -1 & -1 & -1 & 2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}^{-1}$$

In general, the formula $M = A^{-1}$ looks quite similar, and we refer here to chapter 4.

With these ingredients in hand, let us go back to probability. We first have:

DEFINITION 12.10. *We define quantities $M_\pi(f), k_\pi(f)$, depending on partitions*

$$\pi \in P(k)$$

by starting with $M_n(f), k_n(f)$, and using multiplicativity over the blocks.

To be more precise, the convention here is that for the one-block partition $1_n \in P(n)$, the corresponding moment and cumulant are the usual ones, namely:

$$M_{1_n}(f) = M_n(f) \quad , \quad k_{1_n}(f) = k_n(f)$$

Then, for an arbitrary partition $\pi \in P(k)$, we decompose this partition into blocks, having sizes b_1, \dots, b_s , and we set, by multiplicativity over blocks:

$$M_\pi(f) = M_{b_1}(f) \dots M_{b_s}(f) \quad , \quad k_\pi(f) = k_{b_1}(f) \dots k_{b_s}(f)$$

With this convention, following Rota and others, we can now formulate a key result, fully clarifying the relation between moments and cumulants, as follows:

THEOREM 12.11. *We have the moment-cumulant formulae*

$$M_n(f) = \sum_{\nu \in P(n)} k_\nu(f) \quad , \quad k_n(f) = \sum_{\nu \in P(n)} \mu(\nu, 1_n) M_\nu(f)$$

or, equivalently, we have the moment-cumulant formulae

$$M_\pi(f) = \sum_{\nu \leq \pi} k_\nu(f) \quad , \quad k_\pi(f) = \sum_{\nu \leq \pi} \mu(\nu, \pi) M_\nu(f)$$

where μ is the Möbius function of $P(n)$.

PROOF. There are several things going on here, the idea being as follows:

(1) First, it is clear from our conventions, from Definition 12.10, that the first set of formulae is equivalent to the second set of formulae, by multiplicativity over blocks.

(2) The other observation is that, due to the Möbius inversion formula, from Theorem 12.9, in the second set of formulae, the two formulae there are in fact equivalent.

(3) Summarizing, the 4 formulae in the statement are all equivalent. In what follows we will focus on the first 2 formulae, which are the most useful, in practice.

(4) Let us first work out some examples. At $n = 1, 2, 3$ the moment formula gives the following equalities, which are in tune with the findings from Proposition 12.4:

$$M_1 = k_{|} = k_1$$

$$M_2 = k_{||} + k_{\sqcap} = k_1^2 + k_2$$

$$M_3 = k_{|||} + k_{\sqcap|} + k_{|\sqcap} + k_{\sqcap\sqcap} = k_1^3 + 3k_1k_2 + k_3$$

At $n = 4$ now, which is a case which is of particular interest for certain considerations to follow, the computation is as follows, again in tune with Proposition 12.4:

$$\begin{aligned} M_4 &= k_{||||} + \underbrace{(k_{\sqcap||} + \dots)}_{6 \text{ terms}} + \underbrace{(k_{\sqcap\sqcap} + \dots)}_{3 \text{ terms}} + \underbrace{(k_{|\sqcap|} + \dots)}_{4 \text{ terms}} + k_{\sqcap\sqcap\sqcap} \\ &= k_1^4 + 6k_1^2k_2 + 3k_2^2 + 4k_1k_3 + k_4 \end{aligned}$$

As for the cumulant formula, at $n = 1, 2, 3$ this gives the following formulae for the cumulants, again in tune with the findings from Proposition 12.4:

$$k_1 = M_{|} = M_1$$

$$k_2 = (-1)M_{||} + M_{\sqcap} = -M_1^2 + M_2$$

$$k_3 = 2M_{|||} + (-1)M_{\sqcap|} + (-1)M_{|\sqcap} + (-1)M_{\sqcap\sqcap} + M_{\sqcap\sqcap\sqcap} = 2M_1^3 - 3M_1M_2 + M_3$$

Finally, at $n = 4$, after computing the Möbius function of $P(4)$, we obtain the following formula for the fourth cumulant, again in tune with Proposition 12.4:

$$\begin{aligned} k_4 &= (-6)M_{||||} + \underbrace{2(M_{\sqcap||} + \dots)}_{6 \text{ terms}} + (-1)\underbrace{(M_{\sqcap\sqcap} + \dots)}_{3 \text{ terms}} + (-1)\underbrace{(M_{\sqcap\sqcap|} + \dots)}_{4 \text{ terms}} + M_{\sqcap\sqcap\sqcap\sqcap} \\ &= -6M_1^4 + 12M_1^2M_2 - 3M_2^2 - 4M_1M_3 + M_4 \end{aligned}$$

(5) After all these preliminaries, time now to get to work, and prove the result. As mentioned above, our formulae are all equivalent, and it is enough to prove just one of them. We will prove in what follows the first formula, namely:

$$M_n(f) = \sum_{\nu \in P(n)} k_\nu(f)$$

(6) In order to do this, we use the very definition of the cumulants, namely:

$$\log E(e^{\xi f}) = \sum_{s=1}^{\infty} k_s(f) \frac{\xi^s}{s!}$$

By exponentiating, we obtain from this the following formula:

$$E(e^{\xi f}) = \exp \left(\sum_{s=1}^{\infty} k_s(f) \frac{\xi^s}{s!} \right)$$

(7) Let us first compute the function on the left. This is easily done, as follows:

$$\begin{aligned} E(e^{\xi f}) &= E \left(\sum_{n=0}^{\infty} \frac{(\xi f)^n}{n!} \right) \\ &= \sum_{n=0}^{\infty} M_n(f) \frac{\xi^n}{n!} \end{aligned}$$

(8) Regarding now the function on the right, this is given by:

$$\begin{aligned}
 \exp\left(\sum_{s=1}^{\infty} k_s(f) \frac{\xi^s}{s!}\right) &= \sum_{p=0}^{\infty} \frac{\left(\sum_{s=1}^{\infty} k_s(f) \frac{\xi^s}{s!}\right)^p}{p!} \\
 &= \sum_{p=0}^{\infty} \frac{1}{p!} \sum_{s_1=1}^{\infty} k_{s_1}(f) \frac{\xi^{s_1}}{s_1!} \cdots \sum_{s_p=1}^{\infty} k_{s_p}(f) \frac{\xi^{s_p}}{s_p!} \\
 &= \sum_{p=0}^{\infty} \frac{1}{p!} \sum_{s_1=1}^{\infty} \cdots \sum_{s_p=1}^{\infty} k_{s_1}(f) \cdots k_{s_p}(f) \frac{\xi^{s_1+\dots+s_p}}{s_1! \cdots s_p!}
 \end{aligned}$$

(9) The point now is that all this leads us into partitions. Indeed, we are summing over indices $s_1, \dots, s_p \in \mathbb{N}$, which can be thought of as corresponding to a partition of $n = s_1 + \dots + s_p$. So, let us rewrite our sum, as a sum over partitions. For this purpose, recall that the number of partitions $\nu \in P(n)$ having blocks of sizes s_1, \dots, s_p is:

$$\binom{n}{s_1, \dots, s_p} = \frac{n!}{p_1! \cdots p_s!}$$

Also, when resumming over partitions, there will be a $p!$ factor as well, coming from the permutations of s_1, \dots, s_p . Thus, our sum can be rewritten as follows:

$$\begin{aligned}
 \exp\left(\sum_{s=1}^{\infty} k_s(f) \frac{\xi^s}{s!}\right) &= \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{1}{p!} \sum_{s_1+\dots+s_p=n} k_{s_1}(f) \cdots k_{s_p}(f) \frac{\xi^n}{s_1! \cdots s_p!} \\
 &= \sum_{n=0}^{\infty} \frac{\xi^n}{n!} \sum_{p=0}^{\infty} \frac{1}{p!} \sum_{s_1+\dots+s_p=n} \binom{n}{s_1, \dots, s_p} k_{s_1}(f) \cdots k_{s_p}(f) \\
 &= \sum_{n=0}^{\infty} \frac{\xi^n}{n!} \sum_{\nu \in P(n)} k_{\nu}(f)
 \end{aligned}$$

(10) We are now in position to conclude. According to (6,7,9), we have:

$$\sum_{n=0}^{\infty} M_n(f) \frac{\xi^n}{n!} = \sum_{n=0}^{\infty} \frac{\xi^n}{n!} \sum_{\nu \in P(n)} k_{\nu}(f)$$

Thus, we have the following formula, valid for any $n \in \mathbb{N}$:

$$M_n(f) = \sum_{\nu \in P(n)} k_{\nu}(f)$$

We are therefore led to the conclusions in the statement. \square

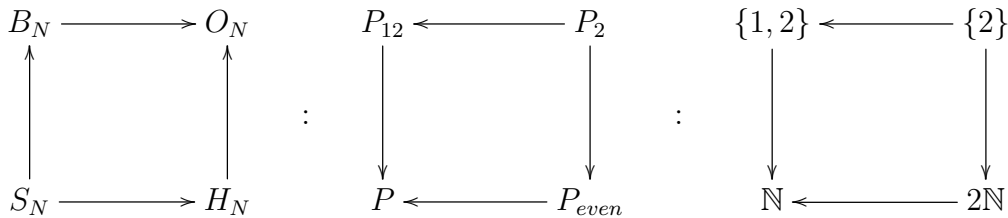
Summarizing, we have now a nice theory of cumulants, or rather a beginning of such a theory, and with this in hand, we can go back to the diagram in Theorem 12.1, see if we can now better understand what is going on there. However, this is a bit tricky:

(1) Our theory of cumulants as developed so far only applies properly to the “real classical” case, that is, to the measures g_t, b_t there. In order to deal with the full classical case, comprising as well the measures G_t, B_t , we would have to upgrade everything into a theory of $*$ -cumulants, and this is something quite technical.

(2) Regarding the “free real” measures γ_t, β_t and their complex analogues Γ_t, \mathfrak{B}_t , here the cumulant theory developed above gives nothing interesting. We will see in the next section, at least in the real case, that of γ_t, β_t , that the relevant theory which applies to them is a substantial modification of what we have, called free cumulant theory.

In short, technical problems in all directions, and we are not ready yet for better understanding Theorem 12.1. As a more modest objective, however, we have the quite reasonable question of understanding the moment formula $M_k = \sum_{\pi \in D(k)} t^{|\pi|}$ there for the measures g_t, b_t , by using the cumulant theory developed above. Which is in fact a non-trivial question too, with the answer involving the following result from [21]:

THEOREM 12.12. *The uniform orthogonal easy groups $G \subset O_N$, and their associated categories of partitions $D \subset P$, all coming from subsets $L \subset \mathbb{N}$, are as follows,*



with D consisting of the partitions $\pi \in P$ whose blocks have lengths belonging to $L \subset \mathbb{N}$.

PROOF. Consider an arbitrary easy group, $S_N \subset G_N \subset O_N$. This group must then come from a category of partitions, as follows:

$$P_2 \subset D \subset P$$

Now if we assume $G = (G_N)$ to be uniform, this category D is uniquely determined by the subset $L \subset \mathbb{N}$ consisting of the sizes of the blocks of the partitions in D . And as explained in [21], one can prove that the admissible sets are those in the statement, corresponding to the categories and the groups in the statement. \square

In relation now with cumulants, we have the following result, also from [21]:

THEOREM 12.13. *The cumulants of the asymptotic truncated characters for the uniform easy groups $G = (G_N)$ are given by the formula*

$$k_n(\chi_t) = t\delta_{n \in L}$$

with $L \subset \mathbb{N}$ being the associated subset, and at the level of asymptotic moments this gives

$$M_k(\chi_t) = \sum_{\pi \in D(k)} t^{|\pi|}$$

with $D \subset P$ being the associated category of partitions.

PROOF. This is clear indeed from Theorem 12.12, by performing a case-by-case analysis, with the cases $G = O, S, H$ corresponding to the computations for g_t, p_t, b_t from Proposition 12.5, and with the remaining case, that of the bistochastic groups, $G = B$, being similar. Again, for details on all this, we refer to [21]. \square

Summarizing, we have now a good understanding of the formula $M_k = \sum_{\pi \in D(k)} t^{|\pi|}$ for the real classical limiting measures, based on cumulants, but with this involving however some more advanced mathematics. It is possible of course to reformulate all the above in terms of categories of partitions only, but this won't lead to any simplifications in the proofs, which are based on categories of partitions anyway, and would rather obscure the final results themselves, which are best thought of in terms of easy groups.

Finally, in order to extend the above results to the general the complex case, the cumulant theory must be upgraded into a $*$ -cumulant theory, which is something quite technical. We will discuss however such questions in chapter 15 below, directly in a more general setting, that of operator-valued noncommutative probability theory, following Speicher and others [71], [79], [80]. In what regards the easy groups, and more generally easy quantum groups, in the general unitary setting, this is again a quite technical subject, and we will be back to this on several occasions, in the remainder of this book.

12c. Free cumulants

In what follows we discuss the free analogues of the above, following Speicher [79], and subsequent work. We first have the following definition:

DEFINITION 12.14. *The free cumulants $\kappa_n(a)$ of a variable $a \in A$ are defined by*

$$R_a(\xi) = \sum_{n=1}^{\infty} \kappa_n(a)\xi^{n-1}$$

with the R -transform being defined as usual by the formula

$$G_a \left(R_a(\xi) + \frac{1}{\xi} \right) = \xi$$

where $G_a(\xi) = \int_{\mathbb{R}} \frac{d\mu(t)}{\xi-t}$ with $\mu = \mu_a$ is the corresponding Cauchy transform.

As before with classical cumulants, we have a number of basic examples and illustrations, and a number of basic general results. Let us start with some numerics:

PROPOSITION 12.15. *The free cumulants $\kappa_1, \kappa_2, \kappa_3, \dots$ appear as a modification of the moments M_1, M_2, M_3, \dots , and uniquely determine μ . We have*

$$\begin{aligned}\kappa_1 &= M_1 \\ \kappa_2 &= -M_1^2 + M_2 \\ \kappa_3 &= 2M_1^3 - 3M_1M_2 + M_3 \\ \kappa_4 &= -5M_1^4 + 10M_1^2M_2 - 2M_2^2 - 4M_1M_3 + M_4 \\ &\vdots\end{aligned}$$

in one sense, and in the other sense we have

$$\begin{aligned}M_1 &= \kappa_1 \\ M_2 &= \kappa_1^2 + \kappa_2 \\ M_3 &= \kappa_1^3 + 3\kappa_1\kappa_2 + \kappa_3 \\ M_4 &= \kappa_1^4 + 6\kappa_1^2\kappa_2 + 2\kappa_2^2 + 4\kappa_1\kappa_3 + \kappa_4 \\ &\vdots\end{aligned}$$

with in both cases the correspondence being polynomial, with integer coefficients.

PROOF. Here all theoretical assertions regarding moments and cumulants are clear from definitions, and the numerics are clear from definitions too, after some computations based on Definition 12.14. Let us actually present these computations, which are quite instructive, more complicated than the classical ones, and that we will need, later on:

(1) We know that the Cauchy transform is the following function:

$$G(\xi) = \sum_{n=0}^{\infty} \frac{M_n}{\xi^{n+1}}$$

Consider the inverse of this Cauchy transform G , with respect to composition:

$$G(K(\xi)) = K(G(\xi)) = \xi$$

According to Definition 12.14, the free cumulants κ_n appear then as follows:

$$K(\xi) = \frac{1}{\xi} + \sum_{n=1}^{\infty} \kappa_n \xi^{n-1}$$

Thus, we can compute moments in terms of free cumulants, and vice versa, by using either of the inversion formulae $G(K(\xi)) = \xi$ and $K(G(\xi)) = \xi$.

(2) This was for the theory. In practice now, playing with the original inversion formula from Definition 12.14, namely $G(K(\xi)) = \xi$, proves to be something quite complicated, so we will choose to use instead the other inversion formula, namely:

$$K(G(\xi)) = \xi$$

Thus, the equation that we want to use is as follows, with $G = G(\xi)$:

$$\frac{1}{G} + \sum_{n=1}^{\infty} \kappa_n G^{n-1} = \xi$$

(3) With $\xi = z^{-1}$ our equation takes the following form, with $G = G(z^{-1})$:

$$\frac{1}{G} + \sum_{n=1}^{\infty} \kappa_n G^{n-1} = z^{-1}$$

Now by multiplying by z , our equation takes the following form:

$$\frac{z}{G} + z \sum_{n=1}^{\infty} \kappa_n G^{n-1} = 1$$

Equivalently, our equation is as follows, with $G = G(z^{-1})$ as before:

$$\frac{z}{G} + \sum_{n=1}^{\infty} \kappa_n z^n \left(\frac{G}{z}\right)^{n-1} = 1$$

(4) Observe now that we have the following formula:

$$\frac{G}{z} = \frac{G(z^{-1})}{z} = \frac{\sum_{n=0}^{\infty} M_n z^{n+1}}{z} = \sum_{n=0}^{\infty} M_n z^n$$

This suggests introducing the following quantity:

$$F = \sum_{n=1}^{\infty} M_n z^n$$

Indeed, we have then $G/z = 1 + F$, and our equation becomes:

$$\frac{1}{1+F} + \sum_{n=1}^{\infty} \kappa_n z_n (1+F)^{n-1} = 1$$

(5) By expanding the fraction on the left, our equation becomes:

$$\sum_{n=0}^{\infty} (-F)^n + \sum_{n=1}^{\infty} \kappa_n z_n (1+F)^{n-1} = 1$$

Moreover, we can cancel the 1 term on both sides, and our equation becomes:

$$\sum_{n=1}^{\infty} (-F)^n + \sum_{n=1}^{\infty} \kappa_n z_n (1+F)^{n-1} = 0$$

Alternatively, we can write our equation as follows:

$$\sum_{n=1}^{\infty} \kappa_n z_n (1+F)^{n-1} = - \sum_{n=1}^{\infty} (-F)^n$$

(6) Good news, this latter equation is something that we are eventually happy with. By remembering that we have $F = \sum_{n=1}^{\infty} M_n z^n$, our equation looks as follows:

$$\begin{aligned} & \kappa_1 z + \kappa_2 z^2 (1 + M_1 z + M_2 z^2 + \dots) + \kappa_3 z^3 (1 + M_1 z + M_2 z^2 + \dots)^2 + \dots \\ = & (M_1 z + M_2 z^2 + \dots) - (M_1 z + M_2 z^2 + \dots)^2 + (M_1 z + M_2 z^2 + \dots)^3 - \dots \end{aligned}$$

(7) This was for the hard part, carefully fine-tuning our equation, as to have it as simple as possible, before getting to numeric work. The rest is routine. Indeed, by looking at the terms of order 1, 2, 3, 4 we obtain, instantly or almost, the formulae of $\kappa_1, \kappa_2, \kappa_3, \kappa_4$ in the statement. As for the formulae for M_1, M_2, M_3, M_4 , these follow from these.

(8) To be more precise, the equations that we get at order 1, 2, 3, 4 are as follows:

$$\begin{aligned} \kappa_1 &= M_1 \\ \kappa_2 &= M_2 - M_1^2 \\ \kappa_2 M_1 + \kappa_3 &= M_3 - 2M_1 M_2 + M_1^3 \\ \kappa_4 + 2\kappa_3 M_1 + \kappa_2 M_2 &= M_4 - 2M_1 M_3 - M_2^2 + 3M_1^2 M_2 - M_1^4 \end{aligned}$$

Thus, we are led to the formulae of $\kappa_1, \kappa_2, \kappa_3, \kappa_4$ in the statement, and then to the formulae of M_1, M_2, M_3, M_4 in the statement, as desired. \square

Observe the similarity with the formulae in Proposition 12.4. In fact, a careful comparison with Proposition 12.4 is worth the effort, leading to the following conclusion:

CONCLUSION 12.16. *The first three classical and free cumulants coincide,*

$$k_1 = \kappa_1 \quad , \quad k_2 = \kappa_2 \quad , \quad k_3 = \kappa_3$$

but the formulae for the fourth classical and free cumulants are different,

$$k_4 = -6M_1^4 + 12M_1^2 M_2 - 3M_2^2 - 4M_1 M_3 + M_4$$

$$\kappa_4 = -5M_1^4 + 10M_1^2 M_2 - 2M_2^2 - 4M_1 M_3 + M_4$$

and the same happens at higher order as well.

This is something quite interesting, and we will back later with a conceptual explanation for this, via partitions, the idea being that all this comes from:

$$P(n) = NC(n) \iff n \leq 3$$

But more on this later. At the level of basic general results, we first have:

THEOREM 12.17. *The free cumulants have the following properties:*

- (1) $\kappa_n(\lambda a) = \lambda^n \kappa_n(a)$.
- (2) $\kappa_n(a + b) = \kappa_n(a) + \kappa_n(b)$, if a, b are free.

PROOF. This is something very standard, the idea being as follows:

(1) We have the following Cauchy transform computation:

$$\begin{aligned} G_{\lambda a}(\xi) &= \int_{\mathbb{R}} \frac{d\mu_{\lambda a}(t)}{\xi - t} \\ &= \int_{\mathbb{R}} \frac{d\mu_a(s)}{\xi - \lambda s} \\ &= \frac{1}{\lambda} \int_{\mathbb{R}} \frac{d\mu_a(s)}{\xi/\lambda - s} \\ &= \frac{1}{\lambda} G_a\left(\frac{\xi}{\lambda}\right) \end{aligned}$$

But this gives the following formula, by using the definition of the R -transform:

$$\begin{aligned} G_{\lambda a}\left(\lambda R_a(\lambda\xi) + \frac{1}{\xi}\right) &= \frac{1}{\lambda} G_a\left(R_a(\lambda\xi) + \frac{1}{\lambda\xi}\right) \\ &= \frac{1}{\lambda} \cdot \lambda\xi \\ &= \xi \end{aligned}$$

Thus we have the formula $R_{\lambda a}(\xi) = \lambda R_a(\lambda\xi)$, which gives (1).

(2) This follows from the standard fact, that we know well from chapter 9, that the R -transform linearizes the free convolution operation. \square

Again in analogy with the classical case, at the level of examples, we have:

THEOREM 12.18. *The sequence of free cumulants $\kappa_1, \kappa_2, \kappa_3, \dots$ is as follows:*

- (1) For $\mu = \delta_c$ the free cumulants are $c, 0, 0, \dots$
- (2) For $\mu = \gamma_t$ the free cumulants are $0, t, 0, 0, \dots$
- (3) For $\mu = \pi_t$ the free cumulants are t, t, t, \dots
- (4) For $\mu = \beta_t$ the free cumulants are $0, t, 0, t, \dots$

Also, for compound free Poisson laws the free cumulants are $k_n(\pi_\nu) = M_n(\nu)$.

PROOF. The proofs are analogous to those from the classical case, as follows:

(1) For $\mu = \delta_c$ we have $G_\mu(\xi) = 1/(\xi - c)$, and so $R_\mu(\xi) = c$, as desired.

(2) For $\mu = \gamma_t$ we have, as computed in chapter 9, $R_\mu(\xi) = t\xi$, as desired.

(3) For $\mu = \pi_t$ we have, also from chapter 11, $R_\mu(\xi) = t/(1 - \xi)$, as desired.

(4) For $\mu = \beta_t$ this follows from the formulae in chapter 11, but the best is to prove directly the last assertion, which generalizes (3,4). With $\nu = \sum_i c_i \delta_{z_i}$ we have:

$$\begin{aligned} R_{\pi_\nu}(\xi) &= \sum_i \frac{c_i z_i}{1 - \xi z_i} \\ &= \sum_i c_i z_i \sum_{n \geq 0} (\xi z_i)^n \\ &= \sum_{n \geq 0} \xi^n \sum_i c_i z_i^{n+1} \\ &= \sum_{n \geq 1} \xi^{n-1} \sum_i c_i z_i^n \\ &= \sum_{n \geq 1} \xi^{n-1} M_n(\nu) \end{aligned}$$

Thus, we are led to the conclusion in the statement. \square

Observe in particular that the last formula in the above statement, $k_n(\pi_\nu) = M_n(\nu)$, which is something quite powerful, clarifies a discussion started in chapter 8, and then continued in chapter 11, in relation with the block-modified Wishart matrices.

As before in the classical case, we can define now generalized free cumulants, $\kappa_\pi(a)$ with $\pi \in P(k)$, by starting with the numeric free cumulants $\kappa_n(a)$, as follows:

DEFINITION 12.19. *We define free cumulants $\kappa_\pi(a)$, depending on partitions*

$$\pi \in P(k)$$

by starting with $\kappa_n(a)$, and using multiplicativity over the blocks.

To be more precise, the convention here is that for the one-block partition $1_n \in P(n)$, the corresponding free cumulant is the usual one, namely:

$$\kappa_{1_n}(a) = \kappa_n(a)$$

Then, for an arbitrary partition $\pi \in P(k)$, we decompose this partition into blocks, having sizes b_1, \dots, b_s , and we set, by multiplicativity over blocks:

$$\kappa_\pi(a) = \kappa_{b_1}(a) \dots \kappa_{b_s}(a)$$

With this convention, we have the following result, due to Speicher [79]:

THEOREM 12.20. *We have the moment-cumulant formulae*

$$M_n(a) = \sum_{\nu \in NC(n)} \kappa_\nu(a) \quad , \quad \kappa_n(a) = \sum_{\nu \in NC(n)} \mu(\nu, 1_n) M_\nu(a)$$

or, equivalently, we have the moment-cumulant formulae

$$M_\pi(a) = \sum_{\nu \leq \pi} \kappa_\nu(a) \quad , \quad \kappa_\pi(a) = \sum_{\nu \leq \pi} \mu(\nu, \pi) M_\nu(a)$$

where μ is the Möbius function of $NC(n)$.

PROOF. As before in the classical case, the 4 formulae in the statement are equivalent, via Möbius inversion. Thus, it is enough to prove one of them, and we will prove the first formula, which in practice is the most useful one. Thus, we must prove that:

$$M_n(a) = \sum_{\nu \in NC(n)} \kappa_\nu(a)$$

(1) In order to prove this formula, let us get back to the construction of the free cumulants, from Definition 12.14. The Cauchy transform of a is the following function:

$$G_a(\xi) = \sum_{n=0}^{\infty} \frac{M_n(a)}{\xi^{n+1}}$$

Consider the inverse of this Cauchy transform G_a , with respect to composition:

$$G_a(K_a(\xi)) = K_a(G_a(\xi)) = \xi$$

According to Definition 12.14, the free cumulants $\kappa_n(a)$ appear then as follows:

$$K_a(\xi) = \frac{1}{\xi} + \sum_{n=1}^{\infty} \kappa_n(a) \xi^{n-1}$$

Thus, we can compute moments in terms of free cumulants by using either of the inversion formulae $G_a(K_a(\xi)) = \xi$ and $K_a(G_a(\xi)) = \xi$.

(2) In practice, as explained in the proof of Proposition 12.15, the best is to use the second inversion formula, $K_a(G_a(\xi)) = \xi$, which after some manipulations reads:

$$\begin{aligned} & \kappa_1 z + \kappa_2 z^2 (1 + M_1 z + M_2 z^2 + \dots) + \kappa_3 z^3 (1 + M_1 z + M_2 z^2 + \dots)^2 + \dots \\ = & (M_1 z + M_2 z^2 + \dots) - (M_1 z + M_2 z^2 + \dots)^2 + (M_1 z + M_2 z^2 + \dots)^3 - \dots \end{aligned}$$

We have already seen, in the proof of Proposition 12.15, how to exploit this formula at order $n = 1, 2, 3, 4$. The same method works in general, and after some computations, this leads to the formula that we want to establish, namely:

$$M_n(a) = \sum_{\nu \in NC(n)} \kappa_\nu(a)$$

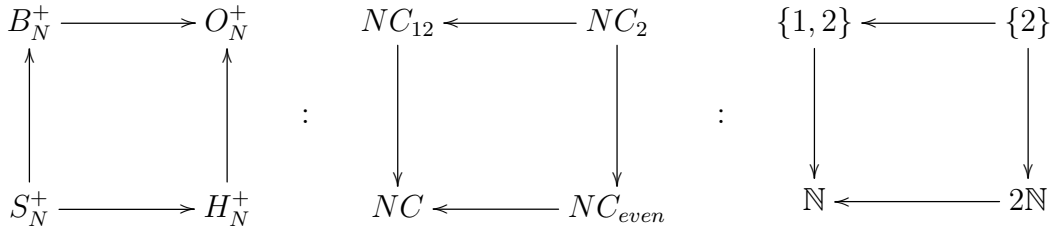
(3) We are therefore led to the conclusions in the statement. All this was of course quite brief, and for details here, we refer for instance to Nica-Speicher [71]. \square

Observe that the above result leads among others to a more conceptual explanation for Conclusion 12.16, with the equalities and non-equalities there simply coming from:

$$P(n) = NC(n) \iff n \leq 3$$

Finally, in what regards more advanced aspects, in relation with the moment formula $M_k = \sum_{\pi \in D(k)} t^{|\pi|}$, this ideally requires quantum groups, and more specifically easy quantum groups, and we will talk about this in chapter 13 below. As an advertisement for that material, however, let us record in advance the following statement:

THEOREM 12.21. *The free uniform orthogonal easy quantum groups $G \subset O_N^+$, and their associated categories of partitions $D \subset P$, all coming from subsets $L \subset \mathbb{N}$, are*



with D consisting of the partitions $\pi \in NC$ whose blocks have lengths belonging to $L \subset \mathbb{N}$. The free cumulants of the corresponding measures are given by the formula

$$\kappa_n = t\delta_{n \in L}$$

and at the level of moments this gives the formula $M_k = \sum_{\pi \in D(k)} t^{|\pi|}$.

PROOF. Obviously, this is something informal, and we will be back to it, with details. However, with the plea of just believing us, the idea is that the easy quantum groups are abstract beasts of type $S_N^+ \subset G \subset O_N^+$, coming from categories $NC_2 \subset D \subset NC$, and so we are left with an algebraic and probabilistic study of these latter categories, which can be done exactly as in the classical case, and which leads to the above conclusions. More on this in a moment, and in the meantime, we refer to [21] for all this. \square

There are many other things that can be said about free cumulants, and we will come back to this later on, in chapter 15 below, directly in a more general setting, that of the operator-valued free probability theory, following [80], when discussing free de Finetti theorems, which crucially use the free cumulant technology.

Importantly, everything that has been said above about free cumulants, be it a bit technical, is a mirror image of what can be said about classical cumulants. But at a more advanced level, things are far more interesting than this, for instance because of the key isomorphism $NC(k) \simeq NC_2(2k)$, that we already met in this book in some other contexts, having no classical counterpart. We will be back to this.

12d. The bijection

With the above classical and free cumulant theory in hand, we can now formulate the following simple definition, making the connection between classical and free:

DEFINITION 12.22. *We say that a real probability measure*

$$m \in \mathcal{P}(\mathbb{R})$$

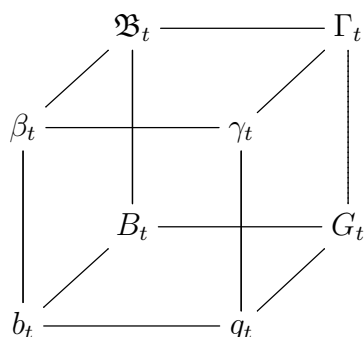
is the classical version of another measure, called its free version, or liberation

$$\mu \in \mathcal{P}(\mathbb{R})$$

when the classical cumulants of m coincide with the free cumulants of μ .

As a first observation, this definition fits with all the classical and free probability theory developed in the above, in this whole book so far, and notably with the measures from the standard cube, and to start with, we have the following result:

THEOREM 12.23. *In the standard cube of basic probability measures,*



the upper measures appear as the free versions of the lower measures.

PROOF. This follows indeed from our various cumulant formulae found above. □

In order to reach now to a more advanced theory, depending this time on a parameter $t > 0$, which is something essential, and whose importance will become clear later on, let us formulate, following Bercovici-Pata [24], and the subsequent work in [71]:

DEFINITION 12.24. *A convolution semigroup of measures*

$$\{m_t\}_{t>0} \quad : \quad m_s * m_t = m_{s+t}$$

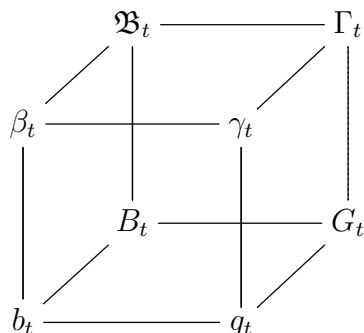
is in Bercovici-Pata bijection with a free convolution semigroup of measures

$$\{\mu_t\}_{t>0} \quad : \quad \mu_s \boxplus \mu_t = \mu_{s+t}$$

when the classical cumulants of m_t coincide with the free cumulants of μ_t .

As before, this fits with all the theory developed so far in this book, and notably with the measures from the standard cube, and we have the following result:

THEOREM 12.25. *In the standard cube of basic semigroups of measures,*



the upper semigroups are in Bercovici-Pata bijection with the lower semigroups.

PROOF. This is a technical improvement of Theorem 12.23, based on the fact that the upper measures in the above diagram form indeed free convolution semigroups, and that the lower measures form indeed classical convolution semigroups, which itself is something that we know well, from the various semigroup results established in above. \square

Back to the examples now, there are many other, and we will be back to this. But, before anything, let us formulate the following surprising result, from [23]:

THEOREM 12.26. *The normal law g_1 is freely infinitely divisible.*

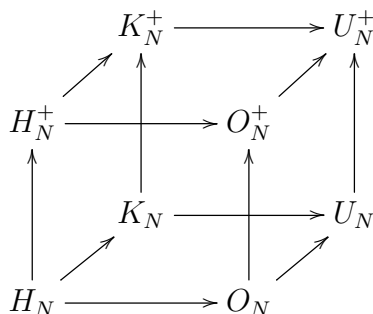
PROOF. This is something tricky, involving all sorts of not very intuitive computations, and for full details here, we refer here to the original paper [23]. \square

The above result shows that the normal law g_1 should have a “classical analogue” in the sense of the Bercovici-Pata bijection. And isn’t that puzzling. The problem, however, is that this latter law is difficult to compute, and interpret. See [23].

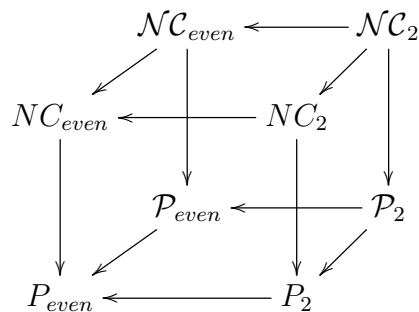
Still in relation with the Bercovici-Pata bijection, let us also mention that there are many interesting analytic aspects, coming from the combinatorics of the infinitely divisible laws, classical or free. For this, and other analytic aspects, we refer to [24].

Finally, as previously promised, let us briefly discuss the axiomatization of the standard cube, using quantum groups. Skipping some details, or rather leaving them for chapter 13 below, the idea is that we have a result as follows:

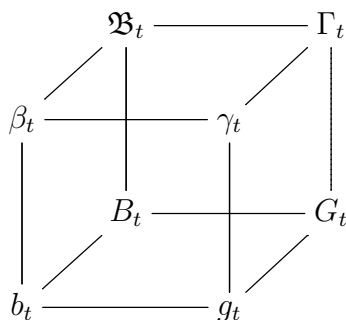
THEOREM 12.27 (Ground Zero). *Under a collection of suitable extra assumptions*



are the unique easy quantum groups. Equivalently, under suitable extra assumptions



are the unique categories of partitions. Also equivalently, under suitable assumptions



are the unique main probability measures.

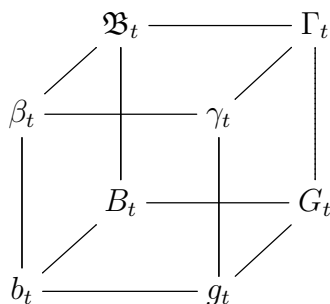
PROOF. There is a long story here, first for formulating the precise statement, which is something non-trivial, and then of course for proving it, and for the whole story here, we refer to [8]. We will be back with more details on all this in chapter 13 below. \square

As a conclusion to all this, with some ideas from combinatorics and quantum groups, we have managed to axiomatize the main laws in classical and free probability. Which is certainly something interesting, because we have now some clear ground, free of traps and abstractions, that we can build upon. We will discuss this a bit, in what follows.

12e. Exercises

There has been a lot of theory in this chapter, and as an exercise here, we have:

EXERCISE 12.28. *Clarify all the details for the standard cube of measures*



by proving that we have indeed the Bercovici-Pata bijection on the vertical.

This is something discussed in the above, but with some details missing, and the problem now, which is very instructive, is that of filling all the details.

Part IV

Quantum algebra

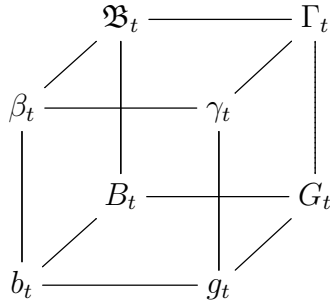
Strangers in the night
Exchanging glances
Wandering in the night
What were the chances

CHAPTER 13

Quantum groups

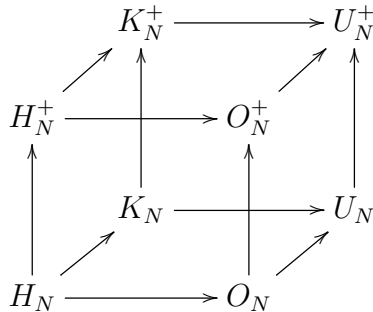
13a. Quantum groups

We have seen so far that classical probability has a “twin sister”, which is Voiculescu’s free probability theory. The relation between the two comes from an almost perfect symmetry between the main limiting theorems in both theories, which can be axiomatized. At a more concrete level, passed a few technical manipulations, the main limiting laws are as follows, with the vertical correspondence being the Bercovici-Pata bijection:



All this remains however a bit abstract. Fortunately, beasts like random matrices and quantum groups are there, providing us with explicit models for the above laws, and for what is going on, in general. In what regards quantum groups, we have:

THEOREM 13.1. *The main limiting laws in classical and free probability come from*



as asymptotic laws, with $N \rightarrow \infty$, of the corresponding truncated characters.

PROOF. This is something that we know from chapter 4, for the lower face of the cube. In what regards the upper face, this is something which remains to be clarified. \square

Our purpose in this chapter and in the next one will be to discuss the details of this result, and then further build on it, by expanding the theory into a more general correspondence between classical geometry and free geometry. Then later, in chapters 15-16, we will discuss invariance questions, and we will add as well to the picture some further beasts, which are of even more tricky type, namely the Jones subfactors.

As a starting point, we have the following key definition, from [98]:

DEFINITION 13.2. *A Woronowicz algebra is a C^* -algebra A , given with a unitary matrix $v \in M_N(A)$ whose coefficients generate A , such that the formulae*

$$\Delta(v_{ij}) = \sum_k v_{ik} \otimes v_{kj} \quad , \quad \varepsilon(v_{ij}) = \delta_{ij} \quad , \quad S(v_{ij}) = v_{ji}^*$$

define morphisms of C^* -algebras $\Delta : A \rightarrow A \otimes A$, $\varepsilon : A \rightarrow \mathbb{C}$, $S : A \rightarrow A^{opp}$.

This definition is in fact a modified version of Woronowicz' main definition in [98], which best fits our purposes here, covering well the objects in Theorem 13.1. More on this later. We say that A is cocommutative when $\Sigma\Delta = \Delta$, where $\Sigma(a \otimes b) = b \otimes a$ is the flip. We have the following result, which justifies the terminology and axioms:

PROPOSITION 13.3. *The following are Woronowicz algebras:*

(1) $C(G)$, with $G \subset U_N$ compact Lie group. Here the structural maps are:

$$\Delta(\varphi) = [(g, h) \rightarrow \varphi(gh)] \quad , \quad \varepsilon(\varphi) = \varphi(1) \quad , \quad S(\varphi) = [g \rightarrow \varphi(g^{-1})]$$

(2) $C^*(\Gamma)$, with $F_N \rightarrow \Gamma$ finitely generated group. Here the structural maps are:

$$\Delta(g) = g \otimes g \quad , \quad \varepsilon(g) = 1 \quad , \quad S(g) = g^{-1}$$

Moreover, we obtain in this way all the commutative/cocommutative algebras.

PROOF. In both cases, we have to indicate a certain matrix v . For the first assertion, we can use the matrix $v = (v_{ij})$ formed by matrix coordinates of G , given by:

$$g = \begin{pmatrix} v_{11}(g) & \dots & v_{1N}(g) \\ \vdots & & \vdots \\ v_{N1}(g) & \dots & v_{NN}(g) \end{pmatrix}$$

As for the second assertion, we can use here the diagonal matrix formed by generators:

$$v = \begin{pmatrix} g_1 & & 0 \\ & \ddots & \\ 0 & & g_N \end{pmatrix}$$

Finally, the last assertion follows from the Gelfand theorem, in the commutative case. In the cocommutative case this follows from the Peter-Weyl theory, explained below. \square

In view of Proposition 13.3, we can formulate the following definition:

DEFINITION 13.4. *Given a Woronowicz algebra A , we formally write*

$$A = C(G) = C^*(\Gamma)$$

and call G compact quantum group, and Γ discrete quantum group.

When A is both commutative and cocommutative, G is a compact abelian group, Γ is a discrete abelian group, and these groups are dual to each other:

$$G = \widehat{\Gamma} \quad , \quad \Gamma = \widehat{G}$$

In general, we still agree to write the formulae $G = \widehat{\Gamma}, \Gamma = \widehat{G}$, but in a formal sense. Finally, let us make as well the following convention:

DEFINITION 13.5. *We identify two Woronowicz algebras (A, v) and (B, w) , as well as the corresponding quantum groups, when we have an isomorphism of $*$ -algebras*

$$\langle v_{ij} \rangle \simeq \langle w_{ij} \rangle$$

mapping standard coordinates to standard coordinates.

This convention is here for avoiding amenability issues, as for any compact or discrete quantum group to correspond to a unique Woronowicz algebra. More on this later.

Moving ahead now, let us call corepresentation of A any unitary matrix $u \in M_n(\mathcal{A})$, where $\mathcal{A} = \langle v_{ij} \rangle$, satisfying the same conditions as those satisfied by u , namely:

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj} \quad , \quad \varepsilon(u_{ij}) = \delta_{ij} \quad , \quad S(u_{ij}) = u_{ji}^*$$

We have the following key result, due to Woronowicz [98]:

THEOREM 13.6. *Any Woronowicz algebra has a unique Haar integration functional,*

$$\left(\int_G \otimes id \right) \Delta = \left(id \otimes \int_G \right) \Delta = \int_G (\cdot) 1$$

which can be constructed by starting with any faithful positive form $\varphi \in A^$, and setting*

$$\int_G = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \varphi^{*k}$$

*where $\phi * \psi = (\phi \otimes \psi)\Delta$. Moreover, for any corepresentation $u \in M_n(\mathbb{C}) \otimes A$ we have*

$$\left(id \otimes \int_G \right) u = P$$

where P is the orthogonal projection onto $Fix(u) = \{\xi \in \mathbb{C}^n | u\xi = \xi\}$.

PROOF. Following [98], this can be done in 3 steps, as follows:

(1) Given $\varphi \in A^*$, our claim is that the following limit converges, for any $a \in A$:

$$\int_{\varphi} a = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \varphi^{*k}(a)$$

Indeed, by linearity we can assume that $a \in A$ is the coefficient of certain corepresentation, $a = (\tau \otimes id)u$. But in this case, an elementary computation gives the following formula, with P_{φ} being the orthogonal projection onto the 1-eigenspace of $(id \otimes \varphi)u$:

$$\left(id \otimes \int_{\varphi} \right) u = P_{\varphi}$$

(2) Since $u\xi = \xi$ implies $[(id \otimes \varphi)u]\xi = \xi$, we have $P_{\varphi} \geq P$, where P is the orthogonal projection onto the fixed point space in the statement, namely:

$$Fix(u) = \left\{ \xi \in \mathbb{C}^n \mid u\xi = \xi \right\}$$

The point now is that when $\varphi \in A^*$ is faithful, by using a standard positivity trick, we can prove that we have $P_{\varphi} = P$, exactly as in the classical case.

(3) With the above formula in hand, the left and right invariance of $\int_G = \int_{\varphi}$ is clear on coefficients, and so in general, and this gives all the assertions. See [98]. \square

We can now develop, again following [98], the Peter-Weyl theory for the corepresentations of A . Consider the dense subalgebra $\mathcal{A} \subset A$ generated by the coefficients of the fundamental corepresentation v , and endow it with the following scalar product:

$$\langle a, b \rangle = \int_G ab^*$$

With this convention, we have the following result, from [98]:

THEOREM 13.7. *We have the following Peter-Weyl type results:*

- (1) *Any corepresentation decomposes as a sum of irreducible corepresentations.*
- (2) *Each irreducible corepresentation appears inside a certain $v^{\otimes k}$.*
- (3) *$\mathcal{A} = \bigoplus_{u \in Irr(A)} M_{\dim(u)}(\mathbb{C})$, the summands being pairwise orthogonal.*
- (4) *The characters of irreducible corepresentations form an orthonormal system.*

PROOF. All these results are from [98], the idea being as follows:

(1) Given $u \in M_n(A)$, the intertwiner algebra $End(u) = \{T \in M_n(\mathbb{C}) \mid Tu = uT\}$ is a finite dimensional C^* -algebra, and so decomposes as $End(u) = M_{n_1}(\mathbb{C}) \oplus \dots \oplus M_{n_r}(\mathbb{C})$. But this gives a decomposition of type $u = u_1 + \dots + u_r$, as desired.

(2) Consider the Peter-Weyl corepresentations, $v^{\otimes k}$ with k colored integer, defined by $v^{\otimes 0} = 1, v^{\otimes \circ} = v, v^{\otimes \bullet} = \bar{v}$ and multiplicativity. The coefficients of these corepresentations span the dense algebra \mathcal{A} , and by using (1), this gives the result.

(3) Here the direct sum decomposition, which is a $*$ -coalgebra isomorphism, follows from (2). As for the second assertion, this follows from the fact that $(id \otimes \int_G)u$ is the orthogonal projection P_u onto the space $Fix(u)$, for any corepresentation u .

(4) Let us define indeed the character of $u \in M_n(A)$ to be the trace, $\chi_u = Tr(u)$. Since this character is a coefficient of u , the orthogonality assertion follows from (3). As for the norm 1 claim, this follows once again from $(id \otimes \int_G)u = P_u$. \square

We can now solve a problem that we left open before, namely:

PROPOSITION 13.8. *The cocommutative Woronowicz algebras appear as the quotients*

$$C^*(\Gamma) \rightarrow A \rightarrow C_{red}^*(\Gamma)$$

given by $A = C_\pi^*(\Gamma)$ with $\pi \otimes \pi \subset \pi$, with Γ being a discrete group.

PROOF. This follows from the Peter-Weyl theory, and clarifies a number of things said before, notably in Proposition 13.3. Indeed, for a cocommutative Woronowicz algebra the irreducible corepresentations are all 1-dimensional, and this gives the results. \square

As another consequence of the above results, once again by following Woronowicz [98], we have the following statement, dealing with functional analysis aspects, and extending what we already knew about the C^* -algebras of the usual discrete groups:

THEOREM 13.9. *Let A_{full} be the enveloping C^* -algebra of A , and A_{red} be the quotient of A by the null ideal of the Haar integration. The following are then equivalent:*

- (1) *The Haar functional of A_{full} is faithful.*
- (2) *The projection map $A_{full} \rightarrow A_{red}$ is an isomorphism.*
- (3) *The counit map $\varepsilon : A_{full} \rightarrow \mathbb{C}$ factorizes through A_{red} .*
- (4) *We have $N \in \sigma(Re(\chi_v))$, the spectrum being taken inside A_{red} .*

If this is the case, we say that the underlying discrete quantum group Γ is amenable.

PROOF. This is well-known in the group dual case, $A = C^*(\Gamma)$, with Γ being a usual discrete group. In general, the result follows by adapting the group dual case proof:

(1) \iff (2) This simply follows from the fact that the GNS construction for the algebra A_{full} with respect to the Haar functional produces the algebra A_{red} .

(2) \iff (3) Here \implies is trivial, and conversely, a counit map $\varepsilon : A_{red} \rightarrow \mathbb{C}$ produces an isomorphism $A_{red} \rightarrow A_{full}$, via a formula of type $(\varepsilon \otimes id)\Phi$. See [98].

(3) \iff (4) Here \implies is clear, coming from $\varepsilon(N - Re(\chi(v))) = 0$, and the converse can be proved by doing some functional analysis. Once again, we refer here to [98]. \square

Let us discuss now some interesting examples. Following Wang [92], we have:

PROPOSITION 13.10. *The following universal algebras are Woronowicz algebras,*

$$C(O_N^+) = C^* \left((v_{ij})_{i,j=1,\dots,N} \mid v = \bar{v}, v^t = v^{-1} \right)$$

$$C(U_N^+) = C^* \left((v_{ij})_{i,j=1,\dots,N} \mid v^* = v^{-1}, v^t = \bar{v}^{-1} \right)$$

so the underlying compact quantum spaces O_N^+, U_N^+ are compact quantum groups.

PROOF. This follows from the elementary fact that if a matrix $v = (v_{ij})$ is orthogonal or biunitary, then so must be the following matrices:

$$v_{ij}^\Delta = \sum_k v_{ik} \otimes v_{kj} \quad , \quad v_{ij}^\varepsilon = \delta_{ij} \quad , \quad v_{ij}^S = v_{ji}^*$$

Thus, we can indeed define morphisms Δ, ε, S as in Definition 13.2, by using the universal properties of $C(O_N^+), C(U_N^+)$, and this gives the result. \square

There is a connection here with group duals, coming from:

PROPOSITION 13.11. *Given a closed subgroup $G \subset U_N^+$, consider its “diagonal torus”, which is the closed subgroup $T \subset G$ constructed as follows:*

$$C(T) = C(G) / \langle v_{ij} = 0 \mid \forall i \neq j \rangle$$

This torus is then a group dual, $T = \widehat{\Lambda}$, where $\Lambda = \langle g_1, \dots, g_N \rangle$ is the discrete group generated by the elements $g_i = v_{ii}$, which are unitaries inside $C(T)$.

PROOF. Since u is unitary, its diagonal entries $g_i = v_{ii}$ are unitaries inside $C(T)$. Moreover, from $\Delta(v_{ij}) = \sum_k v_{ik} \otimes v_{kj}$ we obtain, when passing inside the quotient:

$$\Delta(g_i) = g_i \otimes g_i$$

It follows that we have $C(T) = C^*(\Lambda)$, modulo identifying as usual the C^* -completions of the various group algebras, and so that we have $T = \widehat{\Lambda}$, as claimed. \square

With this notion in hand, we have the following result:

THEOREM 13.12. *The diagonal tori of the basic rotation groups are as follows,*

$$\begin{array}{ccc}
 U_N & \longrightarrow & U_N^+ \\
 \uparrow & & \uparrow \\
 O_N & \longrightarrow & O_N^+
 \end{array}
 \quad : \quad
 \begin{array}{ccc}
 \mathbb{T}^N & \longrightarrow & \widehat{F}_N \\
 \uparrow & & \uparrow \\
 \mathbb{Z}_2^N & \longrightarrow & \widehat{\mathbb{Z}_2^{*N}}
 \end{array}$$

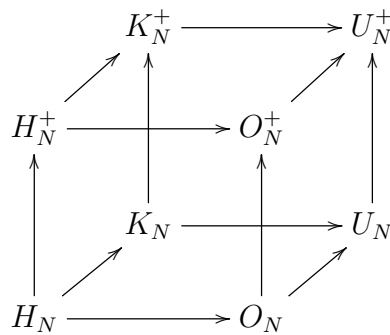
where F_N is the free group on N generators, and $*$ is a group-theoretical free product.

PROOF. This is clear indeed from U_N^+ , and the other results can be obtained by imposing to the generators of F_N the relations defining the corresponding quantum groups. \square

As a conclusion to all this, the above results, coming from [92], suggest developing a theory of “noncommutative geometry”, covering both the classical and the free geometry, by using compact quantum groups. We will be back to this in chapter 14.

Getting now into more examples, we have the following key result, coming from the work in [8], [9], [10], [13], [30], covering the basic rotation and reflection groups:

THEOREM 13.13. *The classical and free, real and complex quantum rotation groups can be complemented with quantum reflection groups, as follows,*



with $H_N = \mathbb{Z}_2 \wr S_N$ and $K_N = \mathbb{T} \wr S_N$ being the hyperoctahedral group and the full complex reflection group, and $H_N^+ = \mathbb{Z}_2 \wr_* S_N^+$ and $K_N^+ = \mathbb{T} \wr_* S_N^+$ being their free versions.

PROOF. This is something quite tricky, the idea being as follows:

(1) The first observation is that S_N , regarded as group of permutations of the N coordinate axes of \mathbb{R}^N , is a group of orthogonal matrices, $S_N \subset O_N$. The corresponding coordinate functions $v_{ij} : S_N \rightarrow \{0, 1\}$ form a matrix $v = (v_{ij})$ which is “magic”, in the sense that its entries are projections, summing up to 1 on each row and each column. In fact, by using the Gelfand theorem, we have the following presentation result:

$$C(S_N) = C_{comm}^* \left((v_{ij})_{i,j=1,\dots,N} \mid v = \text{magic} \right)$$

(2) Based on the above, and following Wang’s paper [93], we can construct the free analogue S_N^+ of the symmetric group S_N via the following formula:

$$C(S_N^+) = C^* \left((v_{ij})_{i,j=1,\dots,N} \mid v = \text{magic} \right)$$

Here the fact that we have indeed a Woronowicz algebra is standard, exactly as for the free rotation groups in Proposition 13.10, because if a matrix $v = (v_{ij})$ is magic, then so are the matrices $v^\Delta, v^\varepsilon, v^S$ constructed there, and this gives the existence of Δ, u, S .

(3) Consider now the group $H_N^s \subset U_N$ consisting of permutation-like matrices having as entries the s -th roots of unity. This group decomposes as follows:

$$H_N^s = \mathbb{Z}_s \wr S_N$$

It is straightforward then to construct a free analogue $H_N^{s+} \subset U_N^+$ of this group, for instance by formulating a definition as follows, with \wr_s being a free wreath product:

$$H_N^{s+} = \mathbb{Z}_s \wr_s S_N^+$$

(4) In order to finish, besides the case $s = 1$, of particular interest are the cases $s = 2, \infty$. Here the corresponding reflection groups are as follows:

$$H_N = \mathbb{Z}_2 \wr S_N \quad , \quad K_N = \mathbb{T} \wr S_N$$

As for the corresponding quantum groups, these are denoted as follows:

$$H_N^+ = \mathbb{Z}_2 \wr_s S_N^+ \quad , \quad K_N^+ = \mathbb{T} \wr_s S_N^+$$

Thus, we are led to the conclusions in the statement. See [9], [10]. □

13b. Diagrams, easiness

Getting now towards easiness, let us start with the following definition:

DEFINITION 13.14. *The Tannakian category associated to a Woronowicz algebra (A, v) is the collection $C_A = (C_A(k, l))$ of vector spaces*

$$C_A(k, l) = Hom(v^{\otimes k}, v^{\otimes l})$$

where the corepresentations $v^{\otimes k}$ with $k = \circ \bullet \bullet \circ \dots$ colored integer, defined by

$$v^{\otimes \emptyset} = 1 \quad , \quad v^{\otimes \circ} = v \quad , \quad v^{\otimes \bullet} = \bar{v}$$

and multiplicativity, $v^{\otimes kl} = v^{\otimes k} \otimes v^{\otimes l}$, are the Peter-Weyl corepresentations.

As a key remark, the fact that $v \in M_N(A)$ is biunitary translates into the following conditions, where $R : \mathbb{C} \rightarrow \mathbb{C}^N \otimes \mathbb{C}^N$ is the linear map given by $R(1) = \sum_i e_i \otimes e_i$:

$$R \in Hom(1, v \otimes \bar{v}) \quad , \quad R \in Hom(1, \bar{v} \otimes v)$$

$$R^* \in Hom(v \otimes \bar{v}, 1) \quad , \quad R^* \in Hom(\bar{v} \otimes v, 1)$$

We are therefore led to the following abstract definition, summarizing the main properties of the categories appearing from Woronowicz algebras:

DEFINITION 13.15. *Let H be a finite dimensional Hilbert space. A tensor category over H is a collection $C = (C(k, l))$ of subspaces*

$$C(k, l) \subset \mathcal{L}(H^{\otimes k}, H^{\otimes l})$$

satisfying the following conditions:

- (1) $S, T \in C$ implies $S \otimes T \in C$.
- (2) If $S, T \in C$ are composable, then $ST \in C$.
- (3) $T \in C$ implies $T^* \in C$.
- (4) Each $C(k, k)$ contains the identity operator.
- (5) $C(\emptyset, \circ \bullet)$ and $C(\emptyset, \bullet \circ)$ contain the operator $R : 1 \rightarrow \sum_i e_i \otimes e_i$.

The point now is that conversely, we can associate a Woronowicz algebra to any tensor category in the sense of Definition 13.15, in the following way:

PROPOSITION 13.16. *Given a tensor category $C = (C(k, l))$ over \mathbb{C}^N , as above,*

$$A_C = C^* \left((v_{ij})_{i,j=1,\dots,N} \middle| T \in \text{Hom}(v^{\otimes k}, v^{\otimes l}), \forall k, l, \forall T \in C(k, l) \right)$$

is a Woronowicz algebra.

PROOF. This is something standard, because the relations $T \in \text{Hom}(v^{\otimes k}, v^{\otimes l})$ determine a Hopf ideal, so they allow the construction of Δ, ε, S as in Definition 13.2. \square

With the above constructions in hand, we have the following result:

THEOREM 13.17. *The Tannakian duality constructions*

$$C \rightarrow A_C \quad , \quad A \rightarrow C_A$$

are inverse to each other, modulo identifying full and reduced versions.

PROOF. The idea is that we have $C \subset C_{A_C}$, for any algebra A , and so we are left with proving that we have $C_{A_C} \subset C$, for any category C . But this follows from a long series of algebraic manipulations, and for details we refer to Malacarne [64], and also to Woronowicz [99], where this result was first proved, by using other methods. \square

In practice now, all this is quite abstract, and we will rather need Brauer type results, for the specific quantum groups that we are interested in. Let us start with:

DEFINITION 13.18. *Let $P(k, l)$ be the set of partitions between an upper colored integer k , and a lower colored integer l . A collection of subsets*

$$D = \bigsqcup_{k,l} D(k, l)$$

with $D(k, l) \subset P(k, l)$ is called a category of partitions when it has the following properties:

- (1) *Stability under the horizontal concatenation, $(\pi, \sigma) \rightarrow [\pi\sigma]$.*
- (2) *Stability under vertical concatenation $(\pi, \sigma) \rightarrow \left[\begin{smallmatrix} \sigma \\ \pi \end{smallmatrix} \right]$, with matching middle symbols.*
- (3) *Stability under the upside-down turning $*$, with switching of colors, $\circ \leftrightarrow \bullet$.*
- (4) *Each set $P(k, k)$ contains the identity partition $|| \dots ||$.*
- (5) *The sets $P(\emptyset, \circ\bullet)$ and $P(\emptyset, \bullet\circ)$ both contain the semicircle \cap .*

In other words, what we have here are the same axioms as in chapter 4, but with the condition that $P(k, \bar{k})$ with $|k| = 2$ must contain the crossing partition \times removed. At the level of examples, there are many of them, and we will get to this in a moment.

Observe the similarity with Definition 13.15. In fact Definition 13.18 is a delinearized version of Definition 13.15, the relation with the Tannakian categories coming from:

PROPOSITION 13.19. *Given a partition $\pi \in P(k, l)$, consider the linear map*

$$T_\pi : (\mathbb{C}^N)^{\otimes k} \rightarrow (\mathbb{C}^N)^{\otimes l}$$

given by the following formula, where e_1, \dots, e_N is the standard basis of \mathbb{C}^N ,

$$T_\pi(e_{i_1} \otimes \dots \otimes e_{i_k}) = \sum_{j_1 \dots j_l} \delta_\pi \begin{pmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_l \end{pmatrix} e_{j_1} \otimes \dots \otimes e_{j_l}$$

and with the Kronecker type symbols $\delta_\pi \in \{0, 1\}$ depending on whether the indices fit or not. The assignment $\pi \rightarrow T_\pi$ is then categorical, in the sense that we have

$$T_\pi \otimes T_\sigma = T_{[\pi\sigma]} \quad , \quad T_\pi T_\sigma = N^{c(\pi, \sigma)} T_{\left[\begin{smallmatrix} \pi \\ \sigma \end{smallmatrix}\right]} \quad , \quad T_\pi^* = T_{\pi^*}$$

where $c(\pi, \sigma)$ are certain integers, coming from the erased components in the middle.

PROOF. The concatenation property follows from the following computation:

$$\begin{aligned} & (T_\pi \otimes T_\sigma)(e_{i_1} \otimes \dots \otimes e_{i_p} \otimes e_{k_1} \otimes \dots \otimes e_{k_r}) \\ &= \sum_{j_1 \dots j_q} \sum_{l_1 \dots l_s} \delta_\pi \begin{pmatrix} i_1 & \dots & i_p \\ j_1 & \dots & j_q \end{pmatrix} \delta_\sigma \begin{pmatrix} k_1 & \dots & k_r \\ l_1 & \dots & l_s \end{pmatrix} e_{j_1} \otimes \dots \otimes e_{j_q} \otimes e_{l_1} \otimes \dots \otimes e_{l_s} \\ &= \sum_{j_1 \dots j_q} \sum_{l_1 \dots l_s} \delta_{[\pi\sigma]} \begin{pmatrix} i_1 & \dots & i_p & k_1 & \dots & k_r \\ j_1 & \dots & j_q & l_1 & \dots & l_s \end{pmatrix} e_{j_1} \otimes \dots \otimes e_{j_q} \otimes e_{l_1} \otimes \dots \otimes e_{l_s} \\ &= T_{[\pi\sigma]}(e_{i_1} \otimes \dots \otimes e_{i_p} \otimes e_{k_1} \otimes \dots \otimes e_{k_r}) \end{aligned}$$

As for the other two formulae in the statement, their proofs are similar. □

In relation with quantum groups, we have the following result, from [21]:

THEOREM 13.20. *Each category of partitions $D = (D(k, l))$ produces a family of compact quantum groups $G = (G_N)$, one for each $N \in \mathbb{N}$, via the following formula:*

$$Hom(v^{\otimes k}, v^{\otimes l}) = span \left(T_\pi \Big|_{\pi \in D(k, l)} \right)$$

To be more precise, the spaces on the right form a Tannakian category, and so produce a certain closed subgroup $G_N \subset U_N^+$, via the Tannakian duality correspondence.

PROOF. This follows indeed from Woronowicz’s Tannakian duality, in its “soft” form from Malacarne [64], as explained in Theorem 13.17. Indeed, let us set:

$$C(k, l) = span \left(T_\pi \Big|_{\pi \in D(k, l)} \right)$$

By using the various axioms in Definition 13.18, and the categorical properties of the operation $\pi \rightarrow T_\pi$, from Proposition 13.19, we deduce that $C = (C(k, l))$ is a Tannakian category. Thus the Tannakian duality applies, and gives the result. □

Philosophically speaking, the quantum groups appearing as in Theorem 13.20 are the simplest, from the perspective of Tannakian duality, so let us formulate:

DEFINITION 13.21. A closed subgroup $G \subset U_N^+$ is called easy when we have

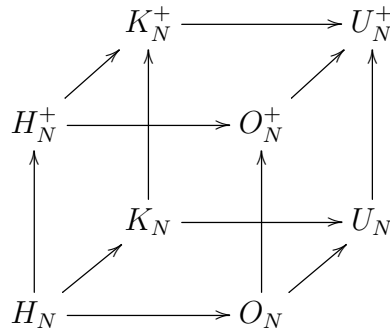
$$\text{Hom}(v^{\otimes k}, v^{\otimes l}) = \text{span} \left(T_\pi \mid \pi \in D(k, l) \right)$$

for any colored integers k, l , for a certain category of partitions $D \subset P$.

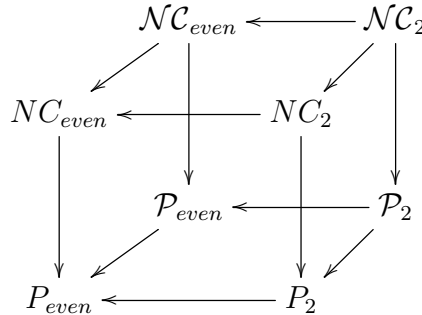
In other words, we adhere here to the same philosophy as before in chapter 4, in the classical case, namely that easiness means easiness at the Tannakian level.

Getting now to examples, we have the following Brauer type result, coming from the work in [8], [9], [10], [13], [30], covering the basic rotation and reflection groups:

THEOREM 13.22. The basic quantum rotation and reflection groups,



are all easy, the corresponding categories of partitions being as follows,



with on top, the symbol NC standing everywhere for noncrossing partitions.

PROOF. We already know, from chapter 4, the results for the lower face of the cube. In what regards the results for the upper face, the idea is as follows:

(1) Let us first discuss the easiness property of O_N^+, U_N^+ . The quantum group U_N^+ is by definition constructed via the following relations:

$$v^* = v^{-1} \quad , \quad v^t = \bar{v}^{-1}$$

Thus, the following operators must be in the associated Tannakian category C :

$$T_\pi, \pi = \begin{array}{c} \cap \\ \circ \bullet \end{array} \quad , \quad T_\pi, \pi = \begin{array}{c} \cap \\ \bullet \circ \end{array}$$

It follows that the associated Tannakian category is $C = \text{span}(T_\pi | \pi \in D)$, with:

$$D = \langle \begin{array}{c} \cap \\ \bullet \bullet \end{array}, \begin{array}{c} \cap \\ \bullet \circ \end{array} \rangle = \mathcal{NC}_2$$

Now by imposing the extra relation $v = \bar{v}$, we obtain the easiness of O_N^+ as well.

(2) In what regards now the easiness property of H_N^+, K_N^+ , this follows again like in the classical case. Indeed, the first observation is that the magic condition satisfied by a matrix v can be reformulated as follows, with $\mu \in P(2, 1)$ being the fork partition:

$$T_\mu \in \text{Hom}(v^{\otimes 2}, v)$$

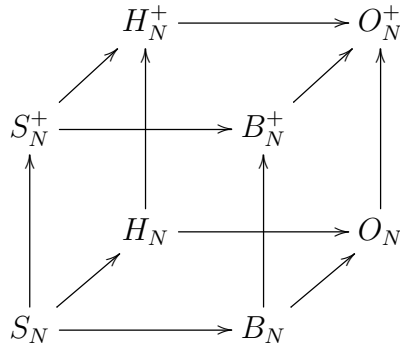
Now by proceeding as in the proof for U_N^+ discussed above, we conclude that the quantum group S_N^+ is indeed easy, the associated category of partitions being:

$$D = \langle \mathcal{NC}_2, \mu \rangle = \mathcal{NC}$$

With this in hand, we can pass to the quantum groups H_N^+, K_N^+ in a standard way, and we are led to easiness, and the categories in the statement. See [9], [10]. \square

There are many other examples of easy quantum groups, as for instance the real and complex, classical and free bishochastic quantum groups B_N, C_N, B_N^+, C_N^+ , or various intermediate liberations $G_N \subset G_N^\times \subset G_N^+$ of the easy groups that we know. However, those in Theorem 13.22 remain the most important ones. In order to discuss this, classification results for the easy quantum groups, let us start with a basic result from [21]:

THEOREM 13.23. *The classical and free uniform orthogonal easy quantum groups, $S_N \subset G \subset O_N^+$, with inclusions between them, are as follows:*



Moreover, this is an intersection/easy generation diagram, in the sense that for any of its square subdiagrams $P \subset Q, R \subset S$ we have $P = Q \cap R$ and $\langle Q, R \rangle = S$.

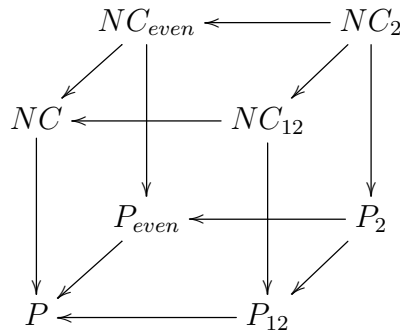
PROOF. There are several things to be proved, the idea being as follows:

(1) To start with, regarding the terminology and notations, the notion of uniformity in the statement is a straightforward compact quantum group extension of the notion of uniformity that we met in chapter 4, for the compact Lie groups.

(2) Also regarding the statement, $B_N \subset O_N$ is the real bistochastic group, consisting of matrices whose entries sum up to 1, on each row and column, and $B_N^+ \subset O_N^+$ is its straightforward liberation, obtained by imposing the condition $v\xi = \xi$, with $\xi \in \mathbb{C}^N$ being the all-one vector. It is routine to check that B_N, B_N^+ are indeed easy, coming respectively from the categories P_{12}, NC_{12} , with 12 standing for “singletons and pairings”.

(3) Finally, the easy generation operation \langle, \rangle is defined by saying that if $G, H \subset U_N^+$ are easy, coming from categories of partitions D_G, D_H , then $\langle G, H \rangle \subset U_N^+$ is the easy quantum group coming from the category of partitions $D = D_G \cap D_H$.

(4) Regarding now the proof, we know that the quantum groups in the statement are indeed easy and uniform, the corresponding categories of partitions being as follows:



Since this latter diagram is an intersection and generation diagram, we conclude that we have an intersection and easy generation diagram of quantum groups, as stated.

(5) Regarding now the classification, consider first an easy group $S_N \subset G_N \subset O_N$. This must come from a certain category $P_2 \subset D \subset P$, and if we assume $G = (G_N)$ to be uniform, then D is uniquely determined by the subset $L \subset \mathbb{N}$ consisting of the sizes of the blocks of the partitions in D . Our claim is that the admissible sets are as follows:

- $L = \{2\}$, producing O_N .
- $L = \{1, 2\}$, producing B_N .
- $L = \{2, 4, 6, \dots\}$, producing H_N .
- $L = \{1, 2, 3, \dots\}$, producing S_N .

(6) Indeed, in one sense, this follows from our easiness results for O_N, B_N, H_N, S_N . In the other sense now, assume that $L \subset \mathbb{N}$ is such that the set P_L consisting of partitions whose sizes of the blocks belong to L is a category of partitions. We know from the axioms of the categories of partitions that the semicircle \cap must be in the category, so we have $2 \in L$. We claim that the following conditions must be satisfied as well:

$$k, l \in L, k > l \implies k - l \in L$$

$$k \in L, k \geq 2 \implies 2k - 2 \in L$$

(7) Indeed, we will prove that both conditions follow from the axioms of the categories of partitions. Let us denote by $b_k \in P(0, k)$ the one-block partition:

$$b_k = \left\{ \begin{array}{cccc} \sqcap \sqcap & \dots & \sqcap & \\ 1 & 2 & \dots & k \end{array} \right\}$$

For $k > l$, we can write b_{k-l} in the following way:

$$b_{k-l} = \left\{ \begin{array}{cccccc} \sqcap \sqcap & \dots & \dots & \dots & \dots & \sqcap \\ 1 & 2 & \dots & l & l+1 & \dots & k \\ \sqcup \sqcup & \dots & \sqcup & | & \dots & | & \\ & & & 1 & \dots & k-l & \end{array} \right\}$$

In other words, we have the following formula:

$$b_{k-l} = (b_l^* \otimes |\otimes^{k-l}|) b_k$$

Since all the terms of this composition are in P_L , we have $b_{k-l} \in P_L$, and this proves our first claim. As for the second claim, this can be proved in a similar way, by capping two adjacent k -blocks with a 2-block, in the middle.

(8) With these conditions in hand, we can conclude in the following way:

Case 1. Assume $1 \in L$. By using the first condition with $l = 1$ we get:

$$k \in L \implies k - 1 \in L$$

This condition shows that we must have $L = \{1, 2, \dots, m\}$, for a certain number $m \in \{1, 2, \dots, \infty\}$. On the other hand, by using the second condition we get:

$$\begin{aligned} m \in L &\implies 2m - 2 \in L \\ &\implies 2m - 2 \leq m \\ &\implies m \in \{1, 2, \infty\} \end{aligned}$$

The case $m = 1$ being excluded by the condition $2 \in L$, we reach to one of the two sets producing the groups S_N, B_N .

Case 2. Assume $1 \notin L$. By using the first condition with $l = 2$ we get:

$$k \in L \implies k - 2 \in L$$

This condition shows that we must have $L = \{2, 4, \dots, 2p\}$, for a certain number $p \in \{1, 2, \dots, \infty\}$. On the other hand, by using the second condition we get:

$$\begin{aligned} 2p \in L &\implies 4p - 2 \in L \\ &\implies 4p - 2 \leq 2p \\ &\implies p \in \{1, \infty\} \end{aligned}$$

Thus L must be one of the two sets producing O_N, H_N , and we are done.

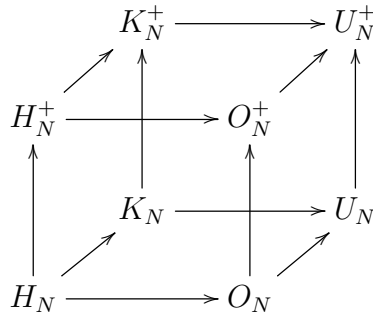
(9) In the free case, $S_N^+ \subset G_N \subset O_N^+$, the situation is quite similar, the admissible sets being once again the above ones, producing this time $O_N^+, B_N^+, H_N^+, S_N^+$. See [21]. \square

The above classification is something quite simple, but when when lifting the uniformity assumption, or when looking at the unitary case, or, more generally, when looking at the unitary case without the uniformity assumption, things become quite complicated. However, a classification is still possible, and we refer here to Tarrago-Weber [82].

This was for the story of the classification of easy quantum groups, in the classical and free cases. When looking at intermediate liberations $G_N \subset G_N^\times \subset G_N^+$ things become quite complicated, and we refer here to Raum-Weber [76] and subsequent papers.

Quite remarkably, however, by tricking a bit, we have the following result:

THEOREM 13.24 (Ground Zero). *Under a collection of suitable extra assumptions*



are the unique easy quantum groups.

PROOF. This is something quite technical, and it is beyond our purposes here to get into the details of the proof, or even into the full details of the statement. Let us mention, however, that in what regards the exact assumptions, these are as follows:

- (1) Easiness. This is the key assumption, bringing into the picture partitions and combinatorics, and classification techniques in the spirit of those used above.
- (2) Uniformity. With this being, as before, the straightforward quantum group extension of the uniformity notion that we met in chapter 4, for the classical groups.
- (3) Twistability. With this meaning that we have an inclusion $H_N \subset G$, which is something which is normally needed, in order to twist G .
- (4) Orientability. With this meaning that $H_N \subset G \subset U_N^+$, which can be thought of as living inside the cube, can be recovered out of its projections on the edges.

So, this was for the general idea. As for the precise statement, and then of course for the proof, and for the whole story in general, with all this, we refer here to [8]. \square

13c. Weingarten formula

With the above understood, let us discuss now the probabilistic consequences of our general easiness theory, in the spirit of the work done in chapter 4, in the classical case. In what regards the asymptotic laws of the main characters, we have here:

THEOREM 13.25. *For an easy quantum group $G = (G_N)$, coming from a category of partitions $D = (D(k, l))$, the asymptotic moments of the character $\chi = \sum_i v_{ii}$ are*

$$\lim_{N \rightarrow \infty} \int_{G_N} \chi^k = |D(k)|$$

where $D(k) = D(\emptyset, k)$, with the limiting sequence on the left consisting of certain integers, and being stationary at least starting from the k -th term.

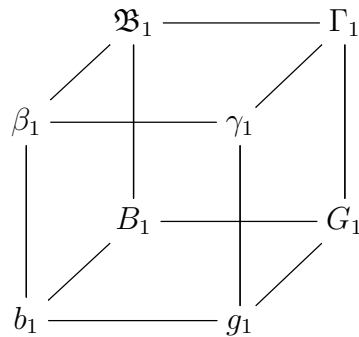
PROOF. This is something elementary, which follows straight from Peter-Weyl theory, by using the linear independence result for the vectors ξ_π from chapter 4, as follows:

$$\begin{aligned} \lim_{N \rightarrow \infty} \int_{G_N} \chi^k &= \lim_{N \rightarrow \infty} \dim (Fix(v^{\otimes k})) \\ &= \lim_{N \rightarrow \infty} \dim (\text{span} (\xi_\pi | \pi \in D(k))) \\ &= |D(k)| \end{aligned}$$

Thus, we are led to the conclusions in the statement. □

In practice now, for the basic rotation and reflection groups, we obtain:

THEOREM 13.26. *The character laws for basic rotation and reflection groups are*



in the $N \rightarrow \infty$ limit, corresponding to the basic probabilistic limiting theorems, at $t = 1$.

PROOF. This follows indeed from Theorem 13.22 and Theorem 13.25, by using the known moment formulae for the laws in the statement, at $t = 1$. □

In the free case, the convergence can be shown to be stationary starting from $N = 4$. The “fix” comes by looking at truncated characters, constructed as follows:

$$\chi_t = \sum_{i=1}^{[tN]} v_{ii}$$

In order to investigate these truncated characters, we can use the Weingarten formula, which is very similar to the one from the classical case, as follows:

THEOREM 13.27. *For an easy quantum group $G \subset_v U_N^+$, coming from a category of partitions $D = (D(k, l))$, we have the Weingarten formula*

$$\int_G v_{i_1 j_1}^{e_1} \cdots v_{i_k j_k}^{e_k} = \sum_{\pi, \nu \in D(k)} \delta_\pi(i) \delta_\nu(j) W_{kN}(\pi, \nu)$$

for any $k = e_1 \dots e_k$ and any i, j , where $D(k) = D(\emptyset, k)$, δ are usual Kronecker type symbols, checking whether the indices match, and $W_{kN} = G_{kN}^{-1}$, with

$$G_{kN}(\pi, \nu) = N^{|\pi \vee \nu|}$$

where $|\cdot|$ is the number of blocks.

PROOF. This is something very standard, coming from the fact that the above integrals form altogether the orthogonal projection P^k onto the following space:

$$Fix(v^{\otimes k}) = span(D(k))$$

Consider indeed the following linear map, with $D(k)$ being as in the statement:

$$E(x) = \sum_{\pi \in D(k)} \langle x, \xi_\pi \rangle \xi_\pi$$

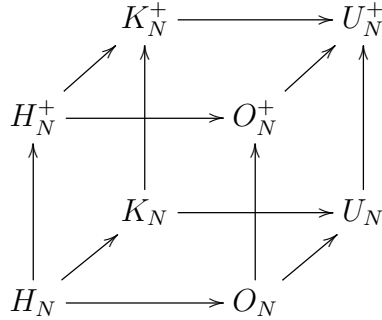
By a standard linear algebra computation, it follows that we have $P = WE$, where W is the inverse of the restriction of E to the following space:

$$K = span\left(T_\pi \Big|_{\pi \in D(k)}\right)$$

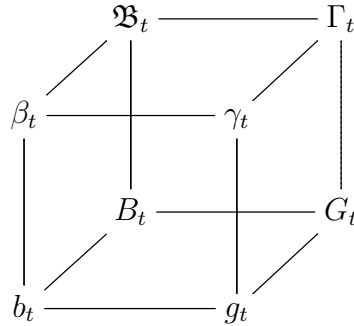
But this restriction is the linear map given by the matrix G_{kN} , and so W is the linear map given by the inverse matrix $W_{kN} = G_{kN}^{-1}$, and this gives the result. \square

Now back to characters, we have the following final result on the subject, with the convergence being non-stationary at $t < 1$, in both the classical and free cases:

THEOREM 13.28. *The truncated characters for the basic quantum groups*



are in the $N \rightarrow \infty$ limit the following laws,



which are the main laws in classical and free probability.

PROOF. As before with other results, this is something that we know from chapter 4 for the lower face of the cube, and the proof for the upper face is similar. To be more precise, the point is that we have in the present quantum group setting we have:

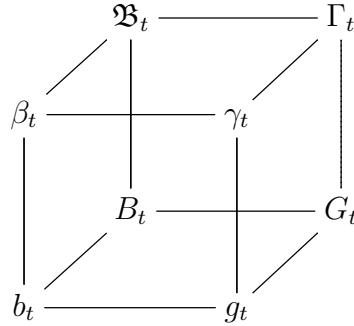
$$\begin{aligned} \int_{G_N} \chi_t^k &\simeq \sum_{\pi \in D(k)} W_{kN}(\pi, \pi) G_{k[tN]}(\pi, \pi) \\ &\simeq \sum_{\pi \in D(k)} N^{-|\pi|} (tN)^{|\pi|} \\ &= \sum_{\pi \in D(k)} t^{|\pi|} \end{aligned}$$

But this leads to the laws in the statement, via results that we already know. □

We refer to [21] and related papers for full details on all the above. Also, we refer to [16], [17], [76], [82] for more general theory for the easy quantum groups.

Finally, as a consequence of this, and of the Ground Zero theorem, we have:

CONCLUSION 13.29. *Under suitable combinatorial assumptions,*



are the unique main laws in noncommutative probability.

To be more precise, this conclusion, while being obviously something a bit informal and philosophical, is in fact, technically speaking, more of a mathematical theorem, coming by putting together Theorem 13.24 and Theorem 13.28. So, very nice all this, we eventually managed to understand how general noncommutative probability works.

13d. Gram determinants

As a last topic for this chapter, let us discuss, following [41] and related papers, the computation of Gram determinants for the easy quantum groups. We already know from chapter 4 that for the group S_N the formula of the Gram determinant is as follows:

THEOREM 13.30. *The determinant of the Gram matrix of S_N is given by*

$$\det(G_{kN}) = \prod_{\pi \in P(k)} \frac{N!}{(N - |\pi|)!}$$

with the convention that in the case $N < k$ we obtain 0.

PROOF. This is something that we know from chapter 4, the idea being that G_{kN} decomposes as a product of an upper triangular and lower triangular matrix. \square

For the orthogonal group O_N , the combinatorics is that of the Young diagrams. We denote by $|\cdot|$ the number of boxes, and we use quantity f^λ , which gives the number of standard Young tableaux of shape λ . The result is then as follows:

THEOREM 13.31. *The determinant of the Gram matrix of O_N is given by*

$$\det(G_{kN}) = \prod_{|\lambda|=k/2} f_N(\lambda)^{f^{2\lambda}}$$

where the quantities on the right are $f_N(\lambda) = \prod_{(i,j) \in \lambda} (N + 2j - i - 1)$.

PROOF. This follows from the results of Zinn-Justin in [100]. Indeed, it is known from there that the Gram matrix is diagonalizable, as follows:

$$G_{kN} = \sum_{|\lambda|=k/2} f_N(\lambda) P_{2\lambda}$$

To be more precise, here $1 = \sum P_{2\lambda}$ is the standard partition of unity associated to the Young diagrams having $k/2$ boxes, and the coefficients $f_N(\lambda)$ are those in the statement. Now since we have $Tr(P_{2\lambda}) = f^{2\lambda}$, this gives the result. See [16], [100]. \square

For the free orthogonal and symmetric groups, the results, by Di Francesco [41], are substantially more complicated. But, we can use the following trick:

PROPOSITION 13.32. *The Gram matrices of $NC_2(2k) \simeq NC(k)$ are related by*

$$G_{2k,n}(\pi, \sigma) = n^k (\Delta_{kn}^{-1} G_{k,n^2} \Delta_{kn}^{-1})(\pi', \sigma')$$

where $\pi \rightarrow \pi'$ is the shrinking operation, and Δ_{kn} is the diagonal of G_{kn} .

PROOF. In the context of the standard bijection $NC_2(2k) \simeq NC(k)$, we have:

$$|\pi \vee \sigma| = k + 2|\pi' \vee \sigma'| - |\pi'| - |\sigma'|$$

We therefore have the following formula, valid for any $n \in \mathbb{N}$:

$$n^{|\pi \vee \sigma|} = n^{k+2|\pi' \vee \sigma'| - |\pi'| - |\sigma'|}$$

Thus, we are led to the formula in the statement. \square

Now back to determinants, let us begin with some examples. We first have:

PROPOSITION 13.33. *The first Gram matrices and determinants for O_N^+ are*

$$\det \begin{pmatrix} N^2 & N \\ N & N^2 \end{pmatrix} = N^2(N^2 - 1)$$

$$\det \begin{pmatrix} N^3 & N^2 & N^2 & N^2 & N \\ N^2 & N^3 & N & N & N^2 \\ N^2 & N & N^3 & N & N^2 \\ N^2 & N & N & N^3 & N^2 \\ N & N^2 & N^2 & N^2 & N^3 \end{pmatrix} = N^5(N^2 - 1)^4(N^2 - 2)$$

with the matrices being written by using the lexicographic order on $NC_2(2k)$.

PROOF. The formula at $k = 2$, where $NC_2(4) = \{\square\square, \sqcup\}$, is clear. At $k = 3$ however, things are tricky. We have $NC(3) = \{|||, \square|, \sqcup, |\square, \square\square\}$, and the corresponding Gram

matrix and its determinant are, according to Theorem 13.30:

$$\det \begin{pmatrix} N^3 & N^2 & N^2 & N^2 & N \\ N^2 & N^2 & N & N & N \\ N^2 & N & N^2 & N & N \\ N^2 & N & N & N^2 & N \\ N & N & N & N & N \end{pmatrix} = N^5(N-1)^4(N-2)$$

By using Proposition 13.32, the Gram determinant of $NC_2(6)$ is given by:

$$\begin{aligned} \det(G_{6N}) &= \frac{1}{N^2\sqrt{N}} \times N^{10}(N^2-1)^4(N^2-2) \times \frac{1}{N^2\sqrt{N}} \\ &= N^5(N^2-1)^4(N^2-2) \end{aligned}$$

Thus, we have obtained the formula in the statement. □

In general, such tricks won't work, because $NC(k)$ is strictly smaller than $P(k)$ at $k \geq 4$. However, following Di Francesco [41], we have the following result:

THEOREM 13.34. *The determinant of the Gram matrix for O_N^+ is given by*

$$\det(G_{kN}) = \prod_{r=1}^{[k/2]} P_r(N)^{d_{k/2,r}}$$

where P_r are the Chebycheff polynomials, given by

$$P_0 = 1 \quad , \quad P_1 = X \quad , \quad P_{r+1} = XP_r - P_{r-1}$$

and $d_{kr} = f_{kr} - f_{k,r+1}$, with f_{kr} being the following numbers, depending on $k, r \in \mathbb{Z}$,

$$f_{kr} = \binom{2k}{k-r} - \binom{2k}{k-r-1}$$

with the convention $f_{kr} = 0$ for $k \notin \mathbb{Z}$.

PROOF. This is something quite technical, obtained by using a decomposition as follows of the Gram matrix G_{kN} , with the matrix T_{kN} being lower triangular:

$$G_{kN} = T_{kN}T_{kN}^t$$

Thus, a bit as in the proof of Theorem 13.30, we obtain the result, but the problem lies however in the construction of T_{kN} , which is non-trivial. See [41]. □

We refer to [16] for further details regarding the above result, including a short proof, based on the bipartite planar algebra combinatorics developed by Jones in [59]. Let us also mention that the Chebycheff polynomials have something to do with all this due to the fact that these are the orthogonal polynomials for the Wigner law. See [16].

Moving ahead now, regarding S_N^+ , we have here the following formula, which is quite similar, obtained via shrinking, also from Di Francesco [41]:

THEOREM 13.35. *The determinant of the Gram matrix for S_N^+ is given by*

$$\det(G_{kN}) = (\sqrt{N})^{a_k} \prod_{r=1}^k P_r(\sqrt{N})^{d_{kr}}$$

where P_r are the Chebycheff polynomials, given by

$$P_0 = 1 \quad , \quad P_1 = X \quad , \quad P_{r+1} = XP_r - P_{r-1}$$

and $d_{kr} = f_{kr} - f_{k,r+1}$, with f_{kr} being the following numbers, depending on $k, r \in \mathbb{Z}$,

$$f_{kr} = \binom{2k}{k-r} - \binom{2k}{k-r-1}$$

with the convention $f_{kr} = 0$ for $k \notin \mathbb{Z}$, and where $a_k = \sum_{\pi \in \mathcal{P}(k)} (2|\pi| - k)$.

PROOF. This follows indeed from Theorem 13.34, by using Proposition 13.32. \square

We refer to [16], [41] and related papers, for more on the above.

13e. Exercises

We had a lot of theory in this chapter, and as a best exercise on all this, quantum groups, nothing is better than spending some time on S_N^+ , and we have:

EXERCISE 13.36. *Futher advance in your understanding of $S_N \rightarrow S_N^+$, as follows:*

- (1) *Prove that $S_3^+ = S_3$, by using a clever method, of your choice.*
- (2) *Prove that $S_4^+ \neq S_4$, again by using a clever method, of your choice.*
- (3) *Prove that S_4^+ is coamenable, while S_5^+ is not coamenable.*
- (4) *Can we talk about quantum permutations of finite quantum spaces?*
- (5) *If yes, can you prove that for M_2 , given by $C(M_2) = M_2(\mathbb{C})$, we get SO_3 ?*
- (6) *Based on this, can we say that S_4^+ should be a kind of twist of SO_3 ?*

Some of these exercises are actually quite tricky, especially those at the end, but do not worry, we will come back to some of them, in what follows.

CHAPTER 14

Free geometry

14a. Spheres and tori

In order to obtain more instances of the Bercovici-Pata bijection, and why not constructing as well some further, related correspondences between classical and free, a very simple and natural idea, inspired by the above, is that of doing “free geometry”. That is, we would like to have free analogues of various classical manifolds that we know, and then compare the probability theory over classical manifolds, and their free versions.

This sounds quite exciting, and we will do this in this chapter. As a piece of advertisement for what we will find, which is something purely probabilistic, we have:

ADVERTISEMENT 14.1. By looking at probability theory over classical manifolds, and their free versions, we will find, among others, an explanation for the Meixner/free Meixner correspondence, which is something not covered by Bercovici-Pata.

But more on this later. Getting started now, it is not very clear what “manifold” should mean, in the above, but since we definitely want to integrate over our manifolds, these manifolds should normally be Riemannian, in some appropriate sense. On the other hand, we know from chapter 5 that the operator algebra theory describes well spaces which are compact. Thus, our manifolds should be compact and Riemannian.

Long story short, these are our goals, and instead of thinking too much, let us just start working, and see later for the philosophy. The simplest compact manifolds that we know are the spheres, and if we want to have free analogues of these spheres, there are not many choices here, the straightforward definition, from [8], being as follows:

DEFINITION 14.2. We have compact quantum spaces, constructed as follows,

$$C(S_{\mathbb{R},+}^{N-1}) = C^* \left(z_1, \dots, z_N \mid z_i = z_i^*, \sum_i z_i^2 = 1 \right)$$
$$C(S_{\mathbb{C},+}^{N-1}) = C^* \left(z_1, \dots, z_N \mid \sum_i z_i z_i^* = \sum_i z_i^* z_i = 1 \right)$$

called respectively the free real sphere, and the free complex sphere.

Here the C^* symbols on the right stand as usual for “universal C^* -algebra generated by”. The fact that such algebras exist indeed follows by considering the corresponding universal $*$ -algebras, and completing with respect to the biggest C^* -norm. Observe that this biggest C^* -norm exists indeed, because the quadratic conditions give:

$$\|z_i\|^2 = \|z_i z_i^*\| \leq \left\| \sum_i z_i z_i^* \right\| = 1$$

Given a compact quantum space X , meaning as usual the abstract space associated to a C^* -algebra, we define its classical version to be the classical space X_{class} obtained by dividing $C(X)$ by its commutator ideal, then applying the Gelfand theorem:

$$C(X_{class}) = C(X)/I \quad , \quad I = \langle [a, b] \rangle$$

Observe that we have an embedding of compact quantum spaces $X_{class} \subset X$. In this situation, we also say that X appears as a “liberation” of X_{class} . We have:

PROPOSITION 14.3. *We have embeddings of compact quantum spaces*

$$\begin{array}{ccc} S_{\mathbb{C}}^{N-1} & \longrightarrow & S_{\mathbb{C},+}^{N-1} \\ \uparrow & & \uparrow \\ S_{\mathbb{R}}^{N-1} & \longrightarrow & S_{\mathbb{R},+}^{N-1} \end{array}$$

and the spaces on the right appear as liberations of the spaces of the left.

PROOF. The embeddings are all clear. For the last assertion, we must establish the following isomorphisms, where C_{comm}^* stands for “universal commutative C^* -algebra”:

$$C(S_{\mathbb{R}}^{N-1}) = C_{comm}^* \left(z_1, \dots, z_N \mid z_i = z_i^*, \sum_i z_i^2 = 1 \right)$$

$$C(S_{\mathbb{C}}^{N-1}) = C_{comm}^* \left(z_1, \dots, z_N \mid \sum_i z_i z_i^* = \sum_i z_i^* z_i = 1 \right)$$

But these isomorphisms are both clear, by using the Gelfand theorem. □

We can now introduce a broad class of compact quantum manifolds, as follows:

DEFINITION 14.4. *A real algebraic submanifold $X \subset S_{\mathbb{C},+}^{N-1}$ is a closed quantum space defined, at the level of the corresponding C^* -algebra, by a formula of type*

$$C(X) = C(S_{\mathbb{C},+}^{N-1}) / \langle f_i(z_1, \dots, z_N) = 0 \rangle$$

for certain noncommutative polynomials $f_i \in \mathbb{C} \langle X_1, \dots, X_N \rangle$.

Observe that such manifolds exist indeed, because the free complex spheres themselves exist, and this due to the fact that the quadratic conditions defining them give:

$$\|z_i\| \leq 1$$

This estimate, explained before, is something extremely important, and any attempt of further extending Definition 14.4, beyond the sphere level, stumbles into this. There are no such things as free analogues of \mathbb{R}^N or \mathbb{C}^N , and the problem comes from this.

In practice now, while our assumption $X \subset S_{\mathbb{C},+}^{N-1}$ is definitely something technical, we are not losing much when imposing it, and we have the following list of examples:

THEOREM 14.5. *The following are algebraic submanifolds $X \subset S_{\mathbb{C},+}^{N-1}$:*

- (1) *The spheres $S_{\mathbb{R}}^{N-1} \subset S_{\mathbb{C}}^{N-1}, S_{\mathbb{R},+}^{N-1} \subset S_{\mathbb{C},+}^{N-1}$.*
- (2) *Any compact Lie group, $G \subset U_n$, when $N = n^2$.*
- (3) *The duals $\widehat{\Gamma}$ of finitely generated groups, $\Gamma = \langle g_1, \dots, g_N \rangle$.*
- (4) *More generally, the closed quantum groups $G \subset U_n^+$, when $N = n^2$.*

PROOF. These facts are all well-known, the proof being as follows:

(1) This is indeed true by definition of our various spheres.

(2) Given a closed subgroup $G \subset U_n$, we have an embedding $G \subset S_{\mathbb{C}}^{N-1}$, with $N = n^2$, given in double indices by $z_{ij} = v_{ij}/\sqrt{n}$, that we can further compose with the standard embedding $S_{\mathbb{C}}^{N-1} \subset S_{\mathbb{C},+}^{N-1}$. As for the fact that we obtain indeed a real algebraic manifold, this is standard too, coming either from Lie theory or from Tannakian duality.

(3) Given a group $\Gamma = \langle g_1, \dots, g_N \rangle$, consider the following variables:

$$z_i = \frac{g_i}{\sqrt{N}}$$

These variables satisfy then the quadratic relations $\sum_i z_i z_i^* = \sum_i z_i^* z_i = 1$ defining $S_{\mathbb{C},+}^{N-1}$, and the algebraicity claim for the manifold $\widehat{\Gamma} \subset S_{\mathbb{C},+}^{N-1}$ is clear.

(4) Given a closed subgroup $G \subset U_n^+$, we have indeed an embedding $G \subset S_{\mathbb{C},+}^{N-1}$, with $N = n^2$, given in double indices by the following formula:

$$z_{ij} = \frac{v_{ij}}{\sqrt{n}}$$

As for the fact that we obtain indeed in this way a real algebraic manifold, this comes from the Tannakian duality results from [64], [99], explained before. □

Summarizing, we have a broad notion of real algebraic manifold, covering all the examples that we met so far in this book. We will use this notion, in what follows. At the level of the general theory, we have the following version of the Gelfand theorem, which is something very useful, that we will use several times in what follows:

THEOREM 14.6. *Assuming that $X \subset S_{\mathbb{C},+}^{N-1}$ is an algebraic manifold, given by*

$$C(X) = C(S_{\mathbb{C},+}^{N-1}) / \left\langle f_i(z_1, \dots, z_N) = 0 \right\rangle$$

for certain noncommutative polynomials $f_i \in \mathbb{C} \langle X_1, \dots, X_N \rangle$, we have

$$X_{class} = \left\{ x \in S_{\mathbb{C}}^{N-1} \mid f_i(z_1, \dots, z_N) = 0 \right\}$$

and X itself appears as a liberation of X_{class} .

PROOF. The proof is similar to the one for spheres, by using the Gelfand theorem. Indeed, if we let $Y \subset S_{\mathbb{C}}^{N-1}$ be the manifold in the statement, then we have a quotient map of C^* -algebras as follows, mapping standard coordinates to standard coordinates:

$$C(X_{class}) \rightarrow C(Y)$$

Conversely, from $X \subset S_{\mathbb{C},+}^{N-1}$ we obtain $X_{class} \subset S_{\mathbb{C}}^{N-1}$, and since the relations defining Y are satisfied by X_{class} , we obtain an inclusion of subspaces $X_{class} \subset Y$. Thus, at the level of algebras of continuous functions, we have a quotient map of C^* -algebras as follows, mapping standard coordinates to standard coordinates:

$$C(Y) \rightarrow C(X_{class})$$

Thus, we have constructed a pair of inverse morphisms, and this finishes the proof. \square

Getting back now to the examples, the above formalism allows us to have a new, more geometric look at the discrete group duals. Let us formulate indeed:

DEFINITION 14.7. *Given a closed subspace $S \subset S_{\mathbb{C},+}^{N-1}$, the subspace $T \subset S$ given by*

$$C(T) = C(S) / \left\langle z_i z_i^* = z_i^* z_i = \frac{1}{N} \right\rangle$$

is called associated torus. In the real case, $S \subset S_{\mathbb{R},+}^{N-1}$, we also call T cube.

As a basic example, for $S = S_{\mathbb{C}}^{N-1}$ the corresponding submanifold $T \subset S$ appears by imposing the relations $|z_i| = \frac{1}{\sqrt{N}}$ to the coordinates, so we obtain a torus:

$$S = S_{\mathbb{C}}^{N-1} \implies T = \left\{ z \in \mathbb{C}^N \mid |z_i| = \frac{1}{\sqrt{N}} \right\}$$

As for the case of the real sphere, $S = S_{\mathbb{R}}^{N-1}$, here the submanifold $T \subset S$ appears by imposing the relations $z_i = \pm \frac{1}{\sqrt{N}}$ to the coordinates, and we obtain a cube:

$$S = S_{\mathbb{R}}^{N-1} \implies T = \left\{ z \in \mathbb{R}^N \mid z_i = \pm \frac{1}{\sqrt{N}} \right\}$$

Observe that we have a relation here with groups, because the complex torus computed above is the group \mathbb{T}^N , and the cube is the group \mathbb{Z}_2^N . In fact, we have:

THEOREM 14.8. *The tori of the basic spheres are all group duals, as follows,*

$$\begin{array}{ccc}
 \mathbb{T}^N & \longrightarrow & \widehat{F}_N \\
 \uparrow & & \uparrow \\
 \mathbb{Z}_2^N & \longrightarrow & \widehat{\mathbb{Z}_2^{*N}}
 \end{array}$$

where F_N is the free group on N generators, and $*$ is a group-theoretical free product.

PROOF. In order to prove this result, let us get back to Definition 14.7, and assume that the subspace there $S \subset S_{\mathbb{C},+}^{N-1}$ is an algebraic manifold, as follows:

$$C(S) = C(S_{\mathbb{C},+}^{N-1}) / \langle f_i(z_1, \dots, z_N) = 0 \rangle$$

In order to get to group algebras, let us rescale the coordinates, $v_i = z_i/\sqrt{N}$. Consider as well the corresponding rescalings of the polynomials f_i , given by:

$$g_i(v_1, \dots, v_N) = f_i(\sqrt{N}v_1, \dots, \sqrt{N}v_N)$$

Since the relations defining $T \subset S$ from Definition 14.7 correspond to the fact that the rescaled coordinates u_i must be unitaries, we obtain the following formula:

$$C(T) = C^* \left(v_1, \dots, v_N \mid v_i^* = v_i^{-1}, g_i(v_1, \dots, v_N) = 0 \right)$$

Now in the case of the 4 main spheres, from Proposition 14.3, we obtain from this that the diagram formed by the corresponding algebras $C(T)$ is as follows:

$$\begin{array}{ccc}
 C^*(\mathbb{Z}^N) & \longleftarrow & C^*(\mathbb{Z}^{*N}) \\
 \downarrow & & \downarrow \\
 C^*(\mathbb{Z}_2^N) & \longleftarrow & C^*(\mathbb{Z}_2^{*N})
 \end{array}$$

We conclude that the diagram formed by the basic tori is as follows:

$$\begin{array}{ccc}
 \widehat{\mathbb{Z}}^N & \longrightarrow & \widehat{F}_N \\
 \uparrow & & \uparrow \\
 \widehat{\mathbb{Z}_2}^N & \longrightarrow & \widehat{\mathbb{Z}_2^{*N}}
 \end{array}$$

Now since $\widehat{\mathbb{Z}} = \mathbb{T}$ and $\widehat{\mathbb{Z}_2} = \mathbb{Z}_2$, we are led to the conclusion in the statement. □

As a last piece of abstract theory, based on the above, we can now formulate a “fix” for the functoriality issues of the Gelfand correspondence, as follows:

DEFINITION 14.9. *The category of the real algebraic submanifolds $X \subset S_{\mathbb{C},+}^{N-1}$ is formed by the compact quantum spaces appearing as follows,*

$$\mathcal{C}(X) = C(S_{\mathbb{C},+}^{N-1}) / \langle f_i(z_1, \dots, z_N) = 0 \rangle$$

with $f_i \in \mathbb{C} \langle X_1, \dots, X_N \rangle$ being noncommutative polynomials, and with the arrows $X \rightarrow Y$ being the $*$ -algebra morphisms between the $*$ -algebras of coordinates

$$\mathcal{C}(Y) \rightarrow \mathcal{C}(X)$$

mapping standard coordinates to standard coordinates.

In other words, what we are doing here is that of proposing a definition for the morphisms between the compact quantum spaces, in the particular case where these compact quantum spaces are algebraic submanifolds of the free complex sphere $S_{\mathbb{C},+}^{N-1}$. And the point is that this “fix” perfectly works for the group duals, as follows:

THEOREM 14.10. *The category of finitely generated groups $\Gamma = \langle g_1, \dots, g_N \rangle$, with the morphisms mapping generators to generators, embeds contravariantly via*

$$\Gamma \rightarrow \widehat{\Gamma}$$

into the category of real algebraic submanifolds $X \subset S_{\mathbb{C},+}^{N-1}$.

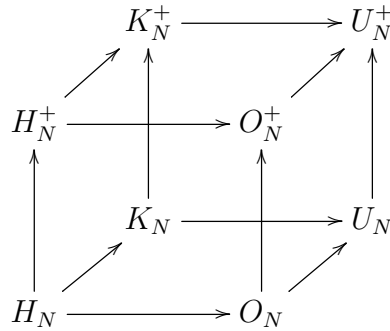
PROOF. We know from Theorem 14.5 that, given an arbitrary finitely generated group $\Gamma = \langle g_1, \dots, g_N \rangle$, we have an embedding $\widehat{\Gamma} \subset S_{\mathbb{C},+}^{N-1}$ given by:

$$z_i = \frac{g_i}{\sqrt{N}}$$

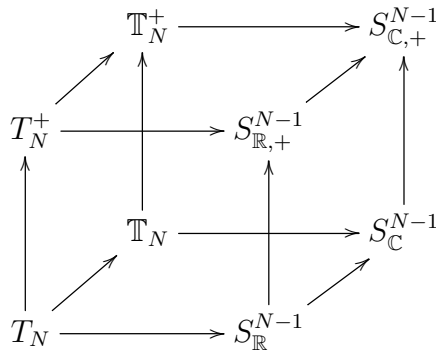
Now since a morphism of $*$ -algebras of coordinates $\mathbb{C}[\Gamma] \rightarrow \mathbb{C}[\Lambda]$ mapping coordinates to coordinates corresponds to a morphism of groups $\Gamma \rightarrow \Lambda$ mapping generators to generators, our notion of isomorphism is indeed the correct one, as claimed. \square

Getting back now to the free spheres and tori, these are related to the quantum rotation and reflection groups, and we have the following result:

THEOREM 14.11. *The spheres and tori associated to the basic quantum groups,*



or rather to the corresponding “quantum geometries” are as follows:

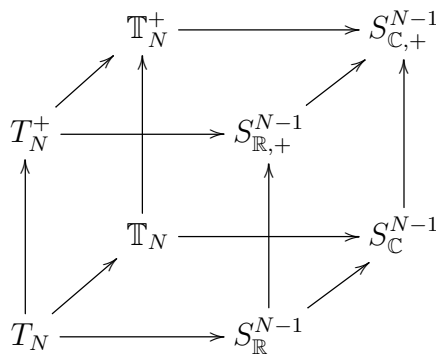


That is, we obtain the various classical and free spheres and tori constructed above.

PROOF. This statement, as formulated, is obviously something a bit informal, but it is possible to have it fully explained and justified. We will not attempt to explain things in detail here. Instead, we refer to book [8], and the related literature. \square

In relation now with probability, we have:

THEOREM 14.12. *The various classical and free spheres and tori,*



all have integration functionals, which can be computed via Weingarten formulae.

PROOF. Again, this statement as formulated is something a bit informal, and for full details, we refer to [8] and the related literature, the idea being as follows:

(1) In what regards the spheres, the idea is that, a bit like in the classical case, the free spheres appear as homogeneous spaces over the corresponding quantum groups, and so the Weingarten formula for the quantum groups applies by restriction to them.

(2) As for the tori, here the integration is something very simple, because we are dealing with group duals, but by using the picture in Theorem 14.11, it is possible to write as well a Weingarten formula for them as well, if we really want to.

(3) So, this was for the story, and for details we refer to [8] and the related literature, as well as to the next section, where we will explain in detail how all this works, for a certain remarkable class of homogeneous spaces, generalizing the spheres. \square

Going back now to the Bercovici-Pata bijection, generally speaking, this bijection should be thought of as being something happening in the $N \rightarrow \infty$ limit. When $N \in \mathbb{N}$ is fixed the situation is more complicated, and we have here many alternative correspondences, coming from quantum groups, or random matrices, which are not obviously related to the Bercovici-Pata bijection, and are sometimes “orthogonal” to it.

Our claim is that we can recover some of these interesting correspondences by using our noncommutative geometry picture. As a basic example here, we have:

THEOREM 14.13. *We have a bijection between the Meixner and free Meixner laws, which appear from the liberation operation for discrete groups*

$$\mathbb{Z}^{\times N} \rightarrow \mathbb{Z}^{*N}$$

by looking at the dual groups, or quantum tori, which are as follows,

$$\mathbb{T}_N \rightarrow \mathbb{T}_N^+$$

and then at the laws of the corresponding main characters.

PROOF. This is something standard, based on the noncommutative geometry picture coming from Theorem 14.11. To be more precise, the truncated characters for the tori $T = \widehat{\Gamma}$, with $\Gamma = \langle g_1, \dots, g_N \rangle$ being a discrete group, are as follows:

$$\chi_t = g_1 + \dots + g_{[tN]}$$

Thus, according to the definition of the Meixner laws, in the classical case we obtain the Meixner laws, and in the free case we obtain the free Meixner laws, as stated. \square

There are many other things that can be said about the correspondence between Meixner laws and free Meixner laws, sometimes of technical probabilistic nature, going beyond the above geometric picture, and we refer here to the literature on the subject, a good reference here, to start with, being the paper of Anshelevich [3].

14b. Quotient spaces

We have seen so far that free geometry is a broad and fluffy subject, with countless potential paths to be taken, and interesting ramifications, and no wonder here, because hundreds of books have been written on classical geometry, and it is probably possible to write as many on free geometry. In practice now, this suggests thinking a bit, and making some good choices for the remainder of this chapter. Our choices will be as follows:

(1) We will first explain how Weingarten integration and the Bercovici-Pata bijection work, for a remarkable class of homogeneous spaces, generalizing the spheres.

(2) Then, we will go back to the question of going beyond Bercovici-Pata, and we will discuss here the free hyperspherical laws, and the free hypergeometric laws.

Getting started now, we would like to find a suitable collection of “free homogeneous spaces”, generalizing at the same time the free spheres S , and the free unitary groups U . This can be done at several levels of generality, and central here is the construction of the free spaces of partial isometries, which can be done in fact for any easy quantum group. In order to explain this, let us start with the classical case. We have here:

DEFINITION 14.14. *Associated to any integers $L \leq M, N$ are the spaces*

$$O_{MN}^L = \left\{ T : E \rightarrow F \text{ isometry} \mid E \subset \mathbb{R}^N, F \subset \mathbb{R}^M, \dim_{\mathbb{R}} E = L \right\}$$

$$U_{MN}^L = \left\{ T : E \rightarrow F \text{ isometry} \mid E \subset \mathbb{C}^N, F \subset \mathbb{C}^M, \dim_{\mathbb{C}} E = L \right\}$$

where the notion of isometry is with respect to the usual real/complex scalar products.

As a first observation, at $L = M = N$ we obtain the groups O_N, U_N :

$$O_{NN}^N = O_N \quad , \quad U_{NN}^N = U_N$$

Another interesting specialization is $L = M = 1$. Here the elements of O_{1N}^1 are the isometries $T : E \rightarrow \mathbb{R}$, with $E \subset \mathbb{R}^N$ one-dimensional. But such an isometry is uniquely determined by $T^{-1}(1) \in \mathbb{R}^N$, which must belong to $S_{\mathbb{R}}^{N-1}$. Thus, we have $O_{1N}^1 = S_{\mathbb{R}}^{N-1}$. Similarly, in the complex case we have $U_{1N}^1 = S_{\mathbb{C}}^{N-1}$, and so our results here are:

$$O_{1N}^1 = S_{\mathbb{R}}^{N-1} \quad , \quad U_{1N}^1 = S_{\mathbb{C}}^{N-1}$$

Yet another interesting specialization is $L = N = 1$. Here the elements of O_{1N}^1 are the isometries $T : \mathbb{R} \rightarrow F$, with $F \subset \mathbb{R}^M$ one-dimensional. But such an isometry is uniquely determined by $T(1) \in \mathbb{R}^M$, which must belong to $S_{\mathbb{R}}^{M-1}$. Thus, we have $O_{M1}^1 = S_{\mathbb{R}}^{M-1}$. Similarly, in the complex case we have $U_{M1}^1 = S_{\mathbb{C}}^{M-1}$, and so our results here are:

$$O_{M1}^1 = S_{\mathbb{R}}^{M-1} \quad , \quad U_{M1}^1 = S_{\mathbb{C}}^{M-1}$$

In general, the most convenient is to view the elements of O_{MN}^L, U_{MN}^L as rectangular matrices, and to use matrix calculus for their study. We have indeed:

PROPOSITION 14.15. *We have identifications of compact spaces*

$$O_{MN}^L \simeq \left\{ U \in M_{M \times N}(\mathbb{R}) \mid UU^t = \text{projection of trace } L \right\}$$

$$U_{MN}^L \simeq \left\{ U \in M_{M \times N}(\mathbb{C}) \mid UU^* = \text{projection of trace } L \right\}$$

with each partial isometry being identified with the corresponding rectangular matrix.

PROOF. We can indeed identify the partial isometries $T : E \rightarrow F$ with their corresponding extensions $U : \mathbb{R}^N \rightarrow \mathbb{R}^M$, $U : \mathbb{C}^N \rightarrow \mathbb{C}^M$, obtained by setting $U_{E^\perp} = 0$. Then, we can identify these latter maps U with the corresponding rectangular matrices. \square

In order to advance, observe now that the isometries $T : E \rightarrow F$, or rather their extensions $U : \mathbb{K}^N \rightarrow \mathbb{K}^M$, with $\mathbb{K} = \mathbb{R}, \mathbb{C}$, obtained by setting $U_{E^\perp} = 0$, can be composed with the isometries of $\mathbb{K}^M, \mathbb{K}^N$, according to the following scheme:

$$\begin{array}{ccccccc}
 \mathbb{K}^N & \xrightarrow{B^*} & \mathbb{K}^N & \cdots \xrightarrow{U} & \mathbb{K}^M & \xrightarrow{A} & \mathbb{K}^M \\
 \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 B(E) & \cdots \cdots \cdots & E & \xrightarrow{T} & F & \cdots \cdots \cdots & A(F)
 \end{array}$$

With the identifications in Proposition 14.15 made, the precise statement here is:

PROPOSITION 14.16. *We have action maps as follows, which are both transitive,*

$$O_M \times O_N \curvearrowright O_{MN}^L \quad , \quad (A, B)U = AUB^t$$

$$U_M \times U_N \curvearrowright U_{MN}^L \quad , \quad (A, B)U = AUB^*$$

whose stabilizers are respectively $O_L \times O_{M-L} \times O_{N-L}$ and $U_L \times U_{M-L} \times U_{N-L}$.

PROOF. We have indeed action maps as in the statement, which are transitive. Let us compute now the stabilizer G of the following point:

$$U = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Since $(A, B) \in G$ satisfy $AU = UB$, their components must be of the following form:

$$A = \begin{pmatrix} x & * \\ 0 & a \end{pmatrix} \quad , \quad B = \begin{pmatrix} x & 0 \\ * & b \end{pmatrix}$$

Now since A, B are unitaries, these matrices follow to be block-diagonal, and so:

$$G = \left\{ (A, B) \mid A = \begin{pmatrix} x & 0 \\ 0 & a \end{pmatrix}, B = \begin{pmatrix} x & 0 \\ 0 & b \end{pmatrix} \right\}$$

The stabilizer of U is parametrized by triples (x, a, b) belonging to $O_L \times O_{M-L} \times O_{N-L}$ and $U_L \times U_{M-L} \times U_{N-L}$, and we are led to the conclusion in the statement. \square

Finally, let us work out the quotient space description of O_{MN}^L, U_{MN}^L . We have here:

THEOREM 14.17. *We have isomorphisms of homogeneous spaces as follows,*

$$\begin{aligned} O_{MN}^L &= (O_M \times O_N)/(O_L \times O_{M-L} \times O_{N-L}) \\ U_{MN}^L &= (U_M \times U_N)/(U_L \times U_{M-L} \times U_{N-L}) \end{aligned}$$

with the quotient maps being given by $(A, B) \rightarrow AUB^*$, where $U = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

PROOF. This is just a reformulation of Proposition 14.16, by taking into account the fact that the fixed point used in the proof there was $U = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. \square

Summarizing, we have here some basic homogeneous spaces, unifying the spheres with the rotation groups. The point now is that we can liberate these spaces, as follows:

DEFINITION 14.18. *Associated to any integers $L \leq M, N$ are the algebras*

$$\begin{aligned} C(O_{MN}^{L+}) &= C^* \left((v_{ij})_{i=1, \dots, M, j=1, \dots, N} \mid v = \bar{v}, vv^t = \text{projection of trace } L \right) \\ C(U_{MN}^{L+}) &= C^* \left((v_{ij})_{i=1, \dots, M, j=1, \dots, N} \mid vv^*, \bar{v}v^t = \text{projections of trace } L \right) \end{aligned}$$

with the trace being by definition the sum of the diagonal entries.

Observe that the above universal algebras are indeed well-defined, as it was previously the case for the free spheres, and this due to the trace conditions, which read:

$$\sum_{ij} v_{ij}v_{ij}^* = \sum_{ij} v_{ij}^*v_{ij} = L$$

We have inclusions between the various spaces constructed so far, as follows:

$$\begin{array}{ccc} O_{MN}^{L+} & \longrightarrow & U_{MN}^{L+} \\ \uparrow & & \uparrow \\ O_{MN}^L & \longrightarrow & U_{MN}^L \end{array}$$

At the level of basic examples now, at $L = M = 1$ and at $L = N = 1$ we obtain the following diagrams, showing that our formalism covers indeed the free spheres:

$$\begin{array}{ccc} S_{\mathbb{R},+}^{N-1} & \longrightarrow & S_{\mathbb{C},+}^{N-1} \\ \uparrow & & \uparrow \\ S_{\mathbb{R}}^{N-1} & \longrightarrow & S_{\mathbb{C}}^{N-1} \end{array} \quad \begin{array}{ccc} S_{\mathbb{R},+}^{M-1} & \longrightarrow & S_{\mathbb{C},+}^{M-1} \\ \uparrow & & \uparrow \\ S_{\mathbb{R}}^{M-1} & \longrightarrow & S_{\mathbb{C}}^{M-1} \end{array}$$

We have as well the following result, in relation with the free rotation groups:

PROPOSITION 14.19. *At $L = M = N$ we obtain the diagram*

$$\begin{array}{ccc} O_N^+ & \longrightarrow & U_N^+ \\ \uparrow & & \uparrow \\ O_N & \longrightarrow & U_N \end{array}$$

consisting of the groups O_N, U_N , and their liberations.

PROOF. According to the above, we have the following presentation results:

$$\begin{aligned} C(O_{NN}^{N \times}) &= C_{\times}^* \left((v_{ij})_{i,j=1,\dots,N} \mid v = \bar{v}, vv^t = \text{projection of trace } N \right) \\ C(U_{NN}^{N \times}) &= C_{\times}^* \left((v_{ij})_{i,j=1,\dots,N} \mid vv^*, \bar{v}v^t = \text{projections of trace } N \right) \end{aligned}$$

We use now the standard fact that if $p = aa^*$ is a projection then $q = a^*a$ is a projection too. We use as well the following formulae:

$$\text{Tr}(vv^*) = \text{Tr}(v^t\bar{v}) \quad , \quad \text{Tr}(\bar{v}v^t) = \text{Tr}(v^*v)$$

We therefore obtain the following formulae:

$$\begin{aligned} C(O_{NN}^{N \times}) &= C_{\times}^* \left((v_{ij})_{i,j=1,\dots,N} \mid v = \bar{v}, vv^t, v^t v = \text{projections of trace } N \right) \\ C(U_{NN}^{N \times}) &= C_{\times}^* \left((v_{ij})_{i,j=1,\dots,N} \mid vv^*, v^*v, \bar{v}v^t, v^t\bar{v} = \text{projections of trace } N \right) \end{aligned}$$

Now observe that, in tensor product notation, the conditions at right are all of the form $(tr \otimes id)p = 1$. Thus, p must be follows, for the above conditions:

$$p = vv^*, v^*v, \bar{v}v^t, v^t\bar{v}$$

We therefore obtain that, for any faithful state φ , we have $(tr \otimes \varphi)(1 - p) = 0$. It follows from this that the following projections must be all equal to the identity:

$$p = vv^*, v^*v, \bar{v}v^t, v^t\bar{v}$$

But this leads to the conclusion in the statement. □

Regarding now the homogeneous space structure of $O_{MN}^{L \times}, U_{MN}^{L \times}$, the situation here is a bit more complicated in the free case than in the classical case, due to a number of algebraic and analytic issues. We first have the following result:

PROPOSITION 14.20. *The spaces $U_{MN}^{L \times}$ have the following properties:*

- (1) *We have an action $U_M^{\times} \times U_N^{\times} \curvearrowright U_{MN}^{L \times}$, given by $v_{ij} \rightarrow \sum_{kl} v_{kl} \otimes a_{ki} \otimes b_{lj}^*$.*
- (2) *We have a map $U_M^{\times} \times U_N^{\times} \rightarrow U_{MN}^{L \times}$, given by $v_{ij} \rightarrow \sum_{r \leq L} a_{ri} \otimes b_{rj}^*$.*

Similar results hold for the spaces $O_{MN}^{L \times}$, with all the $$ exponents removed.*

PROOF. In the classical case, consider the following action and quotient maps:

$$U_M \times U_N \curvearrowright U_{MN}^L \quad , \quad U_M \times U_N \rightarrow U_{MN}^L$$

The transposes of these two maps are as follows, where $J = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$:

$$\begin{aligned} \varphi &\rightarrow ((U, A, B) \rightarrow \varphi(AUB^*)) \\ \varphi &\rightarrow ((A, B) \rightarrow \varphi(AJB^*)) \end{aligned}$$

But with $\varphi = v_{ij}$ we obtain precisely the formulae in the statement. The proof in the orthogonal case is similar. Regarding now the free case, the proof goes as follows:

(1) Assuming $vv^*v = v$, let us set $U_{ij} = \sum_{kl} v_{kl} \otimes a_{ki} \otimes b_{lj}^*$. We have then:

$$\begin{aligned} (UU^*U)_{ij} &= \sum_{pq} \sum_{klmnst} v_{kl} v_{mn}^* v_{st} \otimes a_{ki} a_{mq}^* a_{sq} \otimes b_{lp}^* b_{np} b_{tj}^* \\ &= \sum_{klmt} v_{kl} v_{ml}^* v_{mt} \otimes a_{ki} \otimes b_{tj}^* \\ &= \sum_{kt} v_{kt} \otimes a_{ki} \otimes b_{tj}^* \\ &= U_{ij} \end{aligned}$$

Also, assuming that we have $\sum_{ij} v_{ij} v_{ij}^* = L$, we obtain:

$$\begin{aligned} \sum_{ij} U_{ij} U_{ij}^* &= \sum_{ij} \sum_{klst} v_{kl} v_{st}^* \otimes a_{ki} a_{si}^* \otimes b_{lj}^* b_{tj} \\ &= \sum_{kl} v_{kl} v_{kl}^* \otimes 1 \otimes 1 \\ &= L \end{aligned}$$

(2) Assuming $vv^*v = v$, let us set $V_{ij} = \sum_{r \leq L} a_{ri} \otimes b_{rj}^*$. We have then:

$$\begin{aligned} (VV^*V)_{ij} &= \sum_{pq} \sum_{x,y,z \leq L} a_{xi} a_{yq}^* a_{zq} \otimes b_{xp}^* b_{yp} b_{zj}^* \\ &= \sum_{x \leq L} a_{xi} \otimes b_{xj}^* \\ &= V_{ij} \end{aligned}$$

Finally, assuming that we have $\sum_{ij} u_{ij} u_{ij}^* = L$, we obtain:

$$\sum_{ij} V_{ij} V_{ij}^* = \sum_{ij} \sum_{r,s \leq L} a_{ri} a_{si}^* \otimes b_{rj}^* b_{sj} = \sum_{l \leq L} 1 = L$$

By removing all the $*$ exponents, we obtain as well the orthogonal results. \square

Let us examine now the relation between the above maps. In the classical case, given a quotient space $X = G/H$, the associated action and quotient maps are given by:

$$\begin{cases} a : X \times G \rightarrow X & : (Hg, h) \rightarrow Hgh \\ p : G \rightarrow X & : g \rightarrow Hg \end{cases}$$

Thus we have $a(p(g), h) = p(gh)$. In our context, a similar result holds:

THEOREM 14.21. *With $G = G_M \times G_N$ and $X = G_{MN}^L$, where $G_N = O_N^\times, U_N^\times$, we have*

$$\begin{array}{ccc} G \times G & \xrightarrow{m} & G \\ p \times id \downarrow & & \downarrow p \\ X \times G & \xrightarrow{a} & X \end{array}$$

where a, p are the action map and the map constructed in Proposition 14.20.

PROOF. At the level of the associated algebras of functions, we must prove that the following diagram commutes, where Φ, α are morphisms of algebras induced by a, p :

$$\begin{array}{ccc} C(X) & \xrightarrow{\Phi} & C(X \times G) \\ \alpha \downarrow & & \downarrow \alpha \otimes id \\ C(G) & \xrightarrow{\Delta} & C(G \times G) \end{array}$$

When going right, and then down, the composition is as follows:

$$\begin{aligned} (\alpha \otimes id)\Phi(u_{ij}) &= (\alpha \otimes id) \sum_{kl} v_{kl} \otimes a_{ki} \otimes b_{lj}^* \\ &= \sum_{kl} \sum_{r \leq L} a_{rk} \otimes b_{rl}^* \otimes a_{ki} \otimes b_{lj}^* \end{aligned}$$

On the other hand, when going down, and then right, the composition is as follows, where F_{23} is the flip between the second and the third components:

$$\begin{aligned} \Delta\pi(u_{ij}) &= F_{23}(\Delta \otimes \Delta) \sum_{r \leq L} a_{ri} \otimes b_{rj}^* \\ &= F_{23} \left(\sum_{r \leq L} \sum_{kl} a_{rk} \otimes a_{ki} \otimes b_{rl}^* \otimes b_{lj}^* \right) \end{aligned}$$

Thus the above diagram commutes indeed, and this gives the result. \square

Let us discuss now the integration over the above spaces G_{MN}^L . We first have:

DEFINITION 14.22. *The integration functional of G_{MN}^L is the composition*

$$\int_{G_{MN}^L} : C(G_{MN}^L) \rightarrow C(G_M \times G_N) \rightarrow \mathbb{C}$$

of the representation $v_{ij} \rightarrow \sum_{r \leq L} a_{ri} \otimes b_{rj}^$ with the Haar functional of $G_M \times G_N$.*

As an illustration here, observe that in the case $L = M = N$ we obtain the integration over G_N . Also, at $L = M = 1$, or at $L = N = 1$, we obtain the integration over the sphere. In the general case now, we first have the following result:

PROPOSITION 14.23. *The integration functional of G_{MN}^L has the invariance property*

$$\left(\int_{G_{MN}^L} \otimes id \right) \Phi(x) = \int_{G_{MN}^L} x$$

with respect to the coaction map $\Phi(v_{ij}) = \sum_{kl} v_{kl} \otimes a_{ki} \otimes b_{lj}^$.*

PROOF. We can restrict the attention to the orthogonal case, the proof in the unitary case being similar. We must check the following formula:

$$\left(\int_{G_{MN}^L} \otimes id \right) \Phi(v_{i_1 j_1} \cdots v_{i_s j_s}) = \int_{G_{MN}^L} v_{i_1 j_1} \cdots v_{i_s j_s}$$

Let us compute the left term. This is given by:

$$\begin{aligned} X &= \left(\int_{G_{MN}^L} \otimes id \right) \sum_{k_x l_x} v_{k_1 l_1} \cdots v_{k_s l_s} \otimes a_{k_1 i_1} \cdots a_{k_s i_s} \otimes b_{l_1 j_1}^* \cdots b_{l_s j_s}^* \\ &= \sum_{k_x l_x} \sum_{r_x \leq L} a_{k_1 i_1} \cdots a_{k_s i_s} \otimes b_{l_1 j_1}^* \cdots b_{l_s j_s}^* \int_{G_M} a_{r_1 k_1} \cdots a_{r_s k_s} \int_{G_N} b_{r_1 l_1}^* \cdots b_{r_s l_s}^* \\ &= \sum_{r_x \leq L} \sum_{k_x} a_{k_1 i_1} \cdots a_{k_s i_s} \int_{G_M} a_{r_1 k_1} \cdots a_{r_s k_s} \otimes \sum_{l_x} b_{l_1 j_1}^* \cdots b_{l_s j_s}^* \int_{G_N} b_{r_1 l_1}^* \cdots b_{r_s l_s}^* \end{aligned}$$

By using now the invariance property of the Haar functionals of G_M, G_N , we obtain:

$$\begin{aligned} X &= \sum_{r_x \leq L} \left(\int_{G_M} \otimes id \right) \Delta(a_{r_1 i_1} \cdots a_{r_s i_s}) \otimes \left(\int_{G_N} \otimes id \right) \Delta(b_{r_1 j_1}^* \cdots b_{r_s j_s}^*) \\ &= \sum_{r_x \leq L} \int_{G_M} a_{r_1 i_1} \cdots a_{r_s i_s} \int_{G_N} b_{r_1 j_1}^* \cdots b_{r_s j_s}^* \\ &= \left(\int_{G_M} \otimes \int_{G_N} \right) \sum_{r_x \leq L} a_{r_1 i_1} \cdots a_{r_s i_s} \otimes b_{r_1 j_1}^* \cdots b_{r_s j_s}^* \end{aligned}$$

But this gives the formula in the statement, and we are done. \square

We will prove now that the above functional is in fact the unique positive unital invariant trace on $C(G_{MN}^L)$. For this purpose, we will need the Weingarten formula:

THEOREM 14.24. *We have the Weingarten type formula*

$$\int_{G_{MN}^L} v_{i_1 j_1} \dots v_{i_s j_s} = \sum_{\pi\sigma\tau\nu} L^{|\pi\nu\tau|} \delta_\sigma(i) \delta_\nu(j) W_{sM}(\pi, \sigma) W_{sN}(\tau, \nu)$$

where the matrices on the right are given by $W_{sM} = G_{sM}^{-1}$, with $G_{sM}(\pi, \sigma) = M^{|\pi\nu\sigma|}$.

PROOF. By using the Weingarten formula for G_M, G_N , we obtain:

$$\begin{aligned} \int_{G_{MN}^L} v_{i_1 j_1} \dots v_{i_s j_s} &= \sum_{l_1 \dots l_s \leq L} \int_{G_M} a_{l_1 i_1} \dots a_{l_s i_s} \int_{G_N} b_{l_1 j_1}^* \dots b_{l_s j_s}^* \\ &= \sum_{l_1 \dots l_s \leq L} \sum_{\pi\sigma} \delta_\pi(l) \delta_\sigma(i) W_{sM}(\pi, \sigma) \sum_{\tau\nu} \delta_\tau(l) \delta_\nu(j) W_{sN}(\tau, \nu) \\ &= \sum_{\pi\sigma\tau\nu} \left(\sum_{l_1 \dots l_s \leq L} \delta_\pi(l) \delta_\tau(l) \right) \delta_\sigma(i) \delta_\nu(j) W_{sM}(\pi, \sigma) W_{sN}(\tau, \nu) \end{aligned}$$

The coefficient being $L^{|\pi\nu\tau|}$, we obtain the formula in the statement. □

We can now derive an abstract characterization of the integration, as follows:

THEOREM 14.25. *The integration of G_{MN}^L is the unique positive unital trace*

$$C(G_{MN}^L) \rightarrow \mathbb{C}$$

which is invariant under the action of the quantum group $G_M \times G_N$.

PROOF. This is something very standard, from [18]. Our claim is that we have:

$$\left(id \otimes \int_{G_M} \otimes \int_{G_N} \right) \Phi(v_{i_1 j_1} \dots v_{i_s j_s}) = \int_{G_{MN}^L} v_{i_1 j_1} \dots v_{i_s j_s}$$

Indeed, by using the Weingarten formula, the left term can be written as follows:

$$\begin{aligned} X &= \sum_{k_1 \dots k_s} \sum_{l_1 \dots l_s} v_{k_1 l_1} \dots v_{k_s l_s} \int_{G_M} a_{k_1 i_1} \dots a_{k_s i_s} \int_{G_N} b_{l_1 j_1}^* \dots b_{l_s j_s}^* \\ &= \sum_{k_1 \dots k_s} \sum_{l_1 \dots l_s} v_{k_1 l_1} \dots v_{k_s l_s} \sum_{\pi\sigma} \delta_\pi(k) \delta_\sigma(i) W_{sM}(\pi, \sigma) \sum_{\tau\nu} \delta_\tau(l) \delta_\nu(j) W_{sN}(\tau, \nu) \\ &= \sum_{\pi\sigma\tau\nu} \delta_\sigma(i) \delta_\nu(j) W_{sM}(\pi, \sigma) W_{sN}(\tau, \nu) \sum_{k_1 \dots k_s} \sum_{l_1 \dots l_s} \delta_\pi(k) \delta_\tau(l) u_{k_1 l_1} \dots u_{k_s l_s} \\ &= \sum_{\pi\sigma\tau\nu} L^{|\pi\nu\tau|} \delta_\sigma(i) \delta_\nu(j) W_{sM}(\pi, \sigma) W_{sN}(\tau, \nu) \end{aligned}$$

Now by comparing with the Weingarten formula for G_{MN}^L , this proves our claim. Assume now that $\tau : C(G_{MN}^L) \rightarrow \mathbb{C}$ satisfies the invariance condition. We have then:

$$\tau \left(id \otimes \int_{G_M} \otimes \int_{G_N} \right) \Phi(x) = \left(\int_{G_M} \otimes \int_{G_N} \right) (\tau(x)1) = \tau(x)$$

On the other hand, according to the formula established above, we have as well:

$$\tau \left(id \otimes \int_{G_M} \otimes \int_{G_N} \right) \Phi(x) = \tau(tr(x)1) = tr(x)$$

Thus we obtain $\tau = tr$, and this finishes the proof. □

As a main application of the above results, we have the following quite conceptual statement, making the link with the Bercovici-Pata bijection [24]:

THEOREM 14.26. *In the context of the liberation operations $G_{MN}^L \rightarrow G_{MN}^{L+}$, the laws of the sums of non-overlapping coordinates,*

$$\chi_E = \sum_{(ij) \in E} u_{ij}$$

are in Bercovici-Pata bijection, in the $|E| = \kappa N, L = \lambda N, M = \mu N$ and $N \rightarrow \infty$ regime.

PROOF. We use various formulae from [9], [10], [21]. In terms of $K = |E|$, the moments of the variables in the statement are given by:

$$\begin{aligned} M_s &= \sum_{\pi\sigma\tau\nu} K^{|\pi \vee \tau|} L^{|\sigma \vee \nu|} W_{sM}(\pi, \sigma) W_{sN}(\tau, \nu) \\ &\simeq \sum_{\pi\tau} K^{|\pi \vee \tau|} L^{|\pi \vee \tau|} M^{-|\pi|} N^{-|\tau|} \\ &\simeq \sum_{\pi} K^{|\pi|} L^{|\pi|} M^{-|\pi|} N^{-|\pi|} \\ &= \sum_{\pi} \left(\frac{\kappa\lambda}{\mu} \right)^{|\pi|} \end{aligned}$$

In order to interpret this formula, we use general theory from [9], [10], [21]:

(1) For $G_N = O_N/O_N^+$, the above variables χ_E follow to be asymptotically Gaussian/semicircular, of parameter $\frac{\kappa\lambda}{\mu}$, and hence in Bercovici-Pata bijection.

(2) For $G_N = U_N/U_N^+$ the situation is similar, with χ_E being asymptotically complex Gaussian/circular, of parameter $\frac{\kappa\lambda}{\mu}$, and in Bercovici-Pata bijection. □

There are several possible extensions of the above result, to the discrete case, and by using twisting operations as well. We refer here to [9], [10] and related papers.

14c. Hyperspherical laws

Changing topics now, we know from Theorem 14.13, dealing with the Meixner/free Meixner correspondence, that doing probability in the free geometry setting can lead us to unexplored territory, beyond what the Bercovici-Pata bijection says. As a continuation of that material, we will discuss here the classical and free hyperspherical laws. In the classical case, we will need the following result, that we know well from chapter 1:

THEOREM 14.27. *The even moments of the hyperspherical variables are*

$$\int_{S_{\mathbb{R}}^{N-1}} z_i^k dx = \frac{(N-1)!!k!!}{(N+k-1)!!}$$

and the variables $y_i = \sqrt{N}z_i$ become normal and independent with $N \rightarrow \infty$.

PROOF. The moment formula in the statement is something that we know from chapter 1. Now observe that with $N \rightarrow \infty$ we have the following estimate:

$$\int_{S_{\mathbb{R}}^{N-1}} z_i^k dz \simeq N^{-k/2} \times k!! = N^{-k/2} M_k(g_1)$$

Thus we have, as claimed, $\sqrt{N}z_i \sim g_1$. Finally, the asymptotic independence assertion follows as well from the formulae in chapter 1, via standard probability theory. \square

In the case of the free real sphere now, the computations are substantially more complicated than those in the classical case. Let us start with the following result:

THEOREM 14.28. *For the free sphere $S_{\mathbb{R},+}^{N-1}$, the rescaled coordinates*

$$y_i = \sqrt{N}z_i$$

become semicircular and free, in the $N \rightarrow \infty$ limit.

PROOF. The Weingarten formula for the free sphere, together with the standard fact that the Gram matrix is asymptotically diagonal, gives the following estimate:

$$\int_{S_{\mathbb{R},+}^{N-1}} z_{i_1} \dots z_{i_k} dz \simeq N^{-k/2} \sum_{\sigma \in NC_2(k)} \delta_{\sigma}(i_1, \dots, i_k)$$

With this formula in hand, we can compute the asymptotic moments of each coordinate x_i . Indeed, by setting $i_1 = \dots = i_k = i$, all Kronecker symbols are 1, and we obtain:

$$\int_{S_{\mathbb{R},+}^{N-1}} z_i^k dz \simeq N^{-k/2} |NC_2(k)|$$

Thus the rescaled coordinates $y_i = \sqrt{N}z_i$ become semicircular in the $N \rightarrow \infty$ limit, as claimed. As for the asymptotic freeness result, this follows as well from the above general joint moment estimate, via standard free probability theory. See [8], [13], [18]. \square

Summarizing, we have good results for the free sphere, with $N \rightarrow \infty$. The problem now, which is non-trivial, is that of computing the moments of the coordinates of the free sphere at fixed values of $N \in \mathbb{N}$. The answer here, from [15], which is based on advanced quantum group techniques, that we will briefly explain here, is as follows:

THEOREM 14.29. *The moments of the free hyperspherical law are given by*

$$\int_{S_{\mathbb{R},+}^{N-1}} z_1^{2l} = \frac{1}{(N+1)^l} \cdot \frac{q+1}{q-1} \cdot \frac{1}{l+1} \sum_{r=-l-1}^{l+1} (-1)^r \binom{2l+2}{l+r+1} \frac{r}{1+q^r}$$

where $q \in [-1, 0)$ is such that $q + q^{-1} = -N$.

PROOF. The idea is that $z_1 \in C(S_{\mathbb{R},+}^{N-1})$ has the same law as $v_{11} \in C(O_N^+)$, which has the same law as a certain variable $w \in C(SU_2^q)$, which can be modelled by an explicit operator on $l^2(\mathbb{N})$, whose law can be computed by using advanced calculus.

(1) Let us first explain the relation between O_N^+ and SU_2^q . To any matrix $F \in GL_N(\mathbb{R})$ satisfying $F^2 = 1$ we associate the following universal algebra:

$$C(O_F^+) = C^* \left((v_{ij})_{i,j=1,\dots,N} \mid v = F\bar{v}F = \text{unitary} \right)$$

Observe that we have $O_{I_N}^+ = O_N^+$. In general, the above algebra satisfies Woronowicz' generalized axioms in [98], which do not include the antipode axiom $S^2 = id$.

(2) At $N = 2$ now, up to a trivial equivalence relation on the matrices F , and on the quantum groups O_F^+ , we can assume that F is as follows, with $q \in [-1, 0)$:

$$F = \begin{pmatrix} 0 & \sqrt{-q} \\ 1/\sqrt{-q} & 0 \end{pmatrix}$$

Our claim is that for this matrix we have $O_F^+ = SU_2^q$. Indeed, the relations $v = F\bar{v}F$ tell us that v must be of the following form:

$$v = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$$

Thus $C(O_F^+)$ is the universal algebra generated by two elements α, γ , with the relations making the above matrix v a unitary. But these unitarity conditions are:

$$\begin{aligned} \alpha\gamma &= q\gamma\alpha \quad , \quad \alpha\gamma^* = q\gamma^*\alpha \quad , \quad \gamma\gamma^* = \gamma^*\gamma \\ \alpha^*\alpha + \gamma^*\gamma &= 1 \quad , \quad \alpha\alpha^* + q^2\gamma\gamma^* = 1 \end{aligned}$$

We recognize here the relations in [98] defining the algebra $C(SU_2^q)$, and it follows that we have an isomorphism of Hopf algebras, as follows:

$$C(O_F^+) \simeq C(SU_2^q)$$

(3) Now back to the general case, where $F \in GL_N(\mathbb{R})$ satisfies $F^2 = 1$, let us try to understand the integration over O_F^+ . Given $\pi \in NC_2(2k)$ and $i = (i_1, \dots, i_{2k})$, we set:

$$\delta_\pi^F(i) = \prod_{s \in \pi} F_{i_{s_l} i_{s_r}}$$

Here the product is over all the strings $s = \{s_l \curvearrowright s_r\}$ of π . Our claim is that the following family of vectors, with $\pi \in NC_2(2k)$, spans the space of fixed vectors of $v^{\otimes 2k}$:

$$\xi_\pi = \sum_i \delta_\pi^F(i) e_{i_1} \otimes \dots \otimes e_{i_{2k}}$$

Indeed, having ξ_π fixed by $v^{\otimes 2}$ is equivalent to assuming that $v = F\bar{v}F$ is unitary. By using now these vectors, as in [13], we obtain the following Weingarten formula:

$$\int_{O_F^+} v_{i_1 j_1} \dots v_{i_{2k} j_{2k}} = \sum_{\pi\sigma} \delta_\pi^F(i) \delta_\sigma^F(j) W_{kN}(\pi, \sigma)$$

(4) With these preliminaries in hand, we can now start the computation that we are interested in. Let $N \in \mathbb{N}$, and consider the number $q \in [-1, 0)$ satisfying:

$$q + q^{-1} = -N$$

Our claim is that we have the following formula:

$$\int_{O_N^+} \varphi(\sqrt{N+2} v_{ij}) = \int_{SU_2^q} \varphi(\alpha + \alpha^* + \gamma - q\gamma^*)$$

Indeed, according to the above, the moments of the variable on the left are given by:

$$\int_{O_N^+} v_{ij}^{2k} = \sum_{\pi\sigma} W_{kN}(\pi, \sigma)$$

On the other hand, the moments of the variable on the right, which in terms of the fundamental corepresentation $u = (u_{ij})$ is given by $w = \sum_{ij} u_{ij}$, are as follows:

$$\int_{SU_2^q} w^{2k} = \sum_{ij} \sum_{\pi\sigma} \delta_\pi^F(i) \delta_\sigma^F(j) W_{kN}(\pi, \sigma)$$

We deduce that $w/\sqrt{N+2}$ has the same moments as v_{ij} , which proves our claim.

(5) In order to do the computation over SU_2^q , we can use a well-known matrix model, due to Woronowicz [98], where the standard generators α, γ are mapped as follows:

$$\pi_u(\alpha)e_k = \sqrt{1 - q^{2k}}e_{k-1} \quad , \quad \pi_u(\gamma)e_k = uq^k e_k$$

Here $u \in \mathbb{T}$ is a parameter, and (e_k) is the standard basis of $l^2(\mathbb{N})$. The point with this representation is that it allows the computation of the Haar functional. Indeed, if D

is the diagonal operator given by $D(e_k) = q^{2k}e_k$, then we have the following formula:

$$\int_{SU_2^q} x = (1 - q^2) \int_{\mathbb{T}} \text{tr}(D\pi_u(x)) \frac{du}{2\pi i u}$$

With the above explicit model in hand, we conclude that the law of the variable that we are interested in is subject to the following formula:

$$\int_{SU_2^q} \varphi(\alpha + \alpha^* + \gamma - q\gamma^*) = (1 - q^2) \int_{\mathbb{T}} \text{tr}(D\varphi(M)) \frac{du}{2\pi i u}$$

To be more precise, this formula holds indeed, with M being as follows:

$$M(e_k) = e_{k+1} + q^k(u - qu^{-1})e_k + (1 - q^{2k})e_{k-1}$$

(6) The point now is that the integral on the right in the above can be computed, by using advanced calculus methods, and this gives the result. We refer here to [15]. \square

The computation of the joint free hyperspherical laws remains an open problem. Open as well is the question of finding a more conceptual proof for the above formula.

14d. Hypergeometric laws

Following now [11], let us discuss a remarkable relation of all this with the quantum permutations, and with the free hypergeometric laws. The idea will be that of working out some abstract algebraic results, regarding twists of quantum automorphism groups, which will particularize into results relating quantum rotations and permutations, having no classical counterpart, both at the algebraic and the probabilistic level.

In order to explain this material, from [11], which is quite technical, requiring good algebraic knowledge, let us begin with some generalities. We first have:

DEFINITION 14.30. *A finite quantum space X is the abstract dual of a finite dimensional C^* -algebra B , according to the following formula:*

$$C(X) = B$$

The number of elements of such a space is $|X| = \dim B$. By decomposing the algebra B , we have a formula of the following type:

$$C(X) = M_{n_1}(\mathbb{C}) \oplus \dots \oplus M_{n_k}(\mathbb{C})$$

With $n_1 = \dots = n_k = 1$ we obtain in this way the space $X = \{1, \dots, k\}$. Also, when $k = 1$ the equation is $C(X) = M_n(\mathbb{C})$, and the solution will be denoted $X = M_n$.

We endow each finite quantum space x with its counting measure, corresponding as the algebraic level to the integration functional obtained by applying the regular representation, and then the unique normalized trace of the matrix algebra $\mathcal{L}(C(X))$:

$$\text{tr} : C(X) \subset \mathcal{L}(C(X)) \rightarrow \mathbb{C}$$

Now if we denote by μ, η the multiplication and unit map of the algebra $C(X)$, we have the following standard result, from [8], based on some previous work from [93]:

THEOREM 14.31. *Given a finite quantum space X , there is a universal compact quantum group S_X^+ acting on X , leaving the counting measure invariant. We have*

$$C(S_X^+) = C(U_N^+) / \left\langle \mu \in \text{Hom}(v^{\otimes 2}, v), \eta \in \text{Fix}(v) \right\rangle$$

where $N = |X|$ and where μ, η are the multiplication and unit maps of $C(X)$. Also:

- (1) For $X = \{1, \dots, N\}$ we have $S_X^+ = S_N^+$.
- (2) For $X = M_n$ we have $S_X^+ = PO_n^+ = PU_n^+$.

PROOF. Consider a linear map $\Phi : C(X) \rightarrow C(X) \otimes C(G)$, written as follows, with $\{e_i\}$ being a linear space basis of the algebra $C(X)$, orthonormal with respect to tr :

$$\Phi(e_j) = \sum_i e_i \otimes v_{ij}$$

Then Φ is a coaction precisely when v is a unitary corepresentation, satisfying:

$$\mu \in \text{Hom}(v^{\otimes 2}, v) \quad , \quad \eta \in \text{Fix}(v)$$

But this gives the first assertion. Regarding now the statement about $X = \{1, \dots, N\}$, this is clear. Finally, regarding $X = M_2$, here we have embeddings as followss:

$$PO_n^+ \subset PU_n^+ \subset S_X^+$$

Now since the fusion rules of all these 3 quantum groups are known to be the same as the fusion rules for SO_3 , these inclusions follow to be isomorphisms. See [8]. □

We have as well the following result, also from [8]:

THEOREM 14.32. *The quantum groups S_X^+ have the following properties:*

- (1) *The associated Tannakian categories are $TL(N)$, with $N = |X|$.*
- (2) *The main character follows the Marchenko-Pastur law π_1 , when $N \geq 4$.*
- (3) *The fusion rules for S_X^+ with $|F| \geq 4$ are the same as for SO_3 .*

PROOF. This result is from [8], the idea being as follows:

(1) This follows from the fact that the multiplication and unit of any complex algebra, and in particular of $C(X)$, can be modeled by the following two diagrams:

$$m = |\cup| \quad , \quad u = \cap$$

(2) The proof here is as for S_N^+ , by using moments. To be more precise, according to (1) these moments are the Catalan numbers, which are the moments of π_1 .

(3) Once again same proof as for S_N^+ , by using the fact that the moments of χ are the Catalan numbers, which lead to the Clebsch-Gordan rules. See [8]. □

Let us discuss now a number of more advanced twisting aspects, which will eventually lead us into probability, and hypergeometric laws. Following [11], we have:

THEOREM 14.33. *If G is a finite group and σ is a 2-cocycle on G , the Hopf algebras*

$$C(S_{\widehat{G}}^+) \quad , \quad C(S_{\widehat{G}_\sigma}^+)$$

are 2-cocycle twists of each other.

PROOF. This is something quite technical, requiring a good knowledge of algebraic twisting techniques, and for full details here, we refer to [11]. \square

As an example, let $G = \mathbb{Z}_n^2$, and consider the following map, with $w = e^{2\pi i/n}$:

$$\sigma : G \times G \rightarrow \mathbb{C}^* \quad , \quad \sigma_{(ij)(kl)} = w^{jk}$$

Then σ is a bicharacter, and hence a 2-cocycle on G . Thus, we can apply our twisting result, to this situation. We obtain a concrete result, also from [11], as follows:

THEOREM 14.34. *Let $n \geq 2$ and $w = e^{2\pi i/n}$. Then the formula*

$$\Theta(u_{ij}u_{kl}) = \frac{1}{n} \sum_{ab=0}^{n-1} w^{-a(k-i)+b(l-j)} p_{ia,jb}$$

defines a coalgebra isomorphism $C(PO_n^+) \rightarrow C(S_{n^2}^+)$, commuting with the Haar integrals.

PROOF. This follows indeed from our general twisting result from Theorem 14.33, by using as ingredients the group and the cocycle indicated above. \square

As a probabilistic consequence now, which is of interest for us, we have:

THEOREM 14.35. *The following families of variables have the same joint law,*

- (1) $\{v_{ij}^2\} \in C(O_n^+)$,
- (2) $\{\eta_{ij} = \frac{1}{n} \sum_{ab} p_{ia,jb}\} \in C(S_{n^2}^+)$,

where $v = (v_{ij})$ and $p = (p_{ia,jb})$ are the corresponding fundamental corepresentations.

PROOF. This follows from Theorem 14.34. Alternatively, we can use the Weingarten formula for our quantum groups, and the shrinking operation $\pi \rightarrow \pi'$. Indeed, we have:

$$\int_{O_n^+} v_{ij}^{2k} = \sum_{\pi, \sigma \in NC_2(2k)} W_{2k,n}(\pi, \sigma)$$

$$\int_{S_{n^2}^+} \eta_{ij}^k = \sum_{\pi, \sigma \in NC_2(2k)} n^{|\pi'|+|\sigma'|-k} W_{k,n^2}(\pi', \sigma')$$

By doing now some standard combinatorics, the summands coincide, and so the moments are equal, as desired. The proof for joint moments is similar. See [11]. \square

As an explicit application of the above, also from [11], we have:

THEOREM 14.36. *The free hyperspherical and hypergeometric variables,*

$$z_i^2 \in C(S_{\mathbb{R},+}^{N-1}) \quad , \quad \eta_{ij} = \frac{1}{n} \sum_{a,b=1}^n u_{ia,jb} \in C(S_{n^2}^+)$$

has the same law.

PROOF. This follows indeed from Theorem 14.35, particularized to the case of single variables. For details on all this, and for more, we refer to [11]. \square

As a conclusion, interesting things happen when doing noncommutative geometry. Needless to say, all this is of interest too in relation with physics. For instance in the Connes interpretation of the Standard Model, coming from [38], the probabilistic study of the corresponding free gauge group leads to beasts as above.

14e. Exercises

Things have been quite advanced in this chapter, and as a unique exercise, which is rather elementary, and very instructive, we have:

EXERCISE 14.37. *Work out the asymptotics of the free hypergeometric laws.*

Here the computations are quite standard, and very instructive. In case you are stuck with something, all this is done in [11], so read and write a brief account of that.

CHAPTER 15

Invariance questions

15a. Invariance questions

An interesting question, which often appears in theoretical probability, as well in connection with certain questions coming from physics, is the study of the sequences of random variables $x_1, x_2, x_3, \dots \in L^\infty(X)$ which are exchangeable, in the sense that their joint distribution is invariant under the infinite permutations $\sigma \in S_\infty$:

$$\mu_{x_1, x_2, x_3, \dots} = \mu_{x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, \dots}$$

This question is solved by the classical De Finetti theorem, which basically says that the variables x_1, x_2, x_3, \dots must be i.i.d., in some asymptotic sense. We will see a precise statement of this theorem, along with a complete proof, in a minute.

The De Finetti theorem has many generalizations. One can replace for instance the action of the group $S_\infty = \cup_N S_N$ by the action of the bigger group $O_\infty = \cup_N O_N$, and the sequences $x_1, x_2, x_3, \dots \in L^\infty(X)$ which are invariant in this stronger sense, which are called “rotatable”, can be characterized as well, via a De Finetti type theorem.

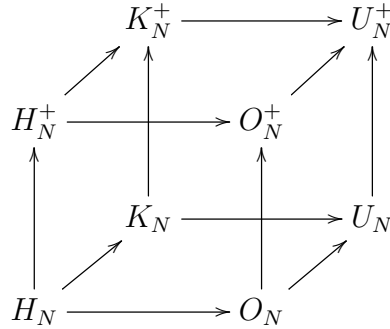
All this is interesting for us, in connection with what we have been doing so far, in this book. On one hand the groups S_N, O_N are easy, and we would like to understand how the above-mentioned De Finetti theorems, involving S_N, O_N , as well as their various technical generalizations, follow from the easiness property of S_N, O_N . On the other hand, we would like to understand as well what happens for S_N^+, O_N^+ .

Long story short, we would like to discuss here probabilistic invariance questions with respect to the basic quantum permutation and rotation groups, namely:

$$\begin{array}{ccc} S_N^+ & \longrightarrow & O_N^+ \\ \uparrow & & \uparrow \\ S_N & \longrightarrow & O_N \end{array}$$

As a second objective, in tune with what we have been doing so far in this book, we would like as well to understand what happens to the invariance questions with respect

to the basic quantum reflection and rotation groups, from our beloved cube, namely:



We will discuss here most of these questions, following the classical theory of the De Finetti theorem, then the foundational paper of K\"ostler and Speicher [62], in the free case, and then the more advanced paper [17], dealing with both the classical and free De Finetti theorems, and their other easy quantum group generalizations.

Let us start by fixing some notations. In order to deal with our first question above, we will use here the formalism of the orthogonal quantum groups, which best covers the main quantum groups that we are interested in. We first have the following definition:

DEFINITION 15.1. *Given a closed subgroup $G \subset O_N^+$, we denote by*

$$\begin{aligned}
 \alpha : \mathbb{C} \langle t_1, \dots, t_N \rangle &\rightarrow \mathbb{C} \langle t_1, \dots, t_N \rangle \otimes C(G) \\
 t_i &\rightarrow \sum_j t_j \otimes v_{ji}
 \end{aligned}$$

the standard coaction of $C(G)$ on the free complex algebra on N variables.

Observe that the map α constructed above is indeed a coaction, in the sense that it satisfies the following standard coassociativity and counitality conditions:

$$\begin{aligned}
 (id \otimes \Delta)\alpha &= (\alpha \otimes id)\alpha \\
 (id \otimes \varepsilon)\alpha &= id
 \end{aligned}$$

With the above notion of coaction in hand, we can now talk about invariant sequences of classical or noncommutative random variables, in the following way:

DEFINITION 15.2. *Let (B, tr) be a C^* -algebra with a trace, and $x_1, \dots, x_N \in B$. We say that $x = (x_1, \dots, x_N)$ is invariant under $G \subset O_N^+$ if the distribution functional*

$$\begin{aligned}
 \mu_x : \mathbb{C} \langle t_1, \dots, t_N \rangle &\rightarrow \mathbb{C} \\
 P &\rightarrow tr(P(x_1, \dots, x_N))
 \end{aligned}$$

is invariant under the coaction α , in the sense that we have

$$(\mu_x \otimes id)\alpha(P) = \mu_x(P)$$

for any noncommuting polynomial $P \in \mathbb{C} \langle t_1, \dots, t_N \rangle$.

In the classical case, where $G \subset O_N$ is a usual group, we recover in this way the usual invariance notion from classical probability. In the general case, where $G \subset O_N^+$ is arbitrary, what we have is a natural generalization of this. For further comments on all this, including examples, and motivations too, we refer to [17], [39], [40], [62], [63].

We have the following equivalent formulation of the above invariance condition:

PROPOSITION 15.3. *Let (B, tr) be a C^* -algebra with a trace, and $x_1, \dots, x_N \in B$. Then $x = (x_1, \dots, x_N)$ is invariant under $G \subset O_N^+$ precisely when*

$$tr(x_{i_1} \dots x_{i_k}) = \sum_{j_1 \dots j_k} tr(x_{j_1} \dots x_{j_k}) v_{j_1 i_1} \dots v_{j_k i_k}$$

as an equality in $C(G)$, for any $k \in \mathbb{N}$, and any $i_1, \dots, i_k \in \{1, \dots, N\}$.

PROOF. By linearity, in order for a sequence $x = (x_1, \dots, x_N)$ to be G -invariant in the sense of Definition 15.2, the formula there must be satisfied for any noncommuting monomial $P \in \mathbb{C} \langle t_1, \dots, t_N \rangle$. But an arbitrary such monomial can be written as follows, for a certain $k \in \mathbb{N}$, and certain indices $i_1, \dots, i_k \in \{1, \dots, N\}$:

$$P = t_{i_1} \dots t_{i_k}$$

Now with this formula for P in hand, we have the following computation:

$$\begin{aligned} (\mu_x \otimes id)\alpha(P) &= (\mu_x \otimes id) \sum_{j_1, \dots, j_k} t_{j_1} \dots t_{j_k} \otimes v_{j_1 i_1} \dots v_{j_k i_k} \\ &= \sum_{j_1, \dots, j_k} \mu_x(t_{j_1} \dots t_{j_k}) v_{j_1 i_1} \dots v_{j_k i_k} \\ &= \sum_{j_1 \dots j_k} tr(x_{j_1} \dots x_{j_k}) v_{j_1 i_1} \dots v_{j_k i_k} \end{aligned}$$

On the other hand, by definition of the distribution μ_x , we have:

$$\mu_x(P) = \mu_x(t_{i_1} \dots t_{i_k}) = tr(x_{i_1} \dots x_{i_k})$$

Thus, we are led to the conclusion in the statement. □

As already mentioned after Definition 15.2, in the classical case, where $G \subset O_N$ is a usual compact group, our notion of G -invariance coincides with the usual G -invariance notion from classical probability. We have in fact the following result:

PROPOSITION 15.4. *In the classical group case, $G \subset O_N$, a sequence (x_1, \dots, x_N) is G -invariant in the above sense if and only if*

$$tr(x_{i_1} \dots x_{i_k}) = \sum_{j_1 \dots j_k} g_{j_1 i_1} \dots g_{j_k i_k} tr(x_{j_1} \dots x_{j_k})$$

for any $k \in \mathbb{N}$, any $i_1, \dots, i_k \in \{1, \dots, N\}$, and any $g = (g_{ij}) \in G$, and this coincides with the usual notion of G -invariance for a sequence of classical random variables.

PROOF. According to Proposition 15.3, the invariance property happens precisely when we have the following equality, for any $k \in \mathbb{N}$, and any $i_1, \dots, i_k \in \{1, \dots, N\}$:

$$\operatorname{tr}(x_{i_1} \dots x_{i_k}) = \sum_{j_1 \dots j_k} \operatorname{tr}(x_{j_1} \dots x_{j_k}) v_{j_1 i_1} \dots v_{j_k i_k}$$

Now by evaluating both sides of this equation at a given $g \in G$, we obtain:

$$\operatorname{tr}(x_{i_1} \dots x_{i_k}) = \sum_{j_1 \dots j_k} g_{j_1 i_1} \dots g_{j_k i_k} \operatorname{tr}(x_{j_1} \dots x_{j_k})$$

Thus, we are led to the conclusion in the statement. \square

Summarizing, what we have so far is a general notion of probabilistic invariance, generalizing the classical notions of exchangeability and rotatability, than we can use for reformulating the classical De Finetti problematics, and its various generalizations.

In order to formulate De Finetti type theorems, that we can try to prove afterwards, we are still in need of a few pieces of general theory. Indeed, in the classical De Finetti theorem, the independence occurs after conditioning. Likewise, we can expect the free De Finetti theorem to be a statement about freeness with amalgamation.

Both these concepts may be expressed in terms of operator-valued probability theory, that we will recall now. There are many things to be said here, and in what follows we will mainly present the main definitions and theorems, with some brief explanations. Following Speicher's paper [80], we first have the following definition:

DEFINITION 15.5. *An operator-valued probability space consists of:*

- (1) *A unital algebra A .*
- (2) *A unital subalgebra $B \subset A$.*
- (3) *An expectation $E : A \rightarrow B$, which must be unital, $E(1) = 1$, and satisfying*

$$E(b_1 a b_2) = b_1 E(a) b_2$$

for any $a \in A$, and any $b_1, b_2 \in B$.

As a basic example, which motivates the whole theory, we have the case where $A = L^\infty(X)$ is a usual algebra of classical random variables, and $B = L^\infty(Y)$ is a subalgebra. Here the expectation $E : A \rightarrow B$ is the usual one from classical probability.

Given an operator-valued probability space as above, the joint distribution of a family of variables $(x_i)_{i \in I}$ in the algebra A is by definition the following functional:

$$\begin{aligned} \mu_x : B \langle (t_i)_{i \in I} \rangle &\rightarrow B \\ P &\rightarrow E(P(x)) \end{aligned}$$

We refer to Speicher's paper [80] and related papers for more on all this, general results and examples, in relation with the operator-valued probability theory.

Next in line, we have the following key definition, also from [80]:

DEFINITION 15.6. *Let (A, B, E) be as above, and $(x_i)_{i \in I}$ be a family of variables.*

- (1) *These variables are called independent if the following algebra is commutative*

$$\langle B, (x_i)_{i \in I} \rangle \subset A$$

and for $i_1, \dots, i_k \in I$ distinct and $P_1, \dots, P_k \in B \langle t \rangle$, we have:

$$E(P_1(x_{i_1}) \dots P_k(x_{i_k})) = E(P_1(x_{i_1})) \dots E(P_k(x_{i_k}))$$

- (2) *These variables are called free if for any $i_1, \dots, i_k \in I$ such that $i_l \neq i_{l+1}$, and any $P_1, \dots, P_k \in B \langle t \rangle$ such that $E(P_l(x_{i_l})) = 0$, we have:*

$$E(P_1(x_{i_1}) \dots P_k(x_{i_k})) = 0$$

The above notions are straightforward extensions of the usual notions of independence and freeness, that we discussed in chapter 9, which correspond to the case $B = \mathbb{C}$.

As in the scalar case, $B = \mathbb{C}$, in order to deal with invariance questions, we will need the theory of classical and free cumulants, in the present setting. Let us start with:

DEFINITION 15.7. *Let (A, B, E) be an operator-valued probability space.*

- (1) *A B -functional is a N -linear map $\rho : A^N \rightarrow B$ such that:*

$$\rho(b_0 a_1 b_1, a_2 b_2 \dots, a_N b_N) = b_0 \rho(a_1, b_1 a_2, \dots, b_{N-1} a_N) b_N$$

Equivalently, ρ is a linear map of the following type

$$A^{\otimes_B N} \rightarrow B$$

where the tensor product is taken with respect to the natural $B - B$ bimodule structure on the algebra A .

- (2) *Suppose that B is commutative. For $k \in \mathbb{N}$ let $\rho^{(k)}$ be a B -functional. Given $\pi \in P(n)$, we define a B -functional $\rho^{(\pi)} : A^N \rightarrow B$ by the formula*

$$\rho^{(\pi)}(a_1, \dots, a_N) = \prod_{V \in \pi} \rho(V)(a_1, \dots, a_N)$$

where if $V = (i_1 < \dots < i_s)$ is a block of π then:

$$\rho(V)(a_1, \dots, a_N) = \rho_s(a_{i_1}, \dots, a_{i_s})$$

As before with the notions of independence and freeness, these are classical extensions of the notions that we discussed in chapter 12 above. See [80].

When B is not commutative, there is no natural order in which to compute the product appearing in the above formula for $\rho^{(\pi)}$. However, the nesting property of the noncrossing partitions allows for a natural definition of $\rho^{(\pi)}$ for $\pi \in NC(N)$, which we now recall:

DEFINITION 15.8. For $k \in \mathbb{N}$ let $\rho^{(k)} : A^k \rightarrow B$ be a B -functional. Given $\pi \in NC(N)$, define a B -functional $\rho^{(N)} : A^N \rightarrow B$ recursively as follows:

- (1) If $\pi = 1_N$ is the partition having one block, define $\rho^{(\pi)} = \rho^{(N)}$.
- (2) Otherwise, let $V = \{l + 1, \dots, l + s\}$ be an interval of π and define:

$$\rho^{(\pi)}(a_1, \dots, a_N) = \rho^{(\pi-V)}(a_1, \dots, a_l \rho^{(s)}(a_{l+1}, \dots, a_{l+s}), a_{l+s+1}, \dots, a_N)$$

As before, we refer to [71], [80] and related work for more on all this.

Finally, we have the following definition:

DEFINITION 15.9. Let $(x_i)_{i \in I}$ be a family of random variables in A .

- (1) The operator-valued classical cumulants $c_E^{(k)} : A^k \rightarrow B$ are the B -functionals defined by the following classical moment-cumulant formula:

$$E(a_1 \dots a_N) = \sum_{\pi \in P(N)} c_E^{(\pi)}(a_1, \dots, a_N)$$

- (2) The operator-valued free cumulants $\kappa_E^{(k)} : A^k \rightarrow B$ are the B -functionals defined by the following free moment-cumulant formula:

$$E(a_1, \dots, a_N) = \sum_{\pi \in NC(N)} \kappa_E^{(\pi)}(a_1, \dots, a_N)$$

As basic illustrations here, in the scalar case, where the subalgebra is $B = \mathbb{C}$, we recover in this way the classical and free cumulants, as discussed in chapter 12 above. In general, we refer to [80] for more on the above notions.

We have the following result, which is well-known in the classical case, due to Rota, and which in the free case is due to Speicher [80]:

THEOREM 15.10. Let $(x_i)_{i \in I}$ a family of random variables in A .

- (1) If the algebra $\langle B, (x_i)_{i \in I} \rangle$ is commutative, then $(x_i)_{i \in I}$ are conditionally independent given B if and only if when there are $1 \leq k, l \leq N$ such that $i_k \neq i_l$:

$$c_E^{(N)}(b_0 x_{i_1} b_1, \dots, x_{i_N} b_N) = 0$$

- (2) The variables $(x_i)_{i \in I}$ are free with amalgamation over B if and only if when there are $1 \leq k, l \leq N$ such that $i_k \neq i_l$:

$$\kappa_E^{(N)}(b_0 x_{i_1} b_1, \dots, x_{i_N} b_N) = 0$$

PROOF. As a first observation, the condition in (1) is equivalent to the statement that if $\pi \in P(N)$, then the following happens, unless $\pi \leq \ker i$:

$$c_E^{(\pi)}(b_0 x_{i_1} b_1, \dots, x_{i_N} b_N) = 0$$

Similarly, the condition (2) above is equivalent to the statement that if $\pi \in NC(N)$, then the following happens, unless $\pi \leq \ker i$:

$$\kappa_E^{(\pi)}(b_0 x_{i_1} b_1, \dots, x_{i_N} b_N) = 0$$

Observe also that in the case $B = \mathbb{C}$ we obtain the usual notions of independence and freeness. In general now, the proof is via standard combinatorics, following the proof from the case $B = \mathbb{C}$, and as before, we refer to [71], [80] for more on all this. \square

Stronger characterizations of the joint distribution of $(x_i)_{i \in I}$ can be given by specifying what types of partitions may contribute to the nonzero cumulants.

To be more precise, we have here the following result, also from [80]:

THEOREM 15.11. *Let $(x_i)_{i \in I}$ be a family of random variables in A .*

- (1) *Suppose that $\langle B, (x_i)_{i \in I} \rangle$ is commutative. The B -valued joint distribution of $(x_i)_{i \in I}$ is independent for $D = P$ and independent centered Gaussian for $D = P_2$ if and only if, for any $\pi \in P(N)$, unless $\pi \in D(N)$ and $\pi \leq \ker i$:*

$$c_E^{(\pi)}(b_0 x_{i_1} b_1, \dots, x_{i_N} b_N) = 0$$

- (2) *The B -valued joint distribution of $(x_i)_{i \in I}$ is freely independent for $D = NC$ and freely independent centered semicircular for $D = NC_2$ if and only if, for any $\pi \in NC(N)$, unless $\pi \in D(N)$ and $\pi \leq \ker i$:*

$$\kappa_E^{(\pi)}(b_0 x_{i_1} b_1, \dots, x_{i_N} b_N) = 0$$

PROOF. These results are indeed well-known, coming from the definition of the classical and free cumulants, in the present setting, via some combinatorics. See [80]. \square

Finally, here is one more basic result that we will need:

THEOREM 15.12. *Let $(x_i)_{i \in I}$ be a family of random variables. Define the B -valued moment functionals $E^{(N)}$ by the following formula:*

$$E^{(N)}(a_1, \dots, a_N) = E(a_1 \dots a_N)$$

- (1) *If B is commutative, then for any $\sigma \in P(N)$ and $a_1, \dots, a_N \in A$ we have:*

$$c_E^{(\sigma)}(a_1, \dots, a_N) = \sum_{\pi \in P(N), \pi \leq \sigma} \mu_{P(N)}(\pi, \sigma) E^{(\pi)}(a_1, \dots, a_N)$$

- (2) *For any $\sigma \in NC(N)$ and $a_1, \dots, a_N \in A$ we have:*

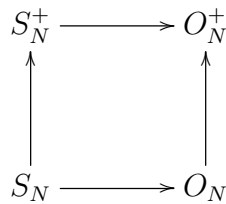
$$\kappa_E^{(\sigma)}(a_1, \dots, a_N) = \sum_{\pi \in NC(N), \pi \leq \sigma} \mu_{NC(N)}(\pi, \sigma) E^{(\pi)}(a_1, \dots, a_N)$$

PROOF. This follows indeed from the Möbius inversion formula. See [71], [80]. \square

This was the general operator-valued free probability theory that we will need, in what follows. For the detailed proofs, examples and comments on all the above, and for more operator-valued free probability in general, we refer to [71], [80].

15b. Reverse De Finetti

With the above ingredients in hand, we can now investigate invariance questions for the sequences of classical or noncommutative random variables, with respect to the main quantum permutation and rotation groups that we are interested in here, namely:



To be more precise, we first have a reverse De Finetti theorem, from [17], as follows:

THEOREM 15.13. *Let (x_1, \dots, x_N) be a sequence in A .*

- (1) *If x_1, \dots, x_N are freely independent and identically distributed with amalgamation over B , then the sequence is S_N^+ -invariant.*
- (2) *If x_1, \dots, x_N are freely independent and identically distributed with amalgamation over B , and have centered semicircular distributions with respect to E , then the sequence is O_N^+ -invariant.*
- (3) *If $\langle B, x_1, \dots, x_N \rangle$ is commutative and x_1, \dots, x_N are conditionally independent and identically distributed given B , then the sequence is S_N -invariant.*
- (4) *If $\langle x_1, \dots, x_N \rangle$ is commutative and x_1, \dots, x_N are conditionally independent and identically distributed given B , and have centered Gaussian distributions with respect to E , then the sequence is O_N -invariant.*

PROOF. Assume that the joint distribution of (x_1, \dots, x_N) satisfies one of the conditions in the statement, and let D be the category of partitions associated to the corresponding easy quantum group. We have then the following computation:

$$\begin{aligned}
 \sum_{j_1 \dots j_k} \text{tr}(x_{j_1} \dots x_{j_k}) v_{j_1 i_1} \dots v_{j_k i_k} &= \sum_{j_1 \dots j_k} \text{tr}(E(x_{j_1} \dots x_{j_k})) v_{j_1 i_1} \dots v_{j_k i_k} \\
 &= \sum_{j_1 \dots j_k} \sum_{\pi \leq \ker j} \text{tr}(\xi_E^{(\pi)}(x_1, \dots, x_1)) v_{j_1 i_1} \dots v_{j_k i_k} \\
 &= \sum_{\pi \in D(k)} \text{tr}(\xi_E^{(\pi)}(x_1, \dots, x_1)) \sum_{\ker j \geq \pi} v_{j_1 i_1} \dots v_{j_k i_k}
 \end{aligned}$$

Here ξ denotes the free and classical cumulants in the cases (1,2) and (3,4) respectively. On the other hand, it follows from a direct computation that if $\pi \in D(k)$ then we have

the following formula, in each of the 4 cases in the statement:

$$\sum_{\ker j \geq \pi} v_{j_1 i_1} \cdots v_{j_k i_k} = \begin{cases} 1 & \text{if } \pi \leq \ker i \\ 0 & \text{otherwise} \end{cases}$$

By using this formula, we can finish our computation, in the following way:

$$\begin{aligned} \sum_{j_1 \cdots j_k} \text{tr}(x_{j_1} \cdots x_{j_k}) v_{j_1 i_1} \cdots v_{j_k i_k} &= \sum_{\pi \in D(k)} \text{tr}(\xi_E^{(\pi)}(x_1, \dots, x_1)) \delta_{\pi \leq \ker i} \\ &= \sum_{\pi \leq \ker i} \text{tr}(\xi_E^{(\pi)}(x_1, \dots, x_1)) \\ &= \text{tr}(x_{i_1} \cdots x_{i_k}) \end{aligned}$$

Thus, we are led to the conclusions in the statement. □

Summarizing, we have so far a reverse De Finetti theorem, for the various quantum groups that we are interested in here. Our goal in what follows will be that of proving the corresponding De Finetti theorems, which are converse to the above theorem.

This will be something quite technical, getting us, among others, into certain technical aspects of the Weingarten integration and combinatorics.

Let us begin with some technical results, in view to establish the above-mentioned converse De Finetti theorems. We will use the following standard fact:

PROPOSITION 15.14. *Assume that a sequence (x_1, \dots, x_N) is G -invariant. Then there is a coaction*

$$\tilde{\alpha} : M_N(\mathbb{C}) \rightarrow M_N(\mathbb{C}) \otimes C(G)$$

determined by the following formula:

$$\tilde{\alpha}(p(x)) = (ev_x \otimes \pi_N)\alpha(p)$$

Moreover, the fixed point algebra of $\tilde{\alpha}$ is the G -invariant subalgebra B_N .

PROOF. This follows indeed after identifying the GNS representation of the algebra $\mathbb{C} \langle t_1, \dots, t_N \rangle$ for the state μ_x with the morphism $ev_x : \mathbb{C} \langle t_1, \dots, t_N \rangle \rightarrow M_N(\mathbb{C})$. □

In order to further advance, we use the fact that there is a natural conditional expectation given by integrating the coaction $\tilde{\alpha}$ with respect to the Haar state, as follows:

$$\begin{aligned} E_N : M_N(\mathbb{C}) &\rightarrow B_N \\ E_N(m) &= \left(id \otimes \int_G \right) \tilde{\alpha}(m) \end{aligned}$$

The point now is that by using the Weingarten formula, we can give a simple combinatorial formula for the moment functionals with respect to E_N , in the case where G is one of the easy quantum groups under consideration.

To be more precise, we have the following result, from [17]:

THEOREM 15.15. *Assume that (x_1, \dots, x_N) is G -invariant, and that either we have $G = O_N^+, S_N^+$, or that $G = O_N, S_N$ and (x_1, \dots, x_N) commute. We have then*

$$E_N^{(\pi)}(b_0x_1b_1, \dots, x_1b_k) = \frac{1}{N^{|\pi|}} \sum_{\pi \leq \ker i} b_0x_{i_1} \dots bx_{i_k}b_k$$

for any π in the partition category $D(k)$ for G , and any $b_0, \dots, b_k \in B_N$.

PROOF. We prove this result by recurrence on the number of blocks of π . First suppose that $\pi = 1_k$ is the partition with only one block. Then:

$$\begin{aligned} E_N^{(1_k)}(b_0x_1b_1, \dots, x_1b_k) &= E_N(b_0x_1 \dots x_1b_k) \\ &= \sum_{i_1 \dots i_k} b_0x_{i_1} \dots x_{i_k}b_k \int_G v_{i_1 1} \dots v_{i_k 1} \end{aligned}$$

Here we have used the fact that the elements b_0, \dots, b_k are fixed by the coaction $\tilde{\alpha}$. Applying now the Weingarten integration formula, we have:

$$\begin{aligned} E_N(b_0x_1 \dots x_1b_k) &= \sum_{i_1 \dots i_k} b_0x_{i_1} \dots x_{i_k}b_k \sum_{\pi \leq \ker i} \sum_{\sigma} W_{kN}(\pi, \sigma) \\ &= \sum_{\pi \in D(k)} \left(\sum_{\sigma \in D(k)} W_{kN}(\pi, \sigma) \right) \sum_{\pi \leq \ker i} b_0x_{i_1} \dots x_{i_k}b_k \end{aligned}$$

Now observe that for any $\sigma \in D(k)$ we have the following formula:

$$G_{kN}(\sigma, 1_k) = N^{|\sigma \vee 1_k|} = N$$

It follows that for any partition $\pi \in D(k)$, we have:

$$\begin{aligned} N \sum_{\sigma \in D(k)} W_{kN}(\pi, \sigma) &= \sum_{\sigma \in D(k)} W_{kN}(\pi, \sigma) G_{kN}(\sigma, 1_k) \\ &= \delta_{\pi 1_k} \end{aligned}$$

Applying this in the above context, we find, as desired:

$$\begin{aligned} E_N(b_0x_1 \dots x_1b_k) &= \sum_{\pi \in D(k)} \frac{1}{N} \delta_{\pi 1_k} \sum_{\pi \leq \ker i} b_0x_{i_1} \dots x_{i_k}b_k \\ &= \frac{1}{N} \sum_{i=1}^N b_0x_i \dots x_i b_k \end{aligned}$$

If the condition (3) or (4) is satisfied, then the general case follows from:

$$E_N^{(\pi)}(b_0x_1b_1, \dots, x_1b_k) = b_1 \dots b_k \prod_{V \in \pi} E_N(V)(x_1, \dots, x_1)$$

Indeed, the one thing that we must check here is that if $\pi \in D(k)$ and V is a block of π with s elements, then $1_s \in D(s)$. But this is easily verified, in each case.

Assume now that the condition (1) or (2) is satisfied. Let $\pi \in D(k)$. Since π is noncrossing, π contains an interval $V = \{l + 1, \dots, l + s + 1\}$, and we have:

$$\begin{aligned} & E_N^{(\pi)}(b_0 x_1 b_1, \dots, x_1 b_k) \\ &= E_N^{(\pi - V)}(b_0 x_1 b_1, \dots, E_N(x_1 b_{l+1} \dots x_1 b_{l+s}) x_1, \dots, x_1 b_k) \end{aligned}$$

To apply induction, we must check that we have $\pi - V \in D(k - s)$ and $1_s \in D(s)$. Indeed, this is easily verified for NC, NC_2 . Applying induction, we have:

$$\begin{aligned} & E_N^{(\pi)}(b_0 x_1 b_1, \dots, x_1 b_k) \\ &= \frac{1}{N^{|\pi|-1}} \sum_{\pi - V \leq_{\ker i}} b_0 x_{i_1} \dots b_l (E_n(x_1 b_{l+1} \dots x_1 b_{l+s})) x_{i_{l+s}} \dots x_{i_k} b_k \\ &= \frac{1}{N^{|\pi|-1}} \sum_{\pi - V \leq_{\ker i}} b_0 x_{i_1} \dots b_l \left(\frac{1}{N} \sum_{i=1}^N x_i b_{l+1} \dots b x_i b_{l+s} \right) x_{i_{l+s}} \dots x_{i_k} b_k \\ &= \frac{1}{N^{|\pi|}} \sum_{\pi \leq_{\ker i}} b_0 x_{i_1} \dots x_{i_k} b_k \end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Summarizing, we have so far reverse De Finetti theorems for the quantum groups that we are interested in here, along with some technical results, connecting the corresponding potential De Finetti theorems to the Weingarten function combinatorics.

15c. Weingarten estimates

In order to advance, we will need some standard Weingarten estimates for our quantum groups, which have their own interest, and that we will discuss now. So, consider the diagram formed by the main quantum permutation and quantum rotation groups:

$$\begin{array}{ccc} S_N^+ & \longrightarrow & O_N^+ \\ \uparrow & & \uparrow \\ S_N & \longrightarrow & O_N \end{array}$$

Regarding the symmetric group S_N , the situation here is very simple, because we can explicitly compute the Weingarten function, and estimate it, as follows:

PROPOSITION 15.16. *For S_N the Weingarten function is given by*

$$W_{kN}(\pi, \nu) = \sum_{\tau \leq \pi \wedge \nu} \mu(\tau, \pi) \mu(\tau, \nu) \frac{(N - |\tau|)!}{N!}$$

and satisfies the following estimate,

$$W_{kN}(\pi, \nu) = N^{-|\pi \wedge \nu|} (\mu(\pi \wedge \nu, \pi) \mu(\pi \wedge \nu, \nu) + O(N^{-1}))$$

with μ being the Möbius function of $P(k)$.

PROOF. The first assertion follows from the usual Weingarten formula, namely:

$$\int_{S_N} v_{i_1 j_1} \cdots v_{i_k j_k} = \sum_{\pi, \nu \in P(k)} \delta_\pi(i) \delta_\nu(j) W_{kN}(\pi, \nu)$$

Indeed, in this formula the integrals on the left are in fact known, from the explicit integration formula over S_N that we established before, namely:

$$\int_{S_N} g_{i_1 j_1} \cdots g_{i_k j_k} = \begin{cases} \frac{(N - |\ker i|)!}{N!} & \text{if } \ker i = \ker j \\ 0 & \text{otherwise} \end{cases}$$

But this allows the computation of the right term, via the Möbius inversion formula, explained before. As for the second assertion, this follows from the first one. \square

The above result is of course something very special, coming from the fact that the Haar integration over the permutation group S_N , save for being just an averaging, this group being finite, is something very simple, combinatorially speaking.

Regarding now the quantum group S_N^+ , that we are particularly interested in here, let us begin with some explicit computations. We first have the following simple and final result at $k = 2, 3$, directly in terms of the quantum group integrals:

PROPOSITION 15.17. *At $k = 2, 3$ we have the following estimate:*

$$\int_{S_N^+} u_{i_1 j_1} \cdots u_{i_k j_k} = \begin{cases} 0 & (\ker i \neq \ker j) \\ \simeq N^{-|\ker i|} & (\ker i = \ker j) \end{cases}$$

PROOF. Since at $k \leq 3$ we have $NC(k) = P(k)$, the Weingarten integration formulae for S_N and S_N^+ coincide, and we obtain, by using the above formula for S_N :

$$\begin{aligned} \int_{S_N^+} v_{i_1 j_1} \cdots v_{i_k j_k} &= \int_{S_N} v_{i_1 j_1} \cdots v_{i_k j_k} \\ &= \delta_{\ker i, \ker j} \frac{(N - |\ker i|)!}{N!} \end{aligned}$$

Thus, we obtain the formula in the statement. \square

In general now, the idea will be that of working out a “master estimate” for the Weingarten function, as above. Before starting, let us record the formulae at $k = 2, 3$, which will be useful later, as illustrations. At $k = 2$, with indices $||, \sqcap$ as usual, and with the convention that \approx means componentwise dominant term, we have:

$$W_{2N} \approx \begin{pmatrix} N^{-2} & -N^{-2} \\ -N^{-2} & N^{-1} \end{pmatrix}$$

At $k = 3$ now, with indices $|||, |\sqcap, \sqcap|, \sqcap| \sqcap|$ as usual, and same meaning for \approx , we have:

$$W_{3N} \approx \begin{pmatrix} N^{-3} & -N^{-3} & -N^{-3} & -N^{-3} & 2N^{-3} \\ -N^{-3} & N^{-2} & N^{-3} & N^{-3} & -N^{-2} \\ -N^{-3} & N^{-3} & N^{-2} & N^{-3} & -N^{-2} \\ -N^{-3} & N^{-3} & N^{-3} & N^{-2} & -N^{-2} \\ 2N^{-3} & -N^{-2} & -N^{-2} & -N^{-2} & N^{-1} \end{pmatrix}$$

These formulae follow indeed from the plain formulae for the Weingarten matrix W_{kN} at $k = 2, 3$ from [13] and related papers, after rearranging the matrix indices as above.

Observe in particular, in the context of the above computations, that we have the following formula, which will be of interest in what follows:

$$W_{3N}(|\sqcap, \sqcap|) \simeq N^{-3}$$

In order to deal now with the general case, let us start with some standard facts:

PROPOSITION 15.18. *The following happen, regarding the partitions in $P(k)$:*

- (1) $|\pi| + |\nu| \leq |\pi \vee \nu| + |\pi \wedge \nu|$.
- (2) $|\pi \vee \tau| + |\tau \vee \nu| \leq |\pi \vee \nu| + |\tau|$.
- (3) $d(\pi, \nu) = \frac{|\pi| + |\nu|}{2} - |\pi \vee \nu|$ is a distance.

PROOF. All this is well-known, the idea being as follows:

- (1) This is well-known, coming from the fact that $P(k)$ is a semi-modular lattice.
- (2) This follows from (1), as explained for instance in the paper [17].
- (3) This follows from (2) above, which says that the following holds:

$$\begin{aligned} & \frac{|\pi| + |\tau|}{2} - d(\pi, \tau) + \frac{|\tau| + |\nu|}{2} - d(\tau, \nu) \\ & \leq \frac{|\pi| + |\nu|}{2} - d(\pi, \nu) + |\tau| \end{aligned}$$

Thus, we obtain in this way the triangle inequality:

$$d(\pi, \tau) + d(\tau, \nu) \geq d(\pi, \nu)$$

As for the other axioms for a distance, these are all clear. □

Actually in what follows we will only need (3) in the above statement. For more on this, and on the geometry and combinatorics of partitions, we refer to [71].

As a main result now regarding the Weingarten functions, we have:

THEOREM 15.19. *The Weingarten matrix W_{kN} has a series expansion in N^{-1} ,*

$$W_{kN}(\pi, \nu) = N^{|\pi \vee \nu| - |\pi| - |\nu|} \sum_{g=0}^{\infty} K_g(\pi, \nu) N^{-g}$$

where the various objects on the right are defined as follows:

(1) A path from π to ν is a sequence as follows:

$$p = [\pi = \tau_0 \neq \tau_1 \neq \dots \neq \tau_r = \nu]$$

(2) The signature of such a path is $+$ when r is even, and $-$ when r is odd.

(3) The geodesicity defect of such a path is:

$$g(p) = \sum_{i=1}^r d(\tau_{i-1}, \tau_i) - d(\pi, \nu)$$

(4) K_g counts the signed paths from π to ν , with geodesicity defect g .

PROOF. We recall that the Weingarten matrix W_{kN} appears as the inverse of the Gram matrix G_{kN} , which is given by the following formula:

$$G_{kN}(\pi, \nu) = N^{|\pi \vee \nu|}$$

Now observe that the Gram matrix can be written in the following way:

$$\begin{aligned} G_{kN}(\pi, \nu) &= N^{|\pi \vee \nu|} \\ &= N^{\frac{|\pi|}{2}} N^{|\pi \vee \nu| - \frac{|\pi| + |\nu|}{2}} N^{\frac{|\nu|}{2}} \\ &= N^{\frac{|\pi|}{2}} N^{-d(\pi, \nu)} N^{\frac{|\nu|}{2}} \end{aligned}$$

This suggests considering the following diagonal matrix:

$$\Delta = \text{diag}(N^{\frac{|\pi|}{2}})$$

So, let us do this, and consider as well the following matrix:

$$H(\pi, \nu) = \begin{cases} 0 & (\pi = \nu) \\ N^{-d(\pi, \nu)} & (\pi \neq \nu) \end{cases}$$

In terms of these two matrices, the above formula for G_{kN} simply reads:

$$G_{kN} = \Delta(1 + H)\Delta$$

Thus, the Weingarten matrix W_{kN} is given by the following formula:

$$W_{kN} = \Delta^{-1}(1 + H)^{-1}\Delta^{-1}$$

In order to compute now the inverse of $1 + H$, we will use the following formula:

$$(1 + H)^{-1} = 1 - H + H^2 - H^3 + \dots$$

Consider indeed the set $P_r(\pi, \nu)$ of length r paths between π and ν . We have:

$$\begin{aligned} H^r(\pi, \nu) &= \sum_{p \in P_r(\pi, \nu)} H(\tau_0, \tau_1) \dots H(\tau_{r-1}, \tau_r) \\ &= \sum_{p \in P_r(\pi, \nu)} N^{-d(\pi, \nu) - g(p)} \end{aligned}$$

Thus by using $(1 + H)^{-1} = 1 - H + H^2 - H^3 + \dots$ we obtain:

$$\begin{aligned} (1 + H)^{-1}(\pi, \nu) &= \sum_{r=0}^{\infty} (-1)^r H^r(\pi, \nu) \\ &= N^{-d(\pi, \nu)} \sum_{r=0}^{\infty} \sum_{p \in P_r(\pi, \nu)} (-1)^r N^{-g(p)} \end{aligned}$$

It follows that the Weingarten matrix is given by the following formula:

$$\begin{aligned} W_{kN}(\pi, \nu) &= \Delta^{-1}(\pi)(1 + H)^{-1}(\pi, \nu)\Delta^{-1}(\nu) \\ &= N^{-\frac{|\pi|}{2} - \frac{|\nu|}{2} - d(\pi, \nu)} \sum_{r=0}^{\infty} \sum_{p \in P_r(\pi, \nu)} (-1)^r N^{-g(p)} \\ &= N^{|\pi \vee \nu| - |\pi| - |\nu|} \sum_{r=0}^{\infty} \sum_{p \in P_r(\pi, \nu)} (-1)^r N^{-g(p)} \end{aligned}$$

Now by rearranging the various terms in the above double sum according to their geodesicity defect $g = g(p)$, this gives the following formula:

$$W_{kN}(\pi, \nu) = N^{|\pi \vee \nu| - |\pi| - |\nu|} \sum_{g=0}^{\infty} K_g(\pi, \nu) N^{-g}$$

Thus, we are led to the conclusion in the statement. □

As an illustration for all this, we have the following explicit estimates:

THEOREM 15.20. *Consider an easy quantum group $G = (G_N)$, coming from a category of partitions $D = (D(k))$. For any $\pi \leq \nu$ we have the estimate*

$$W_{kN}(\pi, \nu) = N^{-|\pi|}(\mu(\pi, \nu) + O(N^{-1}))$$

and for π, ν arbitrary we have

$$W_{kN}(\pi, \nu) = O(N^{|\pi \vee \nu| - |\pi| - |\nu|})$$

with μ being the Möbius function of $D(k)$.

PROOF. We have two assertions here, the idea being as follows:

(1) The first estimate is clear from the general expansion formula established in Theorem 15.19, namely:

$$W_{kN}(\pi, \nu) = N^{|\pi \vee \nu| - |\pi| - |\nu|} \sum_{g=0}^{\infty} K_g(\pi, \nu) N^{-g}$$

(2) In the case $\pi \leq \nu$ it is known that K_0 coincides with the Möbius function of $NC(k)$, as explained for instance in [17], so we obtain once again from Theorem 15.19 the fine estimate in the statement as well, namely:

$$W_{kN}(\pi, \nu) = N^{-|\pi|} (\mu(\pi, \nu) + O(N^{-1})) \quad \forall \pi \leq \nu$$

Observe that, by symmetry of W_{kN} , we obtain as well that we have:

$$W_{kN}(\pi, \nu) = N^{-|\nu|} (\mu(\nu, \pi) + O(N^{-1})) \quad \forall \pi \geq \nu$$

Thus, we are led to the conclusions in the statement. \square

When π, ν are not comparable by \leq , things are quite unclear. The simplest example appears at $k = 3$, where we have the following formula, which is elementary:

$$W_{3N}(|\square, \square|) \simeq N^{-3}$$

Observe that the exponent -3 is precisely the dominant one, and this because:

$$\left| |\square \vee \square| \right| - \left| |\square| \right| - \left| |\square| \right| = 1 - 2 - 2 = -3$$

As for the corresponding coefficient, $K_0(|\square, \square|) = 1$, this is definitely not the Möbius function, which vanishes for partitions which are not comparable by \leq . According to Theorem 15.19, this is rather the number of signed geodesic paths from $|\square$ to $|\square|$.

In relation to all this, observe that geometrically, $NC(5)$ consists of the partitions $|\square, \square|, \sqcap$ which form an equilateral triangle with edges worth 1, and then the partitions $|||, \square\square$, which are at distance 1 apart, and each at distance $1/2$ from each of the vertices of the triangle. It is not obvious how to recover the formula $K_0(|\square, \square|) = 1$ from this.

Finally, also following [17], we will need as well the following result:

PROPOSITION 15.21. *We have the following results:*

- (1) *If $D = NC, NC_2$, then $\mu_{D(k)}(\pi, \nu) = \mu_{NC(k)}(\pi, \nu)$.*
- (2) *If $D = P, P_2$ then $\mu_{D(k)}(\pi, \nu) = \mu_{P(k)}(\pi, \nu)$.*

PROOF. Let $Q = NC, P$ according to the cases (1,2) above. It is easy to see in each case that $D(k)$ is closed under taking intervals in $Q(k)$, in the sense that if $\pi_1, \pi_2 \in D(k)$, $\nu \in Q(k)$ and $\pi_1 < \nu < \pi_2$ then $\nu \in D(k)$. With this observation in hand, the result now follows from the definition of the Möbius function. See [17]. \square

15d. De Finetti theorems

With the above ingredients in hand, let us go back now to invariance questions with respect to the main quantum permutation and rotation groups, namely:

$$\begin{array}{ccc}
 S_N^+ & \longrightarrow & O_N^+ \\
 \uparrow & & \uparrow \\
 S_N & \longrightarrow & O_N
 \end{array}$$

More generally, we would like in fact to have, ideally, de Finetti type theorems for all the easy quantum groups that we know, from the previous chapters. This is of course something quite technical, and time consuming, but we would like at least to understand what happens for the main quantum reflection and rotation groups, namely:

$$\begin{array}{ccccc}
 & & K_N^+ & \longrightarrow & U_N^+ \\
 & \nearrow & \uparrow & & \nearrow \\
 H_N^+ & \longrightarrow & O_N^+ & & \\
 \uparrow & & \uparrow & & \uparrow \\
 & & K_N & \longrightarrow & U_N \\
 \nearrow & & \uparrow & & \nearrow \\
 H_N & \longrightarrow & O_N & &
 \end{array}$$

In order to discuss these questions, or at least some of them, let us start with a basic approximation result for the finite sequences, in the real case, from [17], as follows:

THEOREM 15.22. *Suppose that (x_1, \dots, x_N) is G_N -invariant, and that $G_N = O_N^+, S_N^+$, or that $G_N = O_N, S_N$ and (x_1, \dots, x_N) commute. Let (y_1, \dots, y_N) be a sequence of B_N -valued random variables with B_N -valued joint distribution determined as follows:*

- (1) $G = O^+$: Free semicircular, centered with same variance as x_1 .
- (2) $G = S^+$: Freely independent, y_i has same distribution as x_1 .
- (3) $G = O$: Independent Gaussian, centered with same variance as x_1 .
- (4) $G = S$: Independent, y_i has same distribution as x_1 .

Then if $1 \leq j_1, \dots, j_k \leq N$ and $b_0, \dots, b_k \in B_N$, we have the following estimate,

$$\|E_N(b_0 x_{j_1} \dots x_{j_k} b_k) - E(b_0 y_{j_1} \dots y_{j_k} b_k)\| \leq \frac{C_k(G)}{N} \|x_1\|^k \|b_0\| \dots \|b_k\|$$

with $C_k(G)$ being a constant depending only on k and G .

PROOF. First we note that it suffices to prove the result for N large enough. We will assume that N is sufficiently large, as for the Gram matrix G_{kN} to be invertible.

Let $1 \leq j_1, \dots, j_k \leq N$ and $b_0, \dots, b_k \in B_N$. We have then:

$$\begin{aligned} E_N(b_0 x_{j_1} \dots x_{j_k} b_k) &= \sum_{i_1 \dots i_k} b_0 x_{i_1} \dots x_{i_k} b_k \int v_{i_1 j_1} \dots v_{i_k j_k} \\ &= \sum_{i_1 \dots i_k} b_0 x_{i_1} \dots x_{i_k} b_k \sum_{\pi \leq \ker i} \sum_{\sigma \leq \ker j} W_{kN}(\pi, \sigma) \\ &= \sum_{\sigma \leq \ker j} \sum_{\pi} W_{kN}(\pi, \sigma) \sum_{\pi \leq \ker i} b_0 x_{i_1} \dots x_{i_k} b_k \end{aligned}$$

On the other hand, it follows from our assumptions on (y_1, \dots, y_N) , and from the various moment-cumulant formulae given before, that we have:

$$E(b_0 y_{j_1} \dots y_{j_k} b_k) = \sum_{\sigma \leq \ker j} \xi_{E_N}^{(\sigma)}(b_0 x_1 b_1, \dots, x_1 b_k)$$

Here, and in what follows, ξ denote the relevant free or classical cumulants.

The right hand side can be expanded, via the Möbius inversion formula, in terms of expectation functionals of the following type, with π being a partition in NC, P according to the cases (1,2) or (3,4) in the statement, and with $\pi \leq \sigma$ for some $\sigma \in D(k)$:

$$E_N^{(\pi)}(b_0 x_1 b_1, \dots, x_1 b_k)$$

Now if $\pi \notin D(k)$, we claim that this expectation functional is zero.

Indeed this is only possible if $D = NC_2, P_2$, and if π has a block with an odd number of legs. But it is easy to see that in these cases x_1 has an even distribution with respect to E_N , and therefore we have, as claimed, the following formula:

$$E_N^{(\pi)}(b_0 x_1 b_1, \dots, x_1 b_k) = 0$$

Now this observation allows to to rewrite the above equation as follows:

$$E(b_0 y_{j_1} \dots y_{j_k} b_k) = \sum_{\sigma \leq \ker j} \sum_{\pi \leq \sigma} \mu_{D(k)}(\pi, \sigma) E_N^{(\pi)}(b_0 x_1 b_1, \dots, x_1 b_k)$$

We therefore obtain the following formula:

$$E(b_0 y_{j_1} \dots y_{j_k} b_k) = \sum_{\sigma \leq \ker j} \sum_{\pi \leq \sigma} \mu_{D(k)}(\pi, \sigma) N^{-|\pi|} \sum_{\pi \leq \ker i} b_0 x_{i_1} \dots x_{i_k} b_k$$

Comparing the above two equations, we find that:

$$\begin{aligned} &E_N(b_0 x_{j_1} \dots x_{j_k} b_k) - E(b_0 y_{j_1} \dots y_{j_k} b_k) \\ &= \sum_{\sigma \leq \ker j} \sum_{\pi} (W_{kN}(\pi, \sigma) - \mu_{D(k)}(\pi, \sigma) N^{-|\pi|}) \sum_{\pi \leq \ker i} b_0 x_{i_1} \dots x_{i_k} b_k \end{aligned}$$

Now since x_1, \dots, x_N are identically distributed with respect to the faithful state φ , it follows that these variables have the same norm. Thus, for any $\pi \in D(k)$:

$$\left\| \sum_{\pi \leq \ker i} b_0 x_{i_1} \dots x_{i_k} b_k \right\| \leq N^{|\pi|} \|x_1\|^k \|b_0\| \dots \|b_k\|$$

Combining this with the former equation, we obtain the following estimate:

$$\begin{aligned} & \left| E_N(b_0 x_{j_1} \dots x_{j_k} b_k) - E(b_0 y_{j_1} \dots y_{j_k} b_k) \right| \\ & \leq \sum_{\sigma \leq \ker j} \sum_{\pi} |W_{kN}(\pi, \sigma) N^{|\pi|} - \mu_{D(k)}(\pi, \sigma)| \|x_1\|^k \|b_0\| \dots \|b_k\| \end{aligned}$$

Let us set now, according to the above:

$$C_k(G) = \sup_{N \in \mathbb{N}} \left(N \times \sum_{\sigma, \pi \in D(k)} |W_{kN}(\pi, \sigma) N^{|\pi|} - \mu_{D(k)}(\pi, \sigma)| \right)$$

But this number is finite by our main estimate, which completes the proof. □

We will use in what follows the inclusions $G_N \subset G_M$ for $N < M$, which correspond to the Hopf algebra morphisms $\omega_{N,M} : C(G_M) \rightarrow C(G_N)$ given by:

$$\omega_{N,M}(u_{ij}) = \begin{cases} u_{ij} & \text{if } 1 \leq i, j \leq N \\ \delta_{ij} & \text{if } \max(i, j) > N \end{cases}$$

Still following [17], we begin by extending the notion of G_N -invariance to the infinite sequences of variables, in the following way:

DEFINITION 15.23. *Let $(x_i)_{i \in \mathbb{N}}$ be a sequence in a noncommutative probability space (A, φ) . We say that $(x_i)_{i \in \mathbb{N}}$ is G -invariant if*

$$(x_1, \dots, x_N)$$

is G_N -invariant for each $N \in \mathbb{N}$.

In other words, the condition is that the joint distribution of (x_1, \dots, x_N) should be invariant under the following coaction map, for each $N \in \mathbb{N}$:

$$\alpha_N : \mathbb{C} \langle t_1, \dots, t_N \rangle \rightarrow \mathbb{C} \langle t_1, \dots, t_N \rangle \otimes C(G_N)$$

It is convenient to extend these coactions to a coaction on the algebra of noncommutative polynomials on an infinite number of variables, in the following way:

$$\beta_N : \mathbb{C} \langle t_i | i \in \mathbb{N} \rangle \rightarrow \mathbb{C} \langle t_i | i \in \mathbb{N} \rangle \otimes C(G_N)$$

Indeed, we can define β_N to be the unique unital morphism satisfying:

$$\beta_N(t_j) = \begin{cases} \sum_{i=1}^N t_i \otimes v_{ij} & \text{if } 1 \leq j \leq N \\ t_j \otimes 1 & \text{if } j > N \end{cases}$$

It is clear that β_N as constructed above is a coaction of G_N . Also, we have the following relations, where $\iota_N : \mathbb{C} \langle t_1, \dots, t_N \rangle \rightarrow \mathbb{C} \langle t_i | i \in \mathbb{N} \rangle$ is the natural inclusion:

$$(id \otimes \omega_{N,M})\beta_M = \beta_N$$

$$(\iota_N \otimes id)\alpha_N = \beta_N \iota_N$$

By using these compatibility relations, we obtain the following result:

PROPOSITION 15.24. *An infinite sequence of random variables $(x_i)_{i \in \mathbb{N}}$ is G -invariant if and only if the joint distribution functional*

$$\mu_x : \mathbb{C} \langle t_i | i \in \mathbb{N} \rangle \rightarrow \mathbb{C}$$

$$P \rightarrow tr(P(x))$$

is invariant under the coaction β_N , for each $N \in \mathbb{N}$.

PROOF. This is clear indeed from the above discussion. □

In what follows $(x_i)_{i \in \mathbb{N}}$ will be a sequence of self-adjoint random variables in a von Neumann algebra (M, tr) . We will assume that M is generated by $(x_i)_{i \in \mathbb{N}}$. We denote by $L^2(M, tr)$ the corresponding GNS Hilbert space, with inner product as follows:

$$\langle m_1, m_2 \rangle = tr(m_1 m_2^*)$$

Also, the strong topology on M , that we will use in what follows, will be taken by definition with respect to the faithful representation on the space $L^2(M, tr)$.

We let P_N be the fixed point algebra of the action β_N , and we set:

$$B_N = \left\{ p(x) \mid p \in P_N \right\}''$$

We have then an inclusion $B_{N+1} \subset B_N$, for any $N \geq 1$, and we can then define the G -invariant subalgebra as the common intersection of these algebras:

$$B = \bigcap_{N \geq 1} B_N$$

With these conventions, we have the following result, from [17]:

PROPOSITION 15.25. *If an infinite sequence of random variables $(x_i)_{i \in \mathbb{N}}$ is G -invariant, then for each $N \in \mathbb{N}$ there is a coaction*

$$\tilde{\beta}_N : M \rightarrow M \otimes L^\infty(G_N)$$

determined by the following formula, for any $p \in \mathcal{P}_\infty$:

$$\tilde{\beta}_N(p(x)) = (ev_x \otimes \pi_N)\beta_N(p)$$

The fixed point algebra of $\tilde{\beta}_N$ is then B_N .

PROOF. This is indeed clear from definitions, and from the various compatibility formulae given above, between the coactions α_N and β_N . \square

We have as well the following result, which is clear as well:

PROPOSITION 15.26. *In the above context, that of an infinite sequence of random variables belonging to an arbitrary von Neumann algebra M with a trace*

$$(x_i)_{i \in \mathbb{N}}$$

which is G -invariant, for each $N \in \mathbb{N}$ there is a trace-preserving conditional expectation $E_N : M \rightarrow B_N$ given by integrating the action $\tilde{\beta}_N$:

$$E_N(m) = \left(id \otimes \int_G \right) \tilde{\beta}_N(m)$$

By taking the limit of these expectations as $N \rightarrow \infty$, we obtain a trace-preserving conditional expectation onto the G -invariant subalgebra.

PROOF. Once again, this is clear from definitions, and from the various compatibility formulae given above, between the coactions α_N and β_N . \square

We are now prepared to state and prove the main theorem, from [17], which comes as a complement to the reverse De Finetti theorem that we already established:

THEOREM 15.27. *Let $(x_i)_{i \in \mathbb{N}}$ be a G -invariant sequence of self-adjoint random variables in (M, tr) , and assume that $M = \langle (x_i)_{i \in \mathbb{N}} \rangle$. Then there exists a subalgebra $B \subset M$ and a trace-preserving conditional expectation $E : M \rightarrow B$ such that:*

- (1) *If $G = (S_N)$, then $(x_i)_{i \in \mathbb{N}}$ are conditionally independent and identically distributed given B .*
- (2) *If $G = (S_N^+)$, then $(x_i)_{i \in \mathbb{N}}$ are freely independent and identically distributed with amalgamation over B .*
- (3) *If $G = (O_N)$, then $(x_i)_{i \in \mathbb{N}}$ are conditionally independent, and have Gaussian distributions with mean zero and common variance, given B .*
- (4) *If $G = (O_N^+)$, then $(x_i)_{i \in \mathbb{N}}$ form a B -valued free semicircular family with mean zero and common variance.*

PROOF. We use the various partial results and formulae established above. Let $j_1, \dots, j_k \in \mathbb{N}$ and $b_0, \dots, b_k \in B$. We have then the following computation:

$$\begin{aligned} E(b_0 x_{j_1} \dots x_{j_k} b_k) &= \lim_{N \rightarrow \infty} E_N(b_0 x_{j_1} \dots x_{j_k} b_k) \\ &= \lim_{N \rightarrow \infty} \sum_{\sigma \leq \ker j} \sum_{\pi} W_{kN}(\pi, \sigma) \sum_{\pi \leq \ker i} b_0 x_{i_1} \dots x_{i_k} b_k \\ &= \lim_{N \rightarrow \infty} \sum_{\sigma \leq \ker j} \sum_{\pi \leq \sigma} \mu_{D(k)}(\pi, \sigma) N^{-|\pi|} \sum_{\pi \leq \ker i} b_0 x_{i_1} \dots x_{i_k} b_k \end{aligned}$$

Let us recall now from the above that we have the following compatibility formula, where $\tilde{\iota}_N : W^*(x_1, \dots, x_N) \rightarrow M$ is the canonical inclusion, and $\tilde{\alpha}_N$ is as before:

$$(\tilde{\iota}_N \otimes id)\tilde{\alpha}_N = \tilde{\beta}_N \tilde{\iota}_N$$

By using this formula, and the above cumulant results, we have:

$$E(b_0 x_{j_1} \dots x_{j_k} b_k) = \lim_{N \rightarrow \infty} \sum_{\sigma \leq \ker j} \sum_{\pi \leq \sigma} \mu_{D(k)}(\pi, \sigma) E_N^{(\pi)}(b_0 x_1 b_1, \dots, x_1 b_k)$$

We therefore obtain the following formula:

$$E(b_0 x_{j_1} \dots x_{j_k} b_k) = \sum_{\sigma \leq \ker j} \sum_{\pi \leq \sigma} \mu_{D(k)}(\pi, \sigma) E^{(\pi)}(b_0 x_1 b_1, \dots, x_1 b_k)$$

We can replace the sum of expectation functionals by cumulants, as to obtain:

$$E(b_0 x_{j_1} \dots x_{j_k} b_k) = \sum_{\sigma \leq \ker j} \xi_E^{(\sigma)}(b_0 x_1 b_1, \dots, x_1 b_k)$$

Here and in what follows ξ denotes as usual the relevant free or classical cumulants, depending on the quantum group that we are dealing with, free or classical.

Now since the cumulants are determined by the moment-cumulant formulae, we conclude that we have the following formula:

$$\xi_E^{(\sigma)}(b_0 x_{j_1} b_1, \dots, x_{j_k} b_k) = \begin{cases} \xi_E^{(\sigma)}(b_0 x_1 b_1, \dots, x_1 b_k) & \text{if } \sigma \in D(k) \text{ and } \sigma \leq \ker j \\ 0 & \text{otherwise} \end{cases}$$

With this formula in hand, the result then follows from the characterizations of these joint distributions in terms of cumulants, and we are done. □

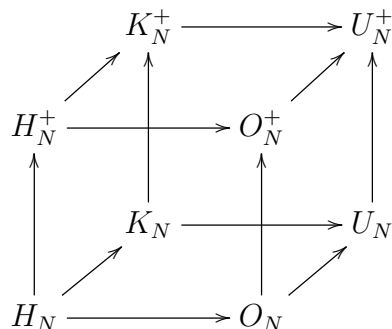
Summarizing, we are done with our first and main objective, namely establishing De Finetti theorems for the main quantum permutation and rotation groups, namely:

$$\begin{array}{ccc} S_N^+ & \longrightarrow & O_N^+ \\ \uparrow & & \uparrow \\ S_N & \longrightarrow & O_N \end{array}$$

The story is of course not over here, and there are many related interesting questions left, which are more technical, in relation with the invariance questions with respect to these quantum groups. We refer here to [17], [39], [40], [62], [63] and related papers.

Regarding now our second objective, which appears as a variation of this, fully in tune with the present book, we would like to understand as well what happens to the invariance

questions with respect to the basic quantum reflection and rotation groups, namely:



Here the answer is more or less known as well from [17], but with the problem however that the paper [17] is extremely general, and in relation with our cube question, more general than needed. In any case, for this and for further aspects of invariance questions, we refer as before to [17], [39], [40], [62], [63] and related papers.

15e. Exercises

Things have been quite technical in this chapter, dealing with advanced probability theory, and so will be our exercises here. As a first exercise, we have:

EXERCISE 15.28. *Formulate and prove the classical De Finetti theorem, concerning sequences which are invariant under S_∞ , without using representation theory methods.*

This is something very standard, and is a must-do exercise, the point being that all the Weingarten technology used in this chapter, which is something quite heavy, was motivated by the fact that we want to deal with several quantum groups at the same time, in a “uniform” way. In the case of the symmetric group itself things are in fact much simpler, and the exercise is about understanding how this works.

EXERCISE 15.29. *Formulate and prove the free De Finetti theorem, concerning sequences which are invariant under (S_N^+) , without using representation theory methods.*

The same comments as for the previous exercise apply, the idea being that, once again, the Weingarten function machinery can be avoided in this case.

EXERCISE 15.30. *Work out the full proof of the explicit formula for the Weingarten function for S_N , namely*

$$W_{kN}(\pi, \nu) = \sum_{\tau \leq \pi \wedge \nu} \mu(\tau, \pi) \mu(\tau, \nu) \frac{(N - |\tau|)!}{N!}$$

then of the main estimate for this function, namely

$$W_{kN}(\pi, \nu) = N^{-|\pi \wedge \nu|} (\mu(\pi \wedge \nu, \pi) \mu(\pi \wedge \nu, \nu) + O(N^{-1}))$$

where μ is the Möbius function of $P(k)$.

This was something that was already discussed in the above, the idea being that all this comes from the explicit knowledge of the integrals over S_N , via the Möbius inversion formula, and the problem now is that of working out all the details.

EXERCISE 15.31. *Work out estimates for the integrals of type*

$$\int_{S_N^+} v_{i_1 j_1} v_{i_2 j_2} v_{i_3 j_3} v_{i_4 j_4}$$

and then for the Weingarten function of S_N^+ at $k = 4$.

Once again, this was something partly discussed in the above, with the comment that things are clear at $k = 2, 3$, due to the formula $P(k) = NC(k)$ valid here. The problem now is that of working out what happens at $k = 4$, where things are non-trivial.

EXERCISE 15.32. *Prove directly that the function*

$$d(\pi, \nu) = \frac{|\pi| + |\nu|}{2} - |\pi \vee \nu|$$

is a distance on $P(k)$.

To be more precise here, this is something that we talked about in the above, with the idea being that this follows from a number of well-known facts regarding the partitions in $P(k)$. The problem now is that of proving directly this result.

CHAPTER 16

Subfactor theory

16a. Factors, subfactors

We have now a quite complete picture of free probability from a combinatorial point of view, in relation with basic questions from random matrices and quantum groups. In this final chapter we go for the real thing, namely discussing the connections between free probability and selected topics from von Neumann algebra theory.

We already know a few things about the algebras of operators $A \subset B(H)$ which are norm closed. The von Neumann algebras will be by definition those such algebras which are weakly closed. In order to discuss this, let us start with a standard result:

PROPOSITION 16.1. *For an algebra $A \subset B(H)$, the following are equivalent:*

- (1) *A is closed under the weak operator topology, making each of the linear maps $T \rightarrow \langle Tx, y \rangle$ continuous.*
- (2) *A is closed under the strong operator topology, making each of the linear maps $T \rightarrow Tx$ continuous.*

In the case where these conditions are satisfied, A is closed under the norm topology.

PROOF. There are several statements here, the proof being as follows:

(1) It is clear that the norm topology is stronger than the strong operator topology, which is in turn stronger than the weak operator topology. Thus, we are left with proving that for any operator algebra $A \subset B(H)$, strongly closed implies weakly closed.

(2) But this latter fact is standard, and can be proved by using an amplification trick. Consider the Hilbert space obtained by summing k times H with itself:

$$H^+ = H \oplus \dots \oplus H$$

The operators over H^+ can be regarded as being square matrices with entries in $B(H)$, and in particular, we have a representation $\pi : B(H) \rightarrow B(H^+)$, given by:

$$\pi(T) = \begin{pmatrix} T & & \\ & \ddots & \\ & & T \end{pmatrix}$$

Assume now that we are given an operator $T \in \bar{A}$, with the bar denoting the weak closure. We have, by using the Hahn-Banach theorem, for any $\xi \in H^+$:

$$\begin{aligned} T \in \bar{A} &\implies \pi(T) \in \overline{\pi(A)} \\ &\implies \pi(T)x \in \overline{\pi(A)\xi} \\ &\implies \pi(T)x \in \overline{\pi(A)\xi}^{\|\cdot\|} \end{aligned}$$

Now observe that the last formula tells us that for any $\xi = (\xi_1, \dots, \xi_k)$, and any $\varepsilon > 0$, we can find an operator $S \in A$ such that the following holds, for any i :

$$\|S\xi_i - T\xi_i\| < \varepsilon$$

It follows that T belongs to the strong operator closure of A , as desired. \square

In the above statement the terminology, while quite standard, is a bit confusing, because the norm topology is stronger than the strong operator topology. As a solution to this issue, we agree in what follows to call the norm topology “strong”, and the weak and strong operator topologies “weak”, whenever these two topologies coincide.

With this convention, the operator algebras $A \subset B(H)$ from Proposition 16.1 are those which are weakly closed, and we can now formulate:

DEFINITION 16.2. *A von Neumann algebra is a *-algebra of operators*

$$A \subset B(H)$$

which is closed under the weak topology.

As basic examples, we have the algebra $B(H)$ itself, then the singly generated von Neumann algebras, $A = \langle T \rangle$, with $T \in B(H)$, and then the multiply generated von Neumann algebras, namely $A = \langle T_i \rangle$, with $T_i \in B(H)$. There are many other examples, and also general methods for constructing examples, and we will discuss this later.

At the level of the general results, we first have the bicommutant theorem of von Neumann, which provides a useful alternative to Definition 16.2, as follows:

THEOREM 16.3. *For a *-algebra $A \subset B(H)$, the following are equivalent:*

- (1) *A is weakly closed, so it is a von Neumann algebra.*
- (2) *A equals its algebraic bicommutant A'' , taken inside $B(H)$.*

PROOF. Since the commutants are weakly closed, it is enough to show that weakly closed implies $A = A''$. For this purpose, we will prove something a bit more general, stating that given a *-algebra of operators $A \subset B(H)$, the following holds, with A'' being the bicommutant inside $B(H)$, and with \bar{A} being the weak closure:

$$A'' = \bar{A}$$

We can prove this by double inclusion, as follows:

“ \supset ” Since any operator commutes with the operators that it commutes with, we have an inclusion $E \subset E''$, valid for any set $E \subset B(H)$. In particular, we have:

$$A \subset A''$$

Our claim now is that the algebra $A'' \subset B(H)$ is closed, with respect to the strong operator topology. Indeed, assuming that we have $T_i \rightarrow T$ in this topology, we have:

$$\begin{aligned} T_i \in A'' &\implies ST_i = T_iS, \forall S \in A' \\ &\implies ST = TS, \forall S \in A' \\ &\implies T \in A \end{aligned}$$

Thus our claim is proved, and together with Proposition 16.1, which allows to pass from the strong to the weak operator topology, this gives the desired inclusion, namely:

$$\bar{A} \subset A''$$

“ \subset ” Here we must prove that we have the following implication, valid for any operator $T \in B(H)$, with the bar denoting as usual the weak operator closure:

$$T \in A'' \implies T \in \bar{A}$$

For this purpose, we use the same amplification trick as in the proof of Proposition 16.1. Consider the Hilbert space obtained by summing k times H with itself:

$$H^+ = H \oplus \dots \oplus H$$

The operators over H^+ can be regarded as being square matrices with entries in $B(H)$, and in particular, we have a representation $\pi : B(H) \rightarrow B(H^+)$, given by:

$$\pi(T) = \begin{pmatrix} T & & \\ & \ddots & \\ & & T \end{pmatrix}$$

The idea will be that of doing the computations in this latter representation. First, in this representation, the image of our algebra $A \subset B(H)$ is given by:

$$\pi(A) = \left\{ \begin{pmatrix} T & & \\ & \ddots & \\ & & T \end{pmatrix} \mid T \in A \right\}$$

We can now compute the commutant of this image, exactly as in the usual scalar matrix case, and we obtain the following formula:

$$\pi(A)' = \left\{ \begin{pmatrix} S_{11} & \dots & S_{1k} \\ \vdots & & \vdots \\ S_{k1} & \dots & S_{kk} \end{pmatrix} \mid S_{ij} \in A' \right\}$$

We conclude from this that, given $T \in A''$ as above, we have:

$$\begin{pmatrix} T & & \\ & \ddots & \\ & & T \end{pmatrix} \in \pi(A)''$$

In other words, the conclusion of all this is that we have the following implication:

$$T \in A'' \implies \pi(T) \in \pi(A)''$$

Now given $\xi \in H^+$, consider the orthogonal projection $P \in B(H^+)$ on the norm closure of the vector space $\pi(A)\xi \subset H^+$. Since the subspace $\pi(A)\xi \subset H^+$ is invariant under the action of $\pi(A)$, so is its norm closure inside H^+ , and we obtain from this:

$$P \in \pi(A)'$$

By combining this with what we found above, we conclude that:

$$T \in A'' \implies \pi(T)P = P\pi(T)$$

Now since this holds for any $\xi \in H^+$, it follows that any $T \in A''$ belongs to the strong operator closure of A . By using now Proposition 16.1, which allows us to pass from the strong to the weak operator closure, we conclude that we have $A'' \subset \bar{A}$, as desired. \square

In order to develop now some general theory for the von Neumann algebras, let us start by investigating the commutative case. The result here is as follows:

THEOREM 16.4. *The commutative von Neumann algebras are the algebras of type*

$$A = L^\infty(X)$$

with X being a measured space.

PROOF. We have two assertions to be proved, the idea being as follows:

(1) In one sense, we must prove that given a measured space X , we can realize the commutative algebra $A = L^\infty(X)$ as a von Neumann algebra, on a certain Hilbert space H . But this can be done as follows, using a probability measure on X :

$$L^\infty(X) \subset B(L^2(X)) \quad , \quad f \rightarrow (g \rightarrow fg)$$

(2) In the other sense, given a commutative von Neumann algebra $A \subset B(H)$, any operator $T \in A$ is normal. So, let us pick a linear space basis $\{T_i\} \subset A$, as to have:

$$A = \langle T_i \rangle$$

The generators $T_i \in B(H)$ are then commuting normal operators, and by using the spectral theorem for such families of operators, we obtain the result. \square

The above result is very interesting, because it shows that an arbitrary von Neumann algebra $A \subset B(H)$ can be thought of as being of the form $A = L^\infty(X)$, with X being a “quantum measured space”. Thus, we have here a connection with the various quantum group and noncommutative geometry considerations made before.

Moving ahead now, we will be interested here in the “free” von Neumann algebras. These algebras, traditionally called factors, can be axiomatized as follows:

DEFINITION 16.5. *A factor is a von Neumann algebra $A \subset B(H)$ whose center*

$$Z(A) = A \cap A'$$

which is a commutative von Neumann algebra, reduces to the scalars, $Z(A) = \mathbb{C}$.

Here the fact that the center is indeed a von Neumann algebra follows from the bicommutant theorem, which shows that the commutant of any $*$ -algebra is a von Neumann algebra. Thus, the intersection $Z(A) = A \cap A'$ is indeed a von Neumann algebra.

Before going further, let us mention that, besides their intuitive freeness, there are some deeper reasons too for the consideration of factors, which among others fully justify the term “factor”, coming from the following advanced theorem of von Neumann:

THEOREM 16.6. *Given a von Neumann algebra $A \subset B(H)$, if we write its center as*

$$Z(A) = L^\infty(X)$$

then we have a decomposition as follows, with the fibers A_x being factors:

$$A = \int_X A_x dx$$

Moreover, in the case where A has a trace, $tr : A \rightarrow \mathbb{C}$, this trace decomposes as

$$tr = \int_X tr_x dx$$

with each $tr_x : A_x \rightarrow \mathbb{C}$ being the restriction of tr to the factor A_x .

PROOF. As a first observation, this is something that we know to hold in finite dimensions, because here the algebra decomposes as follows, with the summands corresponding precisely to the points of the spectrum of the center, $Z(A) \simeq \mathbb{C}^k$:

$$A = M_{N_1}(\mathbb{C}) \oplus \dots \oplus M_{N_k}(\mathbb{C})$$

In general, however, this is something quite difficult to prove, requiring a good knowledge of advanced operator theory and functional analysis. We will not really need this result in what follows, and we refer here to any good operator algebra book. \square

Moving ahead now, in order to do probability on our factors we will need a trace as well. Leaving aside the somewhat trivial case $A = M_N(\mathbb{C})$, we are led in this way to:

DEFINITION 16.7. *A II_1 factor is a von Neumann algebra $A \subset B(H)$ which is infinite dimensional, has trivial center, and has a trace $\text{tr} : A \rightarrow \mathbb{C}$.*

As a first observation, according to Theorem 16.6, such factors are exactly those appearing in the spectral decomposition of the von Neumann algebras $A \subset B(H)$ which have traces, $\text{tr} : A \rightarrow \mathbb{C}$, provided that we add some extra axioms which avoid trivial summands of type $M_N(\mathbb{C})$. Moreover, by results of Connes, adding to those of von Neumann, and which are non-trivial as well, the non-tracial case basically reduces to the tracial case, via certain crossed product type operations, and so the conclusion is that “the II_1 factors are the building blocks of the von Neumann algebra theory”.

Summarizing, some heavy things going on here. In what follows we will be mainly interested in concrete mathematics and combinatorics, and we will take Definition 16.7 as it is, as a simple and intuitive definition for the “free von Neumann algebras”.

There are many things that can be said about the II_1 factors, and of particular interest is the following key result of Murray and von Neumann [70], which clarifies the situation with the various Hilbert space representations $A \subset B(H)$ of a given II_1 factor A :

THEOREM 16.8. *Given a representation of a II_1 factor $A \subset B(H)$, we can talk about the corresponding coupling constant*

$$\dim_A H \in (0, \infty]$$

which for the standard form, where $H = L^2(A)$, takes the value 1, and which in general measures how far is $A \subset B(H)$ from the standard form.

PROOF. There are several proofs for this fact, the idea being as follows:

(1) We can amplify the standard representation of A , on the Hilbert space $L^2(A)$, into a representation on $L^2(A) \otimes l^2(\mathbb{N})$, and then cut it down with a projection. We obtain in this way a whole family of embeddings $A \subset B(H)$, which are quite explicit.

(2) The point now is that of proving, via a technical 2×2 matrix trick, that any representation $A \subset B(H)$ appears in this way. In this picture, the coupling constant appears as the trace of the projection used to cut down $L^2(A) \otimes l^2(\mathbb{N})$.

(3) Thus, we are led to the conclusion in the statement. Alternatively, the coupling constant can be defined as follows, with the number on the right being independent of the choice on a nonzero vector $x \in H$, and with this being the original definition from [70]:

$$\dim_A H = \frac{\text{tr}_A(P_{A'x})}{\text{tr}_{A'}(P_{Ax})}$$

We refer to [70], or for instance to the book [28], for more details here. □

Following Jones [56], given a II_1 factor A_0 , let us discuss now the representations $A_0 \subset A_1$, with A_1 being another II_1 factor. This is a quite natural notion too, and perhaps even more natural than the representations $A_0 \subset B(H)$, because we have decided in the above that the II_1 factors A_1 , and not the full operator algebras $B(H)$, are the correct infinite dimensional generalization of the usual matrix algebras $M_N(\mathbb{C})$.

Given an inclusion of II_1 factors $A_0 \subset A_1$, a first question is that of defining its index, measuring how big is A_1 , when compared to A_0 . This can be done as follows:

THEOREM 16.9. *Given an inclusion of II_1 factors $A_0 \subset A_1$, the number*

$$N = \frac{\dim_{A_0} H}{\dim_{A_1} H}$$

is independent of the ambient Hilbert space H , and is called index.

PROOF. This is standard, with the fact that the index as defined by the above formula is indeed independent of the ambient Hilbert space H coming from the various basic properties of the coupling constant, from Theorem 16.8 and its proof. \square

There are many examples of subfactors coming from groups, and every time we obtain the intuitive index. In general now, following Jones [56], let us start with:

DEFINITION 16.10. *Associated to any subfactor $A_0 \subset A_1$ is the basic construction*

$$A_0 \subset_e A_1 \subset A_2$$

with $A_2 = \langle A_1, e \rangle$ being the algebra generated by A_1 and by the standard projection

$$e : L^2(A_1) \rightarrow L^2(A_0)$$

also called Jones projection, acting on the Hilbert space $L^2(A_1)$.

The idea now, following [56], will be that $A_1 \subset A_2$ appears as a kind of “reflection” of $A_0 \subset A_1$, and also that the basic construction can be iterated, and with all this leading to non-trivial results. Let us start by further studying the basic construction:

PROPOSITION 16.11. *Given a subfactor $A_0 \subset A_1$ having finite index,*

$$[A_1 : A_0] < \infty$$

the basic construction $A_0 \subset_e A_1 \subset A_2$ has the following properties:

- (1) $A_2 = JA'_0J$.
- (2) $A_2 = \overline{A_1 + A_1eb}$.
- (3) A_2 is a II_1 factor.
- (4) $[A_2 : A_1] = [A_1 : A_0]$.
- (5) $eA_2e = A_0e$.
- (6) $\text{tr}(e) = [A_1 : A_0]^{-1}$.
- (7) $\text{tr}(xe) = \text{tr}(x)[A_1 : A_0]^{-1}$, for any $x \in A_1$.

PROOF. This is routine, with $J(T) = T^*$, and we refer here to Jones [56]. □

The above result is quite interesting, potentially leading to some interesting mathematics, so let us perform now twice the basic construction, and see what we get. The result here, which is something more technical, at least at the first glance, is as follows:

PROPOSITION 16.12. *Associated to $A_0 \subset A_1$ is the double basic construction*

$$A_0 \subset_e A_1 \subset_f A_2 \subset A_3$$

with $e : L^2(A_1) \rightarrow L^2(A_0)$ and $f : L^2(A_2) \rightarrow L^2(A_1)$ having the following properties:

$$fef = [A_1 : A_0]^{-1}f \quad , \quad efe = [A_1 : A_0]^{-1}e$$

PROOF. Again, this is standard, and for details, we refer to Jones [56]. □

We can in fact perform the basic construction by recurrence, and we obtain:

THEOREM 16.13. *Associated to any subfactor $A_0 \subset A_1$ is the Jones tower*

$$A_0 \subset_{e_1} A_1 \subset_{e_2} A_2 \subset_{e_3} A_3 \subset \dots$$

with the Jones projections having the following properties:

- (1) $e_i^2 = e_i = e_i^*$.
- (2) $e_i e_j = e_j e_i$ for $|i - j| \geq 2$.
- (3) $e_i e_{i \pm 1} e_i = [A_1 : A_0]^{-1} e_i$.
- (4) $tr(w e_{n+1}) = [A_1 : A_0]^{-1} tr(w)$, for any word $w \in \langle e_1, \dots, e_n \rangle$.

PROOF. This follows from Proposition 16.11 and Proposition 16.12, because the triple basic construction does not need in fact any further study. See [56]. □

The point now is that the relations found in Theorem 16.13 are well-known, from the standard theory of the Temperley-Lieb algebra [83]. Thus, still following Jones' paper [56], we can now reformulate Theorem 16.13 into something more conceptual, as follows:

THEOREM 16.14. *Given a subfactor $A_0 \subset A_1$, construct its the Jones tower:*

$$A_0 \subset_{e_1} A_1 \subset_{e_2} A_2 \subset_{e_3} A_3 \subset \dots$$

The rescaled sequence of projections $e_1, e_2, e_3, \dots \in B(H)$ produces then a representation

$$TL_N \subset B(H)$$

of the Temperley-Lieb algebra of index $N = [A_1 : A_0]$.

PROOF. We know from Theorem 16.13 that the rescaled sequence of projections $e_1, e_2, e_3, \dots \in B(H)$ behaves algebraically exactly as the following TL_N diagrams:

$$\varepsilon_1 = \cup_n \quad , \quad \varepsilon_2 = \left| \cup_n \right. \quad , \quad \varepsilon_3 = \left\| \cup_n \right. \quad , \quad \dots$$

But these diagrams generate TL_N , and so we have an embedding $TL_N \subset B(H)$, where H is the Hilbert space where our subfactor $A_0 \subset A_1$ lives, as claimed. □

16b. Planar algebras

Quite remarkably, the planar algebra structure of TL_N , taken in an intuitive sense, that of composing diagrams, in various possible ways, extends to a planar algebra structure on the sequence of higher relative commutants $P_n = A'_0 \cap A_n$. In order to discuss this, let us start with axioms for the planar algebras. Following Jones [58], we have:

DEFINITION 16.15. *The planar algebras are defined as follows:*

- (1) *We consider rectangles in the plane, with the sides parallel to the coordinate axes, and taken up to planar isotopy, and we call such rectangles boxes.*
- (2) *A labeled box is a box with $2n$ marked points on its boundary, n on its upper side, and n on its lower side, for some integer $n \in \mathbb{N}$.*
- (3) *A tangle is labeled box, containing a number of labelled boxes, with all marked points, on the big and small boxes, being connected by noncrossing strings.*
- (4) *A planar algebra is a sequence of finite dimensional vector spaces $P = (P_n)$, together with linear maps $P_{n_1} \otimes \dots \otimes P_{n_k} \rightarrow P_n$, one for each tangle, such that the gluing of tangles corresponds to the composition of linear maps.*

In this definition we are using rectangles, but everything being up to isotopy, we could have used instead circles with marked points, as in [58]. Our choice for using rectangles comes from the main examples that we have in mind, to be discussed below, where the planar algebra structure is best viewed by using rectangles, as above.

Let us also mention that Definition 16.15 is something quite simplified, based on [58]. As explained in [58], in order for subfactors to produce planar algebras and vice versa, there are quite a number of supplementary axioms that must be added, and in view of this, it is perhaps better to start with something stronger than Definition 16.15, as basic axioms. However, as before with rectangles vs circles, our axiomatic choices here are mainly motivated by the concrete examples that we have in mind. More on this later.

As a basic example of a planar algebra, we have the Temperley-Lieb algebra:

THEOREM 16.16. *The Temperley-Lieb algebra TL_N , viewed as graded algebra*

$$TL_N = (TL_N(n))_{n \in \mathbb{N}}$$

is a planar algebra, with the corresponding linear maps associated to the planar tangles

$$TL_N(n_1) \otimes \dots \otimes TL_N(n_k) \rightarrow TL_N(n)$$

appearing by putting the various $TL_N(n_i)$ diagrams into the small boxes of the given tangle, which produces a $TL_N(n)$ diagram.

PROOF. This is something trivial, which follows from definitions:

- (1) Assume indeed that we are given a planar tangle π , as in Definition 16.15, consisting of a box having $2n$ marked points on its boundary, and containing k small boxes, having

respectively $2n_1, \dots, 2n_k$ marked points on their boundaries, and then a total of $n + \sum n_i$ noncrossing strings, connecting the various $2n + \sum 2n_i$ marked points.

(2) We want to associate to this tangle π a linear map as follows:

$$T_\pi : TL_N(n_1) \otimes \dots \otimes TL_N(n_k) \rightarrow TL_N(n)$$

For this purpose, by linearity, it is enough to construct elements as follows, for any choice of Temperley-Lieb diagrams $\sigma_i \in TL_N(n_i)$, with $i = 1, \dots, k$:

$$T_\pi(\sigma_1 \otimes \dots \otimes \sigma_k) \in TL_N(n)$$

(3) But constructing such an element is obvious, just by putting the various diagrams $\sigma_i \in TL_N(n_i)$ into the small boxes the given tangle π . Indeed, this procedure produces a certain diagram in $TL_N(n)$, that we can call $T_\pi(\sigma_1 \otimes \dots \otimes \sigma_k)$, as above.

(4) Finally, we have to check that everything is well-defined up to planar isotopy, and that the gluing of tangles corresponds to the composition of linear maps. But both these checks are trivial, coming from the definition of TL_N , and we are done. \square

As a conclusion to all this, $P = TL_N$ is indeed a planar algebra, but of somewhat “trivial” type, with the triviality coming from the fact that, in this case, the elements of P are planar diagrams themselves, and so the planar structure appears trivially.

The Temperley-Lieb planar algebra TL_N is however an important planar algebra, because it is the “smallest” one, appearing inside the planar algebra of any subfactor. But more on this later, when talking about planar algebras and subfactors.

Moving ahead now, here is our second basic example of a planar algebra, which is also “trivial” in the above sense, with the elements of the planar algebra being planar diagrams themselves, but which appears in a bit more complicated way:

THEOREM 16.17. *The Fuss-Catalan algebra $FC_{N,M}$, which appears by coloring the Temperley-Lieb diagrams with black/white colors, clockwise, as follows*



and keeping those diagrams whose strings connect either $\circ - \circ$ or $\bullet - \bullet$, is a planar algebra, with again the corresponding linear maps associated to the planar tangles

$$FC_{N,M}(n_1) \otimes \dots \otimes FC_{N,M}(n_k) \rightarrow FC_{N,M}(n)$$

appearing by putting the various $FC_{N,M}(n_i)$ diagrams into the small boxes of the given tangle, which produces a $FC_{N,M}(n)$ diagram.

PROOF. The proof here is nearly identical to the proof of Theorem 16.16, with the only change appearing at the level of the colors. To be more precise:

(1) Forgetting about upper and lower sequences of points, which must be joined by strings, a Temperley-Lieb diagram can be thought of as being a collection of strings, say

black strings, which compose in the obvious way, with the rule that the value of the circle, which is now a black circle, is N . And it is this obvious composition rule that gives the planar algebra structure, as explained in the proof of Theorem 16.16.

(2) Similarly, forgetting about points, a Fuss-Catalan diagram can be thought of as being a collection of strings, which come now in two colors, black and white. These Fuss-Catalan diagrams compose then in the obvious way, with the rule that the value of the black circle is N , and the value of the white circle is M . And it is this obvious composition rule that gives the planar algebra structure, as before for TL_N . \square

Getting back now to generalities, and to Definition 16.15, that of a general planar algebra, we have so far two illustrations for it, which, while both important, are both “trivial”, with the planar structure simply coming from the fact that, in both these cases, the elements of the planar algebra are planar diagrams themselves.

In general, the planar algebras can be more complicated than this, and we will see some further examples in a moment. However, the idea is very simple, namely “the elements of a planar algebra are not necessarily diagrams, but they behave like diagrams”.

In relation now with subfactors, the result, which extends Theorem 16.14, and which was found by Jones in [58], almost 20 years after [56], is as follows:

THEOREM 16.18. *Given a subfactor $A_0 \subset A_1$, the collection $P = (P_n)$ of linear spaces*

$$P_n = A'_0 \cap A_n$$

has a planar algebra structure, extending the planar algebra structure of TL_N .

PROOF. We know from Theorem 16.14 that we have an inclusion as follows, coming from the basic construction, and with TL_N itself being a planar algebra:

$$TL_N \subset P$$

Thus, the whole point is that of proving that the trivial planar algebra structure of TL_N extends into a planar algebra structure of P . But this can be done via a long algebraic study, and for the full computation here, we refer to Jones’ paper [58]. \square

As a first illustration for the above result, we have:

THEOREM 16.19. *We have the following universality results:*

- (1) *The Temperley-Lieb algebra TL_N appears inside the planar algebra of any subfactor $A_0 \subset A_1$ having index N .*
- (2) *The Fuss-Catalan algebra $FC_{N,M}$ appears inside the planar algebra of any subfactor $A_0 \subset A_1$, in the presence of an intermediate subfactor $A_0 \subset B \subset A_1$.*

PROOF. Here the first assertion is something that we already know, from Theorem 16.18, and the second assertion is something quite standard as well, by carefully working out the basic construction for $A_0 \subset A_1$, in the presence of an intermediate subfactor $A_0 \subset B \subset A_1$. For details here, we refer to the paper of Bisch and Jones [27]. \square

As a free probability comment here, the Temperley-Lieb algebra, which appears by definition as the span of NC_2 , is certainly a free probability object, and one way of being more concrete here is by saying that suitable fixed point subfactors associated to S_N^+, O_N^+, U_N^+ have planar algebra equal to TL_N . See [8]. As in what regards the Fuss-Catalan algebra, this is related to the bicolored partitions appearing in the study of H_N^+ , and more generally of H_N^{s+} , and again, the precise subfactor statement about this concerns fixed point subfactors associated to the quantum groups H_N^{s+} . See [9].

The above results raise the question on whether any planar algebra produces a subfactor. The answer here is yes, but with many subtleties, and in order to talk about this, we first need to introduce a certain distinguished II_1 factor R , as follows:

DEFINITION 16.20. *The Murray-von Neumann hyperfinite II_1 factor is*

$$R = \overline{\bigcup_i M_{n_i}(\mathbb{C})}^w$$

independently of the choice of the algebras $M_{n_i}(\mathbb{C})$, and of the embeddings between them.

To be more precise, all this is based on two theorems of Murray and von Neumann [70], stating on one hand that when performing the above inductive limit construction we obtain, after taking the weak closure, a certain II_1 factor, and on the other hand, that the factor that we obtain is independent on the choice of the algebras $M_{n_i}(\mathbb{C})$, and of the embeddings between them. All this is certainly non-trivial, and even less trivial is the following theorem, coming as a continuation of the work in [70], due to Connes [37]:

THEOREM 16.21. *The Murray-von Neumann II_1 factor R is the unique II_1 factor which is amenable, in the sense that we have a conditional expectation as follows:*

$$E : B(H) \rightarrow R$$

In particular, for a discrete group Γ we have $L(\Gamma) = R$ precisely when $\Gamma \neq \{1\}$ has the infinite conjugacy class (ICC) property, and is amenable.

PROOF. This is something fairly complicated, to the point of causing troubles not only to mathematicians, and no surprise here, but to physicists as well. In case you know a good physicist, the best is to ask that physicist, but there is no guarantee here, guy might well be clueless on all this. So, read from time to time operator algebras, say from [28], and once ready go through [37]. And in the meantime do not hesitate to ask around, this being a good test for distinguishing good physicists from first-class physicists. \square

Jokes left aside now, what is difficult in the above is the proof of “amenability implies hyperfiniteness”. Indeed, the converse can only be something standard, namely proving that a certain concrete algebra, R from Definition 16.20, has a certain concrete property. As for the last assertion, this cannot be complicated either, because one of the possible definitions of the amenability of Γ is in terms of an invariant mean $m : l^\infty(\Gamma) \rightarrow \mathbb{C}$, and this makes the connection with the expectation $E : B(l^2(\Gamma)) \rightarrow L(\Gamma)$. See [37].

Getting back now to subfactors, and to our questions regarding the correspondence between subfactors and planar algebras, these are difficult questions too, and the various answers to these questions can be summarized, a bit informally, as follows:

THEOREM 16.22. *We have the following results:*

- (1) *Any planar algebra with positivity produces a subfactor.*
- (2) *In particular, we have TL and FC type subfactors.*
- (3) *In the amenable case, and with $A_1 = R$, the correspondence is bijective.*
- (4) *In general, we must take $A_1 = L(F_\infty)$, and we do not have bijectivity.*
- (5) *The axiomatization of P , in the case $A_1 = R$, is not known.*

PROOF. All this is quite heavy, basically coming from the work of Popa in the 90s, using heavy functional analysis, the idea being as follows:

(1) As already mentioned after Definition 16.15, our planar algebra axioms here are something quite simplified, based on [58]. However, when getting back to Theorem 16.18, the conclusion is that the subfactor planar algebras there satisfy a number of supplementary “positivity” conditions, basically coming from the positivity of the II_1 factor trace. And the point is that, with these positivity conditions axiomatized, we reach to something which is equivalent to Popa’s axiomatization of the lattice of higher relative commutants $A'_i \cap A_j$ of the finite index subfactors [74], obtained in the 90s via heavy functional analysis. For the full story here, and details, we refer to Jones’ paper [58].

(2) The existence of the TL_N subfactors, also known as “ A_∞ subfactors” in the literature, is something which was known for some time, since some early work of Popa on the subject. As for the existence of the $FC_{N,M}$ subfactors, this can be shown by using the intermediate subfactor picture, $A_0 \subset B \subset A_1$, by composing two A_∞ subfactors of suitable indices, $A_0 \subset B$ and $B \subset A_1$. For the full story here, we refer to [27], [58].

(3) This is something fairly heavy, as it is always the case with operator algebra results regarding hyperfiniteness and amenability, due to Popa [73], [74].

(4) This is something a bit more recent, obtained by further building on the above-mentioned constructions of Popa, and we refer here to [50] and related work.

(5) This is the big open question in subfactors. The story here goes back to Jones’ original paper [56], which contains at the end the question, due to Connes, of finding the possible values of the index for the irreducible subfactors of R . This question, which

certainly looks much easier than (5) in the statement, is in fact still open, now 40 years after its formulation, and with on one having any valuable idea in dealing with it. \square

16c. Basic examples

Let us discuss now some basic examples of subfactors, with concrete illustrations for all the above notions, constructions, and general theory. These examples will all come from group actions $G \curvearrowright Q$, which are assumed to be minimal, in the sense that:

$$(Q^G)' \cap Q = \mathbb{C}$$

As a starting point, we have the following result, due to Jones [56]:

PROPOSITION 16.23. *Assuming that G is a compact group, acting minimally on a II_1 factor Q , and that $H \subset G$ is a subgroup of finite index, we have a subfactor*

$$Q^G \subset Q^H$$

having index $N = [G : H]$, called Jones subfactor.

PROOF. This is something standard, the idea being that the factoriality of Q^G, Q^H comes from the minimality of the action, and that the index formula is clear. \square

Along the same lines, we have the following result, due to Ocneanu [72]:

PROPOSITION 16.24. *Assuming that G is a finite group, acting minimally on a II_1 factor Q , we have a subfactor as follows,*

$$Q \subset Q \rtimes G$$

having index $N = |G|$, called Ocneanu subfactor.

PROOF. This is standard as well, the idea being that the factoriality of $Q \rtimes G$ comes from the minimality of the action, and that the index formula is clear. \square

We have as well a third result of the same type, due to Wassermann [94], namely:

PROPOSITION 16.25. *Assuming that G is a compact group, acting minimally on a II_1 factor Q , and that $G \rightarrow \text{PU}_n$ is a projective representation, we have a subfactor*

$$Q^G \subset (M_n(\mathbb{C}) \otimes Q)^G$$

having index $N = n^2$, called Wassermann subfactor.

PROOF. As before, the idea is that the factoriality of $Q^G, (M_n(\mathbb{C}) \otimes Q)^G$ comes from the minimality of the action, and the index formula is clear. \square

The above subfactors look quite related, and indeed they are, due to:

THEOREM 16.26. *The Jones, Ocneanu and Wassermann subfactors are all of the same nature, and can be written as follows,*

$$\begin{aligned} (Q^G \subset Q^H) &\simeq ((\mathbb{C} \otimes Q)^G \subset (l^\infty(G/H) \otimes Q)^G) \\ (Q \subset Q \rtimes G) &\simeq ((l^\infty(G) \otimes Q)^G \subset (\mathcal{L}(l^2(G)) \otimes Q)^G) \\ (Q^G \subset (M_n(\mathbb{C}) \otimes Q)^G) &\simeq ((\mathbb{C} \otimes Q)^G \subset (M_n(\mathbb{C}) \otimes Q)^G) \end{aligned}$$

with standard identifications for the various tensor products and fixed point algebras.

PROOF. This is something standard, from [7], modulo several standard identifications. We will explain all this more in detail later, after unifying these subfactors. \square

In order to unify now the above constructions of subfactors, following [7], [94], the idea is quite clear. Given a compact group G , acting minimally on a II_1 factor Q , and an inclusion of finite dimensional algebras $B_0 \subset B_1$, endowed as well with an action of G , we would like to construct a kind of generalized Wassermann subfactor, as follows:

$$(B_0 \otimes Q)^G \subset (B_1 \otimes Q)^G$$

In order to do this, we must talk first about the finite dimensional algebras B , and about inclusions of such algebras $B_0 \subset B_1$. Let us start with the following definition:

DEFINITION 16.27. *Associated to any finite dimensional algebra B is its canonical trace, obtained by composing the left regular representation with the trace of $\mathcal{L}(B)$:*

$$\text{tr} : B \subset \mathcal{L}(B) \rightarrow \mathbb{C}$$

We say that an inclusion of finite dimensional algebras $B_0 \subset B_1$ is Markov if it commutes with the canonical traces of B_0, B_1 .

As a basic illustration for this, any inclusion of type $\mathbb{C} \subset B$ is Markov. In general, if we write $B_0 = C(X_0)$ and $B_1 = C(X_1)$, then the inclusion $B_0 \subset B_1$ must come from a certain fibration $X_1 \rightarrow X_0$, and the inclusion $B_0 \subset B_1$ is Markov precisely when the fibration $X_1 \rightarrow X_0$ commutes with the respective counting measures.

We will be back to Markov inclusions and their various properties on several occasions, in what follows. For our next purposes here, we just need the following result:

PROPOSITION 16.28. *Given a Markov inclusion of finite dimensional algebras $B_0 \subset B_1$ we can perform to it the basic construction, as to obtain a Jones tower*

$$B_0 \subset_{e_1} B_1 \subset_{e_2} B_2 \subset_{e_3} B_3 \subset \dots$$

exactly as we did in the above for the inclusions of II_1 factors.

PROOF. This is something standard, from [56], by following the computations in the above, from the case of the II_1 factors, and with everything extending well. It is of course possible to do something more general here, unifying the constructions for the inclusions of II_1 factors $A_0 \subset A_1$, and for the inclusions of Markov inclusions of finite dimensional algebras $B_0 \subset B_1$, but we will not need this degree of generality, in what follows. \square

With these ingredients in hand, getting back now to the Jones, Ocneanu and Wassermann subfactors, from Theorem 16.26, the point is that these constructions can be unified, and then further studied, the final result on the subject being as follows:

THEOREM 16.29. *Let G be a compact group, and $G \rightarrow \text{Aut}(Q)$ be a minimal action on a II_1 factor. Consider a Markov inclusion of finite dimensional algebras*

$$B_0 \subset B_1$$

and let $G \rightarrow \text{Aut}(B_1)$ be an action which leaves invariant B_0 , and which is such that its restrictions to the centers of B_0 and B_1 are ergodic. We have then a subfactor

$$(B_0 \otimes Q)^G \subset (B_1 \otimes Q)^G$$

of index $N = [B_1 : B_0]$, called generalized Wassermann subfactor, whose Jones tower is

$$(B_1 \otimes Q)^G \subset (B_2 \otimes Q)^G \subset (B_3 \otimes Q)^G \subset \dots$$

where $\{B_i\}_{i \geq 1}$ are the algebras in the Jones tower for $B_0 \subset B_1$, with the canonical actions of G coming from the action $G \rightarrow \text{Aut}(B_1)$, and whose planar algebra is given by:

$$P_k = (B'_0 \cap B_k)^G$$

These subfactors generalize the Jones, Ocneanu and Wassermann subfactors.

PROOF. This is something which is routine, from [8], following Wassermann [94], and we will be back to this in a moment, with details, directly in a more general setting. \square

In addition to the Jones, Ocneanu and Wassermann subfactors, discussed and unified in the above, we have the Popa subfactors, which are constructed as follows:

PROPOSITION 16.30. *Given a discrete group $\Gamma = \langle g_1, \dots, g_n \rangle$, acting faithfully via outer automorphisms on a II_1 factor P , we have the following “diagonal” subfactor*

$$\left\{ \left(\begin{array}{ccc} g_1(q) & & \\ & \ddots & \\ & & g_n(q) \end{array} \right) \middle| q \in P \right\} \subset M_n(P)$$

having index $N = n^2$, called Popa subfactor.

PROOF. This is something standard, a bit as for the Jones, Ocneanu and Wassermann subfactors, with the result basically coming from the work of Popa [73], [74]. \square

In order to unify now Theorem 16.29 and Proposition 16.30, observe that the diagonal subfactors can be written in the following way, by using a group dual:

$$(P \rtimes \Gamma)^{\widehat{\Gamma}} \subset (M_n(\mathbb{C}) \otimes (P \rtimes \Gamma))^{\widehat{\Gamma}}$$

Here the group dual $\widehat{\Gamma}$ acts on $Q = P \rtimes \Gamma$ via the dual of the action $\Gamma \subset \text{Aut}(P)$, and on $M_n(\mathbb{C})$ via the adjoint action of the following formal representation:

$$\oplus g_i : \widehat{\Gamma} \rightarrow \mathbb{C}^n$$

Summarizing, we are led into quantum groups. So, let us start with:

DEFINITION 16.31. *A coaction of a Woronowicz algebra A on a finite von Neumann algebra Q is an injective morphism $\Phi : Q \rightarrow Q \otimes A''$ satisfying the following conditions:*

- (1) *Coassociativity:* $(\Phi \otimes id)\Phi = (id \otimes \Delta)\Phi$.
- (2) *Trace equivariance:* $(tr \otimes id)\Phi = tr(\cdot)1$.
- (3) *Smoothness:* $\overline{Q}^w = Q$, where $Q = \Phi^{-1}(Q \otimes_{alg} A)$.

These conditions come from what happens in the commutative case, $A = C(G)$, where they correspond to the usual associativity, trace equivariance and smoothness of the corresponding action $G \curvearrowright Q$. Along the same lines, we have as well:

DEFINITION 16.32. *A coaction $\Phi : Q \rightarrow Q \otimes A''$ as above is called:*

- (1) *Ergodic, if the algebra $Q^\Phi = \{p \in Q \mid \Phi(p) = p \otimes 1\}$ reduces to \mathbb{C} .*
- (2) *Faithful, if the span of $\{(f \otimes id)\Phi(Q) \mid f \in Q_*\}$ is dense in A'' .*
- (3) *Minimal, if it is faithful, and satisfies $(Q^\Phi)' \cap Q = \mathbb{C}$.*

Observe that the minimality of the action implies in particular that the fixed point algebra Q^Φ is a factor. Thus, we are getting here to the case that we are interested in, actions producing factors, via their fixed point algebras. Following [7], we have:

PROPOSITION 16.33. *Consider a Woronowicz algebra $A = (A, \Delta, S)$, and denote by A_σ the Woronowicz algebra $(A, \sigma\Delta, S)$, where σ is the flip. Given two coactions*

$$\beta : B \rightarrow B \otimes A \quad , \quad \pi : Q \rightarrow Q \otimes A_\sigma$$

with B being finite dimensional, the following linear map, while not being multiplicative in general, is coassociative with respect to the comultiplication $\sigma\Delta$ of A_σ ,

$$\beta \circ \pi : B \otimes Q \rightarrow B \otimes Q \otimes A_\sigma$$

$$b \otimes p \rightarrow \pi(p)_{23}((id \otimes S)\beta(b))_{13}$$

and its fixed point space, which is by definition the following linear space,

$$(B \otimes Q)^{\beta \circ \pi} = \left\{ x \in B \otimes Q \mid (\beta \circ \pi)x = x \otimes 1 \right\}$$

is then a von Neumann subalgebra of $B \otimes Q$.

PROOF. This is something standard, which follows from a straightforward algebraic verification, explained in [7]. As mentioned in the statement, to be noted is that the tensor product coaction $\beta \odot \pi$ is not multiplicative in general. See [7]. \square

Our first task is to investigate the factoriality of such algebras, and we have here:

THEOREM 16.34. *If $\beta : B \rightarrow B \otimes A$ is a coaction and $\pi : Q \rightarrow Q \otimes A_\sigma$ is a minimal coaction, then the following conditions are equivalent:*

- (1) *The von Neumann algebra $(B \otimes Q)^{\beta \odot \pi}$ is a factor.*
- (2) *The coaction β is centrally ergodic, $Z(B) \cap B^\beta = \mathbb{C}$.*

PROOF. This is something standard, from [7], the idea being as follows:

(1) Our first claim, which is something whose proof is a routine verification, explained in [7], is that the following diagram is a non-degenerate commuting square:

$$\begin{array}{ccc} Q & \subset & B \otimes Q \\ \cup & & \cup \\ Q^\pi & \subset & (B \otimes Q)^{\beta \odot \pi} \end{array}$$

(2) In order to prove now the result, it is enough to check the following equality, between subalgebras of the von Neumann algebra $B \otimes Q$:

$$Z((B \otimes Q)^{\beta \odot \pi}) = (Z(B) \cap B^\beta) \otimes 1$$

But this follows from the non-degeneracy of the above commuting square. See [7]. \square

With the above results in hand, we can now formulate our main theorem regarding the fixed point subfactors, of the most possible general type, as follows:

THEOREM 16.35. *Let G be a compact quantum group, and $G \rightarrow \text{Aut}(Q)$ be a minimal action on a II_1 factor. Consider a Markov inclusion of finite dimensional algebras*

$$B_0 \subset B_1$$

and let $G \rightarrow \text{Aut}(B_1)$ be an action which leaves invariant B_0 and which is such that its restrictions to the centers of B_0 and B_1 are ergodic. We have then a subfactor

$$(B_0 \otimes Q)^G \subset (B_1 \otimes Q)^G$$

of index $N = [B_1 : B_0]$, called generalized Wassermann subfactor, whose Jones tower is

$$(B_1 \otimes Q)^G \subset (B_2 \otimes Q)^G \subset (B_3 \otimes Q)^G \subset \dots$$

where $\{B_i\}_{i \geq 1}$ are the algebras in the Jones tower for $B_0 \subset B_1$, with the canonical actions of G coming from the action $G \rightarrow \text{Aut}(B_1)$, and whose planar algebra is given by:

$$P_k = (B'_0 \cap B_k)^G$$

These subfactors generalize the Jones, Ocneanu, Wassermann and Popa subfactors.

PROOF. This is something routine, based on the above general theory and results, and for the full story here, and technical details, we refer to [7], [94]. \square

The above result is important in connection with probability questions, because our usual character computations for G , for instance in the case where $G \subset U_N^+$ is easy, take place in the associated planar algebra $P_k = (B'_0 \cap B_k)^G$. More on this later.

This was for the basic theory of the fixed point subfactors. Many more things can be said about them, notably with an axiomatization of the planar algebras that we can obtain in this way, as being the subalgebras of Jones' bipartite graph planar algebras from [59], and also with a number of results and open questions regarding amenability. For more on all this, and for further details on the above, we refer to [7], [59], [81].

16d. Spectral measures

In what follows we discuss various structure and classification questions for the subfactors, all interesting questions, related to physics, regarded from a probabilistic viewpoint. In order to get started, we need invariants for our subfactors. We have the choice here between algebraic and analytic invariants, the situation being as follows:

DEFINITION 16.36. *Associated to any finite index subfactor $A_0 \subset A_1$, having planar algebra $P = (P_n)$, are the following invariants:*

- (1) *Its principal graph Γ , which describes the inclusions $P_0 \subset P_1 \subset P_2 \subset \dots$, with the reflections coming from basic constructions removed.*
- (2) *Its fusion algebra F , which describes the fusion rules for the various types of bimodules that can appear, namely $A_0 - A_0$, $A_0 - A_1$, $A_1 - A_0$, $A_1 - A_1$.*
- (3) *Its Poincaré series f , which is the generating series of the graded components of the planar algebra, $f(z) = \sum_n \dim(P_n)z^n$.*
- (4) *Its spectral measure μ , which is the probability measure having as moments the dimensions of the planar algebra components, $\int x^n d\mu(x) = \dim(P_n)$.*

This definition is of course something a bit informal, and there is certainly some work to be done, in order to fully define all these invariants Γ , F , f , μ , and to work out the precise relation between them. We will be back to this later, but for the moment, let us keep in mind the fact that associated to a given subfactor $A \subset B$ are several invariants, which are not exactly equivalent, but are definitely versions of the same thing, the “combinatorics of the subfactor”, and which come in algebraic or analytic flavors.

More in detail now, let us begin by explaining how the principal graph Γ is constructed. Consider a finite index irreducible subfactor $A_0 \subset A_1$, with associated planar algebra $P_n = A'_0 \cap A_n$, and let us look at the following system of inclusions:

$$P_0 \subset P_1 \subset P_2 \subset \dots$$

By taking the Bratelli diagram of this system of inclusions, and then deleting the reflections coming from basic constructions, we obtain a certain graph Γ , called principal graph of $A_0 \subset A_1$. The main properties of Γ can be summarized as follows:

PROPOSITION 16.37. *The principal graph Γ has the following properties:*

- (1) *The higher relative commutant $P_n = A'_0 \cap A_n$ is isomorphic to the abstract vector space spanned by the $2n$ -loops on Γ based at the root.*
- (2) *In the amenable case, where $A_1 = R$ and when the subfactor is “amenable”, the index of $A_0 \subset A_1$ is given by $N = \|\Gamma\|^2$.*

PROOF. This is something standard, the idea being as follows:

(1) The statement here, which explains among others the relation between the principal graph Γ , and the other subfactor invariants, from Definition 16.36, comes from the definition of the principal graph, as a Bratelli diagram, with the reflections removed.

(2) This is actually a quite subtle statement, but for our purposes here, we can take the equality $N = \|\Gamma\|^2$, which reminds the Kesten amenability condition for discrete groups, as a definition for the amenability of the subfactor.

(3) With the remark that for the Popa diagonal subfactors what we have here is precisely the Kesten amenability condition for the underlying discrete group G .

(4) And with the further remark that, more generally, for the arbitrary generalized Popa or Wassermann subfactors, discussed above, what we have here is precisely the Kesten type amenability condition for the underlying discrete quantum group G . \square

As an illustration for all this, let us first discuss the case of the small index subfactors, $N \in [1, 4]$. Following Jones [56] and related work, we first have the following result:

THEOREM 16.38. *The index of subfactors is subject to the condition*

$$N \in \left\{ 4 \cos^2 \left(\frac{\pi}{n} \right) \mid n \geq 3 \right\} \cup [4, \infty]$$

and at $N \leq 4$, the principal graph must be one of the Coxeter-Dynkin ADE graphs.

PROOF. This comes from the combinatorics of e_1, e_2, e_3, \dots , as follows:

(1) In order to best comment on what happens, when iterating the basic construction, let us record the first few values of the numbers in the statement, namely:

$$4 \cos^2 \left(\frac{\pi}{3} \right) = 1 \quad , \quad 4 \cos^2 \left(\frac{\pi}{4} \right) = 2$$

$$4 \cos^2 \left(\frac{\pi}{5} \right) = \frac{3 + \sqrt{5}}{2} \quad , \quad 4 \cos^2 \left(\frac{\pi}{6} \right) = 3$$

(2) By using a basic construction, we get, by trace manipulations on e_1 :

$$N \notin (1, 2)$$

With a double basic construction, we get, by trace manipulations on $\langle e_1, e_2 \rangle$:

$$N \notin \left(2, \frac{3 + \sqrt{5}}{2} \right)$$

And so on. In short, by doing computations, we are led to the conclusion in the statement, by a kind of recurrence, involving a certain family of orthogonal polynomials.

(3) In practice now, following [56], the most elegant way of proving the result is by using the fact, explained in Theorem 16.14, that that sequence of Jones projections $e_1, e_2, e_3, \dots \subset B(H)$ generates a copy of the Temperley-Lieb algebra of index N :

$$TL_N \subset B(H)$$

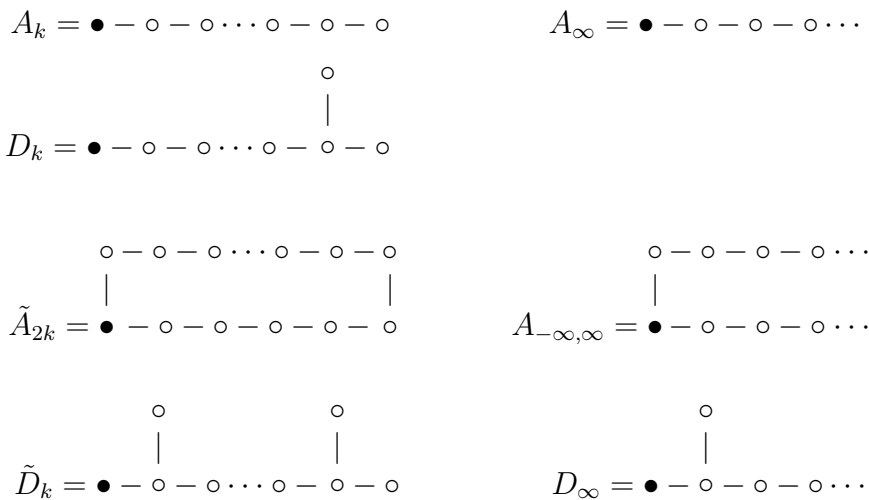
With this result in hand, we must prove that such a representation cannot exist in index $N < 4$, unless we are in the following special situation:

$$N = 4 \cos^2 \left(\frac{\pi}{n} \right)$$

But this can be proved by using some suitable trace and positivity manipulations on TL_N , as in (2) above, and for full details here, we refer to Jones' paper [56].

(4) As for the second assertion in the statement, this comes via a refinement of all this, the key ingredient being the fact that in index $N \leq 4$, and in fact more generally in the amenable case, as discussed before, we must have $N = \|\Gamma\|^2$. See [56]. \square

More in detail now, the usual Coxeter-Dynkin ADE graphs are as follows:



There are as well a number of exceptional Coxeter-Dynkin graphs. First we have:

$$\begin{array}{l}
 E_6 = \bullet - \circ - \circ - \overset{\circ}{\underset{|}{\circ}} - \circ - \circ - \circ \\
 E_7 = \bullet - \circ - \circ - \circ - \overset{\circ}{\underset{|}{\circ}} - \circ - \circ - \circ \\
 E_8 = \bullet - \circ - \circ - \circ - \circ - \circ - \overset{\circ}{\underset{|}{\circ}} - \circ - \circ - \circ
 \end{array}$$

Finally, we have index 4 versions of the above exceptional graphs, as follows:

$$\begin{array}{l}
 \tilde{E}_6 = \bullet - \circ - \circ - \overset{\circ}{\underset{\circ}{\underset{|}{\circ}}} - \circ - \circ - \circ \\
 \tilde{E}_7 = \bullet - \circ - \circ - \circ - \overset{\circ}{\underset{|}{\circ}} - \circ - \circ - \circ - \circ \\
 \tilde{E}_8 = \bullet - \circ - \circ - \circ - \circ - \circ - \circ - \overset{\circ}{\underset{|}{\circ}} - \circ - \circ - \circ
 \end{array}$$

Getting back now to Theorem 16.38, with this list in hand, the story is not over, because we still have to understand which of these graphs can really appear as principal graphs of subfactors. And, for those graphs which can appear, we must understand the structure and classification of the subfactors of R , having them as principal graphs.

In short, still a lot of work to be done, as a continuation of Theorem 16.38. The subfactors of index ≤ 4 were intensively studied in the 80s and early 90s, and about 10 years after Jones' foundational paper [56], a complete classification result was found, with contributions by many authors. A simplified form of this result is as follows:

THEOREM 16.39. *The principal graphs of subfactors of index ≤ 4 are:*

- (1) *Index < 4 graphs:* $A_k, D_{\text{even}}, E_6, E_8$.
- (2) *Index 4 finite graphs:* $\tilde{A}_{2k}, \tilde{D}_k, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8$.
- (3) *Index 4 infinite graphs:* $A_\infty, A_{-\infty, \infty}, D_\infty$.

PROOF. As already mentioned, this is something quite heavy, with contributions by many authors, and notably Ocneanu [72]. Observe that, as a subtlety of subfactor theory, the graphs D_{odd} and E_7 don't appear in the above list. For a discussion, see [73]. \square

With the above understood, we can now have a more conceptual look at the random walk computations from chapter 3. Let us recall indeed from there that we have:

DEFINITION 16.40. *The Poincaré series of a rooted bipartite graph X is*

$$f(z) = \sum_{k=0}^{\infty} L_{2k} z^k$$

where L_{2k} is the number of $2k$ -loops based at the root.

We can see that this is in tune with Definition 16.36, in the sense that the Poincaré series constructed there coincides with the above one, with X being the principal graph. Thus, when looking now at the spectral measures, these coincide too, and we have:

CONCLUSION 16.41. *The spectral measures of ADE graphs that we computed in chapter 3 are, from a subfactor viewpoint, the spectral measures of subfactors of index ≤ 4 .*

Which is certainly something very nice, and for the continuation of the story here, we refer to [12], [45], [60] and related papers. There is as well a certain connection with the Deligne work on the exceptional series of Lie groups, which is not understood yet.

Regarding now the subfactors of index $N \in (4, 5]$, and also of small index above 5, these can be classified, but this is a long and complicated story. Let us just record here the result in index 5, which is something quite easy to formulate, as follows:

THEOREM 16.42. *The principal graphs of the irreducible index 5 subfactors are:*

- (1) A_{∞} , and a non-extremal perturbation of $A_{\infty}^{(1)}$.
- (2) The McKay graphs of $\mathbb{Z}_5, D_5, GA_1(5), A_5, S_5$.
- (3) The twists of the McKay graphs of A_5, S_5 .

PROOF. This is a heavy result, and we refer to [60] and subsequent papers for the whole story, which involved the work of many people, all over the 2000s. \square

Next, in index $N = 6$, the subfactors cannot be classified, at least in general, due to several uncountable families, coming from groups, group duals, and more generally compact quantum groups. The exact assumption to be added is not known yet.

Summarizing, the current small index classification problem meets considerable difficulties in index $N = 6$, and right below. In small index $N > 6$ the situation is largely unexplored. We refer here to [60] and the recent literature on the subject.

So long for small index. In higher index now, $N \in (4, \infty)$, where the Jones result in [60] does apply, the precise correct “blowup” manipulation on the spectral measure is not known yet. Again, we refer here to [12], [45], [60] and related papers.

Finally, one interesting question regards the case of large, uniform index, $N \gg 0$. Here the main examples are those coming from Theorem 16.35, with the underlying compact quantum group G being assumed to be easy. But here, there is no need to do further probability, because we already did this, in chapters 13-14 above.

16e. Exercises

Congratulations for having read this book, and no exercises for this final chapter. However, if looking for a good question, learn more, from Connes, Popa and others about the Murray-von Neumann hyperfinite factor R , and start doing some math, inside it.

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