



# Methods of free probability

Teo Banica

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# Methods of free probability

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ABSTRACT. This is a joint introduction to classical and free probability, which are twin sisters. We first review the foundations of classical probability, notably with the main limiting theorems (CLT, CCLT, PLT, CPLT), and with a look into examples coming from Lie groups and random matrices. Then we present the foundations and main results of free probability, notably with free limiting theorems, and with a look into examples coming from quantum groups and random matrices. We discuss then a number of more advanced aspects, in relation on one hand with free geometry, and on the other hand with questions in operator algebras coming from subfactor theory.

## Preface

Probability theory, and the probabilistic way of thinking, have seen a considerable surge in the last years, with virtually every single branch of mathematics being affected. It goes without saying that everything mathematics coming from quantum mechanics, which actually accounts for a big part of pure mathematics as we know it, has some probability behind, and this has become more and more visible during recent years. The same goes of course for statistical mechanics, once again somehow by definition. As in what regards classical mechanics, randomness of the initial data is certainly a very fruitful idea too. Finally, old branches of pure mathematics, such as number theory, are increasingly becoming more analytic, and more probabilistic too.

At the technical level, probability theory comes in many flavors. However, if there is one thing to be known, having interesting mathematics and physics behind, this is the fact that classical probability theory has a “twin sister”, namely free probability.

Free probability was introduced by Voiculescu in the mid 80s, with motivation coming from general quantum mechanics, and more specifically with a number of operator algebra questions in mind. Among the main discoveries of Voiculescu was the fact that Wigner’s semicircle law, coming from advanced quantum physics and random matrices, appears as the “free analogue” of the normal law. This has led to a lot of interest and activity in free probability, with the subject having now deep ties to operator algebras, random matrices, quantum groups, noncommutative geometry, and virtually any other branch of mathematics coming from quantum mechanics, or statistical mechanics.

The present book is an introduction to free probability, with the aim of keeping things as simple and concrete as possible, while still being complete. Our goals will be on one hand that of explaining the definition and main properties of free probability, in analogy with the definition and main properties of classical probability, and by keeping the presentation as elementary as possible, and on the other hand to go, at least a little bit, into each of the above-mentioned classes of examples and applications, namely operator algebras, random matrices, quantum groups and noncommutative geometry.

The first half of the book contains basic material, all beautiful and useful things, leading to free probability. Part I is concerned with classical probability, or rather with

selected topics from classical probability, which extend well to the free case. These include the standard classical limiting theorems (CLT, CCLT, PLT, CPLT), all done via the moment method and combinatorics, and then a discussion regarding Lie groups, and Weingarten calculus. Part II is an introduction to the random matrices, benefiting from the probability theory learned in Part I, and making a transition towards the free probability theory from Parts III-IV. The main results here are the classical limiting theorems of Wigner and Marchenko-Pastur, both done via the moment method and combinatorics, and with a look into the block-modified random matrices too.

The second half of the book is concerned with free probability itself, and applications. Part III deals with the definition and main properties of free probability, central here being the free analogues of the classical limiting theorems (CLT, CCLT, PLT, CPLT), following Voiculescu. Our approach is based on standard calculus and light operator algebra theory, making somehow a compromise between the two standard approaches, from the books of Voiculescu-Dykema-Nica [90] and Nica-Speicher [70], which are respectively heavier on operator algebras, and using combinatorics instead of calculus. We will explain in particular the Bercovici-Pata bijection, and the block-modified random matrix models for the corresponding main free laws. As for Part IV, this deals with applications to quantum groups, noncommutative geometry, operator algebras and subfactors.

All in all, many things to be discussed, with this book meant to correspond to a 1-year graduate course. At the level of things which are not discussed, for advanced combinatorics you have Nica-Speicher [70]. For advanced applications to operator algebras, there is Voiculescu-Dykema-Nica [90] and Hiai-Petz [53]. As for advanced random matrix theory, you have here Anderson-Guionnet-Zeitouni [1] and Mingo-Speicher [68]. So, in the hope that you will like free probability, and end up learning everything, from here and from [1], [53], [68], [70], [90], with the precise order being more a matter of taste.

I learned myself free probability long ago, as a graduate student, from [90], with my first research paper being a 1996 note on the circular variables [5]. Later I started doing quantum groups, and sometimes random matrices too, with free probability always in mind. I am grateful to Mireille Capitaine, Benoît Collins, Steve Curran, Ion Nechita and Roland Speicher, for substantial joint work on the subject. Many thanks go as well to my cats. No serious science can be done without advice from a cat or two.

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## Part I

# Classical probability

*The Magical Mystery Tour*  
*Is coming to take you away*  
*Coming to take you away*  
*Take you today*

## CHAPTER 1

### The normal law

#### 1a. Probability theory

Generally speaking, probability theory is best learned by flipping coins and throwing dices. At a more advanced level, which is playing cards, we have:

**THEOREM 1.1.** *The probabilities at poker are as follows:*

- (1) *One pair:* 0.533.
- (2) *Two pairs:* 0.120.
- (3) *Three of a kind:* 0.053.
- (4) *Full house:* 0.006.
- (5) *Straight:* 0.005.
- (6) *Four of a kind:* 0.001.
- (7) *Flush:* 0.000.
- (8) *Straight flush:* 0.000.

**PROOF.** Let us consider indeed our deck of 32 cards, 7, 8, 9, 10,  $J$ ,  $Q$ ,  $K$ ,  $A$ . The total number of possibilities for a poker hand is:

$$\binom{32}{5} = \frac{32 \cdot 31 \cdot 30 \cdot 29 \cdot 28}{2 \cdot 3 \cdot 4 \cdot 5} = 32 \cdot 31 \cdot 29 \cdot 7$$

(1) For having a pair, the number of possibilities is:

$$N = \binom{8}{1} \binom{4}{2} \times \binom{7}{3} \binom{4}{1}^3 = 8 \cdot 6 \cdot 35 \cdot 64$$

Thus, the probability of having a pair is:

$$P = \frac{8 \cdot 6 \cdot 35 \cdot 64}{32 \cdot 31 \cdot 29 \cdot 7} = \frac{6 \cdot 5 \cdot 16}{31 \cdot 29} = \frac{480}{899} = 0.533$$

(2) For having two pairs, the number of possibilities is:

$$N = \binom{8}{2} \binom{4}{2}^2 \times \binom{24}{1} = 28 \cdot 36 \cdot 24$$

Thus, the probability of having two pairs is:

$$P = \frac{28 \cdot 36 \cdot 24}{32 \cdot 31 \cdot 29 \cdot 7} = \frac{36 \cdot 3}{31 \cdot 29} = \frac{108}{899} = 0.120$$

(3) For having three of a kind, the number of possibilities is:

$$N = \binom{8}{1} \binom{4}{3} \times \binom{7}{2} \binom{4}{1}^2 = 8 \cdot 4 \cdot 21 \cdot 16$$

Thus, the probability of having three of a kind is:

$$P = \frac{8 \cdot 4 \cdot 21 \cdot 16}{32 \cdot 31 \cdot 29 \cdot 7} = \frac{3 \cdot 16}{31 \cdot 29} = \frac{48}{899} = 0.053$$

(4) For having full house, the number of possibilities is:

$$N = \binom{8}{1} \binom{4}{3} \times \binom{7}{1} \binom{4}{2} = 8 \cdot 4 \cdot 7 \cdot 6$$

Thus, the probability of having full house is:

$$P = \frac{8 \cdot 4 \cdot 7 \cdot 6}{32 \cdot 31 \cdot 29 \cdot 7} = \frac{6}{31 \cdot 29} = \frac{6}{899} = 0.006$$

(5) For having a straight, the number of possibilities is:

$$N = 4 \left[ \binom{4}{1}^4 - 4 \right] = 16 \cdot 63$$

Thus, the probability of having a straight is:

$$P = \frac{16 \cdot 63}{32 \cdot 31 \cdot 29 \cdot 7} = \frac{9}{2 \cdot 31 \cdot 29} = \frac{9}{1798} = 0.005$$

(6) For having four of a kind, the number of possibilities is:

$$N = \binom{8}{1} \binom{4}{4} \times \binom{7}{1} \binom{4}{1} = 8 \cdot 7 \cdot 4$$

Thus, the probability of having four of a kind is:

$$P = \frac{8 \cdot 7 \cdot 4}{32 \cdot 31 \cdot 29 \cdot 7} = \frac{1}{31 \cdot 29} = \frac{1}{899} = 0.001$$

(7) For having a flush, the number of possibilities is:

$$N = 4 \left[ \binom{8}{4} - 4 \right] = 4 \cdot 66$$

Thus, the probability of having a flush is:

$$P = \frac{4 \cdot 66}{32 \cdot 31 \cdot 29 \cdot 7} = \frac{33}{4 \cdot 31 \cdot 29 \cdot 7} = \frac{9}{25172} = 0.000$$

(8) For having a straight flush, the number of possibilities is:

$$N = 4 \cdot 4$$

Thus, the probability of having a straight flush is:

$$P = \frac{4 \cdot 4}{32 \cdot 31 \cdot 29 \cdot 7} = \frac{1}{2 \cdot 31 \cdot 29 \cdot 7} = \frac{1}{12586} = 0.000$$

Thus, we have obtained the numbers in the statement.  $\square$

Summarizing, probability is basically about binomials and factorials, and ultimately about numbers. We will see later on that, in connection with more advanced questions, of continuous nature, some standard calculus comes into play as well.

Let us discuss now the general theory. The fundamental result in probability is the Central Limit Theorem (CLT), and our first task will be that of explaining this. With the idea in mind of doing things a bit abstractly, our starting point will be:

**DEFINITION 1.2.** *Let  $X$  be a probability space, that is, a space with a probability measure, and with the corresponding integration denoted  $\mathbb{E}$ , and called expectation.*

- (1) *The random variables are the real functions  $f \in L^\infty(X)$ .*
- (2) *The moments of such a variable are the numbers  $M_k(f) = \mathbb{E}(f^k)$ .*
- (3) *The law of such a variable is the measure given by  $M_k(f) = \int_{\mathbb{R}} x^k d\mu_f(x)$ .*

Here the fact that  $\mu_f$  exists indeed is not trivial. By linearity, we would like to have a real probability measure making hold the following formula, for any  $P \in \mathbb{R}[X]$ :

$$\mathbb{E}(P(f)) = \int_{\mathbb{R}} P(x) d\mu_f(x)$$

By using a standard continuity argument, it is enough to have this formula for the characteristic functions  $\chi_I$  of the arbitrary measurable sets of real numbers  $I \subset \mathbb{R}$ :

$$\mathbb{E}(\chi_I(f)) = \int_{\mathbb{R}} \chi_I(x) d\mu_f(x)$$

But this latter formula, which reads  $\mathbb{P}(f \in I) = \mu_f(I)$ , can serve as a definition for  $\mu_f$ , and we are done. Alternatively, assuming some familiarity with measure theory,  $\mu_f$  is the push-forward of the probability measure on  $X$ , via the function  $f : X \rightarrow \mathbb{R}$ .

Next in line, we need to talk about independence. We can do this as follows:

**DEFINITION 1.3.** *Two variables  $f, g \in L^\infty(X)$  are called independent when*

$$\mathbb{E}(f^k g^l) = \mathbb{E}(f^k) \mathbb{E}(g^l)$$

*happens, for any  $k, l \in \mathbb{N}$ .*

Again, this definition hides some non-trivial things. Indeed, by linearity, we would like to have a formula as follows, valid for any polynomials  $P, Q \in \mathbb{R}[X]$ :

$$\mathbb{E}[P(f)Q(g)] = \mathbb{E}[P(f)] \mathbb{E}[Q(g)]$$

By a continuity argument, it is enough to have this formula for characteristic functions  $\chi_I, \chi_J$  of the arbitrary measurable sets of real numbers  $I, J \subset \mathbb{R}$ :

$$\mathbb{E}[\chi_I(f)\chi_J(g)] = \mathbb{E}[\chi_I(f)] \mathbb{E}[\chi_J(g)]$$

Thus, we are led to the usual definition of independence, namely:

$$\mathbb{P}(f \in I, g \in J) = \mathbb{P}(f \in I) \mathbb{P}(g \in J)$$

All this might seem a bit abstract, but in practice, the idea is of course that  $f, g$  must be independent, in an intuitive, real-life sense. As a first result now, we have:

**PROPOSITION 1.4.** *Assuming that  $f, g \in L^\infty(X)$  are independent, we have*

$$\mu_{f+g} = \mu_f * \mu_g$$

where  $*$  is the convolution of real probability measures.

**PROOF.** We have the following computation, using the independence of  $f, g$ :

$$\begin{aligned} M_k(f+g) &= \mathbb{E}((f+g)^k) \\ &= \sum_r \binom{k}{r} \mathbb{E}(f^r g^{k-r}) \\ &= \sum_r \binom{k}{r} M_r(f) M_{k-r}(g) \end{aligned}$$

On the other hand, by using the Fubini theorem, we have as well:

$$\begin{aligned} \int_{\mathbb{R}} x^k d(\mu_f * \mu_g)(x) &= \int_{\mathbb{R} \times \mathbb{R}} (x+y)^k d\mu_f(x) d\mu_g(y) \\ &= \sum_r \binom{k}{r} \int_{\mathbb{R}} x^r d\mu_f(x) \int_{\mathbb{R}} y^{k-r} d\mu_g(y) \\ &= \sum_r \binom{k}{r} M_r(f) M_{k-r}(g) \end{aligned}$$

Thus  $\mu_{f+g}$  and  $\mu_f * \mu_g$  have the same moments, so they coincide, as desired.  $\square$

Here is now a second result on independence, which is something more advanced:

**THEOREM 1.5.** *Assuming that  $f, g \in L^\infty(X)$  are independent, we have*

$$F_{f+g} = F_f F_g$$

where  $F_f(x) = \mathbb{E}(e^{ixf})$  is the Fourier transform.

PROOF. We have the following computation, using Proposition 1.4 and Fubini:

$$\begin{aligned}
 F_{f+g}(x) &= \int_{\mathbb{R}} e^{ixz} d\mu_{f+g}(z) \\
 &= \int_{\mathbb{R}} e^{ixz} d(\mu_f * \mu_g)(z) \\
 &= \int_{\mathbb{R} \times \mathbb{R}} e^{ix(z+t)} d\mu_f(z) d\mu_g(t) \\
 &= \int_{\mathbb{R}} e^{ixz} d\mu_f(z) \int_{\mathbb{R}} e^{ixt} d\mu_g(t) \\
 &= F_f(x) F_g(x)
 \end{aligned}$$

Thus, we are led to the conclusion in the statement.  $\square$

This was for the foundations of probability theory, quickly explained. For further reading, a classical book is Feller [45]. A nice, more modern book is Durrett [42].

### 1b. Central limits

The main result in classical probability is the Central Limit Theorem (CLT), that we will explain now. Let us first discuss the normal distributions, that we will see later to appear as limiting laws in the CLT. We will need the following standard result:

THEOREM 1.6. *We have the following formula,*

$$\int_{\mathbb{R}} e^{-x^2} dx = \sqrt{\pi}$$

*called Gauss integral formula.*

PROOF. Let  $I$  be the integral in the statement. By using polar coordinates, namely  $x = r \cos t$ ,  $y = r \sin t$ , with the corresponding Jacobian being  $r$ , we have:

$$\begin{aligned}
 I^2 &= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-x^2-y^2} dx dy \\
 &= \int_0^{2\pi} \int_0^\infty e^{-r^2} r dr dt \\
 &= 2\pi \int_0^\infty \left( -\frac{e^{-r^2}}{2} \right)' dr \\
 &= \pi
 \end{aligned}$$

Thus, we are led to the formula in the statement.  $\square$

We can now introduce the normal distributions, as follows:

DEFINITION 1.7. *The normal law of parameter 1 is the following measure:*

$$g_1 = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

*More generally, the normal law of parameter  $t > 0$  is the following measure:*

$$g_t = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} dx$$

*These are also called Gaussian distributions, with “g” standing for Gauss.*

These laws are usually denoted  $\mathcal{N}(0, 1)$  and  $\mathcal{N}(0, t)$ , but since we will be doing in this book all kinds of probability, we will use simplified notations for all our measures. Let us mention as well that the normal laws traditionally have 2 parameters, the mean and the variance, but here we will not need the mean, our theory using centered laws. Finally, observe that the above laws have indeed mass 1, as they should, due to:

$$\int_{\mathbb{R}} e^{-x^2/2t} dx = \int_{\mathbb{R}} e^{-y^2} \sqrt{2t} dy = \sqrt{2\pi t}$$

Generally speaking, the normal laws appear as bit everywhere, in real life. The reasons behind this come from the Central Limit Theorem (CLT), that we will explain in a moment, after developing some more general theory. As a first result, we have:

PROPOSITION 1.8. *We have the variance formula*

$$V(g_t) = t$$

*valid for any  $t > 0$ .*

PROOF. The first moment is 0, because our normal law  $g_t$  is centered. As for the second moment, this can be computed as follows:

$$\begin{aligned} M_2 &= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} x^2 e^{-x^2/2t} dx \\ &= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} (tx) \left( -e^{-x^2/2t} \right)' dx \\ &= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} t e^{-x^2/2t} dx \\ &= t \end{aligned}$$

We conclude from this that the variance is  $V = M_2 = t$ . □

Here is another result, which is widely useful in practice:

THEOREM 1.9. *We have the following formula, valid for any  $t > 0$ :*

$$F_{g_t}(x) = e^{-tx^2/2}$$

*In particular, the normal laws satisfy  $g_s * g_t = g_{s+t}$ , for any  $s, t > 0$ .*

PROOF. The Fourier transform formula can be established as follows:

$$\begin{aligned}
F_{g_t}(x) &= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} e^{-z^2/2t + izx} dz \\
&= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} e^{-(z/\sqrt{2t} - \sqrt{t/2} iz)^2 - tx^2/2} dz \\
&= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} e^{-y^2 - tx^2/2} \sqrt{2t} dy \\
&= \frac{1}{\sqrt{\pi}} e^{-tx^2/2} \int_{\mathbb{R}} e^{-y^2} dy \\
&= e^{-tx^2/2}
\end{aligned}$$

As for  $g_s * g_t = g_{s+t}$ , this follows from Theorem 1.5,  $\log F_{g_t}$  being linear in  $t$ .  $\square$

We are now ready to state and prove the CLT, as follows:

**THEOREM 1.10 (CLT).** *Given real variables  $f_1, f_2, f_3, \dots \in L^\infty(X)$  which are i.i.d., centered, and with common variance  $t > 0$ , we have*

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n f_i \sim g_t$$

with  $n \rightarrow \infty$ , in moments.

PROOF. In terms of moments, the Fourier transform is given by:

$$\begin{aligned}
F_f(x) &= \mathbb{E} \left( \sum_{r=0}^{\infty} \frac{(ixf)^r}{r!} \right) \\
&= \sum_{r=0}^{\infty} \frac{(ix)^r \mathbb{E}(f^r)}{r!} \\
&= \sum_{r=0}^{\infty} \frac{i^r M_r(f)}{r!} x^r
\end{aligned}$$

Thus, the Fourier transform of the variable in the statement is:

$$\begin{aligned}
F(x) &= \left[ F_f \left( \frac{x}{\sqrt{n}} \right) \right]^n \\
&= \left[ 1 - \frac{tx^2}{2n} + O(n^{-2}) \right]^n \\
&\simeq e^{-tx^2/2}
\end{aligned}$$

But this function being the Fourier transform of  $g_t$ , we obtain the result.  $\square$

Let us discuss now some further properties of the normal law. We first have:

PROPOSITION 1.11. *The even moments of the normal law are the numbers*

$$M_k(g_t) = t^{k/2} \times k!!$$

where  $k!! = (k-1)(k-3)(k-5)\dots$ , and the odd moments vanish.

PROOF. We have the following computation, valid for any integer  $k \in \mathbb{N}$ :

$$\begin{aligned} M_k &= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} y^k e^{-y^2/2t} dy \\ &= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} (ty^{k-1}) \left(-e^{-y^2/2t}\right)' dy \\ &= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} t(k-1)y^{k-2} e^{-y^2/2t} dy \\ &= t(k-1) \times \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} y^{k-2} e^{-y^2/2t} dy \\ &= t(k-1)M_{k-2} \end{aligned}$$

Now recall from the proof of Proposition 1.8 that we have  $M_0 = 1$ ,  $M_1 = 0$ . Thus by recurrence, we are led to the formula in the statement.  $\square$

We have the following alternative formulation of the above result:

PROPOSITION 1.12. *The moments of the normal law are the numbers*

$$M_k(g_t) = t^{k/2} |P_2(k)|$$

where  $P_2(k)$  is the set of pairings of  $\{1, \dots, k\}$ .

PROOF. Let us count the pairings of  $\{1, \dots, k\}$ . In order to have such a pairing, we must pair 1 with one of the numbers  $2, \dots, k$ , and then use a pairing of the remaining  $k-2$  numbers. Thus, we have the following recurrence formula:

$$|P_2(k)| = (k-1)|P_2(k-2)|$$

As for the initial data, this is  $P_1 = 0$ ,  $P_2 = 1$ . Thus, we are led to the result.  $\square$

We are not done yet, and here is one more improvement of the above:

THEOREM 1.13. *The moments of the normal law are the numbers*

$$M_k(g_t) = \sum_{\pi \in P_2(k)} t^{|\pi|}$$

where  $P_2(k)$  is the set of pairings of  $\{1, \dots, k\}$ , and  $|\cdot|$  is the number of blocks.

PROOF. This follows indeed from Proposition 1.12, because the number of blocks of a pairing of  $\{1, \dots, k\}$  is trivially  $k/2$ , independently of the pairing.  $\square$

We will see later on that many other interesting probability distributions are subject to similar formulae regarding their moments, involving partitions. Discussing this key phenomenon will be in fact a main theme of the present book.

### 1c. Spherical integrals

In a purely mathematical context, the simplest way of recovering the normal laws is by looking at the coordinates over the real spheres  $S_{\mathbb{R}}^{N-1}$ , in the  $N \rightarrow \infty$  limit. To start with, at  $N = 2$  the sphere is the unit circle  $\mathbb{T}$ , and with  $z = e^{it}$  the coordinates are  $\cos t, \sin t$ . Let us first integrate powers of these coordinates. We have here:

PROPOSITION 1.14. *We have the following formulae,*

$$\int_0^{\pi/2} \cos^k t \, dt = \int_0^{\pi/2} \sin^k t \, dt = \left(\frac{\pi}{2}\right)^{\varepsilon(k)} \frac{k!!}{(k+1)!!}$$

where  $\varepsilon(k) = 1$  if  $k$  is even, and  $\varepsilon(k) = 0$  if  $k$  is odd.

PROOF. Let us first compute the integral on the left in the statement:

$$I_k = \int_0^{\pi/2} \cos^k t \, dt$$

We do this by partial integration. We have the following formula:

$$\begin{aligned} (\cos^k t \sin t)' &= k \cos^{k-1} t (-\sin t) \sin t + \cos^k t \cos t \\ &= (k+1) \cos^{k+1} t - k \cos^{k-1} t \end{aligned}$$

By integrating between 0 and  $\pi/2$ , we obtain the following formula:

$$(k+1)I_{k+1} = kI_{k-1}$$

Thus we can compute  $I_k$  by recurrence, and we obtain in this way:

$$\begin{aligned} I_k &= \frac{k-1}{k} I_{k-2} \\ &= \frac{k-1}{k} \cdot \frac{k-3}{k-2} I_{k-4} \\ &= \frac{k-1}{k} \cdot \frac{k-3}{k-2} \cdot \frac{k-5}{k-4} I_{k-6} \\ &\vdots \\ &= \frac{k!!}{(k+1)!!} I_{1-\varepsilon(k)} \end{aligned}$$

The initial data being  $I_0 = \pi/2$  and  $I_1 = 1$ , we obtain the result. As for the second formula, this follows from the first one, with the change of variables  $t = \pi/2 - s$ .  $\square$

More generally now, we have the following result:

THEOREM 1.15. *We have the following formula,*

$$\int_0^{\pi/2} \cos^r t \sin^s t dt = \left(\frac{\pi}{2}\right)^{\varepsilon(r)\varepsilon(s)} \frac{r!!s!!}{(r+s+1)!!}$$

where  $\varepsilon(r) = 1$  if  $r$  is even, and  $\varepsilon(r) = 0$  if  $r$  is odd.

PROOF. Consider the integral in the statement, namely:

$$I_{rs} = \int_0^{\pi/2} \cos^r t \sin^s t dt$$

In order to do the partial integration, observe that we have:

$$\begin{aligned} (\cos^r t \sin^s t)' &= r \cos^{r-1} t (-\sin t) \sin^s t + \cos^r t \cdot s \sin^{s-1} t \cos t \\ &= -r \cos^{r-1} t \sin^{s+1} t + s \cos^{r+1} t \sin^{s-1} t \end{aligned}$$

By integrating between 0 and  $\pi/2$ , we obtain, for  $r, s > 0$ :

$$r I_{r-1, s+1} = s I_{r+1, s-1}$$

Thus, we can compute  $I_{rs}$  by recurrence. When  $s$  is even we have:

$$\begin{aligned} I_{rs} &= \frac{s-1}{r+1} I_{r+2, s-2} \\ &= \frac{s-1}{r+1} \cdot \frac{s-3}{r+3} I_{r+4, s-4} \\ &= \frac{s-1}{r+1} \cdot \frac{s-3}{r+3} \cdot \frac{s-5}{r+5} I_{r+6, s-6} \\ &\vdots \\ &= \frac{r!!s!!}{(r+s)!!} I_{r+s} \end{aligned}$$

But the last term comes from Proposition 1.14, and we obtain the result:

$$\begin{aligned} I_{rs} &= \frac{r!!s!!}{(r+s)!!} I_{r+s} \\ &= \frac{r!!s!!}{(r+s)!!} \left(\frac{\pi}{2}\right)^{\varepsilon(r+s)} \frac{(r+s)!!}{(r+s+1)!!} \\ &= \left(\frac{\pi}{2}\right)^{\varepsilon(r)\varepsilon(s)} \frac{r!!s!!}{(r+s+1)!!} \end{aligned}$$

Observe that this gives the result for  $r$  even as well, by symmetry. In the remaining case now, where both the exponents  $r, s$  are odd, we can use once again the formula

$rI_{r-1,s+1} = sI_{r+1,s-1}$  found above, and the recurrence goes as follows:

$$\begin{aligned}
 I_{rs} &= \frac{s-1}{r+1} I_{r+2,s-2} \\
 &= \frac{s-1}{r+1} \cdot \frac{s-3}{r+3} I_{r+4,s-4} \\
 &= \frac{s-1}{r+1} \cdot \frac{s-3}{r+3} \cdot \frac{s-5}{r+5} I_{r+6,s-6} \\
 &\vdots \\
 &= \frac{r!!s!!}{(r+s-1)!!} I_{r+s-1,1}
 \end{aligned}$$

In order to compute the last term, observe that we have:

$$\begin{aligned}
 I_{r1} &= \int_0^{\pi/2} \cos^r t \sin t \, dt \\
 &= -\frac{1}{r+1} \int_0^{\pi/2} (\cos^{r+1} t)' \, dt \\
 &= \frac{1}{r+1}
 \end{aligned}$$

Thus, we obtain the formula in the statement, the exponent of  $\pi/2$  appearing there being  $\varepsilon(r)\varepsilon(s) = 0 \cdot 0 = 0$  in the present case, and this finishes the proof.  $\square$

In order to deal now with the higher spheres, we will use spherical coordinates:

**THEOREM 1.16.** *We have spherical coordinates in  $N$  dimensions,*

$$\begin{cases}
 x_1 &= r \cos t_1 \\
 x_2 &= r \sin t_1 \cos t_2 \\
 \vdots & \\
 x_{N-1} &= r \sin t_1 \sin t_2 \dots \sin t_{N-2} \cos t_{N-1} \\
 x_N &= r \sin t_1 \sin t_2 \dots \sin t_{N-2} \sin t_{N-1}
 \end{cases}$$

*the corresponding Jacobian being given by the following formula:*

$$J(r, t) = r^{N-1} \sin^{N-2} t_1 \sin^{N-3} t_2 \dots \sin^2 t_{N-3} \sin t_{N-2}$$

**PROOF.** The fact that we have indeed spherical coordinates is clear. Regarding the Jacobian, the proof is similar to the one from 2 dimensions, by developing the determinant over the last column, and then by proceeding by recurrence.  $\square$

As a first application, we can compute the volume of the sphere:

THEOREM 1.17. *The volume of the unit sphere in  $\mathbb{R}^N$  is given by*

$$\frac{V}{2^N} = \left(\frac{\pi}{2}\right)^{[N/2]} \frac{1}{(N+1)!!}$$

*with our usual convention  $m!! = (m-1)(m-3)(m-5)\dots$  for double factorials.*

PROOF. If we denote by  $Q$  the positive part of the sphere, obtained by cutting the sphere in  $2^N$  parts, we have, by using Theorems 1.15 and 1.16 and Fubini:

$$\begin{aligned} \frac{V}{2^N} &= \int_0^1 \int_0^{\pi/2} \dots \int_0^{\pi/2} r^{N-1} \sin^{N-2} t_1 \dots \sin t_{N-2} dr dt_1 \dots dt_{N-1} \\ &= \int_0^1 r^{N-1} dr \int_0^{\pi/2} \sin^{N-2} t_1 dt_1 \dots \int_0^{\pi/2} \sin t_{N-2} dt_{N-2} \int_0^{\pi/2} 1 dt_{N-1} \\ &= \frac{1}{N} \times \left(\frac{\pi}{2}\right)^{[N/2]} \times \frac{(N-2)!!}{(N-1)!!} \cdot \frac{(N-3)!!}{(N-2)!!} \dots \frac{2!!}{3!!} \cdot \frac{1!!}{2!!} \cdot 1 \\ &= \left(\frac{\pi}{2}\right)^{[N/2]} \frac{1}{(N+1)!!} \end{aligned}$$

Here we have used the following formula for computing the exponent of  $\pi/2$ , where  $\varepsilon(r) = 1$  if  $r$  is even and  $\varepsilon(r) = 0$  if  $r$  is odd, as in Theorem 1.15:

$$\begin{aligned} \varepsilon(0) + \varepsilon(1) + \varepsilon(2) + \dots + \varepsilon(N-2) &= 1 + 0 + 1 + 0 + \dots + \varepsilon(N-2) \\ &= \left\lfloor \frac{N-2}{2} \right\rfloor + 1 \\ &= \left\lfloor \frac{N}{2} \right\rfloor \end{aligned}$$

Thus, we are led to the conclusion in the statement.  $\square$

Let us discuss now the computation of the arbitrary polynomial integrals, over the spheres of arbitrary dimension. The result here is as follows:

THEOREM 1.18. *The spherical integral of  $x_{i_1} \dots x_{i_r}$  vanishes, unless each index  $a \in \{1, \dots, N\}$  appears an even number of times in the sequence  $i_1, \dots, i_r$ . We have*

$$\int_{S_{\mathbb{R}}^{N-1}} x_{i_1} \dots x_{i_r} dx = \frac{(N-1)!! k_1!! \dots k_N!!}{(N + \sum k_i - 1)!!}$$

*with  $k_a$  being this number of occurrences.*

PROOF. We can restrict attention to the case  $k_a \in 2\mathbb{N}$ , since the other integrals vanish. The integral in the statement can be written in spherical coordinates, as follows:

$$I = \frac{2^N}{V} \int_0^{\pi/2} \dots \int_0^{\pi/2} x_1^{k_1} \dots x_N^{k_N} J dt_1 \dots dt_{N-1}$$

In this formula  $V$  is the volume of the sphere,  $J$  is the Jacobian, and the  $2^N$  factor comes from the restriction to the  $1/2^N$  part of the sphere where all the coordinates are positive. According to the formula in Theorem 1.17, the normalization constant is:

$$\frac{2^N}{V} = \left(\frac{2}{\pi}\right)^{[N/2]} (N+1)!!$$

As for the unnormalized integral, this is given by:

$$\begin{aligned} I' = \int_0^{\pi/2} \dots \int_0^{\pi/2} & (\cos t_1)^{k_1} (\sin t_1 \cos t_2)^{k_2} \\ & \vdots \\ & (\sin t_1 \sin t_2 \dots \sin t_{N-2} \cos t_{N-1})^{k_{N-1}} \\ & (\sin t_1 \sin t_2 \dots \sin t_{N-2} \sin t_{N-1})^{k_N} \\ & \sin^{N-2} t_1 \sin^{N-3} t_2 \dots \sin^2 t_{N-3} \sin t_{N-2} \\ & dt_1 \dots dt_{N-1} \end{aligned}$$

By rearranging the terms, we obtain:

$$\begin{aligned} I' = & \int_0^{\pi/2} \cos^{k_1} t_1 \sin^{k_2+\dots+k_N+N-2} t_1 dt_1 \\ & \int_0^{\pi/2} \cos^{k_2} t_2 \sin^{k_3+\dots+k_N+N-3} t_2 dt_2 \\ & \vdots \\ & \int_0^{\pi/2} \cos^{k_{N-2}} t_{N-2} \sin^{k_{N-1}+k_N+1} t_{N-2} dt_{N-2} \\ & \int_0^{\pi/2} \cos^{k_{N-1}} t_{N-1} \sin^{k_N} t_{N-1} dt_{N-1} \end{aligned}$$

Now by using the formula in Theorem 1.15, this gives:

$$\begin{aligned} I' = & \frac{k_1!!(k_2 + \dots + k_N + N - 2)!!}{(k_1 + \dots + k_N + N - 1)!!} \left(\frac{\pi}{2}\right)^{\varepsilon(N-2)} \\ & \frac{k_2!!(k_3 + \dots + k_N + N - 3)!!}{(k_2 + \dots + k_N + N - 2)!!} \left(\frac{\pi}{2}\right)^{\varepsilon(N-3)} \\ & \vdots \\ & \frac{k_{N-2}!!(k_{N-1} + k_N + 1)!!}{(k_{N-2} + k_{N-1} + k_N + 2)!!} \left(\frac{\pi}{2}\right)^{\varepsilon(1)} \\ & \frac{k_{N-1}!!k_N!!}{(k_{N-1} + k_N + 1)!!} \left(\frac{\pi}{2}\right)^{\varepsilon(0)} \end{aligned}$$

Now observe that the various double factorials multiply up to quantity in the statement, modulo a  $(N - 1)!!$  factor, and that the  $\pi/2$  factors multiply up to:

$$F = \left(\frac{\pi}{2}\right)^{[N/2]}$$

Thus by multiplying by the normalization constant, we obtain the result.  $\square$

We can now recover the normal laws, geometrically, as follows:

**THEOREM 1.19.** *The moments of the hyperspherical variables are*

$$\int_{S_{\mathbb{R}}^{N-1}} x_i^p dx = \frac{(N-1)!!p!!}{(N+p-1)!!}$$

and the rescaled variables  $y_i = \frac{x_i}{\sqrt{N}}$  become normal and independent with  $N \rightarrow \infty$ .

**PROOF.** The moment formula in the statement follows from Theorem 1.18. As a consequence, with  $N \rightarrow \infty$  we have the following estimate:

$$\begin{aligned} \int_{S_{\mathbb{R}}^{N-1}} x_i^p dx &\simeq N^{p/2} \times p!! \\ &= N^{p/2} M_p(g_1) \end{aligned}$$

Thus, the rescaled variables  $x_i/\sqrt{N}$  become normal with  $N \rightarrow \infty$ , as claimed. As for the proof of the asymptotic independence, this is standard too, once again by using the formula in Theorem 1.18. Indeed, the joint moments of  $x_1, \dots, x_N$  are given by:

$$\begin{aligned} \int_{S_{\mathbb{R}}^{N-1}} x_1^{k_1} \dots x_N^{k_N} dx &= \frac{(N-1)!!k_1!! \dots k_N!!}{(N + \sum k_i - 1)!!} \\ &\simeq N^{\sum k_i} \times k_1!! \dots k_N!! \end{aligned}$$

By rescaling, the joint moments of the variables  $y_i = x_i/\sqrt{N}$  are given by:

$$\int_{S_{\mathbb{R}}^{N-1}} y_1^{k_1} \dots y_N^{k_N} dx \simeq k_1!! \dots k_N!!$$

Thus, we have multiplicativity, and so independence with  $N \rightarrow \infty$ , as claimed.  $\square$

As a last result about the normal laws, we can recover these as well in connection with the rotation groups. Indeed, we have the following reformulation of Theorem 1.19:

**THEOREM 1.20.** *We have the integration formula*

$$\int_{O_N} U_{ij}^p dU = \frac{(N-1)!!p!!}{(N+p-1)!!}$$

and the rescaled variables  $V_{ij} = \frac{U_{ij}}{\sqrt{N}}$  become normal and independent with  $N \rightarrow \infty$ .

PROOF. We use the basic fact that the rotations  $U \in O_N$  act on the points of the real sphere  $z \in S_{\mathbb{R}}^{N-1}$ , with the stabilizer of  $z = (1, 0, \dots, 0)$  being the subgroup  $O_{N-1} \subset O_N$ . In algebraic terms, this gives an identification as follows:

$$S_{\mathbb{R}}^{N-1} = O_N / O_{N-1}$$

In functional analytic terms, this result provides us with an embedding as follows, for any  $i$ , which makes correspond the respective integration functionals:

$$C(S_{\mathbb{R}}^{N-1}) \subset C(O_N) \quad , \quad x_i \rightarrow U_{1i}$$

With this identification made, the result follows from Theorem 1.19.  $\square$

We will see later, following [35], [95], that the relation between the orthogonal group  $O_N$  and the normal laws goes well beyond Theorem 1.20. And we will see as well, following [13], [6] and related papers, that there are also “free versions” of all this.

### 1d. Complex variables

We have seen so far a number of interesting results regarding the normal laws, and their geometric interpretation. As a last topic of this chapter, let us discuss now the complex analogues of all this. To start with, we have the following definition:

DEFINITION 1.21. *The complex Gaussian law of parameter  $t > 0$  is*

$$G_t = \text{law} \left( \frac{1}{\sqrt{2}}(a + ib) \right)$$

where  $a, b$  are independent, each following the law  $g_t$ .

As in the real case, these measures form convolution semigroups:

THEOREM 1.22. *The complex Gaussian laws have the property*

$$G_s * G_t = G_{s+t}$$

for any  $s, t > 0$ , and so they form a convolution semigroup.

PROOF. This follows indeed from the real result, namely  $g_s * g_t = g_{s+t}$ , established in Theorem 1.9 above, simply by taking real and imaginary parts.  $\square$

We have as well the following complex analogue of the CLT:

THEOREM 1.23 (CCLT). *Given complex variables  $f_1, f_2, f_3, \dots \in L^\infty(X)$  which are i.i.d., centered, and with common variance  $t > 0$ , we have*

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n f_i \sim G_t$$

with  $n \rightarrow \infty$ , in moments.

PROOF. This follows indeed from the real CLT, established in Theorem 1.10 above, simply by taking real and imaginary parts.  $\square$

Regarding now the moments, the situation is more complicated than in the real case, because in order to have good results, we have to deal with both the complex variables, and their conjugates. Let us formulate the following definition:

DEFINITION 1.24. *The moments of a complex variable  $f \in L^\infty(X)$  are the numbers*

$$M_k = \mathbb{E}(f^k)$$

*depending on colored integers  $k = \circ \bullet \bullet \circ \dots$ , with the conventions*

$$f^\emptyset = 1 \quad , \quad f^\circ = f \quad , \quad f^\bullet = \bar{f}$$

*and multiplicativity, in order to define the colored powers  $f^k$ .*

Observe that, since  $f, \bar{f}$  commute, we can permute terms, and restrict the attention to exponents of type  $k = \dots \circ \circ \circ \bullet \bullet \bullet \dots$ , if we want to. However, our result about the complex Gaussian laws, and other complex laws, later on, will actually look better without doing is, and so we will use Definition 1.24 as stated.

Given a colored integer  $k = \circ \bullet \bullet \circ \dots$ , we say that a pairing  $\pi \in P_2(k)$  is matching when it pairs  $\circ - \bullet$  symbols. With these conventions, we have the following result:

THEOREM 1.25. *The moments of the complex normal law are the numbers*

$$M_k(G_t) = \sum_{\pi \in \mathcal{P}_2(k)} t^{|\pi|}$$

*where  $\mathcal{P}_2(k)$  are the matching pairings of  $\{1, \dots, k\}$ , and  $|\cdot|$  is the number of blocks.*

PROOF. This is something basic and well-known, and there are many possible proofs, which are all instructive. We present here a rather elementary proof:

(1) We recall from the above that the moments of the real Gaussian law  $g_1$ , with respect to integer exponents  $k \in \mathbb{N}$ , are the following numbers:

$$m_k = |P_2(k)|$$

Numerically, we have the following formula, explained as well in the above:

$$m_k = \begin{cases} k!! & (k \text{ even}) \\ 0 & (k \text{ odd}) \end{cases}$$

(2) We will show here that in what concerns the complex Gaussian law  $G_1$ , similar results hold. Numerically, we will prove that we have the following formula, where a

colored integer  $k = \circ \bullet \bullet \circ \dots$  is called uniform when it contains the same number of  $\circ$  and  $\bullet$ , and where  $|k| \in \mathbb{N}$  is the length of such a colored integer:

$$M_k = \begin{cases} (|k|/2)! & (k \text{ uniform}) \\ 0 & (k \text{ not uniform}) \end{cases}$$

Now since the matching partitions  $\pi \in \mathcal{P}_2(k)$  are counted by exactly the same numbers, and this for trivial reasons, we will obtain the formula in the statement, namely:

$$M_k = |\mathcal{P}_2(k)|$$

(3) This was for the plan. In practice now, we must compute the moments, with respect to colored integer exponents  $k = \circ \bullet \bullet \circ \dots$ , of the variable in the statement:

$$f = \frac{1}{\sqrt{2}}(a + ib)$$

As a first observation, in the case where such an exponent  $k = \circ \bullet \bullet \circ \dots$  is not uniform in  $\circ, \bullet$ , a rotation argument shows that the corresponding moment of  $f$  vanishes. To be more precise, the variable  $f' = wf$  can be shown to be complex Gaussian too, for any  $w \in \mathbb{C}$ , and from  $M_k(f) = M_k(f')$  we obtain  $M_k(f) = 0$ , in this case.

(4) In the uniform case now, where the exponent  $k = \circ \bullet \bullet \circ \dots$  consists of  $p$  copies of  $\circ$  and  $p$  copies of  $\bullet$ , the corresponding moment can be computed as follows:

$$\begin{aligned} M_k &= \int (f \bar{f})^p \\ &= \frac{1}{2^p} \int (a^2 + b^2)^p \\ &= \frac{1}{2^p} \sum_r \binom{p}{r} \int a^{2r} \int b^{2p-2r} \\ &= \frac{1}{2^p} \sum_r \binom{p}{r} (2r)!! (2p-2r)!! \\ &= \frac{1}{2^p} \sum_r \frac{p!}{r!(p-r)!} \cdot \frac{(2r)!}{2^r r!} \cdot \frac{(2p-2r)!}{2^{p-r}(p-r)!} \\ &= \frac{p!}{4^p} \sum_r \binom{2r}{r} \binom{2p-2r}{p-r} \end{aligned}$$

(5) In order to finish now the computation, let us recall that we have the following formula, coming from the generalized binomial formula, or from the Taylor formula:

$$\frac{1}{\sqrt{1+t}} = \sum_{q=0}^{\infty} \binom{2q}{q} \left(\frac{-t}{4}\right)^q$$

By taking the square of this series, we obtain the following formula:

$$\begin{aligned} \frac{1}{1+t} &= \sum_{qr} \binom{2q}{q} \binom{2r}{r} \left(\frac{-t}{4}\right)^{q+r} \\ &= \sum_p \left(\frac{-t}{4}\right)^p \sum_r \binom{2r}{r} \binom{2p-2r}{p-r} \end{aligned}$$

Now by looking at the coefficient of  $t^p$  on both sides, we conclude that the sum on the right equals  $4^p$ . Thus, we can finish the moment computation in (4), as follows:

$$M_k = \frac{p!}{4^p} \times 4^p = p!$$

(6) As a conclusion, if we denote by  $|k|$  the length of a colored integer  $k = \circ \bullet \bullet \circ \dots$ , the moments of the variable  $f$  in the statement are given by:

$$M_k = \begin{cases} (|k|/2)! & (k \text{ uniform}) \\ 0 & (k \text{ not uniform}) \end{cases}$$

On the other hand, the numbers  $|\mathcal{P}_2(k)|$  are given by exactly the same formula. Indeed, in order to have a matching pairing of  $k$ , our exponent  $k = \circ \bullet \bullet \circ \dots$  must be uniform, consisting of  $p$  copies of  $\circ$  and  $p$  copies of  $\bullet$ , with  $p = |k|/2$ . But then the matching pairings of  $k$  correspond to the permutations of the  $\bullet$  symbols, as to be matched with  $\circ$  symbols, and so we have  $p!$  such pairings. Thus, we have the same formula as for the moments of  $f$ , and we are led to the conclusion in the statement.  $\square$

In practice, we also need to know how to compute joint moments of independent normal variables. We have here the following result, to be heavily used later on:

**THEOREM 1.26** (Wick formula). *Given independent variables  $f_i$ , each following the complex normal law  $G_t$ , with  $t > 0$  being a fixed parameter, we have the formula*

$$\mathbb{E}(f_{i_1}^{k_1} \dots f_{i_s}^{k_s}) = t^{s/2} \# \left\{ \pi \in \mathcal{P}_2(k) \mid \pi \leq \ker(i) \right\}$$

where  $k = k_1 \dots k_s$  and  $i = i_1 \dots i_s$ , for the joint moments of these variables.

**PROOF.** This is something well-known, and the basis for all possible computations with complex normal variables, which can be proved in two steps, as follows:

(1) Let us first discuss the case where we have a single variable  $f$ , which amounts in taking  $f_i = f$  for any  $i$  in the formula in the statement. What we have to compute here are the moments of  $f$ , with respect to colored integer exponents  $k = \circ \bullet \bullet \circ \dots$ , and the formula in the statement tells us that these moments must be:

$$\mathbb{E}(f^k) = t^{|k|/2} |\mathcal{P}_2(k)|$$

But this is the formula in Theorem 1.25, so we are done with this case.

(2) In general now, when expanding the product  $f_{i_1}^{k_1} \dots f_{i_s}^{k_s}$  and rearranging the terms, we are left with doing a number of computations as in (1), and then making the product of the expectations that we found. But this amounts in counting the partitions in the statement, with the condition  $\pi \leq \ker(i)$  there standing for the fact that we are doing the various type (1) computations independently, and then making the product.  $\square$

The above statement is one of the possible formulations of the Wick formula, and there are in fact many more formulations, which are all useful. Here is an alternative such formulation, which is quite popular, and that we will often use in what follows:

**THEOREM 1.27** (Wick formula 2). *Given independent variables  $f_i$ , each following the complex normal law  $G_t$ , with  $t > 0$  being a fixed parameter, we have the formula*

$$\mathbb{E}(f_{i_1} \dots f_{i_k} f_{j_1}^* \dots f_{j_k}^*) = t^k \# \left\{ \pi \in S_k \mid i_{\pi(r)} = j_r, \forall r \right\}$$

for the non-vanishing joint moments of these variables.

**PROOF.** This follows from the usual Wick formula, from Theorem 1.26. With some changes in the indices and notations, the formula there reads:

$$\mathbb{E}(f_{I_1}^{K_1} \dots f_{I_s}^{K_s}) = t^{s/2} \# \left\{ \sigma \in \mathcal{P}_2(K) \mid \sigma \leq \ker(I) \right\}$$

Now observe that we have  $\mathcal{P}_2(K) = \emptyset$ , unless the colored integer  $K = K_1 \dots K_s$  is uniform, in the sense that it contains the same number of  $\circ$  and  $\bullet$  symbols. Up to permutations, the non-trivial case, where the moment is non-vanishing, is the case where the colored integer  $K = K_1 \dots K_s$  is of the following special form:

$$K = \underbrace{\circ \circ \dots \circ}_k \underbrace{\bullet \bullet \dots \bullet}_k$$

So, let us focus on this case, which is the non-trivial one. Here we have  $s = 2k$ , and we can write the multi-index  $I = I_1 \dots I_s$  in the following way:

$$I = i_1 \dots i_k \ j_1 \dots j_k$$

With these changes made, the above usual Wick formula reads:

$$\mathbb{E}(f_{i_1} \dots f_{i_k} f_{j_1}^* \dots f_{j_k}^*) = t^k \# \left\{ \sigma \in \mathcal{P}_2(K) \mid \sigma \leq \ker(ij) \right\}$$

The point now is that the matching pairings  $\sigma \in \mathcal{P}_2(K)$ , with  $K = \circ \dots \circ \bullet \dots \bullet$ , of length  $2k$ , as above, correspond to the permutations  $\pi \in S_k$ , in the obvious way. With this identification made, the above modified usual Wick formula becomes:

$$\mathbb{E}(f_{i_1} \dots f_{i_k} f_{j_1}^* \dots f_{j_k}^*) = t^k \# \left\{ \pi \in S_k \mid i_{\pi(r)} = j_r, \forall r \right\}$$

Thus, we have reached to the formula in the statement, and we are done.  $\square$

Finally, here is one more formulation of the Wick formula, which is useful as well:

THEOREM 1.28 (Wick formula 3). *Given independent variables  $f_i$ , each following the complex normal law  $G_t$ , with  $t > 0$  being a fixed parameter, we have the formula*

$$\mathbb{E}(f_{i_1} f_{j_1}^* \dots f_{i_k} f_{j_k}^*) = t^k \# \left\{ \pi \in S_k \mid i_{\pi(r)} = j_r, \forall r \right\}$$

for the non-vanishing joint moments of these variables.

PROOF. This follows from our second Wick formula, from Theorem 1.27, simply by permuting the terms, as to have an alternating sequence of plain and conjugate variables. Alternatively, we can start with Theorem 1.26, and then perform the same manipulations as in the proof of Theorem 1.27, but with the exponent being this time as follows:

$$K = \underbrace{\circ \bullet \bullet \dots \bullet \circ}_{2k}$$

Thus, we are led to the conclusion in the statement.  $\square$

There are many other things that can be said about the complex normal laws, and we will be back to this, on several occasions. Getting now to geometric aspects, we have:

THEOREM 1.29. *We have the following integration formula over the complex sphere  $S_{\mathbb{C}}^{N-1} \subset \mathbb{R}^N$ , with respect to the normalized uniform measure,*

$$\int_{S_{\mathbb{C}}^{N-1}} |z_1|^{2k_1} \dots |z_N|^{2k_N} dz = 4^{\sum k_i} \frac{(2N-1)! k_1! \dots k_n!}{(2N + \sum k_i - 1)!}$$

valid for any exponents  $k_i \in \mathbb{N}$ . As for the other polynomial integrals in  $z_1, \dots, z_N$  and their conjugates  $\bar{z}_1, \dots, \bar{z}_N$ , these all vanish.

PROOF. Consider an arbitrary polynomial integral over  $S_{\mathbb{C}}^{N-1}$ , written as follows:

$$I = \int_{S_{\mathbb{C}}^{N-1}} z_{i_1} \bar{z}_{i_2} \dots z_{i_{2k-1}} \bar{z}_{i_{2k}} dz$$

By using transformations of type  $p \rightarrow \lambda p$  with  $|\lambda| = 1$ , we see that this integral  $I$  vanishes, unless each  $z_a$  appears as many times as  $\bar{z}_a$  does, and this gives the last assertion. So, assume now that we are in the non-vanishing case. Then the  $k_a$  copies of  $z_a$  and the  $k_a$  copies of  $\bar{z}_a$  produce by multiplication a factor  $|z_a|^{2k_a}$ , so we have:

$$I = \int_{S_{\mathbb{C}}^{N-1}} |z_1|^{2k_1} \dots |z_N|^{2k_N} dz$$

Now by using the standard identification  $S_{\mathbb{C}}^{N-1} \simeq S_{\mathbb{R}}^{2N-1}$ , we obtain:

$$\begin{aligned} I &= \int_{S_{\mathbb{R}}^{2N-1}} (x_1^2 + y_1^2)^{k_1} \dots (x_N^2 + y_N^2)^{k_N} d(x, y) \\ &= \sum_{r_1 \dots r_N} \binom{k_1}{r_1} \dots \binom{k_N}{r_N} \int_{S_{\mathbb{R}}^{2N-1}} x_1^{2k_1-2r_1} y_1^{2r_1} \dots x_N^{2k_N-2r_N} y_N^{2r_N} d(x, y) \end{aligned}$$

By using the formula in Theorem 1.18, we obtain:

$$\begin{aligned}
& I \\
&= \sum_{r_1 \dots r_N} \binom{k_1}{r_1} \dots \binom{k_N}{r_N} \frac{(2N-1)!!(2r_1)!! \dots (2r_N)!!(2k_1-2r_1)!! \dots (2k_N-2r_N)!!}{(2N+2\sum k_i-1)!!} \\
&= \sum_{r_1 \dots r_N} \binom{l_1}{r_1} \dots \binom{l_N}{r_N} \frac{(2N-1)!(2r_1)! \dots (2r_N)!(2k_1-2r_1)! \dots (2k_N-2r_N)!}{(2N+\sum k_i-1)!r_1! \dots r_N!(k_1-r_1)! \dots (k_N-r_N)!}
\end{aligned}$$

Now observe that can rewrite this quantity in the following way:

$$\begin{aligned}
& I \\
&= \sum_{r_1 \dots r_N} \frac{k_1! \dots k_N! (2N-1)!(2r_1)! \dots (2r_N)!(2k_1-2r_1)! \dots (2k_N-2r_N)!}{(2N+\sum k_i-1)!(r_1! \dots r_N!(k_1-r_1)! \dots (k_N-r_N)!)^2} \\
&= \sum_{r_1} \binom{2r_1}{r_1} \binom{2k_1-2r_1}{k_1-r_1} \dots \sum_{r_N} \binom{2r_N}{r_N} \binom{2k_N-2r_N}{k_N-r_N} \frac{(2N-1)!k_1! \dots k_N!}{(2N+\sum k_i-1)!} \\
&= 4^{k_1} \times \dots \times 4^{k_N} \times \frac{(2N-1)!k_1! \dots k_N!}{(2N+\sum k_i-1)!}
\end{aligned}$$

Thus, we obtain the formula in the statement.  $\square$

Regarding now the hyperspherical variables, investigated in the above in the real case, we have similar results for the complex spheres, as follows:

**THEOREM 1.30.** *The rescalings  $z_i/\sqrt{N}$  of the complex sphere coordinates*

$$z_i : S_{\mathbb{C}}^{N-1} \rightarrow \mathbb{C}$$

*as well as the rescalings  $U_{ij}/\sqrt{N}$  of the unitary group coordinates*

$$U_{ij} : U_N \rightarrow \mathbb{C}$$

*become complex Gaussian and independent with  $N \rightarrow \infty$ .*

**PROOF.** We have two assertions to be proved, the idea being as follows:

(1) In what regards the result about the complex spheres, this can be deduced from the formula in Theorem 1.29, exactly as in the real case.

(2) As for the result about the unitary groups, this follows from the result for the spheres, by using the same quotient space argument as in the real case.  $\square$

As already mentioned in the real context, it is possible to get beyond such results, by using advanced group theory. We will be back to this, in chapters 3-4 below. It is also possible to formulate “free versions” of all the above, and we will do this later.

So long for the basics of probability theory, quickly explained. For further theory, the best is to go to a dedicated probability book, such as the one of Feller [45], or Durrett

[42]. Alternatively, you can learn more probability from the preliminary chapters of more specialized probability books, and with the comment here that, among probabilists, the random matrix people know well their job, and are very close to what we will be doing in this book. Well-known introductions to random matrices include the classical and delightful book by Mehta [65], the more modern and rock-solid book by Anderson, Guionnet and Zeitouni [1], the book by Mingo and Speicher [68], and many more.

Needless to say, you can also learn good probability from physicists, or other scientists. In fact, probability theory was accepted only recently, in the late 20th century, as a respectable branch of mathematics, and if there are some scientists who have taken probability seriously, and this since ever, these are the physicists.

### 1e. Exercises

Things have been quite classical in this opening chapter, and there are just a few further things that need to be learned. First, in connection with the CLT, we have:

EXERCISE 1.31. *Look up the CLT, which was done here in moments, learn how the convergence can be improved, and write a brief account of that.*

This is a bit vague, but at this stage, learning more theory would be a good thing.

EXERCISE 1.32. *Prove that the area of the unit sphere in  $\mathbb{R}^N$  is given by*

$$A = \left(\frac{\pi}{2}\right)^{[N/2]} \frac{2^N}{(N-1)!!}$$

*with our usual convention  $N!! = (N-1)(N-3)(N-5)\dots$  for double factorials.*

Here you can either recycle our proof for  $V$ , by making changes where needed, or deduce the result from our result for  $V$ . In any case, think first at  $N = 2$ .

EXERCISE 1.33. *Establish the following integration formula over  $S_{\mathbb{R}}^{N-1} \subset \mathbb{R}^N$ , with respect to the normalized measure, valid for any exponents  $p_i \in \mathbb{N}$ ,*

$$\int_{S_{\mathbb{R}}^{N-1}} |x_1^{p_1} \dots x_N^{p_N}| dx = \left(\frac{2}{\pi}\right)^{\Sigma(p_1, \dots, p_N)} \frac{(N-1)!! p_1!! \dots p_N!!}{(N + \Sigma p_i - 1)!!}$$

*where  $\Sigma = [\text{odds}/2]$  if  $N$  is odd and  $\Sigma = [(\text{odds}+1)/2]$  if  $N$  is even, where “odds” denotes the number of odd numbers in the sequence  $p_1, \dots, p_N$ .*

Observe that this generalizes the integration formula for monomials that we established in the above, because odd powers lead to 0 integrals. The proof can only be similar.

EXERCISE 1.34. *Compute the density of the hyperspherical law at  $N = 4$ , that is, the law of one of the coordinates over the unit sphere  $S_{\mathbb{R}}^3 \subset \mathbb{R}^4$ .*

If you find something very interesting, as an answer here, do not be surprised. After all,  $S_{\mathbb{R}}^3$  is the sphere of space-time, having its own magic. We will be back to this.

## CHAPTER 2

### The Poisson law

#### 2a. Poisson limits

We have seen so far that the centered normal laws  $g_t$  and their complex analogues  $G_t$ , which appear from the Central Limit Theorem (CLT), have interesting combinatorial properties, and appear in several group-theoretical and geometric contexts.

We discuss now the discrete counterpart of these results. The mathematics here will involve the Poisson laws  $p_t$ , which appear via the Poisson Limit Theorem (PLT), and their generalized versions  $p_\nu$ , called compound Poisson laws, which appear via the Compound Poisson Limit Theorem (CPLT). Let us start with the following definition:

DEFINITION 2.1. *The Poisson law of parameter 1 is the following measure,*

$$p_1 = \frac{1}{e} \sum_{k \in \mathbb{N}} \frac{\delta_k}{k!}$$

*and the Poisson law of parameter  $t > 0$  is the following measure,*

$$p_t = e^{-t} \sum_{k \in \mathbb{N}} \frac{t^k}{k!} \delta_k$$

*with the letter “p” standing for Poisson.*

We are using here, as before, some simplified notations for these laws, which are in tune with the notation  $g_t$  that we used for the centered Gaussian laws. Observe that these laws have indeed mass 1, as they should, due to the following formula:

$$e^t = \sum_{k \in \mathbb{N}} \frac{t^k}{k!}$$

We will see in the moment why these measures appear a bit everywhere, in discrete contexts, the reasons behind this coming from the Poisson Limit Theorem (PLT). Let us first develop some general theory. We first have the following result:

THEOREM 2.2. *We have the following formula, for any  $s, t > 0$ ,*

$$p_s * p_t = p_{s+t}$$

*so the Poisson laws form a convolution semigroup.*

PROOF. By using  $\delta_k * \delta_l = \delta_{k+l}$  and the binomial formula, we obtain:

$$\begin{aligned}
 p_s * p_t &= e^{-s} \sum_k \frac{s^k}{k!} \delta_k * e^{-t} \sum_l \frac{t^l}{l!} \delta_l \\
 &= e^{-s-t} \sum_n \delta_n \sum_{k+l=n} \frac{s^k t^l}{k! l!} \\
 &= e^{-s-t} \sum_n \frac{\delta_n}{n!} \sum_{k+l=n} \frac{n!}{k! l!} s^k t^l \\
 &= e^{-s-t} \sum_n \frac{(s+t)^n}{n!} \delta_n \\
 &= p_{s+t}
 \end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Next in line, we have the following result, which is fundamental as well:

**THEOREM 2.3.** *The Poisson laws appear as formal exponentials*

$$p_t = \sum_k \frac{t^k (\delta_1 - \delta_0)^{*k}}{k!}$$

with respect to the convolution of measures  $*$ .

PROOF. By using the binomial formula, the measure on the right is:

$$\begin{aligned}
 \mu &= \sum_k \frac{t^k}{k!} \sum_{r+s=k} (-1)^s \frac{k!}{r! s!} \delta_r \\
 &= \sum_k t^k \sum_{r+s=k} (-1)^s \frac{\delta_r}{r! s!} \\
 &= \sum_r \frac{t^r \delta_r}{r!} \sum_s \frac{(-1)^s}{s!} \\
 &= \frac{1}{e} \sum_r \frac{t^r \delta_r}{r!} \\
 &= p_t
 \end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Regarding now the Fourier transform computation, this is as follows:

**THEOREM 2.4.** *The Fourier transform of  $p_t$  is given by*

$$F_{p_t}(y) = \exp((e^{iy} - 1)t)$$

for any  $t > 0$ .

PROOF. We have indeed the following computation:

$$\begin{aligned}
 F_{p_t}(y) &= e^{-t} \sum_k \frac{t^k}{k!} F_{\delta_k}(y) \\
 &= e^{-t} \sum_k \frac{t^k}{k!} e^{iky} \\
 &= e^{-t} \sum_k \frac{(e^{iy}t)^k}{k!} \\
 &= \exp(-t) \exp(e^{iy}t) \\
 &= \exp((e^{iy} - 1)t)
 \end{aligned}$$

Thus, we obtain the formula in the statement.  $\square$

Observe that the above formula gives an alternative proof for Theorem 2.2, by the using the fact that the logarithm of the Fourier transform linearizes the convolution. As another application, we can now establish the Poisson Limit Theorem, as follows:

THEOREM 2.5 (PLT). *We have the following convergence, in moments,*

$$\left( \left( 1 - \frac{t}{n} \right) \delta_0 + \frac{t}{n} \delta_1 \right)^{*n} \rightarrow p_t$$

for any  $t > 0$ .

PROOF. Let us denote by  $\nu_n$  the measure under the convolution sign, namely:

$$\nu_n = \left( 1 - \frac{t}{n} \right) \delta_0 + \frac{t}{n} \delta_1$$

We have the following computation, for the Fourier transform of the limit:

$$\begin{aligned}
 F_{\delta_r}(y) = e^{iry} &\implies F_{\nu_n}(y) = \left( 1 - \frac{t}{n} \right) + \frac{t}{n} e^{iy} \\
 &\implies F_{\nu_n^{*n}}(y) = \left( \left( 1 - \frac{t}{n} \right) + \frac{t}{n} e^{iy} \right)^n \\
 &\implies F_{\nu_n^{*n}}(y) = \left( 1 + \frac{(e^{iy} - 1)t}{n} \right)^n \\
 &\implies F(y) = \exp((e^{iy} - 1)t)
 \end{aligned}$$

Thus, we obtain indeed the Fourier transform of  $p_t$ , as desired.  $\square$

### 2b. Bell numbers

At the level of moments now, things are quite subtle for the Poisson laws, combinatorially speaking, and more complicated than for the normal laws. We first have the following result, dealing with the simplest case, where the parameter is  $t = 1$ :

**THEOREM 2.6.** *The moments of  $p_1$  are the Bell numbers,*

$$M_k(p_1) = |P(k)|$$

where  $P(k)$  is the set of partitions of  $\{1, \dots, k\}$ .

**PROOF.** The moments of  $p_1$  are given by the following formula:

$$M_k = \frac{1}{e} \sum_r \frac{r^k}{r!}$$

We therefore have the following recurrence formula for these moments:

$$\begin{aligned} M_{k+1} &= \frac{1}{e} \sum_r \frac{(r+1)^{k+1}}{(r+1)!} \\ &= \frac{1}{e} \sum_r \frac{(r+1)^k}{r!} \\ &= \frac{1}{e} \sum_r \frac{r^k}{r!} \left(1 + \frac{1}{r}\right)^k \\ &= \frac{1}{e} \sum_r \frac{r^k}{r!} \sum_s \binom{k}{s} r^{-s} \\ &= \sum_s \binom{k}{s} \cdot \frac{1}{e} \sum_r \frac{r^{k-s}}{r!} \\ &= \sum_s \binom{k}{s} M_{k-s} \end{aligned}$$

With this done, let us try now to find a recurrence for the Bell numbers:

$$B_k = |P(k)|$$

A partition of  $\{1, \dots, k+1\}$  appears by choosing  $s$  neighbors for 1, among the  $k$  numbers available, and then partitioning the  $k-s$  elements left. Thus, we have:

$$B_{k+1} = \sum_s \binom{k}{s} B_{k-s}$$

Thus, our moments  $M_k$  satisfy the same recurrence as the numbers  $B_k$ . Regarding now the initial values, in what concerns the first moment of  $p_1$ , we have:

$$M_1 = \frac{1}{e} \sum_r \frac{r}{r!} = 1$$

Also, by using the above recurrence for the numbers  $M_k$ , we obtain from this:

$$M_2 = \sum_s \binom{1}{s} M_{k-s} = 1 + 1 = 2$$

On the other hand, the initial values for the Bell numbers are  $B_1 = 1$  and  $B_2 = 2$ . Thus we obtain by recurrence  $M_k = B_k$ , as claimed.  $\square$

More generally now, we have the following result, dealing with the case  $t > 0$ :

**THEOREM 2.7.** *The moments of  $p_t$  with  $t > 0$  are given by*

$$M_k(p_t) = \sum_{\pi \in P(k)} t^{|\pi|}$$

where  $|\cdot|$  is the number of blocks.

**PROOF.** The moments of the Poisson law  $p_t$  with  $t > 0$  are given by:

$$M_k = e^{-t} \sum_r \frac{t^r r^k}{r!}$$

We have the following recurrence formula for these moments:

$$\begin{aligned} M_{k+1} &= e^{-t} \sum_r \frac{t^{r+1} (r+1)^{k+1}}{(r+1)!} \\ &= e^{-t} \sum_r \frac{t^{r+1} (r+1)^k}{r!} \\ &= e^{-t} \sum_r \frac{t^{r+1} r^k}{r!} \left(1 + \frac{1}{r}\right)^k \\ &= e^{-t} \sum_r \frac{t^{r+1} r^k}{r!} \sum_s \binom{k}{s} r^{-s} \\ &= \sum_s \binom{k}{s} \cdot e^{-t} \sum_r \frac{t^{r+1} r^{k-s}}{r!} \\ &= t \sum_s \binom{k}{s} M_{k-s} \end{aligned}$$

Regarding now the initial values, the first moment of  $p_t$  is given by:

$$\begin{aligned} M_1 &= e^{-t} \sum_r \frac{t^r r}{r!} \\ &= e^{-t} \sum_r \frac{t^r}{(r-1)!} \\ &= t \end{aligned}$$

Now by using the above recurrence we obtain from this:

$$\begin{aligned} M_2 &= t \sum_s \binom{1}{s} M_{k-s} \\ &= t(1+t) \\ &= t + t^2 \end{aligned}$$

On the other hand, consider the numbers in the statement, namely:

$$S_k = \sum_{\pi \in P(k)} t^{|\pi|}$$

Since a partition of  $\{1, \dots, k+1\}$  appears by choosing  $s$  neighbors for 1, among the  $k$  numbers available, and then partitioning the  $k-s$  elements left, we have:

$$S_{k+1} = t \sum_s \binom{k}{s} S_{k-s}$$

As for the initial values of these numbers, these are  $S_1 = t$ ,  $S_2 = t + t^2$ . Thus the initial values coincide, and so these numbers are the moments of  $p_t$ , as stated.  $\square$

As a conclusion to all this, the Poisson laws  $p_t$  are now entitled to join the collection of “interesting” probability measures that we have, formed by the real and complex Gaussian laws  $g_t$  and  $G_t$ . Indeed, not only all these measures appear via key limiting theorems, and form convolution semigroups, but at the level of moments, we have:

**THEOREM 2.8.** *The moments of  $\mu_t = p_t, g_t, G_t$  are given by the formula*

$$M_k(\mu_t) = \sum_{\pi \in D(k)} t^{|\pi|}$$

where  $D = P, P_2, \mathcal{P}_2$  respectively, and  $|\cdot|$  is the number of blocks.

**PROOF.** This follows indeed from Theorem 2.7, and from the results in chapter 1.  $\square$

The above result raises a whole string of interesting questions. Are there more measures of this type? Is a classification of such measures possible? Are the convolution semigroup results consequences of the moment formula? What about the limiting theorems? And so on. All these questions will be answered, in due time.

## 2c. Derangements

In relation now with groups, and with applications to pure mathematics in general, let us start with the following well-known, beautiful and fundamental result:

**THEOREM 2.9.** *The probability for a random  $\sigma \in S_N$  to have no fixed points is*

$$P \simeq \frac{1}{e}$$

*in the  $N \rightarrow \infty$  limit, where  $e = 2.718 \dots$  is the usual constant from analysis.*

**PROOF.** This is best viewed by using the inclusion-exclusion principle. Let us set:

$$S_N^k = \left\{ \sigma \in S_N \mid \sigma(k) = k \right\}$$

The set of permutations having no fixed points, called derangements, is then:

$$X_N = \left( \bigcup_k S_N^k \right)^c$$

In order to compute now the cardinality  $|X_N|$ , consider as well the following sets, depending on indices  $k_1 < \dots < k_r$ , obtained by taking intersections:

$$S_N^{k_1 \dots k_r} = S_N^{k_1} \cap \dots \cap S_N^{k_r}$$

Observe that we have the following formula:

$$S_N^{k_1 \dots k_r} = \left\{ \sigma \in S_N \mid \sigma(k_1) = k_1, \dots, \sigma(k_r) = k_r \right\}$$

Now the inclusion-exclusion principle tells us that we have:

$$\begin{aligned} & \left| \left( \bigcup_k S_N^k \right)^c \right| \\ &= |S_N| - \sum_k |S_N^k| + \sum_{k < l} |S_N^k \cap S_N^l| - \dots + (-1)^N \sum_{k_1 < \dots < k_N} |S_N^{k_1} \cup \dots \cup S_N^{k_N}| \\ &= |S_N| - \sum_k |S_N^k| + \sum_{k < l} |S_N^{kl}| - \dots + (-1)^N \sum_{k_1 < \dots < k_N} |S_N^{k_1 \dots k_N}| \end{aligned}$$

Thus, the probability that we are interested in, for a random permutation  $\sigma \in S_N$  to have no fixed points, is given by an alternating sum, as follows:

$$P = \frac{1}{N!} \left( |S_N| - \sum_k |S_N^k| + \sum_{k < l} |S_N^{kl}| - \dots + (-1)^N \sum_{k_1 < \dots < k_N} |S_N^{k_1 \dots k_N}| \right)$$

Now observe that for any  $k_1 < \dots < k_r$  we have the following formula:

$$|S_N^{k_1 \dots k_r}| = (N - r)!$$

We obtain the following formula, for the probability that we want to compute:

$$\begin{aligned}
P &= \frac{1}{N!} \sum_{r=0}^N (-1)^r \sum_{k_1 < \dots < k_r} |S_N^{k_1 \dots k_r}| \\
&= \frac{1}{N!} \sum_{r=0}^N (-1)^r \sum_{k_1 < \dots < k_r} (N-r)! \\
&= \frac{1}{N!} \sum_{r=0}^N (-1)^r \binom{N}{r} (N-r)! \\
&= \sum_{r=0}^N \frac{(-1)^r}{r!} \\
&= 1 - \frac{1}{1!} + \frac{1}{2!} - \dots + (-1)^{N-1} \frac{1}{(N-1)!} + (-1)^N \frac{1}{N!}
\end{aligned}$$

Since on the right we have the expansion of  $1/e$ , this gives the result.  $\square$

In order to refine now the above result, as to reach to Poisson laws, we will need some basic notions from group theory. Let us start with the following standard definition:

DEFINITION 2.10. *Given a closed subgroup  $G \subset U_N$ , the function*

$$\chi : G \rightarrow \mathbb{C} \quad , \quad \chi(g) = \sum_{i=1}^N g_{ii}$$

*is called main character of  $G$ .*

We will see later a number for motivations for the study of such characters, the idea being that a compact group  $G$  can have several representations  $\pi : G \subset U_N$ , which can be studied via their characters  $\chi_\pi : G \rightarrow \mathbb{C}$ . For the moment, we will not need any kind of abstract motivations for the study of the group characters, and this because for the simplest group that we know, namely  $S_N$ , we have the following beautiful result:

THEOREM 2.11. *Consider the symmetric group  $S_N$ , regarded as the permutation group,  $S_N \subset O_N$ , of the  $N$  coordinate axes of  $\mathbb{R}^N$ .*

- (1) *The main character  $\chi \in C(S_N)$  counts the number of fixed points.*
- (2) *The law of  $\chi \in C(S_N)$  becomes Poisson (1), in the  $N \rightarrow \infty$  limit.*

PROOF. We have two things to be proved here, the idea being as follows:

(1) The permutation matrices  $\sigma \in O_N$ , which give the embedding  $S_N \subset O_N$  in the statement, being given by  $\sigma_{ij} = \delta_{i\sigma(j)}$ , we have the following computation:

$$\begin{aligned}\chi(\sigma) &= \sum_i \delta_{\sigma(i)i} \\ &= \# \left\{ i \in \{1, \dots, N\} \mid \sigma(i) = i \right\}\end{aligned}$$

(2) In order to establish now the asymptotic result in the statement, we must prove the following formula, for any  $r \in \mathbb{N}$ , in the  $N \rightarrow \infty$  limit:

$$P(\chi = r) \simeq \frac{1}{r!e}$$

We already know, from Theorem 2.9, that this formula holds at  $r = 0$ . In the general case now, we have to count the permutations  $\sigma \in S_N$  having exactly  $r$  points. Now since having such a permutation amounts in choosing  $r$  points among  $1, \dots, N$ , and then permuting the  $N - r$  points left, without fixed points allowed, we have:

$$\begin{aligned}\# \left\{ \sigma \in S_N \mid \chi(\sigma) = r \right\} &= \binom{N}{r} \# \left\{ \sigma \in S_{N-r} \mid \chi(\sigma) = 0 \right\} \\ &= \frac{N!}{r!(N-r)!} \# \left\{ \sigma \in S_{N-r} \mid \chi(\sigma) = 0 \right\} \\ &= N! \times \frac{1}{r!} \times \frac{\# \left\{ \sigma \in S_{N-r} \mid \chi(\sigma) = 0 \right\}}{(N-r)!}\end{aligned}$$

By dividing everything by  $N!$ , we obtain from this the following formula:

$$\frac{\# \left\{ \sigma \in S_N \mid \chi(\sigma) = r \right\}}{N!} = \frac{1}{r!} \times \frac{\# \left\{ \sigma \in S_{N-r} \mid \chi(\sigma) = 0 \right\}}{(N-r)!}$$

Now by using the computation at  $r = 0$ , that we already have, from Theorem 2.9, it follows that with  $N \rightarrow \infty$  we have the following estimate:

$$\begin{aligned}P(\chi = r) &\simeq \frac{1}{r!} \cdot P(\chi = 0) \\ &\simeq \frac{1}{r!} \cdot \frac{1}{e}\end{aligned}$$

Thus, we obtain as limiting measure the Poisson law of parameter 1, as stated.  $\square$

As a next step, let us try now to generalize what we have, namely Theorem 2.11, as to reach to the Poisson laws of arbitrary parameter  $t > 0$ . We will need:

DEFINITION 2.12. *Given a closed subgroup  $G \subset U_N$ , the function*

$$\chi : G \rightarrow \mathbb{C} \quad , \quad \chi_t(g) = \sum_{i=1}^{[tN]} g_{ii}$$

*is called main truncated character of  $G$ , of parameter  $t \in (0, 1]$ .*

As before with the plain characters, there is some theory behind this definition, and we will discuss this later on, in chapters 3-4 below. In relation with the present considerations, we actually already met such truncated characters, but in a disguised form, in chapter 1, when talking about  $O_N, U_N$ . Indeed, the results there show that we have:

PROPOSITION 2.13. *For the orthogonal and unitary groups  $O_N, U_N$ , the rescalings*

$$\chi = \frac{\chi_{1/N}}{\sqrt{N}}$$

*become respectively real and complex Gaussian, in the  $N \rightarrow \infty$  limit.*

PROOF. According to our conventions, given a closed subgroup  $G \subset U_N$ , the main character truncated at  $t = 1/N$  is simply the first coordinate:

$$\chi_{1/N}(g) = g_{11}$$

With this remark made, the conclusions from the statement follow from the computations from chapter 1, for the laws of coordinates on  $O_N, U_N$ .  $\square$

Getting back now to the symmetric groups, we have the following result, generalizing Theorem 2.11, and which will be our final result on the subject:

THEOREM 2.14. *Consider the symmetric group  $S_N$ , regarded as the permutation group,  $S_N \subset O_N$ , of the  $N$  coordinate axes of  $\mathbb{R}^N$ .*

- (1) *The variable  $\chi_t$  counts the number of fixed points among  $1, \dots, [tN]$ .*
- (2) *The law of this variable  $\chi_t$  becomes Poisson ( $t$ ), in the  $N \rightarrow \infty$  limit.*

PROOF. We already know from Theorem 2.11 that the results hold at  $t = 1$ . In general, the proof is similar, the idea being as follows:

- (1) We have indeed the following computation, coming from definitions:

$$\begin{aligned} \chi_t(\sigma) &= \sum_{i=1}^{[tN]} \delta_{\sigma(i)i} \\ &= \# \left\{ i \in \{1, \dots, [tN]\} \mid \sigma(i) = i \right\} \end{aligned}$$

- (2) Consider indeed the following sets, as in the proof of Theorem 2.9:

$$S_N^k = \left\{ \sigma \in S_N \mid \sigma(k) = k \right\}$$

The set of permutations having no fixed points among  $1, \dots, [tN]$  is then:

$$X_N = \left( \bigcup_{k \leq [tN]} S_N^k \right)^c$$

In order to compute now the cardinality  $|X_N|$ , consider as well the following sets, depending on indices  $k_1 < \dots < k_r$ , obtained by taking intersections:

$$S_N^{k_1 \dots k_r} = S_N^{k_1} \cap \dots \cap S_N^{k_r}$$

As before in the proof of Theorem 2.9, we obtain by inclusion-exclusion that:

$$\begin{aligned} P(\chi_t = 0) &= \frac{1}{N!} \sum_{r=0}^{[tN]} (-1)^r \sum_{k_1 < \dots < k_r < [tN]} |S_N^{k_1 \dots k_r}| \\ &= \frac{1}{N!} \sum_{r=0}^{[tN]} (-1)^r \sum_{k_1 < \dots < k_r < [tN]} (N-r)! \\ &= \frac{1}{N!} \sum_{r=0}^{[tN]} (-1)^r \binom{[tN]}{r} (N-r)! \\ &= \sum_{r=0}^{[tN]} \frac{(-1)^r}{r!} \cdot \frac{[tN]! (N-r)!}{N! ([tN]-r)!} \end{aligned}$$

Now with  $N \rightarrow \infty$ , we obtain from this the following estimate:

$$\begin{aligned} P(\chi_t = 0) &\simeq \sum_{r=0}^{[tN]} \frac{(-1)^r}{r!} \cdot t^r \\ &= \sum_{r=0}^{[tN]} \frac{(-t)^r}{r!} \\ &\simeq e^{-t} \end{aligned}$$

More generally, by counting the permutations  $\sigma \in S_N$  having exactly  $r$  fixed points among  $1, \dots, [tN]$ , as in the proof of Theorem 2.11, we obtain:

$$P(\chi_t = r) \simeq \frac{t^r}{r! e^t}$$

Thus, we obtain in the limit a Poisson law of parameter  $t$ , as stated.  $\square$

The above result is quite fundamental, and worth proving a second time, by using an alternative method. We can indeed use the following formula:

THEOREM 2.15. *Consider the symmetric group  $S_N$ , with its standard coordinates:*

$$u_{ij} = \chi \left( \sigma \in S_N \mid \sigma(j) = i \right)$$

*We have then the following integration formula,*

$$\int_{S_N} u_{i_1 j_1} \dots u_{i_r j_r} = \begin{cases} \frac{(N - |\ker i|)!}{N!} & \text{if } \ker i = \ker j \\ 0 & \text{otherwise} \end{cases}$$

*where  $\ker i$  denotes the partition of  $\{1, \dots, r\}$  whose blocks collect the equal indices of  $i$ , and where  $|\cdot|$  denotes the number of blocks.*

PROOF. Observe first that the above formula computes all the integrals over  $S_N$ , and this because the coordinates  $u_{ij}$  separate the points of  $S_N$ . In what regards the proof, according to the definition of  $u_{ij}$ , the integrals in the statement are given by:

$$\int_{S_N} u_{i_1 j_1} \dots u_{i_r j_r} = \frac{1}{N!} \# \left\{ \sigma \in S_N \mid \sigma(j_1) = i_1, \dots, \sigma(j_r) = i_r \right\}$$

Now observe that the existence of  $\sigma \in S_N$  as above requires:

$$i_k = i_l \iff j_k = j_l$$

Thus, the integral in the statement vanishes if  $\ker i \neq \ker j$ . As for the case left, namely  $\ker i = \ker j$ , if we denote by  $b \in \{1, \dots, r\}$  the number of blocks of this partition  $\ker i = \ker j$ , then we have  $N - b$  points to be sent bijectively to  $N - b$  points, and so  $(N - b)!$  solutions, and our integral follows to be  $(N - b)!/N!$ , as claimed.  $\square$

As an illustration for the above formula, we can now recover the computation of the asymptotic laws of the truncated characters  $\chi_t$ . We have indeed:

THEOREM 2.16. *For the symmetric group  $S_N \subset O_N$ , regarded as group of matrices,  $S_N \subset O_N$ , via the standard permutation matrices, the truncated character*

$$\chi_t = \sum_{i=1}^{[tN]} u_{ii}$$

*counts the number of fixed points among  $\{1, \dots, [tN]\}$ , and its law with respect to the counting measure becomes, with  $N \rightarrow \infty$ , a Poisson law of parameter  $t$ .*

PROOF. The first assertion is something trivial, that we already know. Regarding now the second assertion, we can use here Theorem 2.15. With  $S_{r,b}$  being the Stirling numbers,

counting the partitions of  $\{1, \dots, r\}$  having  $b$  blocks, we have:

$$\begin{aligned}
 \int_{S_N} \chi_t^r &= \sum_{i_1=1}^{[tN]} \dots \sum_{i_r=1}^{[tN]} \int_{S_N} u_{i_1 i_1} \dots u_{i_r i_r} \\
 &= \sum_{\pi \in P(r)} \frac{[tN]!}{([tN] - |\pi|)!} \cdot \frac{(N - |\pi|)!}{N!} \\
 &= \sum_{b=1}^{[tN]} \frac{[tN]!}{([tN] - b)!} \cdot \frac{(N - b)!}{N!} \cdot S_{rb}
 \end{aligned}$$

In particular with  $N \rightarrow \infty$  we obtain the following formula:

$$\lim_{N \rightarrow \infty} \int_{S_N} \chi_t^r = \sum_{b=1}^r S_{rb} t^b$$

But this is a Poisson ( $t$ ) moment, according to our formula for the moments of  $p_t$ , which in terms of Stirling numbers is the above one, and so we are done.  $\square$

As a conclusion to all this, the Poisson laws  $p_t$  appear to be quite similar to the real and complex Gaussian laws  $g_t$  and  $G_t$ , in the sense that:

- (1) All these laws appear via basic limiting theorems.
- (2) They form semigroups with respect to the convolution.
- (3) Their moments can be computed by counting certain partitions.
- (4) There is a relation with pure mathematics as well, involving  $S_N, O_N, U_N$ .

All this remains of course to be further discussed. We will be back to this in chapters 3-4 below, following [13], [35], [95] and related papers, and later on as well.

## 2d. Bessel laws

Let us keep doing computations for subgroups  $G \subset U_N$ . An obvious choice here is the hyperoctahedral group  $H_N$ , whose definition and basic properties are as follows:

**THEOREM 2.17.** *Consider the hyperoctahedral group  $H_N \subset O_N$ , consisting of the various symmetries of the hypercube in  $\mathbb{R}^N$ .*

- (1)  $H_N$  is the symmetry group of the  $N$  coordinate axes of  $\mathbb{R}^N$ .
- (2)  $H_N$  consists of the permutation-like matrices over  $\{-1, 0, 1\}$ .
- (3) We have the cardinality formula  $|H_N| = 2^N N!$ .
- (4) We have a crossed product decomposition  $H_N = S_N \rtimes \mathbb{Z}_2^N$ .
- (5) We have a wreath product decomposition  $H_N = \mathbb{Z}_2 \wr S_N$ .

PROOF. Consider indeed the standard cube in  $\mathbb{R}^N$ , which is by definition centered at 0, and having as vertices the points having coordinates  $\pm 1$ .

(1) With the above picture of the cube in hand, it is clear that the symmetries of the cube coincide with the symmetries of the  $N$  coordinate axes of  $\mathbb{R}^N$ .

(2) Each of the permutations  $\sigma \in S_N$  of the  $N$  coordinate axes of  $\mathbb{R}^N$  can be further “decorated” by a sign vector  $\varepsilon \in \{\pm 1\}^N$ , consisting of the possible  $\pm 1$  flips which can be applied to each coordinate axis, at the arrival. In matrix terms, this gives the result.

(3) By using the above interpretation of  $H_N$ , we have the following formula:

$$|H_N| = |S_N| \cdot |\mathbb{Z}_2^N| = N! \cdot 2^N$$

(4) We know from (3) that at the level of cardinalities we have  $|H_N| = |S_N \times \mathbb{Z}_2^N|$ , and with a bit more work, we obtain that we have  $H_N = S_N \rtimes \mathbb{Z}_2^N$ , as claimed.

(5) This is simply a reformulation of (4), in terms of wreath products.  $\square$

Getting back now to our character computations, following [10], we have:

THEOREM 2.18. *For the hyperoctahedral group  $H_N \subset O_N$ , the law of the variable*

$$\chi = g_{11} + \dots + g_{ss}$$

*with  $s = [tN]$  is, in the  $N \rightarrow \infty$  limit, the following measure,*

$$b_t = e^{-t} \sum_{r=-\infty}^{\infty} \delta_r \sum_{p=0}^{\infty} \frac{(t/2)^{|r|+2p}}{(|r|+p)!p!}$$

*called Bessel law of parameter  $t > 0$ .*

PROOF. We regard  $H_N$  as being the symmetry group of the graph  $I_N = \{I^1, \dots, I^N\}$  formed by  $n$  segments. The diagonal coefficients are then given by:

$$u_{ii}(g) = \begin{cases} 0 & \text{if } g \text{ moves } I^i \\ 1 & \text{if } g \text{ fixes } I^i \\ -1 & \text{if } g \text{ returns } I^i \end{cases}$$

Let us denote by  $F_g, R_g$  the number of segments among  $\{I^1, \dots, I^s\}$  which are fixed, respectively returned by an element  $g \in H_N$ . With this notation, we have:

$$u_{11} + \dots + u_{ss} = F_g - R_g$$

We denote by  $P_N$  probabilities computed over  $H_N$ . The density of the law of the variable  $u_{11} + \dots + u_{ss}$  at a point  $r \geq 0$  is then given by the following formula:

$$\begin{aligned} D(r) &= P_N(F_g - R_g = r) \\ &= \sum_{p=0}^{\infty} P_N(F_g = r + p, R_g = p) \end{aligned}$$

Assume first that we are in the case  $t = 1$ . We have the following computation:

$$\begin{aligned}
\lim_{N \rightarrow \infty} D(r) &= \lim_{N \rightarrow \infty} \sum_{p=0}^{\infty} (1/2)^{r+2p} \binom{r+2p}{r+p} P_N(F_g + R_g = r+2p) \\
&= \sum_{p=0}^{\infty} (1/2)^{r+2p} \binom{r+2p}{r+p} \frac{1}{e(r+2p)!} \\
&= \frac{1}{e} \sum_{p=0}^{\infty} \frac{(1/2)^{r+2p}}{(r+p)!p!}
\end{aligned}$$

The general case  $0 < t \leq 1$  follows by performing some modifications in the above computation. Indeed, the asymptotic density can be computed as follows:

$$\begin{aligned}
\lim_{N \rightarrow \infty} D(r) &= \lim_{N \rightarrow \infty} \sum_{p=0}^{\infty} (1/2)^{r+2p} \binom{r+2p}{r+p} P_N(F_g + R_g = r+2p) \\
&= \sum_{p=0}^{\infty} (1/2)^{r+2p} \binom{r+2p}{r+p} \frac{t^{r+2p}}{e^t(r+2p)!} \\
&= e^{-t} \sum_{p=0}^{\infty} \frac{(t/2)^{r+2p}}{(r+p)!p!}
\end{aligned}$$

Together with  $D(-r) = D(r)$ , this gives the formula in the statement.  $\square$

The above result is quite interesting, because the densities that we found there are the following functions, called Bessel functions of the first kind:

$$f_r(t) = \sum_{p=0}^{\infty} \frac{t^{|r|+2p}}{(|r|+p)!p!}$$

Due to this fact, the limiting measures are called Bessel laws, as mentioned in Theorem 2.18. Let us study now these Bessel laws. We first have the following result, from [10]:

**THEOREM 2.19.** *The Bessel laws  $b_t$  have the property*

$$b_s * b_t = b_{s+t}$$

*so they form a truncated one-parameter semigroup with respect to convolution.*

**PROOF.** With  $f_r(t)$  being the Bessel functions of the first kind, we have:

$$b_t = e^{-t} \sum_{r=-\infty}^{\infty} \delta_r f_r(t/2)$$

The Fourier transform of this measure  $b_t$  is given by:

$$F_{b_t}(y) = e^{-t} \sum_{r=-\infty}^{\infty} e^{iry} f_r(t/2)$$

We compute now the derivative with respect to the variable  $t$ :

$$F_{b_t}(y)' = -F_{b_t}(y) + \frac{e^{-t}}{2} \sum_{r=-\infty}^{\infty} e^{iry} f'_r(t/2)$$

On the other hand, the derivative of  $f_r$  with  $r \geq 1$  is given by:

$$\begin{aligned} f'_r(t) &= \sum_{p=0}^{\infty} \frac{(r+2p)t^{r+2p-1}}{(r+p)!p!} \\ &= \sum_{p=0}^{\infty} \frac{(r+p)t^{r+2p-1}}{(r+p)!p!} + \sum_{p=0}^{\infty} \frac{p t^{r+2p-1}}{(r+p)!p!} \\ &= \sum_{p=0}^{\infty} \frac{t^{r+2p-1}}{(r+p-1)!p!} + \sum_{p=1}^{\infty} \frac{t^{r+2p-1}}{(r+p)!(p-1)!} \\ &= \sum_{p=0}^{\infty} \frac{t^{(r-1)+2p}}{((r-1)+p)!p!} + \sum_{p=1}^{\infty} \frac{t^{(r+1)+2(p-1)}}{((r+1)+(p-1))!(p-1)!} \\ &= f_{r-1}(t) + f_{r+1}(t) \end{aligned}$$

This computation works in fact for any  $r$ , and we obtain in this way:

$$\begin{aligned} F_{b_t}(y)' &= -F_{b_t}(y) + \frac{e^{-t}}{2} \sum_{r=-\infty}^{\infty} e^{iry} (f_{r-1}(t/2) + f_{r+1}(t/2)) \\ &= -F_{b_t}(y) + \frac{e^{-t}}{2} \sum_{r=-\infty}^{\infty} e^{i(r+1)y} f_r(t/2) + e^{i(r-1)y} f_r(t/2) \\ &= -F_{b_t}(y) + \frac{e^{iy} + e^{-iy}}{2} F_{b_t}(y) \\ &= \left( \frac{e^{iy} + e^{-iy}}{2} - 1 \right) F_{b_t}(y) \end{aligned}$$

By integrating, we obtain from this the following formula:

$$F_{b_t}(y) = \exp \left( \left( \frac{e^{iy} + e^{-iy}}{2} - 1 \right) t \right)$$

Thus the log of the Fourier transform is linear in  $t$ , and we get the assertion.  $\square$

In order to further discuss all this, and extend the above results, we will need a number of standard probabilistic preliminaries. We have the following notion, extending the Poisson limit theory developed in the beginning of the present chapter:

DEFINITION 2.20. *Associated to any compactly supported positive measure  $\nu$  on  $\mathbb{C}$ , not necessarily of mass 1, is the probability measure*

$$p_\nu = \lim_{n \rightarrow \infty} \left( \left( 1 - \frac{t}{n} \right) \delta_0 + \frac{1}{n} \nu \right)^{*n}$$

where  $t = \text{mass}(\nu)$ , called *compound Poisson law*.

In what follows we will be mainly interested in the case where the measure  $\nu$  is discrete, as is for instance the case for  $\nu = t\delta_1$  with  $t > 0$ , which produces the Poisson laws. The following standard result allows one to detect compound Poisson laws:

PROPOSITION 2.21. *For a discrete measure, written as*

$$\nu = \sum_{i=1}^s t_i \delta_{z_i}$$

with  $t_i > 0$  and  $z_i \in \mathbb{C}$ , we have

$$F_{p_\nu}(y) = \exp \left( \sum_{i=1}^s t_i (e^{iyz_i} - 1) \right)$$

where  $F$  denotes the Fourier transform.

PROOF. Let  $\eta_n$  be the measure in Definition 2.20, under the convolution sign:

$$\eta_n = \left( 1 - \frac{t}{n} \right) \delta_0 + \frac{1}{n} \nu$$

We have then the following computation:

$$\begin{aligned} F_{\eta_n}(y) &= \left( 1 - \frac{t}{n} \right) + \frac{1}{n} \sum_{i=1}^s t_i e^{iyz_i} \\ \implies F_{\eta_n^{*n}}(y) &= \left( \left( 1 - \frac{t}{n} \right) + \frac{1}{n} \sum_{i=1}^s t_i e^{iyz_i} \right)^n \\ \implies F_{p_\nu}(y) &= \exp \left( \sum_{i=1}^s t_i (e^{iyz_i} - 1) \right) \end{aligned}$$

Thus, we have obtained the formula in the statement.  $\square$

We have as well the following result, providing an alternative to Definition 2.20, and which will be our formulation here of the Compound Poisson Limit Theorem:

THEOREM 2.22 (CPLT). *For a discrete measure, written as*

$$\nu = \sum_{i=1}^s t_i \delta_{z_i}$$

*with  $t_i > 0$  and  $z_i \in \mathbb{C}$ , the corresponding compound Poisson law appears as*

$$p_\nu = \text{law} \left( \sum_{i=1}^s z_i \alpha_i \right)$$

*where the variables  $\alpha_i$  are Poisson ( $t_i$ ), independent.*

PROOF. Let  $\alpha$  be the sum of Poisson variables in the statement, namely:

$$\alpha = \sum_{i=1}^s z_i \alpha_i$$

By using standard Fourier transform formulae, we have:

$$\begin{aligned} F_{\alpha_i}(y) = \exp(t_i(e^{iy} - 1)) &\implies F_{z_i \alpha_i}(y) = \exp(t_i(e^{iy z_i} - 1)) \\ &\implies F_\alpha(y) = \exp \left( \sum_{i=1}^s t_i(e^{iy z_i} - 1) \right) \end{aligned}$$

Thus we have indeed the same formula as in Proposition 2.21.  $\square$

Summarizing, we have now a full generalization of the PLT. Getting back now to the Poisson and Bessel laws, with the above formalism in hand, we have:

THEOREM 2.23. *The Poisson and Bessel laws are compound Poisson laws,*

$$p_t = p_{t\delta_1} \quad , \quad b_t = p_{t\varepsilon}$$

*where  $\delta_1$  is the Dirac mass at 1, and  $\varepsilon$  is the centered Bernoulli law,  $\varepsilon = (\delta_1 + \delta_{-1})/2$ .*

PROOF. We have two assertions here, the idea being as follows:

(1) The first assertion, regarding the Poisson law  $p_t$ , is clear from Definition 2.20, which for  $\nu = t\delta_1$  takes the following form:

$$p_\nu = \lim_{n \rightarrow \infty} \left( \left( 1 - \frac{t}{n} \right) \delta_0 + \frac{t}{n} \delta_1 \right)^{*n}$$

Indeed, according to the PLT, the limit on the right produces the Poisson law  $p_t$ , as desired. Alternatively, the result follows as well from Proposition 2.21, which gives:

$$F_{p_\nu}(y) = \exp(t(e^{iy} - 1))$$

But the simplest way of proving the result is by invoking Theorem 2.22, which tells us that for  $\nu = t\delta_1$  we have  $p_\nu = \text{law}(\alpha)$ , with  $\alpha$  being Poisson ( $t$ ).

(2) Regarding the second assertion, concerning  $b_t$ , the most convenient here is to use the formula of the Fourier transform found in the proof of Theorem 2.19, namely:

$$F_{b_t}(y) = \exp \left( t \left( \frac{e^{iy} + e^{-iy}}{2} - 1 \right) \right)$$

On the other hand, the formula in Proposition 2.21 gives, for  $\nu = t\varepsilon$ :

$$F_{p_\nu}(y) = \exp \left( \frac{t}{2}(e^{iy} - 1) + \frac{t}{2}(e^{-iy} - 1) \right)$$

Thus, with  $\nu = t\varepsilon$  we have  $p_\nu = b_t$ , as claimed.  $\square$

As a conclusion to all this, we can add the Bessel laws  $b_t$  to the family of “interesting” probability measures that we have, consisting so far of the real and complex Gaussian laws  $g_t$  and  $G_t$ , and the Poisson laws  $p_t$ . Indeed, the measures  $p_t, b_t, g_t, G_t$  all appear via basic limiting theorems, they form convolution semigroups, and they are related to group theory as well, and more specifically to the groups  $S_N, H_N, O_N, U_N$ .

Still missing, however, for  $b_t$  is a combinatorial formula for the moments, in the spirit of the formulae that we have for  $p_t, g_t, G_t$ . This is something quite tricky, and the formula is as follows, with  $P_{even}$  standing for the partitions all whose blocks have even size:

$$M_k(b_t) = \sum_{\pi \in P_{even}(k)} t^{|\pi|}$$

It is possible to prove this out of what we have, for instance by taking the generating function of the above numbers, then converting this series into a Fourier one, with the conclusion that we obtain indeed the Fourier transform  $F_{b_t}$  computed above. However, the computations are quite complex, and this even in the simplest case,  $t = 1$ , and instead of embarking into this, we will leave it for later, when we will have better tools.

Moving ahead, Theorem 2.23 suggests formulating the following definition:

DEFINITION 2.24. *The Bessel law of level  $s \in \mathbb{N} \cup \{\infty\}$  and parameter  $t > 0$  is*

$$b_t^s = p_{t\varepsilon_s}$$

*with  $\varepsilon_s$  being the uniform measure on the  $s$ -th roots of unity.*

Observe that at  $s = 1, 2$  we obtain the Poisson and the usual Bessel laws  $p_t, b_t$ . Another important particular case is  $s = \infty$ , where we obtain a measure which is actually not discrete, called purely complex Bessel law, and denoted as follows:

$$B_t = b_t^\infty$$

Here we use the same convention as in the continuous case, namely that capital letters stand for complexifications. And this latter measure will stand as a motivation for Definition 2.24, because we will see that  $B_t$  is a complexification of  $b_t$ , in a suitable sense,

so that the couple  $b_t/B_t$  stands as the correct “discrete analogue” of the couple  $g_t/G_t$ . Which is itself something quite interesting, philosophically speaking.

In practice now, we first have to study the measures  $b_t^s$  in our standard way, meaning density, moments, Fourier, semigroup property, limiting theorems, and relation with group theory. In what regards limiting theorems, the measures  $b_t^s$  appear by definition via the CPLT, so done with that. As a consequence of this, however, let us record:

PROPOSITION 2.25. *The Bessel laws are given by*

$$b_t^s = \text{law} \left( \sum_{k=1}^s w^k a_k \right)$$

where  $a_1, \dots, a_s$  are Poisson ( $t$ ) independent, and  $w = e^{2\pi i/s}$ .

PROOF. At  $s = 1, 2$  this is something that we already know, coming from Theorem 2.23 and its proof. In general, this follows from Theorem 2.22.  $\square$

Following [8], where the laws  $b_t^s$  were introduced and studied, let us discuss now Fourier transforms and the semigroup property. Consider the level  $s$  exponential function:

$$\exp_s z = \sum_{k=0}^{\infty} \frac{z^{sk}}{(sk)!}$$

We have then the following formula, in terms of  $w = e^{2\pi i/s}$ :

$$\exp_s z = \frac{1}{s} \sum_{k=1}^s \exp(w^k z)$$

Observe that  $\exp_1 = \exp$  and  $\exp_2 = \cosh$ . We have the following result:

THEOREM 2.26. *The Fourier transform of  $b_t^s$  is given by*

$$\log F_t^s(z) = t (\exp_s z - 1)$$

where  $\exp_s z$  is as above. In particular we have the formula

$$b_t^s * b_{t'}^s = b_{t+t'}^s$$

so the measures  $b_t^s$  form a one-parameter convolution semigroup.

PROOF. Consider, as in Proposition 2.25, the variable  $a = \sum_{k=1}^s w^k a_k$ . We have then the following Fourier transform computation:

$$\begin{aligned} \log F_a(z) &= \sum_{k=1}^s \log F_{a_k}(w^k z) \\ &= \sum_{k=1}^s \frac{t}{s} (\exp(w^k z) - 1) \end{aligned}$$

But this gives the following formula:

$$\begin{aligned}\log F_a(z) &= t \left( \left( \frac{1}{s} \sum_{k=1}^s \exp(w^k z) \right) - 1 \right) \\ &= t (\exp_s z - 1)\end{aligned}$$

Now since  $b_t^s$  is the law of  $a$ , this gives the formula in the statement. As for the last assertion, this comes from the fact that the log of the Fourier transform is linear in  $t$ .  $\square$

Still following [8], we can compute the density of  $b_t^s$ , as follows:

**THEOREM 2.27.** *We have the formula*

$$b_t^s = e^{-t} \sum_{c_1=0}^{\infty} \cdots \sum_{c_s=0}^{\infty} \frac{1}{c_1! \cdots c_s!} \left( \frac{t}{s} \right)^{c_1+\dots+c_s} \delta \left( \sum_{k=1}^s w^k c_k \right)$$

where  $w = e^{2\pi i/s}$ , and the  $\delta$  symbol is a Dirac mass.

**PROOF.** The Fourier transform of the measure on the right is given by:

$$\begin{aligned}F(z) &= e^{-t} \sum_{c_1=0}^{\infty} \cdots \sum_{c_s=0}^{\infty} \frac{1}{c_1! \cdots c_s!} \left( \frac{t}{s} \right)^{c_1+\dots+c_s} F \delta \left( \sum_{k=1}^s w^k c_k \right) (z) \\ &= e^{-t} \sum_{c_1=0}^{\infty} \cdots \sum_{c_s=0}^{\infty} \frac{1}{c_1! \cdots c_s!} \left( \frac{t}{s} \right)^{c_1+\dots+c_s} \exp \left( \sum_{k=1}^s w^k c_k z \right) \\ &= e^{-t} \sum_{r=0}^{\infty} \left( \frac{t}{s} \right)^r \sum_{\Sigma c_i=r} \frac{\exp \left( \sum_{k=1}^s w^k c_k z \right)}{c_1! \cdots c_s!}\end{aligned}$$

We multiply now by  $e^t$ , and we compute the derivative with respect to  $t$ :

$$\begin{aligned}(e^t F(z))' &= \sum_{r=1}^{\infty} \frac{r}{s} \left( \frac{t}{s} \right)^{r-1} \sum_{\Sigma c_i=r} \frac{\exp \left( \sum_{k=1}^s w^k c_k z \right)}{c_1! \cdots c_s!} \\ &= \frac{1}{s} \sum_{r=1}^{\infty} \left( \frac{t}{s} \right)^{r-1} \sum_{\Sigma c_i=r} \left( \sum_{l=1}^s c_l \right) \frac{\exp \left( \sum_{k=1}^s w^k c_k z \right)}{c_1! \cdots c_s!} \\ &= \frac{1}{s} \sum_{r=1}^{\infty} \left( \frac{t}{s} \right)^{r-1} \sum_{\Sigma c_i=r} \sum_{l=1}^s \frac{\exp \left( \sum_{k=1}^s w^k c_k z \right)}{c_1! \cdots c_{l-1}! (c_l - 1)! c_{l+1}! \cdots c_s!}\end{aligned}$$

By using the variable  $u = r - 1$ , we obtain in this way:

$$\begin{aligned}
(e^t F(z))' &= \frac{1}{s} \sum_{u=0}^{\infty} \left(\frac{t}{s}\right)^u \sum_{\Sigma d_i=u} \sum_{l=1}^s \frac{\exp(w^l z + \sum_{k=1}^s w^k d_k z)}{d_1! \dots d_s!} \\
&= \left(\frac{1}{s} \sum_{l=1}^s \exp(w^l z)\right) \left(\sum_{u=0}^{\infty} \left(\frac{t}{s}\right)^u \sum_{\Sigma d_i=u} \frac{\exp(\sum_{k=1}^s w^k d_k z)}{d_1! \dots d_s!}\right) \\
&= (\exp_s z)(e^t F(z))
\end{aligned}$$

On the other hand,  $\Phi(t) = \exp(t \exp_s z)$  satisfies the same equation, namely:

$$\Phi'(t) = (\exp_s z) \Phi(t)$$

Thus, we have the  $e^t F(z) = \Phi(t)$ , which gives the following formula:

$$\begin{aligned}
\log F &= \log(e^{-t} \exp(t \exp_s z)) \\
&= \log(\exp(t(\exp_s z - 1))) \\
&= t(\exp_s z - 1)
\end{aligned}$$

Thus, we obtain the formulae in the statement.  $\square$

Regarding now the questions which are left, namely moments and relation with groups, these are quite technical, and related. Let us start by discussing the relation with groups. Obviously we need here a generalization of the groups  $S_N, H_N$ , involving a parameter  $s \in \mathbb{N} \cup \{\infty\}$ , and the answer to this question is straightforward, as follows:

**DEFINITION 2.28.** *The complex reflection group  $H_N^s \subset U_N$  is the group of permutations of  $N$  copies of the  $s$ -simplex. Equivalently, we have*

$$H_N^s = M_N(\mathbb{Z}_s \cup \{0\}) \cap U_N$$

*telling us that  $H_N^s$  consists of the permutation-type matrices with  $s$ -th roots of unity as entries. Also equivalently, we have the formula  $H_N^s = \mathbb{Z}_s \wr S_N$ .*

Here the equivalence between the various viewpoints on  $H_N^s$  comes as in Theorem 2.17, which corresponds to the case  $s = 2$ . In fact, the basic examples are as follows:

- (1)  $s = 1$ . Here  $H_N^1 = S_N$ , trivially, no matter which viewpoint we take.
- (2)  $s = 2$ . Here  $H_N^2 = H_2$ , with this coming from Theorem 2.17.
- (3)  $s = \infty$ . Here  $H_N^\infty = K_N$  is an interesting group, and more on it later.

In general,  $H_N^s$  are well-known in group theory, the idea being that, up to a number of exceptional examples, the complex reflection groups are exactly these groups  $H_N^s$ , and their versions  $H_N^{sd}$  obtained by adding the supplementary condition  $(\det U)^d = 1$ .

In relation with the Bessel laws, we have the following result, from [8]:

THEOREM 2.29. *For the complex reflection group  $H_N^s$  we have, with  $N \rightarrow \infty$ :*

$$\chi_t \sim b_t^s$$

*Moreover, the asymptotic moments of this variable are the numbers*

$$M_k(b_t^s) = \sum_{\pi \in P^s(k)} t^{|\pi|}$$

*where  $P^s(k)$  are the partitions of  $\{1, \dots, k\}$  satisfying  $\# \circ = \# \bullet(s)$ , in each block.*

PROOF. This is something quite long, that we will discuss in detail in chapters 3-4 below, when systematically doing representation theory, the idea being as follows:

(1) At  $s = 1$  the reflection group is  $H_N^1 = S_N$ , the Bessel law is the Poisson law,  $b_t^1 = p_t$ , and the formula  $\chi_t \sim p_t$  with  $N \rightarrow \infty$  is something that we know. As for the moment formula, where  $P^1 = P$ , this is something that we know too.

(2) At  $s = 2$  the reflection group is  $H_N^2 = H_N$ , the Bessel law is  $b_t^2 = b_t$ , and the formula  $\chi_t \sim p_t$  with  $N \rightarrow \infty$  is something that we know. As for the moment formula, where  $P^2 = P_{\text{even}}$ , this is something more technical, which remains to be discussed.

(3) In the general case, where  $s \in \mathbb{N} \cup \{\infty\}$ , the formula  $\chi_t \sim p_t$  with  $N \rightarrow \infty$  can be established a bit like for  $S_N, H_N$ , and the moment formula is something quite technical. We will discuss both questions in chapters 3-4 below, using more advanced tools.  $\square$

As a conclusion to all this, what we did so far in this book, the Gaussian and Poisson laws, and their various versions, have interesting combinatorics. All the above was an introduction to this combinatorics, following the classical theory, and [8], [10], [35] and related papers. We will be back to these laws and results on numerous occasions.

## 2e. Exercises

There has been a lot of non-trivial material in this chapter, especially in relation with the probabilistic aspects of the finite groups, which were sometimes only briefly explained, and this is because we will come back to this later, with more powerful tools. However, before that, let us start with a standard and beautiful exercise:

EXERCISE 2.30. *Prove that the Bell numbers  $B_k = |P(k)|$ , which are the moments of the Poisson law  $p_1$ , have the following properties:*

$$B_{k+1} = \sum_{r=0}^k \binom{k}{r} B_r \quad , \quad B_k = \frac{1}{e} \sum_{r=0}^{\infty} \frac{r^k}{r!}$$

$$\sum_{k=0}^{\infty} \frac{B_k}{k!} z^k = e^{e^z - 1} \quad , \quad B_k = \frac{k!}{2\pi i e} \int_{|z|=1} \frac{e^{e^z}}{z^{k+1}} dz$$

*Also, prove as well that we have  $\log B_k/k \simeq \log k - \log \log k - 1$ .*

Here some of the formulae are things that we already know, from the above, some other formulae are fairly easy, and some other are more difficult.

EXERCISE 2.31. *Prove that for the cyclic group  $\mathbb{Z}_N \subset O_N$  we have:*

$$\text{law}(\chi) = \left(1 - \frac{1}{N}\right) \delta_0 + \frac{1}{N} \delta_N$$

This looks quite elementary, and indeed it is, matter of having things started, in relation with character computations, beyond what has been said in the above.

EXERCISE 2.32. *Prove that for the dihedral group  $D_N \subset S_N$  we have:*

$$\text{law}(\chi) = \begin{cases} \left(\frac{3}{4} - \frac{1}{2N}\right) \delta_0 + \frac{1}{4} \delta_2 + \frac{1}{2N} \delta_N & (N \text{ even}) \\ \left(\frac{1}{2} - \frac{1}{2N}\right) \delta_0 + \frac{1}{2} \delta_1 + \frac{1}{2N} \delta_N & (N \text{ odd}) \end{cases}$$

Again, this is something which can only be quite elementary. As a conclusion to this, the character laws for  $\mathbb{Z}_N, D_N$  have no interesting asymptotics.

EXERCISE 2.33. *Prove that, if  $g_{ij}$  are the standard coordinates of  $S_N \subset O_N$ ,*

$$\text{law}(g_{11} + \dots + g_{ss}) = \frac{s!}{N!} \sum_{p=0}^s \frac{(N-p)!}{(s-p)!} \cdot \frac{(\delta_1 - \delta_0)^{*p}}{p!}$$

*and deduce from this that such variables become Poisson, with  $N \rightarrow \infty$ .*

As a bonus exercise, you can try to work out all the missing details for the various computations involving the complex reflection groups  $H_N^s$ , and the Bessel laws  $b_i^s$ . And if all this looks too complicated, don't worry, because we will be back to this, later.

## CHAPTER 3

### Compact groups

#### 3a. Representation theory

We discuss in this chapter and in the next one a unification and extension of the various results obtained in the above, dealing with finite or compact groups of matrices  $G \subset U_N$ . Following the tradition, starting with the influential work of Weyl [96], we will be mainly interested in group representations, and their characters:

**DEFINITION 3.1.** *A representation of a compact group  $G$  is a continuous group morphism, which can be faithful or not, into a unitary group:*

$$v : G \rightarrow U_N$$

*The character of such a representation is the function  $\chi : G \rightarrow \mathbb{C}$  given by*

$$g \rightarrow \text{Tr}(v_g)$$

*where  $\text{Tr}$  is the usual, unnormalized trace of the  $N \times N$  matrices.*

At the level of examples, most of the compact groups that we met so far, finite or continuous, naturally appear as closed subgroups  $G \subset U_N$ . In this case, the embedding  $G \subset U_N$  is of course a representation, called fundamental representation.

Let us discuss the various operations on the representations. We have here:

**PROPOSITION 3.2.** *The representations of a compact group  $G$  are subject to:*

- (1) *Making sums. Given representations  $v, w$ , of dimensions  $N, M$ , their sum is the  $N + M$ -dimensional representation  $v + w = \text{diag}(v, w)$ .*
- (2) *Making products. Given representations  $v, w$ , of dimensions  $N, M$ , their product is the  $NM$ -dimensional representation  $(v \otimes w)_{ia,jb} = v_{ij}w_{ab}$ .*
- (3) *Taking conjugates. Given a  $N$ -dimensional representation  $v$ , its conjugate is the  $N$ -dimensional representation  $(\bar{v})_{ij} = \bar{v}_{ij}$ .*
- (4) *Spinning by unitaries. Given a  $N$ -dimensional representation  $v$ , and a unitary  $U \in U_N$ , we can spin  $v$  by this unitary,  $v \rightarrow UvU^*$ .*

**PROOF.** The fact that the operations in the statement are indeed well-defined, among morphisms from  $G$  to unitary groups, is indeed clear from definitions.  $\square$

In relation now with characters, we have the following result:

PROPOSITION 3.3. *We have the following formulae, regarding characters*

$$\chi_{v+w} = \chi_v + \chi_w \quad , \quad \chi_{v \otimes w} = \chi_v \chi_w \quad , \quad \chi_{\bar{v}} = \bar{\chi}_v \quad , \quad \chi_{UvU^*} = \chi_v$$

*in relation with the basic operations for the representations.*

PROOF. All these assertions are elementary, by using the following well-known trace formulae, valid for any square matrices  $V, W$ , and any unitary  $U$ :

$$Tr(diag(V, W)) = Tr(V) + Tr(W)$$

$$Tr(V \otimes W) = Tr(V)Tr(W)$$

$$Tr(\bar{V}) = \overline{Tr(V)}$$

$$Tr(UVU^*) = Tr(V)$$

Thus, we are led to the formulae in the statement.  $\square$

Assume now that we are given a closed subgroup  $G \subset U_N$ . By using the above operations, we can construct a whole family of representations of  $G$ , as follows:

DEFINITION 3.4. *Given a closed subgroup  $G \subset U_N$ , its Peter-Weyl representations are the various tensor products between the fundamental representation and its conjugate:*

$$v : G \subset U_N \quad , \quad \bar{v} : G \subset U_N$$

*We denote these tensor products  $v^{\otimes k}$ , with  $k = \circ \bullet \circ \dots$  being a colored integer, with the colored tensor powers being defined according to the rules*

$$v^{\otimes \circ} = v \quad , \quad v^{\otimes \bullet} = \bar{v} \quad , \quad v^{\otimes kl} = v^{\otimes k} \otimes v^{\otimes l}$$

*and with the convention that  $v^{\otimes \emptyset}$  is the trivial representation  $1 : G \rightarrow U_1$ .*

Here are a few examples of such representations, namely those coming from the colored integers of length 2, which will often appear in what follows:

$$v^{\otimes \circ \circ} = v \otimes v \quad , \quad v^{\otimes \circ \bullet} = v \otimes \bar{v}$$

$$v^{\otimes \bullet \circ} = \bar{v} \otimes v \quad , \quad v^{\otimes \bullet \bullet} = \bar{v} \otimes \bar{v}$$

In relation now with characters, we have the following result:

PROPOSITION 3.5. *The characters of the Peter-Weyl representations are given by*

$$\chi_{v^{\otimes k}} = (\chi_v)^k$$

*with the colored powers being given by  $\chi^\circ = \chi$ ,  $\chi^\bullet = \bar{\chi}$  and multiplicativity.*

PROOF. This follows indeed from the additivity, multiplicativity and conjugation formulae from Proposition 3.3, via the conventions in Definition 3.4.  $\square$

Getting back now to our motivations, we can see the interest in the above constructions. Indeed, the joint moments of the main character  $\chi = \chi_v$  and its adjoint  $\bar{\chi} = \chi_{\bar{v}}$  are the expectations of the characters of various Peter-Weyl representations:

$$\int_G \chi^k = \int_G \chi_{v^{\otimes k}}$$

In order to advance, we must develop some general theory. Let us start with:

DEFINITION 3.6. *Given a compact group  $G$ , and two of its representations,*

$$v : G \rightarrow U_N \quad , \quad w : G \rightarrow U_M$$

*we define the space of intertwiners between these representations as being*

$$\text{Hom}(v, w) = \left\{ T \in M_{M \times N}(\mathbb{C}) \mid Tv(g) = w(g)T, \forall g \in G \right\}$$

*and we use the following conventions:*

- (1) *We use the notations  $\text{Fix}(v) = \text{Hom}(1, v)$ , and  $\text{End}(v) = \text{Hom}(v, v)$ .*
- (2) *We write  $v \sim w$  when  $\text{Hom}(v, w)$  contains an invertible element.*
- (3) *We say that  $v$  is irreducible, and write  $v \in \text{Irr}(G)$ , when  $\text{End}(v) = \mathbb{C}1$ .*

The terminology here is standard, with  $\text{Fix}$ ,  $\text{Hom}$ ,  $\text{End}$  standing for fixed points, homomorphisms and endomorphisms. We will see later that irreducible means indecomposable, in a suitable sense. Here are now a few basic results, regarding these spaces:

PROPOSITION 3.7. *The spaces of intertwiners have the following properties:*

- (1)  $T \in \text{Hom}(v, w), S \in \text{Hom}(w, z) \implies ST \in \text{Hom}(v, z)$ .
- (2)  $S \in \text{Hom}(v, w), T \in \text{Hom}(z, t) \implies S \otimes T \in \text{Hom}(v \otimes z, w \otimes t)$ .
- (3)  $T \in \text{Hom}(v, w) \implies T^* \in \text{Hom}(w, v)$ .

*In abstract terms, we say that the  $\text{Hom}$  spaces form a tensor category.*

PROOF. All the formulae in the statement are clear from definitions, via elementary computations. As for the last assertion, this is something coming from (1,2,3). We will be back to tensor categories later on, with more details on this latter fact.  $\square$

As a main consequence of the above result, we have:

PROPOSITION 3.8. *Given a representation  $v : G \rightarrow U_N$ , the linear space*

$$\text{End}(v) \subset M_N(\mathbb{C})$$

*is a  $*$ -algebra, with respect to the usual involution of the matrices.*

PROOF. By definition,  $\text{End}(v)$  is a linear subspace of  $M_N(\mathbb{C})$ . We know from Proposition 3.7 (1) that this subspace  $\text{End}(v)$  is a subalgebra of  $M_N(\mathbb{C})$ , and then we know as well from Proposition 3.7 (3) that this subalgebra is stable under the involution  $*$ . Thus, what we have here is a  $*$ -subalgebra of  $M_N(\mathbb{C})$ , as claimed.  $\square$

In order to exploit the above fact, we will need a basic result from linear algebra, stating that any  $*$ -algebra  $A \subset M_N(\mathbb{C})$  decomposes as a direct sum, as follows:

$$A \simeq M_{N_1}(\mathbb{C}) \oplus \dots \oplus M_{N_k}(\mathbb{C})$$

Indeed, let us write the unit  $1 \in A$  as  $1 = p_1 + \dots + p_k$ , with  $p_i \in A$  being central minimal projections. Then each of the spaces  $A_i = p_i A p_i$  is a subalgebra of  $A$ , and we have a decomposition  $A = A_1 \oplus \dots \oplus A_k$ . But since each central projection  $p_i \in A$  was chosen minimal, we have  $A_i \simeq M_{N_i}(\mathbb{C})$ , with  $N_i = \text{rank}(p_i)$ , as desired.

We can now formulate our first Peter-Weyl type theorem, as follows:

**THEOREM 3.9 (PW1).** *Let  $v : G \rightarrow U_N$  be a group representation, consider the algebra  $A = \text{End}(v)$ , and write its unit  $1 = p_1 + \dots + p_k$ , as above. We have then*

$$v = v_1 + \dots + v_k$$

*with each  $v_i$  being an irreducible representation, obtained by restricting  $v$  to  $\text{Im}(p_i)$ .*

**PROOF.** This basically follows from Proposition 3.8, as follows:

(1) We first associate to our representation  $v : G \rightarrow U_N$  the corresponding action map on  $\mathbb{C}^N$ . If a linear subspace  $W \subset \mathbb{C}^N$  is invariant, the restriction of the action map to  $W$  is an action map too, which must come from a subrepresentation  $w \subset v$ .

(2) Consider now a projection  $p \in \text{End}(v)$ . From  $pv = vp$  we obtain that the linear space  $W = \text{Im}(p)$  is invariant under  $v$ , and so this space must come from a subrepresentation  $w \subset v$ . It is routine to check that the operation  $p \rightarrow w$  maps subprojections to subrepresentations, and minimal projections to irreducible representations.

(3) With these preliminaries in hand, let us decompose the algebra  $\text{End}(v)$  as above, by using the decomposition  $1 = p_1 + \dots + p_k$  into central minimal projections. If we denote by  $v_i \subset v$  the subrepresentation coming from the vector space  $V_i = \text{Im}(p_i)$ , then we obtain in this way a decomposition  $v = v_1 + \dots + v_k$ , as in the statement.  $\square$

Here is now our second Peter-Weyl theorem, complementing Theorem 3.9:

**THEOREM 3.10 (PW2).** *Given a closed subgroup  $G \subset_v U_N$ , any of its irreducible smooth representations*

$$w : G \rightarrow U_M$$

*appears inside a tensor product of the fundamental representation  $v$  and its adjoint  $\bar{v}$ .*

**PROOF.** Given a representation  $w : G \rightarrow U_M$ , we define the space of coefficients  $C_w \subset C(G)$  of this representation as being the following linear space:

$$C_w = \text{span} \left[ g \rightarrow w(g)_{ij} \right]$$

With this notion in hand, the result can be deduced as follows:

(1) The construction  $w \rightarrow C_w$  is functorial, in the sense that it maps subrepresentations into linear subspaces. This is indeed something which is routine to check.

(2) A closed subgroup  $G \subset_v U_N$  is a Lie group, and a representation  $w : G \rightarrow U_M$  is smooth when we have an inclusion  $C_w \subset \langle C_v \rangle$ . This is indeed well-known.

(3) By definition of the Peter-Weyl representations, as arbitrary tensor products between the fundamental representation  $v$  and its conjugate  $\bar{v}$ , we have:

$$\langle C_v \rangle = \sum_k C_{v^{\otimes k}}$$

(4) Now by putting together the above observations (2,3) we conclude that we must have an inclusion as follows, for certain exponents  $k_1, \dots, k_p$ :

$$C_w \subset C_{v^{\otimes k_1} \oplus \dots \oplus v^{\otimes k_p}}$$

(5) By using now (1), we deduce that we have an inclusion  $w \subset v^{\otimes k_1} \oplus \dots \oplus v^{\otimes k_p}$ , and by applying Theorem 3.9, this leads to the conclusion in the statement.  $\square$

### 3b. Haar integration

In order to further advance, we need to talk about integration over  $G$ . This is something quite technical, the idea being that the uniform measure  $\mu$  over  $G$  can be constructed by starting with an arbitrary probability measure  $\eta$ , and setting:

$$\mu = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \eta^{*k}$$

Thus, our next task will be that of proving this result. It is convenient, for this purpose, to work with the integration functionals with respect to the various measures on  $G$ , instead of the measures themselves. Let us begin with the following key result:

**PROPOSITION 3.11.** *Given a unital positive linear form  $\psi : C(G) \rightarrow \mathbb{C}$ , the limit*

$$\int_{\psi} f = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \psi^{*k}(f)$$

*exists, and for a coefficient of a representation  $f = (\tau \otimes id)w$  we have*

$$\int_{\psi} f = \tau(P)$$

*where  $P$  is the orthogonal projection onto the 1-eigenspace of  $(id \otimes \psi)w$ .*

**PROOF.** By linearity it is enough to prove the second assertion. More precisely, we can have the whole result proved if we can establish the following formula:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \psi^{*k}(f) = \tau(P)$$

In order to prove this latter formula, observe that we have:

$$\begin{aligned}\psi^{*k}(f) &= (\tau \otimes \psi^{*k})w \\ &= \tau((id \otimes \psi^{*k})w)\end{aligned}$$

Consider the matrix  $M = (id \otimes \psi)w$ . In terms of this matrix, we have:

$$\begin{aligned}((id \otimes \psi^{*k})w)_{i_0 i_{k+1}} &= \sum_{i_1 \dots i_k} M_{i_0 i_1} \dots M_{i_k i_{k+1}} \\ &= (M^k)_{i_0 i_{k+1}}\end{aligned}$$

Thus we have the following formula, valid for any integer  $k \in \mathbb{N}$ :

$$(id \otimes \psi^{*k})w = M^k$$

It follows that our Cesàro limit is given by the following formula:

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \psi^{*k}(f) &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \tau(M^k) \\ &= \tau \left( \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n M^k \right)\end{aligned}$$

Now since  $w$  is unitary we have  $\|w\| = 1$ , and we obtain from this that we have:

$$\|M\| \leq 1$$

Thus, in the above expression, the Cesàro limit on the right converges, and equals the orthogonal projection onto the 1-eigenspace of  $M$ :

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n M^k = P$$

Thus our initial Cesàro limit converges as well, to  $\tau(P)$ , as desired.  $\square$

When the linear  $\psi$  is chosen faithful, we have the following finer result:

**PROPOSITION 3.12.** *Given a faithful unital linear form  $\psi \in C(G)^*$ , the limit*

$$\int_{\psi} f = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \psi^{*k}(f)$$

*exists, and is independent of  $\psi$ , given on coefficients of representations by*

$$\left( id \otimes \int_{\psi} \right) w = P$$

*where  $P$  is the orthogonal projection onto  $Fix(w) = \{\xi \in \mathbb{C}^n \mid w\xi = \xi\}$ .*

PROOF. In view of Proposition 3.11, it remains to prove that when  $\psi$  is faithful, the 1-eigenspace of  $M = (id \otimes \psi)w$  equals the space  $Fix(w)$ .

“ $\supset$ ” This inclusion is clear, and for any  $\psi$ , because  $w\xi = \xi \implies M\xi = \xi$ .

“ $\subset$ ” Here we must prove that, if  $\psi$  is faithful, we have  $M\xi = \xi \implies w\xi = \xi$ . For this purpose, assume that we have  $M\xi = \xi$ , and consider the following function:

$$f = \sum_i \left( \sum_j w_{ij} \xi_j - \xi_i \right) \left( \sum_k w_{ik} \xi_k - \xi_i \right)^*$$

We must prove that we have  $f = 0$ . Since  $w$  is unitary, we have:

$$\begin{aligned} f &= \sum_i \left( \sum_j \left( w_{ij} \xi_j - \frac{1}{N} \xi_i \right) \right) \left( \sum_k \left( w_{ik}^* \bar{\xi}_k - \frac{1}{N} \bar{\xi}_i \right) \right) \\ &= \sum_{ijk} w_{ij} w_{ik}^* \xi_j \bar{\xi}_k - \frac{1}{N} w_{ij} \xi_j \bar{\xi}_i - \frac{1}{N} w_{ik}^* \xi_i \bar{\xi}_k + \frac{1}{N^2} \xi_i \bar{\xi}_i \\ &= \sum_j |\xi_j|^2 - \sum_{ij} w_{ij} \xi_j \bar{\xi}_i - \sum_{ik} w_{ik}^* \xi_i \bar{\xi}_k + \sum_i |\xi_i|^2 \\ &= \|\xi\|^2 - \langle w\xi, \xi \rangle - \overline{\langle w\xi, \xi \rangle} + \|\xi\|^2 \\ &= 2(\|\xi\|^2 - \operatorname{Re}(\langle w\xi, \xi \rangle)) \end{aligned}$$

By using now our assumption  $M\xi = \xi$ , we obtain from this that we have:

$$\begin{aligned} \psi(f) &= 2\psi(\|\xi\|^2 - \operatorname{Re}(\langle w\xi, \xi \rangle)) \\ &= 2(\|\xi\|^2 - \operatorname{Re}(\langle M\xi, \xi \rangle)) \\ &= 2(\|\xi\|^2 - \|\xi\|^2) \\ &= 0 \end{aligned}$$

Now since  $\psi$  is faithful, this gives  $f = 0$ , and so  $w\xi = \xi$ , as claimed.  $\square$

We can now formulate a main result, as follows:

**THEOREM 3.13.** *Any compact group  $G$  has a unique Haar integration, which can be constructed by starting with any faithful positive unital form  $\psi \in C(G)^*$ , and setting:*

$$\int_G = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \psi^{*k}$$

Moreover, for any representation  $w$  we have the formula

$$\left( id \otimes \int_G \right) w = P$$

where  $P$  is the orthogonal projection onto  $Fix(w) = \{\xi \in \mathbb{C}^n \mid w\xi = \xi\}$ .

PROOF. Let us first go back to the general context of Proposition 3.11. Since convolving one more time with  $\psi$  will not change the Cesàro limit appearing there, the functional  $\int_\psi \in C(G)^*$  constructed there has the following invariance property:

$$\int_\psi * \psi = \psi * \int_\psi = \int_\psi$$

In the case where  $\psi$  is assumed to be faithful, as in Proposition 3.12, our claim is that we have the following formula, valid this time for any  $\varphi \in C(G)^*$ :

$$\int_\psi * \varphi = \varphi * \int_\psi = \varphi(1) \int_\psi$$

Indeed, it is enough to prove this formula on a coefficient of a corepresentation:

$$f = (\tau \otimes id)w$$

In order to do so, consider the following two matrices:

$$P = \left( id \otimes \int_\psi \right) w \quad , \quad Q = (id \otimes \varphi)w$$

We have then the following formulae, which all follow from definitions:

$$\left( \int_\psi * \varphi \right) f = \tau(PQ) \quad , \quad \left( \varphi * \int_\psi \right) f = \tau(QP) \quad , \quad \varphi(1) \int_\psi f = \varphi(1) \tau(P)$$

Thus, in order to prove our claim, it is enough to establish the following formula:

$$PQ = QP = \psi(1)P$$

But this latter formula follows from the fact, coming from Proposition 3.12, that  $P = (id \otimes \int_\psi)w$  equals the orthogonal projection onto  $Fix(w)$ . Thus, we have proved our claim, namely that we have the following formula, valid for any  $\varphi \in C(G)^*$ :

$$\int_\psi * \varphi = \varphi * \int_\psi = \varphi(1) \int_\psi$$

Now observe that, with  $\Delta f(g \otimes h) = f(gh)$ , this formula can be written as follows:

$$\varphi \left( \int_\psi \otimes id \right) \Delta = \varphi \left( id \otimes \int_\psi \right) \Delta = \varphi \int_\psi (.)1$$

This formula being true for any  $\varphi \in C(G)^*$ , we can simply delete  $\varphi$ , and we conclude that  $\int_G = \int_\psi$  has the required left and right invariance property, namely:

$$\left( \int_G \otimes id \right) \Delta = \left( id \otimes \int_G \right) \Delta = \int_G (.)1$$

Finally, the uniqueness is clear as well, because if we have two invariant integrals  $\int_G, \int'_G$ , then their convolution equals on one hand  $\int_G$ , and on the other hand,  $\int'_G$ .  $\square$

Summarizing, we can now integrate over  $G$ . As a first application, we have:

**THEOREM 3.14.** *Given a compact group  $G$ , we have the following formula, valid for any unitary representation  $w : G \rightarrow U_M$ :*

$$\int_G \chi_w = \dim(\text{Fix}(w))$$

*In particular, in the case  $G \subset_v U_N$ , the moments of the main character  $\chi = \chi_v$  are*

$$\int_G \chi^k = \dim(\text{Fix}(v^{\otimes k}))$$

*so knowing  $\text{law}(\chi)$  is the same as knowing the numbers  $\dim(\text{Fix}(v^{\otimes k}))$ .*

**PROOF.** We have several assertions here, the idea being as follows:

(1) Given a unitary representation  $w : G \rightarrow U_M$  as in the statement, its character  $\chi_w$  is a coefficient, so we can use the integration formula for coefficients in Theorem 3.13. If we denote by  $P$  the projection onto  $\text{Fix}(w)$ , this formula gives, as desired:

$$\begin{aligned} \int_G \chi_w &= \text{Tr}(P) \\ &= \dim(\text{Im}(P)) \\ &= \dim(\text{Fix}(w)) \end{aligned}$$

(2) This follows from (1), applied to the Peter-Weyl representations, as follows:

$$\begin{aligned} \int_G \chi^k &= \int_G \chi_v^k \\ &= \int_G \chi_{v^{\otimes k}} \\ &= \dim(\text{Fix}(v^{\otimes k})) \end{aligned}$$

(3) The last assertion is clear,  $\text{law}(\chi)$  being uniquely determined by its moments.  $\square$

Beyond characters, we have the following result, called Weingarten formula:

**THEOREM 3.15.** *The Haar integration over a closed subgroup  $G \subset U_N$  is given by*

$$\int_G g_{i_1 j_1}^{e_1} \dots g_{i_k j_k}^{e_k} dg = \sum_{\pi, \nu \in D(k)} \delta_\pi(i) \delta_\nu(j) W_k(\pi, \nu)$$

*for any colored integer  $k = e_1 \dots e_k$  and any multi-indices  $i, j$ , where  $D(k)$  is a linear basis of  $\text{Fix}(v^{\otimes k})$ , the associated generalized Kronecker symbols are given by*

$$\delta_\pi(i) = \langle \pi, e_{i_1} \otimes \dots \otimes e_{i_k} \rangle$$

*and  $W_k = G_k^{-1}$  is the inverse of the Gram matrix,  $G_k(\pi, \nu) = \langle \pi, \nu \rangle$ .*

PROOF. This is something old and classical, known to generations of mathematicians in various forms, probably since Weyl himself. In modern times the interest in such formulae came from Weingarten's paper [95], later on systematically used by Collins in [30], and some time after, further axiomatized and used in [35], then in [13]. In the above precise formulation, the proof of the Weingarten formula is as follows:

(1) We know from Peter-Weyl theory that the integrals in the statement form altogether the orthogonal projection  $P^k$  onto the following space:

$$\text{Fix}(v^{\otimes k}) = \text{span}(D(k))$$

(2) Consider now the following linear map, with  $D(k) = \{\xi_k\}$  being as above:

$$E(x) = \sum_{\pi \in D(k)} \langle x, \xi_\pi \rangle \xi_\pi$$

(3) By a standard linear algebra computation, it follows that we have  $P = WE$ , where  $W$  is the inverse of the restriction of  $E$  to the following space:

$$K = \text{span} \left( T_\pi \Big|_{\pi \in D(k)} \right)$$

(4) But this restriction is the linear map given by the Gram matrix  $G_k$ , and so  $W$  is the linear map given by the Weingarten matrix  $W_k = G_k^{-1}$ , and this gives the result.  $\square$

Getting back now to algebra, in order to further develop the Peter-Weyl theory, which is obviously something very useful, we will need the following result:

PROPOSITION 3.16. *We have a Frobenius type isomorphism*

$$\text{Hom}(v, w) \simeq \text{Fix}(v \otimes \bar{w})$$

*valid for any two representations  $v, w$ .*

PROOF. According to definitions, we have the following equivalences:

$$\begin{aligned} T \in \text{Hom}(v, w) &\iff Tv = wT \\ &\iff \sum_i T_{ai} v_{ij} = \sum_b w_{ab} T_{bj}, \forall a, j \end{aligned}$$

On the other hand, we have as well the following equivalences:

$$\begin{aligned} T \in \text{Fix}(v \otimes \bar{w}) &\iff (v \otimes \bar{w})T = \xi \\ &\iff \sum_{bi} v_{ji} \bar{w}_{ab} T_{bi} = T_{aj} \forall a, j \end{aligned}$$

With these formulae in hand, both inclusions follow from the unitarity of  $v, w$ .  $\square$

We can now formulate our third Peter-Weyl theorem, as follows:

THEOREM 3.17 (PW3). *The dense subalgebra  $\mathcal{C}(G) \subset C(G)$  generated by the coefficients of the fundamental representation decomposes as a direct sum*

$$\mathcal{C}(G) = \bigoplus_{w \in \text{Irr}(G)} M_{\dim(w)}(\mathbb{C})$$

*with the summands being pairwise orthogonal with respect to the scalar product*

$$\langle f, g \rangle = \int_G f \bar{g}$$

*where  $\int_G$  is the Haar integration over  $G$ .*

PROOF. By combining the previous two Peter-Weyl results, we deduce that we have a linear space decomposition as follows:

$$\mathcal{C}(G) = \sum_{w \in \text{Irr}(G)} C_w = \sum_{w \in \text{Irr}(G)} M_{\dim(w)}(\mathbb{C})$$

Thus, in order to conclude, it is enough to prove that for any two irreducible representations  $v, w \in \text{Irr}(G)$ , the corresponding spaces of coefficients are orthogonal:

$$v \not\sim w \implies C_v \perp C_w$$

But this follows from Theorem 3.13, via Proposition 3.16. Let us set indeed:

$$P_{ia,jb} = \int_G v_{ij} \bar{w}_{ab}$$

Then  $P$  is the orthogonal projection onto the following vector space:

$$\text{Fix}(v \otimes \bar{w}) \simeq \text{Hom}(v, w) = \{0\}$$

Thus we have  $P = 0$ , and this gives the result.  $\square$

Finally, we have the following result, completing the Peter-Weyl theory:

THEOREM 3.18 (PW4). *The characters of irreducible representations belong to*

$$\mathcal{C}(G)_{\text{central}} = \left\{ f \in \mathcal{C}(G) \mid f(gh) = f(hg) \right\}$$

*called algebra of central functions on  $G$ , and form an orthonormal basis of it.*

PROOF. Observe first that  $\mathcal{C}(G)_{\text{central}}$  is indeed an algebra, which contains all the characters. Conversely, consider a function  $f \in \mathcal{C}(G)$ , written as follows:

$$f = \sum_{w \in \text{Irr}(G)} f_w$$

The condition  $f \in \mathcal{C}(G)_{\text{central}}$  states then that for any  $w \in \text{Irr}(G)$ , we must have:

$$f_w \in \mathcal{C}(G)_{\text{central}}$$

But this means that  $f_w$  must be a scalar multiple of  $\chi_w$ , so the characters form a basis of  $\mathcal{C}(G)_{\text{central}}$ , as stated. Also, the fact that we have an orthogonal basis follows from Theorem 3.17. As for the fact that the characters have norm 1, this follows from:

$$\begin{aligned} \int_G \chi_w \bar{\chi}_w &= \sum_{ij} \int_G w_{ii} \bar{w}_{jj} \\ &= \sum_i \frac{1}{M} \\ &= 1 \end{aligned}$$

Here we have used the fact, coming from Theorem 3.13 and Proposition 3.16, that the integrals  $\int_G w_{ij} \bar{w}_{kl}$  form the orthogonal projection onto the following vector space:

$$\text{Fix}(w \otimes \bar{w}) \simeq \text{End}(w) = \mathbb{C}1$$

Thus, the proof of our theorem is now complete.  $\square$

All the above was quite brief, but further details on all this, Haar measure and Peter-Weyl theory, can be found in any group theory book, such as Weyl [96]. Also, importantly, many of the results discussed above have substantial refinements in the case where the group  $G$  under investigation is finite, and again we refer here to the literature.

### 3c. Diagrams, easiness

We discuss now more advanced aspects of representation theory, going towards probability theory, based on Brauer's paper [29], and on the seemingly endless flurry of papers based on it, ancient or more modern, including [20], [35], [95], to name a few.

In view of the above results, no matter on what we want to do with our group, we must compute the spaces  $\text{Fix}(v^{\otimes k})$ . It is technically convenient to slightly enlarge the class of spaces to be computed, by talking about Tannakian categories, as follows:

**DEFINITION 3.19.** *The Tannakian category associated to a closed subgroup  $G \subset_v U_N$  is the collection  $C_G = (C_G(k, l))$  of vector spaces*

$$C_G(k, l) = \text{Hom}(v^{\otimes k}, v^{\otimes l})$$

where the representations  $v^{\otimes k}$  with  $k = \circ \bullet \bullet \circ \dots$  colored integer, defined by

$$v^{\otimes \emptyset} = 1 \quad , \quad v^{\otimes \circ} = v \quad , \quad v^{\otimes \bullet} = \bar{v}$$

and multiplicativity,  $v^{\otimes kl} = v^{\otimes k} \otimes v^{\otimes l}$ , are the Peter-Weyl representations.

Let us make a summary of what we have so far, regarding these spaces  $C_G(k, l)$ . In order to formulate our result, let us start with the following definition:

DEFINITION 3.20. Let  $H$  be a finite dimensional Hilbert space. A tensor category over  $H$  is a collection  $C = (C(k, l))$  of linear spaces

$$C(k, l) \subset \mathcal{L}(H^{\otimes k}, H^{\otimes l})$$

satisfying the following conditions:

- (1)  $S, T \in C$  implies  $S \otimes T \in C$ .
- (2) If  $S, T \in C$  are composable, then  $ST \in C$ .
- (3)  $T \in C$  implies  $T^* \in C$ .
- (4)  $C(k, k)$  contains the identity operator.
- (5)  $C(\emptyset, k)$  with  $k = \circ\bullet, \bullet\circ$  contain the operator  $R : 1 \rightarrow \sum_i e_i \otimes e_i$ .
- (6)  $C(kl, lk)$  with  $k, l = \circ, \bullet$  contain the flip operator  $\Sigma : a \otimes b \rightarrow b \otimes a$ .

Here the tensor power Hilbert spaces  $H^{\otimes k}$ , with  $k = \circ\bullet\bullet\circ\dots$  being a colored integer, are defined by the following formulae, and multiplicativity:

$$H^{\otimes \emptyset} = \mathbb{C} \quad , \quad H^{\otimes \circ} = H \quad , \quad H^{\otimes \bullet} = \bar{H} \simeq H$$

With these conventions, we have the following result, summarizing our knowledge on the subject, coming from the results established in the above:

THEOREM 3.21. For a closed subgroup  $G \subset_v U_N$ , the associated Tannakian category

$$C_G(k, l) = \text{Hom}(v^{\otimes k}, v^{\otimes l})$$

is a tensor category over the Hilbert space  $H = \mathbb{C}^N$ .

PROOF. We know that the fundamental representation  $v$  acts on the Hilbert space  $H = \mathbb{C}^N$ , and that its conjugate  $\bar{v}$  acts on the Hilbert space  $\bar{H} = \mathbb{C}^N$ . Now by multiplicativity we conclude that any Peter-Weyl representation  $v^{\otimes k}$  acts on the Hilbert space  $H^{\otimes k}$ , and so that we have embeddings as in Definition 3.20, as follows:

$$C_G(k, l) \subset \mathcal{L}(H^{\otimes k}, H^{\otimes l})$$

Regarding now the fact that the axioms (1-6) in Definition 3.20 are indeed satisfied, this is something that we basically already know. To be more precise, (1-4) are clear, and (5) follows from the fact that each element  $g \in G$  is a unitary, which gives:

$$R \in \text{Hom}(1, g \otimes \bar{g})$$

$$R \in \text{Hom}(1, \bar{g} \otimes g)$$

As for (6), this is something trivial, coming from the fact that the matrix coefficients  $g \rightarrow g_{ij}$  and their complex conjugates  $g \rightarrow \bar{g}_{ij}$  commute with each other.  $\square$

Our purpose now will be that of showing that any closed subgroup  $G \subset U_N$  is uniquely determined by its Tannakian category  $C_G = (C_G(k, l))$ . This result, known as Tannakian duality, is something quite deep, and extremely useful. Indeed, the idea is that what we would have here is a “linearization” of  $G$ , allowing us to do combinatorics, and to ultimately reach to concrete and powerful results, regarding  $G$  itself.

The construction in the other sense is something simple as well, as follows:

**THEOREM 3.22.** *Given a tensor category  $C = (C(k, l))$  over a finite dimensional Hilbert space  $H \simeq \mathbb{C}^N$ , the following construction,*

$$G_C = \left\{ g \in U_N \mid Tg^{\otimes k} = g^{\otimes l}T, \forall k, l, \forall T \in C(k, l) \right\}$$

*produces a closed subgroup  $G_C \subset U_N$ .*

**PROOF.** This is something elementary, with the fact that the closed subset  $G_C \subset U_N$  constructed in the statement is indeed stable under the multiplication, unit and inversion operation for the unitary matrices  $g \in U_N$  being clear from definitions.  $\square$

We can now formulate the Tannakian duality result, as follows:

**THEOREM 3.23.** *The above Tannakian constructions*

$$G \rightarrow C_G$$

$$C \rightarrow G_C$$

*are bijective, and inverse to each other.*

**PROOF.** This is something quite technical, obtained by doing some abstract algebra, and for full details here, we refer to the Tannakian duality literature. The whole subject is actually, in modern times, for the most part of quantum algebra, and you can consult here [63], [99], both quantum group papers, for details on the above.  $\square$

In order to reach now to more concrete things, following Brauer's philosophy in [29], and more specifically the more modern paper [20], based on it, we have:

**DEFINITION 3.24.** *Let  $P(k, l)$  be the set of partitions between an upper colored integer  $k$ , and a lower colored integer  $l$ . A collection of subsets*

$$D = \bigsqcup_{k, l} D(k, l)$$

*with  $D(k, l) \subset P(k, l)$  is called a category of partitions when it has the following properties:*

- (1) *Stability under the horizontal concatenation,  $(\pi, \sigma) \rightarrow [\pi\sigma]$ .*
- (2) *Stability under vertical concatenation  $(\pi, \sigma) \rightarrow \left[ \begin{smallmatrix} \sigma \\ \pi \end{smallmatrix} \right]$ , with matching middle symbols.*
- (3) *Stability under the upside-down turning  $*$ , with switching of colors,  $\circ \leftrightarrow \bullet$ .*
- (4) *Each set  $P(k, k)$  contains the identity partition  $|| \dots ||$ .*
- (5) *The sets  $P(\emptyset, \circ\bullet)$  and  $P(\emptyset, \bullet\circ)$  both contain the semicircle  $\cap$ .*
- (6) *The sets  $P(k, \bar{k})$  with  $|k| = 2$  contain the crossing partition  $\bowtie$ .*

Let us formulate as well the following definition, also from [20]:

DEFINITION 3.25. *Given a partition  $\pi \in P(k, l)$  and an integer  $N \in \mathbb{N}$ , we can construct a linear map between tensor powers of  $\mathbb{C}^N$ ,*

$$T_\pi : (\mathbb{C}^N)^{\otimes k} \rightarrow (\mathbb{C}^N)^{\otimes l}$$

*by the following formula, with  $e_1, \dots, e_N$  being the standard basis of  $\mathbb{C}^N$ ,*

$$T_\pi(e_{i_1} \otimes \dots \otimes e_{i_k}) = \sum_{j_1 \dots j_l} \delta_\pi \begin{pmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_l \end{pmatrix} e_{j_1} \otimes \dots \otimes e_{j_l}$$

*and with the coefficients on the right being Kronecker type symbols,*

$$\delta_\pi \begin{pmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_l \end{pmatrix} \in \{0, 1\}$$

*whose values depend on whether the indices fit or not.*

To be more precise, we put the indices of  $i, j$  on the legs of  $\pi$ , in the obvious way. In case all the blocks of  $\pi$  contain equal indices of  $i, j$ , we set  $\delta_\pi \binom{i}{j} = 1$ . Otherwise, we set  $\delta_\pi \binom{i}{j} = 0$ . The relation with the Tannakian categories comes from:

PROPOSITION 3.26. *The assignment  $\pi \rightarrow T_\pi$  is categorical, in the sense that*

$$T_\pi \otimes T_\nu = T_{[\pi\nu]}$$

$$T_\pi T_\nu = N^{c(\pi, \nu)} T_{[\pi]^\nu}$$

$$T_\pi^* = T_{\pi^*}$$

*where  $c(\pi, \nu)$  are certain integers, coming from the erased components in the middle.*

PROOF. This is something elementary, the computations being as follows:

(1) The concatenation axiom can be checked as follows:

$$\begin{aligned} & (T_\pi \otimes T_\nu)(e_{i_1} \otimes \dots \otimes e_{i_p} \otimes e_{k_1} \otimes \dots \otimes e_{k_r}) \\ &= \sum_{j_1 \dots j_q} \sum_{l_1 \dots l_s} \delta_\pi \begin{pmatrix} i_1 & \dots & i_p \\ j_1 & \dots & j_q \end{pmatrix} \delta_\nu \begin{pmatrix} k_1 & \dots & k_r \\ l_1 & \dots & l_s \end{pmatrix} e_{j_1} \otimes \dots \otimes e_{j_q} \otimes e_{l_1} \otimes \dots \otimes e_{l_s} \\ &= \sum_{j_1 \dots j_q} \sum_{l_1 \dots l_s} \delta_{[\pi\nu]} \begin{pmatrix} i_1 & \dots & i_p & k_1 & \dots & k_r \\ j_1 & \dots & j_q & l_1 & \dots & l_s \end{pmatrix} e_{j_1} \otimes \dots \otimes e_{j_q} \otimes e_{l_1} \otimes \dots \otimes e_{l_s} \\ &= T_{[\pi\nu]}(e_{i_1} \otimes \dots \otimes e_{i_p} \otimes e_{k_1} \otimes \dots \otimes e_{k_r}) \end{aligned}$$

(2) The composition axiom can be checked as follows:

$$\begin{aligned}
& T_\pi T_\nu(e_{i_1} \otimes \dots \otimes e_{i_p}) \\
&= \sum_{j_1 \dots j_q} \delta_\nu \begin{pmatrix} i_1 & \dots & i_p \\ j_1 & \dots & j_q \end{pmatrix} \sum_{k_1 \dots k_r} \delta_\pi \begin{pmatrix} j_1 & \dots & j_q \\ k_1 & \dots & k_r \end{pmatrix} e_{k_1} \otimes \dots \otimes e_{k_r} \\
&= \sum_{k_1 \dots k_r} N^{c(\pi, \nu)} \delta_{[\pi]}^{[\nu]} \begin{pmatrix} i_1 & \dots & i_p \\ k_1 & \dots & k_r \end{pmatrix} e_{k_1} \otimes \dots \otimes e_{k_r} \\
&= N^{c(\pi, \nu)} T_{[\pi]}^{[\nu]}(e_{i_1} \otimes \dots \otimes e_{i_p})
\end{aligned}$$

(3) Finally, the involution axiom can be checked as follows:

$$\begin{aligned}
& T_\pi^*(e_{j_1} \otimes \dots \otimes e_{j_q}) \\
&= \sum_{i_1 \dots i_p} \langle T_\pi^*(e_{j_1} \otimes \dots \otimes e_{j_q}), e_{i_1} \otimes \dots \otimes e_{i_p} \rangle e_{i_1} \otimes \dots \otimes e_{i_p} \\
&= \sum_{i_1 \dots i_p} \delta_\pi \begin{pmatrix} i_1 & \dots & i_p \\ j_1 & \dots & j_q \end{pmatrix} e_{i_1} \otimes \dots \otimes e_{i_p} \\
&= T_{\pi^*}(e_{j_1} \otimes \dots \otimes e_{j_q})
\end{aligned}$$

Summarizing, our correspondence is indeed categorical.  $\square$

In relation now with the groups, we have the following result, from [20]:

**THEOREM 3.27.** *Each category of partitions  $D = (D(k, l))$  produces a family of compact groups  $G = (G_N)$ , with  $G_N \subset_v U_N$ , via the formula*

$$Hom(v^{\otimes k}, v^{\otimes l}) = span \left( T_\pi \Big| \pi \in D(k, l) \right)$$

*and the Tannakian duality correspondence.*

**PROOF.** Given an integer  $N \in \mathbb{N}$ , consider the correspondence  $\pi \rightarrow T_\pi$  constructed in Definition 3.25, and then the collection of linear spaces in the statement, namely:

$$C(k, l) = span \left( T_\pi \Big| \pi \in D(k, l) \right)$$

According to Proposition 3.26, and to our axioms for the categories of partitions, from Definition 3.24, this collection of spaces  $C = (C(k, l))$  satisfies the axioms for the Tannakian categories, from Definition 3.20. Thus the Tannakian duality result, Theorem 3.23 above, applies, and provides us with a closed subgroup  $G_N \subset_v U_N$  such that:

$$C(k, l) = Hom(v^{\otimes k}, v^{\otimes l})$$

Thus, we are led to the conclusion in the statement.  $\square$

We can now formulate a key definition, as follows:

DEFINITION 3.28. A closed subgroup  $G \subset_v U_N$  is called easy when we have

$$\text{Hom}(v^{\otimes k}, v^{\otimes l}) = \text{span} \left( T_\pi \Big| \pi \in D(k, l) \right)$$

for any colored integers  $k, l$ , for a certain category of partitions  $D \subset P$ .

The notion of easiness goes back to the results of Brauer in [29] regarding the orthogonal group  $O_N$ , and the unitary group  $U_N$ , which reformulate as follows:

THEOREM 3.29. We have the following results:

- (1) The unitary group  $U_N$  is easy, coming from the category  $\mathcal{P}_2$ .
- (2) The orthogonal group  $O_N$  is easy as well, coming from the category  $P_2$ .

PROOF. This is something very standard, the idea being as follows:

(1) The group  $U_N$  being defined via the relations  $v^* = v^{-1}$ ,  $v^t = \bar{v}^{-1}$ , the associated Tannakian category is  $C = \text{span}(T_\pi | \pi \in D)$ , with:

$$D = \langle \begin{smallmatrix} \cap \\ \circ \bullet \end{smallmatrix}, \begin{smallmatrix} \cap \\ \bullet \circ \end{smallmatrix} \rangle = \mathcal{P}_2$$

(2) The group  $O_N \subset U_N$  being defined by imposing the relations  $v_{ij} = \bar{v}_{ij}$ , the associated Tannakian category is  $C = \text{span}(T_\pi | \pi \in D)$ , with:

$$D = \langle \mathcal{P}_2, \begin{smallmatrix} \updownarrow \\ \bullet \end{smallmatrix}, \begin{smallmatrix} \updownarrow \\ \circ \end{smallmatrix} \rangle = P_2$$

Thus, we are led to the conclusion in the statement.  $\square$

Beyond this, a first natural question is that of computing the easy group associated to the category  $P$  itself, and we have here the following Brauer type theorem:

THEOREM 3.30. The symmetric group  $S_N$ , regarded as group of unitary matrices,

$$S_N \subset O_N \subset U_N$$

via the permutation matrices, is easy, coming from the category of all partitions  $P$ .

PROOF. Consider the easy group  $G \subset O_N$  coming from the category of all partitions  $P$ . Since  $P$  is generated by the one-block partition  $\mu \in P(2, 1)$ , we have:

$$C(G) = C(O_N) / \left\langle T_\mu \in \text{Hom}(v^{\otimes 2}, v) \right\rangle$$

The linear map associated to  $\mu$  is given by the following formula:

$$T_\mu(e_i \otimes e_j) = \delta_{ij} e_i$$

Thus, the relation defining the above group  $G \subset O_N$  reformulates as follows:

$$T_\mu \in \text{Hom}(v^{\otimes 2}, v) \iff v_{ij} v_{ik} = \delta_{jk} v_{ij}, \forall i, j, k$$

In other words, the elements  $v_{ij}$  must be projections, and these projections must be pairwise orthogonal on the rows of  $v = (v_{ij})$ . We conclude that  $G \subset O_N$  is the subgroup of matrices  $g \in O_N$  having the property  $g_{ij} \in \{0, 1\}$ . Thus we have  $G = S_N$ , as claimed.  $\square$

In fact, we have the following general easiness result, from [8], regarding the series of complex reflection groups  $H_N^s \subset U_N$ , that we introduced in chapter 2:

**THEOREM 3.31.** *The group  $H_N^s = \mathbb{Z}_s \wr S_N$  is easy, the corresponding category  $P^s$  consisting of the partitions satisfying  $\# \circ = \# \bullet (s)$  in each block. In particular:*

- (1)  $S_N$  is easy, coming from the category  $P$ .
- (2)  $H_N$  is easy, coming from the category  $P_{\text{even}}$ .
- (3)  $K_N$  is easy, coming from the category  $\mathcal{P}_{\text{even}}$ .

**PROOF.** This is something that we already know at  $s = 1$ , from Theorem 3.30. In general, the proof is similar, based on Tannakian duality. To be more precise, in what regards the main assertion, the idea here is that the one-block partition  $\pi \in P(s)$ , which generates the category  $P^s$  in the statement, implements the relations producing the subgroup  $H_N^s \subset U_N$ . As for the last assertions, these follow from the following observations:

(1) At  $s = 1$  we know that we have  $H_N^1 = S_N$ . Regarding now the corresponding category, here the condition  $\# \circ = \# \bullet (1)$  is automatic, and so  $P^1 = P$ .

(2) At  $s = 2$  we know that we have  $H_N^2 = H_N$ . Regarding now the corresponding category, here the condition  $\# \circ = \# \bullet (2)$  reformulates as follows:

$$\# \circ + \# \bullet = 0(2)$$

Thus each block must have even size, and we obtain, as claimed,  $P^2 = P_{\text{even}}$ .

(3) At  $s = \infty$  we know that we have  $H_N^\infty = K_N$ . Regarding now the corresponding category, here the condition  $\# \circ = \# \bullet (\infty)$  reads:

$$\# \circ = \# \bullet$$

But this is the condition defining  $\mathcal{P}_{\text{even}}$ , and so  $P^\infty = \mathcal{P}_{\text{even}}$ , as claimed.  $\square$

Summarizing, we have many examples. In fact, our list of easy groups is already quite big, and here is a selection of the main results that we have so far:

**THEOREM 3.32.** *We have a diagram as follows,*

$$\begin{array}{ccc} K_N & \longrightarrow & U_N \\ \uparrow & & \uparrow \\ H_N & \longrightarrow & O_N \end{array}$$

*and all these groups are easy.*

PROOF. This follows from the above results. To be more precise, we know that the above groups are all easy, the corresponding categories of partitions being as follows:

$$\begin{array}{ccc} \mathcal{P}_{even} & \longleftarrow & \mathcal{P}_2 \\ \downarrow & & \downarrow \\ P_{even} & \longleftarrow & P_2 \end{array}$$

Thus, we are led to the conclusion in the statement.  $\square$

We refer to [13], [20], [75], [81] for more on the easy groups, in the above formulation. In fact, any Lie group or Lie algebra text will do too, because up to formalism, notations, and so on, all the modern theory comes from Brauer's paper [29] anyway.

### 3d. Asymptotic characters

We go back now to probability questions, with the aim of applying the above, to questions regarding characters. As a first objective, for the remainder of this chapter, we would like to compute the asymptotic laws of main characters  $\chi = \chi_v$  for the main examples of easy groups. And the situation here is as follows:

(1) Given a closed subgroup  $G \subset_v U_N$ , we know from Theorem 3.14 that the moments of the main character count the fixed points of the representations  $v^{\otimes k}$ .

(2) On the other hand, assuming that our group  $G \subset_v U_N$  is easy, coming from a category of partitions  $D = (D(k, l))$ , the space formed by these fixed points is spanned by the following vectors, indexed by partitions  $\pi$  belonging to the set  $D(k) = D(0, k)$ :

$$\xi_\pi = \sum_{i_1 \dots i_k} \delta_\pi(i_1 \dots i_k) e_{i_1} \otimes \dots \otimes e_{i_k}$$

(3) Thus, we are left with investigating linear independence questions for the vectors  $\xi_\pi$ , and once these questions solved, to compute the moments of  $\chi$ .

In order to investigate linear independence questions for the vectors  $\xi_\pi$ , we will use the Gram matrix of these vectors. Let us begin with some standard definitions:

DEFINITION 3.33. Let  $P(k)$  be the set of partitions of  $\{1, \dots, k\}$ , and let  $\pi, \nu \in P(k)$ .

- (1) We write  $\pi \leq \nu$  if each block of  $\pi$  is contained in a block of  $\nu$ .
- (2) We let  $\pi \vee \nu \in P(k)$  be the partition obtained by superposing  $\pi, \nu$ .

As an illustration here, at  $k = 2$  we have  $P(2) = \{||, \square\}$ , and the order is:

$$|| \leq \square$$

At  $k = 3$  we have  $P(3) = \{|||, |\square|, |\square|, |\square|, \square\square\}$ , and the order relation is as follows:

$$||| \leq |\square|, |\square|, |\square| \leq \square\square$$

Observe also that we have  $\pi, \nu \leq \pi \vee \nu$ . In fact,  $\pi \vee \nu$  is the smallest partition with this property, called supremum of  $\pi, \nu$ . Now back to the easy groups, we have:

PROPOSITION 3.34. *The Gram matrix  $G_{kN}(\pi, \nu) = \langle \xi_\pi, \xi_\nu \rangle$  is given by*

$$G_{kN}(\pi, \nu) = N^{|\pi \vee \nu|}$$

where  $|\cdot|$  is the number of blocks.

PROOF. According to our formula of the vectors  $\xi_\pi$ , we have:

$$\begin{aligned} \langle \xi_\pi, \xi_\nu \rangle &= \sum_{i_1 \dots i_k} \delta_\pi(i_1, \dots, i_k) \delta_\nu(i_1, \dots, i_k) \\ &= \sum_{i_1 \dots i_k} \delta_{\pi \vee \nu}(i_1, \dots, i_k) \\ &= N^{|\pi \vee \nu|} \end{aligned}$$

Thus, we have obtained the formula in the statement.  $\square$

In order to study the Gram matrix, and more specifically to compute its determinant, we will need several standard facts about the partitions. We first have:

DEFINITION 3.35. *The Möbius function of any lattice, and so of  $P$ , is given by*

$$\mu(\pi, \nu) = \begin{cases} 1 & \text{if } \pi = \nu \\ -\sum_{\pi \leq \tau < \nu} \mu(\pi, \tau) & \text{if } \pi < \nu \\ 0 & \text{if } \pi \not\leq \nu \end{cases}$$

with the construction being performed by recurrence.

As an illustration here, let us go back to the set of 2-point partitions,  $P(2) = \{||, \square\}$ . Here we have by definition:

$$\mu(||, ||) = \mu(\square, \square) = 1$$

Also, we know that we have  $|| < \square$ , with no intermediate partition in between, and so the above recurrence procedure gives the following formular:

$$\mu(||, \square) = -\mu(||, ||) = -1$$

Finally, we have  $\square \not\leq ||$ , which gives  $\mu(\square, ||) = 0$ . Thus, as a conclusion, the Möbius matrix  $M_{\pi\nu} = \mu(\pi, \nu)$  of the lattice  $P(2) = \{||, \square\}$  is as follows:

$$M = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

The interest in the Möbius function comes from the Möbius inversion formula:

$$f(\nu) = \sum_{\pi \leq \nu} g(\pi) \implies g(\nu) = \sum_{\pi \leq \nu} \mu(\pi, \nu) f(\pi)$$

In linear algebra terms, the statement and proof of this formula are as follows:

**THEOREM 3.36.** *The inverse of the adjacency matrix of  $P$ , given by*

$$A_{\pi\nu} = \begin{cases} 1 & \text{if } \pi \leq \nu \\ 0 & \text{if } \pi \not\leq \nu \end{cases}$$

*is the Möbius matrix of  $P$ , given by  $M_{\pi\nu} = \mu(\pi, \nu)$ .*

**PROOF.** This is well-known, coming for instance from the fact that  $A$  is upper triangular. Thus, when inverting, we are led into the recurrence from Definition 3.35.  $\square$

As an illustration here, for  $P(2)$  the formula  $M = A^{-1}$  appears as follows:

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1}$$

Now back to our Gram matrix considerations, we have the following result:

**PROPOSITION 3.37.** *The Gram matrix is given by  $G_{kN} = AL$ , where*

$$L(\pi, \nu) = \begin{cases} N(N-1) \dots (N - |\pi| + 1) & \text{if } \nu \leq \pi \\ 0 & \text{otherwise} \end{cases}$$

*and where  $A = M^{-1}$  is the adjacency matrix of  $P(k)$ .*

**PROOF.** We have the following computation:

$$\begin{aligned} N^{|\pi \vee \nu|} &= \# \left\{ i_1, \dots, i_k \in \{1, \dots, N\} \mid \ker i \geq \pi \vee \nu \right\} \\ &= \sum_{\tau \geq \pi \vee \nu} \# \left\{ i_1, \dots, i_k \in \{1, \dots, N\} \mid \ker i = \tau \right\} \\ &= \sum_{\tau \geq \pi \vee \nu} N(N-1) \dots (N - |\tau| + 1) \end{aligned}$$

According to Proposition 3.34 and to the definition of  $A, L$ , this formula reads:

$$\begin{aligned} (G_{kN})_{\pi\nu} &= \sum_{\tau \geq \pi} L_{\tau\nu} \\ &= \sum_{\tau} A_{\pi\tau} L_{\tau\nu} \\ &= (AL)_{\pi\nu} \end{aligned}$$

Thus, we obtain the formula in the statement.  $\square$

As a concrete illustration for the above result, at  $k = 2$  we have  $P(2) = \{||, \sqcap\}$ , and the above formula  $G_{kN} = AL$  appears as follows:

$$\begin{pmatrix} N^2 & N \\ N & N \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} N^2 - N & 0 \\ N & N \end{pmatrix}$$

In general, the formula  $G_k = A_k L_k$  appears a bit in the same way, with  $A_k$  being binary and upper triangular, and with  $L_k$  depending on  $N$ , and being lower triangular.

With the above result in hand, we can now investigate the linear independence properties of the vectors  $\xi_\pi$ . To be more precise, we have the following result:

**THEOREM 3.38.** *The determinant of the Gram matrix  $G_{kN}$  is given by*

$$\det(G_{kN}) = \prod_{\pi \in P(k)} \frac{N!}{(N - |\pi|)!}$$

*and in particular, for  $N \geq k$ , the vectors  $\{\xi_\pi | \pi \in P(k)\}$  are linearly independent.*

**PROOF.** According to the formula in Proposition 3.37, we have:

$$\det(G_{kN}) = \det(A) \det(L)$$

Now if we order  $P(k)$  as above, with respect to the number of blocks, and then lexicographically, we see that  $A$  is upper triangular, and that  $L$  is lower triangular. Thus  $\det(A)$  can be computed simply by making the product on the diagonal, and we obtain 1. As for  $\det(L)$ , this can be computed as well by making the product on the diagonal, and we obtain the number in the statement, with the technical remark that in the case  $N < k$  the convention is that we obtain a vanishing determinant.  $\square$

We should mention that there is a whole story with the above formula, originally due to Lindström, and its various generalizations. We refer here to [16], [40], [47], and we will be back to these interesting topics, related to physics, later on in this book.

Now back to the laws of characters, we can formulate:

**THEOREM 3.39.** *For an easy group  $G = (G_N)$ , coming from a category of partitions  $D = (D(k, l))$ , the asymptotic moments of the main character are given by*

$$\lim_{N \rightarrow \infty} \int_{G_N} \chi^k = \#D(k)$$

*where  $D(k) = D(\emptyset, k)$ , with the limiting sequence on the left consisting of certain integers, and being stationary at least starting from the  $k$ -th term.*

**PROOF.** This follows indeed from the Peter-Weyl theory, by using the linear independence result for the vectors  $\xi_\pi$  coming from Theorem 3.38.  $\square$

With these preliminaries in hand, we can now state and prove:

**THEOREM 3.40.** *In the  $N \rightarrow \infty$  limit, the laws of the main character for the main easy groups, real and complex, and discrete and continuous, are as follows,*

$$\begin{array}{ccc}
 K_N & \longrightarrow & U_N \\
 \uparrow & & \uparrow \\
 H_N & \longrightarrow & O_N
 \end{array}
 \quad : \quad
 \begin{array}{ccc}
 B_1 & \longrightarrow & G_1 \\
 \uparrow & & \uparrow \\
 b_1 & \longrightarrow & g_1
 \end{array}$$

*with these laws, namely the real and complex Gaussian and Bessel laws, being the main limiting laws in real and complex, and discrete and continuous probability.*

**PROOF.** This follows from the above results. To be more precise, we know that the above groups are all easy, the corresponding categories of partitions being as follows:

$$\begin{array}{ccc}
 \mathcal{P}_{\text{even}} & \longleftarrow & \mathcal{P}_2 \\
 \downarrow & & \downarrow \\
 P_{\text{even}} & \longleftarrow & P_2
 \end{array}$$

Thus, we can use Theorem 3.39, are we are led into counting partitions, and then recovering the measures via their moments, which can be done as follows:

(1) For  $O_N$  the associated category of partitions is  $P_2$ , so the asymptotic moments of the main character are as follows, with the convention  $k!! = 0$  when  $k$  is odd:

$$M_k = \#P_2(k) = k!!$$

Thus, we obtain the real Gaussian law, as desired.

(2) For  $U_N$ , this follows from some combinatorics. To be more precise, the asymptotic moments of the main character, with respect to the colored integers, are as follows:

$$M_k = \#\mathcal{P}_2(k)$$

Thus, we obtain this time the complex Gaussian law, as desired.

(3) For the discrete counterparts  $H_N, K_N$  of the rotation groups  $O_N, U_N$  the situation is similar, and we obtain the real and complex Bessel laws.  $\square$

For more on the above, we refer to [13], [20] and related papers.

### 3e. Exercises

We had a lot of general theory in this chapter, regarding the compact groups and their representations, and most of our exercises will be about this. First, we have:

EXERCISE 3.41. *Compute all the representations of the symmetric group  $S_3$ .*

Here you can use of course Peter-Weyl theory, in order to stop the computations, once you found enough irreducible representations. As a bonus exercise, which is however quite difficult, you can try afterwards  $S_N$  itself, with  $N \in \mathbb{N}$  arbitrary.

EXERCISE 3.42. *Clarify all the details for the decomposition result*

$$A = M_{N_1}(\mathbb{C}) \oplus \dots \oplus M_{N_k}(\mathbb{C})$$

*for the  $*$ -subalgebras of  $M_N(\mathbb{C})$ , used in the proof of the Peter-Weyl theorems.*

This is something that we have discussed in the above, and normally the details are all quite routine. Alternatively, you can try proving Peter-Weyl without using this.

EXERCISE 3.43. *Prove that the bistochastic groups  $B_N \subset O_N$  and  $C_N \subset U_N$ , consisting of matrices having sum 1 on each row and column, are both easy, the corresponding categories being  $P_{12}$  and  $\mathcal{P}_{12}$ , with 12 standing for “singletons and pairings”.*

This looks quite routine, by suitably adapting the proofs for  $O_N$  and  $U_N$ .

EXERCISE 3.44. *Prove that the group  $H_N^s = \mathbb{Z}_s \wr S_N$  is indeed easy, the corresponding category being  $P^s$ , consisting of the partitions satisfying  $\# \circ = \# \bullet(s)$ , in each block.*

As explained in the above, this is something that we know at  $s = 1$ , where the group in question is  $S_N$ , and the proof at  $s = 2, \infty$  is not that complicated either. The problem is that of working out the general case,  $s \in \mathbb{N} \cup \{\infty\}$ , and with full details.

## CHAPTER 4

### Weingarten calculus

#### 4a. Weingarten formula

We have seen in the previous chapter that some conceptual probability theory, based on the notion of easiness, and generalizing several ad-hoc computations from chapters 1-2, can be developed for the main examples of rotation and reflection groups:

$$\begin{array}{ccc}
 K_N & \longrightarrow & U_N \\
 \uparrow & & \uparrow \\
 H_N & \longrightarrow & O_N
 \end{array}$$

Our purpose here will be that of further building on this. Based on the notion of easiness, we will develop an advanced integration theory for the easy groups. This theory, known as “Weingarten calculus”, following [95], will allow us in particular to extend our  $t = 1$  character results to the general case, involving a parameter  $t \in (0, 1]$ .

Let us start with a general formula that we know from chapter 3, namely:

**THEOREM 4.1.** *The Haar integration over a closed subgroup  $G \subset_v U_N$  is given on the dense subalgebra of smooth functions by the Weingarten type formula*

$$\int_G g_{i_1 j_1}^{e_1} \cdots g_{i_k j_k}^{e_k} dg = \sum_{\pi, \nu \in D(k)} \delta_\pi(i) \delta_\nu(j) W_k(\pi, \nu)$$

*valid for any colored integer  $k = e_1 \dots e_k$  and any multi-indices  $i, j$ , where  $D(k)$  is a linear basis of  $\text{Fix}(v^{\otimes k})$ , the associated generalized Kronecker symbols are given by*

$$\delta_\pi(i) = \langle \pi, e_{i_1} \otimes \dots \otimes e_{i_k} \rangle$$

*and  $W_k = G_k^{-1}$  is the inverse of the Gram matrix,  $G_k(\pi, \nu) = \langle \pi, \nu \rangle$ .*

**PROOF.** We know this from chapter 3, but let us recall the proof. The integrals in the statement form altogether the orthogonal projection  $P^k$  on the following space:

$$\text{Fix}(v^{\otimes k}) = \text{span}(D(k))$$

Consider now the following linear map, with  $D(k) = \{\xi_k\}$  being as above:

$$E(x) = \sum_{\pi \in D(k)} \langle x, \xi_\pi \rangle \xi_\pi$$

By a standard linear algebra computation, it follows that we have  $P = WE$ , where  $W$  is the inverse of the restriction of  $E$  to the following space:

$$K = \text{span} \left( T_\pi \Big|_{\pi \in D(k)} \right)$$

But this restriction is the linear map given by the Gram matrix  $G_k$ , and so  $W$  is the linear map given by the Weingarten matrix  $W_k = G_k^{-1}$ , and this gives the result.  $\square$

In the easy case, we have the following more concrete result:

**THEOREM 4.2.** *For an easy group  $G \subset U_N$ , coming from a category of partitions  $D = (D(k, l))$ , we have the Weingarten formula*

$$\int_G g_{i_1 j_1}^{e_1} \cdots g_{i_k j_k}^{e_k} dg = \sum_{\pi, \nu \in D(k)} \delta_\pi(i) \delta_\nu(j) W_{kN}(\pi, \nu)$$

for any  $k = e_1 \dots e_k$  and any  $i, j$ , where  $D(k) = D(\emptyset, k)$ ,  $\delta$  are usual Kronecker type symbols, checking whether the indices match, and  $W_{kN} = G_{kN}^{-1}$ , with

$$G_{kN}(\pi, \nu) = N^{|\pi \vee \nu|}$$

where  $|\cdot|$  is the number of blocks.

**PROOF.** We use the abstract Weingarten formula, from Theorem 4.1. The Kronecker type symbols there are then the usual ones, as shown by:

$$\begin{aligned} \delta_{\xi_\pi}(i) &= \langle \xi_\pi, e_{i_1} \otimes \dots \otimes e_{i_k} \rangle \\ &= \delta_\pi(i_1, \dots, i_k) \end{aligned}$$

The Gram matrix being as well the correct one, we obtain the result.  $\square$

As a toy example for the Weingarten formula, let us first work out the case of the symmetric group  $S_N$ . Here there is no really need for the Weingarten formula, because we have the following elementary result, which completely solves the problem:

**PROPOSITION 4.3.** *The integrals over  $S_N \subset O_N$  are given by*

$$\int_{S_N} g_{i_1 j_1} \cdots g_{i_k j_k} dg = \begin{cases} \frac{(N - |\ker i|)!}{N!} & \text{if } \ker i = \ker j \\ 0 & \text{otherwise} \end{cases}$$

where  $|\cdot|$  denotes as usual the number of blocks.

PROOF. This is something that we know from chapter 2, but let us recall the proof. Since the embedding  $S_N \subset O_N$  is given by  $g_{ij} = \delta_{\sigma(j)i}$ , we have:

$$\int_{S_N} g_{i_1 j_1} \cdots g_{i_k j_k} dg = \frac{1}{N!} \# \left\{ g \in S_N \mid g(j_1) = i_1, \dots, g(j_k) = i_k \right\}$$

In the case  $\ker i \neq \ker j$  there is no  $g \in S_N$  as above, and the integral vanishes. As for the case left, namely  $\ker i = \ker j$ , here if we denote by  $b \in \{1, \dots, k\}$  the number of blocks of this partition  $\ker i = \ker j$ , we have  $N - b$  points to be sent bijectively to  $N - b$  points, and so  $(N - b)!$  solutions, and the integral is  $\frac{(N-b)!}{N!}$ , as claimed.  $\square$

Getting back now to Weingarten matrices, the point is that Proposition 4.3 allows their precise computation, and evaluation, the result being as follows:

THEOREM 4.4. *For  $S_N$  the Weingarten function is given by*

$$W_{kN}(\pi, \nu) = \sum_{\tau \leq \pi \wedge \nu} \mu(\tau, \pi) \mu(\tau, \nu) \frac{(N - |\tau|)!}{N!}$$

and satisfies the following estimate,

$$W_{kN}(\pi, \nu) = N^{-|\pi \wedge \nu|} (\mu(\pi \wedge \nu, \pi) \mu(\pi \wedge \nu, \nu) + O(N^{-1}))$$

with  $\mu$  being the Möbius function of  $P(k)$ .

PROOF. The first assertion follows from the Weingarten formula, namely:

$$\int_{S_N} g_{i_1 j_1} \cdots g_{i_k j_k} dg = \sum_{\pi, \nu \in P(k)} \delta_\pi(i) \delta_\nu(j) W_{kN}(\pi, \nu)$$

Indeed, in this formula the integrals on the left are known, from the explicit integration formula over  $S_N$  that we established in Proposition 4.3, namely:

$$\int_{S_N} g_{i_1 j_1} \cdots g_{i_k j_k} dg = \begin{cases} \frac{(N - |\ker i|)!}{N!} & \text{if } \ker i = \ker j \\ 0 & \text{otherwise} \end{cases}$$

But this allows the computation of the right term, via the Möbius inversion formula, from chapter 3. As for the second assertion, this follows from the first one. See [16].  $\square$

The above result is of course something very special, coming from the fact that the integration over  $S_N$  is something very simple. For other groups, such as the orthogonal group  $O_N$  or the unitary group  $U_N$ , we will see that things are far more complicated.

### 4b. Basic estimates

Let us discuss now the computation of the polynomial integrals over the orthogonal group  $O_N$ . These are best introduced in a rectangular way, as follows:

DEFINITION 4.5. *Associated to any matrix  $a \in M_{p \times q}(\mathbb{N})$  is the integral*

$$I(a) = \int_{O_N} \prod_{i=1}^p \prod_{j=1}^q v_{ij}^{a_{ij}} dv$$

with respect to the Haar measure of  $O_N$ , where  $N \geq p, q$ .

We can of course complete our matrix with 0 values, as to always deal with square matrices,  $a \in M_N(\mathbb{N})$ . However, the parameters  $p, q$  are very useful, because they measure the complexity of the problem, as shown by our various results below.

Let us set as usual  $m!! = (m-1)(m-3)(m-5)\dots$ , with the product ending at 1 or 2, depending on the parity of  $m$ . With this convention, we have the following result:

PROPOSITION 4.6. *At  $p = 1$  we have the formula*

$$I(a_1 \dots a_q) = \varepsilon \cdot \frac{(N-1)!! a_1!! \dots a_q!!}{(N + \sum a_i - 1)!!}$$

where  $\varepsilon = 1$  if all  $a_i$  are even, and  $\varepsilon = 0$  otherwise.

PROOF. This follows from the fact, already used in chapter 1, that the first slice of  $O_N$  is isomorphic to the real sphere  $S_{\mathbb{R}}^{N-1}$ . Indeed, this gives the following formula:

$$I(a_1 \dots a_q) = \int_{S_{\mathbb{R}}^{N-1}} x_1^{a_1} \dots x_q^{a_q} dx$$

But this latter integral can be computed by using polar coordinates, as explained in chapter 1, and we obtain the formula in the statement.  $\square$

Another simple computation, as well of trigonometric nature, is the one at  $N = 2$ . We have here the following result, which completely solves the problem in this case:

PROPOSITION 4.7. *At  $N = 2$  we have the formula*

$$I\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \varepsilon \cdot \frac{(a+d)!!(b+c)!!}{(a+b+c+d+1)!!}$$

where  $\varepsilon = 1$  if  $a, b, c, d$  are even,  $\varepsilon = -1$  if  $a, b, c, d$  are odd, and  $\varepsilon = 0$  if not.

PROOF. When computing the integral over  $O_2$ , we can restrict the integration to  $SO_2 = \mathbb{T}$ , then further restrict the attention to the first quadrant. We obtain:

$$I\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \varepsilon \cdot \frac{2}{\pi} \int_0^{\pi/2} (\cos t)^{a+d} (\sin t)^{b+c} dt$$

But this gives the formula in the statement, via the formulae in chapter 1.  $\square$

The above computations tend to suggest that  $I(a)$  decomposes as a product of factorials. This is far from being true, but in the  $2 \times 2$  case it is known that  $I(a)$  decomposes as a quite reasonable sum of products of factorials. We will be back to this.

In general, we can compute the integrals  $I(a)$  by using the Weingarten formula:

THEOREM 4.8. *We have the Weingarten formula*

$$\int_{O_N} v_{i_1 j_1} \dots v_{i_{2k} j_{2k}} dv = \sum_{\pi, \nu \in D_k} \delta_\pi(i) \delta_\nu(j) W_{kN}(\pi, \nu)$$

where the objects on the right are as follows:

- (1)  $D_k$  is the set of pairings of  $\{1, \dots, 2k\}$ .
- (2) The delta symbols are 1 or 0, depending on whether indices fit or not.
- (3) The Weingarten matrix is  $W_{kN} = G_{kN}^{-1}$ , where  $G_{kN}(\pi, \nu) = N^{|\pi \vee \nu|}$ .

PROOF. This is indeed the usual Weingarten formula, for  $G = O_N$ .  $\square$

As an example, the integrals of quantities of type  $v_{i_1 j_1} v_{i_2 j_2} v_{i_3 j_3} u_{i_4 j_4}$  appear as sums of coefficients of the Weingarten matrix  $W_{2N}$ , which is given by:

$$W_{2N} = \begin{pmatrix} N^2 & N & N \\ N & N^2 & N \\ N & N & N^2 \end{pmatrix}^{-1} = \frac{1}{N(N-1)(N+2)} \begin{pmatrix} N+1 & -1 & -1 \\ -1 & N+1 & -1 \\ -1 & -1 & N+1 \end{pmatrix}$$

More precisely, the various consequences at  $k = 2$  can be summarized as follows:

PROPOSITION 4.9. *We have the following results:*

- (1)  $I\begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} = 3/(N(N+2))$ .
- (2)  $I\begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix} = 1/(N(N+2))$ .
- (3)  $I\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = (N+1)/(N(N-1)(N+2))$ .

PROOF. These results all follow from the Weingarten formula, by using the above numeric values for the entries of  $W_{2N}$ , the computations being as follows:

$$\begin{aligned} I\begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} &= \int v_{11} v_{11} v_{11} v_{11} = \sum_{\pi \sigma} W_{2N}(\pi, \sigma) = \frac{3(N+1) - 6}{N(N-1)(N+2)} = \frac{3}{N(N+2)} \\ I\begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix} &= \int v_{11} v_{11} v_{12} v_{12} = \sum_{\pi} W_{2N}(\pi, \cap \cap) = \frac{(N+1) - 2}{N(N-1)(N+2)} = \frac{1}{N(N+2)} \\ I\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} &= \int v_{11} v_{11} v_{22} v_{22} = W_{2N}(\cap \cap, \cap \cap) = \frac{N+1}{N(N-1)(N+2)} \end{aligned}$$

Observe that the first two formulae follow in fact as well from Proposition 4.6.  $\square$

In terms of the integrals  $I(a)$ , the Weingarten formula reformulates as follows:

THEOREM 4.10. *We have the Weingarten formula*

$$I(a) = \sum_{\pi, \nu} \delta_{\pi}(a_l) \delta_{\nu}(a_r) W_{kN}(\pi, \nu)$$

where  $k = \Sigma a_{ij}/2$ , and where the multi-indices  $a_l/a_r$  are defined as follows:

- (1) Start with  $a \in M_{p \times q}(\mathbb{N})$ , and replace each  $ij$ -entry by  $a_{ij}$  copies of  $i/j$ .
- (2) Read this matrix in the usual way, as to get the multi-indices  $a_l/a_r$ .

PROOF. This is simply a reformulation of the usual Weingarten formula. Indeed, according to our definitions, the integral in the statement is given by:

$$I(a) = \int_{O_N} \underbrace{v_{11} \dots v_{11}}_{a_{11}} \underbrace{v_{12} \dots v_{12}}_{a_{12}} \dots \underbrace{v_{pq} \dots v_{pq}}_{a_{pq}} du$$

Thus what we have here is an integral as in Theorem 4.8, the multi-indices being:

$$\begin{aligned} a_l &= (\underbrace{1 \dots 1}_{a_{11}} \underbrace{1 \dots 1}_{a_{12}} \dots \underbrace{p \dots p}_{a_{pq}}) \\ a_r &= (\underbrace{1 \dots 1}_{a_{11}} \underbrace{2 \dots 2}_{a_{12}} \dots \underbrace{q \dots q}_{a_{pq}}) \end{aligned}$$

With this observation, the result follows now from the Weingarten formula.  $\square$

We can now extend the various vanishing results appearing before, as follows:

PROPOSITION 4.11. *We have  $I(a) = 0$ , unless the matrix  $a$  is “admissible”, in the sense that all  $p + q$  sums on its rows and columns are even numbers.*

PROOF. Observe first that the left multi-index associated to  $a$  consists of  $k_1 = \Sigma a_{1j}$  copies of 1,  $k_2 = \Sigma a_{2j}$  copies of 2, and so on, up to  $k_p = \Sigma a_{pj}$  copies of  $p$ . In the case where one of these numbers is odd we have  $\delta_{\pi}(a) = 0$  for any  $\pi$ , and so  $I(a) = 0$ . A similar argument with the right multi-index associated to  $a$  shows that the sums on the columns of  $a$  must be even as well, and we are done.  $\square$

A natural question now is whether the converse of Proposition 4.11 holds, and if so, the question of computing the sign of  $I(a)$  appears as well. These are both subtle questions, and we begin our investigations with a  $N \rightarrow \infty$  study. We have here:

THEOREM 4.12. *The Weingarten matrix is asymptotically diagonal, in the sense that:*

$$W_{kN}(\pi, \nu) = N^{-k}(\delta_{\pi\nu} + O(N^{-1}))$$

Moreover, the  $O(N^{-1})$  remainder is asymptotically smaller than  $(2k/e)^k N^{-1}$ .

PROOF. It is convenient, for the purposes of this proof, to drop the indices  $k, N$ . We know that the Gram matrix is given by  $G(\pi, \nu) = N^{|\pi \vee \nu|}$ , so we have:

$$G(\pi, \nu) = \begin{cases} N^k & \text{for } \pi = \nu \\ N, N^2, \dots, N^{k-1} & \text{for } \pi \neq \nu \end{cases}$$

Thus the Gram matrix is of the following form, with  $\|H\|_\infty \leq N^{-1}$ :

$$G = N^k(I + H)$$

Now recall that for any complex  $K \times K$  matrix  $A$ , we have the following lineup of standard inequalities, which all follow from definitions:

$$\|A\|_\infty \leq \|A\| \leq \|A\|_2 \leq K\|A\|_\infty$$

In the case of our matrix  $H$ , the size of this matrix is  $K = (2k)!!$ , so we have:

$$\|H\| \leq KN^{-1}$$

We can perform the inversion operation, by using the following formula:

$$(I + H)^{-1} = I - H + H^2 - H^3 + \dots$$

We obtain in this way the following estimate:

$$\|I - (I + H)^{-1}\| \leq \frac{\|H\|}{1 - \|H\|}$$

Thus, we have the following estimate:

$$\begin{aligned} \|I - N^k W\|_\infty &= \|I - (I + H)^{-1}\|_\infty \\ &\leq \|I - (I + H)^{-1}\| \\ &\leq \frac{\|H\|}{1 - \|H\|} \\ &\leq \frac{KN^{-1}}{1 - KN^{-1}} \\ &= \frac{K}{N - K} \end{aligned}$$

Together with the Stirling estimate  $K \approx (2k/e)^k$ , this gives the result.  $\square$

Regarding now the integrals themselves, we have here the following result:

THEOREM 4.13. *We have the estimate*

$$I(a) = N^{-k} \left( \prod_{i=1}^p \prod_{j=1}^q a_{ij}!! + O(N^{-1}) \right)$$

when all  $a_{ij}$  are even, and  $I(a) = O(N^{-k-1})$  otherwise.

PROOF. By using the above results, we have the following estimate:

$$\begin{aligned}
I(a) &= \sum_{\pi, \nu} \delta_{\pi}(a_l) \delta_{\nu}(a_r) W_{kN}(\pi, \nu) \\
&= N^{-k} \sum_{\pi, \nu} \delta_{\pi}(a_l) \delta_{\nu}(a_r) (\delta_{\pi\nu} + O(N^{-1})) \\
&= N^{-k} (\#\{\pi | \delta_{\pi}(a_l) = \delta_{\pi}(a_r) = 1\} + O(N^{-1}))
\end{aligned}$$

In order to count now the partitions appearing in the last set, let us go back to the multi-indices  $a_l, a_r$ , as described in Theorem 4.10. It is convenient to view both these multi-indices in a rectangular way, in the following way:

$$a_l = \begin{pmatrix} \underbrace{1 \dots 1}_{a_{11}} & \dots & \underbrace{1 \dots 1}_{a_{1q}} \\ \dots & \dots & \dots \\ \underbrace{p \dots p}_{a_{p1}} & \dots & \underbrace{p \dots p}_{a_{pq}} \end{pmatrix}, \quad a_r = \begin{pmatrix} \underbrace{1 \dots 1}_{a_{11}} & \dots & \underbrace{q \dots q}_{a_{1q}} \\ \dots & \dots & \dots \\ \underbrace{1 \dots 1}_{a_{p1}} & \dots & \underbrace{p \dots p}_{a_{pq}} \end{pmatrix}$$

In other words, the multi-indices  $a_l/a_r$  are simply obtained from the matrix  $a$  by “dropping” from each entry  $a_{ij}$  a sequence of  $a_{ij}$  numbers, all equal to  $i/j$ . With this picture, the pairings  $\pi$  which contribute are simply those connecting sequences of indices “dropped” from the same  $a_{ij}$ , and this gives the following results, as desired:

(1) If one of the entries  $a_{ij}$  is odd, there is no pairing that can contribute to the leading term under consideration, so we have  $I(a) = O(N^{-k-1})$ , and we are done.

(2) If all the entries  $a_{ij}$  are even, the pairings that contribute to the leading term are those connecting points inside the  $pq$  “dropped” sets, i.e. are made out of a pairing of  $a_{11}$  points, a pairing of  $a_{12}$  points, and so on, up to a pairing of  $a_{pq}$  points. Now since an  $x$ -point set has  $x!!$  pairings, this gives the formula in the statement.  $\square$

In order to further advance, let  $d(\pi, \nu) = k - |\pi \vee \nu|$ . It is well-known, and elementary to check, that this is a distance function on  $D_k$ . With this convention, we have:

THEOREM 4.14. *The Weingarten function  $W_{kN}$  has a series expansion of the form*

$$W_{kN}(\pi, \nu) = N^{-k-d(\pi, \nu)} \sum_{g=0}^{\infty} K_g(\pi, \nu) N^{-g}$$

where the objects on the right are defined as follows:

- (1) A path from  $\pi$  to  $\nu$  is a sequence  $p = [\pi = \tau_0 \neq \tau_1 \neq \dots \neq \tau_r = \nu]$ .
- (2) The signature of such a path is  $+$  when  $r$  is even, and  $-$  when  $r$  is odd.
- (3) The geodesicity defect of such a path is  $g(p) = \sum_{i=1}^r d(\tau_{i-1}, \tau_i) - d(\pi, \nu)$ .
- (4)  $K_g$  counts the signed paths from  $\pi$  to  $\nu$ , with geodesicity defect  $g$ .

PROOF. Let us go back to the proof of our main estimate so far. We can write:

$$G_{kn} = N^{-k}(I + H)$$

In terms of the Brauer space distance, the formula of the matrix  $H$  is simply:

$$H(\pi, \nu) = \begin{cases} 0 & \text{for } \pi = \sigma \\ N^{-d(\pi, \nu)} & \text{for } \pi \neq \nu \end{cases}$$

Consider now the set  $P_r(\pi, \nu)$  of  $r$ -paths between  $\pi$  and  $\nu$ . According to the usual rule of matrix multiplication, the powers of  $H$  are given by:

$$\begin{aligned} H^r(\pi, \nu) &= \sum_{p \in P_r(\pi, \nu)} H(\tau_0, \tau_1) \dots H(\tau_{r-1}, \tau_r) \\ &= \sum_{p \in P_r(\pi, \nu)} N^{-d(\pi, \nu) - g(p)} \end{aligned}$$

We can use now  $(1 + H)^{-1} = 1 - H + H^2 - H^3 + \dots$ , and we obtain:

$$\begin{aligned} W_{kN}(\pi, \nu) &= N^{-k} \sum_{r=0}^{\infty} (-1)^r H^r(\pi, \nu) \\ &= N^{-k-d(\pi, \nu)} \sum_{r=0}^{\infty} \sum_{p \in P_r(\pi, \nu)} (-1)^r N^{-g(p)} \end{aligned}$$

Now by rearranging the various terms of the double sum according to the value of their geodesicity defect  $g = g(p)$ , this gives the formula in the statement.  $\square$

In order to discuss now the  $I(a)$  reformulation of the above result, it is convenient to use the total length of a path, defined as follows:

$$d(p) = \sum_{i=1}^r d(\tau_{i-1}, \tau_i)$$

Observe that we have  $d(p) = d(\pi, \sigma) + g(p)$ . With these conventions, we have:

**THEOREM 4.15.** *The integral  $I(a)$  has a series expansion in  $N^{-1}$  of the form*

$$I(a) = N^{-k} \sum_{d=0}^{\infty} H_d(a) N^{-d}$$

where the coefficient on the right can be interpreted as follows:

- (1) Starting from  $a \in M_{p \times q}(\mathbb{N})$ , construct the multi-indices  $a_l, a_r$  as usual.
- (2) Call a path “ $a$ -admissible” if its endpoints satisfy  $\delta_\pi(a_l) = 1$  and  $\delta_\sigma(a_r) = 1$ .
- (3) Then  $H_d(a)$  counts all  $a$ -admissible signed paths in  $D_k$ , of total length  $d$ .

PROOF. By combining the above results, we obtain, with our various notations:

$$\begin{aligned} I(a) &= \sum_{\pi, \nu} \delta_{\pi}(a_l) \delta_{\nu}(a_r) W_{kN}(\pi, \nu) \\ &= N^{-k} \sum_{\pi, \nu} \delta_{\pi}(a_l) \delta_{\nu}(a_r) \sum_{g=0}^{\infty} K_g(\pi, \nu) N^{-d(\pi, \nu) - g} \end{aligned}$$

Now let  $H_d(\pi, \nu)$  be the number of signed paths between  $\pi$  and  $\nu$ , of total length  $d$ . In terms of the new variable  $d = d(\pi, \nu) + g$ , the above expression becomes:

$$\begin{aligned} I(a) &= N^{-k} \sum_{\pi, \nu} \delta_{\pi}(a_l) \delta_{\nu}(a_r) \sum_{d=0}^{\infty} H_d(\pi, \nu) N^{-d} \\ &= N^{-k} \sum_{d=0}^{\infty} \left( \sum_{\pi, \nu} \delta_{\pi}(a_l) \delta_{\nu}(a_r) H_d(\pi, \nu) \right) N^{-d} \end{aligned}$$

We recognize in the middle the quantity  $H_d(a)$ , and this gives the result.  $\square$

Let us derive now some concrete consequences from the abstract results established above. First, we have the following result, due to Collins and Śniady [35]:

THEOREM 4.16. *We have the estimate*

$$W_{kN}(\pi, \nu) = N^{-k-d(\pi, \nu)} (\mu(\pi, \nu) + O(N^{-1}))$$

where  $\mu$  is the Möbius function.

PROOF. We know from the above that we have the following estimate:

$$W_{kN}(\pi, \nu) = N^{-k-d(\pi, \nu)} (K_0(\pi, \nu) + O(N^{-1}))$$

Now since one of the possible definitions of the Möbius function  $\mu$  is that this counts the signed geodesic paths, we have  $K_0 = \mu$ , and we are done.  $\square$

Let us go back now to our integrals  $I(a)$ . We have here the following result:

THEOREM 4.17. *We have the estimate*

$$I(a) = N^{-k-e(a)} (\mu(a) + O(N^{-1}))$$

where the objects on the right are as follows:

- (1)  $e(a) = \min\{d(\pi, \nu) | \pi, \nu \in D_k, \delta_{\pi}(a_l) = \delta_{\nu}(a_r) = 1\}$ .
- (2)  $\mu(a)$  counts all  $a$ -admissible signed paths in  $D_k$ , of total length  $e(a)$ .

PROOF. We know that we have an estimate of the following type:

$$I(a) = N^{-k-e} (H_e(a) + O(N^{-1}))$$

Here, according to the various notations above,  $e \in \mathbb{N}$  is the smallest total length of an  $a$ -admissible path, and  $H_e(a)$  counts all signed  $a$ -admissible paths of total length  $e$ .

Now since the smallest total length of such a path is attained when the path is just a segment, we have  $e = e(a)$  and  $H_e(a) = \mu(a)$ , and we are done.  $\square$

At a more advanced level, we have the following formula, due to Collins-Matsumoto [31] and Zinn-Justin [100], which uses the theory of zonal spherical functions:

THEOREM 4.18. *We have the formula*

$$W_{kN}(\pi, \nu) = \frac{\sum_{\lambda \vdash k, l(\lambda) \leq k} \chi^{2\lambda}(1_k) w^\lambda(\pi^{-1}\nu)}{(2k)!! \prod_{(i,j) \in \lambda} (N + 2j - i - 1)}$$

where the various objects on the right are as follows:

- (1) The sum is over all partitions of  $\{1, \dots, 2k\}$  of length  $l(\lambda) \leq k$ .
- (2)  $w^\lambda$  is the corresponding zonal spherical function of  $(S_{2k}, H_k)$ .
- (3)  $\chi^{2\lambda}$  is the character of  $S_{2k}$  associated to  $2\lambda = (2\lambda_1, 2\lambda_2, \dots)$ .
- (4) The product is over all squares of the Young diagram of  $\lambda$ .

PROOF. This is something advanced, and we refer here to [31], [100].  $\square$

In relation with the integrals  $I(a)$ , let us just record the following consequence:

PROPOSITION 4.19. *The possible poles of  $I(a)$  can be at the numbers*

$$-(k-1), -(k-2), \dots, 2k-1, 2k$$

where  $k \in \mathbb{N}$  associated to the admissible matrix  $a \in M_{p \times q}(\mathbb{N})$  is given by  $k = \Sigma a_{ij}/2$ .

PROOF. We know from Theorem 4.10 that the possible poles of  $I(a)$  can only come from those of the Weingarten function. On the other hand, Theorem 4.18 tells us that these latter poles are located at the numbers of the form  $-2j + i + 1$ , with  $(i, j)$  ranging over all possible squares of all possible Young diagrams, and this gives the result.  $\square$

We will be back to integration over  $O_N$  at the end of the present chapter.

#### 4c. Truncated characters

Let us go back now to the general easy groups  $G \subset U_N$ , with the idea in mind of computing the laws of truncated characters. First, we have the following formula:

PROPOSITION 4.20. *The moments of truncated characters are given by the formula*

$$\int_G (g_{11} + \dots + g_{ss})^k dg = \text{Tr}(W_{kN} G_{ks})$$

where  $G_{kN}$  and  $W_{kN} = G_{kN}^{-1}$  are the associated Gram and Weingarten matrices.

PROOF. We have indeed the following computation:

$$\begin{aligned}
\int_G (g_{11} + \dots + g_{ss})^k dg &= \sum_{i_1=1}^s \dots \sum_{i_k=1}^s \int_G g_{i_1 i_1} \dots g_{i_k i_k} dg \\
&= \sum_{\pi, \nu \in D(k)} W_{kN}(\pi, \nu) \sum_{i_1=1}^s \dots \sum_{i_k=1}^s \delta_\pi(i) \delta_\nu(i) \\
&= \sum_{\pi, \nu \in D(k)} W_{kN}(\pi, \nu) G_{ks}(\nu, \pi) \\
&= \text{Tr}(W_{kN} G_{ks})
\end{aligned}$$

Thus, we have reached to the formula in the statement.  $\square$

In order to process now the above formula, and reach to concrete results, we must impose on our group a uniformity condition. Let us start with:

PROPOSITION 4.21. *For an easy group  $G = (G_N)$ , coming from a category of partitions  $D \subset P$ , the following conditions are equivalent:*

- (1)  $G_{N-1} = G_N \cap U_{N-1}$ , via the embedding  $U_{N-1} \subset U_N$  given by  $u \rightarrow \text{diag}(u, 1)$ .
- (2)  $G_{N-1} = G_N \cap U_{N-1}$ , via the  $N$  possible diagonal embeddings  $U_{N-1} \subset U_N$ .
- (3)  $D$  is stable under the operation which consists in removing blocks.

If these conditions are satisfied, we say that  $G = (G_N)$  is uniform.

PROOF. We use the general easiness theory from chapter 3, as follows:

(1)  $\iff$  (2) This is standard, coming from the inclusion  $S_N \subset G_N$ , which makes everything  $S_N$ -invariant. The result follows as well from the proof of (1)  $\iff$  (3) below, which can be converted into a proof of (2)  $\iff$  (3), in the obvious way.

(1)  $\iff$  (3) Given a subgroup  $K \subset U_{N-1}$ , with fundamental representation  $v$ , consider the matrix  $u = \text{diag}(v, 1)$ . Our claim is that for any  $\pi \in P(k)$  we have:

$$\xi_\pi \in \text{Fix}(u^{\otimes k}) \iff \xi_{\pi'} \in \text{Fix}(u^{\otimes k'}), \forall \pi' \in P(k'), \pi' \subset \pi$$

In order to prove this claim, we must study the condition on the left. We have:

$$\begin{aligned}
\xi_\pi \in \text{Fix}(u^{\otimes k}) &\iff (u^{\otimes k} \xi_\pi)_{i_1 \dots i_k} = (\xi_\pi)_{i_1 \dots i_k}, \forall i \\
&\iff \sum_j (u^{\otimes k})_{i_1 \dots i_k, j_1 \dots j_k} (\xi_\pi)_{j_1 \dots j_k} = (\xi_\pi)_{i_1 \dots i_k}, \forall i \\
&\iff \sum_j \delta_\pi(j_1, \dots, j_k) u_{i_1 j_1} \dots u_{i_k j_k} = \delta_\pi(i_1, \dots, i_k), \forall i
\end{aligned}$$

Now let us recall that our representation has the special form  $u = \text{diag}(v, 1)$ . We conclude from this that for any index  $a \in \{1, \dots, k\}$ , we have:

$$i_a = N \implies j_a = N$$

With this observation in hand, if we denote by  $i', j'$  the multi-indices obtained from  $i, j$  obtained by erasing all the above  $i_a = j_a = N$  values, and by  $k' \leq k$  the common length of these new multi-indices, our condition becomes:

$$\sum_{j'} \delta_\pi(j_1, \dots, j_k)(u^{\otimes k'})_{i'j'} = \delta_\pi(i_1, \dots, i_k), \forall i$$

Here the index  $j$  is by definition obtained from the index  $j'$  by filling with  $N$  values. In order to finish now, we have two cases, depending on  $i$ , as follows:

Case 1. Assume that the index set  $\{a | i_a = N\}$  corresponds to a certain subpartition  $\pi' \subset \pi$ . In this case, the  $N$  values will not matter, and our formula becomes:

$$\sum_{j'} \delta_\pi(j'_1, \dots, j'_{k'})(u^{\otimes k'})_{i'j'} = \delta_\pi(i'_1, \dots, i'_{k'})$$

Case 2. Assume now the opposite, namely that the set  $\{a | i_a = N\}$  does not correspond to a subpartition  $\pi' \subset \pi$ . In this case the indices mix, and our formula reads  $0 = 0$ .

Thus we have  $\xi_{\pi'} \in \text{Fix}(u^{\otimes k'})$ , for any subpartition  $\pi' \subset \pi$ , as desired.  $\square$

Now back to the laws of truncated characters, we have the following result:

**THEOREM 4.22.** *For a uniform easy group  $G = (G_N)$ , we have the formula*

$$\lim_{N \rightarrow \infty} \int_{G_N} \chi_t^k = \sum_{\pi \in D(k)} t^{|\pi|}$$

with  $D \subset P$  being the associated category of partitions.

**PROOF.** We use Proposition 4.20. With  $s = [tN]$ , the formula there becomes:

$$\int_{G_N} \chi_t^k = \text{Tr}(W_{kN} G_{k[tN]})$$

The point now is that in the uniform case the Gram matrix, and so the Weingarten matrix too, is asymptotically diagonal. Thus, we obtain the following estimate:

$$\begin{aligned} \int_{G_N} \chi_t^k &\simeq \sum_{\pi \in D(k)} W_{kN}(\pi, \pi) G_{k[tN]}(\pi, \pi) \\ &= \sum_{\pi \in D(k)} N^{-|\pi|} [tN]^{|\pi|} \\ &\simeq \sum_{\pi \in D(k)} N^{-|\pi|} (tN)^{|\pi|} \\ &= \sum_{\pi \in D(k)} t^{|\pi|} \end{aligned}$$

Thus, we are led to the formula in the statement.  $\square$

We can now enlarge our collection of truncated character results, and we have:

**THEOREM 4.23.** *With  $N \rightarrow \infty$ , the laws of truncated characters are as follows:*

- (1) *For  $O_N$  we obtain the Gaussian law  $g_t$ .*
- (2) *For  $U_N$  we obtain the complex Gaussian law  $G_t$ .*
- (3) *For  $S_N$  we obtain the Poisson law  $p_t$ .*
- (4) *For  $H_N$  we obtain the Bessel law  $b_t$ .*
- (5) *For  $H_N^s$  we obtain the generalized Bessel law  $b_t^s$ .*
- (6) *For  $K_N$  we obtain the complex Bessel law  $B_t$ .*

**PROOF.** We use the general formula for the asymptotic moments of the truncated characters found in Theorem 4.22, namely:

$$\lim_{N \rightarrow \infty} \int_{G_N} \chi_t^k = \sum_{\pi \in D(k)} t^{|\pi|}$$

By doing now some standard moment combinatorics, which was actually already done in the above, in all cases under consideration, at  $t = 1$ , and was done too in the general situation  $t > 0$ , in most of the cases under consideration, this gives the results.  $\square$

As a main consequence of the above result, we have:

**THEOREM 4.24.** *In the  $N \rightarrow \infty$  limit, the laws of truncated characters for the main easy groups, real and complex, and discrete and continuous, are as follows,*

$$\begin{array}{ccc} K_N & \longrightarrow & U_N \\ \uparrow & & \uparrow \\ H_N & \longrightarrow & O_N \end{array} \quad : \quad \begin{array}{ccc} B_t & \longrightarrow & G_t \\ \uparrow & & \uparrow \\ b_t & \longrightarrow & g_t \end{array}$$

*with these laws, namely the real and complex Gaussian and Bessel laws, being the main limiting laws in real and complex, and discrete and continuous probability.*

**PROOF.** This is something that we already know from chapter 3 for usual characters,  $t = 1$ , and which follows from Theorem 4.23 in the general case,  $t \in (0, 1]$ .  $\square$

There are many other things that can be said about the Weingarten matrices, as well as many other applications of the Weingarten formula. We will be back to this.

#### 4d. Rotation groups

Following [14], we go back here to the group  $O_N$ , with a number of more advanced results. The interpretation of the Weingarten matrix that we will need is in terms of the 0-1-2 matrices having sum 2 on each column, that we call “elementary”, as follows:

PROPOSITION 4.25. *The Weingarten matrix entries are given by*

$$W_{kN}(\pi, \nu) = I(a)$$

where  $a \in M_k(\mathbb{N})$  is the elementary matrix obtained as follows:

- (1) Label  $\pi_1, \dots, \pi_k$  the strings of  $\pi$ .
- (2) Label  $\nu_1, \dots, \nu_k$  the strings of  $\nu$ .
- (3) Set  $a_{ij} = \#\{r \in \{1, \dots, 2k\} | r \in \pi_i, r \in \nu_j\}$ .

PROOF. Consider the multi-indices  $i, j \in \{1, \dots, k\}^{2k}$  given by  $i_r \in \pi_r$  and  $j_r \in \nu_r$ , for any  $r \in \{1, \dots, k\}$ . We have  $\delta_{\pi'}(i) = \delta_{\pi\pi'}$  and  $\delta_{\nu'}(j) = \delta_{\nu\nu'}$  for any pairings  $\pi', \nu'$ , so if we apply the Weingarten formula to the quantity  $v_{i_1 j_1} \dots v_{i_{2k} j_{2k}}$ , we obtain:

$$\begin{aligned} \int_{O_N} v_{i_1 j_1} \dots v_{i_{2k} j_{2k}} dv &= \sum_{\pi' \nu'} \delta_{\pi'}(i) \delta_{\nu'}(j) W_{kN}(\pi', \nu') \\ &= \sum_{\pi' \nu'} \delta_{\pi\pi'} \delta_{\nu\nu'} W_{kN}(\pi', \nu') \\ &= W_{kN}(\pi, \nu) \end{aligned}$$

The integral on the left can be written in the form  $I(a)$ , for a certain matrix  $a$ . Our choice of  $i, j$  shows that  $a$  is the elementary matrix in the statement, and we are done.  $\square$

As an illustration for the above result, consider the partitions  $\pi = \cap \cap \cap$  and  $\nu = \cap \cap$ . We have  $i = (112233)$  and  $j = (122133)$ , and we obtain:

$$\begin{aligned} W_{3N}(\pi, \nu) &= \int_{O_N} v_{11} v_{12} v_{22} v_{21} v_{33} v_{33} dv \\ &= \int_{O_N} v_{11} v_{12} v_{22} v_{21} v_{33}^2 dv \\ &= I \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \end{aligned}$$

In general now, we would like to have a better understanding of the integrals  $I(a)$ . It is convenient to make the following normalization:

DEFINITION 4.26. *For  $a, b$  vectors with even entries we make the normalization*

$$I \begin{pmatrix} a \\ b \end{pmatrix} = I_{N-1}(a) I_{N-1}(b) J \begin{pmatrix} a \\ b \end{pmatrix}$$

where  $I_{N-1}$  is the integration, in the sense of Definition 4.5, over the group  $O_{N-1}$ .

The new quantity  $J$  is just a normalization of the usual integral  $I$ . More precisely, by using the formula in Proposition 4.6 we have the following alternative definition:

PROPOSITION 4.27. *We have the following formula:*

$$J \begin{pmatrix} a \\ b \end{pmatrix} = \frac{(\sum a_i + N - 2)!! (\sum b_i + N - 2)!!}{(N - 2)!! (N - 2)!! \prod a_i!! \prod b_i!!} I \begin{pmatrix} a \\ b \end{pmatrix}$$

PROOF. This follows indeed from the one-row formula in Proposition 4.6.  $\square$

As a first, basic example, for any one-row vector  $a$  we have  $J \begin{pmatrix} a \\ 0 \end{pmatrix} = I_N(a)/I_{N-1}(a)$ , and according to Proposition 4.6, this gives the following formula:

$$J \begin{pmatrix} a \\ 0 \end{pmatrix} = \frac{(N - 1)!!}{(N - 2)!!} \cdot \frac{(\sum a_i + N - 2)!!}{(\sum a_i + N - 1)!!}$$

The advantage of using  $J$  instead of  $I$  comes from a number of invariance properties at the general level, to be established later. For the moment, let us find some rules for computing  $J$ . For  $k, x \in \mathbb{N}$  we let  $k^x = k \dots k$  ( $x$  times). Following [14], we have:

THEOREM 4.28. *We have the “elementary expansion” formula*

$$J \begin{pmatrix} 2a \\ 2b \end{pmatrix} = \sum_{r_1 \dots r_q} \prod_{i=1}^q \frac{4^{r_i} a_i! b_i!}{(2r_i)! (a_i - r_i)! (b_i - r_i)!} J \begin{pmatrix} 1^{2R} & 2^{A-R} & 0^{B-R} \\ 1^{2R} & 0^{A-R} & 2^{B-R} \end{pmatrix}$$

where the sum is over  $r_i = 0, 1, \dots, \min(a_i, b_i)$ , and  $A = \sum a_i, B = \sum b_i, R = \sum r_i$ .

PROOF. Let us apply the Weingarten formula to the integral in the statement:

$$\begin{aligned} I \begin{pmatrix} 2a \\ 2b \end{pmatrix} &= \int_{O_N} v_{11}^{2a_1} \dots v_{1q}^{2a_q} v_{21}^{2b_1} \dots v_{2q}^{2b_q} dv \\ &= \sum_{\pi \nu} \delta_\pi(1^{2A} 2^{2B}) \delta_\nu(1^{2a_1} \dots q^{2a_q} 1^{2b_1} \dots q^{2b_q}) W_{kN}(\pi, \nu) \\ &= \sum_{\nu} \delta_\nu(1^{2a_1} \dots q^{2a_q} 1^{2b_1} \dots q^{2b_q}) \sum_{\pi} \delta_\pi(1^{2A} 2^{2B}) W_{kN}(\pi, \nu) \end{aligned}$$

Now let us look at  $\nu$ . In order for  $\delta_\nu$  not to vanish,  $\nu$  must connect between themselves the  $2a_1 + 2b_1$  copies of 1, the  $2a_2 + 2b_2$  copies of 2, and so on, up to the  $2a_q + 2b_q$  copies of  $q$ . So, for any  $i \in \{1, \dots, q\}$ , let us denote by  $2r_i \in \{0, 2, \dots, \min(2a_i, 2b_i)\}$  the number of “type  $a$ ” copies of  $i$  coupled with “type  $b$ ” copies of  $i$ . Our claim is that when these parameters  $r_1, \dots, r_q$  are fixed, the sum on the right does not depend on  $\sigma$ , and provides us with a decomposition of the following type:

$$I \begin{pmatrix} 2a \\ 2b \end{pmatrix} = \sum_{r_1 \dots r_q} N_r(a, b) I_r(a, b)$$

Indeed, let us label  $\nu_1, \dots, \nu_k$  the strings of  $\nu$ , and consider the multi-index  $j \in \{1, \dots, k\}^{2k}$  given by  $j_r \in \nu_r$ , for any  $r \in \{1, \dots, k\}$ . We have  $\delta_{\nu'}(j) = \delta_{\nu\nu'}$  for any

pairing  $\nu'$ , so by applying once again the Weingarten formula we obtain:

$$\begin{aligned} \int_{O_N} v_{1j_1} \cdots v_{1j_{2A}} v_{2j_{2A+1}} \cdots v_{2j_{2A+2B}} dv &= \sum_{\pi \nu'} \delta_{\pi}(1^{2A} 2^{2B}) \delta_{\nu'}(j) W_{kN}(\pi, \nu') \\ &= \sum_{\pi \nu'} \delta_{\pi}(1^{2A} 2^{2B}) \delta_{\nu \nu'} W_{kN}(\pi, \nu') \\ &= \sum_{\pi} \delta_{\pi}(1^{2A} 2^{2B}) W_{kN}(\pi, \nu) \end{aligned}$$

Now let us look at the integral on the left. This can be written in the form  $I(m)$ , for a certain matrix  $m$ , the procedure being simply to group together, by using exponents, the identical terms in the product of  $u_{ij}$ . Now by getting back to the definition of the multi-index  $j$ , we conclude that this procedure leads to the following formula:

$$\int_{O_N} v_{1j_1} \cdots v_{1j_{2A}} v_{2j_{2A+1}} \cdots v_{2j_{2A+2B}} dv = I \begin{pmatrix} 1^{2R} & 2^{A-R} & 0^{B-R} \\ 1^{2R} & 0^{A-R} & 2^{B-R} \end{pmatrix}$$

Thus  $I_r(a, b)$  is the integral in the statement. That is, we have proved the following formula, where  $N_r(a, b)$  is the number of pairings  $\nu$  as those considered above:

$$I \begin{pmatrix} 2a \\ 2b \end{pmatrix} = \sum_{r_1 \dots r_q} N_r(a, b) I \begin{pmatrix} 1^{2R} & 2^{A-R} & 0^{B-R} \\ 1^{2R} & 0^{A-R} & 2^{B-R} \end{pmatrix}$$

Let us compute now  $N_r(a, b)$ . The pairings  $\nu$  as above are obtained as follows: (1) pick  $2r_i$  elements among  $2a_i$  elements, (2) pick  $2r_i$  elements among  $2b_i$  elements, (3) couple the “type  $a$ ”  $2r_i$  elements to the “type  $b$ ”  $2r_i$  elements, (4) couple the remaining  $2a_i - 2r_i$  elements, (5) couple the remaining  $2b_i - 2r_i$  elements. Thus we have:

$$\begin{aligned} N_r(a, b) &= \prod_{i=1}^q \binom{2a_i}{2r_i} \binom{2b_i}{2r_i} (2r_i)! (2a_i - 2r_i)! (2b_i - 2r_i)! \\ &= \prod_{i=1}^q \frac{(2a_i)! (2b_i)! (2r_i)! (2a_i - 2r_i)! (2b_i - 2r_i)!}{(2r_i)! (2a_i - 2r_i)! (2r_i)! (2b_i - 2r_i)!} \\ &= \prod_{i=1}^q \frac{(2a_i)! (2b_i)!}{(2r_i)! (2a_i - 2r_i + 1)! (2b_i - 2r_i + 1)!} \end{aligned}$$

Summing up, we have proved the following formula:

$$I \begin{pmatrix} 2a \\ 2b \end{pmatrix} = \sum_{r_1 \dots r_q} \prod_{i=1}^q \frac{(2a_i)! (2b_i)!}{(2r_i)! (2a_i - 2r_i + 1)! (2b_i - 2r_i + 1)!} I \begin{pmatrix} 1^{2R} & 2^{A-R} & 0^{B-R} \\ 1^{2R} & 0^{A-R} & 2^{B-R} \end{pmatrix}$$

By applying now Proposition 4.27 twice, we obtain:

$$J \begin{pmatrix} 2a \\ 2b \end{pmatrix} = \frac{(2A + N - 2)!!(2B + N - 2)!!}{(N - 2)!!(N - 2)!! \prod (2a_i)!! \prod (2b_i)!!} I \begin{pmatrix} 2a \\ 2b \end{pmatrix}$$

$$J \begin{pmatrix} 1^{2R} & 2^{A-R} & 0^{B-R} \\ 1^{2R} & 0^{A-R} & 2^{B-R} \end{pmatrix} = \frac{(2A + N - 2)!!(2B + N - 2)!!}{(N - 2)!!(N - 2)!!} I \begin{pmatrix} 1^{2R} & 2^{A-R} & 0^{B-R} \\ 1^{2R} & 0^{A-R} & 2^{B-R} \end{pmatrix}$$

Thus when passing to  $J$  quantities, the only thing that happens is that the numeric coefficient gets divided by  $\prod (2a_i)!! \prod (2b_i)!!$ . So, this coefficient becomes:

$$\begin{aligned} N'_r(a, b) &= \prod_{i=1}^q \frac{1}{(2a_i)!!(2b_i)!!} \prod_{i=1}^q \frac{(2a_i)!(2b_i)!}{(2r_i)!(2a_i - 2r_i + 1)!!(2b_i - 2r_i + 1)!!} \\ &= \prod_{i=1}^q \frac{(2a_i + 1)!!(2b_i + 1)!!}{(2r_i)!(2a_i - 2r_i + 1)!!(2b_i - 2r_i + 1)!!} \\ &= \prod_{i=1}^q \frac{4^{r_i} a_i! b_i!}{(2r_i)!(a_i - r_i)!(b_i - r_i)!} \end{aligned}$$

Thus we have obtained the formula in the statement, and we are done.  $\square$

As a first consequence, we have the following result, also from [14]:

**THEOREM 4.29.** *We have the “compression formula”*

$$J \begin{pmatrix} a & c \\ b & 0 \end{pmatrix} = J \begin{pmatrix} a & \Sigma c_i \\ b & 0 \end{pmatrix}$$

valid for any vectors with even entries  $a, b \in \mathbb{N}^p$  and  $c \in \mathbb{N}^q$ .

**PROOF.** It is convenient to replace  $a, b, c$  with their doubles  $2a, 2b, 2c$ . Consider now the elementary expansion formula for the matrix in the statement:

$$J \begin{pmatrix} 2a & 2c \\ 2b & 0 \end{pmatrix} = \sum_{r_1 \dots r_q} \prod_{i=1}^q \frac{4^{r_i} a_i! b_i!}{(2r_i)!(a_i - r_i)!(b_i - r_i)!} J \begin{pmatrix} 1^{2R} & 2^{A+C-R} & 0^{B-R} \\ 1^{2R} & 0^{A+C-R} & 2^{B-R} \end{pmatrix}$$

Since the numeric coefficient does not depend on  $c$ , and the function on the right depends only on  $C = \Sigma c_i$ , this gives the formula in the statement.  $\square$

We explore now a problematics which is somehow opposite to the “compression principle”: what happens when “extending” the original matrix  $\begin{pmatrix} a \\ b \end{pmatrix}$  with a  $\begin{pmatrix} c \\ 0 \end{pmatrix}$  component? Let us begin with a basic result, also from [14], as follows:

**PROPOSITION 4.30.** *We have the “basic extension” formula*

$$J \begin{pmatrix} a & 2 \\ b & 0 \end{pmatrix} = \frac{1}{N - q} \left( (\Sigma a_i + N - 1) J \begin{pmatrix} a \\ b \end{pmatrix} - \sum_{s=1}^q (a_s + 1) J \begin{pmatrix} a^{(s)} \\ b \end{pmatrix} \right)$$

for any  $a, b \in (2\mathbb{N})^q$ , where  $a^{(s)} = (a_1, \dots, a_{s-1}, a_s + 2, a_{s+1}, \dots, a_q)$ .

PROOF. By using the trivial identity  $\Sigma v_{1i}^2 = 1$ , we obtain the following formula:

$$I \begin{pmatrix} a \\ b \end{pmatrix} = \sum_{s=1}^q I \begin{pmatrix} a^{(s)} \\ b \end{pmatrix} + (N - q) I \begin{pmatrix} a & 2 \\ b & 0 \end{pmatrix}$$

On the other hand, according to Proposition 4.27, we have:

$$\begin{aligned} J \begin{pmatrix} a \\ b \end{pmatrix} &= \frac{(\Sigma a_i + N - 2)!! (\Sigma b_i + N - 2)!!}{(N - 2)!! (N - 2)!! \prod a_i!! \prod b_i!!} I \begin{pmatrix} a \\ b \end{pmatrix} \\ J \begin{pmatrix} a^{(s)} \\ b \end{pmatrix} &= \frac{(\Sigma a_i + N)!! (\Sigma b_i + N - 2)!!}{(N - 2)!! (N - 2)!! \prod a_i!! \prod b_i!! (a_s + 1)} I \begin{pmatrix} a^{(s)} \\ b \end{pmatrix} \\ J \begin{pmatrix} a & 2 \\ b & 0 \end{pmatrix} &= \frac{(\Sigma a_i + N)!! (\Sigma b_i + N - 2)!!}{(N - 2)!! (N - 2)!! \prod a_i!! \prod b_i!!} I \begin{pmatrix} a & 2 \\ b & 0 \end{pmatrix} \end{aligned}$$

Thus our above formula translates as follows:

$$(\Sigma a_i + N - 1) J \begin{pmatrix} a \\ b \end{pmatrix} = \sum_{s=1}^q (a_s + 1) J \begin{pmatrix} a^{(s)} \\ b \end{pmatrix} + (N - q) J \begin{pmatrix} a & 2 \\ b & 0 \end{pmatrix}$$

But this gives the formula in the statement.  $\square$

We have as well a recursive version of the above result, as follows:

PROPOSITION 4.31. *We have the “recursive extension” formula*

$$J \begin{pmatrix} a & c + 2 \\ b & 0 \end{pmatrix} = \frac{1}{N + c - q} \left( (\Sigma a_i + c + N - 1) J \begin{pmatrix} a & c \\ b & 0 \end{pmatrix} - \sum_{s=1}^q (a_s + 1) J \begin{pmatrix} a^{(s)} & c \\ b & 0 \end{pmatrix} \right)$$

valid for any two vectors  $a, b \in (2\mathbb{N})^q$ , and any  $c \in 2\mathbb{N}$ .

PROOF. We use the compression formula. This gives:

$$J \begin{pmatrix} a & c + 2 \\ b & 0 \end{pmatrix} = J \begin{pmatrix} a & c & 2 \\ b & 0 & 0 \end{pmatrix}$$

Now if we denote the quantity on the left by  $K$ , and we apply to the quantity on the right the basic extension formula, we obtain:

$$K = \frac{1}{n - q - 1} \left( (\Sigma a_i + c + N - 1) J \begin{pmatrix} a & c \\ b & 0 \end{pmatrix} - \sum_{s=1}^q (a_s + 1) J \begin{pmatrix} a^{(s)} & c \\ b & 0 \end{pmatrix} - (c + 1) K \right)$$

But this gives the formula of  $K$  in the statement.  $\square$

As a first consequence of our results, we can establish now a number of concrete formulae. The first such formula computes all the joint moments of  $v_{11}, v_{12}, v_{21}$ :

THEOREM 4.32. *We have the “triangular formula”*

$$J \begin{pmatrix} a & c \\ b & 0 \end{pmatrix} = \frac{(N-1)!!}{(N-2)!!} \cdot \frac{(a+c+N-2)!!(b+c+N-2)!!}{(c+N-2)!!(a+b+c+N-1)!!}$$

valid for any  $a, b, c \in 2\mathbb{N}$ .

PROOF. We prove this by recurrence over  $c \in 2\mathbb{N}$ . At  $c = 0$  this follows from the 1-row formula, so assume that this is true at  $c$ . By using Proposition 4.31, we get:

$$J \begin{pmatrix} a & c+2 \\ b & 0 \end{pmatrix} = \frac{1}{N+c-1} \left( (a+c+N-1)J \begin{pmatrix} a & c \\ b & 0 \end{pmatrix} - (a+1)J \begin{pmatrix} a+2 & c \\ b & 0 \end{pmatrix} \right)$$

Let us call  $L - R$  the above expression. According to the recurrence, we have:

$$L = \frac{(N-1)!!}{(N-2)!!} \cdot \frac{(a+c+N)!!(b+c+N-2)!!}{(c+N)!!(a+b+c+N-1)!!}$$

$$R = (a+1) \frac{(N-1)!!}{(N-2)!!} \cdot \frac{(a+c+N)!!(b+c+N-2)!!}{(c+N)!!(a+b+c+N+1)!!}$$

Thus we obtain the following formula:

$$\begin{aligned} & J \begin{pmatrix} a & c+2 \\ b & 0 \end{pmatrix} \\ &= \frac{(N-1)!!}{(N-2)!!} \cdot \frac{(a+c+N)!!(b+c+N-2)!!}{(c+N)!!(a+b+c+N+1)!!} ((a+b+c+N) - (a+1)) \\ &= \frac{(N-1)!!}{(N-2)!!} \cdot \frac{(a+c+N)!!(b+c+N-2)!!}{(c+N)!!(a+b+c+N+1)!!} (b+c+N-1) \\ &= \frac{(N-1)!!}{(N-2)!!} \cdot \frac{(a+c+N)!!(b+c+N)!!}{(c+N)!!(a+b+c+N+1)!!} \end{aligned}$$

Thus the formula to be proved is true at  $c+2$ , and we are done.  $\square$

As a first observation, by combining the above formula with the compression formula we obtain the following result, also from [14], fully generalizing Proposition 4.6:

PROPOSITION 4.33. *We have the formula*

$$J \begin{pmatrix} a & c_1 & \dots & c_q \\ b & 0 & \dots & 0 \end{pmatrix} = \frac{(N-1)!!}{(N-2)!!} \cdot \frac{(a+\Sigma c_i+N-2)!!(b+\Sigma c_i+N-2)!!}{(\Sigma c_i+N-2)!!(a+b+\Sigma c_i+N-1)!!}$$

valid for any even numbers  $a, b$  and  $c_1, \dots, c_q$ .

PROOF. This follows indeed from Theorem 4.32 and from the compression principle. Observe that with  $b = 0$  we recover indeed the formula in Proposition 4.6.  $\square$

As a second observation, at  $a = 0$  the triangular formula computes all the joint moments of  $v_{12}, v_{21}$ , and we obtain an interesting formula here, as follows:

PROPOSITION 4.34. *The joint moments of 2 orthogonal group coordinates  $x, y \in \{u_{ij}\}$ , chosen in generic position, not on the same row or column, are given by*

$$\int_{O_N} x^\alpha y^\beta dv = \frac{(N-2)!\alpha!!\beta!!(\alpha+\beta+N-2)!!}{(\alpha+N-2)!!(\beta+N-2)!!(\alpha+\beta+N-1)!!}$$

for  $\alpha, \beta$  even, and vanish if one of  $\alpha, \beta$  is odd.

PROOF. By symmetry we may assume that our coordinates are  $x = v_{12}$  and  $y = v_{21}$ , and the result follows from Theorem 4.32, with  $a = 0$ ,  $c = \alpha$ ,  $b = \beta$ .  $\square$

Moving ahead, we would like to understand what happens to  $J\binom{a}{b}$  when flipping a column of  $\binom{a}{b}$ . Let us begin with the case of the elementary matrices:

PROPOSITION 4.35. *We have the formula*

$$J\begin{pmatrix} 2^a & 0^b \\ 0^a & 2^b \end{pmatrix} = \frac{(N-1)!!}{(N-2)!!} \cdot \frac{(2a+2b+N-2)!!}{(2a+2b+N-1)!!}$$

valid for any  $a, b \in \mathbb{N}$ .

PROOF. Indeed, by using the compression principle, we obtain:

$$J\begin{pmatrix} 2^a & 0^b \\ 0^a & 2^b \end{pmatrix} = J\begin{pmatrix} 2a & 0 \\ 0 & 2b \end{pmatrix} = J\begin{pmatrix} 0 & 2a \\ 2b & 0 \end{pmatrix}$$

On the other hand, by applying the triangular formula, we obtain:

$$J\begin{pmatrix} 0 & 2a \\ 2b & 0 \end{pmatrix} = \frac{(N-1)!!}{(N-2)!!} \cdot \frac{(2a+N-2)!!(2a+2b+N-2)!!}{(2a+N-2)!!(2a+2b+N-1)!!}$$

By simplifying the fraction, we obtain the formula in the statement.  $\square$

We have the following generalization of the above result, still following [14]:

PROPOSITION 4.36. *We have the “elementary flipping” formula*

$$J\begin{pmatrix} 1^{2s} & 2^a & 0^b \\ 1^{2s} & 0^a & 2^b \end{pmatrix} = J\begin{pmatrix} 1^{2s} & 2^c & 0^d \\ 1^{2s} & 0^c & 2^d \end{pmatrix}$$

valid for any  $s \in \mathbb{N}$  and any  $a, b, c, d \in \mathbb{N}$  satisfying  $a + b = c + d$ .

PROOF. We prove this result by recurrence over  $s$ . At  $s = 0$  this follows from Proposition 4.35, because the right term there depends only on  $a + b$ . So, assume that the result is true at  $s \in \mathbb{N}$ . We use the following equality, coming from the triangular formula:

$$J\begin{pmatrix} 2a & 2c \\ 2b & 0 \end{pmatrix} = J\begin{pmatrix} 2a & 0 \\ 2b & 2c \end{pmatrix}$$

Assume  $a \geq b$  and consider the elementary expansion of the above two quantities, where  $K_r(a, b)$  denotes the coefficient appearing in the elementary expansion formula:

$$J \begin{pmatrix} 2a & 2c \\ 2b & 0 \end{pmatrix} = \sum_{r=0}^b K_r(a, b) J \begin{pmatrix} 1^{2r} & 2^{a+c-r} & 0^{b-r} \\ 1^{2r} & 0^{a+c-r} & 2^{b-r} \end{pmatrix}$$

$$J \begin{pmatrix} 2a & 0 \\ 2b & 2c \end{pmatrix} = \sum_{r=0}^b K_r(a, b) J \begin{pmatrix} 1^{2r} & 2^{a-r} & 0^{b+c-r} \\ 1^{2r} & 0^{a-r} & 2^{b+c-r} \end{pmatrix}$$

We know that the sums on the right are equal, for any  $a, b, c$  with  $a \geq b$ . With the choice  $b = s$ , this equality becomes:

$$\sum_{r=0}^s K_r(a, s) J \begin{pmatrix} 1^{2r} & 2^{a+c-r} & 0^{s-r} \\ 1^{2r} & 0^{a+c-r} & 2^{s-r} \end{pmatrix} = \sum_{r=0}^s K_r(a, s) J \begin{pmatrix} 1^{2r} & 2^{a-r} & 0^{s+c-r} \\ 1^{2r} & 0^{a-r} & 2^{s+c-r} \end{pmatrix}$$

Now by the induction assumption, the first  $r$  terms of the above two sums coincide. So, the above equality tells us that the last terms ( $r = s$ ) of the two sums are equal:

$$J \begin{pmatrix} 1^{2s} & 2^{a+c-s} \\ 1^{2s} & 0^{a+c-s} \end{pmatrix} = J \begin{pmatrix} 1^{2s} & 2^{a-s} & 0^c \\ 1^{2s} & 0^{a-s} & 2^c \end{pmatrix}$$

Since this equality holds for any  $a \geq s$  and any  $c$ , this shows that the elementary flipping formula holds at  $s$ , and we are done.  $\square$

We can now formulate a main result, from [14], as follows:

**THEOREM 4.37.** *We have the “flipping formula”*

$$J \begin{pmatrix} a & c \\ b & d \end{pmatrix} = J \begin{pmatrix} a & d \\ b & c \end{pmatrix}$$

valid for any vectors  $a, b \in \mathbb{N}^p$  and  $c, d \in \mathbb{N}^q$ .

**PROOF.** Consider the elementary expansion of the two quantities in the statement, where  $K_r(a, b)$  are the coefficients appearing in the elementary expansion formula:

$$J \begin{pmatrix} 2a & 2c \\ 2b & 2d \end{pmatrix} = \sum_{r_i s_j} \prod_{ij} K_{r_i}(a_i, b_i) K_{s_j}(c_j, d_j) J \begin{pmatrix} 1^{2R+2S} & 2^{A+C-R-S} & 0^{B+D-R-S} \\ 1^{2R+2S} & 0^{A+C-R-S} & 2^{B+D-R-S} \end{pmatrix}$$

$$J \begin{pmatrix} 2a & 2d \\ 2b & 2c \end{pmatrix} = \sum_{r_i s_j} \prod_{ij} K_{r_i}(a_i, b_i) K_{s_j}(d_j, c_j) J \begin{pmatrix} 1^{2R+2S} & 2^{A+D-R-S} & 0^{B+C-R-S} \\ 1^{2R+2S} & 0^{A+D-R-S} & 2^{B+C-R-S} \end{pmatrix}$$

Our claim is that two formulae are in fact identical. Indeed, the first remark is that the various indices vary in the same sets. Also, since the function  $K_r(a, b)$  is symmetric in  $a, b$ , the numeric coefficients are the same. As for the  $J$  terms on the left, these are equal as well, due to elementary flipping formula, so we are done.  $\square$

As an application of the above, we will work out now a concrete formula for the arbitrary two-row integrals. We already know that these integrals are subject to an “elementary expansion” formula, so what is left to do is to compute the values of the elementary integrals. These values are given by the following technical result:

PROPOSITION 4.38. *For any  $a, b, r$  we have:*

$$J \begin{pmatrix} 1^{2r} & 2^a & 0^b \\ 1^{2r} & 0^a & 2^b \end{pmatrix} = (-1)^r \frac{(N-1)!!}{(N-2)!!} \cdot \frac{(2r)!!(2a+2b+2r+N-2)!!}{(2a+2b+4r+N-1)!!}$$

PROOF. As a first observation, at  $r = 0$  the result follows from Proposition 4.36. In general now, consider the elementary expansion formula, with  $a, b \in \mathbb{N}$ ,  $a \geq b$ :

$$J \begin{pmatrix} 2a \\ 2b \end{pmatrix} = \sum_{r=0}^b \frac{4^r a! b!}{(2r)!(a-r)!(b-r)!} J \begin{pmatrix} 1^{2r} & 2^{a-r} & 0^{b-r} \\ 1^{2r} & 0^{a-r} & 2^{b-r} \end{pmatrix}$$

By using the “flipping principle”, this formula becomes:

$$J \begin{pmatrix} 2a \\ 2b \end{pmatrix} = \sum_{r=0}^b \frac{4^r a! b!}{(2r)!(a-r)!(b-r)!} J \begin{pmatrix} 1^{2r} & 2^{a+b-2r} \\ 1^{2r} & 0^{a+b-2r} \end{pmatrix}$$

The point is that the quantity on the left is known, and this allows the computation of the integrals on the right. More precisely, let us introduce the following function:

$$\psi_r(a) = J \begin{pmatrix} 1^{2r} & 2^a \\ 1^{2r} & 0^a \end{pmatrix}$$

Then the above equality translates into the following equation:

$$J \begin{pmatrix} 2a \\ 2b \end{pmatrix} = \sum_{r=0}^b \frac{4^r a! b!}{(2r)!(a-r)!(b-r)!} \psi_r(a+b-2r)$$

According to Proposition 4.6 and Proposition 4.37, the values on the left are given by:

$$J \begin{pmatrix} 2a \\ 2b \end{pmatrix} = \frac{(N-1)!!(2a+N-2)!!(2b+N-2)!!}{(N-2)!!(N-2)!!(2a+2b+N-1)!!}$$

Now by taking  $b = 0, 1, 2, \dots$ , the above equations will succesively produce the values of  $\psi_r(a)$  for  $r = 0, 1, 2, \dots$ , so we have here an algorithm for computing these values. On the other hand, a direct computation based on standard summation formulae shows that our system is solved by the values of  $\psi_r(a)$  given in the statement, namely:

$$\psi_r(a) = (-1)^r \frac{(N-1)!!}{(N-2)!!} \cdot \frac{(2r)!!(2a+2r+N-2)!!}{(2a+4r+N-1)!!}$$

Now by using one more time the flipping principle, the knowledge of the quantities  $\psi_r(a)$  fully recovers the general formula in the statement, and we are done.  $\square$

We are now in position of stating and proving a main result, from [14], as follows:

THEOREM 4.39. *The 2-row integrals over  $O_N$  are given by the formula*

$$J \begin{pmatrix} 2a \\ 2b \end{pmatrix} = \frac{(N-1)!!}{(N-2)!!} \sum_{r_1, \dots, r_q} (-1)^R \prod_{i=1}^q \frac{4^{r_i} a_i! b_i!}{(2r_i)! (a_i - r_i)! (b_i - r_i)!} \cdot \frac{(2R)!! (2S - 2R + N - 2)!!}{(2S + N - 1)!!}$$

where the sum is over  $r_i = 0, 1, \dots, \min(a_i, b_i)$ , and  $S = \Sigma a_i + \Sigma b_i$ ,  $R = \Sigma r_i$ .

PROOF. This follows from the elementary expansion formula, by plugging in the explicit values for the elementary integrals, that we found in Proposition 4.38.  $\square$

For more complicated integrals, involving 3 rows or coordinates or more, the situation is quite complex, and we refer here to [13], [14], [31], [35], [100] and related papers.

#### 4e. Exercises

There has been a lot of non-trivial theory in this chapter, sometimes erring on the research side, and our exercises here will be not simple either. First, however, we have:

EXERCISE 4.40. *Compute the Gram and Weingarten matrices for the various easy quantum groups that you know, at small values of  $k \in \mathbb{N}$ .*

This is something very instructive, and there are countless computations that you can do here, either by hand or by using a computer, and the more you do, the better it is. Ideally, spend a few days and nights on all this, until you reach the black belt.

EXERCISE 4.41. *Look up the full theory for the symplectic group  $Sp_N \subset U_N$ , namely Brauer theorem, super-easiness, Weingarten formula, and perhaps some numerics for the Weingarten matrices too, and write down a brief account of what you found.*

Here the main reference for theory is the paper by Collins-Śniady [35], but you will still have to adapt the material there, which is quite advanced, as to fit with what you learned from here, as for everything to be complete. As for the numerics, do some by yourself, as before, and for more on all this, you can look up the random matrix literature as well, where the Weingarten formula, in particular for  $Sp_N$ , is heavily used.

## Part II

# Random matrices

*Life is a mystery  
Everyone must stand alone  
I hear you call my name  
And it feels like home*

## CHAPTER 5

### Linear algebra

#### 5a. Random matrices

We have seen so far some interesting probability theory, dealing with usual random variables, which are by definition functions as follows, real or complex:

$$f \in L^\infty(X)$$

The main problem regarding such variables was the computation of the moments, and of the distribution, which is the probability measure  $\mu_f \in \mathcal{P}(\mathbb{C})$  given by:

$$\mathbb{E}(f^k) = \int_{\mathbb{C}} x^k d\mu_f(x)$$

We have seen in particular that interesting phenomena (CLT, PLT..) appear in the  $N \rightarrow \infty$  limit, when averaging i.i.d. variables, and looking at the distribution.

We discuss in what follows more advanced aspects of probability theory, which are of rather “noncommutative” nature, in relation with the random matrices. This will be an introduction to the subject, with more advanced books including the classical book of Mehta [65], the modern reference book of Anderson, Guionnet and Zeitouni [1], the equally modern book by Mingo and Speicher [68], and many more.

The random matrices are simple and fundamental mathematical objects, virtually appearing in all areas of mathematics and physics. They are defined as follows:

DEFINITION 5.1. *A random matrix is a square matrix of type*

$$Z \in M_N(L^\infty(X))$$

*with  $X$  being a probability space, and  $N \in \mathbb{N}$  being an integer.*

As basic examples, we have the usual matrices  $Z \in M_N(\mathbb{C})$ , obtained by taking  $X = \{\cdot\}$ . Also, we have the usual random variables  $Z \in L^\infty(X)$ , obtained by taking  $N = 1$ . In general, what we have is a kind of combination of these 2 situations.

As will we see, the subject is extremely interesting, and several beautiful results, such as the Wigner [97] and Marchenko-Pastur [64] convergence theorems, regarding the self-adjoint Gaussian and complex Wishart random matrices, can be established with the technology that we have, namely the moment method, and regular combinatorics.

Let us begin by specifying the precise classes of matrices that we are interested in. First we have the complex Gaussian matrices, which are constructed as follows:

DEFINITION 5.2. *A complex Gaussian matrix is a random matrix of type*

$$Z \in M_N(L^\infty(X))$$

*which has i.i.d. complex normal entries.*

We will be interested as well in the Wigner random matrices, which are the self-adjoint versions of these matrices. These are constructed as follows:

DEFINITION 5.3. *A Wigner matrix is a random matrix of type*

$$Z \in M_N(L^\infty(X))$$

*which has i.i.d. complex normal entries, up to the constraint  $Z = Z^*$ .*

In other words, a Wigner matrix must be as follows, with  $r_i$  being real normal variables,  $s_{ij}$  being complex normal variables, and all these variables being independent:

$$Z = \begin{pmatrix} r_1 & s_{12} & \dots & \dots & s_{1N} \\ \bar{s}_{12} & r_2 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & r_{N-1} & s_{N-1,N} \\ \bar{s}_{1N} & \dots & \dots & \bar{s}_{N-1,N} & r_N \end{pmatrix}$$

Finally, we will be interested as well in the complex Wishart matrices, which are the positive versions of these matrices, which are constructed as follows:

DEFINITION 5.4. *A complex Wishart matrix is a random matrix of type*

$$Z = YY^* \in M_N(L^\infty(X))$$

*with  $Y$  being a complex Gaussian matrix.*

As a first task, we must understand what the distribution of a random matrix is. This is something non-trivial, which will take some time. Once done all this, in this chapter, and in the next chapter too, we will come back to the Gaussian, Wigner and Wishart matrices, and other explicit random matrices, with results about them.

## 5b. Scalar matrices

Let us begin with a discussion concerning the usual matrices  $A \in M_N(\mathbb{C})$ . These are not exactly random variables in the usual sense, but we can still talk about their moments and their distributions, by being a bit abstract. Let us begin with:

DEFINITION 5.5. *The moments of a usual complex matrix  $A \in M_N(\mathbb{C})$  are the following numbers, indexed by the integers  $k \in \mathbb{N}$ ,*

$$M_k = \text{tr}(A^k)$$

with  $\text{tr} = N^{-1} \cdot \text{Tr}$  being the normalized matrix trace, satisfying  $\text{tr}(1) = 1$ .

As a basic example here, consider the case of a diagonal matrix:

$$A = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix}$$

The powers of  $A$ , with respect to integer exponents  $k \in \mathbb{N}$ , are as follows:

$$A^k = \begin{pmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_N^k \end{pmatrix}$$

Thus the moments in the sense of Definition 5.5 are given by the following formula:

$$M_k = \sum_i \lambda_i^k$$

Regarding now the distribution, things are a bit more tricky here. In view of some further generalizations, let us formulate things here as follows:

DEFINITION 5.6. *The distribution, or law, of a usual complex matrix  $A \in M_N(\mathbb{C})$  is the following abstract functional  $\mu_A : \mathbb{C}[X] \rightarrow \mathbb{C}$ , with  $\text{tr} = N^{-1} \cdot \text{Tr}$ :*

$$P \rightarrow \text{tr}(P(A))$$

In the case where we have a probability measure  $\mu_A \in \mathcal{P}(\mathbb{C})$  such that

$$\text{tr}(P(A)) = \int_{\mathbb{C}} P(x) d\mu_A(x)$$

we identify this complex measure with the distribution of  $A$ .

Observe that, by linearity, the distribution is determined by the moments. Indeed, if we write our polynomial as  $P = \sum_k c_k X^k$ , then we have the following formula:

$$\begin{aligned} \text{tr}(P(A)) &= \text{tr} \left( \sum_k c_k A^k \right) \\ &= \sum_k c_k M_k \end{aligned}$$

In fact, knowing the distribution is the same as knowing the moments.

Once again, for illustrating this notion, consider the case of a diagonal matrix:

$$A = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix}$$

We have then the following formula, valid for any polynomial  $P \in \mathbb{C}[X]$ :

$$P(A) = \begin{pmatrix} P(\lambda_1) & & \\ & \ddots & \\ & & P(\lambda_N) \end{pmatrix}$$

Now by applying the normalized trace, we obtain from this:

$$\begin{aligned} \text{tr}(P(A)) &= \frac{1}{N}(P(\lambda_1) + \dots + P(\lambda_N)) \\ &= \frac{1}{N} \int_{\mathbb{C}} P(x) d(\delta_{\lambda_1} + \dots + \delta_{\lambda_N})(x) \\ &= \int_{\mathbb{C}} P(x) d\left(\frac{1}{N}(\delta_{\lambda_1} + \dots + \delta_{\lambda_N})\right)(x) \end{aligned}$$

Thus, according to Definition 5.6, the law of  $A$  is the following measure:

$$\mu_A = \frac{1}{N}(\delta_{\lambda_1} + \dots + \delta_{\lambda_N})$$

Quite remarkably, the distribution always exists as a probability measure on  $\mathbb{C}$ , and is given by the above formula, as the average of the eigenvalues. However, this is not trivial, and for proving this we will need some density results, as follows:

**THEOREM 5.7.** *The following happen, inside  $M_N(\mathbb{C})$ :*

- (1) *The invertible matrices are dense.*
- (2) *The matrices having distinct eigenvalues are dense.*
- (3) *The diagonalizable matrices are dense.*

**PROOF.** These are quite advanced linear algebra results, which can be proved with the technology that we have so far, the idea being as follows:

(1) The first assertion is clear, because the invertible matrices are given by the condition  $\det A \neq 0$ . Thus, the set formed by the invertible matrices appears as the complement of the surface  $\det A = 0$ , and so must be dense inside  $M_N(\mathbb{C})$ , as claimed.

(2) For the second assertion we can use a similar argument. Let us first recall that given two polynomials  $P, Q \in \mathbb{C}[X]$ , we can construct their resultant  $R(P, Q)$ , which is a polynomial in the coefficients of  $P, Q$ , with integer coefficients, having the property that  $P, Q$  have a common root precisely when this resultant vanishes:

$$R(P, Q) = 0$$

Indeed, let us write our polynomials  $P, Q$  as follows:

$$P = c(X - p_1) \dots (X - p_k)$$

$$Q = d(X - q_1) \dots (X - q_l)$$

We can define then  $R(P, Q)$  as follows, with the fact that this is indeed a polynomial in the coefficients of  $P, Q$ , with integer coefficients, coming via standard algebra:

$$R(P, Q) = c^l d^k \prod_{ij} (p_i - q_j)$$

(3) All this is a bit abstract, and as an alternative definition, which is quite often more convenient in practice, let us write our polynomials  $P, Q$  as follows:

$$P = a_k X^k + \dots + a_1 X + a_0$$

$$Q = b_l X^l + \dots + b_1 X + b_0$$

Their resultant  $R(P, Q)$  appears then as a determinant, as follows, with the matrix having size  $k + l$ , and having 0 coefficients at the blank spaces:

$$R(P, Q) = \begin{vmatrix} a_k & & & b_l & & \\ & \ddots & & \vdots & \ddots & \\ a_0 & & a_k & b_0 & & b_l \\ & & & \vdots & & \vdots \\ & & & a_0 & & b_0 \end{vmatrix}$$

(4) In any case, with either of the above constructions, we have now a definition for the resultant  $R(P, Q)$ , which vanishes precisely when  $P, Q$  have a common root. But this allows us to prove the second assertion in the statement, simply by saying that the matrices  $A$  having distinct eigenvalues appear as the complement of the surface  $R(P_A, P'_A) = 0$ , with  $P_A$  being the characteristic polynomial, and so they are dense.

(5) Even more elegantly, given a polynomial  $P(X) = cX^N + dX^{N-1} + \dots$ , we can define its discriminant as being the following quantity, with the normalization factor being there as for  $P = aX^2 + bX + c$  to have discriminant  $\Delta(P) = b^2 - 4ac$ :

$$\Delta(P) = \frac{(-1)^{\binom{N}{2}}}{c} R(P, P')$$

This discriminant is then a polynomial in the coefficients of  $P$ , with integer coefficients, and  $P$  has a double root precisely when  $\Delta(P) = 0$ . Thus, back now to our question, the matrices  $A$  having distinct eigenvalues appears as the complement of the surface given by  $\Delta(P_A) = 0$ , and so must be dense inside  $M_N(\mathbb{C})$ , as claimed.

(6) Finally, the last assertion follows from the second one, by using the fact that the matrices having distinct eigenvalues are diagonalizable. There are of course some other proofs as well, for instance by putting the matrix in Jordan form.  $\square$

Now back to our probability questions, we have the following result:

**THEOREM 5.8.** *For any matrix  $A \in M_N(\mathbb{C})$  we have the formula*

$$\text{tr}(P(A)) = \frac{1}{N}(P(\lambda_1) + \dots + P(\lambda_N))$$

where  $\lambda_1, \dots, \lambda_N \in \mathbb{C}$  are the eigenvalues of  $A$ . Thus the complex measure

$$\mu_A = \frac{1}{N}(\delta_{\lambda_1} + \dots + \delta_{\lambda_N})$$

is the distribution of  $A$ , in the abstract sense of Definition 5.6.

**PROOF.** There are several proofs for this fact, and a particularly instructive proof, relying on the above density arguments, which are good to know, is as follows:

(1) Consider first the simplest case, that of a diagonal matrix:

$$A = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix}$$

Here we know from the above discussion that the result holds indeed:

$$\mu_A = \frac{1}{N}(\delta_{\lambda_1} + \dots + \delta_{\lambda_N})$$

(2) More generally now, let us discuss the case where our matrix  $A$  is diagonalizable. Here we must have a formula as follows, with  $D$  being diagonal:

$$A = PDP^{-1}$$

Now observe that the moments of  $A$  are given by the following formula:

$$\begin{aligned} \text{tr}(A^k) &= \text{tr}(PDP^{-1} \cdot PDP^{-1} \dots PDP^{-1}) \\ &= \text{tr}(PD^kP^{-1}) \\ &= \text{tr}(D^k) \end{aligned}$$

We conclude, by linearity, that the matrices  $A, D$  have the same distribution:

$$\mu_A = \mu_D$$

On the other hand,  $A = PDP^{-1}$  shows that  $A, D$  have the same eigenvalues. Thus, if we denote by  $\lambda_1, \dots, \lambda_N \in \mathbb{C}$  these eigenvalues, we obtain, by using (1):

$$\mu_A = \frac{1}{N}(\delta_{\lambda_1} + \dots + \delta_{\lambda_N})$$

(3) In general now, the result follows from what we know about the diagonalizable matrices, from (2), by using the fact that these diagonalizable matrices are dense.  $\square$

Summarizing, we have a nice theory for the matrices  $A \in M_N(\mathbb{C})$ , paralleling that of the random variables  $f \in L^\infty(X)$ . It is tempting at this point to try to go further, and to unify the matrices and the random variables, by talking about random matrices:

$$Z \in M_N(L^\infty(X))$$

However, we will not do this right away, because our matrix theory has a flaw. Indeed, all what has been said above, namely Definitions 5.5 and 5.6 and Theorem 5.8, although being nice and conceptual, does not take into account the adjoint matrix:

$$A^* = (\bar{A}_{ji})$$

Before getting into this, a few words on  $A^*$ . The idea is that the matrices  $A \in M_N(\mathbb{C})$  do not come alone, but rather in pairs  $(A, A^*)$ , and this because no matter what we want to do with  $A$ , of advanced type, we will run at some point into its adjoint  $A^*$ . Here is a basic result of this type, showing that the adjoints  $*$  are indeed ubiquitous:

**PROPOSITION 5.9.** *The following happen:*

- (1)  $\langle Ax, y \rangle = \langle x, A^*y \rangle$ , for any vectors  $x, y \in \mathbb{C}^N$ .
- (2)  $T(x) = Ux$  with  $U \in M_N(\mathbb{C})$  is an isometry precisely when  $U^* = U^{-1}$ .
- (3)  $T(x) = Px$  with  $P \in M_N(\mathbb{C})$  is a projection precisely when  $P = P^2 = P^*$ .

**PROOF.** All this is standard linear algebra, the idea being as follows:

- (1) This is clear indeed from definition of the adjoint matrix,  $A^* = (\bar{A}_{ji})$ .

(2) Let us first recall that the lengths, or norms, of the vectors  $x \in \mathbb{C}^N$  can be recovered from the knowledge of the scalar products, as follows:

$$\|x\| = \sqrt{\langle x, x \rangle}$$

Conversely, we can recover the scalar products out of norms, by using the following formula, called complex polarization identity:

$$\begin{aligned} & \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2 \\ = & \|x\|^2 + \|y\|^2 - \|x\|^2 - \|y\|^2 + i\|x\|^2 + i\|y\|^2 - i\|x\|^2 - i\|y\|^2 \\ & + 2\operatorname{Re}(\langle x, y \rangle) + 2\operatorname{Re}(\langle x, y \rangle) + 2i\operatorname{Im}(\langle x, y \rangle) + 2i\operatorname{Im}(\langle x, y \rangle) \\ = & 4\langle x, y \rangle \end{aligned}$$

Now given a matrix  $U \in M_N(\mathbb{C})$ , we have indeed the following equivalences, with the first one coming from the polarization identity, and the other ones being clear:

$$\begin{aligned} \|Ux\| = \|x\| & \iff \langle Ux, Ux \rangle = \langle x, x \rangle \\ & \iff \langle U^*Ux, x \rangle = \langle x, x \rangle \\ & \iff U^*U = 1 \\ & \iff U^* = U^{-1} \end{aligned}$$

(3) Given a matrix  $P \in M_N(\mathbb{C})$ , in order for  $x \rightarrow Px$  to be an oblique projection, we must have  $P^2 = P$ . Now observe that this projection is orthogonal when:

$$\begin{aligned} \langle Px - x, Py \rangle = 0 &\iff \langle P^*Px - P^*x, y \rangle = 0 \\ &\iff P^*P = P^* \end{aligned}$$

The point now is that by conjugating the last formula, we obtain  $P^*P = P$ . Thus we must have  $P = P^*$ , and this gives the result.  $\square$

Summarizing, as claimed, the matrices  $A \in M_N(\mathbb{C})$  do not come alone, but rather in pairs  $(A, A^*)$ . Now by getting back to our probabilistic questions, we must talk about the moments and distribution of the pair  $(A, A^*)$ . This can be done as follows:

DEFINITION 5.10. *The generalized moments of a complex matrix  $A \in M_N(\mathbb{C})$  are the following numbers, indexed by the colored integers  $k = \circ \bullet \bullet \circ \dots$*

$$M_k = \text{tr}(A^k)$$

with  $A^k$  being defined by the following formulae and multiplicativity,  $A^{kl} = A^k A^l$ ,

$$A^\emptyset = 1 \quad , \quad A^\circ = A \quad , \quad A^\bullet = A^*$$

and with  $\text{tr} = N^{-1} \cdot \text{Tr}$  being as usual the normalized matrix trace.

All this might seem a bit complicated, but there is no other way of dealing with such things. Indeed, since the variables  $A, A^*$  do not commute, unless the matrix is normal,  $AA^* = A^*A$ , which is something special, which does not happen in general, we are led to colored exponents  $k = \circ \bullet \bullet \circ \dots$  and to the above definition for the moments.

Regarding now the distribution, we can use here a similar idea, as follows:

DEFINITION 5.11. *The generalized distribution, or law, of a matrix  $A \in M_N(\mathbb{C})$  is the abstract functional  $\mu_A : \mathbb{C} \langle X, X^* \rangle \rightarrow \mathbb{C}$  given by:*

$$P \rightarrow \text{tr}(P(A))$$

In the case where we have a probability measure  $\mu_A \in \mathcal{P}(\mathbb{C})$  such that

$$\text{tr}(P(A)) = \int_{\mathbb{C}} P(x) d\mu_A(x)$$

we identify this complex measure with the distribution of  $A$ .

As before, knowing the distribution is the same as knowing the moments, because if we write our noncommutative polynomial as  $P = \sum_k c_k X^k$ , then we have:

$$\begin{aligned} \text{tr}(P(A)) &= \text{tr} \left( \sum_k c_k A^k \right) \\ &= \sum_k c_k M_k \end{aligned}$$

Also, the same comments as those after Definition 5.10 apply. To be more precise, since the variables  $A, A^*$  do not commute, unless the matrix is normal,  $AA^* = A^*A$ , which is something special, that does not happen in general, we are led to noncommutative polynomials  $P \in \mathbb{C} \langle X, X^* \rangle$ , and to the above definition for the distribution.

As a first result now, coming from Theorem 5.8, we have:

**THEOREM 5.12.** *Given a matrix  $A \in M_N(\mathbb{C})$  which is self-adjoint,  $A = A^*$ , we have the following formula, valid for any polynomial  $P \in \mathbb{C} \langle X, X^* \rangle$ ,*

$$\text{tr}(P(A)) = \frac{1}{N}(P(\lambda_1) + \dots + P(\lambda_N))$$

where  $\lambda_1, \dots, \lambda_N \in \mathbb{C}$  are the eigenvalues of  $A$ . Thus the complex measure

$$\mu_A = \frac{1}{N}(\delta_{\lambda_1} + \dots + \delta_{\lambda_N})$$

is the distribution of  $A$ , in the abstract sense of Definition 5.11.

**PROOF.** This follows indeed from Theorem 5.8, because due to our self-adjointness assumption  $A = A^*$ , the adjoint matrix plays no role in all this.  $\square$

Summarizing, we have our fix for Definition 5.6, taking into account the adjoint matrix  $A^*$ , and with this being certainly a good thing, potentially leading into interesting mathematics. However, at the level of results, Theorem 5.8, known to hold for all the matrices  $A \in M_N(\mathbb{C})$ , fully extends, and in a trivial way, only in the self-adjoint case  $A = A^*$ , leading to Theorem 5.12. The general case still remains to be investigated.

### 5c. Normal matrices

Quite remarkably, Theorem 5.12 extends to the normal case. This is something non-trivial, that we will explain now, after some linear algebra. Let us start with:

**THEOREM 5.13.** *Any matrix  $A \in M_N(\mathbb{C})$  which is self-adjoint,  $A = A^*$ , is diagonalizable, with the diagonalization being of the following type,*

$$A = UDU^*$$

with  $U \in U_N$ , and with  $D \in M_N(\mathbb{R})$  diagonal. The converse holds too.

**PROOF.** As a first remark, the converse holds indeed, because if we take a matrix of the form  $A = UDU^*$ , with  $U$  unitary and  $D$  diagonal and real, then we have:

$$\begin{aligned} A^* &= UD^*U^* \\ &= UDU^* \\ &= A \end{aligned}$$

In the other sense now, assume that  $A$  is self-adjoint,  $A = A^*$ . Our first claim is that the eigenvalues of  $A$  are real. Indeed, assuming  $Ax = \lambda x$ , we have:

$$\begin{aligned}\lambda \langle x, x \rangle &= \langle Ax, x \rangle \\ &= \langle x, Ax \rangle \\ &= \bar{\lambda} \langle x, x \rangle\end{aligned}$$

Thus we obtain  $\lambda \in \mathbb{R}$ , as claimed. Our next claim now is that the eigenspaces corresponding to different eigenvalues are pairwise orthogonal. Assume indeed that:

$$Ax = \lambda x \quad , \quad Ay = \mu y$$

We have then the following computation, by using  $\lambda, \mu \in \mathbb{R}$ :

$$\begin{aligned}\lambda \langle x, y \rangle &= \langle Ax, y \rangle \\ &= \langle x, Ay \rangle \\ &= \mu \langle x, y \rangle\end{aligned}$$

Thus  $\lambda \neq \mu$  implies  $x \perp y$ , as claimed. In order now to finish, it remains to prove that the eigenspaces span the whole  $\mathbb{C}^N$ . For this purpose, we will use a recurrence method. Let us pick an eigenvector of our matrix,  $Ax = \lambda x$ . Assuming  $x \perp y$ , we have:

$$\begin{aligned}\langle Ay, x \rangle &= \langle y, Ax \rangle \\ &= \langle y, \lambda x \rangle \\ &= \lambda \langle y, x \rangle \\ &= 0\end{aligned}$$

Thus, if  $x$  is an eigenvector of  $A$ , then the vector space  $x^\perp$  is invariant under  $A$ . On the other hand, since a square matrix  $A$  is self-adjoint precisely when  $\langle Ax, x \rangle \in \mathbb{R}$ , we conclude that the restriction of our matrix  $A$  to the vector space  $x^\perp$  is self-adjoint. Thus, we can proceed by recurrence, and we obtain in this way the result.  $\square$

As basic examples of self-adjoint matrices, we have the positive matrices. The basic theory of such matrices is as follows:

**PROPOSITION 5.14.** *For a matrix  $A \in M_N(\mathbb{C})$  the following conditions are equivalent, and if they are satisfied, we say that  $A$  is positive:*

- (1)  $A = B^2$ , with  $B = B^*$ .
- (2)  $A = CC^*$ , for some  $C \in M_N(\mathbb{C})$ .
- (3)  $\langle Ax, x \rangle \geq 0$ , for any vector  $x \in \mathbb{C}^N$ .
- (4)  $A = A^*$ , and the eigenvalues are positive,  $\lambda_i \geq 0$ .
- (5)  $A = UDU^*$ , with  $U \in U_N$  and with  $D \in M_N(\mathbb{R}_+)$  diagonal.

**PROOF.** The idea is that the equivalences in the statement basically follow from some elementary computations, with only Theorem 5.13 needed, at some point. Indeed:

(1)  $\implies$  (2) This is clear, because we can simply take  $C = B$ .

(2)  $\implies$  (3) This follows indeed from the following computation:

$$\begin{aligned} \langle Ax, x \rangle &= \langle CC^*x, x \rangle \\ &= \langle C^*x, C^*x \rangle \\ &\geq 0 \end{aligned}$$

(3)  $\implies$  (4) By using the fact that  $\langle Ax, x \rangle$  is real, we have:

$$\begin{aligned} \langle Ax, x \rangle &= \langle x, A^*x \rangle \\ &= \langle A^*x, x \rangle \end{aligned}$$

Thus we have  $A = A^*$ , and the remaining assertion, regarding the eigenvalues, follows from the following computation, by assuming  $Ax = \lambda x$ :

$$\begin{aligned} \langle Ax, x \rangle &= \langle \lambda x, x \rangle \\ &= \lambda \langle x, x \rangle \\ &\geq 0 \end{aligned}$$

(4)  $\implies$  (5) This follows indeed by using Theorem 5.13.

(5)  $\implies$  (1) Assuming  $A = UDU^*$ , with  $U \in U_N$ , and with  $D \in M_N(\mathbb{R}_+)$  being diagonal, we can set  $B = U\sqrt{D}U^*$ . We have then  $B = B^*$ , and:

$$\begin{aligned} B^2 &= U\sqrt{D}U^* \cdot U\sqrt{D}U^* \\ &= UDU^* \\ &= A \end{aligned}$$

Thus, we are led to the conclusion in the statement.  $\square$

Let us record as well the following technical version of the above result:

**PROPOSITION 5.15.** *For a matrix  $A \in M_N(\mathbb{C})$  the following conditions are equivalent, and if they are satisfied, we say that  $A$  is strictly positive:*

- (1)  $A = B^2$ , with  $B = B^*$ , invertible.
- (2)  $A = CC^*$ , for some  $C \in M_N(\mathbb{C})$  invertible.
- (3)  $\langle Ax, x \rangle > 0$ , for any nonzero vector  $x \in \mathbb{C}^N$ .
- (4)  $A = A^*$ , and the eigenvalues are strictly positive,  $\lambda_i > 0$ .
- (5)  $A = UDU^*$ , with  $U \in U_N$  and with  $D \in M_N(\mathbb{R}_+^*)$  diagonal.

**PROOF.** This follows from Proposition 5.14, by adding the extra assumptions in the statement, or from the proof of Proposition 5.14, by modifying where needed.  $\square$

Let us discuss now the case of the unitary matrices. We have here:

**THEOREM 5.16.** *Any matrix  $U \in M_N(\mathbb{C})$  which is unitary,  $U^* = U^{-1}$ , is diagonalizable, with the eigenvalues being on  $\mathbb{T}$ . More precisely we have*

$$U = VDV^*$$

with  $V \in U_N$ , and with  $D \in M_N(\mathbb{T})$  diagonal. The converse holds too.

**PROOF.** As a first remark, the converse holds indeed, because given a matrix of type  $U = VDV^*$ , with  $V \in U_N$ , and with  $D \in M_N(\mathbb{T})$  being diagonal, we have:

$$\begin{aligned} U^* &= VD^*V^* \\ &= VD^{-1}V^{-1} \\ &= (VDV^*)^{-1} \\ &= U^{-1} \end{aligned}$$

Let us prove now the first assertion, stating that the eigenvalues of a unitary matrix  $U \in U_N$  belong to the unit circle  $\mathbb{T}$ . Indeed, by assuming  $Ux = \lambda x$ , we have:

$$\begin{aligned} \langle x, x \rangle &= \langle U^*Ux, x \rangle \\ &= \langle Ux, Ux \rangle \\ &= \langle \lambda x, \lambda x \rangle \\ &= |\lambda|^2 \langle x, x \rangle \end{aligned}$$

Thus we obtain  $\lambda \in \mathbb{T}$ , as desired. Our next claim now is that the eigenspaces corresponding to different eigenvalues are pairwise orthogonal. Assume indeed that:

$$Ux = \lambda x \quad , \quad Uy = \mu y$$

We have then the following computation, by using  $U^* = U^{-1}$  and  $\lambda, \mu \in \mathbb{T}$ :

$$\begin{aligned} \lambda \langle x, y \rangle &= \langle Ux, y \rangle \\ &= \langle x, U^*y \rangle \\ &= \langle x, U^{-1}y \rangle \\ &= \langle x, \mu^{-1}y \rangle \\ &= \mu \langle x, y \rangle \end{aligned}$$

Thus  $\lambda \neq \mu$  implies  $x \perp y$ , as claimed. In order now to finish, it remains to prove that the eigenspaces span the whole  $\mathbb{C}^N$ . For this purpose, we will use a recurrence method. Let us pick an eigenvector,  $Ux = \lambda x$ . Assuming  $x \perp y$ , we have:

$$\begin{aligned} \langle Uy, x \rangle &= \langle y, U^*x \rangle \\ &= \langle y, U^{-1}x \rangle \\ &= \langle y, \lambda^{-1}x \rangle \\ &= \lambda \langle y, x \rangle \\ &= 0 \end{aligned}$$

Thus, if  $x$  is an eigenvector of  $U$ , then the vector space  $x^\perp$  is invariant under  $U$ . Now since  $U$  is an isometry, so is its restriction to this space  $x^\perp$ . Thus this restriction is a unitary, and so we can proceed by recurrence, and we obtain the result.  $\square$

As a basic illustration for the above result, we have:

PROPOSITION 5.17. *The rotation of angle  $t \in \mathbb{R}$  in the plane diagonalizes as:*

$$\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} e^{-it} & 0 \\ 0 & e^{it} \end{pmatrix} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

*Over the reals this is impossible, unless  $t = 0, \pi$ , where the rotation is diagonal.*

PROOF. Observe first that, unlike we are in the case  $t = 0, \pi$ , where our rotation is  $\pm 1$ , our rotation is a “true” rotation, obviously having no eigenvectors in the plane. Fortunately the complex numbers come to the rescue, and we have:

$$\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} \cos t - i \sin t \\ i \cos t + \sin t \end{pmatrix} = e^{-it} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

We have as well a second eigenvector, coming from:

$$\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} \cos t + i \sin t \\ -i \cos t + \sin t \end{pmatrix} = e^{it} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

Thus, we are led to the conclusion in the statement.  $\square$

The self-adjoint matrices and the unitary matrices are particular cases of the general notion of a “normal matrix”, and we have here the following key result:

THEOREM 5.18. *Any matrix  $A \in M_N(\mathbb{C})$  which is normal,  $AA^* = A^*A$ , is diagonalizable, with the diagonalization being of the following type,*

$$A = UDU^*$$

*with  $U \in U_N$ , and with  $D \in M_N(\mathbb{C})$  diagonal. The converse holds too.*

PROOF. As a first remark, the converse holds indeed, because if we take a matrix of the form  $A = UDU^*$ , with  $U$  unitary and with  $D$  diagonal, then we have:

$$\begin{aligned} AA^* &= UDD^*U^* \\ &= UD^*DU^* \\ &= A^*A \end{aligned}$$

In the other sense now, this is something more technical. Our first claim is that a matrix  $A$  is normal precisely when the following is satisfied, for any vector  $x$ :

$$\|Ax\| = \|A^*x\|$$

Indeed, this equality can be written in the following way, which gives  $AA^* = A^*A$ :

$$\langle AA^*x, x \rangle = \langle A^*Ax, x \rangle$$

Our claim now is that  $A, A^*$  have the same eigenvectors, with conjugate eigenvalues:

$$Ax = \lambda x \implies A^*x = \bar{\lambda}x$$

Indeed, this follows from the following computation, and from the trivial fact that if  $A$  is normal, then so is any matrix of type  $A - \lambda 1_N$ , with  $\lambda \in \mathbb{C}$ :

$$\begin{aligned} \|(A^* - \bar{\lambda}1_N)x\| &= \|(A - \lambda 1_N)^*x\| \\ &= \|(A - \lambda 1_N)x\| \\ &= 0 \end{aligned}$$

Let us prove now, by using this fact, that the eigenspaces of  $A$  are pairwise orthogonal. Assuming  $Ax = \lambda x$  and  $Ay = \mu y$  with  $\lambda \neq \mu$ , we have:

$$\begin{aligned} \lambda \langle x, y \rangle &= \langle Ax, y \rangle \\ &= \langle x, A^*y \rangle \\ &= \langle x, \bar{\mu}y \rangle \\ &= \mu \langle x, y \rangle \end{aligned}$$

Thus  $\lambda \neq \mu$  implies  $x \perp y$ , as desired. In order to finish now the proof, it remains to prove that the eigenspaces of  $A$  span the whole  $\mathbb{C}^N$ . This is something quite tricky, and our plan here will be that of proving that the eigenspaces of  $AA^*$  are eigenspaces of  $A$ . In order to do so, let us pick two eigenvectors  $x, y$  of the matrix  $AA^*$ , corresponding to different eigenvalues,  $\lambda \neq \mu$ . The eigenvalue equations are then as follows:

$$AA^*x = \lambda x \quad , \quad AA^*y = \mu y$$

We have the following computation, by using the normality condition  $AA^* = A^*A$ , and the fact that the eigenvalues of  $AA^*$ , and in particular  $\mu$ , are real:

$$\begin{aligned} \lambda \langle Ax, y \rangle &= \langle A\lambda x, y \rangle \\ &= \langle AAA^*x, y \rangle \\ &= \langle AA^*Ax, y \rangle \\ &= \langle Ax, AA^*y \rangle \\ &= \langle Ax, \mu y \rangle \\ &= \mu \langle Ax, y \rangle \end{aligned}$$

We conclude that we have  $\langle Ax, y \rangle = 0$ . But this reformulates as follows:

$$\lambda \neq \mu \implies A(E_\lambda) \perp E_\mu$$

Now since the eigenspaces of  $AA^*$  are pairwise orthogonal, and span the whole  $\mathbb{C}^N$ , we deduce that these eigenspaces are invariant under  $A$ :

$$A(E_\lambda) \subset E_\lambda$$

But with this result in hand, we can now finish. Indeed, we can decompose the problem, and the matrix  $A$  itself, following these eigenspaces of  $AA^*$ , which in practice amounts in saying that we can assume that we only have 1 eigenspace. By rescaling, this is the same as assuming that we have  $AA^* = 1$ , and so we are now into the unitary case, that we know how to solve, as explained in Theorem 5.16.  $\square$

We will be back to spectral theorems later on, in chapter 6 below, with some results for the infinite matrices too, and for other matrices in the infinite dimensional setting.

### 5d. Matrix laws

Getting back now to the laws of matrices, Theorem 5.12 extends to the normal case,  $AA^* = A^*A$ . This is something non-trivial, the result being as follows:

**THEOREM 5.19.** *Given a matrix  $A \in M_N(\mathbb{C})$  which is normal,  $AA^* = A^*A$ , we have the following formula, valid for any polynomial  $P \in \mathbb{C} \langle X, X^* \rangle$ ,*

$$\text{tr}(P(A)) = \frac{1}{N}(P(\lambda_1) + \dots + P(\lambda_N))$$

where  $\lambda_1, \dots, \lambda_N \in \mathbb{C}$  are the eigenvalues of  $A$ . Thus the complex measure

$$\mu_A = \frac{1}{N}(\delta_{\lambda_1} + \dots + \delta_{\lambda_N})$$

is the distribution of  $A$ , in the abstract sense of Definition 5.11.

**PROOF.** There are several proofs for this fact, one of them being as follows:

(1) Let us first consider the case where the matrix is diagonal:

$$A = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix}$$

The powers of  $A$ , with respect to colored integer exponents  $k = \circ \bullet \bullet \circ \dots$  as in Definition 5.10, are then given by the following formula, with the convention that the numbers  $\lambda^k$  are given by  $\lambda^\circ = \lambda$ ,  $\lambda^\bullet = \bar{\lambda}$  and multiplicativity:

$$A^k = \begin{pmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_N^k \end{pmatrix}$$

Thus, the moments of  $A$  are given by the following formula:

$$M_k = \frac{1}{N}(\lambda_1^k + \dots + \lambda_N^k)$$

Regarding now the distribution, this by definition given by:

$$\mu_A : \mathbb{C} \langle X, X^* \rangle \rightarrow \mathbb{C} \quad , \quad P \rightarrow \text{tr}(P(A))$$

Since the matrix is normal,  $AA^* = A^*A$ , knowing this distribution is the same as knowing its restriction to the usual polynomials in two variables:

$$\mu_A : \mathbb{C}[X, X^*] \rightarrow \mathbb{C} \quad , \quad P \rightarrow \text{tr}(P(A))$$

By using now the fact that  $A$  is diagonal, we conclude that the distribution is:

$$\mu_A : \mathbb{C}[X, X^*] \rightarrow \mathbb{C} \quad , \quad P \rightarrow \frac{1}{N}(P(\lambda_1) + \dots + P(\lambda_N))$$

But this functional corresponds to integrating  $P$  with respect to the following complex measure, that we agree to still denote by  $\mu_A$ , and call distribution of  $A$ :

$$\mu_A = \frac{1}{N}(\delta_{\lambda_1} + \dots + \delta_{\lambda_N})$$

Summarizing, modulo a number of standard identifications, the distribution of a diagonal matrix  $A \in M_N(\mathbb{C})$  is a complex probability measure, given by the above formula.

(2) In the general case now, where  $A \in M_N(\mathbb{C})$  is normal and arbitrary, we can use Theorem 5.18, which tells us that  $A$  is diagonalizable, and in fact that  $A, A^*$  are jointly diagonalizable. To be more precise, let us write, as in Theorem 5.18:

$$A = UDU^*$$

Here  $U \in U_N$ , and  $D \in M_N(\mathbb{C})$  is diagonal. The adjoint matrix is then given by:

$$A^* = UD^*U$$

As before in the diagonal matrix case, since our matrix is normal,  $AA^* = A^*A$ , knowing its distribution in the abstract sense of Definition 5.11 is the same as knowing the restriction of this abstract distribution to the usual polynomials in two variables:

$$\mu_A : \mathbb{C}[X, X^*] \rightarrow \mathbb{C} \quad , \quad P \rightarrow \text{tr}(P(A))$$

In order now to compute this functional, we can change the basis via the above unitary matrix  $U \in U_N$ , which in practice means that we can assume  $U = 1$ . Thus, by using now (1), if we denote by  $\lambda_1, \dots, \lambda_N$  the diagonal entries of  $D$ , which are the eigenvalues of  $A$ , the distribution that we are looking for is the following functional:

$$\mu_A : \mathbb{C}[X, X^*] \rightarrow \mathbb{C} \quad , \quad P \rightarrow \frac{1}{N}(P(\lambda_1) + \dots + P(\lambda_N))$$

As before, this functional corresponds to integrating  $P$  with respect to the following complex measure, that we agree to still denote by  $\mu_A$ , and call distribution of  $A$ :

$$\mu_A = \frac{1}{N}(\delta_{\lambda_1} + \dots + \delta_{\lambda_N})$$

Thus, we are led to the conclusion in the statement. □

Importantly now, let us mention that the normality assumption in the above results is really needed. Indeed, we have the following basic counterexample:

PROPOSITION 5.20. *The following matrix, which is not normal,*

$$J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

*has no distribution  $\mu_J \in \mathcal{P}(\mathbb{C})$  in the sense of Definition 5.11.*

PROOF. The adjoint of the matrix in the statement is given by:

$$J^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

We have the following product formulae, which show indeed that  $J$  is not normal:

$$\begin{aligned} JJ^* &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ J^*J &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Now observe that the eigenvalues of  $J$  are 0 and 0. Thus the  $*$ -law formula in Theorem 5.19 has no chance to extend to this setting, simply because we have:

$$\text{tr}(JJ^*) = \text{tr}(J^*J) = \frac{1}{2}$$

Even worse, let us prove now that, as claimed, our matrix  $J$  has no distribution  $\mu_J \in \mathcal{P}(\mathbb{C})$ , in the sense of Definition 5.11. For this purpose, observe that we have:

$$\begin{aligned} \text{tr}(JJ^*JJ^*) &= \text{tr}((JJ^*)^2) \\ &= \text{tr} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ &= \frac{1}{2} \end{aligned}$$

On the other hand, we have as well the following formula:

$$\begin{aligned} \text{tr}(JJJ^*J^*) &= \text{tr} \left( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right) \\ &= \text{tr} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ &= 0 \end{aligned}$$

Since the above numbers are different, we cannot obtain them by integrating with respect to a measure  $\mu_J \in \mathcal{P}(\mathbb{C})$ , and this leads to the above conclusion.  $\square$

We have in fact the following general result, which fully clarifies the situation:

THEOREM 5.21. *A matrix  $A \in M_N(\mathbb{C})$  has a complex measure as distribution,*

$$\mu_A \in \mathcal{P}(\mathbb{C})$$

*in the sense of Definition 5.11, precisely when it is normal,  $AA^* = A^*A$ .*

PROOF. In one sense, this is something that we know from Theorem 5.19. In the other sense now, assume that  $AA^* \neq A^*A$ . In order to show that the law  $\mu_A$  does not exist as a complex measure, we can use a positivity trick, as follows:

$$\begin{aligned} AA^* - A^*A \neq 0 &\implies (AA^* - A^*A)^2 > 0 \\ &\implies AA^*AA^* - AA^*A^*A - A^*AAA^* + A^*AA^*A > 0 \\ &\implies \text{tr}(AA^*AA^* - AA^*A^*A - A^*AAA^* + A^*AA^*A) > 0 \\ &\implies \text{tr}(AA^*AA^* + A^*AA^*A) > \text{tr}(AA^*A^*A + A^*AAA^*) \\ &\implies \text{tr}(AA^*AA^*) > \text{tr}(AAA^*A^*) \end{aligned}$$

Now assuming that the law of  $A$ , in the sense of Definition 5.11, comes from a complex measure  $\mu_A \in \mathcal{P}(\mathbb{C})$ , the above two different numbers would have to both appear by integrating  $|z|^2$  with respect to this measure  $\mu_A$ , which is contradictory, as desired.  $\square$

In what follows we will mainly be interested in normal matrices. Let us record here:

PROPOSITION 5.22. *The laws of normal matrices  $A \in M_N(\mathbb{C})$  are as follows:*

- (1) *In the case  $A = A^*$ , we have  $\text{supp}(\mu_A) \subset \mathbb{R}$ .*
- (2) *In the case  $A \geq 0$ , we have  $\text{supp}(\mu_A) \subset [0, \infty)$ .*
- (3) *In the case  $A > 0$ , we have  $\text{supp}(\mu_A) \subset (0, \infty)$ .*
- (4) *In the case  $A^* = A^{-1}$ , we have  $\text{supp}(\mu_A) \subset \mathbb{T}$ .*

PROOF. This follows indeed from Theorem 5.19, which shows that the support of the law is the set of eigenvalues,  $\text{supp}(\mu_A) = \{\lambda_1, \dots, \lambda_N\}$ , and from our various results regarding the self-adjoint, positive, strictly positive and unitary matrices.  $\square$

Before getting into the normal case, let us have a look into the non-normal case too. Things here are quite tricky, because there are some interesting non-normal matrices having interesting laws, but this is quite advanced, for later, with no simple example available, at this point. As a basic exploration of the subject, however, we have:

PROPOSITION 5.23. *The colored moments of the  $2 \times 2$  Jordan blocks*

$$B = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

*with  $\lambda \in \mathbb{C}$  are given by the formula*

$$M_k(B) = \frac{1}{2} \sum_{l \in k} \lambda^{k-l}$$

*with the sum being over all alternating subwords  $l \in k$  having even length.*

PROOF. This is standard combinatorics, the idea being as follows:

(1) Let us first discuss the case  $\lambda = 0$ . Here our Jordan block is the matrix  $J$  from Proposition 5.20, and since we have  $J^2 = 0$ , and so  $(J^*)^2 = 0$  as well, the only non-vanishing moments  $M_k(J)$  are those coming from alternating words  $k$ . Now in order to compute these latter moments, observe that we have the following formulae:

$$\begin{aligned} JJ^* \dots JJ^* &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad , \quad JJ^* \dots JJ^* J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ J^* J \dots J^* J &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad J^* J \dots J^* J J^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

Thus the alternating products having even length have trace  $1/2$ , and the alternating products having odd length have trace  $0$ , and so the general moment formula is:

$$M_k(J) = \begin{cases} 1/2 & \text{if } k \text{ is alternating, of even length} \\ 0 & \text{otherwise} \end{cases}$$

(2) In the general case  $\lambda \in \mathbb{C}$ , we have  $B = \lambda + J$ , and so the moment formula is as follows, with “ae” standing for alternating, of even length:

$$\begin{aligned} M_k(B) &= M_k((\lambda + J)^k) \\ &= \sum_{l \in k} \lambda^{k-l} M_l(J) \\ &= \frac{1}{2} \sum_{l \in k, l=ae} \lambda^{k-l} \end{aligned}$$

Thus, we are led to the conclusion in the statement.  $\square$

Observe that the above result does not close the question for the  $2 \times 2$  matrices, because we know that we have  $\text{law}(A) = \text{law}(UAU^{-1})$  for any unitary  $U$ , but not for any invertible  $U$ . Thus, the  $2 \times 2$  matrix question remains open. As for the case of the  $3 \times 3$  matrices, things here are quite wild even for the simplest Jordan block, namely:

$$J = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Indeed, we have  $J^3 = 0$ , and so  $(J^*)^3 = 0$  too, so the only non-vanishing moments  $M_k(J)$  are those coming from the words  $k$  containing no subwords of type  $JJJ$  or  $J^*J^*J^*$ . But this is pretty much all we can say about vanishing, of trivial type, and the combinatorics obviously gets quite complicated, and we will end our discussion here.

As a final comment, however, we will see later on that there are some interesting examples of non-normal matrices, or rather interesting examples of non-normal random

matrices, having interesting laws, or rather interesting asymptotic laws, with the main examples here being the Gaussian matrices, having as limiting distribution the Voiculescu circular law. But this is something quite advanced, and more on this later.

Speaking random matrices, with all the above linear algebra preliminaries and results in hand, we can now go ahead and discuss, eventually, the case of the random matrices, where things become truly interesting. We can extend Definition 5.11, as follows:

DEFINITION 5.24. *The colored moments of a random matrix*

$$Z \in M_N(L^\infty(X))$$

*are the following numbers, indexed by the colored integers  $k = \circ \bullet \bullet \circ \dots$*

$$M_k = \int_X \text{tr}(Z^k)$$

*with the powers  $Z^k$  being defined by  $Z^\circ = Z$ ,  $Z^\bullet = Z^*$  and multiplicativity.*

Observe that this notion extends indeed the notion from Definition 5.11 for the usual matrices  $Z \in M_N(\mathbb{C})$ , which can be recovered with  $X = \{.\}$ . Also, in the case  $N = 1$ , where our matrix is just a random variable  $Z \in L^\infty(X)$ , we recover in this way the usual moments, or rather the joint moments of the random variables  $Z, \bar{Z}$ .

Regarding now the distribution, we can use here a similar extension, as follows:

DEFINITION 5.25. *The distribution of a random matrix  $Z \in M_N(L^\infty(X))$  is the abstract functional  $\mu_Z : \mathbb{C} \langle X, X^* \rangle \rightarrow \mathbb{C}$  given by:*

$$P \rightarrow \int_X \text{tr}(P(Z))$$

*In the case where we have a probability measure  $\mu_Z \in \mathcal{P}(\mathbb{C})$  such that*

$$\text{tr}(P(Z)) = \int_{\mathbb{C}} P(x) d\mu_Z(x)$$

*we identify this measure with the distribution, or law of  $Z$ .*

Observe that by linearity, the distribution is uniquely determined by the moments. In fact, knowing the distribution is the same thing as knowing the moments. As basic examples, for the usual matrices  $Z \in M_N(\mathbb{C})$ , obtained by taking  $X = \{.\}$ , we obtain the previous notion of distribution, from Definition 5.11.

Also, for the usual random variables  $Z \in L^\infty(X)$ , obtained by taking  $N = 1$ , we obtain in this way the previous notion of distribution, from chapters 1-2 above. Indeed, these variables are normal,  $ZZ^* = Z^*Z$ , and so the corresponding distributions, in the above abstract sense, can be restricted to usual polynomials  $P \in \mathbb{C}[X, X^*]$ , and then identified with the usual distributions, in the sense of probability theory.

Summarizing, what we have so far are some good definitions. At the theoretical level, what is mostly needed now is an extension of Theorem 5.19, dealing with the normal matrices, to the case of the normal random matrices. This is something quite tricky, and the whole next chapter, 6 below, will be dedicated to solving this question. Then afterwards, in chapters 7-8 below, we will go back to the Gaussian, Wigner and Wishart matrices, with computations and concrete results about them.

### 5e. Exercises

Things have been a bit abstract in the present chapter, and more abstractions to come in the next one, the problem with all this being the fact that, before getting into concrete and beautiful computations regarding the various classes of random matrices, we must be quite familiar with the notion of law of a random matrix, which is something rather advanced, with the intuition not helping that much. As a first exercise, we have:

EXERCISE 5.26. *Compute the first few moments of the complex Gaussian, Wigner and Wishart matrices, with “first few” meaning that the length of the colored integer  $k = \circ \bullet \circ \bullet \dots$  used as exponent is small,  $|k| = 1, 2, 3 \dots$*

This exercise is very instructive, and if you are patient enough, and observe a pattern from the many computations that you made, you can switch directly to the general case, which should not be that complicated, once you have a good guess.

EXERCISE 5.27. *Prove that the symmetry with respect to the  $Ox$  axis rotated by an angle  $t/2 \in \mathbb{R}$  is given by the matrix*

$$S_t = \begin{pmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{pmatrix}$$

*and then diagonalize this matrix, and if possible without computations.*

Here the first part can only be clear on pictures, and by the way, prior to this, do not forget to verify as well that our formula of  $R_t$  is the good one.

EXERCISE 5.28. *Prove that the isometries of  $\mathbb{R}^2$  are rotations or symmetries,*

$$R_t = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \quad , \quad S_t = \begin{pmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{pmatrix}$$

*and then try as well to find a formula for the isometries of  $\mathbb{R}^3$ .*

Here for the first question you should look first at the determinant of such an isometry. As for the second question, this is something quite difficult.

EXERCISE 5.29. *Prove that the isometries of  $\mathbb{C}^2$  of determinant 1 are*

$$U = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \quad , \quad |a|^2 + |b|^2 = 1$$

*then work out as well the general case, of arbitrary determinant.*

As a comment here, if done with this exercise about  $\mathbb{C}^2$ , but not yet with the previous one about  $\mathbb{R}^3$ , you can go back to that exercise, by using a  $\mathbb{C}^2 \simeq \mathbb{R}^4$  trick. And in case this trick leads to tough computations and big headache, look it up.

EXERCISE 5.30. *Prove that the flat matrix, which is the all-one  $N \times N$  matrix, diagonalizes over the complex numbers as follows,*

$$\begin{pmatrix} 1 & \dots & 1 \\ \vdots & & \vdots \\ 1 & \dots & 1 \end{pmatrix} = \frac{1}{N} F_N \begin{pmatrix} N & & \\ & 0 & \\ & & \ddots \\ & & & 0 \end{pmatrix} F_N^*$$

where  $F_N = (w^{ij})_{ij}$  with  $w = e^{2\pi i/N}$  is the Fourier matrix, with the convention that the indices are taken to be  $i, j = 0, 1, \dots, N-1$ .

This is something very instructive. Normally you have to look for eigenvectors for the flat matrix, and you are led in this way to the equation  $x_0 + \dots + x_{N-1} = 0$ . The problem however is that this equation, while looking very gentle, has no “canonical” solutions over the real numbers. Thus you are led to the complex numbers, and more specifically to the roots of unity, and their magic, leading to the above result. Enjoy.

EXERCISE 5.31. *Prove that we have the following formula, as an equality of subsets of  $\mathbb{C}$ , valid for any square matrix  $A \in M_N(\mathbb{C})$ ,*

$$\sigma(A^*) = \overline{\sigma(A)}$$

where  $\sigma$  is the set of eigenvalues, and deduce a geometric interpretation of the adjoint matrix  $A^* \in M_N(\mathbb{C})$ , starting from this.

This looks quite standard, but in order to solve the last question, regarding the geometric interpretation of  $A^*$ , you will probably have to upgrade first your result about  $\sigma$  into a result about  $\sigma^+$ , the set of eigenvalues taken with multiplicities.

EXERCISE 5.32. *Figure out what a symplectic random matrix should mean, and then develop a bit of theory for such matrices, such as the computation of small moments.*

This is a bit vague, but looking at how our central examples of random matrices, namely the Wigner ones, were constructed out of the complex Gaussian ones leads to the conclusion that the groups  $O_N, U_N$  are probably involved in all this. And the point now is that, when talking about  $O_N, U_N$ , the symplectic group  $Sp_N$  should be not far.

## CHAPTER 6

### Spectral theory

#### 6a. Linear operators

We have seen in chapter 5 that probability theory and matrix theory have a common generalization, namely random matrix theory, where the variables are matrices  $Z \in M_N(L^\infty(X))$ . However, things were quite tricky in order to define the distribution of a random matrix, or even of a usual matrix, due to the fact such matrices come in pairs  $(Z, Z^*)$ , and so we must talk about the distribution of such a pair  $(Z, Z^*)$ .

In order to further clarify all this, and to discuss as well what happens in the non-normal case, we will need an extension of the theory that we have, going beyond the random matrix setting, by using some basic functional analysis and spectral theory. All this will be useful for better understanding the material in chapters 7-8, concerned with explicit random matrix computations, and also, importantly, will be of key importance in chapters 9-12 and afterwards, when doing free probability.

In order to get started, let us formulate the following definition:

**DEFINITION 6.1.** *A Hilbert space is a complex vector space  $H$  given with a scalar product  $\langle x, y \rangle$ , satisfying the following conditions:*

- (1)  $\langle x, y \rangle$  is linear in  $x$ , and antilinear in  $y$ .
- (2)  $\overline{\langle x, y \rangle} = \langle y, x \rangle$ , for any  $x, y$ .
- (3)  $\langle x, x \rangle \geq 0$ , for any  $x \neq 0$ .
- (4)  $H$  is complete with respect to the norm  $\|x\| = \sqrt{\langle x, x \rangle}$ .

Here the fact that  $\|\cdot\|$  is indeed a norm comes from the Cauchy-Schwarz inequality, which states that if the conditions (1,2,3) above are satisfied, then we have:

$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$$

Indeed, this inequality comes from the fact that the following degree 2 polynomial, with  $t \in \mathbb{R}$  and  $w \in \mathbb{T}$ , being positive, its discriminant must be negative:

$$f(t) = \|x + twy\|^2$$

At the level of the examples, we first have the Hilbert space  $H = \mathbb{C}^N$ , with its usual scalar product, taken linear at left, namely:

$$\langle x, y \rangle = \sum_i x_i \bar{y}_i$$

Observe that this makes a link with linear algebra, because the linear maps  $T : H \rightarrow H$  correspond in this case to the square matrices  $A \in M_N(\mathbb{C})$ . More on this later. In what regards now the link with probability theory, this comes from:

**THEOREM 6.2.** *Given a measured space  $X$ , the functions  $f : X \rightarrow \mathbb{C}$ , taken up to equality almost everywhere, which are square-summable,*

$$\int_X |f(x)|^2 dx < \infty$$

*form a Hilbert space  $L^2(X)$ , with the following scalar product:*

$$\langle f, g \rangle = \int_X f(x) \overline{g(x)} dx$$

*In the case where  $X = I$  is a set endowed with its counting measure, we obtain the space  $l^2(I)$  of square-summable sequences  $\{x_i\}_{i \in I} \subset \mathbb{C}$ , with  $\langle x, y \rangle = \sum_i x_i \bar{y}_i$ .*

**PROOF.** There are several things to be proved, as follows:

(1) Our first claim is that  $L^2(X)$  is a vector space, and here we must prove that  $f, g \in L^2(X)$  implies  $f + g \in L^2(X)$ . But this leads us into proving  $\|f + g\| \leq \|f\| + \|g\|$ , where  $\|f\| = \sqrt{\langle f, f \rangle}$ . Now since this inequality holds on each subspace  $\mathbb{C}^N \subset L^2(X)$  coming from step functions, this inequality holds everywhere, as desired.

(2) Our second claim is that  $\langle, \rangle$  is well-defined on  $L^2(X)$ . But this follows from the Cauchy-Schwarz inequality,  $|\langle f, g \rangle| \leq \|f\| \cdot \|g\|$ , which can be established by truncating, a bit like we established the Minkowski inequality in (1) above.

(3) It is also clear that  $\langle, \rangle$  is a scalar product on  $L^2(X)$ , with the remark here that if we want to have  $\langle f, f \rangle > 0$  for  $f \neq 0$ , we must declare that  $f = 0$  when  $f = 0$  almost everywhere, and so that  $f = g$  when  $f = g$  almost everywhere, as stated.

(4) It remains to prove that  $L^2(X)$  is complete with respect to  $\|f\| = \sqrt{\langle f, f \rangle}$ . But this is clear, because if we pick a Cauchy sequence  $\{f_n\}_{n \in \mathbb{N}} \subset L^2(X)$ , then we can construct a pointwise, and hence  $L^2$  limit,  $f_n \rightarrow f$ , almost everywhere.

(5) Finally, the last assertion is clear, because the integration with respect to the counting measure is by definition a sum, and so we have  $L^2(I) = l^2(I)$ .  $\square$

Observe that the space  $\mathbb{C}^N$  is covered by the construction in Theorem 6.2, appearing as  $l^2(I)$ , with  $I = \{1, \dots, N\}$ . Quite remarkably, any Hilbert space must be of the form  $L^2(X)$ , and even of the form  $l^2(I)$ . This follows indeed from the following key result:

THEOREM 6.3. *Let  $H$  be a Hilbert space.*

- (1) *Any algebraic basis of this space  $\{f_i\}_{i \in I}$  can be turned into an orthonormal basis  $\{e_i\}_{i \in I}$ , by using the Gram-Schmidt procedure.*
- (2) *Thus,  $H$  has an orthonormal basis, and so we have  $H \simeq l^2(I)$ , with  $I$  being the indexing set for this orthonormal basis.*

PROOF. All this is standard by Gram-Schmidt, the idea being as follows:

(1) First of all, in finite dimensions an orthonormal basis  $\{e_i\}_{i \in I}$  is by definition a usual algebraic basis, satisfying  $\langle e_i, e_j \rangle = \delta_{ij}$ . But the existence of such a basis follows by applying the Gram-Schmidt procedure to any algebraic basis  $\{f_i\}_{i \in I}$ , as claimed.

(2) In infinite dimensions, a first issue comes from the fact that the standard basis  $\{\delta_i\}_{i \in \mathbb{N}}$  of the space  $l^2(\mathbb{N})$  is not an algebraic basis in the usual sense, with the finite linear combinations of the functions  $\delta_i$  producing only a dense subspace of  $l^2(\mathbb{N})$ , that of the functions having finite support. Thus, we must fine-tune our definition of “basis”.

(3) But this can be done in two ways, by saying that  $\{f_i\}_{i \in I}$  is a basis of  $H$  when the functions  $f_i$  are linearly independent, and when either the finite linear combinations of these functions  $f_i$  form a dense subspace of  $H$ , or the linear combinations with  $l^2(I)$  coefficients of these functions  $f_i$  form the whole  $H$ . For orthogonal bases  $\{e_i\}_{i \in I}$  these definitions are equivalent, and in any case, our statement makes now sense.

(4) Regarding now the proof, in infinite dimensions, this follows again from Gram-Schmidt, exactly as in the finite dimensional case, but by using this time a tool from logic, called Zorn lemma, in order to correctly do the recurrence.  $\square$

The above result, and its relation with Theorem 6.2, is something quite subtle, so let us further get into this. First, we have the following definition, based on the above:

DEFINITION 6.4. *A Hilbert space  $H$  is called separable when the following equivalent conditions are satisfied:*

- (1)  *$H$  has a countable algebraic basis  $\{f_i\}_{i \in \mathbb{N}}$ .*
- (2)  *$H$  has a countable orthonormal basis  $\{e_i\}_{i \in \mathbb{N}}$ .*
- (3) *We have  $H \simeq l^2(\mathbb{N})$ , isomorphism of Hilbert spaces.*

As a first observation, according to the above, there is up to isomorphism only one separable Hilbert space, namely:

$$H = l^2(\mathbb{N})$$

This is, however, quite tricky, and can be a bit misleading. Consider for instance the space  $H = L^2[0, 1]$  of square-summable functions  $f : [0, 1] \rightarrow \mathbb{C}$ , with:

$$\langle f, g \rangle = \int_0^1 f(x) \overline{g(x)} dx$$

This space is of course separable, because we can use the basis  $f_n = x^n$  with  $n \in \mathbb{N}$ , orthogonalized by Gram-Schmidt. However, the orthogonalization procedure is something non-trivial, so the isomorphism  $H \simeq l^2(\mathbb{N})$  that we obtain is non-trivial as well.

Let us get now into the study of linear operators, which generalize the usual matrices, in the infinite dimensional setting. We will be interested in the operators  $T : H \rightarrow H$  which are bounded. Regarding such operators, we first have the following result:

**PROPOSITION 6.5.** *For a linear operator  $T : H \rightarrow H$ , the following are equivalent:*

- (1)  $T$  is continuous.
- (2)  $T$  is continuous at 0.
- (3)  $T(B) \subset cB$  for some  $c < \infty$ , where  $B \subset H$  is the unit ball.
- (4)  $T$  is bounded, in the sense that  $\|T\| = \sup_{\|x\| \leq 1} \|Tx\|$  satisfies  $\|T\| < \infty$ .

**PROOF.** This is elementary, with (1)  $\iff$  (2) coming from the linearity of  $T$ , then (2)  $\iff$  (3) coming from definitions, and finally (3)  $\iff$  (4) coming from the fact that the number  $\|T\|$  from (4) is the infimum of the numbers  $c$  making (3) work.  $\square$

Next in line, we have the following key result:

**THEOREM 6.6.** *Given a Hilbert space  $H$ , the linear operators  $T : H \rightarrow H$  which are bounded, in the sense that the quantity*

$$\|T\| = \sup_{\|x\| \leq 1} \|Tx\|$$

*is finite, form a complex algebra  $B(H)$ , having the following properties:*

- (1)  $B(H)$  is complete with respect to  $\|\cdot\|$ , and so we have a Banach algebra.
- (2)  $B(H)$  has an involution  $T \rightarrow T^*$ , given by  $\langle Tx, y \rangle = \langle x, T^*y \rangle$ .

*In addition, the norm and the involution are related by the formula  $\|TT^*\| = \|T\|^2$ .*

**PROOF.** The fact that we have indeed an algebra follows from:

$$\|S + T\| \leq \|S\| + \|T\|$$

$$\|\lambda T\| = |\lambda| \cdot \|T\|$$

$$\|ST\| \leq \|S\| \cdot \|T\|$$

(1) Assuming that  $\{T_k\} \subset B(H)$  is a Cauchy sequence, the sequence  $\{T_k x\}$  is Cauchy for any  $x \in H$ , so we can define the limit  $T = \lim_{k \rightarrow \infty} T_k$  by setting:

$$Tx = \lim_{k \rightarrow \infty} T_k x$$

It is routine then to check that this formula defines indeed an operator  $T \in B(H)$ , and that we have  $T_k \rightarrow T$  in norm, and this gives the result.

(2) The existence of  $T^*$  comes from the fact that  $\psi(x) = \langle Tx, y \rangle$  being a linear map  $H \rightarrow \mathbb{C}$ , we must have a formula as follows, for a certain vector  $T^*y \in H$ :

$$\psi(x) = \langle x, T^*y \rangle$$

Moreover, since this vector  $T^*y$  is unique,  $T^*$  is unique too, and we have as well:

$$(S + T)^* = S^* + T^* \quad , \quad (\lambda T)^* = \bar{\lambda} T^*$$

$$(ST)^* = T^* S^* \quad , \quad (T^*)^* = T$$

Observe also that we have indeed  $T^* \in B(H)$ , due to the following equality:

$$\begin{aligned} \|T\| &= \sup_{\|x\|=1} \sup_{\|y\|=1} \langle Tx, y \rangle \\ &= \sup_{\|y\|=1} \sup_{\|x\|=1} \langle x, T^*y \rangle \\ &= \|T^*\| \end{aligned}$$

(3) Regarding now the last assertion, observe first that we have:

$$\|TT^*\| \leq \|T\| \cdot \|T^*\| = \|T\|^2$$

On the other hand, we have as well the following estimate:

$$\begin{aligned} \|T\|^2 &= \sup_{\|x\|=1} |\langle Tx, Tx \rangle| \\ &= \sup_{\|x\|=1} |\langle x, T^*Tx \rangle| \\ &\leq \|T^*T\| \end{aligned}$$

Now by replacing in this formula  $T \rightarrow T^*$  we obtain  $\|T\|^2 \leq \|TT^*\|$ . Thus, we have proved both the needed inequalities, and we are done.  $\square$

In the case where  $H$  comes with a basis  $\{e_i\}_{i \in I}$ , we can talk about the infinite matrices  $M \in M_I(\mathbb{C})$ , with the remark that the multiplication of such matrices is not always defined, in the case  $|I| = \infty$ . In this context, we have the following result:

**PROPOSITION 6.7.** *Let  $H$  be a Hilbert space, with orthonormal basis  $\{e_i\}_{i \in I}$ . The bounded operators  $T \in B(H)$  can be then identified with matrices  $M \in M_I(\mathbb{C})$  via*

$$Tx = Mx \quad , \quad M_{ij} = \langle Te_j, e_i \rangle$$

*and we obtain in this way an embedding as follows, which is multiplicative:*

$$B(H) \subset M_I(\mathbb{C})$$

*In the case  $H = \mathbb{C}^N$  we obtain in this way the usual isomorphism  $B(H) \simeq M_N(\mathbb{C})$ . In the separable case we obtain in this way a proper embedding  $B(H) \subset M_\infty(\mathbb{C})$ .*

PROOF. We have several assertions to be proved, the idea being as follows:

(1) Regarding the first assertion, given a bounded operator  $T : H \rightarrow H$ , let us associate to it a matrix  $M \in M_I(\mathbb{C})$  as in the statement, by the following formula:

$$M_{ij} = \langle Te_j, e_i \rangle$$

It is clear that this correspondence  $T \rightarrow M$  is linear, and also that its kernel is  $\{0\}$ . Thus, we have an embedding of linear spaces  $B(H) \subset M_I(\mathbb{C})$ .

(2) Our claim now is that this embedding is multiplicative. But this is clear too, because if we denote by  $T \rightarrow M_T$  our correspondence, we have:

$$\begin{aligned} (M_{ST})_{ij} &= \langle STE_j, e_i \rangle \\ &= \left\langle S \sum_k \langle Te_j, e_k \rangle e_k, e_i \right\rangle \\ &= \sum_k \langle Se_k, e_i \rangle \langle Te_j, e_k \rangle \\ &= \sum_k (M_S)_{ik} (M_T)_{kj} \\ &= (M_S M_T)_{ij} \end{aligned}$$

(3) Finally, we must prove that the original operator  $T : H \rightarrow H$  can be recovered from its matrix  $M \in M_I(\mathbb{C})$  via the formula in the statement, namely  $Tx = Mx$ . But this latter formula holds for the vectors of the basis,  $x = e_j$ , because we have:

$$\begin{aligned} (Te_j)_i &= \langle Te_j, e_i \rangle \\ &= M_{ij} \\ &= (Me_j)_i \end{aligned}$$

Now by linearity we obtain from this that the formula  $Tx = Mx$  holds everywhere, on any vector  $x \in H$ , and this finishes the proof of the first assertion.

(4) In finite dimensions we obtain an isomorphism, because any matrix  $M \in M_N(\mathbb{C})$  determines an operator  $T : \mathbb{C}^N \rightarrow \mathbb{C}^N$ , according to the formula  $\langle Te_j, e_i \rangle = M_{ij}$ . In infinite dimensions, however, we do not have an isomorphism. For instance on  $H = l^2(\mathbb{N})$  the following matrix does not define an operator:

$$M = \begin{pmatrix} 1 & 1 & \dots \\ 1 & 1 & \dots \\ \vdots & \vdots & \end{pmatrix}$$

Indeed,  $T(e_1)$  should be the all-one vector, which is not square-summable.  $\square$

Observe that in the above picture the adjoint operation  $T \rightarrow T^*$  takes a very simple form, namely  $(A^*)_{ij} = \bar{A}_{ji}$ , at the level of the associated infinite matrices.

We will be interested here in the algebras of operators, rather than in the operators themselves. The axioms here, coming from Theorem 6.6, are as follows:

DEFINITION 6.8. *A  $C^*$ -algebra is a complex algebra with unit  $A$ , having:*

- (1) *A norm  $a \rightarrow \|a\|$ , making it a Banach algebra (the Cauchy sequences converge).*
- (2) *An involution  $a \rightarrow a^*$ , which satisfies  $\|aa^*\| = \|a\|^2$ , for any  $a \in A$ .*

As basic examples here, we have the usual matrix algebras  $M_N(\mathbb{C})$ , with the norm and the involution being the usual matrix norm and involution, given by:

$$\|A\| = \sup_{\|x\|=1} \|Ax\| \quad , \quad (A^*)_{ij} = \bar{A}_{ji}$$

Some other basic examples are the algebras  $L^\infty(X)$  of essentially bounded functions  $f : X \rightarrow \mathbb{C}$  on a measured space  $X$ , with the usual norm and involution, namely:

$$\|f\| = \sup_{x \in X} |f(x)| \quad , \quad f^*(x) = \overline{f(x)}$$

We can put these two basic classes of examples together, as follows:

PROPOSITION 6.9. *The random matrix algebras  $A = M_N(L^\infty(X))$  are  $C^*$ -algebras, with their usual norm and involution, given by:*

$$\|Z\| = \sup_{x \in X} \|Z_x\| \quad , \quad (Z^*)_{ij} = \bar{Z}_{ij}$$

*These algebras generalize both the algebras  $M_N(\mathbb{C})$ , and the algebras  $L^\infty(X)$ .*

PROOF. The fact that the  $C^*$ -algebra axioms are satisfied is clear from definitions. As for the last assertion, this follows by taking  $X = \{.\}$  and  $N = 1$ , respectively.  $\square$

Summarizing, the  $C^*$ -algebras are natural generalizations of the random matrix algebras. In what follows we will develop some general “noncommutative probability” theory for the  $C^*$ -algebras, then come back to the random matrix algebras later on.

In order to study the  $C^*$ -algebras, the key observation is that, due to Theorem 6.6, the algebra  $B(H)$  of bounded linear operators  $T : H \rightarrow H$  on a Hilbert space  $H$  is a  $C^*$ -algebra. More generally, any closed  $*$ -subalgebra  $A \subset B(H)$  is a  $C^*$ -algebra.

It is possible to prove that any  $C^*$ -algebra appears in this way,  $A \subset B(H)$ , and we will be back to this later. For the moment, let us just record the following elementary result, dealing with the random matrix case, that we are mainly interested in here:

THEOREM 6.10. *Any algebra of type  $L^\infty(X)$  is an operator algebra, as follows:*

$$L^\infty(X) \subset B(L^2(X)) \quad , \quad f \rightarrow (g \rightarrow fg)$$

*More generally, any random matrix algebra is an operator algebra, as follows,*

$$M_N(L^\infty(X)) \subset B(\mathbb{C}^N \otimes L^2(X))$$

*with the embedding being the above one, tensored with the identity.*

PROOF. We have two assertions to be proved, the idea being as follows:

(1) Given  $f \in L^\infty(X)$ , consider the following operator, acting on  $H = L^2(X)$ :

$$T_f(g) = fg$$

Observe that  $T_f$  is indeed well-defined, and bounded as well, because:

$$\begin{aligned} \|fg\|_2 &= \sqrt{\int_X |f(x)|^2 |g(x)|^2 d\mu(x)} \\ &\leq \|f\|_\infty \|g\|_2 \end{aligned}$$

The application  $f \rightarrow T_f$  being linear, involutive, continuous, and injective as well, we obtain in this way a  $C^*$ -algebra embedding  $L^\infty(X) \subset B(H)$ , as desired.

(2) Regarding the second assertion, this is best viewed in the following way:

$$\begin{aligned} M_N(L^\infty(X)) &= M_N(\mathbb{C}) \otimes L^\infty(X) \\ &\subset M_N(\mathbb{C}) \otimes B(L^2(X)) \\ &= B(\mathbb{C}^N \otimes L^2(X)) \end{aligned}$$

Here we have used (1), and some standard tensor product identifications.  $\square$

## 6b. Spectral theory

Our purpose in what follows is to develop the spectral theory of the  $C^*$ -algebras, and in particular that of the random matrix algebras  $A = M_N(L^\infty(X))$  that we are interested in, one of our objectives being that of talking about spectral measures, in the normal case, in analogy with what we know about the usual matrices, from chapter 5. There are many things to be done. Let us begin with a key definition, as follows:

DEFINITION 6.11. *The spectrum of an element  $a \in A$  is the set*

$$\sigma(a) = \left\{ \lambda \in \mathbb{C} \mid a - \lambda \notin A^{-1} \right\}$$

*where  $A^{-1} \subset A$  is the set of invertible elements.*

As a basic example, the spectrum of a usual matrix  $M \in M_N(\mathbb{C})$  is the collection of its eigenvalues. Also, the spectrum of a continuous function  $f \in C(X)$  is its image. In the case of the trivial algebra  $A = \mathbb{C}$ , the spectrum of an element is the element itself. As a first, basic result regarding spectra, we have:

PROPOSITION 6.12. *We have the following formula, valid for any  $a, b \in A$ :*

$$\sigma(ab) \cup \{0\} = \sigma(ba) \cup \{0\}$$

Moreover, there are examples where  $\sigma(ab) \neq \sigma(ba)$ .

PROOF. We first prove that  $1 \notin \sigma(ab) \implies 1 \notin \sigma(ba)$ . Assume indeed that the element  $1 - ab$  is invertible, with inverse  $c = (1 - ab)^{-1}$ . We have then:

$$abc = cab = c - 1$$

By using these formulae, we obtain, by some kind of miracle:

$$\begin{aligned} (1 + bca)(1 - ba) &= 1 + bca - ba - bcaba \\ &= 1 + bca - ba - bca + ba \\ &= 1 \end{aligned}$$

A similar computation shows that we have  $(1 - ba)(1 + bca) = 1$ . We conclude that the element  $1 - ba$  is invertible, with inverse  $1 + bca$ , which proves our claim.

By multiplying by scalars, we deduce from this that we have, for any  $\lambda \in \mathbb{C} - \{0\}$ :

$$\lambda \notin \sigma(ab) \implies \lambda \notin \sigma(ba)$$

But this leads to the conclusion in the statement, namely that we have:

$$\sigma(ab) \cup \{0\} = \sigma(ba) \cup \{0\}$$

Regarding now the last claim, let us first recall that for usual matrices  $a, b \in M_N(\mathbb{C})$  we have  $0 \in \sigma(ab) \iff 0 \in \sigma(ba)$ , because  $ab$  is invertible if and only if  $ba$  is. However, this latter fact fails for general operators on Hilbert spaces. As a basic example, we can take  $a, b$  to be the shift  $S(e_i) = e_{i+1}$  on the space  $l^2(\mathbb{N})$ , and its adjoint. Indeed, we have  $S^*S = 1$ , and  $SS^*$  being the projection onto  $e_0^\perp$ , it is not invertible.  $\square$

Given an element  $a \in A$ , and a rational function  $f = P/Q$  having poles outside  $\sigma(a)$ , we can construct the element  $f(a) = P(a)Q(a)^{-1}$ . For simplicity, we write:

$$f(a) = \frac{P(a)}{Q(a)}$$

With this convention, we have the following result:

THEOREM 6.13. *We have the “rational functional calculus” formula*

$$\sigma(f(a)) = f(\sigma(a))$$

*valid for any rational function  $f \in \mathbb{C}(X)$  having poles outside  $\sigma(a)$ .*

PROOF. We can prove this result in two steps, as follows:

(1) Assume first that we are in the usual polynomial case,  $f \in \mathbb{C}[X]$ . We pick a number  $\lambda \in \mathbb{C}$ , and we decompose the polynomial  $f - \lambda$ :

$$f(X) - \lambda = c(X - p_1) \dots (X - p_n)$$

We have then, as desired, the following computation:

$$\begin{aligned} \lambda \notin \sigma(f(a)) &\iff f(a) - \lambda \in A^{-1} \\ &\iff c(a - p_1) \dots (a - p_n) \in A^{-1} \\ &\iff a - p_1, \dots, a - p_n \in A^{-1} \\ &\iff p_1, \dots, p_n \notin \sigma(a) \\ &\iff \lambda \notin f(\sigma(a)) \end{aligned}$$

(2) In the general case now,  $f \in \mathbb{C}(X)$ , we pick  $\lambda \in \mathbb{C}$ , we write  $f = P/Q$ , and we set  $R = P - \lambda Q$ . By using (1) above, we obtain:

$$\begin{aligned} \lambda \in \sigma(f(a)) &\iff R(a) \notin A^{-1} \\ &\iff 0 \in \sigma(R(a)) \\ &\iff 0 \in R(\sigma(a)) \\ &\iff \exists \mu \in \sigma(a), R(\mu) = 0 \\ &\iff \lambda \in f(\sigma(a)) \end{aligned}$$

Thus, we have obtained the formula in the statement.  $\square$

Given an element  $a \in A$ , its spectral radius  $\rho(a)$  is the radius of the smallest disk centered at 0 containing  $\sigma(a)$ . With this convention, we have the following key result:

**THEOREM 6.14.** *Let  $A$  be a  $C^*$ -algebra.*

- (1) *The spectrum of a norm one element is in the unit disk.*
- (2) *The spectrum of a unitary element ( $a^* = a^{-1}$ ) is on the unit circle.*
- (3) *The spectrum of a self-adjoint element ( $a = a^*$ ) consists of real numbers.*
- (4) *The spectral radius of a normal element ( $aa^* = a^*a$ ) is equal to its norm.*

PROOF. We use the various results established above, as follows:

(1) This comes from the following basic formula, valid when  $\|a\| < 1$ :

$$\frac{1}{1 - a} = 1 + a + a^2 + \dots$$

(2) Assuming  $a^* = a^{-1}$ , we have the following computations:

$$\begin{aligned} \|a\| &= \sqrt{\|aa^*\|} = \sqrt{1} = 1 \\ \|a^{-1}\| &= \|a^*\| = \|a\| = 1 \end{aligned}$$

If we denote by  $D$  the unit disk, we obtain from this, by using (1):

$$\sigma(a) \subset D \quad , \quad \sigma(a^{-1}) \subset D$$

On the other hand, by using the function  $f(z) = z^{-1}$ , we have:

$$\sigma(a^{-1}) \subset D \implies \sigma(a) \subset D^{-1}$$

Thus we have  $\sigma(a) \subset D \cap D^{-1} = \mathbb{T}$ , as desired.

(3) This follows by using the result (2), just established above, and Theorem 6.13, with the following rational function, depending on a parameter  $t \in \mathbb{R}$ :

$$f(z) = \frac{z + it}{z - it}$$

Indeed, for  $t \gg 0$  the element  $f(a)$  is well-defined, and we have:

$$\left( \frac{a + it}{a - it} \right)^* = \frac{a - it}{a + it} = \left( \frac{a + it}{a - it} \right)^{-1}$$

Thus the element  $f(a)$  is a unitary, and by using (2) above its spectrum is contained in  $\mathbb{T}$ . We conclude that we have an inclusion as follows:

$$f(\sigma(a)) = \sigma(f(a)) \subset \mathbb{T}$$

Thus, we obtain an inclusion  $\sigma(a) \subset f^{-1}(\mathbb{T}) = \mathbb{R}$ , and we are done.

(4) We already know from (1) that we have the following inequality:

$$\rho(a) \leq \|a\|$$

For the converse, we fix an arbitrary number  $\rho > \rho(a)$ . We have then:

$$\begin{aligned} \int_{|z|=\rho} \frac{z^n}{z-a} dz &= \sum_{k=0}^{\infty} \left( \int_{|z|=\rho} z^{n-k-1} dz \right) a^k \\ &= \sum_{k=0}^{\infty} \delta_{n,k+1} a^k \\ &= a^{n-1} \end{aligned}$$

By applying the norm and taking  $n$ -th roots we obtain from this:

$$\rho \geq \lim_{n \rightarrow \infty} \|a^n\|^{1/n}$$

In the case  $a = a^*$  we have  $\|a^n\| = \|a\|^n$  for any exponent of the form  $n = 2^k$ , and by taking  $n$ -th roots we get  $\rho \geq \|a\|$ . But this gives the missing inequality, namely:

$$\rho(a) \geq \|a\|$$

In the general case  $aa^* = a^*a$  we have  $a^n(a^n)^* = (aa^*)^n$ . Thus  $\rho(a)^2 = \rho(aa^*)$ , and since the element  $aa^*$  is self-adjoint, we obtain  $\rho(aa^*) = \|a\|^2$ , and we are done.  $\square$

We are now in position of proving a key result, due to Gelfand, as follows:

**THEOREM 6.15.** *Any commutative  $C^*$ -algebra is the form*

$$A = C(X)$$

*with its “spectrum”  $X = \text{Spec}(A)$  appearing as the space of characters  $\chi : A \rightarrow \mathbb{C}$ .*

**PROOF.** Given a commutative  $C^*$ -algebra  $A$ , we can define  $X$  to be the set of characters  $\chi : A \rightarrow \mathbb{C}$ , with topology making continuous all evaluation maps  $ev_a : \chi \rightarrow \chi(a)$ . Then  $X$  is a compact space, and  $a \rightarrow ev_a$  is a morphism of algebras, as follows:

$$ev : A \rightarrow C(X)$$

(1) We first prove that  $ev$  is involutive. For this purpose we use the following formula, which is similar to the  $z = \text{Re}(z) + i\text{Im}(z)$  formula for usual complex numbers:

$$a = \frac{a + a^*}{2} + i \cdot \frac{a - a^*}{2i}$$

Thus it is enough to prove the equality  $ev_{a^*} = ev_a^*$  for self-adjoint elements  $a$ . But this is the same as proving that  $a = a^*$  implies that  $ev_a$  is a real function, which is in turn true, because  $ev_a(\chi) = \chi(a)$  is an element of the spectrum  $\sigma(a)$ , contained in  $\mathbb{R}$ .

(2) Since  $A$  is commutative, each element is normal, so  $ev$  is isometric, due to:

$$\|ev_a\| = \rho(a) = \|a\|$$

(3) It remains to prove that  $ev$  is surjective. But this follows from the Stone-Weierstrass theorem, because  $ev(A)$  is a closed subalgebra of  $C(X)$ , which separates the points.  $\square$

As a first consequence of the Gelfand theorem, we can extend Theorem 6.13, in the case of the normal elements ( $aa^* = a^*a$ ), in the following way:

**THEOREM 6.16.** *For any normal element  $a \in A$  we have an identification as follows:*

$$\langle a \rangle = C(\sigma(a))$$

*In addition, given a function  $f \in C(\sigma(a))$ , we can apply it to  $a$ , and we have*

$$\sigma(f(a)) = f(\sigma(a))$$

*which generalizes the previous rational calculus formula, in the normal case.*

**PROOF.** Since  $a$  is normal, the  $C^*$ -algebra  $\langle a \rangle$  that it generates is commutative, so if we denote by  $X$  the space of the characters  $\chi : \langle a \rangle \rightarrow \mathbb{C}$ , we have:

$$\langle a \rangle = C(X)$$

Now since the map  $X \rightarrow \sigma(a)$  given by evaluation at  $a$  is bijective, we obtain:

$$\langle a \rangle = C(\sigma(a))$$

Thus, we are dealing here with usual functions, and this gives all the assertions.  $\square$

In order to get now towards noncommutative probability, we first have to develop the theory of positive elements, and linear forms. First, we have the following result:

**PROPOSITION 6.17.** *For an element  $a \in A$ , the following are equivalent:*

- (1)  *$a$  is positive, in the sense that  $\sigma(a) \subset [0, \infty)$ .*
- (2)  *$a = b^2$ , for some  $b \in A$  satisfying  $b = b^*$ .*
- (3)  *$a = cc^*$ , for some  $c \in A$ .*

**PROOF.** This is something very standard, as follows:

(1)  $\implies$  (2) Observe first that  $\sigma(a) \subset \mathbb{R}$  implies  $a = a^*$ . Thus the algebra  $\langle a \rangle$  is commutative, and by using Theorem 6.16, we can set:

$$b = \sqrt{a}$$

(2)  $\implies$  (3) This is trivial, because we can simply set  $c = b$ .

(2)  $\implies$  (1) This is clear too, because we have:

$$\sigma(a) = \sigma(b^2) = \sigma(b)^2 \subset \mathbb{R}^2 = [0, \infty)$$

(3)  $\implies$  (1) We proceed by contradiction. By multiplying  $c$  by a suitable element of  $\langle cc^* \rangle$ , we are led to the existence of an element  $d \neq 0$  satisfying:

$$-dd^* \geq 0$$

By writing now  $d = x + iy$  with  $x = x^*, y = y^*$  we have:

$$dd^* + d^*d = 2(x^2 + y^2) \geq 0$$

We conclude that we have  $d^*d \geq 0$ . But this contradicts the elementary fact that  $\sigma(dd^*), \sigma(d^*d)$  must coincide outside  $\{0\}$ , coming from Proposition 6.12.  $\square$

We can talk as well about positive linear forms, as follows:

**DEFINITION 6.18.** *Consider a linear map  $\varphi : A \rightarrow \mathbb{C}$ .*

- (1)  *$\varphi$  is called positive when  $a \geq 0 \implies \varphi(a) \geq 0$ .*
- (2)  *$\varphi$  is called faithful and positive when  $a \geq 0, a \neq 0 \implies \varphi(a) > 0$ .*

In the commutative case,  $A = C(X)$ , the positive linear forms appear as follows, with  $\mu$  being positive, and strictly positive if we want  $\varphi$  to be faithful and positive:

$$\varphi(f) = \int_X f(x) d\mu(x)$$

In general, the positive linear forms can be thought of as being integration functionals with respect to some underlying “positive measures”. We will be back to this.

As a final abstract topic, let us review now another fundamental result, which states that any  $C^*$ -algebra appears as an algebra of operators,  $A \subset B(H)$ , over some Hilbert space  $H$ . In the commutative case, the precise statement is as follows:

PROPOSITION 6.19. *Let  $A$  be a commutative  $C^*$ -algebra, write  $A = C(X)$ , with  $X$  being a compact space, and let  $\mu$  be a positive measure on  $X$ . We have then an embedding*

$$A \subset B(H)$$

where  $H = L^2(X)$ , with  $f \in A$  corresponding to the operator  $g \rightarrow fg$ .

PROOF. Given  $f \in C(X)$ , consider the following operator, on  $H = L^2(X)$ :

$$T_f(g) = fg$$

Observe that  $T_f$  is indeed well-defined, and bounded as well, because we have:

$$\begin{aligned} \|fg\|_2 &= \sqrt{\int_X |f(x)|^2 |g(x)|^2 d\mu(x)} \\ &\leq \|f\|_\infty \|g\|_2 \end{aligned}$$

The application  $f \rightarrow T_f$  being linear, involutive, continuous, and injective as well, we obtain in this way a  $C^*$ -algebra embedding  $A \subset B(H)$ , as claimed.  $\square$

In general, the idea will be that of extending the above construction. In order to do so, we will use a functional analysis trick, coming from the Riesz theorem, which amounts in replacing the positive measures  $\mu$  with the corresponding integration functionals. Thus, we are led to the forms in Definition 6.18, that we can use as follows:

PROPOSITION 6.20. *Let  $\varphi : A \rightarrow \mathbb{C}$  be a positive linear form.*

- (1)  $\langle a, b \rangle = \varphi(ab^*)$  defines a generalized scalar product on  $A$ .
- (2) By separating and completing we obtain a Hilbert space  $H$ .
- (3)  $\pi(a) : b \rightarrow ab$  defines a representation  $\pi : A \rightarrow B(H)$ .
- (4) If  $\varphi$  is faithful in the above sense, then  $\pi$  is faithful.

PROOF. Almost everything here is straightforward. To be more precise, (1) is clear from definitions, and from Proposition 6.17, then (2) is a standard procedure, which works for any scalar product, (3) comes from some standard algebraic computations, and (4) follows indeed from  $a \neq 0 \implies \pi(aa^*) \neq 0 \implies \pi(a) \neq 0$ .  $\square$

In order to establish the embedding theorem, it remains to prove that any  $C^*$ -algebra has a faithful and positive linear form  $\varphi : A \rightarrow \mathbb{C}$ . This can be done as follows:

PROPOSITION 6.21. *Let  $A$  be a  $C^*$ -algebra.*

- (1) Any positive linear form  $\varphi : A \rightarrow \mathbb{C}$  is continuous.
- (2) A linear form  $\varphi$  is positive iff there is a norm one  $h \in A_+$  such that  $\|\varphi\| = \varphi(h)$ .
- (3) For any  $a \in A$  there exists a positive norm one form  $\varphi$  such that  $\varphi(aa^*) = \|a\|^2$ .
- (4) If  $A$  is separable there is a faithful positive form  $\varphi : A \rightarrow \mathbb{C}$ .

PROOF. The proof here, which is quite technical, inspired from the existence proof for the probability measures on abstract compact spaces, goes as follows:

(1) This follows from Proposition 6.20, via the following inequality:

$$|\varphi(a)| \leq \|\pi(a)\|\varphi(1) \leq \|a\|\varphi(1)$$

(2) In one sense we can take  $h = 1$ . Conversely, let  $a \in A_+$ ,  $\|a\| \leq 1$ . We have:

$$\begin{aligned} |\varphi(h) - \varphi(a)| &\leq \|\varphi\| \cdot \|h - a\| \\ &\leq \varphi(h)1 \\ &= \varphi(h) \end{aligned}$$

Thus we have  $\operatorname{Re}(\varphi(a)) \geq 0$ , and it remains to prove that we have:

$$a = a^* \implies \varphi(a) \in \mathbb{R}$$

By using  $1 - h \geq 0$  we can apply the above to  $a = 1 - h$ , and we obtain:

$$\operatorname{Re}(\varphi(1 - h)) \geq 0$$

We conclude that  $\operatorname{Re}(\varphi(1)) \geq \operatorname{Re}(\varphi(h)) = \|\varphi\|$ , so  $\varphi(1) = \|\varphi\|$ . Thus, we can assume  $h = 1$ . Now observe that for any self-adjoint element  $a$ , and  $t \in \mathbb{R}$ , we have:

$$\begin{aligned} |\varphi(1 + ita)|^2 &\leq \|\varphi\|^2 \cdot \|1 + ita\|^2 \\ &= \varphi(1)^2 \|1 + t^2 a^2\| \\ &\leq \varphi(1)^2 (1 + t^2 \|a\|^2) \end{aligned}$$

On the other hand, with  $\varphi(a) = x + iy$ , we have the following estimate:

$$\begin{aligned} |\varphi(1 + ita)| &= |\varphi(1) - ty + itx| \\ &\geq (\varphi(1) - ty)^2 \end{aligned}$$

We therefore obtain that for any  $t \in \mathbb{R}$  we have the following estimate:

$$\varphi(1)^2 (1 + t^2 \|a\|^2) \geq (\varphi(1) - ty)^2$$

Thus we have  $y = 0$ , and this finishes the proof of our remaining claim.

(3) Consider the linear subspace of  $A$  spanned by the element  $aa^*$ . We can define on this subspace a linear form by the following formula:

$$\varphi(\lambda aa^*) = \lambda \|a\|^2$$

This linear form has norm one, and by using Hahn-Banach we get a norm one extension to the whole algebra  $A$ . The positivity of  $\varphi$  follows from (2).

(4) Let  $(a_n)$  be a dense sequence inside our algebra  $A$ . For any  $n \in \mathbb{N}$  we can construct as in (3) above a positive form satisfying the following condition:

$$\varphi_n(a_n a_n^*) = \|a_n\|^2$$

We can then define a linear form  $\varphi$  in the following way:

$$\varphi = \sum_{n=1}^{\infty} \frac{\varphi_n}{2^n}$$

Let  $a \in A$  be a nonzero element. Pick  $a_n$  close to  $a$  and consider the pair  $(H, \pi)$  associated to the pair  $(A, \varphi_n)$ , as in Proposition 6.20. We have then:

$$\begin{aligned} \varphi_n(aa^*) &= \|\pi(a)1\|^2 \\ &\geq \|\pi(a_n)1\|^2 - \|a - a_n\|^2 \\ &= \|a_n\|^2 - \|a - a_n\|^2 \\ &> 0 \end{aligned}$$

Thus  $\varphi_n(aa^*) > 0$ , and it follows that we have  $\varphi(aa^*) > 0$ , and we are done.  $\square$

With these ingredients in hand, we can now state and prove:

**THEOREM 6.22.** *Let  $A$  be a  $C^*$ -algebra.*

- (1)  *$A$  appears as a closed  $*$ -subalgebra  $A \subset B(H)$ , for some Hilbert space  $H$ .*
- (2) *When  $A$  is separable (usually the case),  $H$  can be chosen to be separable.*
- (3) *When  $A$  is finite dimensional,  $H$  can be chosen to be finite dimensional.*

**PROOF.** This result, due to Gelfand, Naimark and Segal, and called GNS embedding theorem, follows indeed by combining the general left regular representation construction from Proposition 6.20 with the existence result from Proposition 6.21.  $\square$

### 6c. Spectral measures

With the above operator algebra ingredients in hand, let us discuss now noncommutative probability theory. We first have:

**DEFINITION 6.23.** *Let  $A$  be a  $C^*$ -algebra, given with a positive trace  $tr : A \rightarrow \mathbb{C}$ .*

- (1) *The elements  $a \in A$  are called random variables.*
- (2) *The moments of such a variable are the numbers  $M_k(a) = tr(a^k)$ .*
- (3) *The law of such a variable is the functional  $\mu_a : P \rightarrow tr(P(a))$ .*

Here the exponent  $k = \circ \bullet \bullet \circ \dots$  is by definition a colored integer, and the powers  $a^k$  are defined by the following formulae, and multiplicativity:

$$a^\emptyset = 1 \quad , \quad a^\circ = a \quad , \quad a^\bullet = a^*$$

As for the polynomial  $P$ , this is a noncommuting  $*$ -polynomial in one variable:

$$P \in \mathbb{C} \langle X, X^* \rangle$$

Observe that the law is uniquely determined by the moments, because we have:

$$P(X) = \sum_k \lambda_k X^k \implies \mu_a(P) = \sum_k \lambda_k M_k(a)$$

At the level of the general theory, we have the following key result, extending the various results about the self-adjoint and normal matrices, from chapter 5:

**THEOREM 6.24.** *Let  $A$  be a  $C^*$ -algebra, with a trace  $tr$ , and consider an element  $a \in A$  which is normal, in the sense that  $aa^* = a^*a$ .*

- (1)  $\mu_a$  is a complex probability measure, satisfying  $\text{supp}(\mu_a) \subset \sigma(a)$ .
- (2) In the self-adjoint case,  $a = a^*$ , this measure  $\mu_a$  is real.
- (3) Assuming that  $tr$  is faithful, we have  $\text{supp}(\mu_a) = \sigma(a)$ .

Moreover, the converse of (1) holds, in the sense that if the law of an element  $a \in A$  is a complex probability measure, then this element must be normal,  $aa^* = a^*a$ .

**PROOF.** This is something very standard, that we already know for the usual complex matrices, from chapter 5, and whose proof in general is quite similar, as follows:

(1) In the normal case,  $aa^* = a^*a$ , the Gelfand theorem, or rather the subsequent continuous functional calculus theorem, tells us that we have:

$$\langle a \rangle = C(\sigma(a))$$

Thus the functional  $f(a) \rightarrow tr(f(a))$  can be regarded as an integration functional on the algebra  $C(\sigma(a))$ , and by the Riesz theorem, this latter functional must come from a probability measure  $\mu$  on the spectrum  $\sigma(a)$ , in the sense that we must have:

$$tr(f(a)) = \int_{\sigma(a)} f(z) d\mu(z)$$

We are therefore led to the conclusions in the statement, with the uniqueness assertion coming from the fact that the elements  $a^k$ , taken as usual with respect to colored integer exponents,  $k = \circ \bullet \bullet \circ \dots$ , generate the whole  $C^*$ -algebra  $C(\sigma(a))$ .

- (2) This is something which is clear from definitions.
- (3) Once again, this is something which is clear from definitions.

Finally, regarding the last assertion, assume that we are in the non-normal case,  $aa^* \neq a^*a$ . We can use the same positivity trick as for the usual matrices, namely:

$$\begin{aligned} aa^* - a^*a \neq 0 &\implies (aa^* - a^*a)^2 > 0 \\ &\implies aa^*aa^* - aa^*a^*a - a^*aaa^* + a^*aa^*a > 0 \\ &\implies tr(aa^*aa^* - aa^*a^*a - a^*aaa^* + a^*aa^*a) > 0 \\ &\implies tr(aa^*aa^* + a^*aa^*a) > tr(aa^*a^*a + a^*aaa^*) \\ &\implies tr(aa^*aa^*) > tr(aaa^*a^*) \end{aligned}$$

Now assuming that  $a$  has a law  $\mu \in \mathcal{P}(\mathbb{C})$ , the above numbers would have to both appear by integrating  $|z|^2$  with respect to  $\mu$ , which is contradictory, as desired.  $\square$

We will heavily use Theorem 6.24 in chapters 9-12 below and afterwards, when doing free probability, the framework there being precisely the one that we will need. As a first concrete application now, by getting back to the random matrices, and to the various questions raised in the beginning of chapter 5, we have:

THEOREM 6.25. *Given a random matrix  $Z \in M_N(L^\infty(X))$  which is normal,*

$$ZZ^* = Z^*Z$$

*its law, which is by definition the following abstract functional,*

$$\mu : \mathbb{C} \langle X, X^* \rangle \rightarrow \mathbb{C} \quad , \quad P \rightarrow \frac{1}{N} \int_X \text{tr}(P(Z))$$

*when restricted to the usual polynomials in two variables,*

$$\mu : \mathbb{C}[X, X^*] \rightarrow \mathbb{C} \quad , \quad P \rightarrow \frac{1}{N} \int_X \text{tr}(P(Z))$$

*must come from a probability measure on the spectrum  $\sigma(Z) \subset \mathbb{C}$ , as follows:*

$$\mu(P) = \int_{\sigma(Z)} P(x) d\mu(x)$$

*We agree to use the symbol  $\mu$  for all these notions.*

PROOF. This follows indeed from what we know from Theorem 6.24, applied to the normal element  $a = Z$ , belonging to the  $C^*$ -algebra  $A = M_N(L^\infty(X))$ .  $\square$

Let us record as well, for future reference, the statement in the self-adjoint case:

THEOREM 6.26. *Given a random matrix  $Z \in M_N(L^\infty(X))$  which is self-adjoint,*

$$Z = Z^*$$

*its law, which is by definition the following abstract functional,*

$$\mu : \mathbb{C} \langle X \rangle \rightarrow \mathbb{C} \quad , \quad P \rightarrow \frac{1}{N} \int_X \text{tr}(P(Z))$$

*when restricted to the usual polynomials in one variable,*

$$\mu : \mathbb{C}[X] \rightarrow \mathbb{C} \quad , \quad P \rightarrow \frac{1}{N} \int_X \text{tr}(P(Z))$$

*must come from a probability measure on the spectrum  $\sigma(Z) \subset \mathbb{R}$ , as:*

$$\mu(P) = \int_{\sigma(Z)} P(x) d\mu(x)$$

*We agree to use the symbol  $\mu$  for all these notions.*

PROOF. As before, this follows from what we know from Theorem 6.24, applied to the self-adjoint element  $a = Z$ , belonging to the  $C^*$ -algebra  $A = M_N(L^\infty(X))$ .  $\square$

### 6d. Diagonalization

As a final topic for this chapter, let us go back to operator theory, and explain how the normal operators can be diagonalized. As a first result here, we can improve the rational functional calculus formula, in the case of the normal operators, as follows:

**THEOREM 6.27.** *Given  $T \in B(H)$  normal, we have a unique morphism of algebras as follows, with the exponent standing for “having poles outside  $\sigma(T)$ ”,*

$$\mathbb{C}(X)^T \rightarrow B(H) \quad , \quad f \rightarrow f(T)$$

*having the properties  $\|f(T)\| = \|f|_{\sigma(T)}\|$  and  $\sigma(f(T)) = f(\sigma(T))$ .*

**PROOF.** This is an improvement of what comes out from the rational calculus theorem, in the normal case. To be more precise, the element  $f(T)$  being normal, we can apply to it the spectral radius formula for normal elements, and we obtain:

$$\begin{aligned} \|f(T)\| &= \rho(f(T)) \\ &= \sup_{\lambda \in \sigma(f(T))} |\lambda| \\ &= \sup_{\lambda \in f(\sigma(T))} |\lambda| \\ &= \|f|_{\sigma(T)}\| \end{aligned}$$

Thus, we are led to the conclusions in the statement. □

Importantly now, in the case of normal elements we have some new functional calculus results, using more general functions than those used before. First, we have:

**THEOREM 6.28.** *Given a normal operator  $T \in B(H)$ , we have a unique continuous morphism of algebras*

$$C(\sigma(T)) \rightarrow B(H) \quad , \quad f \rightarrow f(T)$$

*having the properties  $\|f(T)\| = \|f\|$  and  $\sigma(f(T)) = f(\sigma(T))$ .*

**PROOF.** The idea here is to suitably “complete” the morphism in Theorem 6.27. To be more precise, the morphism there is continuous, and in fact isometric, and so by Stone-Weierstrass, we have a unique continuous extension of this morphism, as in the statement. Regarding now  $\sigma(f(T)) = f(\sigma(T))$ , the proof here goes as follows:

“ $\subset$ ” Given a continuous function  $f \in C(\sigma(T))$ , we must prove that we have:

$$\lambda \notin f(\sigma(T)) \implies \lambda \notin \sigma(f(T))$$

For this purpose, consider the following continuous function, which is well-defined:

$$\frac{1}{f - \lambda} \in C(\sigma(T))$$

We can therefore apply this function to our operator  $T$ , and we obtain:

$$\left(\frac{1}{f - \lambda}\right) T = \frac{1}{f(T) - \lambda}$$

In particular  $f(T) - \lambda$  is invertible, so we get  $\lambda \notin \sigma(f(T))$ , as desired.

“ $\supset$ ” Given a continuous function  $f \in C(\sigma(T))$ , we must prove that we have:

$$\mu \in \sigma(T) \implies f(\mu) \in \sigma(f(T))$$

For this purpose, we approximate our function by polynomials,  $f_n \rightarrow f$ , and we examine the following convergence, which follows from the convergence  $f_n \rightarrow f$ :

$$f_n(T) - f_n(\mu) \rightarrow f(T) - f(\mu)$$

We know from standard polynomial functional calculus that we have:

$$f_n(\mu) \in f_n(\sigma(T)) = \sigma(f_n(T))$$

Thus, the operators  $f_n(T) - f_n(\mu)$  are not invertible. On the other hand, we know that the set formed by the invertible operators is open, so its complement is closed. Thus the limit  $f(T) - f(\mu)$  is not invertible either, so  $f(\mu) \in \sigma(f(T))$ , as desired.  $\square$

As a second result now, along the same lines, we can further extend Theorem 6.28 into a measurable functional calculus theorem, as follows:

**THEOREM 6.29.** *Given  $T \in B(H)$  normal, we have a unique continuous morphism of algebras as follows, with  $L^\infty$  standing for abstract measurable functions*

$$L^\infty(\sigma(T)) \rightarrow B(H) \quad , \quad f \rightarrow f(T)$$

*having the properties  $\|f(T)\| = \|f\|$  and  $\sigma(f(T)) = f(\sigma(T))$ .*

**PROOF.** As before, the idea will be that of “completing” what we have. For this purpose, we use a polarization trick. Given  $x \in H$ , consider the following functional:

$$C(\sigma(T)) \rightarrow \mathbb{C} \quad , \quad f \rightarrow \langle f(T)x, x \rangle$$

By using the Riesz theorem, this functional must be the integration with respect to a certain measure  $\mu$  on the space  $\sigma(T)$ . Thus, we have a formula as follows:

$$\langle f(T)x, x \rangle = \int_{\sigma(T)} f(z) d\mu(z)$$

Now with this formula in hand, we can extend the continuous calculus into an abstract measurable calculus, and we are led to the conclusions in the statement.  $\square$

Let us diagonalize now the normal operators. We will do this in 3 steps, first for the self-adjoint operators, then for the families of commuting self-adjoint operators, and finally for the general normal operators, by using a  $T = \text{Re}(T) + i\text{Im}(T)$  trick. For the self-adjoint operators, the statement and proof are as follows:

THEOREM 6.30. *Any self-adjoint operator  $T \in B(H)$  can be diagonalized,*

$$T = U^* M_f U$$

*with  $U : H \rightarrow L^2(X)$  being a unitary operator from  $H$  to a certain  $L^2$  space associated to  $T$ , with  $f : X \rightarrow \mathbb{R}$  being a certain function, once again associated to  $T$ , and with*

$$M_f(g) = fg$$

*being the usual multiplication operator by  $f$ , on the Hilbert space  $L^2(X)$ .*

PROOF. We first prove the result in the case where our operator  $T$  has a cyclic vector  $\xi \in H$ , with this meaning that the following condition holds:

$$\overline{\text{span} \left( T^k \xi \mid n \in \mathbb{N} \right)} = H$$

For this purpose, let us go back to the proof of Theorem 6.29. We will use the following formula from there, with  $\mu$  being the measure on  $X = \sigma(T)$  associated to  $\xi$ :

$$\langle g(T)\xi, \xi \rangle = \int_{\sigma(T)} g(z) d\mu(z)$$

Our claim is that we can define a unitary  $U : H \rightarrow L^2(X)$ , first on the dense part spanned by the vectors  $T^k \xi$ , by the following formula, and by continuity:

$$U[g(T)\xi] = g$$

Indeed, the following computation shows that  $U$  is well-defined, and isometric:

$$\begin{aligned} \|g(T)\xi\|^2 &= \langle |g|^2(T)\xi, \xi \rangle \\ &= \int_{\sigma(T)} |g(z)|^2 d\mu(z) \\ &= \|g\|_2^2 \end{aligned}$$

With this in hand, we can then extend  $U$  by continuity into a unitary  $U : H \rightarrow L^2(X)$ , as claimed. Now observe that we have the following formula:

$$\begin{aligned} UTU^*g &= U[Tg(T)\xi] \\ &= U[(zg)(T)\xi] \\ &= zg \end{aligned}$$

Thus our result is proved in the present case. In general now, by using the Zorn lemma we can write  $H = \bigoplus_i H_i$ , with each  $H_i$  being invariant under  $T$ , and having a cyclic vector  $\xi_i$ . Thus we can make a direct sum of the diagonalization results obtained above, for each of the restrictions  $T|_{H_i}$ , and we obtain the result.  $\square$

We have the following technical generalization of the above result:

**THEOREM 6.31.** *Any family of commuting self-adjoint operators  $T_i \in B(H)$  can be jointly diagonalized,*

$$T_i = U^* M_{f_i} U$$

*with  $U : H \rightarrow L^2(X)$  being a unitary operator from  $H$  to a certain  $L^2$  space associated to  $\{T_i\}$ , with  $f_i : X \rightarrow \mathbb{R}$  being certain functions, once again associated to  $T_i$ , and with*

$$M_{f_i}(g) = f_i g$$

*being the usual multiplication operator by  $f_i$ , on the Hilbert space  $L^2(X)$ .*

**PROOF.** This is similar to the proof of Theorem 6.30, by suitably modifying the measurable calculus formula, and the measure  $\mu$  itself, as to have this formula working for all the operators  $T_i$ . With this modification done, everything extends.  $\square$

We can now discuss the case of arbitrary normal operators, as follows:

**THEOREM 6.32.** *Any normal operator  $T \in B(H)$  can be diagonalized,*

$$T = U^* M_f U$$

*with  $U : H \rightarrow L^2(X)$  being a unitary operator from  $H$  to a certain  $L^2$  space associated to  $T$ , with  $f : X \rightarrow \mathbb{C}$  being a certain function, once again associated to  $T$ , and with*

$$M_f(g) = fg$$

*being the usual multiplication operator by  $f$ , on the Hilbert space  $L^2(X)$ .*

**PROOF.** This follows by applying Theorem 6.31 to the real and imaginary parts of  $T$ , which are constructed as follows, and which are self-adjoint, and commuting:

$$T = \frac{T + T^*}{2} + i \cdot \frac{T - T^*}{2i}$$

Alternatively, we can use methods similar to those that we used in chapter 5, involving the special relation between  $T$  and the operator  $TT^*$ , which is self-adjoint.  $\square$

With the above diagonalization results in hand, we can now “fix” the continuous and measurable functional calculus theorems, with a key complement, as follows:

**THEOREM 6.33.** *Given a normal operator  $T \in B(H)$ , the following hold, for both the functional calculus and the measurable calculus morphisms:*

- (1) *These morphisms are  $*$ -morphisms.*
- (2) *The function  $\bar{z}$  gets mapped to  $T^*$ .*
- (3) *The functions  $\operatorname{Re}(z), \operatorname{Im}(z)$  get mapped to  $\operatorname{Re}(T), \operatorname{Im}(T)$ .*
- (4) *The function  $|z|^2$  gets mapped to  $TT^* = T^*T$ .*
- (5) *If  $f$  is real, then  $f(T)$  is self-adjoint.*

**PROOF.** These assertions are more or less equivalent, with (1) being the main one, which obviously implies everything else. But this assertion (1) follows from the diagonalization result for normal operators, from Theorem 6.32.  $\square$

To summarize now, this was for the basic theory of operators and operator algebras, that we will regularly need, in what follows. We should mention here that, although this will be more than enough for investigating the basic properties of the random matrices, as we will do in chapters 7-8 below, all this is the bare minimum, in order to talk about free probability, and advanced random matrix theory. The point indeed is that a systematic discussion of these topics, in the spirit of Voiculescu's book [90] and of many research papers, both on random matrices and free probability, ideally involves the notion of von Neumann algebra, which is something quite complicated, and that we have chosen to talk about here only much later, in chapters 15-16 below, after developing all the basics by using  $C^*$ -algebra theory, which is quite elementary, as explained in the above.

Thus, we are at a point in this book where some more learning, be that a bit informal, would be ideal. A good book on operator algebras, dealing with both  $C^*$ -algebras and von Neumann algebras, in a concise and pragmatic way, and that we would like to recommend here, is the one by Blackadar [27]. And for more, you can go with Connes [37].

### 6e. Exercises

In analogy with linear algebra, operator theory is a wide area of mathematics, and there are many interesting operators, and exercises about them. We first have:

EXERCISE 6.34. *Find an explicit orthonormal basis for the Hilbert space*

$$H = L^2[0, 1]$$

*by starting with the algebraic basis  $f_n = x^n$  with  $n \in \mathbb{N}$ , and applying Gram-Schmidt.*

This is actually quite non-trivial, and in case you're stuck with complicated computations, better look it up, and then write an account of what you found. And also, don't be surprised if you get in this way into probability theory, the topic of this book.

EXERCISE 6.35. *Clarify whether the linear operators*

$$T : H \rightarrow H$$

*are automatically bounded, or not.*

To be more precise, this was an annoying topic, avoided in the above, but we can now reveal the truth, with the comment that the answer in general is “no”, due to somewhat opaque reasons, coming from logic. Time to have this understood.

EXERCISE 6.36. *Prove that for the usual matrices  $A, B \in M_N(\mathbb{C})$  we have*

$$\sigma^+(AB) = \sigma^+(BA)$$

*where  $\sigma^+$  denotes the set of eigenvalues, taken with multiplicities.*

As a remark, we have seen in the above that  $\sigma(AB) = \sigma(BA)$  holds outside  $\{0\}$ , and the equality on  $\{0\}$  holds as well, because  $AB$  is invertible if and only if  $BA$  is invertible. However, in what regards the eigenvalues taken with multiplicities, things are more tricky here, and the answer should be somewhere inside your linear algebra knowledge.

EXERCISE 6.37. *Clarify, with examples and counterexamples, the relation between the eigenvalues of an operator  $T \in B(H)$ , and its spectrum  $\sigma(T) \subset \mathbb{C}$ .*

Here, as usual, the counterexamples could only come from the shift operator  $S$ , on the space  $H = l^2(\mathbb{N})$ . As a bonus exercise here, try computing the spectrum of  $S$ .

EXERCISE 6.38. *Find and write down a proof for the spectral theorem for normal operators in the spirit of the proof for normal matrices from chapter 5, and vice versa.*

To be more precise, the problem is that the proof of the spectral theorem for the usual matrices, from chapter 5, was using a certain kind of trick, while the proof of the spectral theorem for the arbitrary operators, given in this chapter, was using some other kind of trick. Thus, for full understanding all this, working out more proofs, both for the usual matrices and for the arbitrary operators, is a useful thing.

## CHAPTER 7

### Wigner matrices

#### 7a. Gaussian matrices

We have now all the needed ingredients for launching some explicit random matrix computations. Our goal will be that of computing the asymptotic moments, and then the asymptotic laws, with  $N \rightarrow \infty$ , for the main classes of large random matrices. Let us begin by recalling the precise classes of matrices that we are interested in. First we have the complex Gaussian matrices, which are constructed as follows:

DEFINITION 7.1. *A complex Gaussian matrix is a random matrix of type*

$$Z \in M_N(L^\infty(X))$$

*which has i.i.d. centered complex normal entries.*

Here we use the notion of complex normal variable, introduced and studied in chapter 1. To be more precise, the complex Gaussian law of parameter  $t > 0$  is by definition the following law, with  $a, b$  being independent, each following the law  $g_t$ :

$$G_t = \text{law} \left( \frac{1}{\sqrt{2}}(a + ib) \right)$$

With this notion in hand, the assumption in the above definition is that all matrix entries  $Z_{ij}$  are independent, and follow this law  $G_t$ , for a fixed value of  $t > 0$ . We will see that the above matrices have an interesting, and “central” combinatorics, among all kinds of random matrices, with the study of the other random matrices being usually obtained as a modification of the study of the Gaussian matrices.

As a somewhat surprising remark, using real normal variables in Definition 7.1, instead of the complex ones appearing there, leads nowhere. The correct real versions of the Gaussian matrices are the Wigner random matrices, constructed as follows:

DEFINITION 7.2. *A Wigner matrix is a random matrix of type*

$$Z \in M_N(L^\infty(X))$$

*which has i.i.d. centered complex normal entries, up to the constraint  $Z = Z^*$ .*

This definition is something a bit compacted, and to be more precise here, a Wigner matrix is by definition a random matrix as follows, with the diagonal entries being real

normal variables,  $a_i \sim g_t$ , for some  $t > 0$ , the upper diagonal entries being complex normal variables,  $b_{ij} \sim G_t$ , the lower diagonal entries being the conjugates of the upper diagonal entries, as indicated, and with all the variables  $a_i, b_{ij}$  being independent:

$$Z = \begin{pmatrix} a_1 & b_{12} & \dots & \dots & b_{1N} \\ \bar{b}_{12} & a_2 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & a_{N-1} & b_{N-1,N} \\ \bar{b}_{1N} & \dots & \dots & \bar{b}_{N-1,N} & a_N \end{pmatrix}$$

As a comment here, for many concrete applications the Wigner matrices are in fact the central objects in random matrix theory, and in particular, they are often more important than the Gaussian matrices. In fact, these are the random matrices which were first considered and investigated, a long time ago, by Wigner himself [97].

However, as we will soon discover, the Gaussian matrices are somehow more fundamental than the Wigner matrices, at least from an abstract point of view, and this will be the point of view that we will follow here, with the Gaussian matrices coming first.

Finally, we will be interested as well in the complex Wishart matrices, which are the positive versions of the above random matrices, constructed as follows:

DEFINITION 7.3. *A complex Wishart matrix is a random matrix of type*

$$Z = YY^* \in M_N(L^\infty(X))$$

*with  $Y$  being a complex Gaussian matrix.*

As before with the Gaussian and Wigner matrices, there are many possible comments that can be made here, of technical or historical nature. First, using real Gaussian variables instead of complex ones leads to a less interesting combinatorics. Also, these matrices were introduced and studied by Marchenko and Pastur not long after Wigner, in [64], and so historically came second. Finally, in what regards their combinatorics and applications, these matrices quite often come first, before both the Gaussian and the Wigner ones, with all this being of course a matter of knowledge and taste.

Summarizing, we have three main types of random matrices, which can be thought of as being “complex”, “real” and “positive”, and that we will study in what follows, in this precise order, with this order being the one that fits us best here. Let us also mention that there are many other interesting classes of random matrices, which are more specialized, usually appearing as modifications of the above. More on these later.

In order to compute the asymptotic laws of the Gaussian, Wigner and Wishart matrices, we use the moment method. Given a colored integer  $k = \circ \bullet \bullet \circ \dots$ , we say that

a pairing  $\pi \in P_2(k)$  is matching when it pairs  $\circ - \bullet$  symbols. With this convention, we have the following result, which will be our main tool for computing moments:

**THEOREM 7.4** (Wick formula). *Given independent variables  $X_i$ , each following the complex normal law  $G_t$ , with  $t > 0$  being a fixed parameter, we have the formula*

$$\mathbb{E}(X_{i_1}^{k_1} \dots X_{i_s}^{k_s}) = t^{s/2} \# \left\{ \pi \in \mathcal{P}_2(k) \mid \pi \leq \ker(i) \right\}$$

where  $k = k_1 \dots k_s$  and  $i = i_1 \dots i_s$ , for the joint moments of these variables.

**PROOF.** This is something that we know from chapter 1, the idea being as follows:

(1) In the case where we have a single complex normal variable  $X$ , which amounts in taking  $X_i = X$  for any  $i$  in the formula in the statement, what we have to compute are the moments of  $X$ , with respect to colored integer exponents  $k = \circ \bullet \bullet \circ \dots$ , and the formula in the statement tells us that these moments must be:

$$\mathbb{E}(X^k) = t^{|k|/2} |\mathcal{P}_2(k)|$$

But this is something that we know from chapter 1, the idea being that at  $t = 1$  this follows by doing some combinatorics and calculus, in analogy with the combinatorics and calculus from the real case, where the moment formula is identical, save for the matching pairings  $\mathcal{P}_2$  being replaced by the usual pairings  $P_2$ , and then that the general case  $t > 0$  follows from this, by rescaling. Thus, we are done with this case.

(2) In general now, when expanding the product  $X_{i_1}^{k_1} \dots X_{i_s}^{k_s}$  and rearranging the terms, we are left with doing a number of computations as in (1), and then making the product of the expectations that we found. But this amounts in counting the partitions in the statement, with the condition  $\pi \leq \ker(i)$  there standing for the fact that we are doing the various type (1) computations independently, and then making the product.  $\square$

Now by getting back to the Gaussian matrices, we have the following result:

**THEOREM 7.5.** *Given a sequence of Gaussian random matrices*

$$Z_N \in M_N(L^\infty(X))$$

having independent  $G_t$  variables as entries, for some fixed  $t > 0$ , we have

$$M_k \left( \frac{Z_N}{\sqrt{N}} \right) \simeq t^{|k|/2} |\mathcal{NC}_2(k)|$$

for any colored integer  $k = \circ \bullet \bullet \circ \dots$ , in the  $N \rightarrow \infty$  limit.

**PROOF.** This is something standard, which can be done as follows:

(1) We fix  $N \in \mathbb{N}$ , and we let  $Z = Z_N$ . Let us first compute the trace of  $Z^k$ . With  $k = k_1 \dots k_s$ , and with the convention  $(ij)^\circ = ij$ ,  $(ij)^\bullet = ji$ , we have:

$$\begin{aligned} \text{Tr}(Z^k) &= \text{Tr}(Z^{k_1} \dots Z^{k_s}) \\ &= \sum_{i_1=1}^N \dots \sum_{i_s=1}^N (Z^{k_1})_{i_1 i_2} (Z^{k_2})_{i_2 i_3} \dots (Z^{k_s})_{i_s i_1} \\ &= \sum_{i_1=1}^N \dots \sum_{i_s=1}^N (Z_{(i_1 i_2)^{k_1}})^{k_1} (Z_{(i_2 i_3)^{k_2}})^{k_2} \dots (Z_{(i_s i_1)^{k_s}})^{k_s} \end{aligned}$$

(2) Next, we rescale our variable  $Z$  by a  $\sqrt{N}$  factor, as in the statement, and we also replace the usual trace by its normalized version,  $\text{tr} = \text{Tr}/N$ . Our formula becomes:

$$\text{tr} \left( \left( \frac{Z}{\sqrt{N}} \right)^k \right) = \frac{1}{N^{s/2+1}} \sum_{i_1=1}^N \dots \sum_{i_s=1}^N (Z_{(i_1 i_2)^{k_1}})^{k_1} (Z_{(i_2 i_3)^{k_2}})^{k_2} \dots (Z_{(i_s i_1)^{k_s}})^{k_s}$$

Thus, the moment that we are interested in is given by:

$$M_k \left( \frac{Z}{\sqrt{N}} \right) = \frac{1}{N^{s/2+1}} \sum_{i_1=1}^N \dots \sum_{i_s=1}^N \int_X (Z_{(i_1 i_2)^{k_1}})^{k_1} (Z_{(i_2 i_3)^{k_2}})^{k_2} \dots (Z_{(i_s i_1)^{k_s}})^{k_s}$$

(3) Let us apply now the Wick formula, from Theorem 7.4. We conclude that the moment that we are interested in is given by:

$$\begin{aligned} &M_k \left( \frac{Z}{\sqrt{N}} \right) \\ &= \frac{t^{s/2}}{N^{s/2+1}} \sum_{i_1=1}^N \dots \sum_{i_s=1}^N \# \left\{ \pi \in \mathcal{P}_2(k) \mid \pi \leq \ker((i_1 i_2)^{k_1}, (i_2 i_3)^{k_2}, \dots, (i_s i_1)^{k_s}) \right\} \\ &= t^{s/2} \sum_{\pi \in \mathcal{P}_2(k)} \frac{1}{N^{s/2+1}} \# \left\{ i \in \{1, \dots, N\}^s \mid \pi \leq \ker((i_1 i_2)^{k_1}, (i_2 i_3)^{k_2}, \dots, (i_s i_1)^{k_s}) \right\} \end{aligned}$$

(4) Our claim now is that in the  $N \rightarrow \infty$  limit the combinatorics of the above sum simplifies, with only the noncrossing partitions contributing to the sum, and with each of them contributing precisely with a 1 factor, so that we will have, as desired:

$$\begin{aligned} M_k \left( \frac{Z}{\sqrt{N}} \right) &= t^{s/2} \sum_{\pi \in \mathcal{P}_2(k)} \left( \delta_{\pi \in NC_2(k)} + O(N^{-1}) \right) \\ &\simeq t^{s/2} \sum_{\pi \in \mathcal{P}_2(k)} \delta_{\pi \in NC_2(k)} \\ &= t^{s/2} |\mathcal{NC}_2(k)| \end{aligned}$$

(5) In order to prove this, the first observation is that when  $k$  is not uniform, in the sense that it contains a different number of  $\circ$ ,  $\bullet$  symbols, we have  $\mathcal{P}_2(k) = \emptyset$ , and so:

$$M_k \left( \frac{Z}{\sqrt{N}} \right) = t^{s/2} |\mathcal{NC}_2(k)| = 0$$

(6) Thus, we are left with the case where  $k$  is uniform. Let us examine first the case where  $k$  consists of an alternating sequence of  $\circ$  and  $\bullet$  symbols, as follows:

$$k = \underbrace{\circ \bullet \circ \bullet \dots \circ \bullet}_{2p}$$

In this case it is convenient to relabel our multi-index  $i = (i_1, \dots, i_s)$ , with  $s = 2p$ , in the form  $(j_1, l_1, j_2, l_2, \dots, j_p, l_p)$ . With this done, our moment formula becomes:

$$M_k \left( \frac{Z}{\sqrt{N}} \right) = t^p \sum_{\pi \in \mathcal{P}_2(k)} \frac{1}{N^{p+1}} \# \left\{ j, l \in \{1, \dots, N\}^p \mid \pi \leq \ker(j_1 l_1, j_2 l_1, j_2 l_2, \dots, j_1 l_p) \right\}$$

Now observe that, with  $k$  being as above, we have an identification  $\mathcal{P}_2(k) \simeq S_p$ , obtained in the obvious way. With this done too, our moment formula becomes:

$$M_k \left( \frac{Z}{\sqrt{N}} \right) = t^p \sum_{\pi \in S_p} \frac{1}{N^{p+1}} \# \left\{ j, l \in \{1, \dots, N\}^p \mid j_r = j_{\pi(r)+1}, l_r = l_{\pi(r)}, \forall r \right\}$$

(7) We are now ready to do our asymptotic study, and prove the claim in (4). Let indeed  $\gamma \in S_p$  be the full cycle, which is by definition the following permutation:

$$\gamma = (1 \ 2 \ \dots \ p)$$

In terms of  $\gamma$ , the conditions  $j_r = j_{\pi(r)+1}$  and  $l_r = l_{\pi(r)}$  found above read:

$$\gamma \pi \leq \ker j \quad , \quad \pi \leq \ker l$$

Counting the number of free parameters in our moment formula, we obtain:

$$\begin{aligned} M_k \left( \frac{Z}{\sqrt{N}} \right) &= \frac{t^p}{N^{p+1}} \sum_{\pi \in S_p} N^{|\pi| + |\gamma \pi|} \\ &= t^p \sum_{\pi \in S_p} N^{|\pi| + |\gamma \pi| - p - 1} \end{aligned}$$

(8) The point now is that the last exponent is well-known to be  $\leq 0$ , with equality precisely when the permutation  $\pi \in S_p$  is geodesic, which in practice means that  $\pi$  must come from a noncrossing partition. Thus we obtain, in the  $N \rightarrow \infty$  limit, as desired:

$$M_k \left( \frac{Z}{\sqrt{N}} \right) \simeq t^p |\mathcal{NC}_2(k)|$$

This finishes the proof in the case of the exponents  $k$  which are alternating, and the case where  $k$  is an arbitrary uniform exponent is similar, by permuting everything.  $\square$

As a conclusion, we have obtained as asymptotic law for the Gaussian matrices a certain mysterious distribution, having as moments some numbers which are similar to the moments of the usual normal laws, but with the “underlying matching pairings being now replaced by underlying matching noncrossing pairings”.

Obviously, some interesting things are going on here. We will see in a moment, after doing some more combinatorics, this time in connection with the Wigner matrices, that there are some good reasons for calling the above mysterious law “circular”.

Thus, for ending with our present study with a nice-looking conclusion, we can say that the Gaussian matrices become “asymptotically circular”, with this meaning by definition that the  $N \rightarrow \infty$  moments are those computed above. This is of course something quite bold, and we will be back to it in chapters 9-12 below, when doing free probability.

### 7b. Wigner matrices

Moving ahead now, let us investigate the second class of random matrices that we are interested in, namely the Wigner matrices, which are by definition self-adjoint. Here our results will be far more complete than those for the Gaussian matrices. As a starting point, we have the following simple fact, making the connection with the above:

**PROPOSITION 7.6.** *Given a Gaussian matrix  $Z$ , with independent entries following the centered complex normal law  $G_t$ , with  $t > 0$ , if we write*

$$Z = \frac{1}{\sqrt{2}}(X + iY)$$

*with  $X, Y$  being self-adjoint, then both  $X, Y$  are Wigner matrices, of parameter  $t$ .*

**PROOF.** This is something elementary, which can be done in two steps, as follows:

(1) As a first observation, the result holds at  $N = 1$ . Indeed, here our Gaussian matrix  $Z$  is just a random variable, subject to the condition  $Z \sim G_t$ . But recall that the law  $G_t$  is by definition as follows, with  $X, Y$  being independent, each following the law  $g_t$ :

$$G_t = \text{law} \left( \frac{1}{\sqrt{2}}(X + iY) \right)$$

Thus in this case,  $N = 1$ , the variables  $X, Y$  that we obtain in the statement, as rescaled real and imaginary parts of  $Z$ , are subject to the condition  $X, Y \sim g_t$ , and so are Wigner matrices of size  $N = 1$  and parameter  $t > 0$ , as in Definition 7.2.

(2) In the general case now,  $N \in \mathbb{N}$ , the proof is similar, by using the basic behavior of the real and complex normal variables with respect to sums.  $\square$

The above result is quite interesting for us, because it shows that, in order to investigate the Wigner matrices, we are basically not in need of some new computations, starting from the Wick formula, and doing combinatorics afterwards, but just of some manipulations on the results that we already have, regarding the Gaussian matrices.

To be more precise, by using this method, we obtain the following result, coming by combining the observation in Proposition 7.6 with the formula in Theorem 7.5:

**THEOREM 7.7.** *Given a sequence of Wigner random matrices*

$$Z_N \in M_N(L^\infty(X))$$

*having independent  $G_t$  variables as entries, with  $t > 0$ , up to  $Z_N = Z_N^*$ , we have*

$$M_k \left( \frac{Z_N}{\sqrt{N}} \right) \simeq t^{k/2} |NC_2(k)|$$

*for any integer  $k \in \mathbb{N}$ , in the  $N \rightarrow \infty$  limit.*

**PROOF.** This can be deduced from a direct computation based on the Wick formula, similar to that from the proof of Theorem 7.5, but the best is to deduce this result from Theorem 7.5 itself. Indeed, we know from there that for Gaussian matrices  $Y_N \in M_N(L^\infty(X))$  we have the following formula, valid for any colored integer  $K = \circ \bullet \bullet \circ \dots$ , in the  $N \rightarrow \infty$  limit, with  $\mathcal{NC}_2$  standing for noncrossing matching pairings:

$$M_K \left( \frac{Y_N}{\sqrt{N}} \right) \simeq t^{|K|/2} |\mathcal{NC}_2(K)|$$

By doing some combinatorics, we deduce from this that we have the following formula for the moments of the matrices  $Re(Y_N)$ , with respect to usual exponents,  $k \in \mathbb{N}$ :

$$\begin{aligned} M_k \left( \frac{Re(Y_N)}{\sqrt{N}} \right) &= 2^{-k} \cdot M_k \left( \frac{Y_N}{\sqrt{N}} + \frac{Y_N^*}{\sqrt{N}} \right) \\ &= 2^{-k} \sum_{|K|=k} M_K \left( \frac{Y_N}{\sqrt{N}} \right) \\ &\simeq 2^{-k} \sum_{|K|=k} t^{k/2} |\mathcal{NC}_2(K)| \\ &= 2^{-k} \cdot t^{k/2} \cdot 2^{k/2} |\mathcal{NC}_2(k)| \\ &= 2^{-k/2} \cdot t^{k/2} |NC_2(k)| \end{aligned}$$

Now since the matrices  $Z_N = \sqrt{2} Re(Y_N)$  are of Wigner type, this gives the result.  $\square$

Summarizing, all this brings us into counting noncrossing pairings. So, let us start with some preliminaries here. We first have the following well-known result:

THEOREM 7.8. *The Catalan numbers, which are by definition given by*

$$C_k = |NC_2(2k)|$$

*satisfy the following recurrence formula,*

$$C_{k+1} = \sum_{a+b=k} C_a C_b$$

*their generating series  $f(z) = \sum_{k \geq 0} C_k z^k$  satisfies the equation*

$$zf^2 - f + 1 = 0$$

*and is given by the following explicit formula,*

$$f(z) = \frac{1 - \sqrt{1 - 4z}}{2z}$$

*and we have the following explicit formula for these numbers:*

$$C_k = \frac{1}{k+1} \binom{2k}{k}$$

PROOF. We must count the noncrossing pairings of  $\{1, \dots, 2k\}$ . But such a pairing appears by pairing 1 to an odd number,  $2a + 1$ , and then inserting a noncrossing pairing of  $\{2, \dots, 2a\}$ , and a noncrossing pairing of  $\{2a + 2, \dots, 2l\}$ . We conclude that we have the following recurrence formula for the Catalan numbers:

$$C_k = \sum_{a+b=k-1} C_a C_b$$

Consider now generating series of the Catalan numbers,  $f(z) = \sum_{k \geq 0} C_k z^k$ . In terms of this generating series, the above recurrence gives:

$$\begin{aligned} zf^2 &= \sum_{a,b \geq 0} C_a C_b z^{a+b+1} \\ &= \sum_{k \geq 1} \sum_{a+b=k-1} C_a C_b z^k \\ &= \sum_{k \geq 1} C_k z^k \\ &= f - 1 \end{aligned}$$

Thus  $f$  satisfies  $zf^2 - f + 1 = 0$ , and by solving this equation, and choosing the solution which is bounded at  $z = 0$ , we obtain the following formula:

$$f(z) = \frac{1 - \sqrt{1 - 4z}}{2z}$$

In order to compute this function, we use the generalized binomial formula:

$$(1+t)^p = \sum_{k=0}^{\infty} \binom{p}{k} x^k$$

For the exponent  $p = 1/2$ , the generalized binomial coefficients are:

$$\begin{aligned} \binom{1/2}{k} &= \frac{1/2(-1/2)(-3/2)\dots(3/2-k)}{k!} \\ &= (-1)^{k-1} \frac{1 \cdot 3 \cdot 5 \dots (2k-3)}{2^k k!} \\ &= (-1)^{k-1} \frac{(2k-2)!}{2^{k-1} (k-1)! 2^k k!} \\ &= \frac{(-1)^{k-1}}{2^{2k-1}} \cdot \frac{1}{k} \binom{2k-2}{k-1} \\ &= -2 \left( \frac{-1}{4} \right)^k \cdot \frac{1}{k} \binom{2k-2}{k-1} \end{aligned}$$

Thus the generalized binomial formula at exponent  $p = 1/2$  reads:

$$\sqrt{1+t} = 1 - 2 \sum_{k=1}^{\infty} \frac{1}{k} \binom{2k-2}{k-1} \left( \frac{-t}{4} \right)^k$$

With  $t = -4z$  we obtain from this the following formula:

$$\sqrt{1-4z} = 1 - 2 \sum_{k=1}^{\infty} \frac{1}{k} \binom{2k-2}{k-1} z^k$$

Now back to our series  $f$ , we obtain the following formula for it:

$$\begin{aligned} f(z) &= \frac{1 - \sqrt{1-4z}}{2z} \\ &= \sum_{k=1}^{\infty} \frac{1}{k} \binom{2k-2}{k-1} z^{k-1} \\ &= \sum_{k=0}^{\infty} \frac{1}{k+1} \binom{2k}{k} z^k \end{aligned}$$

Now recall that  $f(z) = \sum_{k \geq 0} C_k z^k$ , by definition. Thus the Catalan numbers are given by the formula the statement, and this finishes the proof.  $\square$

Getting back now to the Wigner matrices, we have the following result:

THEOREM 7.9. *Given a sequence of Wigner random matrices*

$$Z_N \in M_N(L^\infty(X))$$

*having independent  $G_t$  variables as entries, with  $t > 0$ , up to  $Z_N = Z_N^*$ , we have*

$$M_{2k} \left( \frac{Z_N}{\sqrt{N}} \right) \simeq t^k C_k$$

*in the  $N \rightarrow \infty$  limit. As for the asymptotic odd moments, these all vanish.*

PROOF. This follows from Theorem 7.7 and Theorem 7.8. Indeed, according to the results there, the asymptotic even moments are given by:

$$M_{2k} \left( \frac{Z_N}{\sqrt{N}} \right) \simeq t^k |NC_2(2k)| = t^k C_k$$

As for the asymptotic odd moments, once again from Theorem 7.7, we know that these all vanish. Thus, we are led to the conclusion in the statement.  $\square$

Summarizing, we are done with the moment computations, and with the asymptotic study, for both the Gaussian and the Wigner matrices. It remains now to interpret the results that we have. As already explained before, for the Gaussian matrices this is something quite complicated, with the technology that we presently have, and this will have to wait a bit, until we do some free probability. Regarding the Wigner matrices, however, the problems left here are quite elementary, and we will solve them next.

### 7c. Semicircle laws

In order to recapture the asymptotic measure of the Wigner matrices out of the moments, which are the Catalan numbers, there are several methods available, namely:

- (1) Stieltjes inversion.
- (2) Knowledge of  $SU_2$ .
- (3) Cheating.

The first method, which is something straightforward, not requiring any kind of trick, is based on the Stieltjes inversion formula, which is as follows:

THEOREM 7.10. *The density of a real probability measure  $\mu$  can be recaptured from the sequence of moments  $(M_k)$  via the Stieltjes inversion formula*

$$d\mu(x) = \lim_{t \searrow 0} -\frac{1}{\pi} \operatorname{Im} (G(x + it)) \cdot dx$$

*where the function on the right, given in terms of moments by*

$$G(\xi) = \xi^{-1} + M_1 \xi^{-2} + M_2 \xi^{-3} + \dots$$

*is the Cauchy transform of the measure  $\mu$ .*

PROOF. The Cauchy transform of our measure  $\mu$  is given by:

$$\begin{aligned} G(\xi) &= \xi^{-1} \sum_{k=0}^{\infty} M_k \xi^{-k} \\ &= \int_{\mathbb{R}} \frac{\xi^{-1}}{1 - \xi^{-1}y} d\mu(y) \\ &= \int_{\mathbb{R}} \frac{1}{\xi - y} d\mu(y) \end{aligned}$$

Now with  $\xi = x + it$ , we obtain the following formula:

$$\begin{aligned} \operatorname{Im}(G(x + it)) &= \int_{\mathbb{R}} \operatorname{Im} \left( \frac{1}{x - y + it} \right) d\mu(y) \\ &= \int_{\mathbb{R}} \frac{1}{2i} \left( \frac{1}{x - y + it} - \frac{1}{x - y - it} \right) d\mu(y) \\ &= - \int_{\mathbb{R}} \frac{t}{(x - y)^2 + t^2} d\mu(y) \end{aligned}$$

By integrating over  $[a, b]$  we obtain, with the change of variables  $x = y + tz$ :

$$\begin{aligned} \int_a^b \operatorname{Im}(G(x + it)) dx &= - \int_{\mathbb{R}} \int_a^b \frac{t}{(x - y)^2 + t^2} dx d\mu(y) \\ &= - \int_{\mathbb{R}} \int_{(a-y)/t}^{(b-y)/t} \frac{t}{(tz)^2 + t^2} t dz d\mu(y) \\ &= - \int_{\mathbb{R}} \int_{(a-y)/t}^{(b-y)/t} \frac{1}{1 + z^2} dz d\mu(y) \\ &= - \int_{\mathbb{R}} \left( \arctan \frac{b - y}{t} - \arctan \frac{a - y}{t} \right) d\mu(y) \end{aligned}$$

Now observe that with  $t \searrow 0$  we have:

$$\lim_{t \searrow 0} \left( \arctan \frac{b - y}{t} - \arctan \frac{a - y}{t} \right) = \begin{cases} \frac{\pi}{2} - \frac{\pi}{2} = 0 & (y < a) \\ \frac{\pi}{2} - 0 = \frac{\pi}{2} & (y = a) \\ \frac{\pi}{2} - (-\frac{\pi}{2}) = \pi & (a < y < b) \\ 0 - (-\frac{\pi}{2}) = \frac{\pi}{2} & (y = b) \\ -\frac{\pi}{2} - (-\frac{\pi}{2}) = 0 & (y > b) \end{cases}$$

We therefore obtain the following formula:

$$\lim_{t \searrow 0} \int_a^b \operatorname{Im}(G(x + it)) dx = -\pi \left( \mu(a, b) + \frac{\mu(a) + \mu(b)}{2} \right)$$

Thus, we are led to the conclusion in the statement.  $\square$

Before getting further, let us mention that the above result does not fully solve the moment problem, because we still have the question of understanding when a sequence of numbers  $M_1, M_2, M_3, \dots$  can be the moments of a measure  $\mu$ . We have here:

**THEOREM 7.11.** *A sequence of numbers  $M_0, M_1, M_2, M_3, \dots \in \mathbb{R}$ , with  $M_0 = 1$ , is the series of moments of a real probability measure  $\mu$  precisely when:*

$$|M_0| \geq 0 \quad , \quad \begin{vmatrix} M_0 & M_1 \\ M_1 & M_2 \end{vmatrix} \geq 0 \quad , \quad \begin{vmatrix} M_0 & M_1 & M_2 \\ M_1 & M_2 & M_3 \\ M_2 & M_3 & M_4 \end{vmatrix} \geq 0 \quad , \quad \dots$$

*That is, the associated Hankel determinants must be all positive.*

**PROOF.** As a first observation, the positivity conditions in the statement tell us that the following associated linear forms must be positive:

$$\sum_{i,j=1}^n c_i \bar{c}_j M_{i+j} \geq 0$$

But this is something very classical, in one sense the result being elementary, coming from the following computation, which shows that we have positivity indeed:

$$\begin{aligned} \int_{\mathbb{R}} \left| \sum_{i=1}^n c_i x^i \right|^2 d\mu(x) &= \int_{\mathbb{R}} \sum_{i,j=1}^n c_i \bar{c}_j x^{i+j} d\mu(x) \\ &= \sum_{i,j=1}^n c_i \bar{c}_j M_{i+j} \end{aligned}$$

As for the other sense, here the result comes once again from the above formula, this time via some standard functional analysis.  $\square$

Now back to our questions, as a basic application of the Stieltjes formula, we can solve the moment problem for the Catalan numbers, as follows:

**PROPOSITION 7.12.** *The real measure having as even moments the Catalan numbers,  $C_k = \frac{1}{k+1} \binom{2k}{k}$ , and having all odd moments 0 is the measure*

$$\gamma_1 = \frac{1}{2\pi} \sqrt{4 - x^2} dx$$

*called Wigner semicircle law on  $[-2, 2]$ .*

**PROOF.** In order to apply the Stieltjes inversion formula, we need a simple formula for the Cauchy transform. For this purpose, our starting point will be the formula from Theorem 7.8 for the generating series of the Catalan numbers, namely:

$$\sum_{k=0}^{\infty} C_k z^k = \frac{1 - \sqrt{1 - 4z}}{2z}$$

By using this formula with  $z = \xi^{-2}$ , we obtain the following formula:

$$\begin{aligned}
 G(\xi) &= \xi^{-1} \sum_{k=0}^{\infty} C_k \xi^{-2k} \\
 &= \xi^{-1} \cdot \frac{1 - \sqrt{1 - 4\xi^{-2}}}{2\xi^{-2}} \\
 &= \frac{\xi}{2} \left(1 - \sqrt{1 - 4\xi^{-2}}\right) \\
 &= \frac{\xi}{2} - \frac{1}{2} \sqrt{\xi^2 - 4}
 \end{aligned}$$

With this formula in hand, let us apply now the Stieltjes inversion formula, from Theorem 7.10. The study here goes as follows:

(1) According to the general philosophy of this Stieltjes formula, obtained by contemplating it, or rather by playing with it a bit, as to get familiar with it, the first term, namely  $\xi/2$ , which is “trivial”, will not contribute to the density.

(2) As for the second term, which is something non-trivial, this will contribute to the density, the rule here being that the square root  $\sqrt{\xi^2 - 4}$  will be replaced by the “dual” square root  $\sqrt{4 - x^2} dx$ , and that we have to multiply everything by  $-1/\pi$ .

(3) As a conclusion, by Stieltjes inversion we obtain the following density:

$$\begin{aligned}
 d\mu(x) &= -\frac{1}{\pi} \cdot -\frac{1}{2} \sqrt{4 - x^2} dx \\
 &= \frac{1}{2\pi} \sqrt{4 - x^2} dx
 \end{aligned}$$

Thus, we have obtained the measure in the statement, and we are done.  $\square$

More generally now, we have the following result:

**PROPOSITION 7.13.** *Given  $t > 0$ , the real measure having as even moments the numbers  $M_{2k} = t^k C_k$  and having all odd moments 0 is the measure*

$$\gamma_t = \frac{1}{2\pi t} \sqrt{4t - x^2} dx$$

*called Wigner semicircle law on  $[-2\sqrt{t}, 2\sqrt{t}]$ .*

PROOF. This follows by redoing the Stieltjes inversion computation. We have:

$$\begin{aligned}
G(\xi) &= \xi^{-1} \sum_{k=0}^{\infty} t^k C_k \xi^{-2k} \\
&= \xi^{-1} \cdot \frac{1 - \sqrt{1 - 4t\xi^{-2}}}{2t\xi^{-2}} \\
&= \frac{\xi}{2t} \left( 1 - \sqrt{1 - 4t\xi^{-2}} \right) \\
&= \frac{\xi}{2t} - \frac{1}{2t} \sqrt{\xi^2 - 4t}
\end{aligned}$$

Thus, by Stieltjes inversion we obtain the following density:

$$d\mu(x) = \frac{1}{2\pi t} \sqrt{4t - x^2} dx$$

But simplest is in fact, perhaps a bit by cheating, simply using the result at  $t = 1$ , from Proposition 4.12, along with a change of variables.  $\square$

Talking cheating, another way of recovering Proposition 4.12 is as follows:

PROPOSITION 7.14. *The Catalan numbers are the even moments of*

$$\gamma_1 = \frac{1}{2\pi} \sqrt{4 - x^2} dx$$

*called Wigner semicircle law. As for the odd moments of  $\gamma_1$ , these all vanish.*

PROOF. The even moments of the Wigner law can be computed with the change of variable  $x = 2 \cos t$ , and we are led to the following formula:

$$\begin{aligned}
M_{2k} &= \frac{1}{\pi} \int_0^2 \sqrt{4 - x^2} x^{2k} dx \\
&= \frac{1}{\pi} \int_0^{\pi/2} \sqrt{4 - 4 \cos^2 t} (2 \cos t)^{2k} 2 \sin t dt \\
&= \frac{4^{k+1}}{\pi} \int_0^{\pi/2} \cos^{2k} t \sin^2 t dt \\
&= \frac{4^{k+1}}{\pi} \cdot \frac{\pi}{2} \cdot \frac{(2k)!! 2!!}{(2k+3)!!} \\
&= 2 \cdot 4^k \cdot \frac{(2k)! / 2^k k!}{2^{k+1} (k+1)!} \\
&= C_k
\end{aligned}$$

As for the odd moments, these all vanish, because the density of  $\gamma_1$  is an even function. Thus, we are led to the conclusion in the statement.  $\square$

More generally, we have the following result, involving a parameter  $t > 0$ :

PROPOSITION 7.15. *The numbers  $t^k C_k$  are the even moments of*

$$\gamma_t = \frac{1}{2\pi t} \sqrt{4t - x^2} dx$$

*called semicircle law on  $[-2\sqrt{t}, 2\sqrt{t}]$ . As for the odd moments of  $\gamma_t$ , these all vanish.*

PROOF. This follows indeed from Proposition 7.14, with  $x = \sqrt{t}y$ . To be more precise, the even moments of the measure in the statement are given by:

$$\begin{aligned} M_{2k} &= \frac{1}{2\pi t} \int_{-2\sqrt{t}}^{2\sqrt{t}} \sqrt{4t - x^2} x^{2k} dx \\ &= \frac{1}{2\pi t} \int_{-1}^1 \sqrt{4t - ty^2} (\sqrt{t}y)^{2k} \sqrt{t} dy \\ &= \frac{t^k}{2\pi} \int_{-1}^1 \sqrt{4 - y^2} y^{2k} dy \\ &= t^k C_k \end{aligned}$$

As for the odd moments, these all vanish, because the density of  $\gamma_t$  is an even function. Thus, we are led to the conclusion in the statement.  $\square$

Now by putting everything together, we obtain the Wigner theorem, as follows:

THEOREM 7.16. *Given a sequence of Wigner random matrices*

$$Z_N \in M_N(L^\infty(X))$$

*having independent  $G_t$  variables as entries, with  $t > 0$ , up to  $Z_N = Z_N^*$ , we have*

$$\frac{Z_N}{\sqrt{N}} \sim \frac{1}{2\pi t} \sqrt{4t - x^2} dx$$

*in the  $N \rightarrow \infty$  limit, with the limiting measure being Wigner's semicircle law  $\gamma_t$ .*

PROOF. This follows indeed by combining Theorem 7.9 either with Proposition 7.13, and doing here an honest job, or with Proposition 7.15.  $\square$

There are many other things that can be said about the Wigner matrices, which appear as variations of the above, and we refer here to the standard random matrix books [1], [65], [68], [90]. We will be back to this later on in this book, in chapter 10 below.

### 7d. Unitary groups

We discuss here an alternative interpretation of the limiting laws  $\gamma_t$  that we found above, by using Lie groups, the idea being that the standard semicircle law  $\gamma_1$ , and more generally all the laws  $\gamma_t$ , naturally appear in connection with the group  $SU_2$ . This is something quite natural, and good to know, and useful for us later on, and in relation with the above, the knowledge of this fact can be used as an alternative to both Stieltjes inversion, and cheating, in order to establish the Wigner theorem. Let us start with:

THEOREM 7.17. *We have the following formula,*

$$SU_2 = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \mid |\alpha|^2 + |\beta|^2 = 1 \right\}$$

*which makes  $SU_2$  isomorphic to the unit sphere  $S_{\mathbb{C}}^1 \subset \mathbb{C}^2$ .*

PROOF. Consider an arbitrary  $2 \times 2$  matrix, written as follows:

$$U = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

Assuming  $\det U = 1$ , the inverse is then given by:

$$U^{-1} = \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}$$

On the other hand, assuming  $U \in U_2$ , the inverse must be the adjoint:

$$U^{-1} = \begin{pmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta} \end{pmatrix}$$

Thus our matrix must be of the following special form:

$$U = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$$

Since the determinant is 1, we must have  $|\alpha|^2 + |\beta|^2 = 1$ , so we are done with one direction. As for the converse, this is clear, the matrices in the statement being unitaries, and of determinant 1, and so being elements of  $SU_2$ . Finally, we have:

$$S_{\mathbb{C}}^1 = \left\{ (\alpha, \beta) \in \mathbb{C}^2 \mid |\alpha|^2 + |\beta|^2 = 1 \right\}$$

Thus, the final assertion in the statement holds as well. □

We have the following useful reformulation of Theorem 7.17:

THEOREM 7.18. *We have the formula*

$$SU_2 = \left\{ \begin{pmatrix} p+iq & r+is \\ -r+is & p-iq \end{pmatrix} \mid p^2 + q^2 + r^2 + s^2 = 1 \right\}$$

*which makes  $SU_2$  isomorphic to the unit real sphere  $S_{\mathbb{R}}^3 \subset \mathbb{R}^3$ .*

PROOF. We recall from Theorem 7.17 that we have:

$$SU_2 = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \mid |\alpha|^2 + |\beta|^2 = 1 \right\}$$

Now let us write our parameters  $\alpha, \beta \in \mathbb{C}$ , which belong to the complex unit sphere  $S_{\mathbb{C}}^1 \subset \mathbb{C}^2$ , in terms of their real and imaginary parts, as follows:

$$\alpha = p + iq \quad , \quad \beta = r + is$$

In terms of these new parameters  $p, q, r, s \in \mathbb{R}$ , our formula for a generic matrix  $U \in SU_2$ , that we established before, reads:

$$U = \begin{pmatrix} p + iq & r + is \\ -r + is & p - iq \end{pmatrix}$$

As for the condition to be satisfied by the parameters  $p, q, r, s \in \mathbb{R}$ , this comes the condition  $|\alpha|^2 + |\beta|^2 = 1$  to be satisfied by  $\alpha, \beta \in \mathbb{C}$ , which reads:

$$p^2 + q^2 + r^2 + s^2 = 1$$

Thus, we are led to the conclusion in the statement. Regarding now the last assertion, recall that the unit sphere  $S_{\mathbb{R}}^3 \subset \mathbb{R}^4$  is given by:

$$S_{\mathbb{R}}^3 = \left\{ (p, q, r, s) \mid p^2 + q^2 + r^2 + s^2 = 1 \right\}$$

Thus, we have an isomorphism of compact spaces  $SU_2 \simeq S_{\mathbb{R}}^3$ , as claimed.  $\square$

Here is another reformulation of our main result so far, regarding  $SU_2$ , obtained by further building on the parametrization from Theorem 7.18:

THEOREM 7.19. *We have the following formula,*

$$SU_2 = \left\{ p\beta_1 + q\beta_2 + r\beta_3 + s\beta_4 \mid p^2 + q^2 + r^2 + s^2 = 1 \right\}$$

where  $\beta_1, \beta_2, \beta_3, \beta_4$  are the Pauli matrices, given by:

$$\begin{aligned} \beta_1 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad \beta_2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \\ \beta_3 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad , \quad \beta_4 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \end{aligned}$$

PROOF. We recall from Theorem 7.18 that the group  $SU_2$  can be parametrized by the real sphere  $S_{\mathbb{R}}^3 \subset \mathbb{R}^4$ , in the following way:

$$SU_2 = \left\{ \begin{pmatrix} p + iq & r + is \\ -r + is & p - iq \end{pmatrix} \mid p^2 + q^2 + r^2 + s^2 = 1 \right\}$$

But this gives the formula in the statement, with the Pauli matrices  $\beta_1, \beta_2, \beta_3, \beta_4$  being the coefficients of  $p, q, r, s$ , in this parametrization.  $\square$

The above result is often the most convenient one, when dealing with  $SU_2$ . This is because the Pauli matrices have a number of remarkable properties, which are very useful when doing computations. These properties can be summarized as follows:

PROPOSITION 7.20. *The Pauli matrices multiply according to the following formulae,*

$$\beta_2^2 = \beta_3^2 = \beta_4^2 = -1$$

$$\beta_2\beta_3 = -\beta_3\beta_2 = \beta_4$$

$$\beta_3\beta_4 = -\beta_4\beta_3 = \beta_2$$

$$\beta_4\beta_2 = -\beta_2\beta_4 = \beta_3$$

*they conjugate according to the following rules,*

$$\beta_1^* = \beta_1, \beta_2^* = -\beta_2, \beta_3^* = -\beta_3, \beta_4^* = -\beta_4$$

*and they form an orthonormal basis of  $M_2(\mathbb{C})$ , with respect to the scalar product*

$$\langle x, y \rangle = \text{tr}(xy^*)$$

*with  $\text{tr} : M_2(\mathbb{C}) \rightarrow \mathbb{C}$  being the normalized trace of  $2 \times 2$  matrices,  $\text{tr} = \text{Tr}/2$ .*

PROOF. The first two assertions, regarding the multiplication and conjugation rules for the Pauli matrices, follow from some elementary computations. As for the last assertion, this follows by using these rules. Indeed, the fact that the Pauli matrices are pairwise orthogonal follows from computations of the following type, for  $i \neq j$ :

$$\begin{aligned} \langle \beta_i, \beta_j \rangle &= \text{tr}(\beta_i\beta_j^*) \\ &= \text{tr}(\pm\beta_i\beta_j) \\ &= \text{tr}(\pm\beta_k) \\ &= 0 \end{aligned}$$

As for the fact that the Pauli matrices have norm 1, this follows from:

$$\begin{aligned} \langle \beta_i, \beta_i \rangle &= \text{tr}(\beta_i\beta_i^*) \\ &= \text{tr}(\pm\beta_i^2) \\ &= \text{tr}(\beta_1) \\ &= 1 \end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Now back to probability, we can recover our measures, as follows:

THEOREM 7.21. *The main character of  $SU_2$  follows the following law,*

$$\gamma_1 = \frac{1}{2\pi} \sqrt{4 - x^2} dx$$

*which is the Wigner law of parameter 1.*

PROOF. This follows from Theorem 7.18, by identifying  $SU_2$  with the sphere  $S_{\mathbb{R}}^3$ , the variable  $\chi = 2Re(p)$  being semicircular. Indeed, let us write, as in Theorem 7.18:

$$SU_2 = \left\{ \begin{pmatrix} p+iq & r+is \\ -p+iq & r-is \end{pmatrix} \mid p^2 + q^2 + r^2 + s^2 = 1 \right\}$$

In this picture, the main character is given by:

$$\chi \begin{pmatrix} p+iq & r+is \\ -r+is & p-iq \end{pmatrix} = 2p$$

We are therefore left with computing the law of the following variable:

$$p \in C(S_{\mathbb{R}}^3)$$

For this purpose, we use the moment method. We recall from chapter 1 that the polynomial integrals over the real spheres are given by the following formula:

$$\int_{S_{\mathbb{R}}^{N-1}} x_1^{k_1} \dots x_N^{k_N} dx = \frac{(N-1)!! k_1!! \dots k_N!!}{(N + \sum k_i - 1)!!}$$

In our case, where  $N = 4$ , we obtain the following moment formula:

$$\begin{aligned} \int_{S_{\mathbb{R}}^3} p^{2k} &= \frac{3!!(2k)!!}{(2k+3)!!} \\ &= 2 \cdot \frac{3 \cdot 5 \cdot 7 \dots (2k-1)}{2 \cdot 4 \cdot 6 \dots (2k+2)} \\ &= 2 \cdot \frac{(2k)!}{2^k k! 2^{k+1} (k+1)!} \\ &= \frac{1}{4^k} \cdot \frac{1}{k+1} \binom{2k}{k} \\ &= \frac{C_k}{4^k} \end{aligned}$$

Thus the variable  $2p \in C(S_{\mathbb{R}}^3)$  follows the Wigner semicircle law  $\gamma_1$ , as claimed.  $\square$

Summarizing, we have managed to recover the Wigner law  $\gamma_1$  out of purely geometric considerations, and we have here the full theory of this law, with the formula of the density being visible, geometrically, in the  $SU_2 \simeq S_{\mathbb{R}}^3$  setting, and with the formula of the moments coming from the above proof. Moreover, with a change of variable, all these results extend to  $\gamma_t$  with  $t > 0$ . And this is quite interesting, philosophically, because retrospectively thinking, this knowledge would have allowed us to short-circuit the Stieltjes inversion, or worse, the cheating, involved in the proof of the Wigner theorem.

Finally, as physicists say, there is no  $SU_2$  without  $SO_3$ , so let us discuss now as well the group  $SO_3$ , that will certainly appear later, when doing more complicated things. Let us start with the following construction, whose goal will become clear in a moment:

PROPOSITION 7.22. *The adjoint action  $SU_2 \curvearrowright M_2(\mathbb{C})$ , given by*

$$T_U(A) = UAU^*$$

*leaves invariant the following real vector subspace of  $M_2(\mathbb{C})$ ,*

$$\mathbb{R}^4 = \text{span}(\beta_1, \beta_2, \beta_3, \beta_4)$$

*and we obtain in this way a group morphism as follows:*

$$SU_2 \rightarrow GL_4(\mathbb{R})$$

*Moreover, we obtain in fact in this way a group morphism  $SU_2 \rightarrow O_4$ .*

PROOF. Everything here is clear from the multiplication formulae for the Pauli matrices, from Proposition 7.20. In fact, all this will come as well as a consequence of the following result, where the morphism  $SU_2 \rightarrow O_4$  is computed explicitly.  $\square$

The point now is that when computing the morphism constructed in Proposition 7.22, we are led to something quite interesting, namely the group  $SO_3$ , as follows:

PROPOSITION 7.23. *With respect to the standard basis  $\beta_1, \beta_2, \beta_3, \beta_4$  of the vector space  $\mathbb{R}^4 = \text{span}(\beta_1, \beta_2, \beta_3, \beta_4)$ , the morphism  $T : SU_2 \rightarrow GL_4(\mathbb{R})$  is given by:*

$$T_U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p^2 + q^2 - r^2 - s^2 & 2(qr - ps) & 2(pr + qs) \\ 0 & 2(ps + qr) & p^2 + r^2 - q^2 - s^2 & 2(rs - pq) \\ 0 & 2(qs - pr) & 2(pq + rs) & p^2 + s^2 - q^2 - r^2 \end{pmatrix}$$

*Thus, when looking at  $T$  as a group morphism  $SU_2 \rightarrow O_4$ , what we have in fact is a group morphism  $SU_2 \rightarrow O_3$ , and even  $SU_2 \rightarrow SO_3$ .*

PROOF. With notations from Proposition 7.22 and its proof, let us first look at the action  $L : SU_2 \curvearrowright \mathbb{R}^4$  by left multiplication,  $L_U(A) = UA$ . Let us write:

$$U = p\beta_1 + q\beta_2 + r\beta_3 + s\beta_4$$

$$A = a\beta_1 + b\beta_2 + c\beta_3 + d\beta_4$$

By using the multiplication formulae in Proposition 7.20, we obtain:

$$\begin{aligned} UA &= (p\beta_1 + q\beta_2 + r\beta_3 + s\beta_4)(a\beta_1 + b\beta_2 + c\beta_3 + d\beta_4) \\ &= (pa - qb - rc - sd)\beta_1 \\ &\quad + (pb + qa + rd - sc)\beta_2 \\ &\quad + (pc - qd + ra + sb)\beta_3 \\ &\quad + (pd + qc - rb + sa)\beta_4 \end{aligned}$$

We conclude that the matrix of the left action considered above is:

$$L_U = \begin{pmatrix} p & -q & -r & -s \\ q & p & -s & r \\ r & s & p & -q \\ s & -r & q & p \end{pmatrix}$$

Similarly, let us look now at the action  $R : SU_2 \curvearrowright \mathbb{R}^4$  by right multiplication,  $R_U(A) = AU^*$ . In order to compute the matrix of this action, let us write:

$$U = p\beta_1 + q\beta_2 + r\beta_3 + s\beta_4$$

$$A = a\beta_1 + b\beta_2 + c\beta_3 + d\beta_4$$

By using the multiplication formulae in Proposition 7.20, we obtain:

$$\begin{aligned} AU^* &= (a\beta_1 + b\beta_2 + c\beta_3 + d\beta_4)(p\beta_1 - q\beta_2 - r\beta_3 - s\beta_4) \\ &= (ap + bq + cr + ds)\beta_1 \\ &\quad + (-aq + bp - cs + dr)\beta_2 \\ &\quad + (-ar + bs + cp - dq)\beta_3 \\ &\quad + (-as - br + cq + dp)\beta_4 \end{aligned}$$

We conclude that the matrix of the right action considered above is:

$$R_U = \begin{pmatrix} p & q & r & s \\ -q & p & -s & r \\ -r & s & p & -q \\ -s & -r & q & p \end{pmatrix}$$

Now by composing, the matrix of the adjoint matrix in the statement is:

$$\begin{aligned} T_U &= R_U L_U \\ &= \begin{pmatrix} p & q & r & s \\ -q & p & -s & r \\ -r & s & p & -q \\ -s & -r & q & p \end{pmatrix} \begin{pmatrix} p & -q & -r & -s \\ q & p & -s & r \\ r & s & p & -q \\ s & -r & q & p \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p^2 + q^2 - r^2 - s^2 & 2(qr - ps) & 2(pr + qs) \\ 0 & 2(ps + qr) & p^2 + r^2 - q^2 - s^2 & 2(rs - pq) \\ 0 & 2(qs - pr) & 2(pq + rs) & p^2 + s^2 - q^2 - r^2 \end{pmatrix} \end{aligned}$$

Thus, we have the formula in the statement, and this gives the result.  $\square$

We can now formulate a famous result, due to Euler-Rodrigues, as follows:

THEOREM 7.24. *We have the Euler-Rodrigues formula*

$$U = \begin{pmatrix} p^2 + q^2 - r^2 - s^2 & 2(qr - ps) & 2(pr + qs) \\ 2(ps + qr) & p^2 + r^2 - q^2 - s^2 & 2(rs - pq) \\ 2(qs - pr) & 2(pq + rs) & p^2 + s^2 - q^2 - r^2 \end{pmatrix}$$

with  $p^2 + q^2 + r^2 + s^2 = 1$ , for the generic elements of  $SO_3$ .

PROOF. This follows from the formula in Proposition 7.23, with the fact that the morphism  $SU_2 \rightarrow SO_3$  constructed there is indeed surjective coming for instance from the fact that any rotation  $U \in SO_3$  has an axis, and with the fact that we have indeed a double cover map being something elementary, obtained by computing the kernel.  $\square$

Now back to probability, let us formulate the following definition:

DEFINITION 7.25. *The standard Marchenko-Pastur law  $\pi_1$  is given by:*

$$f \sim \gamma_1 \implies f^2 \sim \pi_1$$

That is,  $\pi_1$  is the law of the square of a variable following the semicircle law  $\gamma_1$ .

Here the fact that  $\pi_1$  is indeed well-defined comes from the fact that a measure is uniquely determined by its moments. More explicitly now, we have:

PROPOSITION 7.26. *The density of the Marchenko-Pastur law is*

$$\pi_1 = \frac{1}{2\pi} \sqrt{4x^{-1} - 1} dx$$

and the moments of this measure are the Catalan numbers.

PROOF. There are several proofs here, the simplest being by cheating. Indeed, the moments of  $\pi_1$  can be computed with the change of variable  $x = 4 \cos^2 t$ , as follows:

$$\begin{aligned} M_k &= \frac{1}{2\pi} \int_0^4 \sqrt{4x^{-1} - 1} x^k dx \\ &= \frac{1}{2\pi} \int_0^{\pi/2} \frac{\sin t}{\cos t} \cdot (4 \cos^2 t)^k \cdot 2 \cos t \sin t dt \\ &= \frac{4^{k+1}}{\pi} \int_0^{\pi/2} \cos^{2k} t \sin^2 t dt \\ &= \frac{4^{k+1}}{\pi} \cdot \frac{\pi}{2} \cdot \frac{(2k)!! 2!!}{(2k+3)!!} \\ &= 2 \cdot 4^k \cdot \frac{(2k)! / 2^k k!}{2^{k+1} (k+1)!} \\ &= C_k \end{aligned}$$

Thus, we are led to the conclusion in the statement.  $\square$

We can do now the character computation for  $SO_3$ , as follows:

**THEOREM 7.27.** *The main character of  $SO_3$ , modified by adding 1 to it, given in standard Euler-Rodrigues coordinates by*

$$\chi = 3p^2 - q^2 - r^2 - s^2$$

*follows a squared semicircle law, or Marchenko-Pastur law  $\pi_1$ .*

**PROOF.** This follows by using the quotient map  $SU_2 \rightarrow SO_3$ , and the result for  $SU_2$ . Let us recall indeed that the elements of  $SU_2$  can be parametrized as follows:

$$U = \begin{pmatrix} p + iq & r + is \\ -r + is & p - iq \end{pmatrix}$$

As for the elements of  $SO_3$ , these can be parametrized as follows:

$$V = \begin{pmatrix} p^2 + q^2 - r^2 - s^2 & 2(qr - ps) & 2(pr + qs) \\ 2(ps + qr) & p^2 + r^2 - q^2 - s^2 & 2(rs - pq) \\ 2(qs - pr) & 2(pq + rs) & p^2 + s^2 - q^2 - r^2 \end{pmatrix}$$

The point now is that, by using these formulae, in the context of Theorem 7.21 and its proof, the main character of  $SO_3$ , modified by adding 1 to it, is given by:

$$\chi = 4p^2$$

Now recall from the proof of Theorem 7.21 that we have:

$$2p \sim \gamma_1$$

On the other hand, a quick comparison between the moment formulae for the Wigner and Marchenko-Pastur laws, which are very similar, shows that we have:

$$f \sim \gamma_1 \implies f^2 \sim \pi_1$$

Thus, with  $f = 2p$ , we obtain the result in the statement.  $\square$

All the above is quite interesting, and we will meet the Marchenko-Pastur law  $\pi_1$ , as well as its parametric versions  $\pi_t$  with  $t > 0$ , which are still to be defined, on numerous occasions, in what follows, in connection with all sorts of probability considerations. To be more precise, we will see in the next chapter that these are the asymptotic laws of the Wishart matrices. And later on, in chapters 9-12 below, when doing free probability, we will see that these are the free analogues of the Poisson laws.

## 7e. Exercises

There has been a lot of theory in this chapter, and lots of computations as well, both calculus and combinatorics. As a first instructive exercise on all this, we have:

**EXERCISE 7.28.** *Find a direct proof of the Wigner theorem, without passing via the Gaussian matrices.*

This is actually how this theorem was first found, via direct computations.

EXERCISE 7.29. *Look up the various properties of the Catalan numbers,*

$$C_k = \frac{1}{k+1} \binom{2k}{k}$$

*and write down an account of what you learned, ideally 2 pages or so.*

Here by 2 pages we mean 2 pages of statements only, without proofs, the Catalan numbers being as famous as that.

EXERCISE 7.30. *Try to axiomatize the “circular law”, having as moments the numbers*

$$M_k = |\mathcal{NC}_2(k)|$$

*which should appear as asymptotic law for the Gaussian matrices.*

Obviously, this looks like something quite complicated and abstract, and some good imagination is needed. In case you don’t find, don’t worry, we will be back to this.

EXERCISE 7.31. *Try to find what the  $t > 0$  analogue of the Marchenko-Pastur law*

$$\pi_1 = \frac{1}{2\pi} \sqrt{4x^{-1} - 1} \, dx$$

*should be.*

Again, this looks like something quite complicated and abstract, and some good imagination, and love for exploration, science in general, and mathematics in particular, is needed. And again, in case you don’t find, don’t worry, we will be back to this.

## CHAPTER 8

### Wishart matrices

#### 8a. Marchenko-Pastur

We discuss in this chapter the complex Wishart matrices, which are the positive analogues of the Gaussian and Wigner matrices. These matrices were introduced and studied by Marchenko-Pastur in [64], not long after Wigner’s paper [97], and are of interest in connection with many questions. They are constructed as follows:

DEFINITION 8.1. *A complex Wishart matrix is a random matrix of type*

$$W = YY^* \in M_N(L^\infty(X))$$

*with  $Y$  being a complex Gaussian matrix, with entries following the law  $G_t$ .*

There are in fact several possible definitions for the complex Wishart matrices, with some being more clever and useful than some other. To start with, we will use the above definition, which comes naturally out of what we know about the Gaussian and Wigner matrices. Once such matrices are studied, we will talk about their versions, too.

Observe that, due to the formula  $W = YY^*$ , the complex Wishart matrices are obviously positive, in the sense of the positivity notion from chapter 6:

$$W \geq 0$$

Due to this key positivity property, and to the otherwise “randomness” of  $W$ , such matrices are useful in many down-to-earth contexts. More on this later.

As usual with the random matrices, we will be interested in computing the asymptotic laws of our Wishart matrices  $W$ , suitably rescaled, in the  $N \rightarrow \infty$  limit. Quite surprisingly, the computation here leads to the Catalan numbers, but not exactly in the same way as for the Wigner matrices, the precise result being as follows:

THEOREM 8.2. *Given a sequence of complex Wishart matrices*

$$W_N = Y_N Y_N^* \in M_N(L^\infty(X))$$

*with  $Y_N$  being  $N \times N$  complex Gaussian of parameter  $t > 0$ , we have*

$$M_k \left( \frac{W_N}{N} \right) \simeq t^k C_k$$

*for any exponent  $k \in \mathbb{N}$ , in the  $N \rightarrow \infty$  limit.*

PROOF. There are several possible proofs for this result, as follows:

(1) A first method is by using the result that we have from chapter 7, for the Gaussian matrices  $Y_N$ . Indeed, we know from there that we have the following formula, valid for any colored integer  $K = \circ \bullet \bullet \circ \dots$ , in the  $N \rightarrow \infty$  limit:

$$M_K \left( \frac{Y_N}{\sqrt{N}} \right) \simeq t^{|K|/2} |\mathcal{NC}_2(K)|$$

With  $K = \circ \bullet \circ \bullet \dots$ , alternating word of length  $2k$ , with  $k \in \mathbb{N}$ , this gives:

$$M_k \left( \frac{Y_N Y_N^*}{N} \right) \simeq t^k |\mathcal{NC}_2(K)|$$

Thus, in terms of the Wishart matrix  $W_N = Y_N Y_N^*$  we have, for any  $k \in \mathbb{N}$ :

$$M_k \left( \frac{W_N}{N} \right) \simeq t^k |\mathcal{NC}_2(K)|$$

The point now is that, by doing some combinatorics, we have:

$$|\mathcal{NC}_2(K)| = |\mathcal{NC}_2(2k)| = C_k$$

Thus, we are led to the formula in the statement.

(2) A second method, that we will explain now as well, is by proving the result directly, starting from definitions. The matrix entries of our matrix  $W = W_N$  are given by:

$$W_{ij} = \sum_{r=1}^N Y_{ir} \bar{Y}_{jr}$$

Thus, the normalized traces of powers of  $W$  are given by the following formula:

$$\begin{aligned} \text{tr}(W^k) &= \frac{1}{N} \sum_{i_1=1}^N \dots \sum_{i_k=1}^N W_{i_1 i_2} W_{i_2 i_3} \dots W_{i_k i_1} \\ &= \frac{1}{N} \sum_{i_1=1}^N \dots \sum_{i_k=1}^N \sum_{r_1=1}^N \dots \sum_{r_k=1}^N Y_{i_1 r_1} \bar{Y}_{i_2 r_1} Y_{i_2 r_2} \bar{Y}_{i_3 r_2} \dots Y_{i_k r_k} \bar{Y}_{i_1 r_k} \end{aligned}$$

By rescaling now  $W$  by a  $1/N$  factor, as in the statement, we obtain:

$$\text{tr} \left( \left( \frac{W}{N} \right)^k \right) = \frac{1}{N^{k+1}} \sum_{i_1=1}^N \dots \sum_{i_k=1}^N \sum_{r_1=1}^N \dots \sum_{r_k=1}^N Y_{i_1 r_1} \bar{Y}_{i_2 r_1} Y_{i_2 r_2} \bar{Y}_{i_3 r_2} \dots Y_{i_k r_k} \bar{Y}_{i_1 r_k}$$

By using now the Wick rule, we obtain the following formula for the moments, with  $K = \circ \bullet \circ \bullet \dots$ , alternating word of length  $2k$ , and with  $I = (i_1 r_1, i_2 r_1, \dots, i_k r_k, i_1 r_k)$ :

$$\begin{aligned} M_k \left( \frac{W}{N} \right) &= \frac{t^k}{N^{k+1}} \sum_{i_1=1}^N \dots \sum_{i_k=1}^N \sum_{r_1=1}^N \dots \sum_{r_k=1}^N \# \left\{ \pi \in \mathcal{P}_2(K) \mid \pi \leq \ker(I) \right\} \\ &= \frac{t^k}{N^{k+1}} \sum_{\pi \in \mathcal{P}_2(K)} \# \left\{ i, r \in \{1, \dots, N\}^k \mid \pi \leq \ker(I) \right\} \end{aligned}$$

In order to compute this quantity, we use the standard bijection  $\mathcal{P}_2(K) \simeq S_k$ . By identifying the pairings  $\pi \in \mathcal{P}_2(K)$  with their counterparts  $\pi \in S_k$ , we obtain:

$$M_k \left( \frac{W}{N} \right) = \frac{t^k}{N^{k+1}} \sum_{\pi \in S_k} \# \left\{ i, r \in \{1, \dots, N\}^k \mid i_s = i_{\pi(s)+1}, r_s = r_{\pi(s)}, \forall s \right\}$$

Now let  $\gamma \in S_k$  be the full cycle, which is by definition the following permutation:

$$\gamma = (1 \ 2 \ \dots \ k)$$

The general factor in the product computed above is then 1 precisely when following two conditions are simultaneously satisfied:

$$\gamma\pi \leq \ker i \quad , \quad \pi \leq \ker r$$

Counting the number of free parameters in our moment formula, we obtain:

$$M_k \left( \frac{W}{N} \right) = t^k \sum_{\pi \in S_k} N^{|\pi| + |\gamma\pi| - k - 1}$$

The point now is that the last exponent is well-known to be  $\leq 0$ , with equality precisely when the permutation  $\pi \in S_k$  is geodesic, which in practice means that  $\pi$  must come from a noncrossing partition. Thus we obtain, in the  $N \rightarrow \infty$  limit:

$$M_k \left( \frac{W}{N} \right) \simeq t^k C_k$$

Thus, we are led to the conclusion in the statement.  $\square$

As a consequence of the above result, we have a new look on the Catalan numbers, which is more adapted to our present Wishart matrix considerations, as follows:

**PROPOSITION 8.3.** *The Catalan numbers  $C_k = |NC_2(2k)|$  appear as well as*

$$C_k = |NC(k)|$$

where  $NC(k)$  is the set of all noncrossing partitions of  $\{1, \dots, k\}$ .

**PROOF.** This follows indeed from the proof of Theorem 8.2.  $\square$

The direct explanation for the above formula, relating noncrossing partitions and pairings, comes from the following result, which is very useful, and good to know:

PROPOSITION 8.4. *We have a bijection between noncrossing partitions and pairings*

$$NC(k) \simeq NC_2(2k)$$

*which is constructed as follows:*

- (1) *The application  $NC(k) \rightarrow NC_2(2k)$  is the “fattening” one, obtained by doubling all the legs, and doubling all the strings as well.*
- (2) *Its inverse  $NC_2(2k) \rightarrow NC(k)$  is the “shrinking” application, obtained by collapsing pairs of consecutive neighbors.*

PROOF. The fact that the two operations in the statement are indeed inverse to each other is clear, by computing the corresponding two compositions, with the remark that the construction of the fattening operation requires the partitions to be noncrossing.  $\square$

As a comment here, the above result is something quite remarkable, in view of the total lack of relation between  $P(k)$  and  $P_2(2k)$ . Thus, taking for granted that “classical probability is about partitions, and free probability is about noncrossing partitions”, a general principle that emerges from our study so far, and that we will fully justify later on, we have in Proposition 8.4 above an endless source of things to be done, in the free case, having no classical counterpart. We will keep this discovery in our pocket, and have it pulled out of there, for some magic, on several occasions, in what follows.

Getting back now to Wishart matrices, at  $t = 1$  we are led to the question of finding the law having the Catalan numbers as moments. We already know the answer to this question from chapter 7, and more specifically from our considerations there at the end, regarding  $SO_3$ , but here is as well an independent, pedestrian solution to this question:

PROPOSITION 8.5. *The real measure having the Catalan numbers as moments is*

$$\pi_1 = \frac{1}{2\pi} \sqrt{4x^{-1} - 1} dx$$

*called Marchenko-Pastur law of parameter 1.*

PROOF. As already mentioned, this is something that we already know, because we came upon this when talking about  $SO_3$ . Here are two alternative proofs:

(1) By using the Stieltjes inversion formula. In order to apply this formula, we need a simple formula for the Cauchy transform. For this purpose, our starting point will be the formula from chapter 7 for the generating series of the Catalan numbers, namely:

$$\sum_{k=0}^{\infty} C_k z^k = \frac{1 - \sqrt{1 - 4z}}{2z}$$

By using this formula with  $z = \xi^{-1}$ , we obtain the following formula:

$$\begin{aligned}
 G(\xi) &= \xi^{-1} \sum_{k=0}^{\infty} C_k \xi^{-k} \\
 &= \xi^{-1} \cdot \frac{1 - \sqrt{1 - 4\xi^{-1}}}{2\xi^{-1}} \\
 &= \frac{1}{2} \left( 1 - \sqrt{1 - 4\xi^{-1}} \right) \\
 &= \frac{1}{2} - \frac{1}{2} \sqrt{1 - 4\xi^{-1}}
 \end{aligned}$$

With this formula in hand, let us apply now the Stieltjes inversion formula, from chapter 7. The first term, namely  $1/2$ , which is trivial, will not contribute to the density. As for the second term, which is something non-trivial, this will contribute to the density, the rule here being that the square root  $\sqrt{1 - 4\xi^{-1}}$  will be replaced by the “dual” square root  $\sqrt{4x^{-1} - 1} dx$ , and that we have to multiply everything by  $-1/\pi$ . Thus, by Stieltjes inversion we obtain the density in the statement, namely:

$$\begin{aligned}
 d\mu(x) &= -\frac{1}{\pi} \cdot -\frac{1}{2} \sqrt{4x^{-1} - 1} dx \\
 &= \frac{1}{2\pi} \sqrt{4x^{-1} - 1} dx
 \end{aligned}$$

(2) Alternatively, if the above was too complicated, we can simply cheat, as we actually did in chapter 7, when talking about  $SO_3$ . Indeed, the moments of the law  $\pi_1$  in the statement can be computed with the change of variable  $x = 4 \cos^2 t$ , as follows:

$$\begin{aligned}
 M_k &= \frac{1}{2\pi} \int_0^4 \sqrt{4x^{-1} - 1} x^k dx \\
 &= \frac{1}{2\pi} \int_0^{\pi/2} \frac{\sin t}{\cos t} \cdot (4 \cos^2 t)^k \cdot 2 \cos t \sin t dt \\
 &= \frac{4^{k+1}}{\pi} \int_0^{\pi/2} \cos^{2k} t \sin^2 t dt \\
 &= \frac{4^{k+1}}{\pi} \cdot \frac{\pi}{2} \cdot \frac{(2k)!!2!!}{(2k+3)!!} \\
 &= 2 \cdot 4^k \cdot \frac{(2k)!/2^k k!}{2^{k+1}(k+1)!} \\
 &= C_k
 \end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Now back to the Wishart matrices, we are led to the following result:

THEOREM 8.6. *Given a sequence of complex Wishart matrices*

$$W_N = Y_N Y_N^* \in M_N(L^\infty(X))$$

*with  $Y_N$  being  $N \times N$  complex Gaussian of parameter 1, we have*

$$\frac{W_N}{N} \sim \frac{1}{2\pi} \sqrt{4x^{-1} - 1} dx$$

*with  $N \rightarrow \infty$ , with the limiting measure being the Marchenko-Pastur law  $\pi_1$ .*

PROOF. This follows indeed from Theorem 8.2 and Proposition 8.5. □

We have as well a parametric version of the above result, as follows:

THEOREM 8.7. *Given a sequence of complex Wishart matrices*

$$W_N = Y_N Y_N^* \in M_N(L^\infty(X))$$

*with  $Y_N$  being  $N \times N$  complex Gaussian of parameter  $t > 0$ , we have*

$$\frac{W_N}{tN} \sim \frac{1}{2\pi} \sqrt{4x^{-1} - 1} dx$$

*with  $N \rightarrow \infty$ , with the limiting measure being the Marchenko-Pastur law  $\pi_1$ .*

PROOF. This follows again from Theorem 8.2 and Proposition 8.5. To be more precise, recall the main formula from Theorem 8.2, for the matrices as above, namely:

$$M_k \left( \frac{W_N}{N} \right) \simeq t^k C_k$$

By dividing by  $t^k$ , this formula can be written as follows:

$$M_k \left( \frac{W_N}{tN} \right) \simeq C_k$$

Now by using Proposition 8.5, we are led to the conclusion in the statement. □

Summarizing, we have deduced the Marchenko-Pastur theorem from the theorem regarding the Gaussian matrices, via some moment combinatorics. It is possible as well to be a bit more direct here, by passing through the Wigner theorem, and then recovering the Marchenko-Pastur law directly from the Wigner semicircle law, by performing a kind of square operation, but this is more or less the same thing as we did above.

### 8b. Parametric version

We discuss now a generalization of the above results, motivated by a whole array of concrete questions, and bringing into the picture a “true” parameter  $t > 0$ , which is different from the parameter  $t > 0$  used above, which is something quite trivial.

For this purpose, let us go back to the definition of the Wishart matrices. There were as follows, with  $Y$  being a  $N \times N$  matrix with i.i.d. entries, each following the law  $G_t$ :

$$W = YY^*$$

The point now is that, more generally, we can use in this  $W = YY^*$  construction a  $N \times M$  matrix  $Y$  with i.i.d. entries, each following the law  $G_t$ , with  $M \in \mathbb{N}$  being arbitrary. Thus, we have a new parameter, and by ditching the old parameter  $t > 0$ , which was something not very interesting, we are led to the following definition, which is the “true” definition of the Wishart matrices, from [64] and the subsequent literature:

DEFINITION 8.8. *A complex Wishart matrix is a  $N \times N$  matrix of the form*

$$W = YY^*$$

*where  $Y$  is a  $N \times M$  matrix with i.i.d. entries, each following the law  $G_1$ .*

As before with our previous Wishart matrices, that the new ones generalize, up to setting  $t = 1$ , we have  $W \geq 0$ , by definition. Due to this property, and to the otherwise “randomness” of  $W$ , these matrices are useful in many contexts. More on this later.

In order to see what is going on, combinatorially, let us compute moments. The result here is substantially more interesting than that for the previous Wishart matrices, with the new relevant numeric parameter being now the number  $t = M/N$ , as follows:

THEOREM 8.9. *Given a sequence of complex Wishart matrices*

$$W_N = Y_N Y_N^* \in M_N(L^\infty(X))$$

*with  $Y_N$  being  $N \times M$  complex Gaussian of parameter 1, we have*

$$M_k \left( \frac{W_N}{N} \right) \simeq \sum_{\pi \in NC(k)} t^{|\pi|}$$

*for any exponent  $k \in \mathbb{N}$ , in the  $M = tN \rightarrow \infty$  limit.*

PROOF. This is something which is very standard, as follows:

(1) Before starting, let us clarify the relation with our previous Wishart matrix results. In the case  $M = N$  we have  $t = 1$ , and the formula in the statement reads:

$$M_k \left( \frac{W_N}{N} \right) \simeq |NC(k)|$$

Thus, what we have here is the previous Wishart matrix formula, in full generality, at the value  $t = 1$  of our old parameter  $t > 0$ .

(2) Observe also that by rescaling, we can obtain if we want from this the previous Wishart matrix formula, in full generality, at any value  $t > 0$  of our old parameter. Thus, things fine, we are indeed generalizing what we did before.

(3) In order to prove now the formula in the statement, we proceed as usual, by using the Wick formula. The matrix entries of our Wishart matrix  $W = W_N$  are given by:

$$W_{ij} = \sum_{r=1}^M Y_{ir} \bar{Y}_{jr}$$

Thus, the normalized traces of powers of  $W$  are given by the following formula:

$$\begin{aligned} \text{tr}(W^k) &= \frac{1}{N} \sum_{i_1=1}^N \cdots \sum_{i_k=1}^N W_{i_1 i_2} W_{i_2 i_3} \cdots W_{i_k i_1} \\ &= \frac{1}{N} \sum_{i_1=1}^N \cdots \sum_{i_k=1}^N \sum_{r_1=1}^M \cdots \sum_{r_k=1}^M Y_{i_1 r_1} \bar{Y}_{i_2 r_1} Y_{i_2 r_2} \bar{Y}_{i_3 r_2} \cdots Y_{i_k r_k} \bar{Y}_{i_1 r_k} \end{aligned}$$

By rescaling now  $W$  by a  $1/N$  factor, as in the statement, we obtain:

$$\text{tr} \left( \left( \frac{W}{N} \right)^k \right) = \frac{1}{N^{k+1}} \sum_{i_1=1}^N \cdots \sum_{i_k=1}^N \sum_{r_1=1}^M \cdots \sum_{r_k=1}^M Y_{i_1 r_1} \bar{Y}_{i_2 r_1} Y_{i_2 r_2} \bar{Y}_{i_3 r_2} \cdots Y_{i_k r_k} \bar{Y}_{i_1 r_k}$$

(4) By using now the Wick rule, we obtain the following formula for the moments, with  $K = \circ \bullet \circ \bullet \dots$ , alternating word of length  $2k$ , and  $I = (i_1 r_1, i_2 r_1, \dots, i_k r_k, i_1 r_k)$ :

$$\begin{aligned} M_k \left( \frac{W}{N} \right) &= \frac{1}{N^{k+1}} \sum_{i_1=1}^N \cdots \sum_{i_k=1}^N \sum_{r_1=1}^M \cdots \sum_{r_k=1}^M \# \left\{ \pi \in \mathcal{P}_2(K) \mid \pi \leq \ker(I) \right\} \\ &= \frac{1}{N^{k+1}} \sum_{\pi \in \mathcal{P}_2(K)} \# \left\{ i \in \{1, \dots, N\}^k, r \in \{1, \dots, M\}^k \mid \pi \leq \ker(I) \right\} \end{aligned}$$

(5) In order to compute this quantity, we use the standard bijection  $\mathcal{P}_2(K) \simeq S_k$ . By identifying the pairings  $\pi \in \mathcal{P}_2(K)$  with their counterparts  $\pi \in S_k$ , we obtain:

$$M_k \left( \frac{W}{N} \right) = \frac{1}{N^{k+1}} \sum_{\pi \in S_k} \# \left\{ i \in \{1, \dots, N\}^k, r \in \{1, \dots, M\}^k \mid i_s = i_{\pi(s)+1}, r_s = r_{\pi(s)} \right\}$$

Now let  $\gamma \in S_k$  be the full cycle, which is by definition the following permutation:

$$\gamma = (1 \ 2 \ \dots \ k)$$

The general factor in the product computed above is then 1 precisely when following two conditions are simultaneously satisfied:

$$\gamma\pi \leq \ker i \quad , \quad \pi \leq \ker r$$

Counting the number of free parameters in our expectation formula, we obtain:

$$\begin{aligned} M_k \left( \frac{W}{N} \right) &= \frac{1}{N^{k+1}} \sum_{\pi \in S_k} N^{|\gamma\pi|} M^{|\pi|} \\ &= \sum_{\pi \in S_k} N^{|\gamma\pi| - k - 1} M^{|\pi|} \end{aligned}$$

(6) Now by using the same arguments as in the case  $M = N$ , from the proof of Theorem 8.2 above, we conclude that in the  $M = tN \rightarrow \infty$  limit the permutations  $\pi \in S_k$  which matter are those coming from noncrossing partitions, and so that we have:

$$M_k \left( \frac{W}{N} \right) \simeq \sum_{\pi \in NC(k)} N^{-|\pi|} M^{|\pi|} = \sum_{\pi \in NC(k)} t^{|\pi|}$$

We are therefore led to the conclusion in the statement.  $\square$

In order to recapture now the density out of the moments, we can of course use the Stieltjes inversion formula, but the computations here are a bit opaque. So, inspired from what happens at  $t = 1$ , let us cheat a bit, and formulate a nice definition, as follows:

DEFINITION 8.10. *The Marchenko-Pastur law  $\pi_t$  of parameter  $t > 0$  is given by:*

$$a \sim \gamma_t \implies a^2 \sim \pi_t$$

*That is,  $\pi_t$  the law of the square of a variable following the law  $\gamma_t$ .*

This is certainly very nice, and we know from chapter 7 that at  $t = 1$  we obtain indeed the Marchenko-Pastur law  $\pi_1$ , as constructed above. In general, we have:

PROPOSITION 8.11. *The Marchenko-Pastur law of parameter  $t > 0$  is*

$$\pi_t = \max(1 - t, 0)\delta_0 + \frac{\sqrt{4t - (x - 1 - t)^2}}{2\pi x} dx$$

*the support being  $[0, 4t^2]$ , and the moments of this measure are*

$$M_k = \sum_{\pi \in NC(k)} t^{|\pi|}$$

*exactly as for the asymptotic moments of the complex Wishart matrices.*

PROOF. This follows as usual, by doing some computations, either combinatorics, or calculus. To be more precise, we have three formulae for  $\pi_t$  to be connected, namely the one in Definition 8.10, and the two ones from the present statement, and the connections between them can be established exactly as we did before, at  $t = 1$ .  $\square$

Summarizing, we have now a definition for the Marchenko-Pastur law  $\pi_t$ , which is quite elegant, via Definition 8.10, but which still requires some computations, performed in the proof of Proposition 8.11. We will see later on, in chapters 9-12, an even more elegant definition for  $\pi_t$ , out of its particular case  $\pi_1$  which was well understood, simply obtained by considering the corresponding 1-parameter free convolution semigroup. We will also see that  $\pi_t$  appears as the “free version” of the Poisson law  $p_t$ , and that this can be even taken as a definition for  $\pi_t$ , if one really wants to. More on this later.

Now back to the complex Wishart matrices that we are interested in, in this chapter, we can now formulate a final result regarding them, as follows:

**THEOREM 8.12.** *Given a sequence of complex Wishart matrices*

$$W_N = Y_N Y_N^* \in M_N(L^\infty(X))$$

*with  $Y_N$  being  $N \times M$  complex Gaussian of parameter 1, we have*

$$\frac{W_N}{N} \sim \max(1-t, 0)\delta_0 + \frac{\sqrt{4t - (x-1-t)^2}}{2\pi x} dx$$

*with  $M = tN \rightarrow \infty$ , with the limiting measure being the Marchenko-Pastur law  $\pi_t$ .*

**PROOF.** This follows indeed from Theorem 8.9 and Proposition 8.11.  $\square$

As it was the case with the Gaussian and Wigner matrices, there are many other things that can be said about the complex Wishart matrices, as variations of the above. We refer here to the standard random matrix literature [1], [65], [68], [90]. We will be back to this right below, in the remainder of this chapter, with some wizarding computations from [4], and then more systematically in chapter 11 below, when doing free probability.

### 8c. Block modifications

We discuss in what follows a number of further results, regarding some more specialized random matrices, and more specifically the matrices obtained by performing certain suitable “block modifications” to the complex Wishart matrices. Our main goal here will be that of explaining a surprising result, due to Aubrun [4], stating that when suitably block-transposing the entries of a complex Wishart matrix, we obtain as asymptotic distribution a shifted version of Wigner’s semicircle law.

As before with the usual complex Wishart matrices, there will be some non-trivial combinatorics here, that we will fully understand only later, in chapters 9-12 below, when doing free probability. Thus, the material below will be an introduction to that.

Let us begin with some general block modification considerations, following the more recent papers [18], [19]. We have the following construction:

DEFINITION 8.13. *Given a Wishart  $dn \times dn$  matrix, appearing as*

$$W = YY^* \in M_{dn}(L^\infty(X))$$

*with  $Y$  being a complex Gaussian  $dn \times dm$  matrix, and a linear map*

$$\varphi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$$

*we consider the following matrix, obtained by applying  $\varphi$  to the  $n \times n$  blocks of  $W$ ,*

$$\tilde{W} = (id \otimes \varphi)W \in M_{dn}(L^\infty(X))$$

*and call it block-modified Wishart matrix.*

Here we are using of course some standard tensor product identifications, the details being as follows. Let  $Y$  be a complex Gaussian  $dn \times dm$  matrix, as above:

$$Y \in M_{dn \times dm}(L^\infty(X))$$

We can then form the corresponding complex Wishart matrix, as follows:

$$W = YY^* \in M_{dn}(L^\infty(X))$$

The size of this matrix being a composite number,  $N = dn$ , we can regard this matrix as being a  $n \times n$  matrix, with random  $d \times d$  matrices as entries. Equivalently, by using standard tensor product notations, this amounts in regarding  $W$  as follows:

$$W \in M_d(L^\infty(X)) \otimes M_n(\mathbb{C})$$

With this done, we can come up with our linear map  $\varphi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ , and apply it to the tensors on the right. We obtain in this way a matrix as follows:

$$\tilde{W} = (id \otimes \varphi)W \in M_d(L^\infty(X)) \otimes M_n(\mathbb{C})$$

Finally, we can forget now about tensors, and as a conclusion to all this, we have constructed a matrix as follows, that we can call block-modified Wishart matrix:

$$\tilde{W} \in M_{dn}(L^\infty(X))$$

This was for the detail of Definition 8.13, which is something quite tricky, requiring a good knowledge of the tensor product calculus. In practice now, what we mostly need for fully understanding Definition 8.13 are examples. Following Aubrun [4], and the series of papers by Collins and Nechita [32], [33], [34], we have the following basic examples:

DEFINITION 8.14. *We have the following examples of block-modified Wishart matrices  $\tilde{W} = (id \otimes \varphi)W$ , coming from various linear maps  $\varphi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ :*

- (1) *Wishart matrices:*  $\tilde{W} = W$ , obtained via  $\varphi = id$ .
- (2) *Aubrun matrices:*  $\tilde{W} = (id \otimes t)W$ , with  $t$  being the transposition.
- (3) *Collins-Nechita one:*  $\tilde{W} = (id \otimes \varphi)W$ , with  $\varphi = tr(\cdot)1$ .
- (4) *Collins-Nechita two:*  $\tilde{W} = (id \otimes \varphi)W$ , with  $\varphi$  erasing the off-diagonal part.

These examples, whose mathematical construction is something very elementary, but which appear in a wide context of interesting situations, for the most in connection with various questions in quantum physics [4], [32], [33], [34], [64], will actually serve as a main motivation for what we will be doing, in what follows. More on this later.

Getting back now to the general case, that of Definition 8.13, the linear map  $\varphi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$  there is certainly useful for understanding the construction of the block-modified Wishart matrix  $\tilde{W} = (id \otimes \varphi)W$ , as illustrated by the above examples. In practice, however, we would like to have as block-modification “data” something more concrete, such as a usual matrix. To be more precise, we would like to use:

**PROPOSITION 8.15.** *We have a correspondence between linear maps*

$$\varphi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$$

*and square matrices  $\Lambda \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ , given by the formula*

$$\Lambda_{ab,cd} = \varphi(e_{ac})_{bd}$$

*where  $e_{ab} \in M_n(\mathbb{C})$  are the standard generators of the matrix algebra  $M_n(\mathbb{C})$ , given by the formula  $e_{ab} : e_b \rightarrow e_a$ , with  $\{e_1, \dots, e_n\}$  being the standard basis of  $\mathbb{C}^n$ .*

**PROOF.** This is standard linear algebra. Given a linear map  $\varphi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ , we can associated to it numbers  $\Lambda_{ab,cd} \in \mathbb{C}$  by the formula in the statement, namely:

$$\Lambda_{ab,cd} = \varphi(e_{ac})_{bd}$$

Now by using these  $n^4$  numbers, we can construct a  $n^2 \times n^2$  matrix, as follows:

$$\Lambda = \sum_{abcd} \Lambda_{ab,cd} e_{ac} \otimes e_{bd} \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$$

Thus, we have constructed a correspondence  $\varphi \rightarrow \Lambda$ , and since this correspondence is injective, and the dimensions match, this correspondence is bijective, as claimed.  $\square$

Now by getting back to the block-modified Wishart matrices, we have:

**PROPOSITION 8.16.** *Given a Wishart  $dn \times dn$  matrix  $W = YY^*$ , and a linear map*

$$\varphi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$$

*the entries of the corresponding block-modified matrix  $\tilde{W} = (id \otimes \varphi)W$  are given by*

$$\tilde{W}_{ia,jb} = \sum_{cd} \Lambda_{ca,db} W_{ic,jd}$$

*where  $\Lambda \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$  is the square matrix associated to  $\varphi$ , as above.*

PROOF. Again, this is trivial linear algebra, coming from the following computation:

$$\begin{aligned}\tilde{W}_{ia,jb} &= \sum_{cd} W_{ic,jd} \varphi(e_{cd})_{ab} \\ &= \sum_{cd} \Lambda_{ca,db} W_{ic,jd}\end{aligned}$$

Thus, we are led to the conclusion in the statement.  $\square$

At the level of the main examples, from Definition 8.14, the very basic linear maps  $\varphi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$  used there can only correspond to the most basic examples of matrices  $\Lambda \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ , via the correspondence in Proposition 8.15. This is indeed the case, and we will be back to this in the moment, when discussing these examples.

Going ahead now with probability, we would like to study the distribution of the block-modified Wishart matrices  $\tilde{W} = (id \otimes \varphi)W$ . We will use as usual the moment method. However, things will be more tricky in the present setting, and we will need:

DEFINITION 8.17. *The generalized colored moments of a random matrix*

$$W \in M_N(L^\infty(X))$$

*with respect to a colored integer  $e = e_1 \dots e_p$ , and a permutation  $\sigma \in S_p$ , are the numbers*

$$M_e^\sigma(W) = \frac{1}{N^{|\sigma|}} \mathbb{E} \left( \sum_{i_1, \dots, i_p} W_{i_1 i_{\sigma(1)}}^{e_1} \dots W_{i_p i_{\sigma(p)}}^{e_p} \right)$$

*where  $|\sigma|$  is the number of cycles of  $\sigma$ .*

This is something quite technical, in the spirit of the free probability work in [70], that we will need in what follows. In order to understand how these generalized moments work, consider the standard cycle in  $S_p$ , namely:

$$\gamma = (1 \rightarrow 2 \rightarrow \dots \rightarrow p \rightarrow 1)$$

If we use this cycle  $\gamma \in S_p$  as our permutation  $\sigma \in S_p$  in the above definition, the corresponding generalized moment of a random matrix  $W$  is then the usual moment:

$$\begin{aligned}M_e^\gamma(W) &= \frac{1}{N} \mathbb{E} \left( \sum_{i_1, \dots, i_p} W_{i_1 i_2}^{e_1} \dots W_{i_p i_1}^{e_p} \right) \\ &= (\mathbb{E} \circ tr)(W^{e_1} \dots W^{e_p})\end{aligned}$$

In general, we can decompose the computation of  $M_e^\sigma(W)$  over the cycles of  $\sigma$ , and we obtain in this way a certain product of moments of  $W$ . See [70].

As a second illustration now, in relation with the usual square matrices, and more specifically with the square matrices  $\Lambda \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$  as in Proposition 8.15, we have the following formula, that we will use many times in what follows:

PROPOSITION 8.18. *Given a usual square matrix, of type*

$$\Lambda \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$$

*we have the following generalized moment formula*

$$(M_e^\sigma \otimes M_e^\tau)(\Lambda) = \frac{1}{n^{|\sigma|+|\tau|}} \sum_{i_1, \dots, i_p} \sum_{j_1, \dots, j_p} \Lambda_{i_1 j_1, i_{\sigma(1)} j_{\tau(1)}}^{e_1} \cdots \Lambda_{i_p j_p, i_{\sigma(p)} j_{\tau(p)}}^{e_p}$$

*valid for any two permutations  $\sigma, \tau \in S_p$ , and any colored integer  $e = e_1 \dots e_p$ .*

PROOF. This is something obvious, applying the construction in Definition 8.17 with  $N = n^2$ ,  $X = \{.\}$ ,  $W = \Lambda$ , and then making a tensor product of the corresponding moments  $M_e^\sigma$ ,  $M_e^\tau$ , regarded as linear functionals on  $M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ .  $\square$

Consider now the embedding  $NC_p \subset S_p$  obtained by “cycling inside each block”. That is, each block  $b = \{b_1, \dots, b_k\}$  with  $b_1 < \dots < b_k$  of a given noncrossing partition  $\sigma \in NC_p$  produces by definition the cycle  $(b_1 \dots b_k)$  of the corresponding permutation  $\sigma \in S_p$ . Observe that the one-block partition  $\gamma \in NC_p$  corresponds in this way to the standard cycle  $\gamma \in S_p$ . Also, the number of blocks  $|\sigma|$  of a partition  $\sigma \in NC_p$  corresponds in this way to the number of cycles  $|\sigma|$  of the corresponding permutation  $\sigma \in S_p$ .

With these conventions, we have the following result, from [18], [19], generalizing our various Wishart matrix moment computations that we did so far, in this book:

THEOREM 8.19. *The asymptotic moments of a block-modified Wishart matrix*

$$\tilde{W} = (id \otimes \varphi)W$$

*with parameters  $d, m, n \in \mathbb{N}$ , as above, are given by the formula*

$$\lim_{d \rightarrow \infty} M_e \left( \frac{\tilde{W}}{d} \right) = \sum_{\sigma \in NC_p} (mn)^{|\sigma|} (M_e^\sigma \otimes M_e^\gamma)(\Lambda)$$

*where  $\Lambda \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$  is the square matrix associated to  $\varphi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ .*

PROOF. We use the formula for the matrix entries of  $\tilde{W}$ , directly in terms of the matrix  $\Lambda$  associated to the map  $\varphi$ , from Proposition 8.16, namely:

$$\tilde{W}_{ia,jb} = \sum_{cd} \Lambda_{ca,db} W_{ic,jd}$$

By conjugating this formula, we obtain the following formula for the entries of the adjoint matrix  $\tilde{W}^*$ , that we will need as well:

$$\begin{aligned}\tilde{W}_{ia,jb}^* &= \sum_{cd} \bar{\Lambda}_{db,ca} \bar{W}_{jd,ic} \\ &= \sum_{cd} \Lambda_{ca,db}^* W_{ic,jd}\end{aligned}$$

Thus, we have the following global formula, valid for any exponent  $e \in \{1, *\}$ :

$$\tilde{W}_{ia,jb}^e = \sum_{cd} \Lambda_{ca,db}^e W_{ic,jd}$$

In order to compute the moments of  $\tilde{W}$ , observe first that we have:

$$\begin{aligned}tr(\tilde{W}^{e_1} \dots \tilde{W}^{e_p}) &= \frac{1}{dn} \sum_{i_r a_r} \prod_s \tilde{W}_{i_s a_s, i_{s+1} a_{s+1}}^{e_s} \\ &= \frac{1}{dn} \sum_{i_r a_r c_r d_r} \prod_s \Lambda_{c_s a_s, d_s a_{s+1}}^{e_s} W_{i_s c_s, i_{s+1} d_s} \\ &= \frac{1}{dn} \sum_{i_r a_r c_r d_r j_r b_r} \prod_s \Lambda_{c_s a_s, d_s a_{s+1}}^{e_s} Y_{i_s c_s, j_s b_s} \bar{Y}_{i_{s+1} d_s, j_s b_s}\end{aligned}$$

The average of the general term can be computed by the Wick rule, which gives:

$$\mathbb{E} \left( \prod_s Y_{i_s c_s, j_s b_s} \bar{Y}_{i_{s+1} d_s, j_s b_s} \right) = \# \left\{ \sigma \in S_p \mid i_{\sigma(s)} = i_{s+1}, c_{\sigma(s)} = d_s, j_{\sigma(s)} = j_s, b_{\sigma(s)} = b_s \right\}$$

Let us look now at the above sum. The  $i, j, b$  indices range over sets having respectively  $d, d, m$  elements, and they have to be constant under the action of:

$$\sigma \gamma^{-1}, \sigma, \sigma$$

Thus when summing over these  $i, j, b$  indices we simply obtain a factor as follows:

$$f = d^{|\sigma \gamma^{-1}|} d^{|\sigma|} m^{|\sigma|}$$

Thus, we obtain the following moment formula:

$$(\mathbb{E} \circ tr)(\tilde{W}^{e_1} \dots \tilde{W}^{e_p}) = \frac{1}{dn} \sum_{\sigma \in S_p} d^{|\sigma \gamma^{-1}|} (dm)^{|\sigma|} \sum_{a_r c_r} \prod_s \Lambda_{c_s a_s, c_{\sigma(s)} a_{s+1}}^{e_s}$$

On the other hand, we know from Proposition 8.18 that the generalized moments of the matrix  $\Lambda \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$  are given by the following formula:

$$(M_e^\sigma \otimes M_e^\tau)(\Lambda) = \frac{1}{n^{|\sigma|+|\tau|}} \sum_{i_1 \dots i_p} \sum_{j_1 \dots j_p} \Lambda_{i_1 j_1, i_{\sigma(1)} j_{\tau(1)}}^{e_1} \dots \Lambda_{i_p j_p, i_{\sigma(p)} j_{\tau(p)}}^{e_p}$$

By combining the above two formulae, we obtain the following moment formula:

$$(\mathbb{E} \circ \text{tr})(\tilde{W}^{e_1} \dots \tilde{W}^{e_p}) = \sum_{\sigma \in S_p} d^{|\sigma| + |\sigma\gamma^{-1}| - 1} (mn)^{|\sigma|} (M_e^\sigma \otimes M_e^\gamma)(\Lambda)$$

We use now the standard fact that for  $\sigma \in S_p$  we have an inequality as follows, with equality precisely when  $\sigma \in NC_p$ :

$$|\sigma| + |\sigma\gamma^{-1}| \leq p + 1$$

Thus in the  $d \rightarrow \infty$  limit the sum restricts over the partitions  $\sigma \in NC_p$ , and we get:

$$\lim_{d \rightarrow \infty} M_e(\tilde{W}) = d^p \sum_{\sigma \in NC_p} (mn)^{|\sigma|} (M_e^\sigma \otimes M_e^\gamma)(\Lambda)$$

Thus, we are led to the conclusion in the statement.  $\square$

With the above result in hand, we are left with the question of recovering the asymptotic law of  $\tilde{W} = (id \otimes \varphi)W$ , out of the asymptotic moments found there. The question here only involves the matrix  $\Lambda \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ , and to be more precise, given such a matrix, we would like to find the real or complex probability measure, or abstract distribution, having as colored moments the following numbers:

$$M_e = \sum_{\sigma \in NC_p} (mn)^{|\sigma|} (M_e^\sigma \otimes M_e^\gamma)(\Lambda)$$

Although this is basically a linear algebra problem, the underlying linear algebra is of quite difficult type, and this question cannot really be solved, in general. We will see however that this question can be solved for our basic examples, coming from Definition 8.14 above, and more generally, for a certain joint generalization of all these examples.

In short, we are now into troubled waters, and once again by following [18], [19], let us introduce, as a solution to all this, the following technical notion:

**DEFINITION 8.20.** *We call a square matrix  $\Lambda \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$  multiplicative when*

$$(M_e^\sigma \otimes M_e^\gamma)(\Lambda) = (M_e^\sigma \otimes M_e^\sigma)(\Lambda)$$

*holds for any  $p \in \mathbb{N}$ , any exponents  $e_1, \dots, e_p \in \{1, *\}$ , and any  $\sigma \in NC_p$ .*

This notion is something quite technical, but we will see many examples in what follows. For instance, the square matrices  $\Lambda$  coming from the basic linear maps  $\varphi$  appearing in Definition 8.14 are all multiplicative. More on this later.

Regarding now the output measure, that we want to compute, this can only appear as some kind of modification of the Marchenko-Pastur law  $\pi_t$ . Again by being a bit mysterious, and again following [18], [19], let us formulate as well:

PROPOSITION 8.21. *Given a real probability measure  $\mu$ , define its  $R$ -transform by:*

$$G_\mu(\xi) = \int_{\mathbb{R}} \frac{d\mu(t)}{\xi - t} \implies G_\mu \left( R_\mu(\xi) + \frac{1}{\xi} \right) = \xi$$

*The  $R$ -transform of the Marchenko-Pastur law  $\pi_t$  is then given by:*

$$R_{\pi_t}(\xi) = \frac{t}{1 - \xi}$$

*Based on this, a measure  $\mu$  having as  $R$ -transform a function of type*

$$R_\mu(\xi) = \sum_{i=1}^s \frac{c_i z_i}{1 - \xi z_i}$$

*with  $c_i > 0$  and  $z_i \in \mathbb{R}$ , is called modified Marchenko-Pastur law.*

PROOF. All this might seem a bit mysterious, but we are into difficult mathematics here, so let us just prove the result as stated, and we'll understand later what's behind these computations. We can prove the result in two steps, as follows:

(1) At  $t = 1$ , we know that the moments of  $\pi_1$  are the Catalan numbers,  $M_k = C_k$ . As explained in the proof of Proposition 8.5, we obtain from this that the corresponding Cauchy transform is given by the following formula:

$$G(\xi) = \frac{1}{2} - \frac{1}{2} \sqrt{1 - 4\xi^{-1}}$$

Now with  $R(\xi) = \frac{1}{1-\xi}$  being the function in the statement, at  $t = 1$ , we have:

$$\begin{aligned} G \left( R(\xi) + \frac{1}{\xi} \right) &= G \left( \frac{1}{1-\xi} + \frac{1}{\xi} \right) \\ &= G \left( \frac{1}{\xi - \xi^2} \right) \\ &= \frac{1}{2} - \frac{1}{2} \sqrt{1 - 4\xi + 4\xi^2} \\ &= \frac{1}{2} - \frac{1}{2} (1 - 2\xi) \\ &= \xi \end{aligned}$$

Thus, the function  $R(\xi) = \frac{1}{1-\xi}$  is indeed the  $R$ -transform of  $\pi_1$ , in the above sense.

(2) In the general case,  $t > 0$ , the proof is similar, by using the moment formula for  $\pi_t$ , that we know from the above. We will actually not really need this in what follows, with the present result mostly serving as an illustration for the modified Marchenko-Pastur laws that we want to introduce. Of course, we will be back to this with details when really needed, and more specifically in chapters 9-12 below, when doing free probability.  $\square$

As a comment on the above result, there is a similarity here with the theory of the compound Poisson laws from chapter 2 above. The truth regarding all this is that the Marchenko-Pastur law  $\pi_t$  is the free Poisson law of parameter  $t$ , the modified Marchenko-Pastur laws, as introduced above, are the general compound free Poisson laws, and finally the mysterious  $R$ -transform used above is the Voiculescu  $R$ -transform [86], which is the analogue of the log of the Fourier transform in free probability. More on this later.

Based on this analogy, however, we can label our modified Marchenko-Pastur laws, in the same way as we labelled in chapter 2 the compound Poisson laws, as follows:

DEFINITION 8.22. *We denote by  $\pi_\rho$  the modified Marchenko-Pastur law satisfying*

$$R_\mu(\xi) = \sum_{i=1}^s \frac{c_i z_i}{1 - \xi z_i}$$

with  $c_i > 0$  and  $z_i \in \mathbb{R}$ , with  $\rho$  being the following measure,

$$\rho = \sum_{i=1}^s c_i \delta_{z_i}$$

which is a discrete positive measure in the complex plane, not necessarily of mass 1.

As basic examples here, for  $\rho = \delta_1$  we obtain the Marchenko-Pastur law  $\pi_1$ , and more generally for  $\rho = t\delta_1$  with  $t > 0$  we obtain the Marchenko-Pastur law  $\pi_t$ , as shown by Proposition 8.21 above. More on this later, when doing free probability.

Getting back now to the block-modified Wishart matrices, and to the formula in Theorem 8.19, the above abstract notions, from Definition 8.20 and from Definition 8.22, are exactly what we need for further improving all this. Again by following [18], [19], we have the following result, substantially building on Theorem 8.19:

THEOREM 8.23. *Consider a block-modified Wishart matrix*

$$\tilde{W} = (id \otimes \varphi)W$$

and assume that the matrix  $\Lambda \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$  associated to  $\varphi$  is multiplicative. Then

$$\frac{\tilde{W}}{d} \sim \pi_{mnp}$$

holds, in moments, in the  $d \rightarrow \infty$  limit, where  $\rho = \text{law}(\Lambda)$ .

PROOF. This is something quite tricky, using all the above:

(1) Our starting point is the asymptotic moment formula found in Theorem 8.19, for an arbitrary block-modified Wishart matrix, namely:

$$\lim_{d \rightarrow \infty} M_e \left( \frac{\tilde{W}}{d} \right) = \sum_{\sigma \in NC_p} (mn)^{|\sigma|} (M_e^\sigma \otimes M_e^\gamma)(\Lambda)$$

(2) Since our modification matrix  $\Lambda \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$  was assumed to be multiplicative, in the sense of Definition 8.20, this formula reads:

$$\lim_{d \rightarrow \infty} M_e \left( \frac{\tilde{W}}{d} \right) = \sum_{\sigma \in NC_p} (mn)^{|\sigma|} (M_e^\sigma \otimes M_e^\sigma)(\Lambda)$$

(3) On the other hand, a bit of calculus and combinatorics show that, in the context of Definition 8.22, given a square matrix  $\Lambda \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ , having distribution  $\rho = \text{law}(\Lambda)$ , the moments of the modified Marchenko-Pastur law  $\pi_{mn\rho}$  are given by the following formula, for any choice of the extra parameter  $m \in \mathbb{N}$ :

$$M_e(\pi_{mn\rho}) = \sum_{\sigma \in NC_p} (mn)^{|\sigma|} (M_\sigma^e \otimes M_\sigma^e)(\Lambda)$$

(4) The point now is that with this latter formula in hand, our previous asymptotic moment formula for the block-modified Wishart matrix  $\tilde{W}$  simply reads:

$$\lim_{d \rightarrow \infty} M_e \left( \frac{\tilde{W}}{d} \right) = M_e(\pi_{mn\rho})$$

Thus we have indeed  $\frac{\tilde{W}}{d} \sim \pi_{mn\rho}$ , in the  $d \rightarrow \infty$  limit, as stated.  $\square$

All the above was of course a bit technical, but we will come back later to this, with some further details, once we will have a better understanding of the  $R$ -transform, of the free Poisson limit theorem, and of the other things which are hidden in all the above. In any case, welcome to free probability. Or perhaps to theoretical physics. The above theorem was our first free probability one, in this book, and many other to follow.

Let us we work out now some explicit consequences of Theorem 8.23, by using some special classes of modification maps  $\varphi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ . Let us begin with:

**DEFINITION 8.24.** *Let  $P(k, l)$  be the set of partitions between an upper row of  $k$  points, and a lower row of  $l$  points. Associated to any  $\pi \in P(k, l)$  is the linear map*

$$T_\pi(e_{i_1} \otimes \dots \otimes e_{i_k}) = \sum_{j_1 \dots j_l} \delta_\pi \begin{pmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_l \end{pmatrix} e_{j_1} \otimes \dots \otimes e_{j_l}$$

*between tensor powers of  $\mathbb{C}^N$ , called “easy”, with the Kronecker type symbol on the right being given by  $\delta_\pi = 1$  when the indices fit, and  $\delta_\pi = 0$  otherwise.*

Observe the connection with notion of easy group, from chapters 3-4 above. We will be back to this later in this book, when talking about easy quantum groups.

Now back to our questions, we have the following notion:

DEFINITION 8.25. Associated to any partition  $\pi \in P(2s, 2s)$  is the linear map

$$\varphi_\pi(e_{a_1 \dots a_s, c_1 \dots c_s}) = \sum_{b_1 \dots b_s} \sum_{d_1 \dots d_s} \delta_\pi \begin{pmatrix} a_1 & \dots & a_s & c_1 & \dots & c_s \\ b_1 & \dots & b_s & d_1 & \dots & d_s \end{pmatrix} e_{b_1 \dots b_s, d_1 \dots d_s}$$

obtained from  $T_\pi$  by contracting all the tensors, via the following operation:

$$e_{i_1} \otimes \dots \otimes e_{i_{2s}} \rightarrow e_{i_1 \dots i_s, i_{s+1} \dots i_{2s}}$$

Here, as in Definition 8.24 above,  $\{e_1, \dots, e_N\}$  is the standard basis of  $\mathbb{C}^N$ , with  $N \in \mathbb{N}$  being some fixed integer, and  $\{e_{ij}\}$  is the corresponding basis of  $M_N(\mathbb{C})$ .

In relation with our Wishart matrix considerations, the point is that the above linear map  $\varphi_\pi$  can be viewed as a “block-modification” map, as follows:

$$\varphi_\pi : M_{N^s}(\mathbb{C}) \rightarrow M_{N^s}(\mathbb{C})$$

In order to verify that the corresponding matrices  $\Lambda_\pi$  are multiplicative, we will need to check that all the functions  $\varphi(\sigma, \tau) = (M_\sigma^e \otimes M_\tau^e)(\Lambda_\pi)$  have the following property:

$$\varphi(\sigma, \gamma) = \varphi(\sigma, \sigma)$$

For this purpose, we can use the following result, coming from [19]:

PROPOSITION 8.26. The following functions  $\varphi : NC_p \times NC_p \rightarrow \mathbb{R}$  are “multiplicative”, in the sense that they satisfy the condition  $\varphi(\sigma, \gamma) = \varphi(\sigma, \sigma)$ :

- (1)  $\varphi(\sigma, \tau) = |\sigma\tau^{-1}| - |\tau|$ .
- (2)  $\varphi(\sigma, \tau) = |\sigma\tau| - |\tau|$ .
- (3)  $\varphi(\sigma, \tau) = |\sigma \wedge \tau| - |\tau|$ .

PROOF. All this is elementary, and can be proved as follows:

- (1) This follows indeed from the following computation:

$$\varphi_1(\sigma, \gamma) = |\sigma\gamma^{-1}| - 1 = p - |\sigma| = \varphi_1(\sigma, \sigma)$$

- (2) This follows indeed from the following computation:

$$\varphi_2(\sigma, \gamma) = |\sigma\gamma| - 1 = |\sigma^2| - |\sigma| = \varphi_2(\sigma, \sigma)$$

- (3) This follows indeed from the following computation:

$$\varphi_3(\sigma, \gamma) = |\gamma| - |\gamma| = 0 = |\sigma| - |\sigma| = \varphi_3(\sigma, \sigma)$$

Thus, we are led to the conclusions in the statement.  $\square$

As an illustration, let us discuss the case  $s = 1$ . There are 15 partitions  $\pi \in P(2, 2)$ , and among them, the most “basic” are the 4 partitions  $\pi \in P_{\text{even}}(2, 2)$ . We have:

PROPOSITION 8.27. *The partitions  $\pi \in P_{\text{even}}(2, 2)$  are as follows,*

$$\pi_1 = \begin{bmatrix} \circ & \bullet \\ \circ & \bullet \end{bmatrix} \quad , \quad \pi_2 = \begin{bmatrix} \circ & \bullet \\ \bullet & \circ \end{bmatrix} \quad , \quad \pi_3 = \begin{bmatrix} \circ & \circ \\ \bullet & \bullet \end{bmatrix} \quad , \quad \pi_4 = \begin{bmatrix} \circ & \circ \\ \circ & \circ \end{bmatrix}$$

with the associated linear maps  $\varphi_\pi : M_n(\mathbb{C}) \rightarrow M_N(\mathbb{C})$  being as follows:

$$\varphi_1(A) = A \quad , \quad \varphi_2(A) = A^t \quad , \quad \varphi_3(A) = \text{Tr}(A)1 \quad , \quad \varphi_4(A) = A^\delta$$

The corresponding matrices  $\Lambda_\pi$  are all multiplicative, in the sense of Definition 8.20.

PROOF. We use the general formula in Definition 8.25. In the case  $s = 1$ , that we are interested in here, this formula becomes:

$$\varphi_\pi(e_{ac}) = \sum_{bd} \delta_\pi \begin{pmatrix} a & c \\ b & d \end{pmatrix} e_{bd}$$

In the case of the 4 partitions in the statement, such maps are given by:

$$\varphi_1(e_{ac}) = e_{ac} \quad , \quad \varphi_2(e_{ac}) = e_{ca} \quad , \quad \varphi_3(e_{ac}) = \delta_{ac} \sum_b e_{bb} \quad , \quad \varphi_4(e_{ac}) = \delta_{ac} e_{aa}$$

Thus, we obtain the formulae in the statement. Regarding now the associated square matrices, appearing via  $\Lambda_{ab,cd} = \varphi(e_{ac})_{bd}$ , these are given by:

$$\Lambda_{ab,cd}^1 = \delta_{ab} \delta_{cd} \quad , \quad \Lambda_{ab,cd}^2 = \delta_{ad} \delta_{bc} \quad , \quad \Lambda_{ab,cd}^3 = \delta_{ac} \delta_{bd} \quad , \quad \Lambda_{ab,cd}^4 = \delta_{abcd}$$

Since these matrices are all self-adjoint, we can assume that all the exponents are 1 in Definition 8.20, and the condition there becomes:

$$(M_\sigma \otimes M_\tau)(\Lambda) = (M_\sigma \otimes M_\tau)(\Lambda)$$

In order to check this condition, observe that for the above 4 matrices, we have:

$$\begin{aligned} (M^\sigma \otimes M^\tau)(\Lambda_1) &= \frac{1}{n^{|\sigma|+|\tau|}} \sum_{i_1 \dots i_p} \delta_{i_{\sigma(1)} i_{\tau(1)}} \dots \delta_{i_{\sigma(p)} i_{\tau(p)}} = n^{|\sigma\tau^{-1}| - |\sigma| - |\tau|} \\ (M^\sigma \otimes M^\tau)(\Lambda_2) &= \frac{1}{n^{|\sigma|+|\tau|}} \sum_{i_1 \dots i_p} \delta_{i_1 i_{\sigma\tau(1)}} \dots \delta_{i_p i_{\sigma\tau(p)}} = n^{|\sigma\tau| - |\sigma| - |\tau|} \\ (M^\sigma \otimes M^\tau)(\Lambda_3) &= \frac{1}{n^{|\sigma|+|\tau|}} \sum_{i_1 \dots i_p} \sum_{j_1 \dots j_p} \delta_{i_1 i_{\sigma(1)}} \delta_{j_1 j_{\tau(1)}} \dots \delta_{i_p i_{\sigma(p)}} \delta_{j_p j_{\tau(p)}} = 1 \\ (M^\sigma \otimes M^\tau)(\Lambda_4) &= \frac{1}{n^{|\sigma|+|\tau|}} \sum_{i_1 \dots i_p} \delta_{i_1 i_{\sigma(1)} i_{\tau(1)}} \dots \delta_{i_p i_{\sigma(p)} i_{\tau(p)}} = n^{|\sigma \wedge \tau| - |\sigma| - |\tau|} \end{aligned}$$

By using now the results in Proposition 8.26, this gives the result.  $\square$

Summarizing, the partitions  $\pi \in P_{\text{even}}(2, 2)$  provide us with some concrete input for Theorem 8.23. The point now is that, when using this input, we obtain the main known computations for the block-modified Wishart matrices, from [4], [32], [33], [64]:

THEOREM 8.28. *The asymptotic distribution results for the block-modified Wishart matrices coming from the partitions  $\pi_1, \pi_2, \pi_3, \pi_4 \in P_{\text{even}}(2, 2)$  are as follows:*

- (1) *Marchenko-Pastur:*  $\frac{1}{d}W \sim \pi_t$ , where  $t = m/n$ .
- (2) *Aubrun type:*  $\frac{1}{d}(id \otimes t)W \sim \pi_\nu$ , with  $\nu = \frac{m(n-1)}{2}\delta_{-1} + \frac{m(n+1)}{2}\delta_1$ .
- (3) *Collins-Nechita one:*  $n(id \otimes tr(\cdot)1)W \sim \pi_t$ , where  $t = mn$ .
- (4) *Collins-Nechita two:*  $\frac{1}{d}(id \otimes (\cdot)^\delta)W \sim \pi_m$ .

PROOF. All these results follow from Theorem 8.23, with the maps  $\varphi_1, \varphi_2, \varphi_3, \varphi_4$  in Proposition 8.27 producing the 4 matrices in the statement, modulo some rescalings, and with the computation of the corresponding distributions being as follows:

(1) Here  $\Lambda = \sum_{ac} e_{ac} \otimes e_{ac}$ , and so  $\Lambda = nP$ , where  $P$  is the rank one projection on  $\sum_a e_a \otimes e_a \in \mathbb{C}^n \otimes \mathbb{C}^n$ . Thus we have the following formula, which gives the result:

$$\rho = \frac{n^2 - 1}{n^2} \delta_0 + \frac{1}{n^2} \delta_n$$

(2) Here  $\Lambda = \sum_{ac} e_{ac} \otimes e_{ca}$  is the flip operator,  $\Lambda(e_c \otimes e_a) = e_a \otimes e_c$ . Thus  $\rho = \frac{n-1}{2n} \delta_{-1} + \frac{n+1}{2n} \delta_1$ , and so we have the following formula, which gives the result:

$$mn\rho = \frac{m(n-1)}{2} \delta_{-1} + \frac{m(n+1)}{2} \delta_1$$

(3) Here  $\Lambda = \sum_{ab} e_{aa} \otimes e_{bb}$  is the identity matrix,  $\Lambda = 1$ . Thus in this case we have the following formula, which gives  $\pi_{mn\rho} = \pi_{mn}$ , and so  $n\tilde{W} \sim \pi_{mn}$ , as claimed:

$$\rho = \delta_1$$

(4) Here  $\Lambda = \sum_a e_{aa} \otimes e_{aa}$  is the orthogonal projection on  $\text{span}(e_a \otimes e_a) \subset \mathbb{C}^n \otimes \mathbb{C}^n$ . Thus we have the following formula, which gives the result:

$$\rho = \frac{n-1}{n} \delta_0 + \frac{1}{n} \delta_1$$

Summarizing, we have proved all the assertions in the statement.  $\square$

As a conclusion to all this, the block modification of the complex Wishart matrices, which is a quite innocent-looking operation, leads, somehow out of nothing, to a whole new world, populated by beasts such as the  $R$ -transform, the modified Marchenko-Pastur laws, and many more. Looks like we have opened the Pandora box.

We will see later on, in chapters 9-12 below, and afterwards, that this whole new world, called free probability, is in fact not very different from ours. To be more precise, save for some tyrannosaurs coming from Proposition 8.4, which has no classical counterpart, the classical and free things will be in gentle bijection.

### 8d. Shifted semicircles

Things have become fairly complicated in this book, and time to do some free probability, in order to clarify all this. However, as a last thing before that, we still have to understand what comes out of Theorem 8.28 (2). Following [18], we first have:

PROPOSITION 8.29. *The asymptotic moment generating function of  $\frac{\tilde{W}}{d}$ , with*

$$\tilde{W} = (id \otimes t)W$$

*being a block-transposed Wishart matrix, satisfies the following equation:*

$$(F - 1)(1 - z^2 F^2) = mzF(1 + nzF)$$

PROOF. We know from Theorem 8.28 that we have  $\frac{\tilde{W}}{d} \sim \pi_\nu$  with  $d \rightarrow \infty$ , where:

$$\nu = \frac{m(n-1)}{2} \delta_{-1} + \frac{m(n+1)}{2} \delta_1$$

With some calculus, as in the proof of Proposition 8.21, this gives the result.  $\square$

We can now recover the original result of Aubrun [4], as follows:

THEOREM 8.30. *For a block-transposed Wishart matrix*

$$\tilde{W} = (id \otimes t)W$$

*we have, in the  $n = \beta m \rightarrow \infty$  limit, with  $\beta > 0$  fixed, the formula*

$$\frac{\tilde{W}}{d} \sim \gamma_\beta^1$$

*with  $\gamma_\beta^1$  being the shifted version of the semicircle law  $\gamma_\beta$ , with support centered at 1.*

PROOF. This follows from Proposition 8.29, and some calculus. Consider indeed the equation of  $F$  found in Proposition 8.29, namely:

$$(F - 1)(1 - z^2 F^2) = mzF(1 + nzF)$$

In order to reach to the Stieltjes transform, we use the theory of the  $R$ -transform from Proposition 8.21. With  $z \rightarrow \xi^{-1}$  and  $F \rightarrow \xi G$ , so that  $zF \rightarrow G$ , we obtain:

$$(\xi G - 1)(1 - G^2) = mG(1 + nG)$$

Now with  $\xi \rightarrow K$  and  $G \rightarrow z$  we obtain the following formula:

$$(zK - 1)(1 - z^2) = mz(1 + nz)$$

Finally, with  $K \rightarrow R + z^{-1}$  we obtain the following formula:

$$zR(1 - z^2) = mz(1 + nz)$$

Thus the  $R$ -transform of the asymptotic law of  $mW^\Gamma$  is given by:

$$\begin{aligned} R &= m \cdot \frac{1 + nz}{1 - z^2} \\ &= \frac{m}{2} \left( \frac{n+1}{1-z} - \frac{n-1}{1+z} \right) \end{aligned}$$

Now in the  $n = \beta m \rightarrow \infty$  limit, with  $\beta > 0$  fixed, as in the statement, we are led to the following formula for the corresponding Stieltjes transform:

$$f(x) = \frac{\sqrt{4\beta - (1-x)^2}}{2\beta\pi}$$

But this is the density of the shifted semicircle law having support as follows:

$$S = [1 - 2\sqrt{\beta}, 1 + 2\sqrt{\beta}]$$

Thus, we are led to the conclusion in the statement. See [4], [18].  $\square$

Here we have used some standard free probability results at the end, which can be proved by direct computations, and we will be back to this in chapters 9-12 below.

### 8e. Exercises

There has been a lot of combinatorics in this chapter, in relation with the Wishart matrices and the Marchenko-Pastur laws, and as an exercise here, we have:

**EXERCISE 8.31.** *Work out with full details the proof of the Aubrun result regarding the block-transposed Wishart matrices, directly, out of the Wick formula.*

To be more precise, we have seen a proof of this result, but based on heavy, general methods from [19]. A lighter proof is the one in [18], dealing with the block-transposed Wishart matrices only. And there is an even lighter proof, the one in [4], dealing with the block-transposed Wishart matrices, in the  $n = \beta m \rightarrow \infty$  regime. So, find your favorite proof, fully read and understand it, and write down a brief account of that.

## Part III

# Free probability

*Winterlude, Winterlude, my little daisy*  
*Winterlude by the telephone wire*  
*Winterlude, it's making me lazy*  
*Come on, sit by the logs in the fire*

## CHAPTER 9

### Free probability

#### 9a. Freeness

In this chapter and in the next three ones we discuss the foundations and main results of free probability, in analogy with the foundations and main results of classical probability. The common framework for classical and free probability is “noncommutative probability”. This is something very general, that we already met in connection with the random matrices, in chapters 5-8. We first recall this material. Let us start with:

**DEFINITION 9.1.** *A  $C^*$ -algebra is a complex algebra  $A$ , having a norm  $||\cdot||$  making it a Banach algebra, and an involution  $*$ , related to the norm by the formula*

$$||aa^*|| = ||a||^2$$

*which must hold for any  $a \in A$ .*

As a basic example, the algebra  $B(H)$  of the bounded linear operators  $T : H \rightarrow H$  on a Hilbert space  $H$  is a  $C^*$ -algebra, with the usual norm and involution:

$$||T|| = \sup_{||x||=1} ||Tx|| \quad , \quad \langle Tx, y \rangle = \langle x, T^*y \rangle$$

More generally, any closed  $*$ -subalgebra of  $B(H)$  is a  $C^*$ -algebra. It is possible to prove that any  $C^*$ -algebra appears in this way, as explained in chapter 6:

$$A \subset B(H)$$

In finite dimensions we have  $H = \mathbb{C}^N$ , and so the operator algebra  $B(H)$  is the usual matrix algebra  $M_N(\mathbb{C})$ , with the usual norm and involution, namely:

$$||M|| = \sup_{||x||=1} ||Mx|| \quad , \quad (M^*)_{ij} = \bar{M}_{ji}$$

As explained in chapter 3, some elementary algebra shows that the finite dimensional  $C^*$ -algebras are the direct sums of matrix algebras:

$$A = M_{n_1}(\mathbb{C}) \oplus \dots \oplus M_{n_k}(\mathbb{C})$$

Summarizing, the  $C^*$ -algebra formalism is something in between the  $*$ -algebras, which are purely algebraic objects and whose theory basically leads nowhere, and fully advanced operator algebras, such as the von Neumann algebras. More on this later.

As yet another class of examples now, which are of particular importance for us, we have various algebras of functions  $f : X \rightarrow \mathbb{C}$ . The theory here is as follows:

**THEOREM 9.2.** *The commutative  $C^*$ -algebras are the algebras of type  $C(X)$ , with  $X$  being a compact space, the correspondence being as follows:*

- (1) *Given a compact space  $X$ , the algebra  $C(X)$  of continuous functions  $f : X \rightarrow \mathbb{C}$  is a commutative  $C^*$ -algebra, with norm and involution as follows:*

$$\|f\| = \sup_{x \in X} |f(x)| \quad , \quad f^*(x) = \overline{f(x)}$$

- (2) *Conversely, any commutative  $C^*$ -algebra can be written as  $A = C(X)$ , with its “spectrum” appearing as the space of Banach algebra characters of  $A$ :*

$$X = \{\chi : A \rightarrow \mathbb{C}\}$$

*In view of this, given an arbitrary  $C^*$ -algebra  $A$ , not necessarily commutative, we agree to write  $A = C(X)$ , and call the abstract space  $X$  a compact quantum space.*

**PROOF.** This is something that we know from chapter 6, the idea being as follows:

(1) First of all, the fact that  $C(X)$  is a Banach algebra is clear, because a uniform limit of continuous functions must be continuous. As for the formula  $\|ff^*\| = \|f\|^2$ , this is something trivial for functions, because on both sides we obtain  $\sup_{x \in X} |f(x)|^2$ .

(2) Given a commutative  $C^*$ -algebra  $A$ , the character space  $X = \{\chi : A \rightarrow \mathbb{C}\}$  is indeed compact, and we have an evaluation morphism  $ev : A \rightarrow C(X)$ . The tricky point, which follows from basic spectral theory, is to prove that  $ev$  is indeed isometric.  $\square$

The above result is quite interesting for us, because it allows one to formally write any  $C^*$ -algebra as  $A = C(X)$ , with  $X$  being a noncommutative compact space. This is certainly something very nice, and in order to do now some probability theory over such spaces  $X$ , we would need probability measures  $\mu$ . But, the problem is that these measures  $\mu$  are impossible to define, because our spaces  $X$  have no points in general.

However, we can trick, and do probability theory just by using expectations functionals  $\mathbb{E} : A \rightarrow \mathbb{C}$ , instead of the probability measures  $\mu$  themselves. These expectations are called traces, are denoted  $tr : A \rightarrow \mathbb{C}$ , and their axiomatization is as follows:

**DEFINITION 9.3.** *A trace, or expectation, or integration functional, on a  $C^*$ -algebra  $A$  is a linear form  $tr : A \rightarrow \mathbb{C}$  having the following properties:*

- (1)  *$tr$  is unital, and continuous.*
- (2)  *$tr$  is positive,  $a \geq 0 \implies \varphi(a) \geq 0$ .*
- (3)  *$tr$  has the trace property  $tr(ab) = tr(ba)$ .*

*We call  $tr$  faithful when  $a > 0 \implies \varphi(a) > 0$ .*

In the commutative case,  $A = C(X)$ , the Riesz theorem shows that the positive traces  $tr : A \rightarrow \mathbb{C}$  appear as integration functionals with respect to positive measures  $\mu$ :

$$tr(f) = \int_X f(x) d\mu(x)$$

Moreover, the unitality of  $tr$  corresponds to the fact that  $\mu$  has mass one, and the faithfulness of  $tr$  corresponds to the faithfulness of  $\mu$ . Thus, in general, when  $A$  is no longer commutative, in order to do probability theory on the underlying noncommutative compact space  $X$ , what we need is a faithful trace  $tr : A \rightarrow \mathbb{C}$  as above.

So, this will be our philosophy in what follows, a noncommutative probability space  $(X, \mu)$  being something abstract, corresponding in practice to a pair  $(A, tr)$ . This is of course something a bit simplified, because associated to any space  $X$ , noncommutative or even classical, there are in fact many possible  $C^*$ -algebras of functions  $f : X \rightarrow \mathbb{C}$ , such as  $C(X)$ ,  $L^\infty(X)$  and so on, and for a better theory, we would have to make a choice between these various  $C^*$ -algebras associated to  $X$ . But let us not worry with this for the moment, what we have is good for starting some computations, so let us just do these computations, see what we get, and we will come back later to more about formalism.

Going ahead with definitions, everything in what follows will be based on:

DEFINITION 9.4. *Let  $A$  be a  $C^*$ -algebra, given with a trace  $tr : A \rightarrow \mathbb{C}$ .*

- (1) *The elements  $a \in A$  are called random variables.*
- (2) *The moments of such a variable are the numbers  $M_k(a) = tr(a^k)$ .*
- (3) *The law of such a variable is the functional  $\mu : P \rightarrow tr(P(a))$ .*

Here  $k = \circ \bullet \bullet \circ \dots$  is by definition a colored integer, and the corresponding powers  $a^k$  are defined by the following formulae, and multiplicativity:

$$a^\emptyset = 1 \quad , \quad a^\circ = a \quad , \quad a^\bullet = a^*$$

As for the polynomial  $P$ , this is a noncommuting  $*$ -polynomial in one variable:

$$P \in \mathbb{C} \langle X, X^* \rangle$$

Observe that the law is uniquely determined by the moments, because we have:

$$P(X) = \sum_k \lambda_k X^k \implies \mu(P) = \sum_k \lambda_k M_k(a)$$

Generally speaking, the above definition is something quite abstract, but there is no other way of doing things, at least at this level of generality. However, in certain special cases, the formalism simplifies, and we recover more familiar objects, as follows:

**THEOREM 9.5.** *Assuming that  $a \in A$  is normal,  $aa^* = a^*a$ , its law corresponds to a probability measure on its spectrum  $\sigma(a) \subset \mathbb{C}$ , according to the following formula:*

$$\text{tr}(P(a)) = \int_{\sigma(a)} P(x) d\mu(x)$$

*When the trace is faithful we have  $\text{supp}(\mu) = \sigma(a)$ . Also, in the particular case where the variable is self-adjoint,  $a = a^*$ , this law is a real probability measure.*

**PROOF.** This is something very standard, coming from the continuous functional calculus in  $C^*$ -algebras, explained in chapter 6. In fact, we can deduce from there that more is true, in the sense that the following formula holds, for any  $f \in C(\sigma(a))$ :

$$\text{tr}(f(a)) = \int_{\sigma(a)} f(x) d\mu(x)$$

In addition, assuming that we are in the case  $A \subset B(H)$ , the measurable functional calculus tells us that the above formula holds in fact for any  $f \in L^\infty(\sigma(a))$ .  $\square$

We have the following independence notion, generalizing the one from chapter 1:

**DEFINITION 9.6.** *Two subalgebras  $A, B \subset C$  are called independent when the following condition is satisfied, for any  $a \in A$  and  $b \in B$ :*

$$\text{tr}(ab) = \text{tr}(a)\text{tr}(b)$$

*Equivalently, the following condition must be satisfied, for any  $a \in A$  and  $b \in B$ :*

$$\text{tr}(a) = \text{tr}(b) = 0 \implies \text{tr}(ab) = 0$$

*Also, two variables  $a, b \in C$  are called independent when the algebras that they generate,*

$$A = \langle a \rangle, \quad B = \langle b \rangle$$

*are independent inside  $C$ , in the above sense.*

Observe that the above two independence conditions are indeed equivalent, with this following from the following computation, with the convention  $a' = a - \text{tr}(a)$ :

$$\begin{aligned} \text{tr}(ab) &= \text{tr}[(a' + \text{tr}(a))(b' + \text{tr}(b))] \\ &= \text{tr}(a'b') + \text{tr}(a')\text{tr}(b) + \text{tr}(a)\text{tr}(b') + \text{tr}(a)\text{tr}(b) \\ &= \text{tr}(a'b') + \text{tr}(a)\text{tr}(b) \\ &= \text{tr}(a)\text{tr}(b) \end{aligned}$$

The other remark is that the above notion generalizes indeed the usual notion of independence, from the classical case, the precise result here being as follows:

**THEOREM 9.7.** *Given two compact measured spaces  $X, Y$ , the algebras*

$$C(X) \subset C(X \times Y), \quad C(Y) \subset C(X \times Y)$$

*are independent in the above sense, and a converse of this fact holds too.*

PROOF. We have two assertions here, the idea being as follows:

(1) First of all, given two abstract compact spaces  $X, Y$ , we have embeddings of algebras as in the statement, defined by the following formulae:

$$f \rightarrow [(x, y) \rightarrow f(x)] \quad , \quad g \rightarrow [(x, y) \rightarrow g(y)]$$

In the measured space case now, the Fubini theorems tells us that we have:

$$\int_{X \times Y} f(x)g(y) = \int_X f(x) \int_Y g(y)$$

Thus, the algebras  $C(X), C(Y)$  are independent in the sense of Definition 9.6.

(2) Conversely, assume that  $A, B \subset C$  are independent, with  $C$  being commutative. Let us write our algebras as follows, with  $X, Y, Z$  being certain compact spaces:

$$A = C(X) \quad , \quad B = C(Y) \quad , \quad C = C(Z)$$

In this picture, the inclusions  $A, B \subset C$  must come from quotient maps, as follows:

$$p : Z \rightarrow X \quad , \quad q : Z \rightarrow Y$$

Regarding now the independence condition from Definition 9.6, in the above picture, this tells us that the following equality must happen:

$$\int_Z f(p(z))g(q(z)) = \int_Z f(p(z)) \int_X g(q(z))$$

Thus we are in a Fubini type situation, and we obtain from this:

$$X \times Y \subset Z$$

Thus, the independence of the algebras  $A, B \subset C$  appears as in (1) above.  $\square$

It is possible to develop some theory here, but this is ultimately not very interesting. As a much more interesting notion now, we have Voiculescu's freeness [85]:

DEFINITION 9.8. *Two subalgebras  $A, B \subset C$  are called free when the following condition is satisfied, for any  $a_i \in A$  and  $b_i \in B$ :*

$$tr(a_i) = tr(b_i) = 0 \implies tr(a_1 b_1 a_2 b_2 \dots) = 0$$

*Also, two variables  $a, b \in C$  are called free when the algebras that they generate,*

$$A = \langle a \rangle \quad , \quad B = \langle b \rangle$$

*are free inside  $C$ , in the above sense.*

In short, freeness appears by definition as a kind of “free analogue” of usual independence, taking into account the fact that the variables do not necessarily commute. As a first observation, of theoretical nature, there is actually a certain lack of symmetry

between Definition 9.6 and Definition 9.8, because in contrast to the former, the latter does not include an explicit formula for the quantities of the following type:

$$\text{tr}(a_1 b_1 a_2 b_2 \dots)$$

However, this is not an issue, and is simply due to the fact that the formula in the free case is something more complicated, the precise result being as follows:

**PROPOSITION 9.9.** *Assuming that  $A, B \subset C$  are free, the restriction of  $\text{tr}$  to  $\langle A, B \rangle$  can be computed in terms of the restrictions of  $\text{tr}$  to  $A, B$ . To be more precise,*

$$\text{tr}(a_1 b_1 a_2 b_2 \dots) = P\left(\{\text{tr}(a_{i_1} a_{i_2} \dots)\}_i, \{\text{tr}(b_{j_1} b_{j_2} \dots)\}_j\right)$$

where  $P$  is certain polynomial in several variables, depending on the length of the word  $a_1 b_1 a_2 b_2 \dots$ , and having as variables the traces of products of type

$$a_{i_1} a_{i_2} \dots, \quad b_{j_1} b_{j_2} \dots$$

with the indices being chosen increasing,  $i_1 < i_2 < \dots$  and  $j_1 < j_2 < \dots$

**PROOF.** This is something a bit theoretical, so let us begin with an example. Our claim is that if  $a, b$  are free then, exactly as in the case where we have independence:

$$\text{tr}(ab) = \text{tr}(a)\text{tr}(b)$$

Indeed, let us go back to the computation performed after Definition 9.6, which was as follows, with the convention  $a' = a - \text{tr}(a)$ :

$$\begin{aligned} \text{tr}(ab) &= \text{tr}[(a' + \text{tr}(a))(b' + \text{tr}(b))] \\ &= \text{tr}(a'b') + \text{tr}(a')\text{tr}(b) + \text{tr}(a)\text{tr}(b') + \text{tr}(a)\text{tr}(b) \\ &= \text{tr}(a'b') + \text{tr}(a)\text{tr}(b) \\ &= \text{tr}(a)\text{tr}(b) \end{aligned}$$

Our claim is that this computation perfectly works under the sole freeness assumption. Indeed, the only non-trivial equality is the last one, which follows from:

$$\text{tr}(a') = \text{tr}(b') = 0 \implies \text{tr}(a'b') = 0$$

In general, the situation is of course more complicated than this, but the same trick applies. To be more precise, we can start our computation as follows:

$$\begin{aligned} \text{tr}(a_1 b_1 a_2 b_2 \dots) &= \text{tr}[(a'_1 + \text{tr}(a_1))(b'_1 + \text{tr}(b_1))(a'_2 + \text{tr}(a_2))(b'_2 + \text{tr}(b_2)) \dots] \\ &= \text{tr}(a'_1 b'_1 a'_2 b'_2 \dots) + \text{other terms} \\ &= \text{other terms} \end{aligned}$$

Observe that we have used here the freeness condition, in the following form:

$$\text{tr}(a'_i) = \text{tr}(b'_i) = 0 \implies \text{tr}(a'_1 b'_1 a'_2 b'_2 \dots) = 0$$

Now regarding the “other terms”, those which are left, each of them will consist of a product of traces of type  $tr(a_i)$  and  $tr(b_i)$ , and then a trace of a product still remaining to be computed, which is of the following form, for some elements  $\alpha_i \in A$  and  $\beta_i \in B$ :

$$tr(\alpha_1\beta_1\alpha_2\beta_2\dots)$$

To be more precise, the variables  $\alpha_i \in A$  appear as ordered products of those  $a_i \in A$  not getting into individual traces  $tr(a_i)$ , and the variables  $\beta_i \in B$  appear as ordered products of those  $b_i \in B$  not getting into individual traces  $tr(b_i)$ . Now since the length of each such alternating product  $\alpha_1\beta_1\alpha_2\beta_2\dots$  is smaller than the length of the original product  $a_1b_1a_2b_2\dots$ , we are led into of recurrence, and this gives the result.  $\square$

Let us discuss now some models for independence and freeness. We have the following result, from [85], which clarifies the analogy between independence and freeness:

**THEOREM 9.10.** *Given two algebras  $(A, tr)$  and  $(B, tr)$ , the following hold:*

- (1)  *$A, B$  are independent inside their tensor product  $A \otimes B$ , endowed with its canonical tensor product trace, given on basic tensors by  $tr(a \otimes b) = tr(a)tr(b)$ .*
- (2)  *$A, B$  are free inside their free product  $A * B$ , endowed with its canonical free product trace, given by the formulae in Proposition 9.9.*

**PROOF.** Both the above assertions are clear from definitions, as follows:

(1) This is clear with either of the definitions of the independence, from Definition 9.6, because we have by construction of the product trace:

$$\begin{aligned} tr(ab) &= tr[(a \otimes 1)(1 \otimes b)] \\ &= tr(a \otimes b) \\ &= tr(a)tr(b) \end{aligned}$$

Observe that there is a relation here with Theorem 9.7 as well, due to the following formula for compact spaces, with  $\otimes$  being a topological tensor product:

$$C(X \times Y) = C(X) \otimes C(Y)$$

To be more precise, the present statement generalizes the first assertion in Theorem 9.7, and the second assertion tells us that this generalization is more or less the same thing as the original statement. All this comes of course from basic measure theory.

(2) This is clear too from definitions, the only point being that of showing that the notion of freeness, or the recurrence formulae in Proposition 9.9, can be used in order to construct a canonical free product trace, on the free product of the algebras involved:

$$tr : A * B \rightarrow \mathbb{C}$$

But this can be checked for instance by using a GNS construction. Indeed, consider the GNS constructions for the algebras  $(A, tr)$  and  $(B, tr)$ :

$$A \rightarrow B(l^2(A)) \quad , \quad B \rightarrow B(l^2(B))$$

By taking the free product of these representations, we obtain a representation as follows, with the  $*$  on the right being a free product of pointed Hilbert spaces:

$$A * B \rightarrow B(l^2(A) * l^2(B))$$

Now by composing with the linear form  $T \rightarrow \langle T\xi, \xi \rangle$ , where  $\xi = 1_A = 1_B$  is the common distinguished vector of  $l^2(A)$ ,  $l^2(B)$ , we obtain a linear form, as follows:

$$tr : A * B \rightarrow \mathbb{C}$$

It is routine then to check that  $tr$  is indeed a trace, and this is the “canonical free product trace” from the statement. Then, an elementary computation shows that  $A, B$  are free inside  $A * B$ , with respect to this trace, and this finishes the proof. See [85].  $\square$

### 9b. Free convolution

All the above was quite theoretical, and as a concrete application of the above results, bringing us into probability, we have the following result, from [86]:

**THEOREM 9.11.** *We have a free convolution operation  $\boxplus$  for the distributions*

$$\mu : \mathbb{C} \langle X, X^* \rangle \rightarrow \mathbb{C}$$

*which is well-defined by the following formula, with  $a, b$  taken to be free:*

$$\mu_a \boxplus \mu_b = \mu_{a+b}$$

*This restricts to an operation, still denoted  $\boxplus$ , on the real probability measures.*

**PROOF.** We have several verifications to be performed here, as follows:

(1) We first have to check that given two variables  $a, b$  which live respectively in certain  $C^*$ -algebras  $A, B$ , we can recover inside some  $C^*$ -algebra  $C$ , with exactly the same distributions  $\mu_a, \mu_b$ , as to be able to sum them and talk about  $\mu_{a+b}$ . But this comes from Theorem 9.10, because we can set  $C = A * B$ , as explained there.

(2) The other verification which is needed is that of the fact that if two variables  $a, b$  are free, then the distribution  $\mu_{a+b}$  depends only on the distributions  $\mu_a, \mu_b$ . But for this purpose, we can use the general formula from Proposition 9.9, namely:

$$tr(a_1 b_1 a_2 b_2 \dots) = P\left(\{tr(a_{i_1} a_{i_2} \dots)\}_i, \{tr(b_{j_1} b_{j_2} \dots)\}_j\right)$$

Now by plugging in arbitrary powers of  $a, b$  as variables  $a_i, b_j$ , we obtain a family of formulae of the following type, with  $Q$  being certain polynomials:

$$tr(a^{k_1} b^{l_1} a^{k_2} b^{l_2} \dots) = Q\left(\{tr(a^k)\}_k, \{tr(b^l)\}_l\right)$$

Thus the moments of  $a + b$  depend only on the moments of  $a, b$ , with of course colored exponents in all this, according to our moment conventions, and this gives the result.

(3) Finally, in what regards the last assertion, regarding the real measures, this is clear from the fact that if the variables  $a, b$  are self-adjoint, then so is their sum  $a + b$ .  $\square$

Along the same lines, but with some technical subtleties this time, we can talk as well about multiplicative free convolution, following [87], as follows:

**THEOREM 9.12.** *We have a free convolution operation  $\boxtimes$  for the distributions*

$$\mu : \mathbb{C} \langle X, X^* \rangle \rightarrow \mathbb{C}$$

*which is well-defined by the following formula, with  $a, b$  taken to be free:*

$$\mu_a \boxtimes \mu_b = \mu_{ab}$$

*In the case of the self-adjoint variables, we can equally set*

$$\mu_a \boxtimes \mu_b = \mu_{\sqrt{ab}\sqrt{a}}$$

*and so we have an operation, still denoted  $\boxtimes$ , on the real probability measures.*

**PROOF.** We have two statements here, the idea being as follows:

(1) The verifications for the fact that  $\boxtimes$  as above is indeed well-defined at the general distribution level are identical to those done before for  $\boxplus$ , with the result basically coming from the formula in Proposition 9.9, and with Theorem 9.10 invoked as well, in order to say that we have a model, and so we can indeed use this formula.

(2) Regarding now the last assertion, regarding the real measures, this was something trivial for  $\boxplus$ , but is something trickier now for  $\boxtimes$ , because if we take  $a, b$  to be self-adjoint, thier product  $ab$  will in general not be self-adjoint, and definitely it will be not if we want  $a, b$  to be free, and so the formula  $\mu_a \boxtimes \mu_b = \mu_{ab}$  will apparently makes us exit the world of real probability measures. However, this is not exactly the case. Indeed, let us set:

$$c = \sqrt{ab}\sqrt{a}$$

This new variable is then self-adjoint, and its moments are given by:

$$\begin{aligned} \text{tr}(c^k) &= \text{tr}[(\sqrt{ab}\sqrt{a})^k] \\ &= \text{tr}[\sqrt{ab}a \dots ab\sqrt{a}] \\ &= \text{tr}[\sqrt{a} \cdot \sqrt{ab}a \dots ab] \\ &= \text{tr}[(ab)^k] \end{aligned}$$

Thus, we are led to the conclusion in the statement.  $\square$

We would like now to have linearization results for  $\boxplus$  and  $\boxtimes$ , in the spirit of the known results for  $*$  and  $\times$ . We will do this slowly, in several steps. As a first objective, we would like to convert our one and only modelling result so far, namely Theorem 9.10, which is a rather abstract result, into something more concrete. Let us start with:

**THEOREM 9.13.** *Let  $\Gamma$  be a discrete group, and consider the complex group algebra  $\mathbb{C}[\Gamma]$ , with involution given by the fact that all group elements are unitaries:*

$$g^* = g^{-1} \quad , \quad \forall g \in \Gamma$$

*The maximal  $C^*$ -seminorm on  $\mathbb{C}[\Gamma]$  is then a  $C^*$ -norm, and the closure of  $\mathbb{C}[\Gamma]$  with respect to this norm is a  $C^*$ -algebra, denoted  $C^*(\Gamma)$ . Moreover,*

$$tr(g) = \delta_{g1}$$

*defines a positive unital trace  $tr : C^*(\Gamma) \rightarrow \mathbb{C}$ , which is faithful on  $\mathbb{C}[\Gamma]$ .*

**PROOF.** We have two assertions to be proved, the idea being as follows:

(1) In order to prove the first assertion, regarding the maximal seminorm which is a norm, we must find a  $*$ -algebra embedding as follows, with  $H$  being a Hilbert space:

$$\mathbb{C}[\Gamma] \subset B(H)$$

For this purpose, consider the Hilbert space  $H = l^2(\Gamma)$ , having the family  $\{h\}_{h \in \Gamma}$  as orthonormal basis. Our claim is that we have an embedding, as follows:

$$\pi : \mathbb{C}[\Gamma] \subset B(H) \quad , \quad \pi(g)(h) = gh$$

Indeed, since  $\pi(g)$  maps the basis  $\{h\}_{h \in \Gamma}$  into itself, this operator is well-defined and bounded, and is an isometry. It is also clear from the formula  $\pi(g)(h) = gh$  that  $g \rightarrow \pi(g)$  is a morphism of algebras, and since this morphism maps the unitaries  $g \in \Gamma$  into isometries, this is a morphism of  $*$ -algebras. Finally, the faithfulness of  $\pi$  is clear.

(2) Regarding the second assertion, we can use here once again the above construction. Indeed, we can define a linear form on the image of  $C^*(\Gamma)$ , as follows:

$$tr(T) = \langle T\delta_1, \delta_1 \rangle$$

This functional is then positive, and is easily seen to be a trace. Moreover, on the group elements  $g \in \Gamma$ , this functional is given by the following formula:

$$tr(g) = \delta_{g1}$$

Thus, it remains to show that  $tr$  is faithful on  $\mathbb{C}[\Gamma]$ . But this follows from the fact that  $tr$  is faithful on the image of  $C^*(\Gamma)$ , which contains  $\mathbb{C}[\Gamma]$ .  $\square$

As an illustration, we have the following more precise result, in the abelian case:

**PROPOSITION 9.14.** *Given a discrete abelian group  $\Gamma$ , we have an isomorphism*

$$C^*(\Gamma) \simeq C(G)$$

*where  $G = \widehat{\Gamma}$  is its Pontrjagin dual, formed by the characters  $\chi : \Gamma \rightarrow \mathbb{T}$ . Moreover,*

$$tr(g) = \delta_{g1}$$

*corresponds in this way to the Haar integration over  $G$ .*

PROOF. We have two assertions to be proved, the idea being as follows:

(1) Since  $\Gamma$  is abelian,  $A = C^*(\Gamma)$  is commutative, so by the Gelfand theorem we have  $A = C(X)$ . The spectrum  $X = \text{Spec}(A)$ , consisting of the characters  $\chi : C^*(\Gamma) \rightarrow \mathbb{C}$ , can be then identified with the Pontrjagin dual  $G = \widehat{\Gamma}$ , and this gives the result.

(2) Regarding now the last assertion, we must prove here that we have:

$$\text{tr}(f) = \int_G f(x) dx$$

But this is clear via the above identifications, for instance because the linear form  $\text{tr}(g) = \delta_{g1}$ , when viewed as a functional on  $C(G)$ , is left and right invariant.  $\square$

Getting back now to our questions, we can now formulate a general modelling result for independence and freeness, providing us with large classes of examples, as follows:

**THEOREM 9.15.** *We have the following results, valid for group algebras:*

- (1)  $C^*(\Gamma), C^*(\Lambda)$  are independent inside  $C^*(\Gamma \times \Lambda)$ .
- (2)  $C^*(\Gamma), C^*(\Lambda)$  are free inside  $C^*(\Gamma * \Lambda)$ .

PROOF. In order to prove these results, we have two possible methods:

(1) We can either use the general results in Theorem 9.10, along with the following two isomorphisms, which are both standard:

$$C^*(\Gamma \times \Lambda) = C^*(\Lambda) \otimes C^*(\Gamma)$$

$$C^*(\Gamma * \Lambda) = C^*(\Lambda) * C^*(\Gamma)$$

(2) Or, we can prove this directly, by using the fact that each algebra is spanned by the corresponding group elements. Indeed, this shows that it is enough to check the independence and freeness formulae on group elements, which is in turn trivial.  $\square$

### 9c. R-transform

We have seen so far the foundations of free probability, in analogy with those of classical probability, taken with a functional analysis touch. The idea now is that with a bit of luck, the basic theory from the classical case, namely the Fourier transform, and then the CLT, should have free extensions. Let us begin our discussion with the following definition, from [86], coming from the theory developed in the above:

**DEFINITION 9.16.** *The real probability measures are subject to operations  $*$  and  $\boxplus$ , called classical and free convolution, given by the formulae*

$$\mu_a * \mu_b = \mu_{a+b}$$

$$\mu_\alpha \boxplus \mu_\beta = \mu_{\alpha+\beta}$$

*with  $a, b$  being independent, and  $\alpha, \beta$  being free, and all variables being self-adjoint.*

The problem now is that of linearizing these operations  $*$  and  $\boxplus$ . In what regards  $*$ , we know from chapter 1 that this operation is linearized by the logarithm  $\log F$  of the Fourier transform, which in the present setting, where  $\mathbb{E} = \text{tr}$ , is given by:

$$F_a(x) = \text{tr}(e^{ixa})$$

In order to find a similar result for  $\boxplus$ , we need some efficient models for the pairs of free random variables  $(a, b)$ . This is a priori not a problem, because once we have  $a \in A$  and  $b \in B$ , we can form the free product  $A * B$ , which contains  $a, b$  as free variables.

However, the initial choice, that of the variables  $a \in A, b \in B$  modelling some given laws  $\mu, \nu \in \mathcal{P}(\mathbb{R})$ , matters a lot. Indeed, any kind of abstract choice here would lead us into an abstract algebra  $A * B$ , and so into the abstract combinatorics of the free convolution, that cannot be solved with bare hands, and that we want to avoid.

In short, we must be tricky, at least in what concerns the beginning of our computation. Following [86], the idea will be that of temporarily lifting the self-adjointness assumption on our variables  $a, b$ , and looking instead for random variables  $\alpha, \beta$ , not necessarily self-adjoint, modelling in integer moments our given laws  $\mu, \nu \in \mathcal{P}(\mathbb{R})$ , as follows:

$$\text{tr}(\alpha^k) = M_k(\mu) \quad , \quad \text{tr}(\beta^k) = M_k(\nu)$$

To be more precise, assuming that  $\alpha, \beta$  are indeed not self-adjoint, the above formulae are not the general formulae for  $\alpha, \beta$ , simply because these latter formulae involve colored integers  $k = \circ \bullet \bullet \circ \dots$  as exponents. Thus, in the context of the above formulae,  $\mu, \nu$  are not the distributions of  $\alpha, \beta$ , but just some “parts” of these distributions.

Now with this idea in mind, due to Voiculescu and quite tricky, the solution to the law modelling problem comes in a quite straightforward way, involving the good old Hilbert space  $H = l^2(\mathbb{N})$  and the good old shift operator  $S \in B(H)$ , as follows:

**THEOREM 9.17.** *Consider the shift operator on the space  $H = l^2(\mathbb{N})$ , given by  $S(e_i) = e_{i+1}$ . The variables of the following type, with  $f \in \mathbb{C}[X]$  being a polynomial,*

$$S^* + f(S)$$

*model then in moments, up to finite order, all the distributions  $\mu : \mathbb{C}[X] \rightarrow \mathbb{C}$ .*

**PROOF.** We have already met the shift  $S$  in chapter 6, as the simplest example of an isometry which is not a unitary,  $S^*S = 1, SS^* = 1$ , with this coming from:

$$S^*(e_i) = \begin{cases} e_{i-1} & (i > 0) \\ 0 & (i = 0) \end{cases}$$

Consider now a variable as in the statement, namely:

$$T = S^* + a_0 + a_1S + a_2S^2 + \dots + a_nS^n$$

The computation of the moments of  $T$  is then as follows:

- We first have  $\text{tr}(T) = a_0$ .
- Then the computation of  $\text{tr}(T^2)$  will involve  $a_1$ .
- Then the computation of  $\text{tr}(T^3)$  will involve  $a_2$ .
- And so on.

Thus, we are led to a certain recurrence, that we will not attempt to solve now, with bare hands, but which definitely gives the conclusion in the statement.  $\square$

Before getting further, with free products of such models, let us work out a very basic example, which is something fundamental, that we will need in what follows:

**PROPOSITION 9.18.** *In the context of the above correspondence, the variable*

$$T = S + S^*$$

*follows the Wigner semicircle law,  $\gamma_1 = \frac{1}{2\pi}\sqrt{4-x^2}dx$ .*

**PROOF.** In order to compute the law of variable  $T$  in the statement, we can use the moment method. The moments of this variable are as follows:

$$\begin{aligned} M_k &= \text{tr}(T^k) \\ &= \text{tr}((S + S^*)^k) \\ &= \#(1 \in (S + S^*)^k) \end{aligned}$$

Now since the  $S$  shifts to the right on  $\mathbb{N}$ , and  $S^*$  shifts to the left, while remaining positive, we are left with counting the length  $k$  paths on  $\mathbb{N}$  starting and ending at 0. Since there are no such paths when  $k = 2r + 1$  is odd, the odd moments vanish:

$$M_{2r+1} = 0$$

In the case where  $k = 2r$  is even, such paths on  $\mathbb{N}$  are best represented as paths in the upper half-plane, starting at 0, and going at each step NE or SE, depending on whether the original path on  $\mathbb{N}$  goes at right or left, and finally ending at  $k \in \mathbb{N}$ . With this picture we are led to the following formula for the number of such paths:

$$M_{2r+2} = \sum_s M_{2s} M_{2r-s}$$

But this is exactly the recurrence formula for the Catalan numbers, and so:

$$M_{2r} = \frac{1}{r+1} \binom{2r}{r}$$

Summarizing, the odd moments of  $T$  vanish, and the even moments are the Catalan numbers. But these numbers being the moments of the Wigner semicircle law  $\gamma_1$ , as explained in chapter 4, we are led to the conclusion in the statement.  $\square$

Getting back now to our linearization program for  $\boxplus$ , the next step is that of taking a free product of the model found in Theorem 9.17 with itself. There are two approaches here, one being a bit abstract, and the other one being more concrete. We will explain in what follows both of them. The abstract approach, which is quite nice, making a link with our main modelling result so far, involving group algebras, is as follows:

**PROPOSITION 9.19.** *We can talk about semigroup algebras  $C^*(\Gamma) \subset B(l^2(\Gamma))$ , exactly as we did for the group algebras, and at the level of examples:*

- (1) *With  $\Gamma = \mathbb{N}$  we recover the shift algebra  $A = \langle S \rangle$  on  $H = l^2(\mathbb{N})$ .*
- (2) *With  $\Gamma = \mathbb{N} * \mathbb{N}$ , we obtain the algebra  $A = \langle S_1, S_2 \rangle$  on  $H = l^2(\mathbb{N} * \mathbb{N})$ .*

**PROOF.** We can talk indeed about semigroup algebras  $C^*(\Gamma) \subset B(l^2(\Gamma))$ , exactly as we did for the group algebras, the only difference coming from the fact that the semigroup elements  $g \in \Gamma$  will now correspond to isometries, which are not necessarily unitaries. Now this construction in hand, both the assertions are clear, as follows:

(1) With  $\Gamma = \mathbb{N}$  we recover indeed the shift algebra  $A = \langle S \rangle$  on the Hilbert space  $H = l^2(\mathbb{N})$ , the shift  $S$  itself being the isometry associated to the element  $1 \in \mathbb{N}$ .

(2) With  $\Gamma = \mathbb{N} * \mathbb{N}$  we recover the double shift algebra  $A = \langle S_1, S_2 \rangle$  on the Hilbert space  $H = l^2(\mathbb{N} * \mathbb{N})$ , the two shifts  $S_1, S_2$  themselves being the isometries associated to two copies of the element  $1 \in \mathbb{N}$ , one for each of the two copies of  $\mathbb{N}$  which are present.  $\square$

In what follows we will rather use an equivalent, second approach to our problem, which is exactly the same thing, but formulated in a less abstract way, as follows:

**PROPOSITION 9.20.** *We can talk about the algebra of creation operators*

$$S_x : v \rightarrow x \otimes v$$

*on the free Fock space associated to a real Hilbert space  $H$ , given by*

$$F(H) = \mathbb{C}\Omega \oplus H \oplus H^{\otimes 2} \oplus \dots$$

*and at the level of examples, we have:*

- (1) *With  $H = \mathbb{C}$  we recover the shift algebra  $A = \langle S \rangle$  on  $H = l^2(\mathbb{N})$ .*
- (2) *With  $H = \mathbb{C}^2$ , we obtain the algebra  $A = \langle S_1, S_2 \rangle$  on  $H = l^2(\mathbb{N} * \mathbb{N})$ .*

**PROOF.** We can talk indeed about the algebra  $A(H)$  of creation operators on the free Fock space  $F(H)$  associated to a real Hilbert space  $H$ , with the remark that, in terms of the abstract semigroup notions from Proposition 9.19, we have:

$$A(\mathbb{C}^k) = C^*(\mathbb{N}^{*k}) \quad , \quad F(\mathbb{C}^k) = l^2(\mathbb{N}^{*k})$$

As for the assertions (1,2) in the statement, these are both clear, either directly, or by passing via (1,2) from Proposition 9.19, which were both clear as well.  $\square$

The advantage with this latter model comes from the following result, from [86], which has a very simple formulation, without linear combinations or anything:

PROPOSITION 9.21. *Given a real Hilbert space  $H$ , and two orthogonal vectors  $x \perp y$ , the corresponding creation operators  $S_x$  and  $S_y$  are free with respect to*

$$\text{tr}(T) = \langle T\Omega, \Omega \rangle$$

*called trace associated to the vacuum vector.*

PROOF. In standard tensor product notation for the elements of the free Fock space  $F(H)$ , the formula of a creation operator associated to a vector  $x \in H$  is as follows:

$$S_x(y_1 \otimes \dots \otimes y_n) = x \otimes y_1 \otimes \dots \otimes y_n$$

As for the formula of the adjoint of this creation operator, called annihilation operator associated to the vector  $x \in H$ , this is as follows:

$$S_x^*(y_1 \otimes \dots \otimes y_n) = \langle x, y_1 \rangle y_2 \otimes \dots \otimes y_n$$

We obtain from this the following formula, which holds for any two vectors  $x, y \in H$ :

$$S_x^* S_y = \langle x, y \rangle \text{id}$$

With these formulae in hand, the result follows by doing some elementary computations, in the spirit of those done for the group algebras, in the above.  $\square$

With this technology in hand, let us go back to our linearization program for  $\boxplus$ . We know from Theorem 9.17 how to model the individual distributions  $\mu \in \mathcal{P}(\mathbb{R})$ , and by combining this with Proposition 9.10 and Proposition 9.21, we therefore know how to freely model pairs of distributions  $\mu, \nu \in \mathcal{P}(\mathbb{R})$ , as required by the convolution problem. We are therefore left with doing the sum in the model, and then computing its distribution. And the point here is that, still following [86], we have:

THEOREM 9.22. *Given two polynomials  $f, g \in \mathbb{C}[X]$ , consider the variables*

$$S^* + f(S) \quad , \quad T^* + g(T)$$

*where  $S, T$  are two creation operators, or shifts, associated to a pair of orthogonal norm 1 vectors. These variables are then free, and their sum has the same law as*

$$R^* + (f + g)(R)$$

*with  $R$  being the usual shift on  $l^2(\mathbb{N})$ .*

PROOF. We have two assertions here, the idea being as follows:

(1) The freeness assertion comes from the general freeness result from Proposition 9.21, via the various identifications coming from the previous results.

(2) Regarding the second assertion, the idea is that this comes from a  $45^\circ$  rotation trick. Let us write indeed the two variables in the statement as follows:

$$X = S^* + a_0 + a_1 S + a_2 S^2 + \dots$$

$$Y = T^* + b_0 + b_1 T + a_2 T^2 + \dots$$

Now let us perform the following  $45^\circ$  base change, on the real span of the vectors  $s, t \in H$  producing our two shifts  $S, T$ , as follows:

$$r = \frac{s+t}{\sqrt{2}} \quad , \quad u = \frac{s-t}{\sqrt{2}}$$

The new shifts, associated to these vectors  $r, u \in H$ , are then given by:

$$R = \frac{S+T}{\sqrt{2}} \quad , \quad U = \frac{S-T}{\sqrt{2}}$$

By using now these two new shifts, which are free according to Proposition 9.21 above, we obtain the following equality of distributions:

$$\begin{aligned} X + Y &= S^* + T^* + \sum_k a_k S^k + b_k T^k \\ &= \sqrt{2}R^* + \sum_k a_k \left( \frac{R+U}{\sqrt{2}} \right)^k + b_k \left( \frac{R-U}{\sqrt{2}} \right)^k \\ &\sim \sqrt{2}R^* + \sum_k a_k \left( \frac{R}{\sqrt{2}} \right)^k + b_k \left( \frac{R}{\sqrt{2}} \right)^k \\ &\sim R^* + \sum_k a_k R^k + b_k R^k \end{aligned}$$

To be more precise, here at the end we have used the freeness property of  $R, U$  in order to cut  $U$  from the computation, as it cannot bring anything, and then we did a basic rescaling at the very end. Thus, we are led to the conclusion in the statement.  $\square$

As a conclusion, the operation  $\mu \rightarrow f$  from Theorem 9.17 linearizes  $\boxplus$ . In order to reach now to something concrete, we are left with a computation inside  $C^*(\mathbb{N})$ , which is elementary, and whose conclusion is that  $R_\mu = f$  can be recaptured from  $\mu$  via the Cauchy transform  $G_\mu$ . The precise result here, due to Voiculescu [86], is as follows:

**THEOREM 9.23.** *Given a real probability measure  $\mu$ , define its  $R$ -transform as follows:*

$$G_\mu(\xi) = \int_{\mathbb{R}} \frac{d\mu(t)}{\xi - t} \implies G_\mu \left( R_\mu(\xi) + \frac{1}{\xi} \right) = \xi$$

*The free convolution operation is then linearized by this  $R$ -transform.*

**PROOF.** This can be done by using the above results, in several steps, as follows:

(1) According to Theorem 9.22, the operation  $\mu \rightarrow f$  in Theorem 9.17 above linearizes the free convolution operation  $\boxplus$ . We are therefore left with a computation inside  $C^*(\mathbb{N})$ . To be more precise, consider a variable as in Theorem 9.17:

$$X = S^* + f(S)$$

In order to establish the result, we must prove that the  $R$ -transform of  $X$ , constructed according to the procedure in the statement, is the function  $f$  itself.

(2) In order to do so, we fix  $|z| < 1$  in the complex plane, and we set:

$$q_z = \delta_0 + \sum_{k=1}^{\infty} z_k \delta_k$$

The shift and its adjoint act then on this vector as follows:

$$Sq_z = z^{-1}(q_z - \delta_0) \quad , \quad S^*q_z = zq_z$$

It follows that the adjoint of our operator  $X$  acts on this vector as follows:

$$\begin{aligned} X^*q_z &= (S + f(S^*))q_z \\ &= z^{-1}(q_z - \delta_0) + f(z)q_z \\ &= (z^{-1} + f(z))q_z - z^{-1}\delta_0 \end{aligned}$$

Now observe that the above formula can be written as follows:

$$z^{-1}\delta_0 = (z^{-1} + f(z) - X^*)q_z$$

The point now is that when  $|z|$  is small, the operator appearing on the right is invertible. Thus, we can rewrite the above formula as follows:

$$(z^{-1} + f(z) - X^*)^{-1}\delta_0 = zq_z$$

Now by applying the trace, we are led to the following formula:

$$\begin{aligned} \text{tr} [(z^{-1} + f(z) - X^*)^{-1}] &= \langle (z^{-1} + f(z) - X^*)^{-1}\delta_0, \delta_0 \rangle \\ &= \langle zq_z, \delta_0 \rangle \\ &= z \end{aligned}$$

(3) Let us apply now the procedure in the statement to the real probability measure  $\mu$  modelled by  $X$ . The Cauchy transform  $G_\mu$  is then given by:

$$\begin{aligned} G_\mu(\xi) &= \text{tr}((\xi - X)^{-1}) \\ &= \overline{\text{tr}((\bar{\xi} - X^*)^{-1})} \\ &= \text{tr}((\xi - X^*)^{-1}) \end{aligned}$$

Now observe that, with the choice  $\xi = z^{-1} + f(z)$  for our complex variable, the trace formula found in (2) above tells us that we have:

$$G_\mu(z^{-1} + f(z)) = z$$

Thus, by definition of the  $R$ -transform, we have the following formula:

$$R_\mu(z) = f(z)$$

But this finishes the proof, as explained before in step (1) above. □

Summarizing, the situation in free probability is quite similar to the one in classical probability, the product spaces needed for the basic properties of the Fourier transform being replaced by something “noncommutative”, namely the free Fock space models. This is of course something quite surprising, and the credit for this remarkable discovery, which has drastically changed operator algebras, goes to Voiculescu’s paper [86].

#### 9d. CLT and PLT

With the above linearization technology in hand, we can do now lots of things. First, we can establish the following free analogue of the CLT, due to Voiculescu [86]:

**THEOREM 9.24 (Free CLT).** *Given self-adjoint variables  $x_1, x_2, x_3, \dots$  which are f.i.d., centered, with variance  $t > 0$ , we have, with  $n \rightarrow \infty$ , in moments,*

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n x_i \sim \gamma_t$$

where  $\gamma_t$  is the Wigner semicircle law of parameter  $t$ , having density:

$$\gamma_t = \frac{1}{2\pi t} \sqrt{4t^2 - x^2} dx$$

**PROOF.** We follow the same idea as in the proof of the CLT, from chapter 1:

(1) At  $t = 1$ , the  $R$ -transform of the variable in the statement on the left can be computed by using the linearization property from Theorem 9.23, and is given by:

$$R(\xi) = nR_x \left( \frac{\xi}{\sqrt{n}} \right) \simeq \xi$$

(2) Regarding now the right term, also taken at  $t = 1$ , our first claim here is that the Cauchy transform of the Wigner law  $\gamma_1$  satisfies the following equation:

$$G_{\gamma_1} \left( \xi + \frac{1}{\xi} \right) = \xi$$

Indeed, we know from chapter 7 that the moments of  $\gamma_1$  are the Catalan numbers:

$$\frac{1}{2\pi} \int_0^4 \sqrt{4 - x^2} x^{2k} dx = C_k$$

Consider now the generating series of the Catalan numbers, namely:

$$f(z) = \sum_{k \geq 0} C_k z^k$$

We know from chapter 7 that  $f$  satisfies the following degree 2 equation:

$$zf^2 - f + 1 = 0$$

But this gives the above formula for  $G_{\gamma_1}$ , via a few manipulations.

(3) We conclude from the formula found in (2) that the  $R$ -transform of the Wigner semicircle law  $\gamma_1$  is given by the following formula:

$$R_{\gamma_1}(\xi) = \xi$$

Observe that this follows in fact as well from the following formula, coming from Proposition 9.18, and from the technical details of the  $R$ -transform:

$$S + S^* \sim \gamma_1$$

Thus, the laws in the statement have the same  $R$ -transforms, so they are equal.

(4) Summarizing, we have proved the free CLT at  $t = 1$ . The passage to the general case, where  $t > 0$  is arbitrary, is routine, by some standard dilation computations.  $\square$

Regarding the limiting measures  $\gamma_t$ , one problem that we were having was that of understanding how  $\gamma_t$  exactly appears, out of  $\gamma_1$ . We can now solve this question:

**THEOREM 9.25.** *The Wigner semicircle laws have the property*

$$\gamma_s \boxplus \gamma_t = \gamma_{s+t}$$

*so they form a 1-parameter semigroup with respect to free convolution.*

**PROOF.** This follows either from Theorem 9.24, or from Theorem 9.23, by using the fact that the  $R$ -transform of  $\gamma_t$ , which is given by  $R_{\gamma_t}(\xi) = t\xi$ , is linear in  $t$ .  $\square$

As a conclusion to what we have so far, we have:

**THEOREM 9.26.** *The Gaussian laws  $g_t$  and the Wigner laws  $\gamma_t$ , given by*

$$g_t = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} dx \quad , \quad \gamma_t = \frac{1}{2\pi t} \sqrt{4t^2 - x^2} dx$$

*have the following properties:*

- (1) *They appear via the CLT, and the free CLT.*
- (2) *They form semigroups with respect to  $*$  and  $\boxplus$ .*
- (3) *Their transforms are  $\log F_{g_t}(x) = -tx^2/2$ ,  $R_{\gamma_t}(x) = tx$ .*
- (4) *Their moments are  $M_k = \sum_{\pi \in D(k)} t^{|\pi|}$ , with  $D = P_2, NC_2$ .*

**PROOF.** These are all results that we already know, the idea being that (3,4) follow by doing some combinatorics and calculus, and that (1,2) follow from (3,4).  $\square$

To summarize, our initial purpose for this chapter was to vaguely explore the basics of free probability, but all of a sudden, due to the power of Voiculescu's  $R$ -transform [86], we are now into stating and proving results which are on par with what we have been doing in the first part of this book, namely reasonably advanced probability theory.

So, let us keep going in this direction, by exploring what else we can do with the  $R$ -transform, in analogy with what we did before, by using the Fourier transform. As a next

natural objective, we have the question of formulating the free analogue of the Poisson Limit Theorem (PLT). Although elementary from what we have, this was something not done by Voiculescu himself, and not appearing in the foundational book [90], and only explained later, in the book of Hiai and Petz [53]. The statement is as follows:

THEOREM 9.27 (Free PLT). *The following limit converges, for any  $t > 0$ ,*

$$\lim_{n \rightarrow \infty} \left( \left( 1 - \frac{t}{n} \right) \delta_0 + \frac{t}{n} \delta_1 \right)^{\boxplus n}$$

and we obtain the Marchenko-Pastur law of parameter  $t$ ,

$$\pi_t = \max(1 - t, 0) \delta_0 + \frac{\sqrt{4t - (x - 1 - t)^2}}{2\pi x} dx$$

also called free Poisson law of parameter  $t$ .

PROOF. Consider the measure in the statement, under the convolution sign:

$$\eta = \left( 1 - \frac{t}{n} \right) \delta_0 + \frac{t}{n} \delta_1$$

The Cauchy transform of this measure is easy to compute, and is given by:

$$G_\eta(\xi) = \left( 1 - \frac{t}{n} \right) \frac{1}{\xi} + \frac{t}{n} \cdot \frac{1}{\xi - 1}$$

In order to prove the result, we want to compute the following  $R$ -transform:

$$R = R_{\eta^{\boxplus n}}(y) = nR_\eta(y)$$

According to the formula of  $G_\eta$ , the equation for this function  $R$  is as follows:

$$\left( 1 - \frac{t}{n} \right) \frac{1}{1/y + R/n} + \frac{t}{n} \cdot \frac{1}{1/y + R/n - 1} = y$$

By multiplying both sides by  $n/y$ , this equation can be written as:

$$\frac{t + yR}{1 + yR/n} = \frac{t}{1 + yR/n - y}$$

With  $n \rightarrow \infty$  things simplify, and we obtain the following formula:

$$t + yR = \frac{t}{1 - y}$$

Thus we have the following formula, for the  $R$ -transform that we are interested in:

$$R = \frac{t}{1 - y}$$

But this gives the result, since  $R_{\pi_t}$  is elementary to compute from what we have, by “doubling” the results for the Wigner law  $\gamma_t$ , and is given by the same formula.  $\square$

As in the continuous case, most of the basic theory of  $\pi_t$  was done before, namely in chapter 8, with all this partly coming from the theory of  $SO_3$ , at  $t = 1$ . One thing which was missing there, however, was that of understanding how the law  $\pi_t$ , with parameter  $t > 0$ , exactly appears, out of  $\pi_1$ . We can now solve this question:

**THEOREM 9.28.** *The Marchenko-Pastur laws have the property*

$$\pi_s \boxplus \pi_t = \pi_{s+t}$$

*so they form a 1-parameter semigroup with respect to free convolution.*

**PROOF.** This follows either from Theorem 9.27, or from the fact that the  $R$ -transform of  $\pi_t$ , computed in the proof of Theorem 9.27, is linear in  $t$ .  $\square$

In analogy with Theorem 9.26, dealing with the continuous case, we can now summarize the various discrete results that we have, classical and free, as follows:

**THEOREM 9.29.** *The Poisson laws  $p_t$  and the Marchenko-Pastur laws  $\pi_t$ , given by*

$$p_t = e^{-t} \sum_k \frac{t^k}{k!} \delta_k$$

$$\pi_t = \max(1-t, 0) \delta_0 + \frac{\sqrt{4t - (x-1-t)^2}}{2\pi x} dx$$

*have the following properties:*

- (1) *They appear via the PLT, and the free PLT.*
- (2) *They form semigroups with respect to  $*$  and  $\boxplus$ .*
- (3) *Their transforms are  $\log F_{p_t}(x) = t(e^{ix} - 1)$ ,  $R_{\pi_t}(x) = t/(1-x)$ .*
- (4) *Their moments are  $M_k = \sum_{\pi \in D(k)} t^{|\pi|}$ , with  $D = P, NC$ .*

**PROOF.** These are all results that we already know, from here and from the previous chapters. To be more precise:

- (1) The PLT is from chapter 2, and the FPLT is from here.
- (2) The semigroup properties are from chapter 2, and from here.
- (3) The formula for  $F_{p_t}$  is from chapter 2, and the one for  $R_{\pi_t}$ , from here.
- (4) The moment formulae follow from the formulae of functional transforms.  $\square$

There is an obvious similarity here with Theorem 9.26, and we have:

THEOREM 9.30. *The moments of the various central limiting measures, namely*

$$\begin{array}{ccc} \pi_t & \xrightarrow{\quad} & \gamma_t \\ | & & | \\ p_t & \xrightarrow{\quad} & g_t \end{array}$$

*are always given by the same formula, involving partitions, namely*

$$M_k = \sum_{\pi \in D(k)} t^{|\pi|}$$

*where the sets of partitions  $D(k)$  in question are respectively*

$$\begin{array}{ccc} NC & \xleftarrow{\quad} & NC_2 \\ | & & | \\ P & \xleftarrow{\quad} & P_2 \end{array}$$

*and where  $|\cdot|$  is the number of blocks.*

PROOF. This follows by putting together the various results that we have.  $\square$

We will see later on a more conceptual explanation for the above result, in terms of cumulants, and a number of extensions as well, eventually ending up with a cube.

### 9e. Exercises

There has been a lot of exciting theory in this chapter, for the most in relation with various free product constructions, and as a first exercise on all this, we have:

EXERCISE 9.31. *Prove that given two algebras  $(A, tr)$  and  $(B, tr)$ , these algebras are free inside their free product  $A * B$ , endowed with its canonical free product trace.*

This is something that we already discussed in the above, but with some details missing. Time now to have this done, with all the details.

EXERCISE 9.32. *State and prove a complex analogue of the free CLT, as well as a compound analogue of the free PLT, and study the limiting measures.*

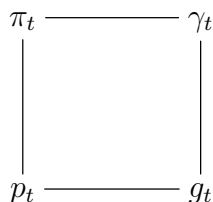
This is something very instructive, and normally all the needed tools, namely the CLT and PLT, and the theory of the  $R$ -transform, are there. Of course, this is more than a regular exercise, and we will be back to this, in what follows, on several occasions.

## CHAPTER 10

### Circular variables

#### 10a. Free CCLT

We have seen so far that free probability theory leads to two key limiting theorems, namely the free CLT and PLT. In these theorems, the limiting measures are the Wigner laws  $\gamma_t$  and the Marchenko-Pastur laws  $\pi_t$ , that we previously met in connection with the random matrices. Together with the Gaussian laws  $g_t$  and the Poisson laws  $p_t$ , appearing from the classical CLT and PLT, these laws form a square diagram, as follows:



Our first purpose here will be that of extending this diagram to the right, with a free analogue of the complex central limiting theorem (CCLT), adding to the classical CCLT, and providing us with free analogues  $\Gamma_t$  of the complex Gaussian laws  $G_t$ .

This will be something quite technical, and in order to get started, let us begin by recalling the theory of the complex Gaussian laws  $G_t$  and of the CCLT, developed at various places in the previous chapters. To start with, we have the following definition:

DEFINITION 10.1. *The complex Gaussian law of parameter  $t > 0$  is*

$$G_t = \text{law} \left( \frac{1}{\sqrt{2}}(a + ib) \right)$$

where  $a, b$  are independent, each following the law  $g_t$ .

There are many things that can be said about these laws, simply by adapting the known results from the real case, regarding the usual normal laws  $g_t$ . As a first such result, the above measures form convolution semigroups:

PROPOSITION 10.2. *The complex Gaussian laws have the property*

$$G_s * G_t = G_{s+t}$$

for any  $s, t > 0$ , and so they form a convolution semigroup.

PROOF. This is something that we know from chapter 1, coming from  $g_s * g_t = g_{s+t}$ , by taking the real and imaginary parts of all variables involved.  $\square$

We have as well the following complex analogue of the CLT:

THEOREM 10.3 (CCLT). *Given complex variables  $f_1, f_2, f_3, \dots \in L^\infty(X)$  which are i.i.d., centered, and with variance  $t > 0$ , we have, with  $n \rightarrow \infty$ , in moments,*

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n f_i \sim G_t$$

where  $G_t$  is the complex Gaussian law of parameter  $t$ .

PROOF. This is something that we know too from chapter 1, which follows from the real CLT, by taking real and imaginary parts. Indeed, let us write:

$$f_i = \frac{1}{\sqrt{2}}(x_i + iy_i)$$

The variables  $x_i$  satisfy then the assumptions of the CLT, so their rescaled averages converge to a normal law  $g_t$ , and the same happens for the variables  $y_i$ . The limiting laws that we obtain being independent, their rescaled sum is complex Gaussian, as desired.  $\square$

Regarding now the moments, we have here the following result:

PROPOSITION 10.4. *The moments of the complex normal law are the numbers*

$$M_k(G_t) = t^{|k|/2} |\mathcal{P}_2(k)|$$

where  $\mathcal{P}_2(k)$  is the set of matching pairings of  $\{1, \dots, k\}$ .

PROOF. This is again something that we know well too, from the above, the idea being as follows, with  $c = \frac{1}{\sqrt{2}}(a + ib)$  being the variable in Definition 10.1:

(1) In the case where  $k$  contains a different number of  $\circ$  and  $\bullet$  symbols, a rotation argument shows that the corresponding moment of  $c$  vanishes. But in this case we also have  $\mathcal{P}_2(k) = \emptyset$ , so the formula in the statement holds indeed, as  $0 = 0$ .

(2) In the case left, where  $k$  consists of  $p$  copies of  $\circ$  and  $p$  copies of  $\bullet$ , the corresponding moment is the  $p$ -th moment of  $|c|^2$ , which by some calculus is  $t^p p!$ . But in this case we have as well  $|\mathcal{P}_2(k)| = p!$ , so the formula in the statement holds indeed, as  $t^p p! = t^p p!$ .  $\square$

As a final result regarding the complex normal laws, we have the Wick formula:

THEOREM 10.5. *Given independent variables  $X_i$ , each following the complex normal law  $G_t$ , with  $t > 0$  being a fixed parameter, we have the Wick formula*

$$\mathbb{E}(X_{i_1}^{k_1} \dots X_{i_s}^{k_s}) = t^{s/2} \# \left\{ \pi \in \mathcal{P}_2(k) \mid \pi \leq \ker(i) \right\}$$

where  $k = k_1 \dots k_s$  and  $i = i_1 \dots i_s$ , for the joint moments of these variables.

PROOF. This is something that we know too, the idea being as follows:

(1) In the case where we have a single complex normal variable  $X$ , we have to compute the moments of  $X$ , with respect to colored integer exponents  $k = \circ \bullet \bullet \circ \dots$ , and the formula in the statement coincides with the one in Theorem 10.4 above, namely:

$$\mathbb{E}(X^k) = t^{|k|/2} |\mathcal{P}_2(k)|$$

(2) In general now, when expanding  $X_{i_1}^{k_1} \dots X_{i_s}^{k_s}$  and rearranging the terms, we are left with doing a number of computations as in (1), then making the product of the numbers that we found. But this amounts in counting the partitions in the statement.  $\square$

Let us discuss now the free analogues of the above results. As in the classical case, there is actually not so much work to be done here, in order to get started, because we can obtain the free convolution and central limiting results, simply by taking the real and imaginary parts of our variables. Following Voiculescu [85], [86], we first have:

DEFINITION 10.6. *The Voiculescu circular law of parameter  $t > 0$  is given by*

$$\Gamma_t = \text{law} \left( \frac{1}{\sqrt{2}}(a + ib) \right)$$

where  $a, b$  are free, each following the Wigner semicircle law  $\gamma_t$ .

In other words, the passage  $\gamma_t \rightarrow \Gamma_t$  is by definition entirely similar to the passage  $g_t \rightarrow G_t$  from the classical case, by taking real and imaginary parts. As before in other similar situations, the fact that  $\Gamma_t$  is indeed well-defined is clear from definitions.

Let us start with a number of straightforward results, obtained by complexifying the free probability theory that we have. As a first result, we have, as announced above:

PROPOSITION 10.7. *The Voiculescu circular laws have the property*

$$\Gamma_s \boxplus \Gamma_t = \Gamma_{s+t}$$

so they form a 1-parameter semigroup with respect to free convolution.

PROOF. This follows from our previous result stating that the Wigner laws  $\gamma_t$  have the free semigroup convolution property, by taking real and imaginary parts.  $\square$

Next in line, also as announced above, and also from [86], we have the following natural free analogue of the complex central limiting theorem (CCLT):

THEOREM 10.8 (Free CCLT). *Given random variables  $x_1, x_2, x_3, \dots$  which are f.i.d., centered, with variance  $t > 0$ , we have, with  $n \rightarrow \infty$ , in moments,*

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n x_i \sim \Gamma_t$$

where  $\Gamma_t$  is the Voiculescu circular law of parameter  $t$ .

PROOF. This follows indeed from the free CLT, established in chapter 9 above, by taking real and imaginary parts. Indeed, let us write:

$$x_i = \frac{1}{\sqrt{2}}(y_i + iz_i)$$

The variables  $y_i$  satisfy then the assumptions of the free CLT, and so their rescaled averages converge to a semicircle law  $\gamma_t$ , and the same happens for the variables  $z_i$ :

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n y_i \sim \gamma_t \quad , \quad \frac{1}{\sqrt{n}} \sum_{i=1}^n z_i \sim \gamma_t$$

Now since the two limiting semicircle laws that we obtain in this way are free, their rescaled sum is circular, in the sense of Definition 10.6, and this gives the result.  $\square$

Summarizing, we have so far complex analogues of both the classical and free CLT, and the basic theory of the limiting measures, including their semigroup property. As a conclusion to all this, let us formulate the following statement:

THEOREM 10.9. *We have classical and free limiting theorems, as follows,*

$$\begin{array}{ccccc} FPLT & \text{---} & FCLT & \text{---} & FCCLT \\ | & & | & & | \\ PLT & \text{---} & CLT & \text{---} & CCLT \end{array}$$

*the limiting laws being the following measures,*

$$\begin{array}{ccccc} \pi_t & \text{---} & \gamma_t & \text{---} & \Gamma_t \\ | & & | & & | \\ p_t & \text{---} & g_t & \text{---} & G_t \end{array}$$

*which form classical and free convolution semigroups.*

PROOF. This follows indeed from the various results established above. To be more precise, the result about the square on the left is from the previous chapter, and the results about the vertical right edge are those discussed in the above.  $\square$

Going ahead with more study of the Voiculescu circular variables, less trivial now is the computation of their moments. We will do this in what follows, among others in order to expand Theorem 10.9 into something much sharper, involving as well moments.

For our computations, we will need explicit models for the circular variables. Following [86], and the material in chapter 9, let us start with the following key result:

PROPOSITION 10.10. *Let  $H$  be the complex Hilbert space having as basis the colored integers  $k = \circ \bullet \bullet \circ \dots$ , and consider the shift operators on this space:*

$$S : k \rightarrow \circ k \quad , \quad T : k \rightarrow \bullet k$$

*We have then the following equalities of distributions,*

$$S + S^* \sim \gamma_1 \quad , \quad S + T^* \sim \Gamma_1$$

*with respect to the state  $\varphi(T) = \langle Te, e \rangle$ , where  $e$  is the empty word.*

PROOF. This is standard free probability, the idea being as follows:

(1) The first formula, namely  $S + S^* \sim \gamma_1$ , is something that we already know, in a slightly different formulation, from chapter 9, when proving the CLT.

(2) As for the second formula,  $S + T^* \sim \Gamma_1$ , this follows from the first formula, by using the freeness results and the rotation tricks established in chapter 9.  $\square$

At the combinatorial level now, we have the following result, which is in analogy with the moment theory of the Wigner semicircle law, developed above:

THEOREM 10.11. *A variable  $a \in A$  follows the law  $\Gamma_1$  precisely when its moments are*

$$\text{tr}(a^k) = |\mathcal{NC}_2(k)|$$

*for any colored integer  $k = \circ \bullet \bullet \circ \dots$ .*

PROOF. By using Proposition 10.10, it is enough to do the computation in the model there. To be more precise, we can use the following explicit formulae for  $S, T$ :

$$S : k \rightarrow \circ k \quad , \quad T : k \rightarrow \bullet k$$

With these formulae in hand, our claim is that we have the following formula:

$$\langle (S + T^*)^k e, e \rangle = |\mathcal{NC}_2(k)|$$

In order to prove this formula, we can proceed as for the semicircle laws, in chapter 9 above. Indeed, let us expand the quantity  $(S + T^*)^k$ , and then apply the state  $\varphi$ .

With respect to the previous computation, from chapter 9, what happens is that the contributions will come this time via the following formulae, which must successively apply, as to collapse the whole product of  $S, S^*, T, T^*$  variables into a 1 quantity:

$$S^*S = 1 \quad , \quad T^*T = 1$$

As before, in the proof for the semicircle laws, from chapter 9, these applications of the rules  $S^*S = 1$ ,  $T^*T = 1$  must appear in a noncrossing manner, but what happens now, in contrast with the computation from the proof in chapter 9 where  $S + S^*$  was self-adjoint, is that at each point where the exponent  $k$  has a  $\circ$  entry we must use  $T^*T = 1$ , and at each point where the exponent  $k$  has a  $\bullet$  entry we must use  $S^*S = 1$ . Thus the contributions, which are each worth 1, are parametrized by the partitions  $\pi \in \mathcal{NC}_2(k)$ . Thus, we obtain the above moment formula, as desired.  $\square$

More generally now, by rescaling, we have the following result:

**THEOREM 10.12.** *A variable  $a \in A$  is circular,  $a \sim \Gamma_t$ , precisely when its moments are given by the formula*

$$\mathrm{tr}(a^k) = t^{|k|/2} |\mathcal{NC}_2(k)|$$

for any colored integer  $k = \circ \bullet \bullet \circ \dots$

**PROOF.** This follows indeed from Theorem 10.11, by rescaling. Alternatively, we can get this as well directly, by suitably modifying Proposition 10.10 first.  $\square$

Even more generally now, we have the following free version of the Wick rule:

**THEOREM 10.13.** *Given free variables  $a_i$ , each following the Voiculescu circular law  $\Gamma_t$ , with  $t > 0$  being a fixed parameter, we have the Wick type formula*

$$\mathrm{tr}(a_{i_1}^{k_1} \dots a_{i_s}^{k_s}) = t^{s/2} \# \left\{ \pi \in \mathcal{NC}_2(k) \mid \pi \leq \ker(i) \right\}$$

where  $k = k_1 \dots k_s$  and  $i = i_1 \dots i_s$ , for the joint moments of these variables, with the inequality  $\pi \leq \ker(i)$  on the right being taken in a technical, appropriate sense.

**PROOF.** This follows a bit as in the classical case, the idea being as follows:

(1) In the case where we have a single complex normal variable  $a$ , we have to compute the moments of  $a$ , with respect to colored integer exponents  $k = \circ \bullet \bullet \circ \dots$ , and the formula in the statement coincides with the one in Theorem 10.12, namely:

$$\mathrm{tr}(a^k) = t^{|k|/2} |\mathcal{NC}_2(k)|$$

(2) In general now, when expanding the product  $a_{i_1}^{k_1} \dots a_{i_s}^{k_s}$  and rearranging the terms, we are left with doing a number of computations as in (1), and then making the product of the expectations that we found. But this amounts precisely in counting the partitions in the statement, with the condition  $\pi \leq \ker(i)$  there standing precisely for the fact that we are doing the various type (1) computations independently.  $\square$

All the above was a bit brief, based on Voiculescu's original paper [86], and on his foundational free probability book with Dykema and Nica [90]. The combinatorics of the free families of circular variables, called “circular systems”, is something quite subtle, and there has been a lot of work developed in this direction. For a complement to the above material, with a systematic study using advanced tools from combinatorics, we refer to the more recent book by Nica and Speicher [70]. We will be actually back to this, in this book too, namely in chapter 12 below, when talking about cumulants.

On the same topic, let us mention as well that various technical extensions and generalizations of the above results can be found, hidden as technical lemmas, throughout the random matrix and operator algebra literature, in connection with free probability, with the notable users of the circular systems including, besides Voiculescu himself, Dykema [43], Mingo, Nica, Speicher [66], [68], [70], [78], [79], and Shlyakhtenko [77].

Getting back now to the case of the single variables, from Theorem 10.12, the formula there has the following more conceptual interpretation:

**THEOREM 10.14.** *The moments of the Voiculescu laws are the numbers*

$$M_k(\Gamma_t) = \sum_{\pi \in \mathcal{NC}_2(k)} t^{|\pi|}$$

with “ $\mathcal{NC}_2$ ” standing for the noncrossing matching pairings.

**PROOF.** This follows from the formula in Theorem 10.12. Indeed, we know from there that a variable  $a \in A$  is circular, of parameter  $t > 0$ , precisely when we have the following formula, for any colored integer  $k = \circ \bullet \bullet \circ \dots$ :

$$\text{tr}(a^k) = t^{|k|/2} |\mathcal{NC}_2(k)|$$

Now since the number of blocks of a pairing  $\pi \in \mathcal{NC}_2(k)$  is given by  $|\pi| = |k|/2$ , this formula can be written in the following alternative way:

$$\text{tr}(a^k) = \sum_{\pi \in \mathcal{NC}_2(k)} t^{|\pi|}$$

Thus, we are led to the conclusion in the statement.  $\square$

All this is quite nice, when compared with the similar results from the classical case, regarding the complex Gaussian laws, that we established above, and with other results of the same type as well. As a conclusion to these considerations, we can now formulate a global result regarding the classical and free complex Gaussian laws, as follows:

**THEOREM 10.15.** *The complex Gaussian laws  $G_t$  and the circular Voiculescu laws  $\Gamma_t$ , given by the formulae*

$$G_t = \text{law} \left( \frac{1}{\sqrt{2}}(a + ib) \right) \quad , \quad \Gamma_t = \text{law} \left( \frac{1}{\sqrt{2}}(\alpha + i\beta) \right)$$

where  $a, b/\alpha, \beta$  are independent/free, following  $g_t/\gamma_t$ , have the following properties:

- (1) They appear via the complex CLT, and the free complex CLT.
- (2) They form semigroups with respect to the operations  $*$  and  $\boxplus$ .
- (3) Their moments are  $M_k = \sum_{\pi \in D(k)} t^{|\pi|}$ , with  $D = \mathcal{P}_2, \mathcal{NC}_2$ .

**PROOF.** This is a summary of results that we know, the idea being as follows:

(1) This is something quite straightforward, by using the linearization results provided by the logarithm of the Fourier transform, and by the  $R$ -transform.

(2) This is quite straightforward, too, once again by using the linearization results provided by the logarithm of the Fourier transform, and by the  $R$ -transform.

(3) This comes by doing some combinatorics and calculus in the classical case, and some combinatorics and operator theory in the free case, as explained above.  $\square$

More generally now, we can put everything together, with some previous results included as well, and we have the following result at the level of the moments of the asymptotic laws that we found so far, in classical and free probability:

**THEOREM 10.16.** *The moments of the various central limiting measures, namely*

$$\begin{array}{ccccc} \pi_t & \text{---} & \gamma_t & \text{---} & \Gamma_t \\ | & & | & & | \\ p_t & \text{---} & g_t & \text{---} & G_t \end{array}$$

*are always given by the same formula, involving partitions, namely*

$$M_k = \sum_{\pi \in D(k)} t^{|\pi|}$$

*where the sets of partitions  $D(k)$  in question are respectively*

$$\begin{array}{ccccc} NC & \longleftarrow & NC_2 & \longleftarrow & \mathcal{NC}_2 \\ | & & | & & | \\ P & \longleftarrow & P_2 & \longleftarrow & \mathcal{P}_2 \end{array}$$

*and where  $|\cdot|$  is the number of blocks.*

**PROOF.** This follows by putting together the various moment results that we have, from the previous chapter, and from Theorem 10.15.  $\square$

Summarizing, we are done with the combinatorial program outlined in the beginning of the present chapter. We will be back to this in the next chapter, 11 below, by adding some new laws to the picture, coming from the classical and free CPLT, and then in the chapter afterwards, 12 below, with full conceptual explanations for all this.

### 10b. Multiplicative results

With the above basic combinatorial study done, let us discuss now a number of more advanced results regarding the Voiculescu circular laws  $\Gamma_t$ , which are of multiplicative nature, and quite often have no classical counterpart. Things here will be quite technical, and all that follows will be rather an introduction to the subject.

In general now, in order to deal with multiplicative questions for the free random variables, we are in need of results regarding the multiplicative free convolution operation  $\boxtimes$ . Let us recall from chapter 9 that we have the following result:

DEFINITION 10.17. *We have a free convolution operation  $\boxtimes$ , constructed as follows:*

- (1) *For abstract distributions, via  $\mu_a \boxtimes \mu_b = \mu_{ab}$ , with  $a, b$  free.*
- (2) *For real measures, via  $\mu_a \boxtimes \mu_b = \mu_{\sqrt{ab}\sqrt{a}}$ , with  $a, b$  self-adjoint and free.*

All this is quite tricky, explained in chapter 9, the idea being that, while (1) is straightforward, (2) is not, and comes by considering the variable  $c = \sqrt{ab}\sqrt{a}$ , which unlike  $ab$  is always self-adjoint, and whose moments are given by:

$$\begin{aligned} \text{tr}(c^k) &= \text{tr}[(\sqrt{ab}\sqrt{a})^k] \\ &= \text{tr}[\sqrt{ab}a \dots ab\sqrt{a}] \\ &= \text{tr}[\sqrt{a} \cdot \sqrt{ab}a \dots ab] \\ &= \text{tr}[(ab)^k] \end{aligned}$$

As a remark here, observe that we have used in the above, and actually for the first time since talking about freeness, the trace property of the trace, namely:

$$\text{tr}(ab) = \text{tr}(ba)$$

This is quite interesting, philosophically speaking, because in the operator algebra world there are many interesting examples of subalgebras  $A \subset B(H)$  coming with natural linear forms  $\varphi : A \rightarrow \mathbb{C}$  which are continuous and positive, but which are not traces. See [27]. It is possible to do a bit of free probability on such algebras, but not much.

Quite remarkably, the free multiplicative convolution operation  $\boxtimes$  can be linearized, in analogy with what happens for the usual multiplicative convolution  $\times$ , and the additive operations  $*, \boxplus$  as well. We have here the following result, due to Voiculescu [87]:

THEOREM 10.18. *The free multiplicative convolution operation  $\boxtimes$  for the real probability measures  $\mu \in \mathcal{P}(\mathbb{R})$  can be linearized as follows:*

- (1) *Start with the sequence of moments  $M_k$ , then compute the moment generating function, or Stieltjes transform of the measure:*

$$f(z) = 1 + M_1 z + M_2 z^2 + M_3 z^3 + \dots$$

- (2) *Perform the following operations to the Stieltjes transform:*

$$\psi(z) = f(z) - 1$$

$$\psi(\chi(z)) = z$$

$$S(z) = \left(1 + \frac{1}{z}\right) \chi(z)$$

- (3) *Then  $\log S$  linearizes the free multiplicative convolution,  $S_{\mu \boxtimes \nu} = S_\mu S_\nu$ .*

PROOF. There are several proofs here, with the original proof of Voiculescu [87] being quite similar to the proof of the  $R$ -transform theorem, using free Fock space models, then with a proof by Haagerup [51], obtained by further improving on this, and finally with the proof from the book of Nica and Speicher [70], using pure combinatorics. The proof of Haagerup [51], which is the most in tune with the present book, is as follows:

(1) According to our conventions from Definition 10.17, we want to prove that, given noncommutative variables  $a, b$  which are free, we have the following formula:

$$S_{\mu_{ab}}(z) = S_{\mu_a}(z)S_{\mu_b}(z)$$

(2) For this purpose, consider the orthogonal shifts  $S, T$  on the free Fock space, as in chapter 9. By using the algebraic arguments from chapter 9, from the proof of the  $R$ -transform theorem, we can assume as there that our variables have a special form, that fits our present objectives, and to be more specifically, the following form:

$$a = (1 + S)f(S^*) \quad , \quad b = (1 + T)g(T^*)$$

Our claim, which will prove the theorem, is that we have the following formulae, for the  $S$ -transforms of the various variables involved:

$$S_{\mu_a}(z) = \frac{1}{f(z)} \quad , \quad S_{\mu_b}(z) = \frac{1}{g(z)} \quad , \quad S_{\mu_{ab}}(z) = \frac{1}{f(z)g(z)}$$

(3) Let us first compute  $S_{\mu_a}$ . We know that we have  $a = (1 + S)f(S^*)$ , with  $S$  being the shift on  $l^2(\mathbb{N})$ . Given  $|z| < 1$ , consider the following vector:

$$p = \sum_{k \geq 0} z^k e_k$$

The shift and its adjoint act on this vector in the following way:

$$Sp = \sum_{k \geq 0} z^k e_{k+1} = \frac{p - e_0}{z}$$

$$S^*p = \sum_{k \geq 1} z^k e_{k-1} = zp$$

Thus  $f(S^*)p = f(z)p$ , and we deduce from this that we have:

$$\begin{aligned} ap &= (1 + S)f(z)p \\ &= f(z)(p + Sp) \\ &= f(z) \left( p + \frac{p - e_0}{z} \right) \\ &= \left( 1 + \frac{1}{z} \right) f(z)p - \frac{f(z)}{z} e_0 \end{aligned}$$

By dividing everything by  $(1 + 1/z)f(z)$ , this formula becomes:

$$\frac{z}{1+z} \cdot \frac{1}{f(z)} ap = p - \frac{e_0}{1+z}$$

We can write this latter formula in the following way:

$$\left(1 - \frac{z}{1+z} \cdot \frac{1}{f(z)} a\right) p = \frac{e_0}{1+z}$$

Now by inverting, we obtain from this the following formula:

$$\left(1 - \frac{z}{1+z} \cdot \frac{1}{f(z)} a\right)^{-1} e_0 = (1+z)p$$

(4) But this gives us the formula of  $S_{\mu_a}$ . Indeed, consider the following function:

$$\rho(z) = \frac{z}{1+z} \cdot \frac{1}{f(z)}$$

With this notation, the formula that we found in (3) becomes:

$$(1 - \rho(z)a)^{-1} e_0 = (1+z)p$$

By using this, in terms of  $\varphi(T) = \langle Te_0, e_0 \rangle$ , we obtain:

$$\begin{aligned} \varphi((1 - \rho(z)a)^{-1}) &= \langle (1 - \rho(z)a)^{-1} e_0, e_0 \rangle \\ &= \langle (1+z)p, e_0 \rangle \\ &= 1+z \end{aligned}$$

Thus the above function  $\rho$  is the inverse of the following function:

$$\psi(z) = \varphi\left(\frac{1}{1-za}\right) - 1$$

But this latter function is the  $\psi$  function from the statement, and so  $\rho$  is the function  $\chi$  from the statement, and we can finish our computation, as follows:

$$\begin{aligned} S_{\mu_a}(z) &= \frac{1+z}{z} \cdot \rho(z) \\ &= \frac{1+z}{z} \cdot \frac{z}{1+z} \cdot \frac{1}{f(z)} \\ &= \frac{1}{f(z)} \end{aligned}$$

(5) A similar computation, or just a symmetry argument, gives  $S_{\mu_b}(z) = 1/g(z)$ . In order to compute now  $S_{\mu_{ab}}(z)$ , we use a similar trick. Consider the following vector of  $l^2(\mathbb{N} * \mathbb{N})$ , with the primes and double primes referring to the two copies of  $\mathbb{N}$ :

$$q = e_0 + \sum_{k \geq 1} (e'_1 + e''_1 + e'_1 \otimes e''_1)^{\otimes k}$$

The adjoints of the shifts  $S, T$  act as follows on this vector:

$$S^*q = z(1 + T)q \quad , \quad T^*q = zq$$

By using these formulae, we have the following computation:

$$\begin{aligned} abq &= (1 + S)f(S^*)(1 + T)g(T^*)q \\ &= (1 + S)f(S^*)(1 + T)g(z)q \\ &= g(z)(1 + S)f(S^*)(1 + T)q \end{aligned}$$

In order to compute the last term, observe that we have:

$$\begin{aligned} S^*(1 + T)q &= (S^* + S^*T)q \\ &= S^*q \\ &= z(1 + T)q \end{aligned}$$

Thus  $f(S^*)(1 + T)q = f(z)(1 + T)q$ , and back to our computation, we have:

$$\begin{aligned} abq &= g(z)(1 + S)f(z)(1 + T)q \\ &= f(z)g(z)(1 + S)(1 + T)q \\ &= f(z)g(z) \left( \frac{1 + z}{z} \cdot q - \frac{e_0}{z} \right) \end{aligned}$$

Now observe that we can write this formula as follows:

$$\left( 1 - \frac{z}{1 + z} \cdot \frac{1}{f(z)g(z)} \cdot ab \right) q = \frac{e_0}{1 + z}$$

By inverting, we obtain from this the following formula:

$$\left( 1 - \frac{z}{1 + z} \cdot \frac{1}{f(z)g(z)} \cdot ab \right)^{-1} e_0 = (1 + z)q$$

(6) But this formula that we obtained is similar to the formula that we obtained at the end of (3) above. Thus, we can use the same argument as in (4), and we obtain:

$$S_{\mu_{ab}}(z) = \frac{1}{f(z)g(z)}$$

We are therefore done with the computations, and this finishes the proof.  $\square$

Getting back now to the circular variables, let us look at the polar decomposition of such variables. In order to discuss this, let us start with a well-known result:

**THEOREM 10.19.** *We have the following results:*

- (1) Any matrix  $T \in M_N(\mathbb{C})$  has a polar decomposition,  $T = U|T|$ .
- (2) Assuming  $T \in A \subset M_N(\mathbb{C})$ , we have  $U, |T| \in A$ .
- (3) Any operator  $T \in B(H)$  has a polar decomposition,  $T = U|T|$ .
- (4) Assuming  $T \in A \subset B(H)$ , we have  $U, |T| \in \bar{A}$ , weak closure.

PROOF. All this is standard, the idea being as follows:

(1) In each case under consideration, the first observation is that the matrix or general operator  $T^*T$  being positive, it has a square root:

$$|T| = \sqrt{T^*T}$$

(2) With this square root extracted, in the invertible case we can compare the action of  $T$  and  $|T|$ , and we conclude that we have  $T = U|T|$ , with  $U$  being a unitary. In the general, non-invertible case, a similar analysis leads to the conclusion that we have as well  $T = U|T|$ , but with  $U$  being this time a partial isometry.

(3) In what regards now algebraic and topological aspects, in finite dimensions the extraction of the square root, and so the polar decomposition itself, takes place over the matrix blocks of the ambient algebra  $A \subset M_N(\mathbb{C})$ , and so takes place inside  $A$  itself.

(4) In infinite dimensions however, we must take the weak closure, an illustrating example here being the functions  $f \in A$  belonging to the algebra  $A = C(X)$ , represented on  $H = L^2(X)$ , whose polar decomposition leads into the bigger algebra  $\bar{A} = L^\infty(X)$ .  $\square$

Summarizing, we have a basic linear algebra result, regarding the polar decomposition of the usual matrices, and in infinite dimensions pretty much the same happens, with the only subtlety coming from the fact that the ambient operator algebra  $A \subset B(H)$  must be taken weakly closed. We will be back to this, with more details, in chapter 15 below, when talking about such algebras  $A \subset B(H)$ , which are called von Neumann algebras.

In connection with our probabilistic questions, we first have the following result:

PROPOSITION 10.20. *The polar decomposition of semicircular variables is  $s = eq$ , with the variables  $e, q$  being as follows:*

- (1)  $e$  has moments  $1, 0, 1, 0, 1, \dots$
- (2)  $q$  is quarter-circular.
- (3)  $e, q$  are independent.

PROOF. It is enough to prove the result in a model of our choice, and the best choice here is the most straightforward model for the semicircular variables, namely:

$$s = x \in L^\infty([-2, 2], \gamma_1)$$

To be more precise, we endow the interval  $[-2, 2]$  with the probability measure  $\gamma_1$ , and we consider here the variable  $s = x = (x \rightarrow x)$ , which is trivially semicircular. The polar decomposition of this variable is then  $s = eq$ , with  $e, q$  being as follows:

$$e = \operatorname{sgn}(x) \quad , \quad q = |x|$$

Now since  $e$  has moments  $1, 0, 1, 0, 1, \dots$ , and also  $q$  is quarter-circular, and finally  $e, q$  are independent, this gives the result in our model, and so in general.  $\square$

Less trivial now is the following result, due to Voiculescu [88]:

**THEOREM 10.21.** *The polar decomposition of circular variables is  $c = uq$ , with the variables  $u, q$  being as follows:*

- (1)  $u$  is a Haar-unitary.
- (2)  $q$  is quarter-circular.
- (3)  $u, q$  are free.

**PROOF.** This is something which looks quite similar to Proposition 10.20, but which is more difficult, and can be however proved, via various techniques:

(1) The original proof, by Voiculescu in [88], uses Gaussian random matrix models for the circular variables. We will discuss this proof at the end of the present chapter, after developing the needed Gaussian random matrix model technology.

(2) A second proof, obtained by pure combinatorics, in the spirit of Theorem 10.13, regarding the free Wick formula, and of Theorem 10.18, regarding the  $S$ -transform, or rather in the spirit of the underlying combinatorics of these results, is the one in [70].

(3) Finally, there is as well a third proof, from [5], more in the spirit of the free Fock space proofs for the  $R$  and  $S$  transform results, from [86], [87], using a suitable generalization of the free Fock spaces. We will discuss this proof right below.  $\square$

### 10c. Semigroup models

We discuss here, following [5], the direct approach to Theorem 10.21, with purely algebraic techniques. We will use semigroup algebras, jointly generalizing the main models that we have, namely group algebras, and free Fock spaces. Let us start with:

**DEFINITION 10.22.** *We call “semigroup” a unital semigroup, embeddable into a group:*

$$M \subset G$$

*For such a semigroup  $M$ , we use the notation*

$$M^{-1} = \left\{ m^{-1} \mid m \in M \right\}$$

*regarded as a subset of some group  $G$  containing  $M$ , as above.*

As a first observation, the above embeddability assumption  $M \subset G$  tells us that the usual group cancellation rules hold in  $M$ , namely:

$$ab = ac \implies b = c$$

$$ba = ca \implies b = c$$

Regarding the precise relation between  $M$  and the various groups  $G$  containing it, it is possible to talk here about the Grothendieck group  $G$  associated to such a semigroup  $M$ . However, we will not need this in what follows, and use Definition 10.22 as such.

With the above definition in hand, we have the following construction, which unifies the main models that we have, namely the group algebras, and the free Fock spaces:

**PROPOSITION 10.23.** *Let  $M$  be a semigroup. By using the left simplifiability of  $M$  we can define, as for the discrete groups, an embedding of semigroups, as follows:*

$$(M, \cdot) \rightarrow (B(l^2(M)), \circ)$$

$$m \rightarrow \lambda_M(m) = [\delta_n \rightarrow \delta_{mn}]$$

*Via this embedding, the  $C^*$ -algebra  $C^*(M) \subset B(l^2(M))$  generated by  $\lambda_M(M)$ , together with the following canonical state, is a noncommutative random variable algebra:*

$$\tau_M(T) = \langle T\delta_e, \delta_e \rangle$$

*Also, the operators in  $\lambda_M(M)$  are isometries, but not necessarily unitaries.*

**PROOF.** Everything here is standard, as for the usual group algebras, with the only subtlety appearing at the level of the isometry property of the operators  $\lambda_M(m)$ . To be more precise, for every  $m \in M$ , the adjoint operator  $\lambda_M(m)^*$  is given by:

$$\lambda_M(m)^*(\delta_n) = \sum_{x \in M} \langle \lambda_M(m)^*\delta_n, \delta_x \rangle \delta_x = \sum_{x \in M} \delta_{n, mx} \delta_x$$

Thus we have indeed the isometry property for these operators, namely:

$$\lambda_M(m)^*\lambda_M(m) = 1$$

As for the unitarity property of the such operators, this definitely holds in the usual discrete group case,  $M = G$ , but not in general. As a basic example here, for the semigroup  $M = \mathbb{N}$ , which satisfies of course the assumptions in Definition 10.22, the operator  $\lambda_M(m)$  associated to the element  $m = 1 \in \mathbb{N}$  is the usual shift:

$$\lambda_{\mathbb{N}}(1) = S \in B(l^2(\mathbb{N}))$$

But this shift  $S$ , that we know well from the above, is an isometry which is not a unitary. Thus, we are led to the conclusions in the statement.  $\square$

At the level of examples now, as announced above, we have:

**PROPOSITION 10.24.** *The construction  $M \rightarrow C^*(M)$  is as follows:*

- (1) *For the discrete groups,  $M = G$ , we obtain in this way the usual discrete group algebras  $C^*(G)$ , as previously constructed in the above.*
- (2) *For a free semigroup,  $M = \mathbb{N}^I$ , we obtain the algebra of creation operators over the full Fock space over  $\mathbb{R}^I$ , with the state associated to the vacuum vector.*

**PROOF.** All this is clear from definitions, with (1) being obvious, and (2) coming via our usual identifications for the free Fock spaces and related algebras.  $\square$

As a key observation now, enabling us to do some probability, we have:

PROPOSITION 10.25. *If  $M \subset N$  are semigroups satisfying the condition*

$$M(N - M) = N - M$$

*then for every family  $\{a_i\}_{i \in I}$  of elements in  $M$ , we have the formula*

$$\{\lambda_N(a_i)\}_{i \in I} \sim \{\lambda_M(a_i)\}_{i \in I}$$

*as an equality of joint distributions, with respect to the canonical states.*

PROOF. Assuming  $M \subset N$  we have  $l^2(M) \subset l^2(N)$ , and for  $m, m' \in M$  we have:

$$\lambda_M(m)\delta_{m'} = \lambda_N(m)\delta_{m'}$$

Thus if we suppose  $M(N - M) = N - M$ , as in the statement, then we have:

$$\begin{aligned} \lambda_M(m)^* \delta_{m'} &= \sum_{x \in M} \delta_{m', mx} \delta_x \\ &= \sum_{x \in N} \delta_{m', mx} \delta_x \\ &= \lambda_N(m)^* \delta_{m'} \end{aligned}$$

In particular, if  $m_1, \dots, m_k \in M$ , and  $\alpha_1, \dots, \alpha_k$  are exponents in  $\{1, *\}$ , then:

$$\lambda_M(m_1)^{\alpha_1} \dots \lambda_M(m_k)^{\alpha_k} \delta_e = \lambda_N(m_1)^{\alpha_1} \dots \lambda_N(m_k)^{\alpha_k} \delta_e$$

Thus, we are led to the conclusion in the statement.  $\square$

Following [5], let us introduce the following technical notion:

DEFINITION 10.26. *Let  $N$  be a semigroup. Consider the following order on it:*

$$a \preceq_N b \iff b \in aN$$

*We say that  $N$  is in the class  $E$  if it satisfies one of the following equivalent conditions:*

- (1) *For  $\preceq_N$  every bounded subset is totally ordered.*
- (2)  *$a \preceq c, b \preceq c \implies a \preceq b$  or  $b \preceq a$ .*
- (3)  *$aN \cap bN \neq \emptyset \implies aN \subset bN$  or  $bN \subset aN$ .*
- (4)  *$NN^{-1} \cap N^{-1}N = N \cup N^{-1}$ .*

Also by following [5], let us introduce as well the following notion, which is something standard in the combinatorial theory of semigroups:

DEFINITION 10.27. *Let  $(a_i)_{i \in I}$  be a family of elements in a semigroup  $N$ .*

- (1) *We say that  $(a_i)_{i \in I}$  is a code if the semigroup  $M \subset N$  generated by the  $a_i$  is isomorphic to  $\mathbb{N}^{*I}$ , via  $a_i \rightarrow e_i$ , and satisfies  $M(N - M) = N - M$ .*
- (2) *We say that  $(a_i)_{i \in I}$  is a prefix if  $a_i \in a_j N \implies i = j$ , which means that the elements  $a_i$  are not comparable via the order relation  $\preceq_N$ .*

In our probabilistic setting, the notion of code is of interest, due to:

PROPOSITION 10.28. *Assuming that  $(a_i, b_i)_{i \in I}$  is a code, the family*

$$\left( \frac{1}{2}(\lambda_N(a_i) + \lambda_N(b_i)^*) \right)_{i \in I}$$

*is a circular family, in the sense of free probability theory.*

PROOF. Let  $(a_i, b_i)_{i \in I}$  be a code, and consider the following family:

$$\left( \lambda_N(a_i), \lambda_N(b_i) \right)_{i \in I} \in B(l^2(N))$$

By using Proposition 10.25, this family has the same distribution as a family of creation operators associated to a family of  $2I$  orthonormal vectors, on the free Fock space:

$$\left( \lambda_{\mathbb{N}^*I}(e_i), \lambda_{\mathbb{N}^*I}(f_i) \right)_{i \in I} \in B(l^2(N^{*I}))$$

Thus, we obtain the result, via the standard facts about the circular systems on free Fock spaces, that we know from chapter 9.  $\square$

In view of this, the following result provides us with a criterion for finding circular systems in the algebras of the semigroups in the class  $E$ , from Definition 10.26:

PROPOSITION 10.29. *For a semigroup  $N \in E$ , a family*

$$(a_i)_{i \in I} \subset N$$

*having at least two elements is a prefix if and only if it is a code.*

PROOF. We have two implications to be proved, as follows:

(1) Let first  $(a_i)_{i \in I}$  be a code which is not a prefix, for instance because we have  $a_i = a_j n$  with  $i \neq j, n \in N$ . Then  $n$  is in the semigroup  $M$  generated by the  $a_k$  and  $a_i = a_j n$  with  $i \neq j$ , so  $M$  cannot be free, and this is a contradiction, as desired.

(2) Conversely, suppose now that  $(a_i)_{i \in I}$  is a prefix and let, with  $m \in N$ :

$$A = a_{i_1}^{\alpha_1} \dots a_{i_n}^{\alpha_n} m = a_{j_1}^{\beta_1} \dots a_{j_s}^{\beta_s}$$

We have then  $a_{i_1} \preceq A$ ,  $a_{j_1} \preceq A$ , and so  $i_1 = j_1$ . We can therefore simplify  $A$  to the left by  $a_{i_1}$ . A recurrence on  $\sum \alpha_i$  shows then that we have  $n \leq s$  and:

$$a_{i_k} = a_{j_k} \quad , \quad \forall k \leq n$$

$$\alpha_k = \beta_k \quad , \quad \forall k < n$$

$$\alpha_n \leq \beta_n$$

$$m = a_{j_n}^{\beta_n - \alpha_n} a_{j_{n+1}}^{\beta_{n+1}} \dots a_{j_s}^{\beta_s}$$

Finally, we know that  $m$  is in the semigroup generated by the  $a_i$ , so we have a code. Moreover, for  $m = e$  we obtain that we have  $n = s$ ,  $a_{j_k} = a_{i_k}$  and  $\alpha_k = \beta_k$  for any  $k \leq n$ . Thus the variables  $a_i$  freely generate the semigroup  $M$ , and so the family  $(a_i)_{i \in I}$  is a code. Thus, we are led to the conclusion in the statement.  $\square$

Summarizing, we have some good freeness results, for our semigroups. Before getting into applications, let us discuss now the examples. We have here the following result:

PROPOSITION 10.30. *The class  $E$  has the following properties:*

- (1) *All the groups are in  $E$ .*
- (2) *The positive parts of totally ordered abelian groups are in  $E$ .*
- (3) *If  $G$  is a group and  $M \in E$ , then  $M \times G \in E$ .*
- (4) *If  $A_1, A_2$  are in  $E$ , then the free product  $A_1 * A_2$  is in  $E$ .*

PROOF. This is something elementary, whose proof goes as follows:

- (1) This is obvious, coming from definitions.
- (2) This is obvious as well, because  $M$  is here totally ordered by  $\preceq_M$ .
- (3) Let  $G$  be a group and  $M \in E$ . We have then, as desired:

$$\begin{aligned}
 & (M \times G)(M \times G)^{-1} \cap (M \times G)^{-1}(M \times G) \\
 = & (M \times G)(M^{-1} \times G) \cap (M^{-1} \times G)(M \times G) \\
 = & (MM^{-1} \times G) \cap (M^{-1}M \times G) \\
 = & (MM^{-1} \cap M^{-1}M) \times G \\
 = & (M \cup M^{-1}) \times G \\
 = & (M \times G) \cup (M^{-1} \times G) \\
 = & (M \times G) \cup (M \times G)^{-1}
 \end{aligned}$$

- (4) Let  $a, b, c \in A_1 * A_2$  such that  $ab = c$ . We write, as reduced words:

$$a = x_1 \dots x_n \quad , \quad b = y_1 \dots y_m \quad , \quad c = z_1 \dots z_p$$

Now let  $s$  be such that the following equalities happen:

$$x_n y_1 = 1 \quad , \quad \dots \quad , \quad x_{n-s+1} y_s = 1 \quad , \quad x_{n-s} y_{s+1} \neq 1$$

Consider now the following element:

$$u = x_{n-s+1} \dots x_n = (y_1 \dots y_s)^{-1}$$

We have then the following computation:

$$c = ab = x_1 \dots x_{n-s} y_{s+1} \dots y_m$$

Now let  $i \in \{1, 2\}$  be such that  $z_{n-s} \in A_i$ . There are two cases:

– If  $x_{n-s} \in A_1$  and  $y_{s+1} \in A_2$  or if  $x_{n-s} \in A_2$  and  $y_{s+1} \in A_1$ , then  $x_1 \dots x_{n-s} y_{s+1} \dots y_m$  is a reduced word. In particular, we have  $x_1 = z_1$ ,  $x_2 = z_2$ , and so on up to  $x_{n-s} = z_{n-s}$ . Thus we have  $a = z_1 \dots z_{n-s} u$ , with  $u$  invertible.

– If  $x_{n-s}, y_{s+1} \in A_i$  then  $x_1 = z_1$  and so on, up to  $x_{n-s-1} = z_{n-s-1}$  and  $x_{n-s} y_{s+1} = z_{n-s}$ . In this case we have  $a = z_1 \dots z_{n-s-1} x_{n-s} u$ , with  $u$  invertible.

Now observe that in both cases we obtained that  $a$  is of the form  $z_1 \dots z_f x u$  for some  $f$ , with  $u$  invertible and such that if  $z_{f+1} \in A_i$ , then there exists  $y \in A_i$  such that:

$$xy = z_{f+1}$$

Indeed, we can take  $f = n - s - 1$  and  $x = z_{n-s}, y = 1$  in the first case, and  $x = x_{n-s}, y = y_{s+1}$  in the second one. Suppose now that  $A_1, A_2 \in E$  and let  $a, b, a', b' \in A_1 * A_2$  such that  $ab = a'b'$ . Let  $z_1 \dots z_p$  be the decomposition of  $ab = a'b'$  as a reduced word. Then we can decompose our words, as above, in the following way:

$$a = z_1 \dots z_f x u \quad , \quad a' = z_1 \dots z_{f'} x' u'$$

We have to show that  $a = a'm$  or that  $a' = am$  for some  $m \in A_1 * A_2$ . But this is clear in all three cases that can appear, namely  $f < f', f' < f, f = f'$ .  $\square$

We can now formulate a main result about semigroup freeness, as follows:

**THEOREM 10.31.** *The following happen:*

- (1) *Given  $M \subset N$ , both in the class  $E$ , satisfying  $M(N - M) = N - M$ , any  $x$  in the  $*$ -algebra generated by  $\lambda(M)$  can be written as follows, with  $p_i, q_i \in M$ :*

$$x = \sum_i a_i \lambda_N(p_i) \lambda_N(q_i)^*$$

- (2) *Assume  $A, B \in E$ , and let  $x$  be an element of the  $*$ -algebra generated by  $\lambda_{A*B}(A)$  such that  $\tau(x) = 0$ . If  $W_A, W_B$  are respectively the sets of reduced words beginning by an element of  $A, B$ , then  $x$  acts as follows:*

$$l^2(W_B \cup \{e\}) \rightarrow l^2(W_A)$$

- (3) *Let  $A, B \in E$ . Then  $\lambda_{A*B}(A)$  and  $\lambda_{A*B}(B)$  are free.*

**PROOF.** This follows from our results so far, the idea being is as follows:

- (1) It is enough to prove this for elements of the form  $x = \lambda(m)^* \lambda(n)$  with  $m, n \in M$ , because the general case will follow easily from this. In order to do so, observe that  $x = \lambda(m)^* \lambda(n)$  is different from 0 precisely when there exist  $a, b \in N$  such that:

$$\langle \lambda(m)^* \lambda(n) \delta_a, \delta_b \rangle \neq 0$$

That is, the following condition must be satisfied:

$$na = mb$$

We know that there exists  $c \in N$  with  $n = mc$  or with  $m = nc$ . Moreover, as  $M(N - M) = N - M$ , it follows that  $c \in M$ . Thus  $x = \lambda(m)^* \lambda(n) \neq 0$  implies that  $x = \lambda(c)$  or  $x = \lambda(c)^*$  with  $c \in M$ , and this finishes the proof.

- (2) We apply (1) with  $M = A$  and  $N = A * B$  for writing, with  $p_i, q_i \in A$ :

$$x = \sum_i a_i \lambda(p_i) \lambda(q_i)^*$$

Consider now the following element:

$$\tau(\lambda(p_i)\lambda(q_i)^*) = \sum_x \delta_{e,p_i x} \delta_{e,q_i x}$$

This element is nonzero precisely when  $p_i = q_i$  is invertible, and in this case:

$$\lambda(p_i)\lambda(q_i)^* = 1$$

Now since we assumed  $\tau(x) = 0$ , it follows that we can write:

$$x = \sum a_i \lambda(p_i)\lambda(q_i)^* \quad , \quad \tau(\lambda(p_i)\lambda(q_i)^*) = 0$$

By linearity, it is enough to prove the result for  $x = \lambda(p_i)\lambda(q_i)^*$ . Let  $m \in W_B \cup \{e\}$  and suppose that  $x\delta_m \neq 0$ . Then  $\lambda(q_i)^*\delta_m \neq 0$  implies that  $m = q_i c$  for some word  $c \in A * B$ . As  $q_i \in A$  and  $m \in W_B \cup \{e\}$ , it follows that  $q_i$  is invertible. Now observe that:

$$p_i q_i^{-1} = 1 \implies \tau(x) = 1$$

It follows that we have, as desired:

$$x\delta_m = \delta_{p_i q_i^{-1} m} \in l^2(W_A)$$

(3) This follows from (2) above. Indeed, let  $P = x_n \dots x_1$  be a product of elements in  $\ker(\tau)$ , such that  $x_{2k}$  is in the  $*$ -algebra generated by  $\lambda(B)$  and  $x_{2k+1}$  is in the  $*$ -algebra generated by  $\lambda(A)$ . Then  $x_1 \delta_e \in l^2(W_A)$ . Thus  $x_2 x_1 \delta_e \in l^2(W_B)$ , and so on. By a recurrence,  $P \delta_e$  is in  $l^2(W_A)$  or in  $l^2(W_B)$ . But this implies that  $\tau(P) = 0$ , as desired.  $\square$

As a main application of the above semigroup technology, we have:

**THEOREM 10.32.** *Consider a Haar-unitary  $u$ , free from a semicircular  $s$ . Then*

$$c = us$$

*is a circular variable.*

**PROOF.** Denote by  $z$  the image of  $1 \in \mathbb{Z}$  and by  $n$  the image of  $1 \in \mathbb{N}$  by the canonical embeddings into the free product  $\mathbb{Z} * \mathbb{N}$ . We know that  $\mathbb{Z} * \mathbb{N} \in E$ . Also  $(zn, nz^{-1})$  is obviously a prefix, so it is a code. Thus, the following variable is circular:

$$c = \frac{1}{2}(\lambda(zn) + \lambda(nz^{-1})^*)$$

The point now is that we have the following formula:

$$\frac{1}{2}(\lambda(zn) + \lambda(nz^{-1})^*) = us$$

But this gives the result, in our model and so in general as well, because  $u = \lambda(z)$  is a Haar-unitary,  $s = 1/2(\lambda(n) + \lambda(n)^*)$  is semicircular, and  $u$  and  $s$  are free.  $\square$

We can now recover the Voiculescu polar decomposition result for the circular variables, obtained in [88], by using random matrix techniques, as follows:

**THEOREM 10.33.** *Consider the polar decomposition of a circular variable, in some von Neumann algebraic probability space with faithful normal state:*

$$x = vb$$

*Then  $v$  is Haar-unitary,  $b$  is quarter-circular, and  $(v, b)$  are free.*

**PROOF.** This follows by suitably manipulating Theorem 10.32, as to replace the semi-circular element there by a quarter-circular. Consider indeed the following group:

$$G = \mathbb{Z} * (\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$$

Let  $z, t, a$  be the images of the following elements, into this group  $G$ :

$$1 \in \mathbb{Z} \quad , \quad (1, \hat{0}) \in \mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z}) \quad , \quad (0, \hat{1}) \in \mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})$$

Let  $u = \lambda_G(z)$ ,  $d = \lambda_G(a)$  and choose a quarter-circular  $q \in C^*(\lambda_G(t))$ . Then  $(q, d)$  are independent, so  $dq$  is semicircular, and so  $c = udq$  is circular, and:

- The module of  $c$  is  $q$ , which is a quarter-circular.
- The polar part of  $c$  is  $ud$ , which is obviously a Haar-unitary.
- Consider the automorphism of  $G$  which is the identity on  $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$  and maps  $z \rightarrow za$ . This extends to a trace-preserving automorphism of  $C^*(G)$  which maps:

$$u \rightarrow ud \quad , \quad q \rightarrow q$$

As  $u, q$  are free, it follows that  $ud, q$  are free too, finishing the proof.  $\square$

### 10d. Gaussian matrices

As an application of the semicircular and circular variable theory developed so far, and of free probability in general, let us go back now to the random matrices. Following Voiculescu's paper [88], we will prove now a number of key freeness results for them, complementing the basic random matrix theory developed in chapters 7-8. As a first result, completing our asymptotic law study for the Gaussian matrices, we have:

**THEOREM 10.34.** *Given a sequence of complex Gaussian matrices*

$$Z_N \in M_N(L^\infty(X))$$

*having independent  $G_t$  variables as entries, with  $t > 0$ , we have*

$$\frac{Z_N}{\sqrt{N}} \sim \Gamma_t$$

*in the  $N \rightarrow \infty$  limit, with the limiting measure being Voiculescu's circular law.*

PROOF. We know from chapter 7, with this having been actually our very first moment computation for random matrices, in this book, that the asymptotic moments of the complex Gaussian matrices are given by the following formula:

$$M_k \left( \frac{Z_N}{\sqrt{N}} \right) \simeq t^{|k|/2} |\mathcal{NC}_2(k)|$$

On the other hand, we also know from the above that an abstract noncommutative variable  $a \in A$  is circular, following the law  $\Gamma_t$ , precisely when its moments are:

$$M_k(a) = t^{|k|/2} |\mathcal{NC}_2(k)|$$

Thus, we are led to the conclusion in the statement.  $\square$

The above result is of course something quite theoretical, and having it formulated as such is certainly something nice. However, and here comes our point, it is actually possible to use free probability theory in order to go well beyond this, with this time some truly “new” results on the random matrices. We will explain this now, following Voiculescu’s paper [88]. Let us begin with the Wigner matrices. We have here:

THEOREM 10.35. *Given a family of sequences of Wigner matrices,*

$$Z_N^i \in M_N(L^\infty(X)) \quad , \quad i \in I$$

*with pairwise independent entries, each following the complex normal law  $G_t$ , with  $t > 0$ , up to the constraint  $Z_N^i = (Z_N^i)^*$ , the rescaled sequences of matrices*

$$\frac{Z_N^i}{\sqrt{N}} \in M_N(L^\infty(X)) \quad , \quad i \in I$$

*become with  $N \rightarrow \infty$  semicircular, each following the Wigner law  $\gamma_t$ , and free.*

PROOF. This is something quite subtle, the idea being as follows:

(1) First of all, we know from chapter 7 that for any  $i \in I$  the corresponding sequence of rescaled Wigner matrices becomes semicircular in the  $N \rightarrow \infty$  limit:

$$\frac{Z_N^i}{\sqrt{N}} \simeq \gamma_t$$

(2) Thus, what is new here, and that we have to prove, is the asymptotic freeness assertion. For this purpose we can assume that we are dealing with the case of 2 sequences of matrices,  $|I| = 2$ . So, assume that we have Wigner matrices as follows:

$$Z_N, Z'_N \in M_N(L^\infty(X))$$

We have to prove that these matrices become asymptotically free, with  $N \rightarrow \infty$ .

(3) But this something that can be proved directly, via various routine computations with partitions, which simplify as usual in the  $N \rightarrow \infty$  limit, and bring freeness.

(4) However, we can prove this as well by using a trick, based on the result in Theorem 10.34. Consider indeed the following random matrix:

$$Y_N = \frac{1}{\sqrt{2}}(Z_N + iZ'_N)$$

This is then a complex Gaussian matrix, and so by using Theorem 10.34, we obtain that in the limit  $N \rightarrow \infty$ , we have:

$$\frac{Y_N}{\sqrt{N}} \simeq \Gamma_t$$

Now recall that the circular law  $\Gamma_t$  was by definition the law of the following variable, with  $a, b$  being semicircular, each following the law  $\gamma_t$ , and free:

$$c = \frac{1}{\sqrt{2}}(a + ib)$$

We are therefore in the situation where the variable  $(Z_N + iZ'_N)/\sqrt{N}$ , which has asymptotically semicircular real and imaginary parts, converges to the distribution of  $a + ib$ , equally having semicircular real and imaginary parts, but with these real and imaginary parts being free. Thus  $Z_N, Z'_N$  become asymptotically free, as desired.  $\square$

Getting now to the complex case, we have a similar result here, as follows:

**THEOREM 10.36.** *Given a family of sequences of complex Gaussian matrices,*

$$Z_N^i \in M_N(L^\infty(X)) \quad , \quad i \in I$$

*with pairwise independent entries, each following the complex normal law  $G_t$ , with  $t > 0$ , the rescaled sequences of matrices*

$$\frac{Z_N^i}{\sqrt{N}} \in M_N(L^\infty(X)) \quad , \quad i \in I$$

*become with  $N \rightarrow \infty$  circular, each following the Voiculescu law  $\Gamma_t$ , and free.*

**PROOF.** This follows from Theorem 10.35, which applies to the real and imaginary parts of our complex Gaussian matrices, and gives the result.  $\square$

The above results are interesting for both free probability and random matrices. As an illustration here, we have the following application to free probability:

**THEOREM 10.37.** *Consider the polar decomposition of a circular variable in some von Neumann algebraic probability space with faithful normal state:*

$$x = vb$$

*Then  $v$  is Haar-unitary,  $b$  is quarter-circular and  $(v, b)$  are free.*

**PROOF.** This is indeed easy to see in the Gaussian matrix model provided by Theorem 10.36 above, and for details here, we refer to Voiculescu's paper [88].  $\square$

There are many other applications along this lines, and conversely, free probability can be used as well for the detailed study of the Wigner and Gaussian matrices. Finally, we have as well a similar result for the Wishart matrices, as follows:

**THEOREM 10.38.** *Given a family of sequences of complex Wishart matrices,*

$$Z_N^i = Y_N^i (Y_N^i)^* \in M_N(L^\infty(X)) \quad , \quad i \in I$$

*with each  $Y_N^i$  being a  $N \times M$  matrix, with entries following the normal law  $G_1$ , and with all these entries being pairwise independent, the rescaled sequences of matrices*

$$\frac{Z_N^i}{N} \in M_N(L^\infty(X)) \quad , \quad i \in I$$

*become with  $M = tN \rightarrow \infty$  Marchenko-Pastur, each following the law  $\pi_t$ , and free.*

**PROOF.** Here the first assertion is the Marchenko-Pastur theorem, and the second assertion follows from the freeness result from Theorem 10.35, or Theorem 10.36.  $\square$

To conclude now, we have seen so far the foundations of free probability, and its basic applications to random matrix theory. We will keep building on all this, in the remainder of this book, but somehow with a preference towards quantum algebra topics.

For further results on the topics discussed above, we recommend, besides Voiculescu's papers [85], [86], [87], [88], [89], and book [90] with Dykema and Nica, [21], [24], [46], [70], [78], [79] for general free probability, [1], [41], [48], [50], [54], [66], [68], [83] for random matrix theory, and [25], [43], [52], [60], [76], [77] for applications to operator algebras. But do not worry, we will be back to some of these topics, in what follows.

## 10e. Exercises

There has been a lot of interesting combinatorics in this chapter, and as an instructive exercise on all this, we have:

**EXERCISE 10.39.** *Try finding the classical analogue of the polar decomposition result of the circular variables, that we found in the above.*

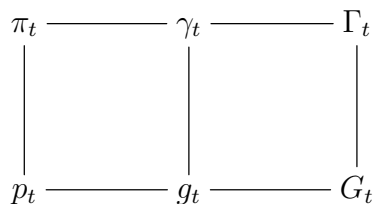
This is something a bit vague, but very instructive. In case you are stuck, try thinking at the passage  $O_N \rightarrow U_N$ , say at the level of the corresponding Lie algebras, and then at the corresponding laws of coordinates, in the  $N \rightarrow \infty$  limit. And if you are still stuck, even with this indication, wait for it: we will be back to this, later on in this book.

## CHAPTER 11

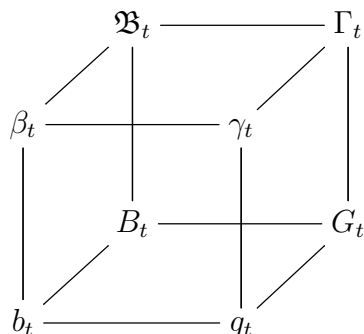
### Poisson limits

#### 11a. Poisson limits

We have seen so far that free probability leads to three key limiting theorems, namely the free analogues of the PLT, CLT and CCLT. The limiting measures are the Marchenko-Pastur laws  $\pi_t$ , the Wigner semicircle laws  $\gamma_t$  and the Voiculescu circular laws  $\Gamma_t$ . Together with the Poisson laws  $p_t$  and the Gaussian laws  $g_t$  and  $G_t$  appearing from the classical PLT, CLT and CCLT, these laws form a rectangular diagram, as follows:



In this chapter we develop some more limiting theorems, by generalizing the free PLT that we know into a free compound Poisson limit theorem (free CPLT). At the level of the above diagram, there are no complex analogues of  $p_t, \pi_t$ , but by using measures found via the classical and free CPLT, namely the real and purely complex Bessel laws  $b_t, B_t$  discussed in chapter 2, and their free analogues  $\beta_t, \mathfrak{B}_t$  to be discussed here, we will be able to modify and then fold the diagram, as to complete it into a cube, as follows:



This is of course quite nice, theoretically speaking, because it leads to a kind of 3D orientation inside the whole subject, classical and free probability, which is useful.

In order to do all this, as already mentioned, we are first in need of a free CPLT. We will follow the CPLT material from chapter 2, by performing modifications where needed, as to replace everywhere classical probability with free probability. Let us start with the following straightforward definition, similar to the one from the classical case:

DEFINITION 11.1. *Associated to any compactly supported positive measure  $\rho$  on  $\mathbb{C}$  is the probability measure*

$$\pi_\rho = \lim_{n \rightarrow \infty} \left( \left(1 - \frac{c}{n}\right) \delta_0 + \frac{1}{n} \rho \right)^{\boxplus n}$$

where  $c = \text{mass}(\rho)$ , called compound free Poisson law.

In what follows we will be mostly interested in the case where  $\rho$  is discrete, as is for instance the case for the measure  $\rho = t\delta_1$  with  $t > 0$ , which produces the free Poisson laws, via the usual Poisson Limit Theorem (PLT), that we learned in chapter 2. The following result allows one to detect compound free Poisson laws:

PROPOSITION 11.2. *For a discrete measure, written as*

$$\rho = \sum_{i=1}^s c_i \delta_{z_i}$$

with  $c_i > 0$  and  $z_i \in \mathbb{C}$ , we have the following formula,

$$R_{\pi_\rho}(y) = \sum_{i=1}^s \frac{c_i z_i}{1 - y z_i}$$

where  $R$  denotes as usual the Voiculescu  $R$ -transform.

PROOF. In order to prove this result, let  $\eta_n$  be the measure appearing in Definition 11.1, under the free convolution sign, namely:

$$\eta_n = \left(1 - \frac{c}{n}\right) \delta_0 + \frac{1}{n} \rho$$

The Cauchy transform of  $\eta_n$  is then given by the following formula:

$$G_{\eta_n}(\xi) = \left(1 - \frac{c}{n}\right) \frac{1}{\xi} + \frac{1}{n} \sum_{i=1}^s \frac{c_i}{\xi - z_i}$$

Consider now the  $R$ -transform of the measure  $\eta_n^{\boxplus n}$ , which is given by:

$$R_{\eta_n^{\boxplus n}}(y) = n R_{\eta_n}(y)$$

By using the general theory of the  $R$ -transform, from chapter 9, the above formula of  $G_{\eta_n}$  shows that the equation for  $R = R_{\eta_n^{\boxplus n}}$  is as follows:

$$\begin{aligned} & \left(1 - \frac{c}{n}\right) \frac{1}{1/y + R/n} + \frac{1}{n} \sum_{i=1}^s \frac{c_i}{1/y + R/n - z_i} = y \\ \implies & \left(1 - \frac{c}{n}\right) \frac{1}{1 + yR/n} + \frac{1}{n} \sum_{i=1}^s \frac{c_i}{1 + yR/n - yz_i} = 1 \end{aligned}$$

Now multiplying by  $n$ , then rearranging the terms, and letting  $n \rightarrow \infty$ , we get:

$$\begin{aligned} \frac{c + yR}{1 + yR/n} &= \sum_{i=1}^s \frac{c_i}{1 + yR/n - yz_i} \implies c + yR_{\pi_\rho}(y) = \sum_{i=1}^s \frac{c_i}{1 - yz_i} \\ &\implies R_{\pi_\rho}(y) = \sum_{i=1}^s \frac{c_i z_i}{1 - yz_i} \end{aligned}$$

Thus, we are led to the conclusion in the statement.  $\square$

We have as well the following result, providing an alternative to Definition 11.1, and which can, together with Definition 11.1, be thought of as being a free CPLT:

**THEOREM 11.3.** *For a discrete measure, written as*

$$\rho = \sum_{i=1}^s c_i \delta_{z_i}$$

with  $c_i > 0$  and  $z_i \in \mathbb{C}$ , we have the formula

$$\pi_\rho = \text{law} \left( \sum_{i=1}^s z_i \alpha_i \right)$$

where the variables  $\alpha_i$  are free Poisson( $c_i$ ), free.

**PROOF.** Let  $\alpha$  be the sum of free Poisson variables in the statement:

$$\alpha = \sum_{i=1}^s z_i \alpha_i$$

In order to prove the result, we will show that the  $R$ -transform of  $\alpha$  is given by the formula in Proposition 11.2. We have the following computation:

$$\begin{aligned} R_{\alpha_i}(y) = \frac{c_i}{1 - y} &\implies R_{z_i \alpha_i}(y) = \frac{c_i z_i}{1 - yz_i} \\ &\implies R_\alpha(y) = \sum_{i=1}^s \frac{c_i z_i}{1 - yz_i} \end{aligned}$$

Thus we have the same formula as in Proposition 11.2, and we are done.  $\square$

All the above is quite general, and in practice, in order to obtain concrete results, the simplest measures that we can use as “input” for the CPLT are the same measures as those that we used in the classical case, namely the measures of type  $\rho = t\varepsilon_s$ , with  $t > 0$ , and with  $\varepsilon_s$  being the uniform measure on the  $s$ -th roots of unity. We discuss this in what follows, basically by following the classical material from chapter 2, and [8].

### 11b. Bessel laws

As mentioned above, for various reasons, including the construction of the “standard cube” discussed in the beginning of this chapter, we are interested in the applications of the free CPLT with the “simplest” input measures, with these simplest measures being those of type  $\rho = t\varepsilon_s$ , with  $t > 0$ , and with  $\varepsilon_s$  being the uniform measure on the  $s$ -th roots of unity. We are led in this way the following class of measures:

**DEFINITION 11.4.** *The Bessel and free Bessel laws, depending on parameters  $s \in \mathbb{N} \cup \{\infty\}$  and  $t > 0$ , are the following compound Poisson and free Poisson laws,*

$$b_t^s = p_{t\varepsilon_s} \quad , \quad \beta_t^s = \pi_{t\varepsilon_s}$$

with  $\varepsilon_s$  being the uniform measure on the  $s$ -th roots of unity. In particular:

- (1) At  $s = 1$  we recover the Poisson laws  $p_t, \pi_t$ .
- (2) At  $s = 2$  we have the real Bessel laws  $b_t, \beta_t$ .
- (3) At  $s = \infty$  we have the complex Bessel laws  $B_t, \mathfrak{B}_t$ .

The terminology here comes from the fact, that we know from chapter 2, that the density of the measure  $b_t$ , appearing at  $s = 2$ , is a Bessel function of the first kind. This was something first discovered in [10], and we refer to that paper, and to the subsequent literature, including [8], for more comments on this phenomenon.

Our next task will be that upgrading our results about the free Poisson law  $\pi_t$  in this setting, using a parameter  $s \in \mathbb{N} \cup \{\infty\}$ . First, we have the following result:

**THEOREM 11.5.** *The free Bessel laws have the property*

$$\beta_t^s \boxplus \beta_{t'}^s = \beta_{t+t'}^s$$

so they form a 1-parameter semigroup with respect to free convolution.

**PROOF.** This follows indeed from the fact that the  $R$ -transform of  $\beta_t^s$  is linear in  $t$ , which is something that we already know, from the above.  $\square$

Let us discuss now, following [8], some more advanced aspects of the free Bessel laws. Given a real probability measure  $\mu$ , one can ask whether the convolution powers  $\mu^{\boxtimes s}$  and  $\mu^{\boxplus t}$  exist, for various values of the parameters  $s, t > 0$ . For the free Poisson law, the answer to these questions is as follows:

PROPOSITION 11.6. *The free convolution powers of the free Poisson law*

$$\pi^{\boxtimes s}, \quad \pi^{\boxplus t}$$

*exist for any positive values of the parameters,  $s, t > 0$ .*

PROOF. We have two measures to be studied, the idea being as follows:

(1) The free Poisson law  $\pi$  is by definition the  $t = 1$  particular case of the free Poisson law of parameter  $t$ , or Marchenko-Pastur law of parameter  $t > 0$ , given by:

$$\pi_t = \max(1 - t, 0)\delta_0 + \frac{\sqrt{4t - (x - 1 - t)^2}}{2\pi x} dx$$

The Cauchy transform of this measure is given by:

$$G(\xi) = \frac{(\xi + 1 - t) + \sqrt{(\xi + 1 - t)^2 - 4\xi}}{2\xi}$$

We can compute now the  $R$  transform, by proceeding as follows:

$$\begin{aligned} \xi G^2 + 1 = (\xi + 1 - t)G &\implies Kz^2 + 1 = (K + 1 - t)z \\ &\implies Rz^2 + z + 1 = (R + 1 - t)z + 1 \\ &\implies Rz = R - t \\ &\implies R = t/(1 - z) \end{aligned}$$

The last expression being linear in  $t$ , the measures  $\pi_t$  form a semigroup with respect to free convolution. Thus we have  $\pi_t = \pi^{\boxplus t}$ , which proves the second assertion.

(2) Regarding now the measure  $\pi^{\boxtimes s}$ , there is no explicit formula for its density. However, we can prove that this measure exists, by using some abstract results.

Indeed, we have the following computation for the  $S$  transform of  $\pi_t$ :

$$\begin{aligned} \xi G^2 + 1 = (\xi + 1 - t)G &\implies zf^2 + 1 = (1 + z - zt)f \\ &\implies z(\psi + 1)^2 + 1 = (1 + z - zt)(\psi + 1) \\ &\implies \chi(z + 1)^2 + 1 = (1 + \chi - \chi t)(z + 1) \\ &\implies \chi(z + 1)(t + z) = z \\ &\implies S = 1/(t + z) \end{aligned}$$

In particular at  $t = 1$  we have the following formula:

$$S(z) = \frac{1}{1 + z}$$

Thus the  $\Sigma$  transform of  $\pi$ , which is by definition  $\Sigma(z) = S(z/(1 - z))$ , is given by:

$$\Sigma(z) = 1 - z$$

On the other hand, it is well-known from the general theory of the  $S$ -transform that the  $\Sigma$  transforms of the probability measures which are  $\boxtimes$ -infinitely divisible are the functions of the form  $\Sigma(z) = e^{v(z)}$ , where  $v : \mathbb{C} - [0, \infty) \rightarrow \mathbb{C}$  is analytic, satisfying:

$$v(\bar{z}) = \bar{v}(z) \quad , \quad v(\mathbb{C}^+) \subset \mathbb{C}^-$$

Now in the case of the free Poisson law, the function  $v(z) = \log(1 - z)$  satisfies these properties, and we are led to the conclusion in the statement. See [8].  $\square$

Getting now towards the free Bessel laws, we have the following remarkable identity, in relation with the above convolution powers of  $\pi$ , also established in [8]:

**THEOREM 11.7.** *We have the formula*

$$\pi^{\boxtimes s-1} \boxtimes \pi^{\boxplus t} = ((1-t)\delta_0 + t\delta_1) \boxtimes \pi^{\boxtimes s}$$

*valid for any  $s \geq 1$ , and any  $t \in (0, 1]$ .*

**PROOF.** We know from the previous proof that the  $S$  transform of the free Poisson law  $\pi$  is given by the following formula:

$$S_1(z) = \frac{1}{1+z}$$

We also know from there that the  $S$  transform of  $\pi^{\boxplus t}$  is given by:

$$S_t(z) = \frac{1}{t+z}$$

Thus the measure on the left in the statement has the following  $S$  transform:

$$S(z) = \frac{1}{(1+z)^{s-1}} \cdot \frac{1}{t+z}$$

The  $S$  transform of  $\alpha_t = (1-t)\delta_0 + t\delta_1$  can be computed as follows:

$$\begin{aligned} f = 1 + tz/(1-z) &\implies \psi = tz/(1-z) \\ &\implies z = t\chi/(1-\chi) \\ &\implies \chi = z/(t+z) \\ &\implies S = (1+z)/(t+z) \end{aligned}$$

Thus the measure on the right in the statement has the following  $S$  transform:

$$S(z) = \frac{1}{(1+z)^s} \cdot \frac{1+z}{t+z}$$

Thus the  $S$  transforms of our two measures are the same, and we are done.  $\square$

The relation with the free Bessel laws, as previously defined, comes from:

THEOREM 11.8. *The free Bessel law is the real probability measure  $\beta_t^s$ , with*

$$(s, t) \in (0, \infty) \times (0, \infty) - (0, 1) \times (1, \infty)$$

*defined concretely as follows:*

- (1) *For  $s \geq 1$  we set  $\beta_t^s = \pi^{\boxtimes s-1} \boxtimes \pi^{\boxplus t}$ .*
- (2) *For  $t \leq 1$  we set  $\beta_t^s = ((1-t)\delta_0 + t\delta_1) \boxtimes \pi^{\boxtimes s}$ .*

PROOF. This follows indeed from the above results. To be more precise, these results show that the measures constructed in the statement exist indeed, and coincide with the free Bessel laws, as previously defined, as compound free Poisson laws.  $\square$

In view of the above, we can regard the free Bessel law  $\beta_t^s$  as being a natural two-parameter generalization of the free Poisson law  $\pi$ , in connection with Voiculescu's free convolution operations  $\boxtimes$  and  $\boxplus$ . Observe that we have the following formulae:

$$\begin{cases} \beta_1^s = \pi^{\boxtimes s} \\ \beta_t^1 = \pi^{\boxplus t} \end{cases}$$

As a comment here, concerning the precise range of the parameters  $(s, t)$ , the above results can be probably improved. The point is that the measure  $\beta_t^s$  still exists for certain points in the critical rectangle  $(0, 1) \times (1, \infty)$ , but not for all of them. To be more precise, the known numeric checks for this question, discussed in [8], show that the critical values of  $(s, t)$  tend to form an algebraic curve contained in  $(0, 1) \times (1, \infty)$ , having  $s = 1$  as an asymptote. However, the case we are the most interested in is  $t \in (0, 1]$ , and here there is no problem, because  $\beta_t^s$  exists for any  $s > 0$ . Thus, we will stop this discussion here.

As before following [8], we have the following result:

PROPOSITION 11.9. *The Stieltjes transform of  $\beta_t^s$  satisfies:*

$$f = 1 + zf^s(f + t - 1)$$

PROOF. We have the following computation:

$$\begin{aligned} S = \frac{1}{(1+z)^{s-1}} \cdot \frac{1}{t+z} &\implies \chi = \frac{z}{(1+z)^s} \cdot \frac{1}{t+z} \\ &\implies z = \frac{\psi}{(1+\psi)^s} \cdot \frac{1}{t+\psi} \\ &\implies z = \frac{f-1}{f^s} \cdot \frac{1}{t+f-1} \end{aligned}$$

Thus, we obtain the equation in the statement.  $\square$

At  $t = 1$ , we have in fact the following result, also from [8], which is more explicit:

THEOREM 11.10. *The Stieltjes transform of  $\beta_1^s$  with  $s \in \mathbb{N}$  is given by*

$$f(z) = \sum_{p \in NC_s} z^{k(p)}$$

where  $NC_s$  is the set of noncrossing partitions all whose blocks have as size multiples of  $s$ , and where  $k : NC_s \rightarrow \mathbb{N}$  is the normalized length.

PROOF. With the notation  $C_k = \#NC_s(k)$ , where  $NC_s(k) \subset NC_s$  consists of the partitions of  $\{1, \dots, sk\}$  belonging to  $NC_s$ , the sum on the right is:

$$f(z) = \sum_k C_k z^k$$

For a given partition  $p \in NC_s(k+1)$  we can consider the last  $s$  legs of the first block, and make cuts at right of them. This gives a decomposition of  $p$  into  $s+1$  partitions in  $NC_s$ , and we obtain in this way the following recurrence formula for the numbers  $C_k$ :

$$C_{k+1} = \sum_{\Sigma k_i = k} C_{k_0} \dots C_{k_s}$$

By multiplying now by  $z^{k+1}$ , and then summing over  $k$ , we obtain that the generating series of these numbers  $C_k$  satisfies the following equation:

$$f - 1 = z f^{s+1}$$

But this is the same as the equation of the Stieltjes transform of  $\beta_1^s$ , found in Proposition 11.9, applied at  $t = 1$ , namely:

$$f = 1 + z f^{s+1}$$

Thus, we are led to the conclusion in the statement.  $\square$

Next, still following [8], we have the following result, dealing with the case  $t > 0$ :

THEOREM 11.11. *The Stieltjes transform of  $\beta_t^s$  with  $s \in \mathbb{N}$  is given by:*

$$f(z) = \sum_{p \in NC_s} z^{k(p)} t^{b(p)}$$

where  $k, b : NC_s \rightarrow \mathbb{N}$  are the normalized length, and the number of blocks.

PROOF. With notations from the previous proof, let  $F_{kb}$  be the number of partitions in  $NC_s(k)$  having  $b$  blocks, and set  $F_{kb} = 0$  for other integer values of  $k, b$ . All sums will be over integer indices  $\geq 0$ . The sum on the right in the statement is then:

$$f(z) = \sum_{kb} F_{kb} z^k t^b$$

The recurrence formula for the numbers  $C_k$  in the previous proof becomes:

$$\sum_b F_{k+1,b} = \sum_{\Sigma k_i=k} \sum_{b_i} F_{k_0 b_0} \dots F_{k_s b_s}$$

In this formula, each term contributes to  $F_{k+1,b}$  with  $b = \Sigma b_i$ , except for those of the form  $F_{00} F_{k_1 b_1} \dots F_{k_s b_s}$ , which contribute to  $F_{k+1,b+1}$ . We get:

$$\begin{aligned} F_{k+1,b} &= \sum_{\Sigma k_i=k} \sum_{\Sigma b_i=b} F_{k_0 b_0} \dots F_{k_s b_s} \\ &+ \sum_{\Sigma k_i=k} \sum_{\Sigma b_i=b-1} F_{k_1 b_1} \dots F_{k_s b_s} \\ &- \sum_{\Sigma k_i=k} \sum_{\Sigma b_i=b} F_{k_1 b_1} \dots F_{k_s b_s} \end{aligned}$$

This gives the following formula for the polynomials  $P_k = \sum_b F_{kb} t^b$ :

$$P_{k+1} = \sum_{\Sigma k_i=k} P_{k_0} \dots P_{k_s} + (t-1) \sum_{\Sigma k_i=k} P_{k_1} \dots P_{k_s}$$

Consider now the following generating function:

$$f = \sum_k P_k z^k$$

In terms of this generating function, we get the following equation:

$$f - 1 = z f^{s+1} + (t-1) z f^s$$

But this is the same as the equation of the Stieltjes transform of  $\beta_t^s$ , namely:

$$f = 1 + z f^s (f + t - 1)$$

Thus, we are led to the conclusion in the statement.  $\square$

Let us discuss now the computation of the moments of the free Bessel laws. The idea will be that of expressing these moments in terms of generalized binomial coefficients. We recall that the coefficient corresponding to  $\alpha \in \mathbb{R}$ ,  $k \in \mathbb{N}$  is:

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1) \dots (\alpha-k+1)}{k!}$$

We denote by  $m_1, m_2, m_3, \dots$  the sequence of moments of a given probability measure. With this convention, we first have the following result, from [8]:

**THEOREM 11.12.** *The moments of  $\beta_1^s$  with  $s > 0$  are*

$$m_k = \frac{1}{sk+1} \binom{sk+k}{k}$$

*which are the Fuss-Catalan numbers.*

PROOF. In the case  $s \in \mathbb{N}$ , we know that we have  $m_k = \#NC_s(k)$ . The formula in the statement follows then by counting such partitions. In the general case  $s > 0$ , observe first that the Fuss-Catalan number in the statement is a polynomial in  $s$ :

$$\frac{1}{sk+1} \binom{sk+k}{k} = \frac{(sk+2)(sk+3)\dots(sk+k)}{k!}$$

Thus, in order to pass from the case  $s \in \mathbb{N}$  to the case  $s > 0$ , it is enough to check that the  $k$ -th moment of  $\pi_{s1}$  is analytic in  $s$ . But this is clear from the equation  $f = 1 + zf^{s+1}$  of the Stieltjes transform of  $\pi_{s1}$ , and this gives the result.  $\square$

We have as well the following result, which deals with the general case  $t > 0$ :

THEOREM 11.13. *The moments of  $\beta_t^s$  with  $s > 0$  are*

$$m_k = \sum_{b=1}^k \frac{1}{b} \binom{k-1}{b-1} \binom{sk}{b-1} t^b$$

*which are the Fuss-Narayana numbers.*

PROOF. In the case  $s \in \mathbb{N}$ , we know from the above that we have the following formula, where  $F_{kb}$  is the number of partitions in  $NC_s(k)$  having  $b$  blocks:

$$m_k = \sum_b F_{kb} t^b$$

With this observation in hand, the formula in the statement follows by counting such partitions, with this count being well-known. This result can be then extended to any parameter  $s > 0$ , by using a standard complex variable argument, as before. See [8].  $\square$

In the case  $s \notin \mathbb{N}$ , the moments of  $\beta_t^s$  can be further expressed in terms of Gamma functions. In the case  $s = 1/2$ , the result, also from [8], is as follows:

THEOREM 11.14. *The moments of  $\beta_1^{1/2}$  are given by the following formulae:*

$$\begin{aligned} m_{2p} &= \frac{1}{p+1} \binom{3p}{p} \\ m_{2p-1} &= \frac{2^{-4p+3} p}{(6p-1)(2p+1)} \cdot \frac{p!(6p)!}{(2p)!(2p)!(3p)!} \end{aligned}$$

PROOF. According to our various results above, the even moments of the free Bessel law  $\beta_t^s$  with  $s = n - 1/2$ ,  $n \in \mathbb{N}$ , are given by:

$$\begin{aligned} m_{2p} &= \frac{1}{(n-1/2)(2p)+1} \binom{(n+1/2)2p}{2p} \\ &= \frac{1}{(2n-1)p+1} \binom{(2n+1)p}{2p} \end{aligned}$$

With  $n = 1$  we get the formula in the statement. Now for the odd moments, we can use here the following well-known identity:

$$\binom{m-1/2}{k} = \frac{4^{-k}}{k!} \cdot \frac{(2m)!}{m!} \cdot \frac{(m-k)!}{(2m-2k)!}$$

With  $m = 2np + p - n$  and  $k = 2p - 1$  we get:

$$\begin{aligned} m_{2p-1} &= \frac{1}{(n-1/2)(2p-1)+1} \binom{(n+1/2)(2p-1)}{2p-1} \\ &= \frac{2}{(2n-1)(2p-1)+2} \binom{(2np+p-n)-1/2}{2p-1} \\ &= \frac{2^{-4p+3}}{(2p-1)!} \cdot \frac{(4np+2p-2n)!}{(2np+p-n)!} \cdot \frac{(2np-p-n+1)!}{(4np-2p-2n+3)!} \end{aligned}$$

In particular with  $n = 1$  we obtain:

$$\begin{aligned} m_{2p-1} &= \frac{2^{-4p+3}}{(2p-1)!} \cdot \frac{(6p-2)!}{(3p-1)!} \cdot \frac{p!}{(2p+1)!} \\ &= \frac{2^{-4p+3}(2p)}{(2p)!} \cdot \frac{(6p)!(3p)}{(3p)!(6p-1)6p} \cdot \frac{p!}{(2p)!(2p+1)} \end{aligned}$$

But this gives the formula in the statement.  $\square$

There are many other interesting things, of both combinatorial and complex analytic nature, that can be said about the free Bessel laws, their moments and their densities, and we refer here to [8]. Also, there is as well a relation with the combinatorics of the intermediate subfactors, and the Fuss-Catalan algebra of Bisch and Jones [26]. All this is a bit technical, and we will be back to this later, when taking about subfactors.

In what follows we will rather focus on the free Bessel laws that we are truly interested in, namely those appearing at  $s = 1, 2, \infty$ . We will be particularly interested in the cases  $s = 2, \infty$ , which can be thought of as being “fully real” and “purely complex”.

Also, instead of insisting on combinatorics and complex analysis, we will rather discuss the question of finding matrix models for the free Bessel laws, which is of key importance, in view of the various random matrix considerations from chapters 5-8.

### 11c. The standard cube

Let us get back now to the fundamental question, mentioned in the beginning of this chapter, of arranging the main probability measures that we know, classical and free, into a cube, and this as for having a kind of 3D orientation, inside probability at large. For this purpose, we will need the following result, coming from the above study:

THEOREM 11.15. *The moments of  $\beta_t^s$  are the numbers*

$$M_k = \sum_{\pi \in NC^s(k)} t^{|\pi|}$$

where  $NC^s$  are the noncrossing partitions satisfying  $\# \circ = \# \bullet(s)$  in each block.

PROOF. At  $t = 1$  the formula to be proved is as follows:

$$M_k(\beta_1^s) = |NC^s(k)|$$

But this can be proved by using Theorem 11.10, via the bijection between the set  $NC_s$  there and the set  $NC^s$  here. At  $t > 0$  now, the formula to be proved is as follows:

$$M_k(\beta_t^s) = \sum_{\pi \in NC^s(k)} t^{|\pi|}$$

But this can be proved again by doing some computations, or by using Theorem 11.11, via the bijection between the set  $NC_s$  there and the set  $NC^s$  here.  $\square$

At the combinatorial level, this is quite interesting, and we have:

THEOREM 11.16. *The various classical and free central limiting measures,*

$$\begin{array}{ccccc} \beta_t^s & \text{---} & \gamma_t & \text{---} & \Gamma_t \\ | & & | & & | \\ b_t^s & \text{---} & g_t & \text{---} & G_t \end{array}$$

have moments always given by the same formula, involving partitions, namely

$$M_k = \sum_{\pi \in D(k)} t^{|\pi|}$$

where the sets of partitions  $D(k)$  in question are respectively

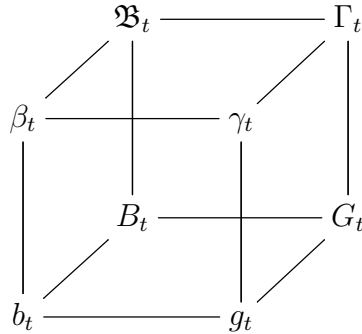
$$\begin{array}{ccccc} NC^s & \longleftarrow & NC_2 & \longleftarrow & \mathcal{NC}_2 \\ | & & | & & | \\ P^s & \longleftarrow & P_2 & \longleftarrow & \mathcal{P}_2 \end{array}$$

and where  $|\cdot|$  is the number of blocks.

PROOF. This follows by putting together the various moment results that we have, namely those from chapter 10, and those from Theorem 11.15.  $\square$

The above result is quite nice, and is complete as well, containing all the moment results that we have established so far, throughout this book. However, forgetting about being as general as possible, we can in fact do better. Nothing in life is better than having some 3D orientation, and as a main application of the above, we can modify a bit the above diagram, as to have a nice-looking cube, as follows:

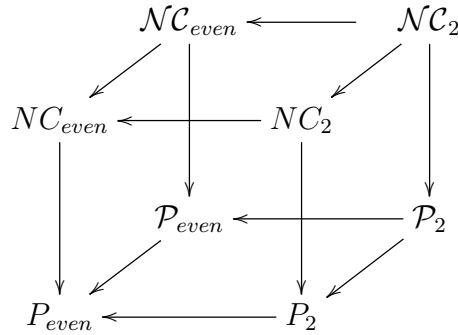
THEOREM 11.17. *The moments of the selected central limiting measures,*



are always given by the same formula, involving partitions, namely

$$M_k = \sum_{\pi \in D(k)} t^{|\pi|}$$

where the sets of partitions  $D(k)$  in question are respectively



and where  $|\cdot|$  is the number of blocks.

PROOF. This follows by putting together the various moment results that we have. To be more precise, the result follows from Theorem 11.16, by restricting the attention on the left to the cases  $s = 2, \infty$ , which can be thought of as being “fully real” and “purely complex”, and then folding the 8-measure diagram into a cube, as above.  $\square$

The above cube, which is something very nice, will basically keep us busy for the rest of this book. Among others, we will see later more conceptual explanations for it.

### 11d. Matrix models

We discuss in what follows a number of random matrix models for the measures  $\beta_t^s$  with  $s \in \mathbb{N}$ , both of multiplicative and of block-modification type. We first restrict attention to the case  $t = 1$ , since  $\beta_t^s = \pi^{\boxtimes s-1} \boxtimes \pi^{\boxplus t}$ , and therefore matrix models for  $\beta_t^s$  will follow from matrix models for  $\pi^{\boxtimes s}$ . Following [8], we first have the following result:

**THEOREM 11.18.** *Let  $G_1, \dots, G_s$  be a family of  $N \times N$  independent matrices formed by independent centered Gaussian variables, of variance  $1/N$ . Then with*

$$M = G_1 \dots G_s$$

*the moments of the spectral distribution of  $MM^*$  converge, up to a normalization, to the corresponding moments of  $\beta_1^s$ , as  $N \rightarrow \infty$ .*

**PROOF.** We proceed by induction. At  $s = 1$  it is well-known that  $MM^*$  is a model for  $\beta_1^1 = \pi$ . So, assume that the result holds for  $s - 1 \geq 1$ . We have:

$$\begin{aligned} \text{tr}(MM^*)^k &= \text{tr}(G_1 \dots G_s G_s^* \dots G_1^*)^k \\ &= \text{tr}(G_1 (G_2 \dots G_s G_s^* \dots G_1^* G_1)^{k-1} G_2 \dots G_s G_s^* \dots G_1^*) \end{aligned}$$

We can pass the first  $G_1$  matrix to the right, and we get:

$$\begin{aligned} \text{tr}(MM^*)^k &= \text{tr}((G_2 \dots G_s G_s^* \dots G_1^* G_1)^{k-1} G_2 \dots G_s G_s^* \dots G_1^* G_1) \\ &= \text{tr}(G_2 \dots G_s G_s^* \dots G_1^* G_1)^k \\ &= \text{tr}((G_2 \dots G_s G_s^* \dots G_2^*)(G_1^* G_1))^k \end{aligned}$$

We know that  $G_1^* G_1$  is a Wishart matrix, hence is a model for  $\pi$ . Also, we know by the induction assumption that  $G_2 \dots G_s G_s^* \dots G_2^*$  gives a matrix model for  $\beta_1^{s-1}$ .

Since the matrices  $G_1^* G_1$  and  $G_2 \dots G_s G_s^* \dots G_2^*$  are asymptotically free, their product gives a matrix model for  $\pi_{s-1,1} \boxtimes \pi_{11} = \beta_1^s$ , and we are done.  $\square$

We should mention that the above result, from [8], has inspired a whole string of extensions and generalizations. We refer here to [8] and the subsequent literature.

Again following [8], we have as well the following result:

**THEOREM 11.19.** *If  $W$  is a complex Wishart matrix of parameters  $(sN, N)$  and*

$$D = \begin{pmatrix} 1_N & 0 & & 0 \\ 0 & w1_N & & 0 \\ & & \ddots & \\ 0 & 0 & & w^{s-1}1_N \end{pmatrix}$$

*with  $w = e^{2\pi i/s}$  then the moments of the spectral distribution of  $(DW)^s$  converge, up to a normalization, to the corresponding moments of  $\beta_1^s$ , as  $N \rightarrow \infty$ .*

PROOF. We use the following complex Wishart matrix formula of Graczyk, Letac and Massam [48], whose proof is via standard combinatorics:

$$\mathbb{E}(Tr(DW)^K) = \sum_{\sigma \in S_K} \frac{M^{\gamma(\sigma^{-1}\pi)}}{M^K} r_{\sigma}(D)$$

Here  $W$  is by definition a complex Wishart matrix of parameters  $(M, N)$ , and  $D$  is a deterministic  $M \times M$  matrix. As for the right term, this is as follows:

- (1)  $\pi$  is the cycle  $(1, \dots, K)$ .
- (2)  $\gamma(\sigma)$  is the number of disjoint cycles of  $\sigma$ .
- (3) If we denote by  $C(\sigma)$  the set of such cycles and for any cycle  $c$ , by  $|c|$  its length, then the function on the right is given by:

$$r_{\sigma}(D) = \prod_{c \in C(\sigma)} Tr(D^{|c|})$$

In our situation we have  $K = sk$  and  $M = sN$ , and we get:

$$\mathbb{E}(Tr(DW)^{sk}) = \sum_{\sigma \in S_{sk}} \frac{(sN)^{\gamma(\sigma^{-1}\pi)}}{(sN)^{sk}} r_{\sigma}(D)$$

Now since  $D$  is uniformly formed by  $s$ -roots of unity, we have:

$$Tr(D^p) = \begin{cases} sN & \text{if } s|p \\ 0 & \text{if } s \nmid p \end{cases}$$

Thus if we denote by  $S_{sk}^s$  the set of permutations  $\sigma \in S_{sk}$  having the property that all the cycles of  $\sigma$  have length multiple of  $s$ , the above formula reads:

$$\mathbb{E}(Tr(DW)^{sk}) = \sum_{\sigma \in S_{sk}^s} \frac{(sN)^{\gamma(\sigma^{-1}\pi)}}{(sN)^{sk}} (sN)^{\gamma(\sigma)}$$

In terms of the normalized trace  $tr$ , we obtain the following formula:

$$\mathbb{E}(tr(DW)^{sk}) = \sum_{\sigma \in S_{sk}^s} (sN)^{\gamma(\sigma^{-1}\pi) + \gamma(\sigma) - sk - 1}$$

The exponent on the right, say  $L_\sigma$ , can be estimated by using the distance on the Cayley graph of  $S_{sk}$ , in the following way:

$$\begin{aligned}
 L_\sigma &= \gamma(\sigma^{-1}\pi) + \gamma(\sigma) - sk - 1 \\
 &= (sk - d(\sigma, \pi)) + (sk - d(e, \sigma)) - sk - 1 \\
 &= sk - 1 - (d(e, \sigma) + d(\sigma, \pi)) \\
 &\leq sk - 1 - d(e, \pi) \\
 &= 0
 \end{aligned}$$

Now when taking the limit  $N \rightarrow \infty$  in the above formula of  $\mathbb{E}(\text{tr}(DW)^{sk})$ , the only terms that count are those coming from permutations  $\sigma \in S_{sk}^s$  having the property  $L_\sigma = 0$ , which each contribute with a 1 value. We therefore obtain:

$$\begin{aligned}
 \lim_{N \rightarrow \infty} \mathbb{E}(\text{tr}(DW)^{sk}) &= \#\{\sigma \in S_{sk}^s \mid L_\sigma = 0\} \\
 &= \#\{\sigma \in S_{sk}^s \mid d(e, \sigma) + d(\sigma, \pi) = d(e, \pi)\} \\
 &= \#\{\sigma \in S_{sk}^s \mid \sigma \in [e, \pi]\}
 \end{aligned}$$

But this number that we obtained is well-known to be the same as the number of noncrossing partitions of  $\{1, \dots, sk\}$  having all blocks of size multiple of  $s$ . Thus we have reached to the sets  $NC_s(k)$  from the above, and we are done.  $\square$

As a consequence of the above random matrix formula, we have the following alternative approach to the free CPLT, in the case of the free Bessel laws, from [8]:

**THEOREM 11.20.** *The moments of the free Bessel law  $\pi_{s1}$  with  $s \in \mathbb{N}$  coincide with those of the variable*

$$\left( \sum_{k=1}^s w^k \alpha_k \right)^s$$

where  $\alpha_1, \dots, \alpha_s$  are free random variables, each of them following the free Poisson law of parameter  $1/s$ , and  $w = e^{2\pi i/s}$ .

**PROOF.** This is something that we already know, coming from the combinatorics of the free CPLT, but we can prove this now by using random matrices as well. For this purpose, let  $G_1, \dots, G_s$  be a family of independent  $sN \times N$  matrices formed by independent, centered complex Gaussian variables, of variance  $1/(sN)$ . The following matrices  $H_1, \dots, H_s$  are then complex Gaussian and independent as well:

$$H_k = \frac{1}{\sqrt{s}} \sum_{p=1}^s w^{kp} G_p$$

Thus the following matrix provides a model for the variable  $\Sigma w^k \alpha_k$ :

$$\begin{aligned}
 M &= \sum_{k=1}^s w^k H_k H_k^* \\
 &= \frac{1}{s} \sum_{k=1}^s \sum_{p=1}^s \sum_{q=1}^s w^{k+kp-kq} G_p G_q^* \\
 &= \sum_{p=1}^s \sum_{q=1}^s \left( \frac{1}{s} \sum_{k=1}^s (w^{1+p-q})^k \right) G_p G_q^* \\
 &= G_1 G_2^* + G_2 G_3^* + \dots + G_{s-1} G_s^* + G_s G_1^*
 \end{aligned}$$

Now observe that this matrix can be written as follows:

$$\begin{aligned}
 M &= (G_1 \ G_2 \ \dots \ G_{s-1} \ G_s) \begin{pmatrix} G_2^* \\ G_3^* \\ \vdots \\ G_s^* \\ G_1^* \end{pmatrix} \\
 &= (G_1 \ G_2 \ \dots \ G_{s-1} \ G_s) \begin{pmatrix} 0 & 1_N & 0 & \dots & 0 \\ 0 & 0 & 1_N & \dots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \dots & 1_N \\ 1_N & 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} G_1^* \\ G_2^* \\ \vdots \\ G_{s-1}^* \\ G_s^* \end{pmatrix} \\
 &= GOG^*
 \end{aligned}$$

In this formula  $G = (G_1 \ \dots \ G_s)$  is the  $sN \times sN$  Gaussian matrix obtained by concatenating  $G_1, \dots, G_s$ , and  $O$  is the matrix in the middle. But this latter matrix is of the form  $O = UDU^*$  with  $U$  unitary, so and we have:

$$M = GUDU^*G^*$$

Now since  $GU$  is a Gaussian matrix,  $M$  has the same law as  $M' = GDG^*$ , and we get:

$$\begin{aligned}
 E \left( \left( \sum_{l=1}^s w^l \alpha_l \right)^{sk} \right) &= \lim_{N \rightarrow \infty} E(\text{tr}(M^{sk})) \\
 &= \lim_{N \rightarrow \infty} E(\text{tr}(GDG^*)^{sk}) \\
 &= \lim_{N \rightarrow \infty} E(\text{tr}(D(G^*G))^{sk})
 \end{aligned}$$

Thus with  $W = G^*G$  we get the result. □

Summarizing, we have applications to the random matrices, and random matrix models for all the 8 basic probability laws, appearing from limiting theorems. As already mentioned, the above results, from [8], have inspired a whole string of extensions and generalizations. We refer here to [8] and the subsequent literature.

As a last topic regarding the free CPLT, which is perhaps the most important, let us review now the results regarding the block-modified Wishart matrices from chapter 8, with free probability tools. We will see in particular that the laws obtained there are free combinations of free Poisson laws, or compound free Poisson laws.

Consider a complex Wishart matrix of parameters  $(dn, dm)$ . In other words, we start with a  $dn \times dm$  matrix  $Y$  having independent complex  $G_1$  entries, and we set:

$$W = YY^*$$

This matrix has size  $dn \times dn$ , and is best thought of as being a  $d \times d$  array of  $n \times n$  matrices. We will be interested here in the study of the block-modified versions of  $W$ , obtained by applying to the  $n \times n$  blocks a given linear map, as follows:

$$\varphi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$$

We recall from chapter 8 that we have the following asymptotic moment formula, extending the usual moment computation for the Wishart matrices:

**THEOREM 11.21.** *The asymptotic moments of a block-modified Wishart matrix*

$$\tilde{W} = (id \otimes \varphi)W$$

*with parameters  $d, m, n \in \mathbb{N}$ , as above, are given by the formula*

$$\lim_{d \rightarrow \infty} M_e \left( \frac{\tilde{W}}{d} \right) = \sum_{\sigma \in NC_p} (mn)^{|\sigma|} (M_e^\sigma \otimes M_e^\gamma)(\Lambda)$$

*where  $\Lambda \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$  is the square matrix associated to  $\varphi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ .*

**PROOF.** This is something that we know well from chapter 8 above, coming from the Wick formula, and with the correspondence between linear maps  $\varphi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$  and square matrices  $\Lambda \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$  being as well explained there.  $\square$

As explained in chapter 8, it is possible to further build on the above result, with some concrete applications, by doing a lot of combinatorics and calculus. That combinatorics and calculus was something a bit ad-hoc in the context of chapter 8, and congratulations of course for having survived that. With the free probability theory that we learned so far, we can now clarify all this. Following [18], [19], we first have the following result:

PROPOSITION 11.22. *Given a square matrix  $\Lambda \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ , having distribution*

$$\rho = \text{law}(\Lambda)$$

*the moments of the compound free Poisson law  $\pi_{mn\rho}$  are given by*

$$M_e(\pi_{mn\rho}) = \sum_{\sigma \in NC_p} (mn)^{|\sigma|} (M_e^\sigma \otimes M_e^\sigma)(\Lambda)$$

*for any choice of the extra parameter  $m \in \mathbb{N}$ .*

PROOF. This can be proved in several ways, as follows:

(1) A first method is by a straightforward computation, based on the general formula of the  $R$ -transform of the compound free Poisson laws, given in the above.

(2) Another method, originally used in [19], is by using the well-known fact, that we will discuss in a moment, in chapter 12 below, that the free cumulants of  $\pi_{mn\rho}$  coincide with the moments of  $mn\rho$ . Thus, these free cumulants are given by:

$$\begin{aligned} \kappa_e(\pi_{mn\rho}) &= M_e(mn\rho) \\ &= mn \cdot M_e(\Lambda) \\ &= mn \cdot (M_e^\gamma \otimes M_e^\gamma)(\Lambda) \end{aligned}$$

By using now Speicher's free moment-cumulant formula, from [70], [78], to be explained in chapter 12 below as well, this gives the result.  $\square$

We can see now an obvious similarity with the formula in Theorem 11.21. In order to exploit this similarity, once again by following [19], let us introduce:

DEFINITION 11.23. *We call a square matrix  $\Lambda \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$  multiplicative when*

$$(M_e^\sigma \otimes M_e^\gamma)(\Lambda) = (M_e^\sigma \otimes M_e^\sigma)(\Lambda)$$

*holds for any  $p \in \mathbb{N}$ , any exponents  $e_1, \dots, e_p \in \{1, *\}$ , and any  $\sigma \in NC_p$ .*

This notion is something quite technical, but we will see many examples in what follows. For instance, the square matrices  $\Lambda$  coming from the basic linear maps  $\varphi$  appearing in chapter 8 are all multiplicative. Now with the above notion in hand, we can formulate an asymptotic result regarding the block-modified Wishart matrices, as follows:

THEOREM 11.24. *Consider a block-modified Wishart matrix*

$$\tilde{W} = (id \otimes \varphi)W$$

*and assume that the matrix  $\Lambda \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$  associated to  $\varphi$  is multiplicative. Then*

$$\frac{\tilde{W}}{d} \sim \pi_{mn\rho}$$

*holds, in moments, in the  $d \rightarrow \infty$  limit, where  $\rho = \text{law}(\Lambda)$ .*

PROOF. By comparing the moment formulae in Theorem 11.21 and in Proposition 11.22, we conclude that the asymptotic formula  $\frac{\bar{W}}{d} \sim \pi_{mn\rho}$  is equivalent to the following equality, which should hold for any  $p \in \mathbb{N}$ , and any exponents  $e_1, \dots, e_p \in \{1, *\}$ :

$$\sum_{\sigma \in NC_p} (mn)^{|\sigma|} (M_e^\sigma \otimes M_e^\sigma)(\Lambda) = \sum_{\sigma \in NC_p} (mn)^{|\sigma|} (M_e^\sigma \otimes M_e^\sigma)(\Lambda)$$

Now by assuming that  $\Lambda$  is multiplicative, in the sense of Definition 11.23, these two sums are trivially equal, and this gives the result.  $\square$

Summarizing, we have now a much better understanding of what is going on with the block-modified Wishart matrices, and in particular with what exactly is behind Theorem 11.24. Still following the material in chapter 8, but armed now with our conceptual free probability knowledge, let us work out now some explicit consequences of Theorem 11.24, by using some special classes of modification maps  $\varphi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ . We recall from chapter 8 that we have the following standard definition:

DEFINITION 11.25. *Associated to any partition  $\pi \in P(2s, 2s)$  is the linear map*

$$\varphi_\pi(e_{a_1 \dots a_s, c_1 \dots c_s}) = \sum_{b_1 \dots b_s} \sum_{d_1 \dots d_s} \delta_\pi \begin{pmatrix} a_1 & \dots & a_s & c_1 & \dots & c_s \\ b_1 & \dots & b_s & d_1 & \dots & d_s \end{pmatrix} e_{b_1 \dots b_s, d_1 \dots d_s}$$

*between tensor powers of  $\mathbb{C}^N$ , with the Kronecker type symbol on the right being given by  $\delta_\pi = 1$  when the indices fit, and  $\delta_\pi = 0$  otherwise.*

As explained in chapter 8, there is a connection here with notion of easy group, from chapter 3. We will be back to this later on, when talking about easy quantum groups. In relation with our Wishart matrix considerations, the point is that the above linear map  $\varphi_\pi$  can be viewed as a “block-modification” map, as follows:

$$\varphi_\pi : M_{N^s}(\mathbb{C}) \rightarrow M_{N^s}(\mathbb{C})$$

In order to verify that the corresponding matrices  $\Lambda_\pi$  are multiplicative, we will need to check that all the functions  $\varphi(\sigma, \tau) = (M_\sigma^e \otimes M_\tau^e)(\Lambda_\pi)$  have the following property:

$$\varphi(\sigma, \gamma) = \varphi(\sigma, \sigma)$$

For this purpose, we can use the following result, coming from [19]:

PROPOSITION 11.26. *The following functions*

$$\varphi : NC_p \times NC_p \rightarrow \mathbb{R}$$

*are “multiplicative”, in the sense that they satisfy the condition  $\varphi(\sigma, \gamma) = \varphi(\sigma, \sigma)$ :*

- (1)  $\varphi(\sigma, \tau) = |\sigma\tau^{-1}| - |\tau|$ .
- (2)  $\varphi(\sigma, \tau) = |\sigma\tau| - |\tau|$ .
- (3)  $\varphi(\sigma, \tau) = |\sigma \wedge \tau| - |\tau|$ .

PROOF. All this is elementary, and can be proved as follows:

(1) This follows indeed from the following computation:

$$\begin{aligned}\varphi_1(\sigma, \gamma) &= |\sigma\gamma^{-1}| - 1 \\ &= p - |\sigma| \\ &= \varphi_1(\sigma, \sigma)\end{aligned}$$

(2) This follows indeed from the following computation:

$$\begin{aligned}\varphi_2(\sigma, \gamma) &= |\sigma\gamma| - 1 \\ &= |\sigma^2| - |\sigma| \\ &= \varphi_2(\sigma, \sigma)\end{aligned}$$

(3) This follows indeed from the following computation:

$$\begin{aligned}\varphi_3(\sigma, \gamma) &= |\gamma| - |\gamma| \\ &= 0 \\ &= |\sigma| - |\sigma| \\ &= \varphi_3(\sigma, \sigma)\end{aligned}$$

Thus, we are led to the conclusions in the statement.  $\square$

In practice now, the above can be quite complicated, related to subtle questions of easiness, so let us first discuss the case  $s = 1$ . There are 15 partitions  $\pi \in P(2, 2)$ , and among them, the most “basic” ones are the 4 partitions  $\pi \in P_{\text{even}}(2, 2)$ . With the standard convention that  $A^\delta \in M_N(\mathbb{C})$  denotes the diagonal of a matrix  $A \in M_N(\mathbb{C})$ , we have the following result, from [19], regarding these partitions:

PROPOSITION 11.27. *The partitions  $\pi \in P_{\text{even}}(2, 2)$  are as follows,*

$$\pi_1 = \begin{bmatrix} \circ & \bullet \\ \circ & \bullet \end{bmatrix}, \quad \pi_2 = \begin{bmatrix} \circ & \bullet \\ \bullet & \circ \end{bmatrix}, \quad \pi_3 = \begin{bmatrix} \circ & \circ \\ \bullet & \bullet \end{bmatrix}, \quad \pi_4 = \begin{bmatrix} \circ & \circ \\ \circ & \circ \end{bmatrix}$$

with the associated linear maps  $\varphi_\pi : M_n(\mathbb{C}) \rightarrow M_N(\mathbb{C})$  being as follows:

$$\varphi_1(A) = A$$

$$\varphi_2(A) = A^t$$

$$\varphi_3(A) = \text{Tr}(A)1$$

$$\varphi_4(A) = A^\delta$$

The corresponding matrices  $\Lambda_\pi$  are all multiplicative, in the sense of Definition 11.23.

PROOF. For the above 4 partitions, the associated linear maps are given by:

$$\begin{aligned}\varphi_1(e_{ac}) &= e_{ac} \\ \varphi_2(e_{ac}) &= e_{ca} \\ \varphi_3(e_{ac}) &= \delta_{ac} \sum_b e_{bb} \\ \varphi_4(e_{ac}) &= \delta_{ac} e_{aa}\end{aligned}$$

Thus, we obtain the formulae in the statement. Regarding now the associated square matrices, appearing via  $\Lambda_{ab,cd} = \varphi(e_{ac})_{bd}$ , these are given by:

$$\begin{aligned}\Lambda_{ab,cd}^1 &= \delta_{ab} \delta_{cd} \\ \Lambda_{ab,cd}^2 &= \delta_{ad} \delta_{bc} \\ \Lambda_{ab,cd}^3 &= \delta_{ac} \delta_{bd} \\ \Lambda_{ab,cd}^4 &= \delta_{abcd}\end{aligned}$$

Since these matrices are all self-adjoint, we can assume that all the exponents are 1 in Definition 11.23, and the condition there becomes:

$$(M^\sigma \otimes M^\tau)(\Lambda) = (M^\sigma \otimes M^\sigma)(\Lambda)$$

In order to check this condition, observe that for the above 4 matrices, we have:

$$\begin{aligned}(M^\sigma \otimes M^\tau)(\Lambda_1) &= \frac{1}{n^{|\sigma|+|\tau|}} \sum_{i_1 \dots i_p} \delta_{i_{\sigma(1)} i_{\tau(1)}} \dots \delta_{i_{\sigma(p)} i_{\tau(p)}} = n^{|\sigma\tau^{-1}| - |\sigma| - |\tau|} \\ (M^\sigma \otimes M^\tau)(\Lambda_2) &= \frac{1}{n^{|\sigma|+|\tau|}} \sum_{i_1 \dots i_p} \delta_{i_1 i_{\sigma\tau(1)}} \dots \delta_{i_p i_{\sigma\tau(p)}} = n^{|\sigma\tau| - |\sigma| - |\tau|} \\ (M^\sigma \otimes M^\tau)(\Lambda_3) &= \frac{1}{n^{|\sigma|+|\tau|}} \sum_{i_1 \dots i_p} \sum_{j_1 \dots j_p} \delta_{i_1 i_{\sigma(1)}} \delta_{j_1 j_{\tau(1)}} \dots \delta_{i_p i_{\sigma(p)}} \delta_{j_p j_{\tau(p)}} = 1 \\ (M^\sigma \otimes M^\tau)(\Lambda_4) &= \frac{1}{n^{|\sigma|+|\tau|}} \sum_{i_1 \dots i_p} \delta_{i_1 i_{\sigma(1)} i_{\tau(1)}} \dots \delta_{i_p i_{\sigma(p)} i_{\tau(p)}} = n^{|\sigma \wedge \tau| - |\sigma| - |\tau|}\end{aligned}$$

By using now the results in Proposition 11.27, this gives the result.  $\square$

Summarizing, the partitions  $\pi \in P_{\text{even}}(2, 2)$  provide us with some concrete input for Theorem 11.24. When using this input, we obtain:

**THEOREM 11.28.** *The asymptotic distribution results for the block-modified Wishart matrices coming from the partitions  $\pi_1, \pi_2, \pi_3, \pi_4 \in P_{\text{even}}(2, 2)$  are as follows:*

- (1) *Marchenko-Pastur:*  $\frac{1}{d}W \sim \pi_t$ , where  $t = m/n$ .
- (2) *Aubrun type:*  $\frac{1}{d}(id \otimes t)W \sim \pi_\nu$ , with  $\nu = \frac{m(n-1)}{2}\delta_{-1} + \frac{m(n+1)}{2}\delta_1$ .
- (3) *Collins-Nechita one:*  $n(id \otimes tr(\cdot)1)W \sim \pi_t$ , where  $t = mn$ .
- (4) *Collins-Nechita two:*  $\frac{1}{d}(id \otimes (\cdot)^\delta)W \sim \pi_m$ .

PROOF. These observations go back to [19]. In our setting, the maps  $\varphi_1, \varphi_2, \varphi_3, \varphi_4$  in Proposition 11.28 give the 4 matrices in the statement, modulo some rescalings, and the computation of the corresponding distributions goes as follows:

(1) Here  $\Lambda = \sum_{ac} e_{ac} \otimes e_{ac}$ , and so  $\Lambda = nP$ , where  $P$  is the rank one projection on  $\sum_a e_a \otimes e_a \in \mathbb{C}^n \otimes \mathbb{C}^n$ . Thus we have the following formula, which gives the result:

$$\rho = \frac{n^2 - 1}{n^2} \delta_0 + \frac{1}{n^2} \delta_n$$

(2) Here  $\Lambda = \sum_{ac} e_{ac} \otimes e_{ca}$  is the flip operator,  $\Lambda(e_c \otimes e_a) = e_a \otimes e_c$ . Thus  $\rho = \frac{n-1}{2n} \delta_{-1} + \frac{n+1}{2n} \delta_1$ , and so we have the following formula, which gives the result:

$$mn\rho = \frac{m(n-1)}{2} \delta_{-1} + \frac{m(n+1)}{2} \delta_1$$

(3) Here  $\Lambda = \sum_{ab} e_{aa} \otimes e_{bb}$  is the identity matrix,  $\Lambda = 1$ . Thus in this case we have the following formula, which gives  $\pi_{mn\rho} = \pi_{mn}$ , and so  $\frac{\tilde{W}}{d} \sim \pi_{mn}$ , as claimed:

$$\rho = \delta_1$$

(4) Here  $\Lambda = \sum_a e_{aa} \otimes e_{aa}$  is the orthogonal projection on  $\text{span}(e_a \otimes e_a) \subset \mathbb{C}^n \otimes \mathbb{C}^n$ . Thus we have the following formula, which gives the result:

$$\rho = \frac{n-1}{n} \delta_0 + \frac{1}{n} \delta_1$$

Summarizing, we have proved all the assertions in the statement.  $\square$

As explained in chapter 8, it is possible to say a bit more about the case (2) in the above result, the one dealing with the block-transposed Wishart matrices, by using all sorts of mysterious complex analysis manipulations on the corresponding functional transforms, allowing us to say more about the corresponding measures  $\pi_\nu$ .

These mysterious complex analysis manipulations correspond, of course, to standard procedures from free probability, related to the combinatorics and analysis of Voiculescu's  $R$ -transform. For the continuation of all this, we refer to [4], [18], [19] and the subsequent literature on the subject, including the more recent papers [3], [47], [67].

In what concerns us, we will rather navigate in what follows towards quantum algebra, but we will be back to random matrix questions on several occasions, and notably in chapter 16 below, in the context of an all-catching final discussion, regarding the relation between Voiculescu's free probability and Jones' subfactor theory.

**11e. Exercises**

Things have been quite technical in this chapter, and as unique exercise here, which is unfortunately even more technical than what has been said above, we have:

EXERCISE 11.29. *Find block-modified matrix models for the free Bessel laws.*

This is something which is not very obvious, and also, needless to say, was not something solved in the above. In case you get stuck with this, of course look it up.

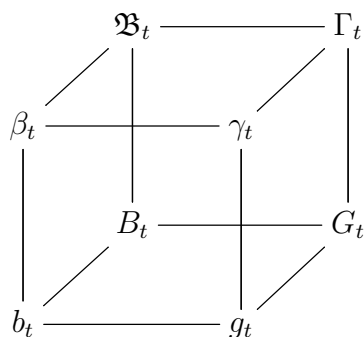
## CHAPTER 12

### The bijection

#### 12a. Cumulants

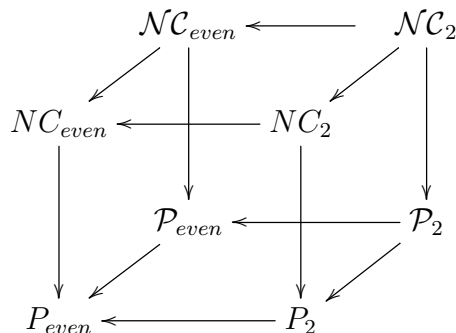
In this chapter we discuss the relation between classical and free probability. As a starting point, we have the following statement, which is something very concrete:

**THEOREM 12.1.** *The moments of the main limiting measures in classical and free probability, real and complex, and discrete and continuous,*



are always given by the same formula,  $M_k = \sum_{\pi \in D(k)} t^{|\pi|}$ , where  $D \subset P$  is a certain set of partitions associated to the measure, and where  $|\cdot|$  is the number of blocks.

**PROOF.** This is something that we know well, the sets of partitions being:



For full details on all this, we refer to the previous chapter. □

What is interesting with the above cube is that it provides us with some 3D orientation in noncommutative probability, taken at large. To be more precise, the 3 “coordinate

axes” that we have, corresponding to the 3 pairs of opposing faces, are real/complex, discrete/continuous, and classical/free. All this is nice, and potentially fruitful.

As a first observation, just by looking at the upper and lower faces of the cube, and how they are connected, we conclude that there should be a bijection between classical and free probability, having something to do with crossing and noncrossing partitions. And this is indeed the case, with this being known since the paper of Bercovici-Pata [23], who discovered this bijection, and explaining all this will be our next task.

We will first need to do some advanced combinatorics, in relation with cumulants, following Rota in the classical case, and Speicher in the free case. Let us start with the classical case. We have here the following well-known, and useful definition:

**DEFINITION 12.2.** *Associated to any real probability measure  $\mu = \mu_f$  is the following modification of the logarithm of the Fourier transform  $F_\mu(\xi) = \mathbb{E}(e^{i\xi f})$ ,*

$$K_\mu(\xi) = \log \mathbb{E}(e^{\xi f})$$

*called cumulant-generating function. The Taylor coefficients  $k_n(\mu)$  of this series, given by*

$$K_\mu(\xi) = \sum_{n=1}^{\infty} k_n(\mu) \frac{\xi^n}{n!}$$

*are called cumulants of the measure  $\mu$ . We also use the notations  $k_f, K_f$  for these cumulants and their generating series, where  $f$  is a variable following the law  $\mu$ .*

In other words, the cumulants are more or less the coefficients of the logarithm of the Fourier transform  $\log F_\mu$ , up to some normalizations. To be more precise, we have  $K_\mu(\xi) = \log F_\mu(-i\xi)$ , so the formula relating  $\log F_\mu$  to the cumulants  $k_n(\mu)$  is:

$$\log F_\mu(-i\xi) = \sum_{n=1}^{\infty} k_n(\mu) \frac{\xi^n}{n!}$$

Equivalently, the formula relating  $\log F_\mu$  to the cumulants  $k_n(\mu)$  is:

$$\log F_\mu(\xi) = \sum_{n=1}^{\infty} k_n(\mu) \frac{(i\xi)^n}{n!}$$

We will see in a moment the reasons for the above normalizations, namely change of variables  $\xi \rightarrow -i\xi$ , and Taylor coefficients instead of plain coefficients, the idea being that for simple laws like  $g_t, p_t$ , we will obtain in this way very simple quantities. Let us also mention that there is a reason for indexing the cumulants by  $n = 1, 2, 3, \dots$  instead of  $n = 0, 1, 2, \dots$ , and more on this later, once we will have some theory and examples.

The interest in cumulants comes from the fact that  $\log F_\mu$ , and so the cumulants  $k_n(\mu)$  too, linearize the convolution. To be more precise, we have the following result:

THEOREM 12.3. *The cumulants have the following properties:*

- (1)  $k_n(cf) = c^n k_n(f)$ .
- (2)  $k_1(f + d) = k_1(f) + d$ , and  $k_n(f + d) = k_n(f)$  for  $n > 1$ .
- (3)  $k_n(f + g) = k_n(f) + k_n(g)$ , if  $f, g$  are independent.

PROOF. Here (1) and (2) are both clear from definitions, because we have the following computation, valid for any  $c, d \in \mathbb{R}$ , which gives the results:

$$\begin{aligned} K_{cf+d}(\xi) &= \log \mathbb{E}(e^{\xi(cf+d)}) \\ &= \log[e^{\xi d} \cdot \mathbb{E}(e^{\xi cf})] \\ &= \xi d + K_f(c\xi) \end{aligned}$$

As for (3), this follows from the fact that the Fourier transform  $F_f(\xi) = \mathbb{E}(e^{i\xi f})$  satisfies the following formula, whenever  $f, g$  are independent random variables:

$$F_{f+g}(\xi) = F_f(\xi)F_g(\xi)$$

Indeed, by applying the logarithm, we obtain the following formula:

$$\log F_{f+g}(\xi) = \log F_f(\xi) + \log F_g(\xi)$$

With the change of variables  $\xi \rightarrow -i\xi$ , we obtain the following formula:

$$K_{f+g}(\xi) = K_f(\xi) + K_g(\xi)$$

Thus, at the level of coefficients, we obtain  $k_n(f + g) = k_n(f) + k_n(g)$ , as claimed.  $\square$

At the level of the main examples now, we have the following result, which shows in particular that the various normalizations that we made when defining the cumulants are indeed the good ones, leading to the simplest possible formulae for these cumulants:

PROPOSITION 12.4. *The sequence of cumulants  $k_1, k_2, k_3, \dots$  is as follows:*

- (1) For  $\mu = \delta_c$  the cumulants are  $c, 0, 0, \dots$
- (2) For  $\mu = g_t$  the cumulants are  $0, t, 0, 0, \dots$
- (3) For  $\mu = p_t$  the cumulants are  $t, t, t, \dots$
- (4) For  $\mu = b_t$  the cumulants are  $0, t, 0, t, \dots$

PROOF. We have 4 computations to be done, the idea being as follows:

- (1) For  $\mu = \delta_c$  we have the following computation:

$$\begin{aligned} K_\mu(\xi) &= \log \mathbb{E}(e^{c\xi}) \\ &= \log(e^{c\xi}) \\ &= c\xi \end{aligned}$$

But the plain coefficients of this series are the numbers  $c, 0, 0, \dots$ , and so the Taylor coefficients of this series are these same numbers  $c, 0, 0, \dots$ , as claimed.

(2) For  $\mu = g_t$  we have the following computation:

$$\begin{aligned} K_\mu(\xi) &= \log F_\mu(-i\xi) \\ &= \log \exp \left[ -t(-i\xi)^2/2 \right] \\ &= t\xi^2/2 \end{aligned}$$

But the plain coefficients of this series are the numbers  $0, t/2, 0, 0, \dots$ , and so the Taylor coefficients of this series are the numbers  $0, t, 0, 0, \dots$ , as claimed.

(3) For  $\mu = p_t$  we have the following computation:

$$\begin{aligned} K_\mu(\xi) &= \log F_\mu(-i\xi) \\ &= \log \exp \left[ (e^{i(-i\xi)} - 1)t \right] \\ &= (e^\xi - 1)t \end{aligned}$$

But the plain coefficients of this series are the numbers  $t/n!$ , and so the Taylor coefficients of this series are the numbers  $t, t, t, \dots$ , as claimed.

(4) For  $\mu = b_t$  we have the following computation:

$$\begin{aligned} K_\mu(\xi) &= \log F_\mu(-i\xi) \\ &= \log \exp \left[ \left( \frac{e^\xi + e^{-\xi}}{2} - 1 \right) t \right] \\ &= \left( \frac{e^\xi + e^{-\xi}}{2} - 1 \right) t \end{aligned}$$

But the plain coefficients of this series are the numbers  $(1 + (-1)^n)t/n!$ , so the Taylor coefficients of this series are the numbers  $0, t, 0, t, \dots$ , as claimed.  $\square$

At a more theoretical level, we have the following result, generalizing (3,4) above:

**THEOREM 12.5.** *For a compound Poisson law  $p_\nu$  we have*

$$k_n(p_\nu) = M_n(\nu)$$

*valid for any integer  $n \geq 1$ .*

**PROOF.** We can assume, by using a continuity argument, that our measure  $\nu$  is discrete, as follows, with  $t_i > 0$  and  $z_i \in \mathbb{R}$ , and with the sum being finite:

$$\nu = \sum_i t_i \delta_{z_i}$$

By using now the Fourier transform formula for  $p_\nu$  from chapter 11, we obtain:

$$\begin{aligned}
 K_{p_\nu}(\xi) &= \log F_{p_\nu}(-i\xi) \\
 &= \log \exp \left[ \sum_i t_i (e^{\xi z_i} - 1) \right] \\
 &= \sum_i t_i \sum_{n \geq 1} \frac{(\xi z_i)^n}{n!} \\
 &= \sum_{n \geq 1} \frac{\xi^n}{n!} \sum_i t_i z_i^n \\
 &= \sum_{n \geq 1} \frac{\xi^n}{n!} M_n(\nu)
 \end{aligned}$$

Thus, we are led to the conclusion in the statement.  $\square$

Getting back to theory now, the sequence of cumulants  $k_1, k_2, k_3, \dots$  appears as a modification of the sequence of moments  $M_1, M_2, M_3, \dots$ , and understanding the relation between moments and cumulants will be our next task. Let us start with:

**PROPOSITION 12.6.** *The sequence of cumulants  $k_1, k_2, k_3, \dots$  appears as a modification of the sequence of moments  $M_1, M_2, M_3, \dots$ , and uniquely determines  $\mu$ . We have*

$$\begin{aligned}
 k_1 &= M_1 \\
 k_2 &= -M_1^2 + M_2 \\
 k_3 &= 2M_1^3 - 3M_1M_2 + M_3 \\
 &\vdots
 \end{aligned}$$

and in the other sense we have

$$\begin{aligned}
 M_1 &= k_1 \\
 M_2 &= k_1^2 + k_2 \\
 M_3 &= k_1^3 + 3k_1k_2 + k_3 \\
 &\vdots
 \end{aligned}$$

with in both cases the correspondence being polynomial, with integer coefficients.

**PROOF.** Here all the theoretical assertions regarding moments and cumulants are clear from definitions, and the numerics are clear from definitions too.  $\square$

In order to understand what exactly is going on, with moments and cumulants, which reminds a bit the Möbius inversion formula, we need to do some combinatorics, in relation with partitions. So, let us go back to the material from chapter 3, where some theory for the partitions was developed. We recall that we have the following definition:

DEFINITION 12.7. *The Möbius function of any lattice, and so of  $P$ , is given by*

$$\mu(\pi, \nu) = \begin{cases} 1 & \text{if } \pi = \nu \\ -\sum_{\pi \leq \tau < \nu} \mu(\pi, \tau) & \text{if } \pi < \nu \\ 0 & \text{if } \pi \not\leq \nu \end{cases}$$

*with the construction being performed by recurrence.*

This is something that we already discussed in chapter 3, and as a first example here, the Möbius matrix  $M_{\pi\nu} = \mu(\pi, \nu)$  of the lattice  $P(2) = \{||, \sqcap\}$  is as follows:

$$M = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

At  $k = 3$  now, we have the following formula for the Möbius matrix  $M_{\pi\nu} = \mu(\pi, \nu)$ , once again written with the indices picked increasing in  $P(3) = \{||, \sqcap, \sqcup, |\sqcap, |\sqcup\}$ :

$$M = \begin{pmatrix} 1 & -1 & -1 & -1 & 2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

In general, as explained in chapter 3, the Möbius matrix of  $P(k)$  looks a bit like the above matrices at  $k = 2, 3$ , being upper triangular, with 1 on the diagonal, and so on.

Back to the general case now, the main interest in the Möbius function comes from the Möbius inversion formula, which states that the following happens:

$$f(\nu) = \sum_{\pi \leq \nu} g(\pi) \quad \implies \quad g(\nu) = \sum_{\pi \leq \nu} \mu(\pi, \nu) f(\pi)$$

This is something elementary, and very useful when dealing with partitions. In linear algebra terms, the statement and proof of this formula are as follows:

THEOREM 12.8. *The inverse of the adjacency matrix of  $P$ , given by*

$$A_{\pi\nu} = \begin{cases} 1 & \text{if } \pi \leq \nu \\ 0 & \text{if } \pi \not\leq \nu \end{cases}$$

*is the Möbius matrix of  $P$ , given by  $M_{\pi\nu} = \mu(\pi, \nu)$ .*

PROOF. This is well-known, coming for the fact that the above adjacency matrix  $A$  is upper triangular. Indeed, when trying to invert this matrix  $A$ , we are led to the recurrence in Definition 12.7, and so to the Möbius matrix  $M$ , as stated.  $\square$

As a first illustration, for  $P(2)$  the formula  $M = A^{-1}$  appears as follows:

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1}$$

At  $k = 3$  now, the formula  $M = A^{-1}$  for  $P(3)$  reads:

$$\begin{pmatrix} 1 & -1 & -1 & -1 & 2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}^{-1}$$

With these ingredients in hand, let us go back to probability. We first have:

DEFINITION 12.9. *We define quantities  $M_\pi(f), k_\pi(f)$ , depending on partitions*

$$\pi \in P(k)$$

*by starting with  $M_n(f), k_n(f)$ , and using multiplicativity over the blocks.*

To be more precise, the convention here is that for the one-block partition  $1_n \in P(n)$ , the corresponding moment and cumulant are the usual ones, namely:

$$M_{1_n}(f) = M_n(f) \quad , \quad k_{1_n}(f) = k_n(f)$$

Then, for an arbitrary partition  $\pi \in P(k)$ , we decompose this partition into blocks, having sizes  $b_1, \dots, b_s$ , and we set, by multiplicativity over blocks:

$$M_\pi(f) = M_{b_1}(f) \dots M_{b_s}(f) \quad , \quad k_\pi(f) = k_{b_1}(f) \dots k_{b_s}(f)$$

With this convention, following Rota and others, we can now formulate a key result, fully clarifying the relation between moments and cumulants, as follows:

THEOREM 12.10. *We have the moment-cumulant formulae*

$$M_n(f) = \sum_{\pi \in P(n)} k_\pi(f)$$

$$k_n(f) = \sum_{\nu \in P(n)} \mu(\nu, 1_n) M_\nu(f)$$

where  $\mu$  is the Möbius function of  $P(n)$ .

PROOF. Let us first work out some examples. At  $n = 1, 2, 3$  the moment formula gives the following equalities, which are in tune with the findings from Proposition 12.6:

$$M_1 = k_{\sqcup} = k_1$$

$$M_2 = k_{\sqcup\sqcup} + k_{\sqcap} = k_1^2 + k_2$$

$$M_3 = k_{\sqcup\sqcup\sqcup} + k_{\sqcup\sqcap} + k_{\sqcap\sqcup} + k_{\sqcap\sqcap} + k_{\sqcap\sqcap\sqcap} = k_1^3 + 3k_1k_2 + k_3$$

As for the cumulant formula, this gives, again in tune with Proposition 12.6:

$$k_1 = M_{|} = M_1$$

$$k_2 = (-1)M_{||} + M_{\sqcap} = -M_1^2 + M_2$$

$$k_3 = 2M_{|||} + (-1)M_{\sqcap|} + (-1)M_{|\sqcap} + (-1)M_{\sqcap\sqcap} + M_{\sqcap\sqcap} = 2M_1^3 - 3M_1M_2 + M_3$$

In general now, the two formulae in the statement are equivalent, via the Möbius inversion formula from Theorem 12.8. Thus, it is enough to prove any of them, and this can be done by starting with Definition 12.2, and doing some standard calculus and combinatorics. For details here, we refer for instance to Nica-Speicher [70].  $\square$

Summarizing, we have now a nice theory of cumulants, or rather a beginning of such a theory, and with this in hand, we can go back to the diagram in Theorem 12.1, see if we can now better understand what is going on there. However, this is a bit tricky:

(1) Our theory of cumulants as developed so far only applies properly to the “real classical” case, namely to the measures  $g_t, b_t$  there. In order to deal with the full “classical” case, comprising as well the measures  $G_t, B_t$ , we would have to upgrade everything into a theory of  $*$ -cumulants, and this is something quite technical.

(2) Regarding the “free real” measures  $\gamma_t, \beta_t$  and their complex analogues  $\Gamma_t, \mathfrak{B}_t$ , here the cumulant theory developed above gives nothing interesting. We will see in the next section, at least in the real case, that of  $\gamma_t, \beta_t$ , that the relevant theory which applies to them is a substantial modification of what we have, called free cumulant theory.

In short, technical problems in all directions, and we are not ready yet for better understanding Theorem 12.1. As a more modest objective, however, we have the quite reasonable question of understanding the moment formula  $M_k = \sum_{\pi \in D(k)} t^{|\pi|}$  there for the measures  $g_t, b_t$ , by using the cumulant theory developed above. Which is in fact a non-trivial question too, with the answer involving the following result from [20]:

**THEOREM 12.11.** *The uniform orthogonal easy groups  $G \subset O_N$ , and their associated categories of partitions  $D \subset P$ , all coming from subsets  $L \subset \mathbb{N}$ , are as follows,*

$$\begin{array}{ccccc} B_N & \longrightarrow & O_N & & P_{12} \longleftarrow P_2 & & \{1, 2\} \longleftarrow \{2\} \\ \uparrow & & \uparrow & : & \downarrow & & \downarrow \\ S_N & \longrightarrow & H_N & & P \longleftarrow P_{\text{even}} & & \mathbb{N} \longleftarrow 2\mathbb{N} \end{array}$$

with  $D$  consisting of the partitions  $\pi \in P$  whose blocks have lengths belonging to  $L \subset \mathbb{N}$ .

PROOF. Consider an arbitrary easy group,  $S_N \subset G_N \subset O_N$ . This group must then come from a category of partitions, as follows:

$$P_2 \subset D \subset P$$

Now if we assume  $G = (G_N)$  to be uniform, this category  $D$  is uniquely determined by the subset  $L \subset \mathbb{N}$  consisting of the sizes of the blocks of the partitions in  $D$ . And as explained in [20], one can prove that the admissible sets are those in the statement, corresponding to the categories and the groups in the statement.  $\square$

In relation now with cumulants, we have the following result, also from [20]:

**THEOREM 12.12.** *The cumulants of the asymptotic truncated characters for the uniform easy groups  $G = (G_N)$  are given by the formula*

$$k_n(\chi_t) = t\delta_{n \in L}$$

with  $L \subset \mathbb{N}$  being the associated subset, and at the level of asymptotic moments this gives

$$M_k(\chi_t) = \sum_{\pi \in D(k)} t^{|\pi|}$$

with  $D \subset P$  being the associated category of partitions.

PROOF. This is clear indeed from Theorem 12.11, by performing a case-by-case analysis, with the cases  $G = O, S, H$  corresponding to the computations for  $g_t, p_t, b_t$  from Proposition 12.4, and with the remaining case, that of the bistochastic groups,  $G = B$ , being similar. Again, for details on all this, we refer to [20].  $\square$

Summarizing, we have now a good understanding of the formula  $M_k = \sum_{\pi \in D(k)} t^{|\pi|}$  for the real classical limiting measures, based on cumulants, but with this involving however some more advanced mathematics. It is possible of course to reformulate all the above in terms of categories of partitions only, but this won't lead to any simplifications in the proofs, which are based on categories of partitions anyway, and would rather obscure the final results themselves, which are best thought of in terms of easy groups.

Finally, in order to extend the above results to the general the complex case, the cumulant theory must be upgraded into a  $*$ -cumulant theory, which is something quite technical. We will discuss however such questions in chapter 14 below, directly in a more general setting, that of operator-valued noncommutative probability theory, following Speicher and others [70], [78], [79]. In what regards the easy groups, and more generally easy quantum groups, in the general unitary setting, this is again a quite technical subject, and we will be back to this on several occasions, in the remainder of this book.

### 12b. Free cumulants

In what follows we discuss the free analogues of the above, following Speicher [78], and subsequent work. We will be quite brief, basically by indicating how the classical theory developed above can be “liberated”. We first have the following definition:

DEFINITION 12.13. *The free cumulants  $\kappa_n(a)$  of a variable  $a \in A$  are defined by*

$$R_a(\xi) = \sum_{n=1}^{\infty} \kappa_n(a) \xi^{n-1}$$

with the  $R$ -transform being defined as usual by the formula

$$G_a \left( R_a(\xi) + \frac{1}{\xi} \right) = \xi$$

where  $G_a(\xi) = \int_{\mathbb{R}} \frac{d\mu(t)}{\xi - t}$  with  $\mu = \mu_a$  is the corresponding Cauchy transform.

As before with classical cumulants, we have a number of basic examples, and a number of basic general results. At the level of basic general results, we first have:

THEOREM 12.14. *The free cumulants have the following properties:*

- (1)  $\kappa_n(\lambda a) = \lambda^n \kappa_n(a)$ .
- (2)  $\kappa_n(a + b) = \kappa_n(a) + \kappa_n(b)$ , if  $a, b$  are free.

PROOF. In what regards (1), we have here the following computation:

$$\begin{aligned} G_{\lambda a}(\xi) &= \int_{\mathbb{R}} \frac{d\mu_{\lambda a}(t)}{\xi - t} \\ &= \int_{\mathbb{R}} \frac{d\mu_a(s)}{\xi - \lambda s} \\ &= \frac{1}{\lambda} \int_{\mathbb{R}} \frac{d\mu_a(s)}{\xi/\lambda - s} \\ &= \frac{1}{\lambda} G_a \left( \frac{\xi}{\lambda} \right) \end{aligned}$$

But this gives the following formula:

$$\begin{aligned} G_{\lambda a} \left( \lambda R_a(\lambda \xi) + \frac{1}{\xi} \right) &= \frac{1}{\lambda} G_a \left( R_a(\lambda \xi) + \frac{1}{\lambda \xi} \right) \\ &= \frac{1}{\lambda} \cdot \lambda \xi \\ &= \xi \end{aligned}$$

Thus  $R_{\lambda a}(\xi) = \lambda R_a(\lambda \xi)$ , which gives (1). As for (2), this follows from the fact, that we know from chapter 9, that the  $R$ -transform linearizes the free convolution operation.  $\square$

Again in analogy with the classical case, at the level of examples, we have:

**THEOREM 12.15.** *The sequence of free cumulants  $\kappa_1, \kappa_2, \kappa_3, \dots$  is as follows:*

- (1) *For  $\mu = \delta_c$  the free cumulants are  $c, 0, 0, \dots$*
- (2) *For  $\mu = \gamma_t$  the free cumulants are  $0, t, 0, 0, \dots$*
- (3) *For  $\mu = \pi_t$  the free cumulants are  $t, t, t, \dots$*
- (4) *For  $\mu = \beta_t$  the free cumulants are  $0, t, 0, t, \dots$*

*Also, for compound free Poisson laws the free cumulants are  $k_n(\pi_\nu) = M_n(\nu)$ .*

**PROOF.** The proofs are analogous to those from the classical case, as follows:

- (1) For  $\mu = \delta_c$  we have  $G_\mu(\xi) = 1/(\xi - c)$ , and so  $R_\mu(\xi) = c$ , as desired.
- (2) For  $\mu = \gamma_t$  we have, as computed in chapter 9,  $R_\mu(\xi) = t\xi$ , as desired.
- (3) For  $\mu = \pi_t$  we have, also from chapter 9,  $R_\mu(\xi) = t/(1 - \xi)$ , as desired.

(4) For  $\mu = \beta_t$  this follows from the formulae in chapter 11, but the best is to prove directly the last assertion, which generalizes (3,4). With  $\nu = \sum_i c_i \delta_{z_i}$  we have:

$$\begin{aligned}
 R_{\pi_\nu}(\xi) &= \sum_i \frac{c_i z_i}{1 - \xi z_i} \\
 &= \sum_i c_i z_i \sum_{n \geq 0} (\xi z_i)^n \\
 &= \sum_{n \geq 0} \xi^n \sum_i c_i z_i^{n+1} \\
 &= \sum_{n \geq 1} \xi^{n-1} \sum_i c_i z_i^n \\
 &= \sum_{n \geq 1} \xi^{n-1} M_n(\nu)
 \end{aligned}$$

Thus, we are led to the conclusion in the statement.  $\square$

Also as before in the classical case, we can define generalized free cumulants  $\kappa_\pi(a)$  with  $\pi \in P(k)$  by starting with the numeric free cumulants  $k_n(a)$  and by using multiplicativity over blocks, and we have the following result, due to Speicher [78]:

**THEOREM 12.16.** *We have the moment-cumulant formulae*

$$\begin{aligned}
 M_n(a) &= \sum_{\pi \in NC(n)} \kappa_\pi(a) \\
 \kappa_n(a) &= \sum_{\nu \in NC(n)} \mu(\nu, 1_n) M_\nu(a)
 \end{aligned}$$

where  $\mu$  is the Möbius function of  $NC(n)$ .

PROOF. The two formulae in the statement are equivalent, via a Möbius inversion formula similar to the one in Theorem 12.8. Thus, it is enough to prove any of them, and this can be done by starting with Definition 12.13, and doing some standard calculus and combinatorics. For details here, we refer for instance to Nica-Speicher [70].  $\square$

Finally, in what regards more advanced aspects, in relation with the moment formula  $M_k = \sum_{\pi \in D(k)} t^{|\pi|}$ , this ideally requires quantum groups, and more specifically easy quantum groups, and we will talk about this at the end of this chapter. As an advertisement for that material, however, let us record in advance the following statement:

**THEOREM 12.17.** *The free uniform orthogonal easy quantum groups  $G \subset O_N^+$ , and their associated categories of partitions  $D \subset P$ , all coming from subsets  $L \subset \mathbb{N}$ , are*

$$\begin{array}{ccccc}
 B_N^+ & \longrightarrow & O_N^+ & & NC_{12} \longleftarrow NC_2 & & \{1, 2\} \longleftarrow \{2\} \\
 \uparrow & & \uparrow & & \downarrow & & \downarrow \\
 S_N^+ & \longrightarrow & H_N^+ & : & NC \longleftarrow NC_{\text{even}} & : & \mathbb{N} \longleftarrow 2\mathbb{N}
 \end{array}$$

with  $D$  consisting of the partitions  $\pi \in NC$  whose blocks have lengths belonging to  $L \subset \mathbb{N}$ . The free cumulants of the corresponding measures are given by the formula

$$\kappa_n = t\delta_{n \in L}$$

and at the level of moments this gives the formula  $M_k = \sum_{\pi \in D(k)} t^{|\pi|}$ .

PROOF. Obviously, this is something informal, and we will be back to it, with details. However, with the plea of just believing us, the idea is that the easy quantum groups are abstract beasts of type  $S_N^+ \subset G \subset O_N^+$ , coming from categories  $NC_2 \subset D \subset NC$ , and so we are left with an algebraic and probabilistic study of these latter categories, which can be done exactly as in the classical case, and which leads to the above conclusions. More on this in a moment, and in the meantime, we refer to [20] for all this.  $\square$

There are many other things that can be said about free cumulants, and we will come back to this later on, in chapter 14 below, directly in a more general setting, that of the operator-valued free probability theory, following [79], when discussing free de Finetti theorems, which crucially use the free cumulant technology.

Importantly, everything that has been said above about free cumulants, be it a bit technical, is a mirror image of what can be said about classical cumulants. But at a more advanced level, things are far more interesting than this, for instance because of the key isomorphism  $NC(k) \simeq NC_2(2k)$ , that we already met in this book in some other contexts, having no classical counterpart. We will be back to this.

### 12c. The bijection

With the above classical and free cumulant theory in hand, we can now formulate the following simple definition, making the connection between classical and free:

DEFINITION 12.18. *We say that a real probability measure*

$$m \in \mathcal{P}(\mathbb{R})$$

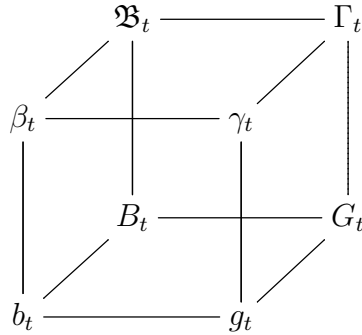
*is the classical version of another measure, called its free version, or liberation*

$$\mu \in \mathcal{P}(\mathbb{R})$$

*when the classical cumulants of  $m$  coincide with the free cumulants of  $\mu$ .*

As a first observation, this definition fits with all the classical and free probability theory developed in the above, in this whole book so far, and notably with the measures from the standard cube, and to start with, we have the following result:

THEOREM 12.19. *In the standard cube of basic probability measures,*



*the upper measures appear as the free versions of the lower measures.*

PROOF. This follows indeed from our various cumulant formulae found above. □

In order to reach now to a more advanced theory, depending this time on a parameter  $t > 0$ , which is something essential, and whose importance will become clear later on, let us formulate, following Bercovici-Pata [23], and the subsequent work in [70]:

DEFINITION 12.20. *A convolution semigroup of measures*

$$\{m_t\}_{t>0} \quad : \quad m_s * m_t = m_{s+t}$$

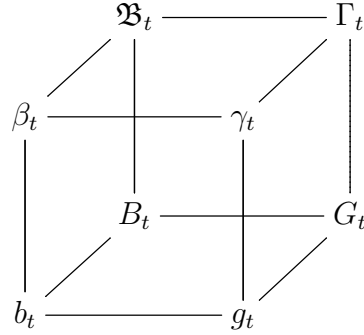
*is in Bercovici-Pata bijection with a free convolution semigroup of measures*

$$\{\mu_t\}_{t>0} \quad : \quad \mu_s \boxplus \mu_t = \mu_{s+t}$$

*when the classical cumulants of  $m_t$  coincide with the free cumulants of  $\mu_t$ .*

As before, this fits with all the theory developed so far in this book, and notably with the measures from the standard cube, and we have the following result:

THEOREM 12.21. *In the standard cube of basic semigroups of measures,*



*the upper semigroups are in Bercovici-Pata bijection with the lower semigroups.*

PROOF. This is a technical improvement of Theorem 12.19, based on the fact that the upper measures in the above diagram form indeed free convolution semigroups, and that the lower measures form indeed classical convolution semigroups, which itself is something that we know well, from the various semigroup results established in above.  $\square$

Back to the examples now, there are many other, and we will be back to this. But, before anything, let us formulate the following surprising result, from [22]:

THEOREM 12.22. *The normal law  $g_1$  is freely infinitely divisible.*

PROOF. This is something tricky, involving all sorts of not very intuitive computations, and for full details here, we refer here to the original paper [22].  $\square$

The above result shows that the normal law  $g_1$  should have a “classical analogue” in the sense of the Bercovici-Pata bijection. And isn’t that puzzling. The problem, however, is thatr this latter law is difficult to compute, and interpret. See [22].

Finally, we refer to [23] for analytic aspects of the Bercovici-Pata bijection.

### 12d. Ground zero

We discuss here the axiomatization and interpretation of the standard cube, using quantum groups. As a starting point, we have the following key definition, from [98]:

DEFINITION 12.23. *A Woronowicz algebra is a  $C^*$ -algebra  $A$ , given with a unitary matrix  $v \in M_N(A)$  whose coefficients generate  $A$ , such that the formulae*

$$\begin{aligned}\Delta(v_{ij}) &= \sum_k v_{ik} \otimes v_{kj} \\ \varepsilon(v_{ij}) &= \delta_{ij} \\ S(v_{ij}) &= v_{ji}^*\end{aligned}$$

*define morphisms of  $C^*$ -algebras  $\Delta : A \rightarrow A \otimes A$ ,  $\varepsilon : A \rightarrow \mathbb{C}$ ,  $S : A \rightarrow A^{opp}$ .*

We say that  $A$  is cocommutative when  $\Sigma\Delta = \Delta$ , where  $\Sigma(a \otimes b) = b \otimes a$  is the flip. We have the following result, which justifies the terminology and axioms:

PROPOSITION 12.24. *The following are Woronowicz algebras:*

(1)  $C(G)$ , with  $G \subset U_N$  compact Lie group. Here the structural maps are:

$$\Delta(\varphi) = (g, h) \rightarrow \varphi(gh)$$

$$\varepsilon(\varphi) = \varphi(1)$$

$$S(\varphi) = g \rightarrow \varphi(g^{-1})$$

(2)  $C^*(\Gamma)$ , with  $F_N \rightarrow \Gamma$  finitely generated group. Here the structural maps are:

$$\Delta(g) = g \otimes g$$

$$\varepsilon(g) = 1$$

$$S(g) = g^{-1}$$

Moreover, we obtain in this way all the commutative/cocommutative algebras.

PROOF. In both cases, we have to indicate a certain matrix  $v$ . For the first assertion, we can use the matrix  $v = (v_{ij})$  formed by matrix coordinates of  $G$ , given by:

$$g = \begin{pmatrix} v_{11}(g) & \dots & v_{1N}(g) \\ \vdots & & \vdots \\ v_{N1}(g) & \dots & v_{NN}(g) \end{pmatrix}$$

As for the second assertion, we can use here the diagonal matrix formed generators:

$$v = \begin{pmatrix} g_1 & & 0 \\ & \ddots & \\ 0 & & g_N \end{pmatrix}$$

Finally, the last assertion follows from the Gelfand theorem, in the commutative case. In the cocommutative case this follows from the Peter-Weyl theory, explained below.  $\square$

In view of Proposition 12.24, we can now formulate the following definition:

DEFINITION 12.25. *Given a Woronowicz algebra  $A$ , we formally write*

$$A = C(G) = C^*(\Gamma)$$

*and call  $G$  compact quantum group, and  $\Gamma$  discrete quantum group.*

When  $A$  is both commutative and cocommutative,  $G$  is a compact abelian group,  $\Gamma$  is a discrete abelian group, and these groups are dual to each other:

$$G = \widehat{\Gamma} \quad , \quad \Gamma = \widehat{G}$$

In general, we still agree to write the formulae  $G = \widehat{\Gamma}, \Gamma = \widehat{G}$ , but in a formal sense.

In general now, the structural maps  $\Delta, \varepsilon, S$  have the following properties:

PROPOSITION 12.26. *Let  $(A, u)$  be a Woronowicz algebra.*

(1)  $\Delta, \varepsilon$  satisfy the usual axioms for a comultiplication and a counit, namely:

$$\begin{aligned} (\Delta \otimes id)\Delta &= (id \otimes \Delta)\Delta \\ (\varepsilon \otimes id)\Delta &= (id \otimes \varepsilon)\Delta = id \end{aligned}$$

(2)  $S$  satisfies the antipode axiom, on the  $*$ -subalgebra generated by entries of  $v$ :

$$m(S \otimes id)\Delta = m(id \otimes S)\Delta = \varepsilon(.)1$$

(3) In addition, the square of the antipode is the identity,  $S^2 = id$ .

PROOF. The two comultiplication axioms follow from:

$$\begin{aligned} (\Delta \otimes id)\Delta(v_{ij}) &= (id \otimes \Delta)\Delta(v_{ij}) = \sum_{kl} v_{ik} \otimes v_{kl} \otimes v_{lj} \\ (\varepsilon \otimes id)\Delta(v_{ij}) &= (id \otimes \varepsilon)\Delta(v_{ij}) = v_{ij} \end{aligned}$$

As for the antipode formulae, the verification here is similar.  $\square$

Summarizing, we have a nice theory of compact and discrete quantum groups going on. As a last ingredient in relation with the formalism and notations, in order to be fully correct, we must complement Definition 12.23 and Definition 12.25 with:

DEFINITION 12.27. *We identify two Woronowicz algebras  $(A, v)$  and  $(B, w)$ , as well as the corresponding quantum groups, when we have an isomorphism of  $*$ -algebras*

$$\langle v_{ij} \rangle \simeq \langle w_{ij} \rangle$$

*mapping standard coordinates to standard coordinates.*

To be more precise, this definition is here in order to avoid amenability issues, as for any compact or discrete quantum group to correspond to a unique Woronowicz algebra. More on this in a moment, when systematically talking about amenability.

Moving ahead, let us call corepresentation of  $A$  any unitary matrix  $u \in M_n(\mathcal{A})$ , where  $\mathcal{A} = \langle v_{ij} \rangle$ , satisfying the same conditions as those satisfied by  $u$ , namely:

$$\begin{aligned} \Delta(u_{ij}) &= \sum_k u_{ik} \otimes u_{kj} \\ \varepsilon(u_{ij}) &= \delta_{ij} \\ S(u_{ij}) &= u_{ji}^* \end{aligned}$$

These corepresentations can be thought of as corresponding to the unitary representations of the underlying compact quantum group  $G$ . As basic examples, we have  $v = (v_{ij})$  itself, its conjugate  $\bar{v} = (v_{ij}^*)$ , as well as any tensor product between  $v, \bar{v}$ .

We have the following key result, due to Woronowicz [98]:

THEOREM 12.28. *Any Woronowicz algebra has a unique Haar integration functional,*

$$\left(\int_G \otimes id\right) \Delta = \left(id \otimes \int_G\right) \Delta = \int_G (\cdot) 1$$

*which can be constructed by starting with any faithful positive form  $\varphi \in A^*$ , and setting*

$$\int_G = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \varphi^{*k}$$

*where  $\phi * \psi = (\phi \otimes \psi)\Delta$ . Moreover, for any corepresentation  $u \in M_n(\mathbb{C}) \otimes A$  we have*

$$\left(id \otimes \int_G\right) u = P$$

*where  $P$  is the orthogonal projection onto  $Fix(u) = \{\xi \in \mathbb{C}^n | u\xi = \xi\}$ .*

PROOF. Following [98], this can be done in 3 standard steps, as follows:

(1) Given  $\varphi \in A^*$ , our claim is that the following limit converges, for any  $a \in A$ :

$$\int_\varphi a = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \varphi^{*k}(a)$$

Indeed, by linearity we can assume that  $a \in A$  is the coefficient of certain corepresentation,  $a = (\tau \otimes id)u$ . But in this case, an elementary computation gives the following formula, with  $P_\varphi$  being the orthogonal projection onto the 1-eigenspace of  $(id \otimes \varphi)u$ :

$$\left(id \otimes \int_\varphi\right) u = P_\varphi$$

(2) Since  $u\xi = \xi$  implies  $[(id \otimes \varphi)u]\xi = \xi$ , we have  $P_\varphi \geq P$ , where  $P$  is the orthogonal projection onto the fixed point space in the statement, namely:

$$Fix(u) = \left\{ \xi \in \mathbb{C}^n \mid u\xi = \xi \right\}$$

The point now is that when  $\varphi \in A^*$  is faithful, by using a standard positivity trick, we can prove that we have  $P_\varphi = P$ , exactly as in the classical case.

(3) With the above formula in hand, the left and right invariance of  $\int_G = \int_\varphi$  is clear on coefficients, and so in general, and this gives all the assertions. See [98].  $\square$

We can now develop, again following [98], the Peter-Weyl theory for the corepresentations of  $A$ . Consider the dense subalgebra  $\mathcal{A} \subset A$  generated by the coefficients of the fundamental corepresentation  $v$ , and endow it with the following scalar product:

$$\langle a, b \rangle = \int_G ab^*$$

With this convention, we have the following result, from [98]:

THEOREM 12.29. *We have the following Peter-Weyl type results:*

- (1) *Any corepresentation decomposes as a sum of irreducible corepresentations.*
- (2) *Each irreducible corepresentation appears inside a certain  $v^{\otimes k}$ .*
- (3)  $\mathcal{A} = \bigoplus_{u \in \text{Irr}(A)} M_{\dim(u)}(\mathbb{C})$ , *the summands being pairwise orthogonal.*
- (4) *The characters of irreducible corepresentations form an orthonormal system.*

PROOF. All these results are from [98], the idea being as follows:

(1) Given  $u \in M_n(A)$ , the intertwiner algebra  $\text{End}(u) = \{T \in M_n(\mathbb{C}) | Tu = uT\}$  is a finite dimensional  $C^*$ -algebra, and so decomposes as  $\text{End}(u) = M_{n_1}(\mathbb{C}) \oplus \dots \oplus M_{n_r}(\mathbb{C})$ . But this gives a decomposition of type  $u = u_1 + \dots + u_r$ , as desired.

(2) Consider the Peter-Weyl corepresentations,  $v^{\otimes k}$  with  $k$  colored integer, defined by  $v^{\otimes \emptyset} = 1$ ,  $v^{\otimes \circ} = v$ ,  $v^{\otimes \bullet} = \bar{v}$  and multiplicativity. The coefficients of these corepresentations span the dense algebra  $\mathcal{A}$ , and by using (1), this gives the result.

(3) Here the direct sum decomposition, which is a  $*$ -coalgebra isomorphism, follows from (2). As for the second assertion, this follows from the fact that  $(\text{id} \otimes \int_G)u$  is the orthogonal projection  $P_u$  onto the space  $\text{Fix}(u)$ , for any corepresentation  $u$ .

(4) Let us define indeed the character of  $u \in M_n(A)$  to be the trace,  $\chi_u = \text{Tr}(u)$ . Since this character is a coefficient of  $u$ , the orthogonality assertion follows from (3). As for the norm 1 claim, this follows once again from  $(\text{id} \otimes \int_G)u = P_u$ .  $\square$

Let us discuss now the basic examples of compact and discrete quantum groups. At the level of the truly “new” examples, following Wang [92], we have:

THEOREM 12.30. *The following universal algebras are Woronowicz algebras,*

$$\begin{aligned} C(O_N^+) &= C^* \left( (v_{ij})_{i,j=1,\dots,N} \middle| v = \bar{v}, v^t = v^{-1} \right) \\ C(U_N^+) &= C^* \left( (v_{ij})_{i,j=1,\dots,N} \middle| v^* = v^{-1}, v^t = \bar{v}^{-1} \right) \end{aligned}$$

*so the underlying spaces  $O_N^+, U_N^+$  are compact quantum groups.*

PROOF. The first assertion follows from the elementary fact that if a matrix  $v = (v_{ij})$  is orthogonal or biunitary, then so must be the following associated matrices:

$$v_{ij}^\Delta = \sum_k v_{ik} \otimes v_{kj} \quad , \quad v_{ij}^\varepsilon = \delta_{ij} \quad , \quad v_{ij}^S = v_{ji}^*$$

Thus, we can define indeed morphisms  $\Delta, \varepsilon, S$  as in Definition 12.23, by using the universality property of the algebras  $C(O_N^+)$ ,  $C(U_N^+)$ .  $\square$

Getting now towards easiness, let us begin with the following definition, from [20]:

DEFINITION 12.31. Let  $P(k, l)$  be the set of partitions between an upper colored integer  $k$ , and a lower colored integer  $l$ . A collection of subsets

$$D = \bigsqcup_{k, l} D(k, l)$$

with  $D(k, l) \subset P(k, l)$  is called a category of partitions when it has the following properties:

- (1) Stability under the horizontal concatenation,  $(\pi, \sigma) \rightarrow [\pi\sigma]$ .
- (2) Stability under vertical concatenation  $(\pi, \sigma) \rightarrow [\pi^\sigma]$ , with matching middle symbols.
- (3) Stability under the upside-down turning  $*$ , with switching of colors,  $\circ \leftrightarrow \bullet$ .
- (4) Each set  $P(k, k)$  contains the identity partition  $|| \dots ||$ .
- (5) The sets  $P(\emptyset, \circ\bullet)$  and  $P(\emptyset, \bullet\circ)$  both contain the semicircle  $\cap$ .

The relation with the quantum groups coming from:

PROPOSITION 12.32. Each partition  $\pi \in P(k, l)$  produces a linear map, given by the following formula, where  $e_1, \dots, e_N$  is the standard basis of  $\mathbb{C}^N$ ,

$$T_\pi(e_{i_1} \otimes \dots \otimes e_{i_k}) = \sum_{j_1 \dots j_l} \delta_\pi \begin{pmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_l \end{pmatrix} e_{j_1} \otimes \dots \otimes e_{j_l}$$

and with the Kronecker type symbols  $\delta_\pi \in \{0, 1\}$  depending on whether the indices fit or not. The assignment  $\pi \rightarrow T_\pi$  is categorical, in the sense that we have

$$T_\pi \otimes T_\sigma = T_{[\pi\sigma]} \quad , \quad T_\pi T_\sigma = N^{c(\pi, \sigma)} T_{[\pi^\sigma]} \quad , \quad T_\pi^* = T_{\pi^*}$$

where  $c(\pi, \sigma)$  are certain integers, coming from the erased components in the middle.

PROOF. This follows from some elementary computations, as in the classical case.  $\square$

In relation with quantum groups, we have the following result, from [20]:

THEOREM 12.33. Each category of partitions  $D = (D(k, l))$  produces a family of compact quantum groups  $G = (G_N)$ , one for each  $N \in \mathbb{N}$ , via the following formula:

$$\text{Hom}(v^{\otimes k}, v^{\otimes l}) = \text{span} \left( T_\pi \Big| \pi \in D(k, l) \right)$$

To be more precise, the spaces on the right form a Tannakian category, and so produce a certain closed subgroup  $G_N \subset U_N^+$ , via the Tannakian duality correspondence.

PROOF. This follows from Woronowicz's Tannakian duality, from [99], best viewed in its "soft" form, from Malacarne's paper [63]. Indeed, let us set:

$$C(k, l) = \text{span} \left( T_\pi \Big| \pi \in D(k, l) \right)$$

By using the axioms in Definition 12.31, and the categorical properties of the operation  $\pi \rightarrow T_\pi$ , from Proposition 12.32, we deduce that  $C = (C(k, l))$  is a Tannakian category. Thus the Tannakian duality applies, and gives the result.  $\square$

Philosophically speaking, the quantum groups appearing in Theorem 12.33 are the simplest from the perspective of Tannakian duality, so let us formulate:

DEFINITION 12.34. A closed subgroup  $G \subset U_N^+$  is called *easy* when we have

$$\text{Hom}(v^{\otimes k}, v^{\otimes l}) = \text{span} \left( T_\pi \mid \pi \in D(k, l) \right)$$

for any colored integers  $k, l$ , for a certain category of partitions  $D \subset P$ .

Getting into examples, we have the following Brauer type result, coming from the work in [6], [8], [10], [13], [29], covering the basic unitary and reflection groups:

THEOREM 12.35. We have quantum unitary and reflection groups as follows,

$$\begin{array}{ccccc} & & K_N^+ & \longrightarrow & U_N^+ \\ & \nearrow & \uparrow & & \nearrow \\ H_N^+ & \longrightarrow & O_N^+ & & \\ \uparrow & & \uparrow & & \uparrow \\ & \nearrow & K_N & \longrightarrow & U_N \\ H_N & \longrightarrow & O_N & & \end{array}$$

which are all easy, the corresponding categories of partitions being as follows:

$$\begin{array}{ccccc} & & \mathcal{NC}_{\text{even}} & \longleftarrow & \mathcal{NC}_2 \\ & \swarrow & \downarrow & & \swarrow \\ \mathcal{NC}_{\text{even}} & \longleftarrow & & \longrightarrow & \mathcal{NC}_2 \\ \downarrow & & \downarrow & & \downarrow \\ & \swarrow & \mathcal{P}_{\text{even}} & \longleftarrow & \mathcal{P}_2 \\ \downarrow & & \downarrow & & \downarrow \\ P_{\text{even}} & \longleftarrow & & \longrightarrow & P_2 \end{array}$$

PROOF. The quantum group  $U_N^+$  is constructed via the following relations:

$$v^* = v^{-1} \quad , \quad v^t = \bar{v}^{-1}$$

Thus, the following operators must be in the associated Tannakian category  $C$ :

$$T_\pi \quad , \quad \pi = \begin{array}{c} \cap \\ \circ \bullet \end{array} \quad , \quad T_\pi \quad , \quad \pi = \begin{array}{c} \cap \\ \bullet \circ \end{array}$$

It follows that the associated Tannakian category is  $C = \text{span}(T_\pi \mid \pi \in D)$ , with:

$$D = \langle \begin{array}{c} \cap \\ \circ \bullet \end{array} , \begin{array}{c} \cap \\ \bullet \circ \end{array} \rangle = \mathcal{NC}_2$$

The results for  $O_N, U_N, O_N^+$  follow in a similar way. Regarding now the quantum groups on the left, their construction and study is standard as well, as follows:

(1) The first observation is that  $S_N$ , regarded as group of permutations of the  $N$  coordinate axes of  $\mathbb{R}^N$ , is a group of orthogonal matrices,  $S_N \subset O_N$ . The corresponding coordinate functions  $v_{ij} : S_N \rightarrow \{0, 1\}$  form a matrix  $v = (v_{ij})$  which is “magic”, in the sense that its entries are projections, summing up to 1 on each row and each column. In fact, by using the Gelfand theorem, we have the following presentation result:

$$C(S_N) = C_{comm}^* \left( (v_{ij})_{i,j=1,\dots,N} \middle| v = \text{magic} \right)$$

(2) Still in relation with the symmetric group  $S_N$ , it is straightforward to check that this group is easy, coming from the category of all the partitions  $P$ . This is indeed well-known, coming from instance from the fact that the magic condition satisfied by  $v = (v_{ij})$  can be reformulated as follows, with  $\mu \in P(2, 1)$  being the fork partition:

$$T_\mu \in \text{Hom}(v^{\otimes 2}, v)$$

Thus, by proceeding as in the proof for  $U_N^+$  given above, we conclude that  $S_N$  is indeed easy, with the associated category of partitions being, as claimed:

$$D = \langle P_2, \mu \rangle = P$$

(3) Based on the above, and following now Wang’s paper [93], we can construct the free analogue  $S_N^+$  of the symmetric group  $S_N$  via the following formula:

$$C(S_N^+) = C^* \left( (v_{ij})_{i,j=1,\dots,N} \middle| v = \text{magic} \right)$$

Here the fact that we have indeed a Woronowicz algebra is standard, exactly as in the proof of Theorem 12.30, because if a matrix  $v = (v_{ij})$  is magic, then so are the matrices  $v^\Delta, v^\varepsilon, v^S$  constructed as there, and this gives the existence of  $\Delta, u, S$ .

(4) Also based on the above, and following now [6], [20], the quantum permutation group  $S_N^+$  follows to be easy, with the corresponding category of partitions being:

$$D = \langle NC_2, \mu \rangle = NC$$

This is indeed clear by following the proof from the classical case, for the group  $S_N$ , with some modifications only needed at the end, when finally computing  $D$ .

(5) Moving ahead now, we can generalize all the above by adding to the picture a parameter  $s \in \mathbb{N} \cup \{\infty\}$ . Consider indeed the group  $H_N^s \subset U_N$  consisting of permutation-like matrices having as entries the  $s$ -th roots of unity, that we already met in chapter 2 above. This group has by definition a wreath product decomposition as follows:

$$H_N^s = \mathbb{Z}_s \wr S_N$$

It is straightforward then to construct a free analogue  $H_N^{s+} \subset U_N^+$  of this group, for instance by formulating a definition as follows, with  $\wr_*$  being a free wreath product:

$$H_N^{s+} = \mathbb{Z}_s \wr_* S_N^+$$

All this is very standard, and we refer here to [8] and related papers.

(6) What happens now is that both the group  $H_N^s$  and the quantum group  $H_N^{s+}$  are easy, coming from the categories  $P^s, NC^s$  or partitions, and noncrossing partitions, which are subject to the following condition, which must happen in each block:

$$\# \circ = \# \bullet (s)$$

(7) In order to finish, besides the case  $s = 1$ , of particular interest are the cases  $s = 2, \infty$ . Here the corresponding groups, that we know from chapter 2, are as follows:

$$H_N = \mathbb{Z}_2 \wr S_N \quad , \quad K_N = \mathbb{T} \wr S_N$$

As for the corresponding quantum groups, from [8], [10], these are denoted as follows:

$$H_N^+ = \mathbb{Z}_2 \wr_* S_N^+ \quad , \quad K_N^+ = \mathbb{T} \wr_* S_N^+$$

In relation with easiness now, the point is that at  $s = 2$  we have:

$$P^2 = P_{\text{even}} \quad , \quad NC^2 = NC_{\text{even}}$$

Also, at  $s = \infty$  we have the following formulae, which are clear too:

$$P^\infty = \mathcal{P}_{\text{even}} \quad , \quad NC^\infty = \mathcal{NC}_{\text{even}}$$

Thus, we are led to the conclusions in the statement. See [8]. □

Let us discuss now probabilistic consequences of the above. We have:

**THEOREM 12.36.** *The truncated characters for the basic quantum groups, namely*

$$\begin{array}{ccccc}
 & & K_N^+ & \longrightarrow & U_N^+ \\
 & \nearrow & \uparrow & & \nearrow \\
 H_N^+ & \longrightarrow & O_N^+ & & \\
 \uparrow & & \uparrow & & \uparrow \\
 & \nearrow & K_N & \longrightarrow & U_N \\
 H_N & \longrightarrow & O_N & & \\
 & \nearrow & & & \nearrow
 \end{array}$$

are with  $N \rightarrow \infty$  the main laws in classical and free probability, namely:

$$\begin{array}{ccccc}
 & & \mathfrak{B}_t & \longrightarrow & \Gamma_t \\
 & \nearrow & \uparrow & & \nearrow \\
 \beta_t & \longrightarrow & \gamma_t & & \\
 \uparrow & & \uparrow & & \uparrow \\
 & \nearrow & B_t & \longrightarrow & G_t \\
 b_t & \longrightarrow & g_t & & \\
 & \nearrow & & & \nearrow
 \end{array}$$

PROOF. We first need linear independence results for the vectors  $\xi_\pi$  associated to the partitions  $\pi \in P(k)$ , and all this comes indeed from the following formula:

$$\det(G_{kN}) = \prod_{\pi \in P(k)} \frac{N!}{(N - |\pi|)!}$$

The point now is that the Gram and Weingarten matrices are asymptotically diagonal, in all cases under consideration, and this gives:

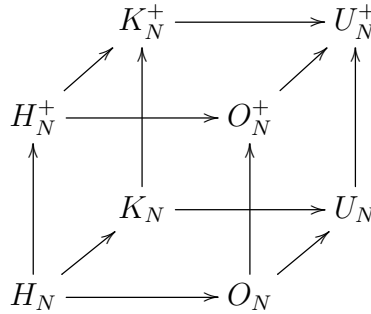
$$\lim_{N \rightarrow \infty} \int_{G_N} \chi_t^k = \sum_{\pi \in D(k)} t^{|\pi|}$$

But this leads to the laws in the statement, via results that we already know.  $\square$

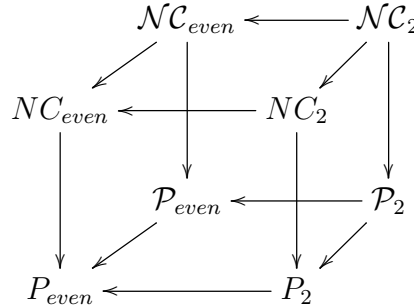
We refer to [20] and related papers for full details on all the above. Also, we refer to [16], [17], [75], [80], [81] for more general theory for the easy quantum groups.

With the above theory developed, let us go now to pure combinatorics, with of course some quantum group ideas in mind. We have here the following result:

THEOREM 12.37 (Ground Zero). *Under a collection of suitable extra assumptions*



*are the unique easy quantum groups. Equivalently, under suitable extra assumptions*



*are the unique categories of partitions.*

PROOF. There is a long story here, first for formulating the precise statement, which is something non-trivial, and then of course for proving it, and we refer here to [6].  $\square$

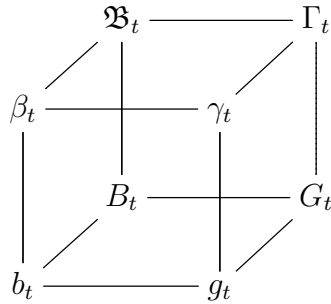
As a conclusion to all this, with some ideas from combinatorics and quantum groups helping, we have managed to axiomatize the main laws in classical and free probability. To be more precise, by bombarding everything with a massive number of suitable axioms, only 8 objects survive, be them quantum groups, categories or partitions, or semigroups of probability measures. And, as a pleasant surprise, these 8 objects form a lovely cube, providing us with some 3D orientation, with the 3 coordinate axes corresponding to the real/complex, discrete/continuous and classical/free dichotomies.

This is certainly interesting, and we have here some clear ground, free of traps and abstractions, that we can build upon. We will discuss this a bit, in what follows.

### 12e. Exercises

There has been a lot of theory in this chapter, and as an exercise here, we have:

EXERCISE 12.38. *Clarify all the details for the standard cube of measures*



*by proving that we have indeed the Bercovici-Pata bijection on the vertical.*

This is something discussed in the above, but with some details missing, and the problem now, which is very instructive, is that of filling all the details.

## Part IV

# Quantum algebra

*Strangers in the night*  
*Exchanging glances*  
*Wandering in the night*  
*What were the chances*

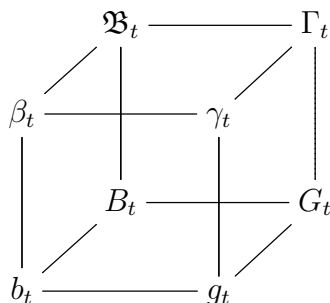
## CHAPTER 13

### Free geometry

#### 13a. Free manifolds

We have seen so far that classical probability has a “twin sister”, which is Voiculescu’s free probability theory. The relation between the two comes from an almost perfect symmetry between the main limiting theorems in both theories. It is even possible to axiomatize the relation between the corresponding limiting laws, and between classical and free probability in general, the main result here being the Bercovici-Pata bijection.

At a more concrete level, passed a few technical manipulations, the main limiting laws are as follows, with the vertical correspondence being the Bercovici-Pata bijection:



All this remains however a bit abstract, because in order to have explicit instances of the various classical and free limiting theorems, we must find in the real life explicit examples of i.i.d. or f.i.d. sequences of random variables, satisfying the needed extra assumptions of that classical and free limiting theorems, and this is not an easy task.

Fortunately, beasts like random matrices and quantum groups are there, providing us with explicit models for the above laws, and for what is going on, in general. Our purpose in this chapter is to further build on the quantum group results, by expanding what we already have into a more general correspondence between classical geometry and free geometry, at the probabilistic level. Then later on, in the next chapters, we will get back to random matrices as well, and we will also add to the picture some further beasts, which are of even more tricky type, namely the Jones subfactors.

In order to get started, let us recall that the above laws appear as laws of truncated characters for the main rotation and reflection groups, the result being follows:

**THEOREM 13.1.** *The main limiting laws in classical and free probability appear from the main quantum rotation and reflection groups, classical and free,*

$$\begin{array}{ccccc}
 & & K_N^+ & \longrightarrow & U_N^+ \\
 & \nearrow & \uparrow & & \nearrow \\
 H_N^+ & \longrightarrow & O_N^+ & & \\
 \uparrow & & \uparrow & & \uparrow \\
 & \nearrow & K_N & \longrightarrow & U_N \\
 H_N & \longrightarrow & O_N & & 
 \end{array}$$

as asymptotic laws, with  $N \rightarrow \infty$ , of the corresponding truncated characters.

**PROOF.** This is something that we know well from chapter 4, and then chapter 12, the idea being that everything comes from easiness, and the Weingarten formula.  $\square$

In order to obtain more instances of the Bercovici-Pata bijection, and why not constructing as well some further, related correspondences between classical and free, a very simple and natural idea, inspired by this, is that of doing “free geometry”. That is, we would like to have free analogues of various classical manifolds that we know, and then compare the probability theory over the classical manifolds, and their free versions.

Here we are a bit vague about what “manifold” should mean, but since we want to integrate over our manifolds, these manifolds should be Riemannian, in some appropriate sense. Also, we know from chapter 6 that the operator algebra theory describes well spaces which are compact, so our manifolds should be compact and Riemannian.

Long story short, these are our goals, and instead of thinking too much, let us just start working, and see later for the philosophy. The simplest compact manifolds that we know are the spheres, and if we want to have free analogues of these spheres, there are not many choices here, the straightforward definition, from [6], being as follows:

**DEFINITION 13.2.** *We have compact quantum spaces, constructed as follows,*

$$C(S_{\mathbb{R},+}^{N-1}) = C^* \left( z_1, \dots, z_N \mid z_i = z_i^*, \sum_i z_i^2 = 1 \right)$$

$$C(S_{\mathbb{C},+}^{N-1}) = C^* \left( z_1, \dots, z_N \mid \sum_i z_i z_i^* = \sum_i z_i^* z_i = 1 \right)$$

called respectively the free real sphere, and the free complex sphere.

Here the  $C^*$  symbols on the right stand for “universal  $C^*$ -algebra generated by”. The fact that such universal  $C^*$ -algebras exist indeed follows by considering the corresponding universal  $*$ -algebras, then completing with respect to the biggest  $C^*$ -norm. Observe that this biggest  $C^*$ -norm exists indeed, because the above quadratic conditions give:

$$\|z_i\|^2 = \|z_i z_i^*\| \leq \left\| \sum_i z_i z_i^* \right\| = 1$$

Given a compact quantum space  $X$ , meaning as usual the abstract space associated to a  $C^*$ -algebra, we define its classical version to be the classical space  $X_{class}$  obtained by dividing  $C(X)$  by its commutator ideal, then applying the Gelfand theorem:

$$C(X_{class}) = C(X)/I \quad : \quad I = \langle [a, b] \rangle$$

Observe that we have an embedding of compact quantum spaces  $X_{class} \subset X$ . In this situation, we also say that  $X$  appears as a “liberation” of  $X$ . We have:

PROPOSITION 13.3. *We have embeddings of compact quantum spaces*

$$\begin{array}{ccc} S_{\mathbb{C}}^{N-1} & \longrightarrow & S_{\mathbb{C},+}^{N-1} \\ \uparrow & & \uparrow \\ S_{\mathbb{R}}^{N-1} & \longrightarrow & S_{\mathbb{R},+}^{N-1} \end{array}$$

and the spaces on the right appear as liberations of the spaces of the left.

PROOF. The embeddings are all clear. For the last assertion, we must establish the following isomorphisms, where  $C_{comm}^*$  stands for “universal commutative  $C^*$ -algebra”:

$$\begin{aligned} C(S_{\mathbb{R}}^{N-1}) &= C_{comm}^* \left( z_1, \dots, z_N \mid z_i = z_i^*, \sum_i z_i^2 = 1 \right) \\ C(S_{\mathbb{C}}^{N-1}) &= C_{comm}^* \left( z_1, \dots, z_N \mid \sum_i z_i z_i^* = \sum_i z_i^* z_i = 1 \right) \end{aligned}$$

But these isomorphisms are both clear, by using the Gelfand theorem.  $\square$

We can now introduce a broad class of compact quantum manifolds, as follows:

DEFINITION 13.4. *A real algebraic submanifold  $X \subset S_{\mathbb{C},+}^{N-1}$  is a closed quantum space defined, at the level of the corresponding  $C^*$ -algebra, by a formula of type*

$$C(X) = C(S_{\mathbb{C},+}^{N-1}) / \left\langle f_i(z_1, \dots, z_N) = 0 \right\rangle$$

for certain noncommutative polynomials  $f_i \in \mathbb{C} \langle X_1, \dots, X_N \rangle$ .

Observe that such manifolds exist indeed, because the free complex spheres themselves exist, and this due to the fact that the quadratic conditions defining them give:

$$||z_i|| \leq 1$$

This estimate, explained before, is something extremely important, and any attempt of further extending Definition 13.4, beyond the sphere level, stumbles into this. There are no such things as free analogues of  $\mathbb{R}^N$  or  $\mathbb{C}^N$ , and the problem comes from this.

In practice now, while our assumption  $X \subset S_{\mathbb{C},+}^{N-1}$  is definitely something technical, we are not losing much when imposing it, and we have the following list of examples:

**THEOREM 13.5.** *The following are algebraic submanifolds  $X \subset S_{\mathbb{C},+}^{N-1}$ :*

- (1) *The spheres  $S_{\mathbb{R}}^{N-1} \subset S_{\mathbb{C}}^{N-1}$ ,  $S_{\mathbb{R},+}^{N-1} \subset S_{\mathbb{C},+}^{N-1}$ .*
- (2) *Any compact Lie group,  $G \subset U_n$ , when  $N = n^2$ .*
- (3) *The duals  $\widehat{\Gamma}$  of finitely generated groups,  $\Gamma = \langle g_1, \dots, g_N \rangle$ .*
- (4) *More generally, the closed quantum groups  $G \subset U_n^+$ , when  $N = n^2$ .*

**PROOF.** These facts are all well-known, the proof being as follows:

- (1) This is indeed true by definition of our various spheres.

(2) Given a closed subgroup  $G \subset U_n$ , we have an embedding  $G \subset S_{\mathbb{C}}^{N-1}$ , with  $N = n^2$ , given in double indices by  $z_{ij} = v_{ij}/\sqrt{n}$ , that we can further compose with the standard embedding  $S_{\mathbb{C}}^{N-1} \subset S_{\mathbb{C},+}^{N-1}$ . As for the fact that we obtain indeed a real algebraic manifold, this is standard too, coming either from Lie theory or from Tannakian duality.

- (3) Given a group  $\Gamma = \langle g_1, \dots, g_N \rangle$ , consider the following variables:

$$z_i = \frac{g_i}{\sqrt{N}}$$

These variables satisfy then the quadratic relations  $\sum_i z_i z_i^* = \sum_i z_i^* z_i = 1$  defining  $S_{\mathbb{C},+}^{N-1}$ , and the algebricity claim for the manifold  $\widehat{\Gamma} \subset S_{\mathbb{C},+}^{N-1}$  is clear.

- (4) Given a closed subgroup  $G \subset U_n^+$ , we have indeed an embedding  $G \subset S_{\mathbb{C},+}^{N-1}$ , with  $N = n^2$ , given in double indices by the following formula:

$$z_{ij} = \frac{v_{ij}}{\sqrt{n}}$$

As for the fact that we obtain indeed in this way a real algebraic manifold, this comes from the Tannakian duality results from [63], [99], explained in chapter 12.  $\square$

Summarizing, we have a broad notion of real algebraic manifold, covering all the examples that we met so far in this book. We will use this notion, in what follows. At the level of the general theory, we have the following version of the Gelfand theorem, which is something very useful, that we will use several times in what follows:

THEOREM 13.6. *Assuming that  $X \subset S_{\mathbb{C},+}^{N-1}$  is an algebraic manifold, given by*

$$C(X) = C(S_{\mathbb{C},+}^{N-1}) / \left\langle f_i(z_1, \dots, z_N) = 0 \right\rangle$$

*for certain noncommutative polynomials  $f_i \in \mathbb{C} \langle X_1, \dots, X_N \rangle$ , we have*

$$X_{class} = \left\{ x \in S_{\mathbb{C}}^{N-1} \mid f_i(z_1, \dots, z_N) = 0 \right\}$$

*and  $X$  itself appears as a liberation of  $X_{class}$ .*

PROOF. This is something that already met, in the context of the free spheres. In general, the proof is similar, by using the Gelfand theorem. Indeed, if we let  $Y \subset S_{\mathbb{C}}^{N-1}$  be the manifold constructed in the statement, then we have a quotient map of  $C^*$ -algebras as follows, mapping standard coordinates to standard coordinates:

$$C(X_{class}) \rightarrow C(Y)$$

Conversely, from  $X \subset S_{\mathbb{C},+}^{N-1}$  we obtain  $X_{class} \subset S_{\mathbb{C}}^{N-1}$ , and since the relations defining  $Y$  are satisfied by  $X_{class}$ , we obtain an inclusion of subspaces  $X_{class} \subset Y$ . Thus, at the level of algebras of continuous functions, we have a quotient map of  $C^*$ -algebras as follows, mapping standard coordinates to standard coordinates:

$$C(Y) \rightarrow C(X_{class})$$

Thus, we have constructed a pair of inverse morphisms, and this finishes the proof.  $\square$

Getting back now to the examples, the above formalism allows us to have a new, more geometric look at the discrete group duals. Let us formulate indeed:

DEFINITION 13.7. *Given a closed subspace  $S \subset S_{\mathbb{C},+}^{N-1}$ , the subspace  $T \subset S$  given by*

$$C(T) = C(S) / \left\langle z_i z_i^* = z_i^* z_i = \frac{1}{N} \right\rangle$$

*is called associated torus. In the real case,  $S \subset S_{\mathbb{R},+}^{N-1}$ , we also call  $T$  cube.*

As a basic example, for  $S = S_{\mathbb{C}}^{N-1}$  the corresponding submanifold  $T \subset S$  appears by imposing the relations  $|z_i| = \frac{1}{\sqrt{N}}$  to the coordinates, so we obtain a torus:

$$S = S_{\mathbb{C}}^{N-1} \implies T = \left\{ z \in \mathbb{C}^N \mid |z_i| = \frac{1}{\sqrt{N}} \right\}$$

As for the case of the real sphere,  $S = S_{\mathbb{R}}^{N-1}$ , here the submanifold  $T \subset S$  appears by imposing the relations  $z_i = \pm \frac{1}{\sqrt{N}}$  to the coordinates, and we obtain a cube:

$$S = S_{\mathbb{R}}^{N-1} \implies T = \left\{ z \in \mathbb{R}^N \mid z_i = \pm \frac{1}{\sqrt{N}} \right\}$$

Observe that we have a relation here with groups, because the complex torus computed above is the group  $\mathbb{T}^N$ , and the cube is the group  $\mathbb{Z}_2^N$ . In fact, we have:

THEOREM 13.8. *The tori of the basic spheres are all group duals, as follows,*

$$\begin{array}{ccc} \mathbb{T}^N & \longrightarrow & \widehat{F_N} \\ \uparrow & & \uparrow \\ \mathbb{Z}_2^N & \longrightarrow & \widehat{\mathbb{Z}_2^{*N}} \end{array}$$

where  $F_N$  is the free group on  $N$  generators, and  $*$  is a group-theoretical free product.

PROOF. In order to prove this result, let us get back to Definition 13.7, and assume that the subspace there  $S \subset S_{\mathbb{C},+}^{N-1}$  is an algebraic manifold, as follows:

$$C(S) = C(S_{\mathbb{C},+}^{N-1}) / \langle f_i(z_1, \dots, z_N) = 0 \rangle$$

In order to get to group algebras, let us rescale the coordinates,  $v_i = z_i / \sqrt{N}$ . Consider as well the corresponding rescalings of the polynomials  $f_i$ , given by:

$$g_i(v_1, \dots, v_N) = f_i(\sqrt{N}v_1, \dots, \sqrt{N}v_N)$$

Since the relations defining  $T \subset S$  from Definition 13.7 correspond to the fact that the rescaled coordinates  $u_i$  must be unitaries, we obtain the following formula:

$$C(T) = C^* \left( v_1, \dots, v_N \mid v_i^* = v_i^{-1}, g_i(v_1, \dots, v_N) = 0 \right)$$

Now in the case of the 4 main spheres, from Proposition 13.3, we obtain from this that the diagram formed by the corresponding algebras  $C(T)$  is as follows:

$$\begin{array}{ccc} C^*(\mathbb{Z}^N) & \longleftarrow & C^*(\mathbb{Z}^{*N}) \\ \downarrow & & \downarrow \\ C^*(\mathbb{Z}_2^N) & \longleftarrow & C^*(\mathbb{Z}_2^{*N}) \end{array}$$

We are therefore led to the conclusion in the statement.  $\square$

All the above is very nice, and not using Hilbert spaces and the GNS theorem, or any kind of advanced mathematics. As a conclusion to these considerations, the Gelfand theorem alone produces out of “nothing”, or at least out of some basic common sense, some potentially interesting mathematics. We will see in what follows that this new mathematics can be useful in relation with our present probability purposes.

As a last piece of abstract theory, based on the above, we can now formulate a “fix” for the functoriality issues of the Gelfand correspondence, as follows:

DEFINITION 13.9. *The category of the real algebraic submanifolds  $X \subset S_{\mathbb{C},+}^{N-1}$  is formed by the compact quantum spaces appearing as follows,*

$$\mathcal{C}(X) = C(S_{\mathbb{C},+}^{N-1}) / \left\langle f_i(z_1, \dots, z_N) = 0 \right\rangle$$

*with  $f_i \in \mathbb{C} \langle X_1, \dots, X_N \rangle$  being noncommutative polynomials, and with the arrows  $X \rightarrow Y$  being the  $*$ -algebra morphisms between the  $*$ -algebras of coordinates*

$$\mathcal{C}(Y) \rightarrow \mathcal{C}(X)$$

*mapping standard coordinates to standard coordinates.*

In other words, what we are doing here is that of formulating a definition for the morphisms between the compact quantum spaces, in the particular case where these compact quantum spaces are algebraic submanifolds of the free complex sphere  $S_{\mathbb{C},+}^{N-1}$ .

The point is that Definition 13.9 works fine for the discrete group duals, which is exactly the point where the Gelfand correspondence was having a “bug”, due to amenability issues, the precise result about the discrete group duals being as follows:

PROPOSITION 13.10. *The category of the finitely generated groups*

$$\Gamma = \langle g_1, \dots, g_N \rangle$$

*with the morphisms mapping generators to generators, embeds contravariantly via*

$$\Gamma \rightarrow \widehat{\Gamma}$$

*into the category of real algebraic submanifolds  $X \subset S_{\mathbb{C},+}^{N-1}$ .*

PROOF. We know from Proposition 13.5 that, given an arbitrary finitely generated group  $\Gamma = \langle g_1, \dots, g_N \rangle$ , we have an embedding  $\widehat{\Gamma} \subset S_{\mathbb{C},+}^{N-1}$  given by:

$$z_i = \frac{g_i}{\sqrt{N}}$$

Now since a morphism of  $*$ -algebras of coordinates  $\mathbb{C}[\Gamma] \rightarrow \mathbb{C}[\Lambda]$  mapping coordinates to coordinates corresponds to a morphism of groups  $\Gamma \rightarrow \Lambda$  mapping generators to generators, our notion of isomorphism is indeed the correct one, as claimed.  $\square$

More generally, Definition 13.9 is compatible with the compact and discrete quantum group conventions from chapter 12, with the compact quantum Lie groups being algebraic manifolds in our sense, and with each quantum group corresponding to a unique algebra. Thus, we have a good, solid axiomatization here, both for the objects and for the arrows, and so a good and broad category, that we can effectively use.

Getting now back to the free spheres and tori, these are related to the quantum rotation and reflection groups, and we have the following result:

THEOREM 13.11. *The spheres and tori associated to the basic quantum groups,*

$$\begin{array}{ccccc}
 & & K_N^+ & \longrightarrow & U_N^+ \\
 & \nearrow & \uparrow & & \nearrow \\
 H_N^+ & \longrightarrow & O_N^+ & & \\
 \uparrow & & \uparrow & & \uparrow \\
 & \nearrow & K_N & \longrightarrow & U_N \\
 H_N & \longrightarrow & O_N & & 
 \end{array}$$

or rather to the corresponding “quantum geometries” are as follows:

$$\begin{array}{ccccc}
 & & \mathbb{T}_N^+ & \longrightarrow & S_{\mathbb{C},+}^{N-1} \\
 & \nearrow & \uparrow & & \nearrow \\
 T_N^+ & \longrightarrow & S_{\mathbb{R},+}^{N-1} & & \\
 \uparrow & & \uparrow & & \uparrow \\
 & \nearrow & \mathbb{T}_N & \longrightarrow & S_{\mathbb{C}}^{N-1} \\
 T_N & \longrightarrow & S_{\mathbb{R}}^{N-1} & & 
 \end{array}$$

PROOF. This statement, as formulated, is something a bit informal, but it is possible to have it fully explained and justified, and we will not attempt to explain things in detail here. Instead, we refer to book [6], and the related literature.  $\square$

In relation now with probability, we have:

THEOREM 13.12. *The various classical and free spheres and tori, namely*

$$\begin{array}{ccccc}
 & & \mathbb{T}_N^+ & \longrightarrow & S_{\mathbb{C},+}^{N-1} \\
 & \nearrow & \uparrow & & \nearrow \\
 T_N^+ & \longrightarrow & S_{\mathbb{R},+}^{N-1} & & \\
 \uparrow & & \uparrow & & \uparrow \\
 & \nearrow & \mathbb{T}_N & \longrightarrow & S_{\mathbb{C}}^{N-1} \\
 T_N & \longrightarrow & S_{\mathbb{R}}^{N-1} & & 
 \end{array}$$

all have integration functionals, that can be computed via Weingarten type formulae.

PROOF. Again, this statement as formulated is something a bit informal, and for full details, we refer to [6] and the related literature, the idea being as follows:

(1) In what regards the spheres, the idea is that, a bit like in the classical case, the free spheres appear as homogeneous spaces over the corresponding quantum groups, and so the Weingarten formula for the quantum groups applies by restriction to them.

(2) As for the tori, here the integration is something very simple, because we are dealing with group duals, but by using the picture in Theorem 13.11, it is possible to write as well a Weingarten formula for them as well, if we really want to.  $\square$

More generally, it is possible to integrate over suitable homogeneous spaces of type  $G/H$ , and this unifies the Weingarten integration results for the quantum groups and for the spheres. Again, we refer here to [6] and the related literature.

As a final comment here, there is a relation of all the above with the noncommutative geometry of Connes [37], at least at the level of the general philosophy, because both what we are doing and what Connes is doing is based on the following two ideas:

(1) The noncommutative manifolds should come from operator algebras. Here there is perfect agreement between Connes and us, with our motivations coming from quantum mechanics, and more specifically from the point of view of von Neumann [91].

(2) The noncommutative manifolds should be Riemannian. Again, we are in perfect agreement here with Connes, with our motivations coming from the fact that, in order to do some serious mathematics, we would like to integrate on our manifolds.

In practice now, passed these two ideas which are both very good and healthy, and that we surely share with Connes, having learned them from him, there are several ways of doing things, and axiomatizing noncommutative geometry. At the level of the main examples, Connes was mostly motivated by crossed products, deformations, and other manifolds which are finally not that far from classical geometry, and his axiomatization is something which is very close too to the classical geometry. In what concerns us, the main examples that we have in mind are the free manifolds as above, which are quite far from the classical world, and to which the Connes axiomatization does not apply.

Long story short, there is a bit of controversy here, and if you are into free probability, as we strongly believe, as a reader of the present book, of course stay with us.

### 13b. Meixner laws

Going back now to the Bercovici-Pata bijection, generally speaking, this bijection should be thought of as being something happening in the  $N \rightarrow \infty$  limit. When  $N \in \mathbb{N}$  is fixed the situation is more complicated, and we have here many alternative correspondences, coming from quantum groups, or random matrices, which are not obviously related to the Bercovici-Pata bijection, and sometimes are “orthogonal” to it.

Our claim is that we can recover some of these interesting correspondences by using our noncommutative geometry picture. This is certainly true for the main examples of the Bercovici-Pata bijection, with Theorem 13.1 being now solidly incorporated into our noncommutative geometry program, coming from Theorem 13.11.

So, our claim now is that much more can be done, along these lines. All this is quite long and technical, and we will only discuss below a few selected topics. As a first, famous example for our claim above, we have case of the Meixner laws. The result here, making the link with geometry, and stated a bit informally, is as follows:

**THEOREM 13.13.** *We have a bijection between the Meixner and free Meixner laws, which appear from the liberation operation for discrete groups*

$$\mathbb{Z}^{\times N} \rightarrow \mathbb{Z}^{*N}$$

*by looking at the dual groups, or quantum tori, which are as follows,*

$$\mathbb{T}_N \rightarrow \mathbb{T}_N^+$$

*and then at the laws of the corresponding main characters.*

**PROOF.** This is something standard, based on the noncommutative geometry picture coming from Theorem 13.11. To be more precise, the truncated characters for the tori  $T = \widehat{\Gamma}$ , with  $\Gamma = \langle g_1, \dots, g_N \rangle$  being a discrete group, are as follows:

$$\chi_t = g_1 + \dots + g_{[tN]}$$

Thus, according to the definition of the Meixner laws, in the classical case we obtain the Meixner laws, and in the free case we obtain the free Meixner laws, as stated.  $\square$

There are many other things that can be said about the correspondence between Meixner laws and free Meixner laws, sometimes of technical probabilistic nature, going beyond the above geometric picture, and we refer here to the literature on the subject, a good reference here, to start with, being the paper of Anshelevich [2].

Moving ahead now, by using the same geometric picture coming from Theorem 13.11, it is possible to talk as well about free hyperspherical laws, as follows:

**THEOREM 13.14.** *We have a bijection between hyperspherical and free hyperspherical laws, which appear from the liberation operation for real spheres*

$$S_{\mathbb{R}}^{N-1} \rightarrow S_{\mathbb{R},+}^{N-1}$$

*as well as from the liberation operation for complex spheres*

$$S_{\mathbb{C}}^{N-1} \rightarrow S_{\mathbb{C},+}^{N-1}$$

*by looking at the laws of the coordinates.*

**PROOF.** This is something standard, based on the geometric picture coming from Theorem 13.11. In practice, there are many other things that can be said about this correspondence, and we will be back to it in a moment, with a detailed study.  $\square$

Finally, it is possible to talk as well about hypergeometric and free hypergeometric laws, with a number of quite surprising results, the idea being as follows:

**THEOREM 13.15.** *We have a bijection between hypergeometric and free hypergeometric laws, which appear from the liberation operation for permutation groups*

$$S_N \rightarrow S_N^+$$

*and in the free case, the free hypergeometric laws are in fact related to the free hyperspherical laws, via a subtle twisting procedure, having no classical counterpart.*

**PROOF.** Again, in what concerns the generalities, and more specifically the first assertion, this is something standard, based on the geometric picture coming from Theorem 13.11. As for explicit computations, and also for the second assertion, which is something non-trivial, we will be back to this in a moment, with a detailed study. Let us just mention here that the reasons behind the latter correspondence in the statement comes from the following remarkable identification, having no classical counterpart:

$$NC_2(2k) \simeq NC(k)$$

As already mentioned, more on this later, when systematically studying this.  $\square$

Summarizing, our noncommutative geometry picture is something quite successful, enabling us to go well beyond the Bercovici-Pata results from Theorem 13.1, with results about the Meixner laws, and with potentially interesting results on the hyperspherical and hypergeometric laws. We will explore all this, in the remainder of this chapter.

### 13c. Hyperspherical laws

We discuss here the classical and free hyperspherical laws. In the classical case, we will need the following result, that we know well from chapter 1:

PROPOSITION 13.16. *The spherical integral of  $z_{i_1} \dots z_{i_k}$  vanishes, unless each index  $a \in \{1, \dots, N\}$  appears an even number of times in the sequence  $i_1, \dots, i_k$ . We have*

$$\int_{S_{\mathbb{R}}^{N-1}} z_{i_1} \dots z_{i_k} dz = \frac{(N-1)!! l_1!! \dots l_N!!}{(N + \sum l_i - 1)!!}$$

with  $l_a$  being this number of occurrences.

PROOF. As a first observation, the result holds indeed at  $N = 2$ , due to the following formula, from chapter 1, where  $\varepsilon(p) = 1$  when  $p$  is even, and  $\varepsilon(p) = 0$  when  $p$  is odd:

$$\int_0^{\pi/2} \cos^p t \sin^q t dt = \left(\frac{\pi}{2}\right)^{\varepsilon(p)\varepsilon(q)} \frac{p!!q!!}{(p+q+1)!!}$$

In general, when  $N \in \mathbb{N}$  is arbitrary, the result follows by using spherical coordinates and Fubini, which reduces everything to the case  $N = 2$ , and so to the above formula. We refer to chapter 1 for details on all this.  $\square$

In connection now with our probabilistic questions, we have:

THEOREM 13.17. *The even moments of the hyperspherical variables are*

$$\int_{S_{\mathbb{R}}^{N-1}} z_i^k dx = \frac{(N-1)!!k!!}{(N+k-1)!!}$$

and the variables  $y_i = z_i/\sqrt{N}$  become normal and independent with  $N \rightarrow \infty$ .

PROOF. The moment formula in the statement follows from Proposition 13.16. Now observe that with  $N \rightarrow \infty$  we have the following estimate:

$$\begin{aligned} \int_{S_{\mathbb{R}}^{N-1}} z_i^k dz &= \frac{(N-1)!!}{(N+k-1)!!} \times k!! \\ &\simeq N^{k/2} \times k!! \\ &= N^{k/2} M_k(g_1) \end{aligned}$$

Thus we have, as claimed, the following asymptotic formula:

$$\frac{z_i}{\sqrt{N}} \sim g_1$$

Finally, the independence assertion follows as well from the formula in Proposition 13.16, via standard probability theory. Again, we refer here to chapter 1.  $\square$

In the case of the free real sphere now, the computations are substantially more complicated than those in the classical case. Let us start with the following result, that we basically know from the above, and that we will recall now:

THEOREM 13.18. *For the free sphere  $S_{\mathbb{R},+}^{N-1}$ , the rescaled coordinates*

$$y_i = \sqrt{N} z_i$$

*become semicircular and free, in the  $N \rightarrow \infty$  limit.*

PROOF. As explained in the above, the Weingarten formula for the free sphere, together with the standard fact that the Gram matrix, and hence the Weingarten matrix too, is asymptotically diagonal, gives the following estimate:

$$\int_{S_{\mathbb{R},+}^{N-1}} z_{i_1} \dots z_{i_k} dz \simeq N^{-k/2} \sum_{\sigma \in NC_2(k)} \delta_\sigma(i_1, \dots, i_k)$$

With this formula in hand, we can compute the asymptotic moments of each coordinate  $x_i$ . Indeed, by setting  $i_1 = \dots = i_k = i$ , all Kronecker symbols are 1, and we obtain:

$$\int_{S_{\mathbb{R},+}^{N-1}} z_i^k dz \simeq N^{-k/2} |NC_2(k)|$$

Thus the rescaled coordinates  $y_i = \sqrt{N} z_i$  become semicircular in the  $N \rightarrow \infty$  limit, as claimed. As for the asymptotic freeness result, this follows as well from the above general joint moment estimate, via standard free probability theory. See [13], [6].  $\square$

Summarizing, we have good results for the free sphere, with  $N \rightarrow \infty$ . The problem now, which is non-trivial, is that of computing the moments of the coordinates of the free sphere at fixed values of  $N \in \mathbb{N}$ . The answer here, from [15], which is based on advanced quantum group techniques, that we will briefly explain here, is as follows:

THEOREM 13.19. *The moments of the free hyperspherical law are given by*

$$\int_{S_{\mathbb{R},+}^{N-1}} z_1^{2l} = \frac{1}{(N+1)^l} \cdot \frac{q+1}{q-1} \cdot \frac{1}{l+1} \sum_{r=-l-1}^{l+1} (-1)^r \binom{2l+2}{l+r+1} \frac{r}{1+q^r}$$

where  $q \in [-1, 0)$  is such that  $q + q^{-1} = -N$ .

PROOF. The idea is that  $z_1 \in C(S_{\mathbb{R},+}^{N-1})$  has the same law as  $v_{11} \in C(O_N^+)$ , which has the same law as a certain variable  $w \in C(SU_2^q)$ , which can be modelled by an explicit operator on  $l^2(\mathbb{N})$ , and whose law can be computed by using advanced calculus.

Let us first explain the relation between  $O_N^+$  and  $SU_2^q$ . To any matrix  $F \in GL_N(\mathbb{R})$  satisfying  $F^2 = 1$  we associate the following universal algebra:

$$C(O_F^+) = C^* \left( (v_{ij})_{i,j=1,\dots,N} \middle| v = F \bar{v} F = \text{unitary} \right)$$

Observe that we have  $O_{I_N}^+ = O_N^+$ . In general, the above algebra satisfies Woronowicz's generalized axioms in [98], which do not include the antipode axiom  $S^2 = id$ .

At  $N = 2$  now, up to a trivial equivalence relation on the matrices  $F$ , and on the quantum groups  $O_F^+$ , we can assume that  $F$  is as follows, with  $q \in [-1, 0)$ :

$$F = \begin{pmatrix} 0 & \sqrt{-q} \\ 1/\sqrt{-q} & 0 \end{pmatrix}$$

Our claim is that for this matrix we have the following formula:

$$O_F^+ = SU_2^q$$

Indeed, the relations  $v = F\bar{v}F$  tell us that  $v$  must be of the following form:

$$v = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$$

Thus  $C(O_F^+)$  is the universal algebra generated by two elements  $\alpha, \gamma$ , with the relations making the above matrix  $v$  a unitary. But these unitarity conditions are:

$$\alpha\gamma = q\gamma\alpha$$

$$\alpha\gamma^* = q\gamma^*\alpha$$

$$\gamma\gamma^* = \gamma^*\gamma$$

$$\alpha^*\alpha + \gamma^*\gamma = 1$$

$$\alpha\alpha^* + q^2\gamma\gamma^* = 1$$

We recognize here the relations in [98] defining the algebra  $C(SU_2^q)$ , and it follows that we have an isomorphism of Hopf algebras, as follows:

$$C(O_F^+) \simeq C(SU_2^q)$$

Now back to the general case, where  $F \in GL_N(\mathbb{R})$  satisfies  $F^2 = 1$ , let us try to understand the integration over  $O_F^+$ . Given  $\pi \in NC_2(2k)$  and  $i = (i_1, \dots, i_{2k})$ , we set:

$$\delta_\pi^F(i) = \prod_{s \in \pi} F_{i_{s_l} i_{s_r}}$$

Here the product is over all the strings  $s = \{s_l \curvearrowright s_r\}$  of  $\pi$ . Our claim is that the following family of vectors, with  $\pi \in NC_2(2k)$ , spans the space of fixed vectors of  $v^{\otimes 2k}$ :

$$\xi_\pi = \sum_i \delta_\pi^F(i) e_{i_1} \otimes \dots \otimes e_{i_{2k}}$$

Indeed, having  $\xi_\pi$  fixed by  $v^{\otimes 2}$  is equivalent to assuming that  $v = F\bar{v}F$  is unitary. By using now these vectors, as in [13], we obtain the following Weingarten formula:

$$\int_{O_F^+} v_{i_1 j_1} \dots v_{i_{2k} j_{2k}} = \sum_{\pi \sigma} \delta_\pi^F(i) \delta_\sigma^F(j) W_{kN}(\pi, \sigma)$$

With these preliminaries in hand, we can now start the computation that we are interested in. Let  $N \in \mathbb{N}$ , and consider the number  $q \in [-1, 0)$  satisfying:

$$q + q^{-1} = -N$$

Our claim is that we have the following formula:

$$\int_{O_N^+} \varphi(\sqrt{N+2} v_{ij}) = \int_{SU_2^q} \varphi(\alpha + \alpha^* + \gamma - q\gamma^*)$$

Indeed, according to the above, the moments of the variable on the left are given by:

$$\int_{O_N^+} v_{ij}^{2k} = \sum_{\pi\sigma} W_{kN}(\pi, \sigma)$$

On the other hand, the moments of the variable on the right, which in terms of the fundamental corepresentation  $u = (u_{ij})$  is given by  $w = \sum_{ij} u_{ij}$ , are as follows:

$$\int_{SU_2^q} w^{2k} = \sum_{ij} \sum_{\pi\sigma} \delta_\pi^F(i) \delta_\sigma^F(j) W_{kN}(\pi, \sigma)$$

We deduce that  $w/\sqrt{N+2}$  has the same moments as  $v_{ij}$ , which proves our claim.

In order to do now the computation over  $SU_2^q$ , we can use a well-known matrix model, due to Woronowicz [98], where the standard generators  $\alpha, \gamma$  are mapped as follows:

$$\begin{aligned} \pi_u(\alpha)e_k &= \sqrt{1 - q^{2k}}e_{k-1} \\ \pi_u(\gamma)e_k &= uq^k e_k \end{aligned}$$

Here  $u \in \mathbb{T}$  is a parameter, and  $(e_k)$  is the standard basis of  $l^2(\mathbb{N})$ . The point with this representation is that it allows the computation of the Haar functional. Indeed, if  $D$  is the diagonal operator given by  $D(e_k) = q^{2k}e_k$ , then we have the following formula:

$$\int_{SU_2^q} x = (1 - q^2) \int_{\mathbb{T}} \text{tr}(D\pi_u(x)) \frac{du}{2\pi i u}$$

With the above explicit model in hand, we conclude that the law of the variable that we are interested in is subject to the following formula:

$$\int_{SU_2^q} \varphi(\alpha + \alpha^* + \gamma - q\gamma^*) = (1 - q^2) \int_{\mathbb{T}} \text{tr}(D\varphi(M)) \frac{du}{2\pi i u}$$

To be more precise, this formula holds indeed, with  $M$  being as follows:

$$M(e_k) = e_{k+1} + q^k(u - qu^{-1})e_k + (1 - q^{2k})e_{k-1}$$

The point now is that the integral on the right in the above can be computed, by using advanced calculus methods, and this gives the result. We refer here to [15].  $\square$

The computation of the joint free hyperspherical laws remains an open problem. Open as well is the question of finding a more conceptual proof for the above formula.

### 13d. Hypergeometric laws

Following now [11], let us discuss a remarkable relation of all this with the quantum permutations, and with the free hypergeometric laws. The idea will be that of working out some abstract algebraic results, regarding twists of quantum automorphism groups, which will particularize into results relating quantum rotations and permutations, having no classical counterpart, both at the algebraic and the probabilistic level.

In order to explain this material, from [11], which is quite technical, requiring good algebraic knowledge, let us begin with some generalities. We first have:

**DEFINITION 13.20.** *A finite quantum space  $X$  is the abstract dual of a finite dimensional  $C^*$ -algebra  $B$ , according to the following formula:*

$$C(X) = B$$

*The number of elements of such a space is  $|X| = \dim B$ . By decomposing the algebra  $B$ , we have a formula of the following type:*

$$C(X) = M_{n_1}(\mathbb{C}) \oplus \dots \oplus M_{n_k}(\mathbb{C})$$

*With  $n_1 = \dots = n_k = 1$  we obtain in this way the space  $X = \{1, \dots, k\}$ . Also, when  $k = 1$  the equation is  $C(X) = M_n(\mathbb{C})$ , and the solution will be denoted  $X = M_n$ .*

We endow each finite quantum space  $x$  with its counting measure, corresponding as the algebraic level to the integration functional obtained by applying the regular representation, and then the unique normalized trace of the matrix algebra  $\mathcal{L}(C(X))$ :

$$tr : C(X) \subset \mathcal{L}(C(X)) \rightarrow \mathbb{C}$$

Now if we denote by  $\mu, \eta$  the multiplication and unit map of the algebra  $C(X)$ , we have the following standard result, from [6]:

**PROPOSITION 13.21.** *Consider a linear map  $\Phi : C(X) \rightarrow C(X) \otimes C(G)$ , written as*

$$\Phi(e_i) = \sum_j e_j \otimes v_{ji}$$

*with  $\{e_i\}$  being a linear space basis of  $C(X)$ , orthonormal with respect to  $tr$ .*

- (1)  $\Phi$  is a linear space coaction  $\iff v$  is a corepresentation.
- (2)  $\Phi$  is multiplicative  $\iff \mu \in Hom(v^{\otimes 2}, v)$ .
- (3)  $\Phi$  is unital  $\iff \eta \in Hom(1, v)$ .
- (4)  $\Phi$  leaves invariant  $tr$   $\iff \eta \in Hom(1, v^*)$ .
- (5) If these conditions hold,  $\Phi$  is involutive  $\iff v$  is unitary.

**PROOF.** This is a bit similar to the proof for  $S_N^+$  from chapter 12, via some routine computations, and for full details here, we refer to [6].  $\square$

Again following [6], we have the following result, extending the basic theory of  $S_N^+$  from chapter 12 to the present finite quantum space setting:

**THEOREM 13.22.** *Given a finite quantum space  $X$ , there is a universal compact quantum group  $S_X^+$  acting on  $X$ , leaving the counting measure invariant. We have*

$$C(S_X^+) = C(U_N^+) / \left\langle \mu \in \text{Hom}(v^{\otimes 2}, v), \eta \in \text{Fix}(v) \right\rangle$$

where  $N = |X|$  and where  $\mu, \eta$  are the multiplication and unit maps of  $C(X)$ . Also:

- (1) For  $X = \{1, \dots, N\}$  we have  $S_X^+ = S_N^+$ .
- (2) For  $X = M_n$  we have  $S_X^+ = PO_n^+ = PU_n^+$ .

**PROOF.** Consider a linear map  $\Phi : C(X) \rightarrow C(X) \otimes C(G)$ , written as follows, with  $\{e_i\}$  being a linear space basis of the algebra  $C(X)$ , orthonormal with respect to  $tr$ :

$$\Phi(e_j) = \sum_i e_i \otimes v_{ij}$$

Then  $\Phi$  is a coaction precisely when  $v$  is a unitary corepresentation, satisfying:

$$\mu \in \text{Hom}(v^{\otimes 2}, v) \quad , \quad \eta \in \text{Fix}(v)$$

But this gives the first assertion. Regarding now the statement about  $X = \{1, \dots, N\}$ , this is clear. Finally, regarding  $X = M_2$ , here we have embeddings as followss:

$$PO_n^+ \subset PU_n^+ \subset S_X^+$$

Now since the fusion rules of all these 3 quantum groups are known to be the same as the fusion rules for  $SO_3$ , these inclusions follow to be isomorphisms. See [6].  $\square$

We have as well the following result, also from [6]:

**THEOREM 13.23.** *The quantum groups  $S_X^+$  have the following properties:*

- (1) *The associated Tannakian categories are  $TL(N)$ , with  $N = |X|$ .*
- (2) *The main character follows the Marchenko-Pastur law  $\pi_1$ , when  $N \geq 4$ .*
- (3) *The fusion rules for  $S_X^+$  with  $|F| \geq 4$  are the same as for  $SO_3$ .*

**PROOF.** This result is from [6], the idea being as follows:

(1) Our first claim is that the fundamental representation is equivalent to its adjoint,  $v \sim \bar{v}$ . Indeed, let us go back to the coaction formula in Proposition 13.21, namely:

$$\Phi(e_i) = \sum_j e_j \otimes v_{ji}$$

We can pick our orthogonal basis  $\{e_i\}$  to be the standard multimatrix basis of  $C(X)$ , so that we have, for a certain involution  $i \rightarrow i^*$  on the index set:

$$e_i^* = e_{i^*}$$

With this convention made, by conjugating the formula of  $\Phi(e_i)$ , we obtain:

$$\Phi(e_{i^*}) = \sum_j e_{j^*} \otimes v_{ji}^*$$

Now by interchanging  $i \leftrightarrow i^*$  and  $j \leftrightarrow j^*$ , this latter formula becomes:

$$\Phi(e_i) = \sum_j e_j \otimes v_{j^*i^*}^*$$

We therefore conclude, by comparing with the original formula, that we have:

$$v_{ji}^* = v_{j^*i^*}$$

But this shows that we have an equivalence of corepresentations as follows, as claimed:

$$v \sim \bar{v}$$

Now with this in hand, the proof goes as for the proof for  $S_N^+$ . To be more precise, the result follows from the fact that the multiplication and unit of any complex algebra, and in particular of  $C(X)$ , can be modelled by the following two diagrams:

$$m = |\cup| \quad , \quad u = \cap$$

(2) The proof here is as for  $S_N^+$ , by using moments. To be more precise, according to (1) these moments are the Catalan numbers, which are the moments of  $\pi_1$ .

(3) Once again same proof as for  $S_N^+$ , by using the fact that the moments of  $\chi$  are the Catalan numbers, which lead to the Clebsch-Gordan rules. See [6].  $\square$

Let us discuss now a number of more advanced twisting aspects, which will eventually lead us into probability, and hypergeometric laws. Following [11], we have:

**PROPOSITION 13.24.** *Given a finite group  $G$ , the algebra  $C(S_G^+)$  is isomorphic to the algebra presented by generators  $w_{gh}$  with  $g, h \in G$ , with the following relations:*

$$w_{1g} = w_{g1} = \delta_{1g} \quad , \quad w_{s,gh} = \sum_{t \in G} w_{st^{-1},g} w_{th} \quad , \quad w_{gh,s} = \sum_{t \in G} w_{gt^{-1}} w_{h,ts}$$

*The comultiplication, counit and antipode are given by the formulae*

$$\Delta(w_{gh}) = \sum_{s \in G} w_{gs} \otimes w_{sh} \quad , \quad \varepsilon(w_{gh}) = \delta_{gh} \quad , \quad S(w_{gh}) = w_{h^{-1}g^{-1}}$$

*on the standard generators  $x_{gh}$ .*

**PROOF.** This follows indeed from a direct verification, based either on Theorem 13.22, or on its equivalent formulation from Wang's paper [93].  $\square$

Let us discuss now the twisted version of the above result. Consider a 2-cocycle on our group  $G$ , which is by definition a map  $\sigma : G \times G \rightarrow \mathbb{C}^*$  satisfying:

$$\sigma_{gh,s}\sigma_{gh} = \sigma_{g,hs}\sigma_{hs} \quad , \quad \sigma_{g1} = \sigma_{1g} = 1$$

Given such a cocycle, we can construct the associated twisted group algebra  $C(\widehat{G}_\sigma)$ , as being the vector space  $C(\widehat{G}) = C^*(G)$ , with product defined as follows:

$$e_g e_h = \sigma_{gh} e_{gh}$$

We have then the following generalization of Proposition 13.24:

**PROPOSITION 13.25.** *The algebra  $C(S_{\widehat{G}_\sigma}^+)$  is isomorphic to the abstract algebra presented by generators  $w_{gh}$  with  $g, h \in G$ , with the relations  $w_{1g} = w_{g1} = \delta_{1g}$  and:*

$$\begin{aligned} \sigma_{gh} w_{s,gh} &= \sum_{t \in G} \sigma_{st^{-1},t} w_{st^{-1},g} w_{th} \\ \sigma_{gh}^{-1} w_{gh,s} &= \sum_{t \in G} \sigma_{t^{-1},ts}^{-1} w_{gt^{-1}} w_{h,ts} \end{aligned}$$

*The comultiplication, counit and antipode are given by the formulae*

$$\Delta(w_{gh}) = \sum_{s \in G} w_{gs} \otimes w_{sh}$$

$$\varepsilon(w_{gh}) = \delta_{gh}$$

$$S(w_{gh}) = \sigma_{h^{-1}h} \sigma_{g^{-1}g}^{-1} w_{h^{-1}g^{-1}}$$

*on the standard generators  $w_{gh}$ .*

**PROOF.** Once again, this follows from a direct verification. Note that by using the cocycle identities we obtain the following formula, needed in the proof:

$$\sigma_{gg^{-1}} = \sigma_{g^{-1}g}$$

Thus we are led to the conclusion in the statement.  $\square$

Now let  $H$  be an arbitrary Hopf algebra. We recall that a left 2-cocycle on  $H$  is a convolution invertible linear map  $\sigma : H \otimes H \rightarrow \mathbb{C}$  satisfying the following conditions:

$$\sigma_{x_1 y_1} \sigma_{x_2 y_2, z} = \sigma_{y_1 z_1} \sigma_{x, y_2 z_2} \quad , \quad \sigma_{x1} = \sigma_{1x} = \varepsilon(x)$$

Given a left 2-cocycle  $\sigma$  on  $H$ , we can form the 2-cocycle twist  $H^\sigma$  as follows. As a coalgebra,  $H^\sigma = H$ , and an element  $x \in H$ , when considered in  $H^\sigma$ , is denoted  $[x]$ . The product in  $H^\sigma$  is defined, in Sweedler notation, by the following formula:

$$[x][y] = \sum \sigma_{x_1 y_1} \sigma_{x_3 y_3}^{-1} [x_2 y_2]$$

With these conventions, following [11], we have the following result:

THEOREM 13.26. *If  $G$  is a finite group and  $\sigma$  is a 2-cocycle on  $G$ , the Hopf algebras*

$$C(S_{\widehat{G}}^+) \quad , \quad C(S_{\widehat{G}_\sigma}^+)$$

*are 2-cocycle twists of each other, in the above sense.*

PROOF. In order to prove this result, we use the following Hopf algebra map:

$$\pi : C(S_{\widehat{G}}^+) \rightarrow C(\widehat{G}) \quad , \quad w_{gh} \rightarrow \delta_{gh} e_g$$

Our 2-cocycle  $\sigma : G \times G \rightarrow \mathbb{C}^*$  can be extended by linearity into a linear map as follows, which is both a left and right 2-cocycle in the above sense:

$$\sigma : C(\widehat{G}) \otimes C(\widehat{G}) \rightarrow \mathbb{C}$$

Consider now the following composition of maps:

$$\alpha = \sigma(\pi \otimes \pi) : C(S_{\widehat{G}}^+) \otimes C(S_{\widehat{G}}^+) \rightarrow C(\widehat{G}) \otimes C(\widehat{G}) \rightarrow \mathbb{C}$$

Then  $\alpha$  is a left and right 2-cocycle, because it is induced by a cocycle on a group algebra, and so its convolution inverse is  $\alpha^{-1}$ . Thus we can construct the twisted algebra  $C(S_{\widehat{G}}^+)^{\alpha^{-1}}$ , and inside this algebra we have the following computation:

$$\begin{aligned} [w_{gh}][w_{rs}] &= \alpha^{-1}(w_g, w_r) \alpha(w_h, w_s) [w_{gh} w_{rs}] \\ &= \sigma_{gr}^{-1} \sigma_{hs} [w_{gh} w_{rs}] \end{aligned}$$

By using this, we obtain next the following formula:

$$\begin{aligned} \sum_{t \in G} \sigma_{st^{-1}, t} [w_{st^{-1}, g}] [w_{th}] &= \sum_{t \in G} \sigma_{st^{-1}, t} \sigma_{st^{-1}, t}^{-1} \sigma_{gh} [w_{st^{-1}, g} w_{th}] \\ &= \sigma_{gh} [w_{s, gh}] \end{aligned}$$

Similarly, we have the following formula, obtained in the same way:

$$\sum_{t \in G} \sigma_{t^{-1}, ts}^{-1} [w_{g, t^{-1}}] [w_{h, ts}] = \sigma_{gh}^{-1} [w_{gh, s}]$$

We deduce from the above formulae that we have a Hopf algebra map, as follows:

$$\Phi : C(S_{\widehat{G}_\sigma}^+) \rightarrow C(S_{\widehat{G}}^+)^{\alpha^{-1}} \quad , \quad w_{gh} \rightarrow [w_{g, h}]$$

But this map is clearly surjective, and is injective as well, as desired, by a standard fusion semiring argument, because both Hopf algebras have the same fusion semiring.  $\square$

Let us discuss now some concrete applications of the above general result. Consider the group  $G = \mathbb{Z}_n^2$ , let  $w = e^{2\pi i/n}$ , and consider the following map:

$$\sigma : G \times G \rightarrow \mathbb{C}^* \quad , \quad \sigma_{(ij)(kl)} = w^{jk}$$

Then  $\sigma$  is a bicharacter, and hence a 2-cocycle on  $G$ . Thus, we can apply our general twisting result, to this situation. We obtain a concrete result, from [11], as follows:

THEOREM 13.27. *Let  $n \geq 2$  and  $w = e^{2\pi i/n}$ . Then the formula*

$$\Theta(u_{ij}u_{kl}) = \frac{1}{n} \sum_{ab=0}^{n-1} w^{-a(k-i)+b(l-j)} p_{ia,jb}$$

*defines a coalgebra isomorphism, as follows*

$$C(PO_n^+) \rightarrow C(S_{n^2}^+)$$

*which commutes with the corresponding Haar integrals.*

PROOF. This follows indeed from our general twisting result from Theorem 13.26, by using as ingredients the group and the cocycle indicated above.  $\square$

As a probabilistic consequence now, which is of interest for us, we have:

THEOREM 13.28. *The following families of variables have the same joint law,*

- (1)  $\{v_{ij}^2\} \in C(O_n^+)$ ,
- (2)  $\{\eta_{ij} = \frac{1}{n} \sum_{ab} p_{ia,jb}\} \in C(S_{n^2}^+)$ ,

*where  $v = (v_{ij})$  and  $p = (p_{ia,jb})$  are the corresponding fundamental corepresentations.*

PROOF. This follows indeed from Theorem 13.27, because the variables in the statement are in correspondence, via the correspondence established there. An alternative approach is by using the Weingarten formula for our two quantum groups, and the shrinking operation  $\pi \rightarrow \pi'$ . Indeed, we obtain the following moment formulae:

$$\begin{aligned} \int_{O_n^+} v_{ij}^{2k} &= \sum_{\pi, \sigma \in NC_2(2k)} W_{2k,n}(\pi, \sigma) \\ \int_{S_{n^2}^+} \eta_{ij}^k &= \sum_{\pi, \sigma \in NC_2(2k)} n^{|\pi'|+|\sigma'|-k} W_{k,n^2}(\pi', \sigma') \end{aligned}$$

But by doing some standard combinatorics, the summands coincide, and so the moments are equal, as desired. The proof for joint moments is similar. See [11].  $\square$

In order to derive now some explicit consequences from the above, let us formulate:

DEFINITION 13.29. *The noncommutative random variable*

$$\eta_{nmN} = \sum_{i=1}^n \sum_{j=1}^m v_{ij} \in C(S_N^+)$$

*is called free hypergeometric, of parameters  $(n, m, N)$ .*

The terminology comes from the fact that the variable  $\eta'_{nmN}$ , defined as above, but over the algebra  $C(S_N)$ , follows a hypergeometric law of parameters  $(n, m, N)$ . Now back to Theorem 13.28, this has as consequence the following quite surprising result, which is of purely free probabilistic nature, having no classical counterpart:

THEOREM 13.30. *The free hypergeometric variable*

$$\eta_{ij} = \frac{1}{n} \sum_{a,b=1}^n u_{ia,jb} \in C(S_{n^2}^+)$$

*has the same law as the squared free hyperspherical variable, namely:*

$$z_i^2 \in C(S_{\mathbb{R},+}^{N-1})$$

PROOF. This follows indeed from Theorem 13.28, particularized to the case of single variables. For details on all this, and for more, we refer to [11].  $\square$

As a conclusion to all this, very interesting things when doing noncommutative geometry, on one hand with explicit models for all the basic instances of the Bercovici-Pata bijection, on the other hand with some new bijections, such as the Meixner/free Meixner one, and on the other other hand some new phenomena, which are of purely free nature, such as the above one, relating the free hypergeometric and free hyperspherical laws.

Needless to say, all this is of interest in relation with physics. For instance in the Connes interpretation of the Standard Model, coming from [37], the probabilistic study of the corresponding free gauge group leads to beasts as above. Also, it is believed that QCD should appear as some kind of “free electrodynamics”, with free geometry and free probability playing a key role in its study, although all this is not confirmed yet.

### 13e. Exercises

Things have been quite advanced in this chapter, and as a unique exercise, which is rather elementary, and very instructive, we have:

EXERCISE 13.31. *Prove that the free hypergeometric laws are as follows:*

- (1) *Let  $n, m, N \rightarrow \infty$ , with  $\frac{nm}{N} \rightarrow t \in (0, \infty)$ . Then the law of*

$$\eta_{nmN}$$

*converges to Marchenko-Pastur law  $\pi_t$ .*

- (2) *Let  $n, m, N \rightarrow \infty$ , with  $\frac{n}{N} \rightarrow \nu \in (0, 1)$  and  $\frac{m}{N} \rightarrow 0$ . Then the law of*

$$\frac{\eta_{nmN} - m\nu}{\sqrt{m\nu(1-\nu)}}$$

*converges to the semicircle law  $\gamma_1$ .*

Here the computations are quite standard, and very instructive. In case you are stuck with something, all this is done in [11], so read and write a brief account of that.

## CHAPTER 14

### Invariance questions

#### 14a. Invariance questions

An interesting question, which often appears in theoretical probability, as well in connection with certain questions coming from physics, is the study of the sequences of random variables  $x_1, x_2, x_3, \dots \in L^\infty(X)$  which are exchangeable, in the sense that their joint distribution is invariant under the infinite permutations  $\sigma \in S_\infty$ :

$$\mu_{x_1, x_2, x_3, \dots} = \mu_{x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, \dots}$$

This question is solved by the classical De Finetti theorem, which basically says that the variables  $x_1, x_2, x_3, \dots$  must be i.i.d., in some asymptotic sense. We will see a precise statement of this theorem, along with a complete proof, in a minute.

The De Finetti theorem has many generalizations. One can replace for instance the action of the group  $S_\infty = \cup_N S_N$  by the action of the bigger group  $O_\infty = \cup_N O_N$ , and the sequences  $x_1, x_2, x_3, \dots \in L^\infty(X)$  which are invariant in this stronger sense, which are called “rotatable”, can be characterized as well, via a De Finetti type theorem.

All this is interesting for us, in connection with what we have been doing so far, in this book. On one hand the groups  $S_N, O_N$  are easy, and we would like to understand how the above-mentioned De Finetti theorems, involving  $S_N, O_N$ , as well as their various technical generalizations, follow from the easiness property of  $S_N, O_N$ . On the other hand, we would like to understand as well what happens for  $S_N^+, O_N^+$ .

Long story short, we would like to discuss here probabilistic invariance questions with respect to the basic quantum permutation and rotation groups, namely:

$$\begin{array}{ccc} S_N^+ & \longrightarrow & O_N^+ \\ \uparrow & & \uparrow \\ S_N & \longrightarrow & O_N \end{array}$$

As a second objective, in tune with what we have been doing so far in this book, we would like as well to understand what happens to the invariance questions with respect

to the basic quantum reflection and rotation groups, from our beloved cube, namely:

$$\begin{array}{ccccc}
 & & K_N^+ & \longrightarrow & U_N^+ \\
 & \nearrow & \uparrow & & \nearrow \\
 H_N^+ & \longrightarrow & O_N^+ & & \\
 \uparrow & & \uparrow & & \uparrow \\
 & \nearrow & K_N & \longrightarrow & U_N \\
 H_N & \longrightarrow & O_N & & \nearrow
 \end{array}$$

We will discuss here most of these questions, following the classical theory of the De Finetti theorem, then the foundational paper of K\"ostler and Speicher [61], in the free case, and then the more advanced paper [17], deling with both the classical and free De Finetti theorems, and their other easy quantum group generalizations.

Let us start by fixing some notations. In order to deal with our first question above, we will use here the formalism of the orthogonal quantum groups, which best covers the main quantum groups that we are interested in. We first have the following definition:

DEFINITION 14.1. *Given a closed subgroup  $G \subset O_N^+$ , we denote by*

$$\begin{aligned}
 \alpha : \mathbb{C} \langle t_1, \dots, t_N \rangle &\rightarrow \mathbb{C} \langle t_1, \dots, t_N \rangle \otimes C(G) \\
 t_i &\rightarrow \sum_j t_j \otimes v_{ji}
 \end{aligned}$$

*the standard coaction of  $C(G)$  on the free complex algebra on  $N$  variables.*

Observe that the map  $\alpha$  constructed above is indeed a coaction, in the sense that it satisfies the following standard coassociativity and counitality conditions:

$$\begin{aligned}
 (id \otimes \Delta)\alpha &= (\alpha \otimes id)\alpha \\
 (id \otimes \varepsilon)\alpha &= id
 \end{aligned}$$

With the above notion of coaction in hand, we can now talk about invariant sequences of classical or noncommutative random variables, in the following way:

DEFINITION 14.2. *Let  $(B, tr)$  be a  $C^*$ -algebra with a trace, and  $x_1, \dots, x_N \in B$ . We say that  $x = (x_1, \dots, x_N)$  is invariant under  $G \subset O_N^+$  if the distribution functional*

$$\begin{aligned}
 \mu_x : \mathbb{C} \langle t_1, \dots, t_N \rangle &\rightarrow \mathbb{C} \\
 P &\rightarrow tr(P(x_1, \dots, x_N))
 \end{aligned}$$

*is invariant under the coaction  $\alpha$ , in the sense that we have*

$$(\mu_x \otimes id)\alpha(P) = \mu_x(P)$$

*for any noncommuting polynomial  $P \in \mathbb{C} \langle t_1, \dots, t_N \rangle$ .*

In the classical case, where  $G \subset O_N$  is a usual group, we recover in this way the usual invariance notion from classical probability. In the general case, where  $G \subset O_N^+$  is arbitrary, what we have is a natural generalization of this. For further comments on all this, including examples, and motivations too, we refer to [17], [38], [39], [61], [62].

We have the following equivalent formulation of the above invariance condition:

**PROPOSITION 14.3.** *Let  $(B, tr)$  be a  $C^*$ -algebra with a trace, and  $x_1, \dots, x_N \in B$ . Then  $x = (x_1, \dots, x_N)$  is invariant under  $G \subset O_N^+$  precisely when*

$$tr(x_{i_1} \dots x_{i_k}) = \sum_{j_1 \dots j_k} tr(x_{j_1} \dots x_{j_k}) v_{j_1 i_1} \dots v_{j_k i_k}$$

as an equality in  $C(G)$ , for any  $k \in \mathbb{N}$ , and any  $i_1, \dots, i_k \in \{1, \dots, N\}$ .

**PROOF.** By linearity, in order for a sequence  $x = (x_1, \dots, x_N)$  to be  $G$ -invariant in the sense of Definition 14.2, the formula there must be satisfied for any noncommuting monomial  $P \in \mathbb{C} \langle t_1, \dots, t_N \rangle$ . But an arbitrary such monomial can be written as follows, for a certain  $k \in \mathbb{N}$ , and certain indices  $i_1, \dots, i_k \in \{1, \dots, N\}$ :

$$P = t_{i_1} \dots t_{i_k}$$

Now with this formula for  $P$  in hand, we have the following computation:

$$\begin{aligned} (\mu_x \otimes id)\alpha(P) &= (\mu_x \otimes id) \sum_{j_1, \dots, j_k} t_{j_1} \dots t_{j_k} \otimes v_{j_1 i_1} \dots v_{j_k i_k} \\ &= \sum_{j_1, \dots, j_k} \mu_x(t_{j_1} \dots t_{j_k}) v_{j_1 i_1} \dots v_{j_k i_k} \\ &= \sum_{j_1 \dots j_k} tr(x_{j_1} \dots x_{j_k}) v_{j_1 i_1} \dots v_{j_k i_k} \end{aligned}$$

On the other hand, by definition of the distribution  $\mu_x$ , we have:

$$\mu_x(P) = \mu_x(t_{i_1} \dots t_{i_k}) = tr(x_{i_1} \dots x_{i_k})$$

Thus, we are led to the conclusion in the statement.  $\square$

As already mentioned after Definition 14.2, in the classical case, where  $G \subset O_N$  is a usual compact group, our notion of  $G$ -invariance coincides with the usual  $G$ -invariance notion from classical probability. We have in fact the following result:

**PROPOSITION 14.4.** *In the classical group case,  $G \subset O_N$ , a sequence  $(x_1, \dots, x_N)$  is  $G$ -invariant in the above sense if and only if*

$$tr(x_{i_1} \dots x_{i_k}) = \sum_{j_1 \dots j_k} g_{j_1 i_1} \dots g_{j_k i_k} tr(x_{j_1} \dots x_{j_k})$$

for any  $k \in \mathbb{N}$ , any  $i_1, \dots, i_k \in \{1, \dots, N\}$ , and any  $g = (g_{ij}) \in G$ , and this coincides with the usual notion of  $G$ -invariance for a sequence of classical random variables.

PROOF. According to Proposition 14.3, the invariance property happens precisely when we have the following equality, for any  $k \in \mathbb{N}$ , and any  $i_1, \dots, i_k \in \{1, \dots, N\}$ :

$$\text{tr}(x_{i_1} \dots x_{i_k}) = \sum_{j_1 \dots j_k} \text{tr}(x_{j_1} \dots x_{j_k}) v_{j_1 i_1} \dots v_{j_k i_k}$$

Now by evaluating both sides of this equation at a given  $g \in G$ , we obtain:

$$\text{tr}(x_{i_1} \dots x_{i_k}) = \sum_{j_1 \dots j_k} g_{j_1 i_1} \dots g_{j_k i_k} \text{tr}(x_{j_1} \dots x_{j_k})$$

Thus, we are led to the conclusion in the statement.  $\square$

Summarizing, what we have so far is a general notion of probabilistic invariance, generalizing the classical notions of exchangeability and rotatability, than we can use for reformulating the classical De Finetti problematics, and its various generalizations.

In order to formulate De Finetti type theorems, that we can try to prove afterwards, we are still in need of a few pieces of general theory. Indeed, in the classical De Finetti theorem, the independence occurs after conditioning. Likewise, we can expect the free De Finetti theorem to be a statement about freeness with amalgamation.

Both these concepts may be expressed in terms of operator-valued probability theory, that we will recall now. There are many things to be said here, and in what follows we will mainly present the main definitions and theorems, with some brief explanations. Following Speicher's paper [79], we first have the following definition:

DEFINITION 14.5. *An operator-valued probability space consists of:*

- (1) *A unital algebra  $A$ .*
- (2) *A unital subalgebra  $B \subset A$ .*
- (3) *An expectation  $E : A \rightarrow B$ , which must be unital,  $E(1) = 1$ , and satisfying*

$$E(b_1 a b_2) = b_1 E(a) b_2$$

*for any  $a \in A$ , and any  $b_1, b_2 \in B$ .*

As a basic example, which motivates the whole theory, we have the case where  $A = L^\infty(X)$  is a usual algebra of classical random variables, and  $B = L^\infty(Y)$  is a subalgebra. Here the expectation  $E : A \rightarrow B$  is the usual one from classical probability.

Given an operator-valued probability space as above, the joint distribution of a family of variables  $(x_i)_{i \in I}$  in the algebra  $A$  is by definition the following functional:

$$\begin{aligned} \mu_x : B < (t_i)_{i \in I} > \rightarrow B \\ P &\rightarrow E(P(x)) \end{aligned}$$

We refer to Speicher's paper [79] and related papers for more on all this, general results and examples, in relation with the operator-valued probability theory.

Next in line, we have the following key definition, also from [79]:

DEFINITION 14.6. *Let  $(A, B, E)$  be as above, and  $(x_i)_{i \in I}$  be a family of variables.*

- (1) *These variables are called independent if the following algebra is commutative*

$$\langle B, (x_i)_{i \in I} \rangle \subset A$$

*and for  $i_1, \dots, i_k \in I$  distinct and  $P_1, \dots, P_k \in B \langle t \rangle$ , we have:*

$$E(P_1(x_{i_1}) \dots P_k(x_{i_k})) = E(P_1(x_{i_1})) \dots E(P_k(x_{i_k}))$$

- (2) *These variables are called free if for any  $i_1, \dots, i_k \in I$  such that  $i_l \neq i_{l+1}$ , and any  $P_1, \dots, P_k \in B \langle t \rangle$  such that  $E(P_l(x_{i_l})) = 0$ , we have:*

$$E(P_1(x_{i_1}) \dots P_k(x_{i_k})) = 0$$

The above notions are straightforward extensions of the usual notions of independence and freeness, that we discussed in chapter 9, which correspond to the case  $B = \mathbb{C}$ .

As in the scalar case,  $B = \mathbb{C}$ , in order to deal with invariance questions, we will need the theory of classical and free cumulants, in the present setting. Let us start with:

DEFINITION 14.7. *Let  $(A, B, E)$  be an operator-valued probability space.*

- (1) *A  $B$ -functional is a  $N$ -linear map  $\rho : A^N \rightarrow B$  such that:*

$$\rho(b_0 a_1 b_1, a_2 b_2 \dots, a_N b_N) = b_0 \rho(a_1, b_1 a_2, \dots, b_{N-1} a_N) b_N$$

*Equivalently,  $\rho$  is a linear map of the following type*

$$A^{\otimes_B N} \rightarrow B$$

*where the tensor product is taken with respect to the natural  $B - B$  bimodule structure on the algebra  $A$ .*

- (2) *Suppose that  $B$  is commutative. For  $k \in \mathbb{N}$  let  $\rho^{(k)}$  be a  $B$ -functional. Given  $\pi \in P(n)$ , we define a  $B$ -functional  $\rho^{(\pi)} : A^N \rightarrow B$  by the formula*

$$\rho^{(\pi)}(a_1, \dots, a_N) = \prod_{V \in \pi} \rho(V)(a_1, \dots, a_N)$$

*where if  $V = (i_1 < \dots < i_s)$  is a block of  $\pi$  then:*

$$\rho(V)(a_1, \dots, a_N) = \rho_s(a_{i_1}, \dots, a_{i_s})$$

As before with the notions of independence and freeness, these are classical extensions of the notions that we discussed in chapter 12 above. See [79].

When  $B$  is not commutative, there is no natural order in which to compute the product appearing in the above formula for  $\rho^{(\pi)}$ . However, the nesting property of the noncrossing partitions allows for a natural definition of  $\rho^{(\pi)}$  for  $\pi \in NC(N)$ , which we now recall:

DEFINITION 14.8. For  $k \in \mathbb{N}$  let  $\rho^{(k)} : A^k \rightarrow B$  be a  $B$ -functional. Given  $\pi \in NC(N)$ , define a  $B$ -functional  $\rho^{(\pi)} : A^N \rightarrow B$  recursively as follows:

- (1) If  $\pi = 1_N$  is the partition having one block, define  $\rho^{(\pi)} = \rho^{(N)}$ .
- (2) Otherwise, let  $V = \{l+1, \dots, l+s\}$  be an interval of  $\pi$  and define:

$$\rho^{(\pi)}(a_1, \dots, a_N) = \rho^{(\pi-V)}(a_1, \dots, a_l \rho^{(s)}(a_{l+1}, \dots, a_{l+s}), a_{l+s+1}, \dots, a_N)$$

As before, we refer to [70], [79] and related work for more on all this.

Finally, we have the following definition:

DEFINITION 14.9. Let  $(x_i)_{i \in I}$  be a family of random variables in  $A$ .

- (1) The operator-valued classical cumulants  $c_E^{(k)} : A^k \rightarrow B$  are the  $B$ -functionals defined by the following classical moment-cumulant formula:

$$E(a_1 \dots a_N) = \sum_{\pi \in P(N)} c_E^{(\pi)}(a_1, \dots, a_N)$$

- (2) The operator-valued free cumulants  $\kappa_E^{(k)} : A^k \rightarrow B$  are the  $B$ -functionals defined by the following free moment-cumulant formula:

$$E(a_1, \dots, a_N) = \sum_{\pi \in NC(N)} \kappa_E^{(\pi)}(a_1, \dots, a_N)$$

As basic illustrations here, in the scalar case, where the subalgebra is  $B = \mathbb{C}$ , we recover in this way the classical and free cumulants, as discussed in chapter 12 above. In general, we refer to [79] for more on the above notions.

We have the following result, which is well-known in the classical case, due to Rota, and which in the free case is due to Speicher [79]:

THEOREM 14.10. Let  $(x_i)_{i \in I}$  a family of random variables in  $A$ .

- (1) If the algebra  $\langle B, (x_i)_{i \in I} \rangle$  is commutative, then  $(x_i)_{i \in I}$  are conditionally independent given  $B$  if and only if when there are  $1 \leq k, l \leq N$  such that  $i_k \neq i_l$ :

$$c_E^{(N)}(b_0 x_{i_1} b_1, \dots, x_{i_N} b_N) = 0$$

- (2) The variables  $(x_i)_{i \in I}$  are free with amalgamation over  $B$  if and only if when there are  $1 \leq k, l \leq N$  such that  $i_k \neq i_l$ :

$$\kappa_E^{(N)}(b_0 x_{i_1} b_1, \dots, x_{i_N} b_N) = 0$$

PROOF. As a first observation, the condition in (1) is equivalent to the statement that if  $\pi \in P(N)$ , then the following happens, unless  $\pi \leq \ker i$ :

$$c_E^{(\pi)}(b_0 x_{i_1} b_1, \dots, x_{i_N} b_N) = 0$$

Similarly, the condition (2) above is equivalent to the statement that if  $\pi \in NC(N)$ , then the following happens, unless  $\pi \leq \ker i$ :

$$\kappa_E^{(\pi)}(b_0 x_{i_1} b_1, \dots, x_{i_N} b_N) = 0$$

Observe also that in the case  $B = \mathbb{C}$  we obtain the usual notions of independence and freeness. In general now, the proof is via standard combinatorics, following the proof from the case  $B = \mathbb{C}$ , and as before, we refer to [70], [79] for more on all this.  $\square$

Stronger characterizations of the joint distribution of  $(x_i)_{i \in I}$  can be given by specifying what types of partitions may contribute to the nonzero cumulants.

To be more precise, we have here the following result, also from [79]:

**THEOREM 14.11.** *Let  $(x_i)_{i \in I}$  be a family of random variables in  $A$ .*

- (1) *Suppose that  $\langle B, (x_i)_{i \in I} \rangle$  is commutative. The  $B$ -valued joint distribution of  $(x_i)_{i \in I}$  is independent for  $D = P$  and independent centered Gaussian for  $D = P_2$  if and only if, for any  $\pi \in P(N)$ , unless  $\pi \in D(N)$  and  $\pi \leq \ker i$ :*

$$c_E^{(\pi)}(b_0 x_{i_1} b_1, \dots, x_{i_N} b_N) = 0$$

- (2) *The  $B$ -valued joint distribution of  $(x_i)_{i \in I}$  is freely independent for  $D = NC$  and freely independent centered semicircular for  $D = NC_2$  if and only if, for any  $\pi \in NC(N)$ , unless  $\pi \in D(N)$  and  $\pi \leq \ker i$ :*

$$\kappa_E^{(\pi)}(b_0 x_{i_1} b_1, \dots, x_{i_N} b_N) = 0$$

**PROOF.** These results are indeed well-known, coming from the definition of the classical and free cumulants, in the present setting, via some combinatorics. See [79].  $\square$

Finally, here is one more basic result that we will need:

**THEOREM 14.12.** *Let  $(x_i)_{i \in I}$  be a family of random variables. Define the  $B$ -valued moment functionals  $E^{(N)}$  by the following formula:*

$$E^{(N)}(a_1, \dots, a_N) = E(a_1 \dots a_N)$$

- (1) *If  $B$  is commutative, then for any  $\sigma \in P(N)$  and  $a_1, \dots, a_N \in A$  we have:*

$$c_E^{(\sigma)}(a_1, \dots, a_N) = \sum_{\pi \in P(N), \pi \leq \sigma} \mu_{P(N)}(\pi, \sigma) E^{(\pi)}(a_1, \dots, a_N)$$

- (2) *For any  $\sigma \in NC(N)$  and  $a_1, \dots, a_N \in A$  we have:*

$$\kappa_E^{(\sigma)}(a_1, \dots, a_N) = \sum_{\pi \in NC(N), \pi \leq \sigma} \mu_{NC(N)}(\pi, \sigma) E^{(\pi)}(a_1, \dots, a_N)$$

**PROOF.** This follows indeed from the Möbius inversion formula. See [70], [79].  $\square$

This was the general operator-valued free probability theory that we will need, in what follows. For the detailed proofs, examples and comments on all the above, and for more operator-valued free probability in general, we refer to [70], [79].

### 14b. Reverse De Finetti

With the above ingredients in hand, we can now investigate invariance questions for the sequences of classical or noncommutative random variables, with respect to the main quantum permutation and rotation groups that we are interested in here, namely:

$$\begin{array}{ccc} S_N^+ & \longrightarrow & O_N^+ \\ \uparrow & & \uparrow \\ S_N & \longrightarrow & O_N \end{array}$$

To be more precise, we first have a reverse De Finetti theorem, from [17], as follows:

**THEOREM 14.13.** *Let  $(x_1, \dots, x_N)$  be a sequence in  $A$ .*

- (1) *If  $x_1, \dots, x_N$  are freely independent and identically distributed with amalgamation over  $B$ , then the sequence is  $S_N^+$ -invariant.*
- (2) *If  $x_1, \dots, x_N$  are freely independent and identically distributed with amalgamation over  $B$ , and have centered semicircular distributions with respect to  $E$ , then the sequence is  $O_N^+$ -invariant.*
- (3) *If  $\langle B, x_1, \dots, x_N \rangle$  is commutative and  $x_1, \dots, x_N$  are conditionally independent and identically distributed given  $B$ , then the sequence is  $S_N$ -invariant.*
- (4) *If  $\langle x_1, \dots, x_N \rangle$  is commutative and  $x_1, \dots, x_N$  are conditionally independent and identically distributed given  $B$ , and have centered Gaussian distributions with respect to  $E$ , then the sequence is  $O_N$ -invariant.*

**PROOF.** Assume that the joint distribution of  $(x_1, \dots, x_N)$  satisfies one of the conditions in the statement, and let  $D$  be the category of partitions associated to the corresponding easy quantum group. We have then the following computation:

$$\begin{aligned} \sum_{j_1 \dots j_k} \text{tr}(x_{j_1} \dots x_{j_k}) v_{j_1 i_1} \dots v_{j_k i_k} &= \sum_{j_1 \dots j_k} \text{tr}(E(x_{j_1} \dots x_{j_k})) v_{j_1 i_1} \dots v_{j_k i_k} \\ &= \sum_{j_1 \dots j_k} \sum_{\pi \leq \ker j} \text{tr}(\xi_E^{(\pi)}(x_1, \dots, x_1)) v_{j_1 i_1} \dots v_{j_k i_k} \\ &= \sum_{\pi \in D(k)} \text{tr}(\xi_E^{(\pi)}(x_1, \dots, x_1)) \sum_{\ker j \geq \pi} v_{j_1 i_1} \dots v_{j_k i_k} \end{aligned}$$

Here  $\xi$  denotes the free and classical cumulants in the cases (1,2) and (3,4) respectively. On the other hand, it follows from a direct computation that if  $\pi \in D(k)$  then we have

the following formula, in each of the 4 cases in the statement:

$$\sum_{\ker j \geq \pi} v_{j_1 i_1} \cdots v_{j_k i_k} = \begin{cases} 1 & \text{if } \pi \leq \ker i \\ 0 & \text{otherwise} \end{cases}$$

By using this formula, we can finish our computation, in the following way:

$$\begin{aligned} \sum_{j_1 \dots j_k} \text{tr}(x_{j_1} \dots x_{j_k}) v_{j_1 i_1} \dots v_{j_k i_k} &= \sum_{\pi \in D(k)} \text{tr}(\xi_E^{(\pi)}(x_1, \dots, x_1)) \delta_{\pi \leq \ker i} \\ &= \sum_{\pi \leq \ker i} \text{tr}(\xi_E^{(\pi)}(x_1, \dots, x_1)) \\ &= \text{tr}(x_{i_1} \dots x_{i_k}) \end{aligned}$$

Thus, we are led to the conclusions in the statement.  $\square$

Summarizing, we have so far a reverse De Finetti theorem, for the various quantum groups that we are interested in here. Our goal in what follows will be that of proving the corresponding De Finetti theorems, which are converse to the above theorem.

This will be something quite technical, getting us, among others, into certain technical aspects of the Weingarten integration and combinatorics.

Let us begin with some technical results, in view to establish the above-mentioned converse De Finetti theorems. We will use the following standard fact:

**PROPOSITION 14.14.** *Assume that a sequence  $(x_1, \dots, x_N)$  is  $G$ -invariant. Then there is a coaction*

$$\tilde{\alpha} : M_N(\mathbb{C}) \rightarrow M_N(\mathbb{C}) \otimes C(G)$$

*determined by the following formula:*

$$\tilde{\alpha}(p(x)) = (ev_x \otimes \pi_N) \alpha(p)$$

*Moreover, the fixed point algebra of  $\tilde{\alpha}$  is the  $G$ -invariant subalgebra  $B_N$ .*

**PROOF.** This follows indeed after identifying the GNS representation of the algebra  $\mathbb{C} \langle t_1, \dots, t_N \rangle$  for the state  $\mu_x$  with the morphism  $ev_x : \mathbb{C} \langle t_1, \dots, t_N \rangle \rightarrow M_N(\mathbb{C})$ .  $\square$

In order to further advance, we use the fact that there is a natural conditional expectation given by integrating the coaction  $\tilde{\alpha}$  with respect to the Haar state, as follows:

$$\begin{aligned} E_N : M_N(\mathbb{C}) &\rightarrow B_N \\ E_N(m) &= \left( id \otimes \int_G \right) \tilde{\alpha}(m) \end{aligned}$$

The point now is that by using the Weingarten formula, we can give a simple combinatorial formula for the moment functionals with respect to  $E_N$ , in the case where  $G$  is one of the easy quantum groups under consideration.

To be more precise, we have the following result, from [17]:

**THEOREM 14.15.** *Assume that  $(x_1, \dots, x_N)$  is  $G$ -invariant, and that either we have  $G = O_N^+, S_N^+$ , or that  $G = O_N, S_N$  and  $(x_1, \dots, x_N)$  commute. We have then*

$$E_N^{(\pi)}(b_0 x_1 b_1, \dots, x_1 b_k) = \frac{1}{N^{|\pi|}} \sum_{\pi \leq \ker i} b_0 x_{i_1} \dots b x_{i_k} b_k$$

for any  $\pi$  in the partition category  $D(k)$  for  $G$ , and any  $b_0, \dots, b_k \in B_N$ .

**PROOF.** We prove this result by recurrence on the number of blocks of  $\pi$ . First suppose that  $\pi = 1_k$  is the partition with only one block. Then:

$$\begin{aligned} E_N^{(1_k)}(b_0 x_1 b_1, \dots, x_1 b_k) &= E_N(b_0 x_1 \dots x_1 b_k) \\ &= \sum_{i_1 \dots i_k} b_0 x_{i_1} \dots x_{i_k} b_k \int_G v_{i_1 1} \dots v_{i_k 1} \end{aligned}$$

Here we have used the fact that the elements  $b_0, \dots, b_k$  are fixed by the coaction  $\tilde{\alpha}$ . Applying now the Weingarten integration formula, we have:

$$\begin{aligned} E_N(b_0 x_1 \dots x_1 b_k) &= \sum_{i_1 \dots i_k} b_0 x_{i_1} \dots x_{i_k} b_k \sum_{\pi \leq \ker i} \sum_{\sigma} W_{kN}(\pi, \sigma) \\ &= \sum_{\pi \in D(k)} \left( \sum_{\sigma \in D(k)} W_{kN}(\pi, \sigma) \right) \sum_{\pi \leq \ker i} b_0 x_{i_1} \dots x_{i_k} b_k \end{aligned}$$

Now observe that for any  $\sigma \in D(k)$  we have the following formula:

$$G_{kN}(\sigma, 1_k) = N^{|\sigma \vee 1_k|} = N$$

It follows that for any partition  $\pi \in D(k)$ , we have:

$$\begin{aligned} N \sum_{\sigma \in D(k)} W_{kN}(\pi, \sigma) &= \sum_{\sigma \in D(k)} W_{kN}(\pi, \sigma) G_{kN}(\sigma, 1_k) \\ &= \delta_{\pi 1_k} \end{aligned}$$

Applying this in the above context, we find, as desired:

$$\begin{aligned} E_N(b_0 x_1 \dots x_1 b_k) &= \sum_{\pi \in D(k)} \frac{1}{N} \delta_{\pi 1_k} \sum_{\pi \leq \ker i} b_0 x_{i_1} \dots x_{i_k} b_k \\ &= \frac{1}{N} \sum_{i=1}^N b_0 x_i \dots x_i b_k \end{aligned}$$

If the condition (3) or (4) is satisfied, then the general case follows from:

$$E_N^{(\pi)}(b_0 x_1 b_1, \dots, x_1 b_k) = b_1 \dots b_k \prod_{V \in \pi} E_N(V)(x_1, \dots, x_1)$$

Indeed, the one thing that we must check here is that if  $\pi \in D(k)$  and  $V$  is a block of  $\pi$  with  $s$  elements, then  $1_s \in D(s)$ . But this is easily verified, in each case.

Assume now that the condition (1) or (2) is satisfied. Let  $\pi \in D(k)$ . Since  $\pi$  is noncrossing,  $\pi$  contains an interval  $V = \{l+1, \dots, l+s+1\}$ , and we have:

$$\begin{aligned} & E_N^{(\pi)}(b_0 x_1 b_1, \dots, x_1 b_k) \\ &= E_N^{(\pi-V)}(b_0 x_1 b_1, \dots, E_N(x_1 b_{l+1} \dots x_1 b_{l+s}) x_1, \dots, x_1 b_k) \end{aligned}$$

To apply induction, we must check that we have  $\pi - V \in D(k-s)$  and  $1_s \in D(s)$ . Indeed, this is easily verified for  $NC, NC_2$ . Applying induction, we have:

$$\begin{aligned} & E_N^{(\pi)}(b_0 x_1 b_1, \dots, x_1 b_k) \\ &= \frac{1}{N^{|\pi|-1}} \sum_{\pi-V \leq_{\ker i} i} b_0 x_{i_1} \dots b_l (E_N(x_1 b_{l+1} \dots x_1 b_{l+s})) x_{i_{l+s}} \dots x_{i_k} b_k \\ &= \frac{1}{N^{|\pi|-1}} \sum_{\pi-V \leq_{\ker i} i} b_0 x_{i_1} \dots b_l \left( \frac{1}{N} \sum_{i=1}^N x_i b_{l+1} \dots b x_i b_{l+s} \right) x_{i_{l+s}} \dots x_{i_k} b_k \\ &= \frac{1}{N^{|\pi|}} \sum_{\pi \leq_{\ker i} i} b_0 x_{i_1} \dots x_{i_k} b_k \end{aligned}$$

Thus, we are led to the conclusion in the statement.  $\square$

Summarizing, we have so far reverse De Finetti theorems for the quantum groups that we are interested in here, along with some technical results, connecting the corresponding potential De Finetti theorems to the Weingarten function combinatorics.

### 14c. Weingarten estimates

In order to advance, we will need some standard Weingarten estimates for our quantum groups, which have their own interest, and that we will discuss now. So, consider the diagram formed by the main quantum permutation and quantum rotation groups:

$$\begin{array}{ccc} S_N^+ & \longrightarrow & O_N^+ \\ \uparrow & & \uparrow \\ S_N & \longrightarrow & O_N \end{array}$$

Regarding the symmetric group  $S_N$ , the situation here is very simple, because we can explicitly compute the Weingarten function, and estimate it, as follows:

PROPOSITION 14.16. *For  $S_N$  the Weingarten function is given by*

$$W_{kN}(\pi, \nu) = \sum_{\tau \leq \pi \wedge \nu} \mu(\tau, \pi) \mu(\tau, \nu) \frac{(N - |\tau|)!}{N!}$$

*and satisfies the following estimate,*

$$W_{kN}(\pi, \nu) = N^{-|\pi \wedge \nu|} (\mu(\pi \wedge \nu, \pi) \mu(\pi \wedge \nu, \nu) + O(N^{-1}))$$

*with  $\mu$  being the Möbius function of  $P(k)$ .*

PROOF. The first assertion follows from the usual Weingarten formula, namely:

$$\int_{S_N} v_{i_1 j_1} \cdots v_{i_k j_k} = \sum_{\pi, \nu \in P(k)} \delta_\pi(i) \delta_\nu(j) W_{kN}(\pi, \nu)$$

Indeed, in this formula the integrals on the left are in fact known, from the explicit integration formula over  $S_N$  that we established before, namely:

$$\int_{S_N} g_{i_1 j_1} \cdots g_{i_k j_k} = \begin{cases} \frac{(N - |\ker i|)!}{N!} & \text{if } \ker i = \ker j \\ 0 & \text{otherwise} \end{cases}$$

But this allows the computation of the right term, via the Möbius inversion formula, explained before. As for the second assertion, this follows from the first one.  $\square$

The above result is of course something very special, coming from the fact that the Haar integration over the permutation group  $S_N$ , save for being just an averaging, this group being finite, is something very simple, combinatorially speaking.

Regarding now the quantum group  $S_N^+$ , that we are particularly interested in here, let us begin with some explicit computations. We first have the following simple and final result at  $k = 2, 3$ , directly in terms of the quantum group integrals:

PROPOSITION 14.17. *At  $k = 2, 3$  we have the following estimate:*

$$\int_{S_N^+} u_{i_1 j_1} \cdots u_{i_k j_k} = \begin{cases} 0 & (\ker i \neq \ker j) \\ \simeq N^{-|\ker i|} & (\ker i = \ker j) \end{cases}$$

PROOF. Since at  $k \leq 3$  we have  $NC(k) = P(k)$ , the Weingarten integration formulae for  $S_N$  and  $S_N^+$  coincide, and we obtain, by using the above formula for  $S_N$ :

$$\begin{aligned} \int_{S_N^+} v_{i_1 j_1} \cdots v_{i_k j_k} &= \int_{S_N} v_{i_1 j_1} \cdots v_{i_k j_k} \\ &= \delta_{\ker i, \ker j} \frac{(N - |\ker i|)!}{N!} \end{aligned}$$

Thus, we obtain the formula in the statement.  $\square$

In general now, the idea will be that of working out a “master estimate” for the Weingarten function, as above. Before starting, let us record the formulae at  $k = 2, 3$ , which will be useful later, as illustrations. At  $k = 2$ , with indices  $||, \sqcap$  as usual, and with the convention that  $\approx$  means componentwise dominant term, we have:

$$W_{2N} \approx \begin{pmatrix} N^{-2} & -N^{-2} \\ -N^{-2} & N^{-1} \end{pmatrix}$$

At  $k = 3$  now, with indices  $|||, |\sqcap, \sqcap|, \sqcap| \sqcap|$  as usual, and same meaning for  $\approx$ , we have:

$$W_{3N} \approx \begin{pmatrix} N^{-3} & -N^{-3} & -N^{-3} & -N^{-3} & 2N^{-3} \\ -N^{-3} & N^{-2} & N^{-3} & N^{-3} & -N^{-2} \\ -N^{-3} & N^{-3} & N^{-2} & N^{-3} & -N^{-2} \\ -N^{-3} & N^{-3} & N^{-3} & N^{-2} & -N^{-2} \\ 2N^{-3} & -N^{-2} & -N^{-2} & -N^{-2} & N^{-1} \end{pmatrix}$$

These formulae follow indeed from the plain formulae for the Weingarten matrix  $W_{kN}$  at  $k = 2, 3$  from [13] and related papers, after rearranging the matrix indices as above.

Observe in particular, in the context of the above computations, that we have the following formula, which will be of interest in what follows:

$$W_{3N}(|\sqcap, \sqcap|) \simeq N^{-3}$$

In order to deal now with the general case, let us start with some standard facts:

**PROPOSITION 14.18.** *The following happen, regarding the partitions in  $P(k)$ :*

- (1)  $|\pi| + |\nu| \leq |\pi \vee \nu| + |\pi \wedge \nu|$ .
- (2)  $|\pi \vee \tau| + |\tau \vee \nu| \leq |\pi \vee \nu| + |\tau|$ .
- (3)  $d(\pi, \nu) = \frac{|\pi| + |\nu|}{2} - |\pi \vee \nu|$  is a distance.

**PROOF.** All this is well-known, the idea being as follows:

- (1) This is well-known, coming from the fact that  $P(k)$  is a semi-modular lattice.
- (2) This follows from (1), as explained for instance in the paper [17].
- (3) This follows from (2) above, which says that the following holds:

$$\begin{aligned} & \frac{|\pi| + |\tau|}{2} - d(\pi, \tau) + \frac{|\tau| + |\nu|}{2} - d(\tau, \nu) \\ & \leq \frac{|\pi| + |\nu|}{2} - d(\pi, \nu) + |\tau| \end{aligned}$$

Thus, we obtain in this way the triangle inequality:

$$d(\pi, \tau) + d(\tau, \nu) \geq d(\pi, \nu)$$

As for the other axioms for a distance, these are all clear. □

Actually in what follows we will only need (3) in the above statement. For more on this, and on the geometry and combinatorics of partitions, we refer to [70].

As a main result now regarding the Weingarten functions, we have:

**THEOREM 14.19.** *The Weingarten matrix  $W_{kN}$  has a series expansion in  $N^{-1}$ ,*

$$W_{kN}(\pi, \nu) = N^{|\pi \vee \nu| - |\pi| - |\nu|} \sum_{g=0}^{\infty} K_g(\pi, \nu) N^{-g}$$

where the various objects on the right are defined as follows:

(1) A path from  $\pi$  to  $\nu$  is a sequence as follows:

$$p = [\pi = \tau_0 \neq \tau_1 \neq \dots \neq \tau_r = \nu]$$

(2) The signature of such a path is  $+$  when  $r$  is even, and  $-$  when  $r$  is odd.

(3) The geodesicity defect of such a path is:

$$g(p) = \sum_{i=1}^r d(\tau_{i-1}, \tau_i) - d(\pi, \nu)$$

(4)  $K_g$  counts the signed paths from  $\pi$  to  $\nu$ , with geodesicity defect  $g$ .

**PROOF.** We recall that the Weingarten matrix  $W_{kN}$  appears as the inverse of the Gram matrix  $G_{kN}$ , which is given by the following formula:

$$G_{kN}(\pi, \nu) = N^{|\pi \vee \nu|}$$

Now observe that the Gram matrix can be written in the following way:

$$\begin{aligned} G_{kN}(\pi, \nu) &= N^{|\pi \vee \nu|} \\ &= N^{\frac{|\pi|}{2}} N^{|\pi \vee \nu| - \frac{|\pi| + |\nu|}{2}} N^{\frac{|\nu|}{2}} \\ &= N^{\frac{|\pi|}{2}} N^{-d(\pi, \nu)} N^{\frac{|\nu|}{2}} \end{aligned}$$

This suggests considering the following diagonal matrix:

$$\Delta = \text{diag}(N^{\frac{|\pi|}{2}})$$

So, let us do this, and consider as well the following matrix:

$$H(\pi, \nu) = \begin{cases} 0 & (\pi = \nu) \\ N^{-d(\pi, \nu)} & (\pi \neq \nu) \end{cases}$$

In terms of these two matrices, the above formula for  $G_{kN}$  simply reads:

$$G_{kN} = \Delta(1 + H)\Delta$$

Thus, the Weingarten matrix  $W_{kN}$  is given by the following formula:

$$W_{kN} = \Delta^{-1}(1 + H)^{-1}\Delta^{-1}$$

In order to compute now the inverse of  $1 + H$ , we will use the following formula:

$$(1 + H)^{-1} = 1 - H + H^2 - H^3 + \dots$$

Consider indeed the set  $P_r(\pi, \nu)$  of length  $r$  paths between  $\pi$  and  $\nu$ . We have:

$$\begin{aligned} H^r(\pi, \nu) &= \sum_{p \in P_r(\pi, \nu)} H(\tau_0, \tau_1) \dots H(\tau_{r-1}, \tau_r) \\ &= \sum_{p \in P_r(\pi, \nu)} N^{-d(\pi, \nu) - g(p)} \end{aligned}$$

Thus by using  $(1 + H)^{-1} = 1 - H + H^2 - H^3 + \dots$  we obtain:

$$\begin{aligned} (1 + H)^{-1}(\pi, \nu) &= \sum_{r=0}^{\infty} (-1)^r H^r(\pi, \nu) \\ &= N^{-d(\pi, \nu)} \sum_{r=0}^{\infty} \sum_{p \in P_r(\pi, \nu)} (-1)^r N^{-g(p)} \end{aligned}$$

It follows that the Weingarten matrix is given by the following formula:

$$\begin{aligned} W_{kN}(\pi, \nu) &= \Delta^{-1}(\pi)(1 + H)^{-1}(\pi, \nu)\Delta^{-1}(\nu) \\ &= N^{-\frac{|\pi|}{2} - \frac{|\nu|}{2} - d(\pi, \nu)} \sum_{r=0}^{\infty} \sum_{p \in P_r(\pi, \nu)} (-1)^r N^{-g(p)} \\ &= N^{|\pi \vee \nu| - |\pi| - |\nu|} \sum_{r=0}^{\infty} \sum_{p \in P_r(\pi, \nu)} (-1)^r N^{-g(p)} \end{aligned}$$

Now by rearranging the various terms in the above double sum according to their geodesicity defect  $g = g(p)$ , this gives the following formula:

$$W_{kN}(\pi, \nu) = N^{|\pi \vee \nu| - |\pi| - |\nu|} \sum_{g=0}^{\infty} K_g(\pi, \nu) N^{-g}$$

Thus, we are led to the conclusion in the statement.  $\square$

As an illustration for all this, we have the following explicit estimates:

**THEOREM 14.20.** *Consider an easy quantum group  $G = (G_N)$ , coming from a category of partitions  $D = (D(k))$ . For any  $\pi \leq \nu$  we have the estimate*

$$W_{kN}(\pi, \nu) = N^{-|\pi|}(\mu(\pi, \nu) + O(N^{-1}))$$

and for  $\pi, \nu$  arbitrary we have

$$W_{kN}(\pi, \nu) = O(N^{|\pi \vee \nu| - |\pi| - |\nu|})$$

with  $\mu$  being the Möbius function of  $D(k)$ .

PROOF. We have two assertions here, the idea being as follows:

(1) The first estimate is clear from the general expansion formula established in Theorem 14.19 above, namely:

$$W_{kN}(\pi, \nu) = N^{|\pi \vee \nu| - |\pi| - |\nu|} \sum_{g=0}^{\infty} K_g(\pi, \nu) N^{-g}$$

(2) In the case  $\pi \leq \nu$  it is known that  $K_0$  coincides with the Möbius function of  $NC(k)$ , as explained for instance in [17], so we obtain once again from Theorem 14.19 the fine estimate in the statement as well, namely:

$$W_{kN}(\pi, \nu) = N^{-|\pi|}(\mu(\pi, \nu) + O(N^{-1})) \quad \forall \pi \leq \nu$$

Observe that, by symmetry of  $W_{kN}$ , we obtain as well that we have:

$$W_{kN}(\pi, \nu) = N^{-|\nu|}(\mu(\nu, \pi) + O(N^{-1})) \quad \forall \pi \geq \nu$$

Thus, we are led to the conclusions in the statement.  $\square$

When  $\pi, \nu$  are not comparable by  $\leq$ , things are quite unclear. The simplest example appears at  $k = 3$ , where we have the following formula, which is elementary:

$$W_{3N}(|\square, \square|) \simeq N^{-3}$$

Observe that the exponent  $-3$  is precisely the dominant one, and this because:

$$||\square \vee \square| - |\square| - |\square|| = 1 - 2 - 2 = -3$$

As for the corresponding coefficient,  $K_0(|\square, \square|) = 1$ , this is definitely not the Möbius function, which vanishes for partitions which are not comparable by  $\leq$ . According to Theorem 14.19, this is rather the number of signed geodesic paths from  $|\square$  to  $|\square|$ .

In relation to all this, observe that geometrically,  $NC(5)$  consists of the partitions  $|\square, \square|, |\square|$ , which form an equilateral triangle with edges worth 1, and then the partitions  $|||, |\square|$ , which are at distance 1 apart, and each at distance  $1/2$  from each of the vertices of the triangle. It is not obvious how to recover the formula  $K_0(|\square, \square|) = 1$  from this.

Finally, also following [17], we will need as well the following result:

PROPOSITION 14.21. *We have the following results:*

- (1) If  $D = NC, NC_2$ , then  $\mu_{D(k)}(\pi, \nu) = \mu_{NC(k)}(\pi, \nu)$ .
- (2) If  $D = P, P_2$  then  $\mu_{D(k)}(\pi, \nu) = \mu_{P(k)}(\pi, \nu)$ .

PROOF. Let  $Q = NC, P$  according to the cases (1,2) above. It is easy to see in each case that  $D(k)$  is closed under taking intervals in  $Q(k)$ , in the sense that if  $\pi_1, \pi_2 \in D(k)$ ,  $\nu \in Q(k)$  and  $\pi_1 < \nu < \pi_2$  then  $\nu \in D(k)$ . With this observation in hand, the result now follows from the definition of the Möbius function. See [17].  $\square$

### 14d. De Finetti theorems

With the above ingredients in hand, let us go back now to invariance questions with respect to the main quantum permutation and rotation groups, namely:

$$\begin{array}{ccc} S_N^+ & \longrightarrow & O_N^+ \\ \uparrow & & \uparrow \\ S_N & \longrightarrow & O_N \end{array}$$

More generally, we would like in fact to have, ideally, de Finetti type theorems for all the easy quantum groups that we know, from the previous chapters. This is of course something quite technical, and time consuming, but we would like at least to understand what happens for the main quantum reflection and rotation groups, namely:

$$\begin{array}{ccccc} & & K_N^+ & \longrightarrow & U_N^+ \\ & \nearrow & \uparrow & & \nearrow \\ H_N^+ & \longrightarrow & O_N^+ & & \\ \uparrow & & \uparrow & & \uparrow \\ & \nearrow & K_N & \longrightarrow & U_N \\ H_N & \longrightarrow & O_N & & \end{array}$$

In order to discuss these questions, or at least some of them, let us start with a basic approximation result for the finite sequences, in the real case, from [17], as follows:

**THEOREM 14.22.** *Suppose that  $(x_1, \dots, x_N)$  is  $G_N$ -invariant, and that  $G_N = O_N^+, S_N^+$ , or that  $G_N = O_N, S_N$  and  $(x_1, \dots, x_N)$  commute. Let  $(y_1, \dots, y_N)$  be a sequence of  $B_N$ -valued random variables with  $B_N$ -valued joint distribution determined as follows:*

- (1)  $G = O^+$ : Free semicircular, centered with same variance as  $x_1$ .
- (2)  $G = S^+$ : Freely independent,  $y_i$  has same distribution as  $x_1$ .
- (3)  $G = O$ : Independent Gaussian, centered with same variance as  $x_1$ .
- (4)  $G = S$ : Independent,  $y_i$  has same distribution as  $x_1$ .

Then if  $1 \leq j_1, \dots, j_k \leq N$  and  $b_0, \dots, b_k \in B_N$ , we have the following estimate,

$$||E_N(b_0 x_{j_1} \dots x_{j_k} b_k) - E(b_0 y_{j_1} \dots y_{j_k} b_k)|| \leq \frac{C_k(G)}{N} ||x_1||^k ||b_0|| \dots ||b_k||$$

with  $C_k(G)$  being a constant depending only on  $k$  and  $G$ .

**PROOF.** First we note that it suffices to prove the result for  $N$  large enough. We will assume that  $N$  is sufficiently large, as for the Gram matrix  $G_{kN}$  to be invertible.

Let  $1 \leq j_1, \dots, j_k \leq N$  and  $b_0, \dots, b_k \in B_N$ . We have then:

$$\begin{aligned} E_N(b_0 x_{j_1} \dots x_{j_k} b_k) &= \sum_{i_1 \dots i_k} b_0 x_{i_1} \dots x_{i_k} b_k \int v_{i_1 j_1} \dots v_{i_k j_k} \\ &= \sum_{i_1 \dots i_k} b_0 x_{i_1} \dots x_{i_k} b_k \sum_{\pi \leq \ker i} \sum_{\sigma \leq \ker j} W_{kN}(\pi, \sigma) \\ &= \sum_{\sigma \leq \ker j} \sum_{\pi} W_{kN}(\pi, \sigma) \sum_{\pi \leq \ker i} b_0 x_{i_1} \dots x_{i_k} b_k \end{aligned}$$

On the other hand, it follows from our assumptions on  $(y_1, \dots, y_N)$ , and from the various moment-cumulant formulae given before, that we have:

$$E(b_0 y_{j_1} \dots y_{j_k} b_k) = \sum_{\sigma \leq \ker j} \xi_{E_N}^{(\sigma)}(b_0 x_1 b_1, \dots, x_1 b_k)$$

Here, and in what follows,  $\xi$  denote the relevant free or classical cumulants.

The right hand side can be expanded, via the Möbius inversion formula, in terms of expectation functionals of the following type, with  $\pi$  being a partition in  $NC, P$  according to the cases (1,2) or (3,4) in the statement, and with  $\pi \leq \sigma$  for some  $\sigma \in D(k)$ :

$$E_N^{(\pi)}(b_0 x_1 b_1, \dots, x_1 b_k)$$

Now if  $\pi \notin D(k)$ , we claim that this expectation functional is zero.

Indeed this is only possible if  $D = NC_2, P_2$ , and if  $\pi$  has a block with an odd number of legs. But it is easy to see that in these cases  $x_1$  has an even distribution with respect to  $E_N$ , and therefore we have, as claimed, the following formula:

$$E_N^{(\pi)}(b_0 x_1 b_1, \dots, x_1 b_k) = 0$$

Now this observation allows to to rewrite the above equation as follows:

$$E(b_0 y_{j_1} \dots y_{j_k} b_k) = \sum_{\sigma \leq \ker j} \sum_{\pi \leq \sigma} \mu_{D(k)}(\pi, \sigma) E_N^{(\pi)}(b_0 x_1 b_1, \dots, x_1 b_k)$$

We therefore obtain the following formula:

$$E(b_0 y_{j_1} \dots y_{j_k} b_k) = \sum_{\sigma \leq \ker j} \sum_{\pi \leq \sigma} \mu_{D(k)}(\pi, \sigma) N^{-|\pi|} \sum_{\pi \leq \ker i} b_0 x_{i_1} \dots x_{i_k} b_k$$

Comparing the above two equations, we find that:

$$\begin{aligned} &E_N(b_0 x_{j_1} \dots x_{j_k} b_k) - E(b_0 y_{j_1} \dots y_{j_k} b_k) \\ &= \sum_{\sigma \leq \ker j} \sum_{\pi} (W_{kN}(\pi, \sigma) - \mu_{D(k)}(\pi, \sigma) N^{-|\pi|}) \sum_{\pi \leq \ker i} b_0 x_{i_1} \dots x_{i_k} b_k \end{aligned}$$

Now since  $x_1, \dots, x_N$  are identically distributed with respect to the faithful state  $\varphi$ , it follows that these variables have the same norm. Thus, for any  $\pi \in D(k)$ :

$$\left\| \sum_{\pi \leq \ker i} b_0 x_{i_1} \dots x_{i_k} b_k \right\| \leq N^{|\pi|} \|x_1\|^k \|b_0\| \dots \|b_k\|$$

Combining this with the former equation, we obtain the following estimate:

$$\begin{aligned} & \|E_N(b_0 x_{j_1} \dots x_{j_k} b_k) - E(b_0 y_{j_1} \dots y_{j_k} b_k)\| \\ & \leq \sum_{\sigma \leq \ker j} \sum_{\pi} |W_{kN}(\pi, \sigma) N^{|\pi|} - \mu_{D(k)}(\pi, \sigma)| \|x_1\|^k \|b_0\| \dots \|b_k\| \end{aligned}$$

Let us set now, according to the above:

$$C_k(G) = \sup_{N \in \mathbb{N}} \left( N \times \sum_{\sigma, \pi \in D(k)} |W_{kN}(\pi, \sigma) N^{|\pi|} - \mu_{D(k)}(\pi, \sigma)| \right)$$

But this number is finite by our main estimate, which completes the proof.  $\square$

We will use in what follows the inclusions  $G_N \subset G_M$  for  $N < M$ , which correspond to the Hopf algebra morphisms  $\omega_{N,M} : C(G_M) \rightarrow C(G_N)$  given by:

$$\omega_{N,M}(u_{ij}) = \begin{cases} u_{ij} & \text{if } 1 \leq i, j \leq N \\ \delta_{ij} & \text{if } \max(i, j) > N \end{cases}$$

Still following [17], we begin by extending the notion of  $G_N$ -invariance to the infinite sequences of variables, in the following way:

**DEFINITION 14.23.** *Let  $(x_i)_{i \in \mathbb{N}}$  be a sequence in a noncommutative probability space  $(A, \varphi)$ . We say that  $(x_i)_{i \in \mathbb{N}}$  is  $G$ -invariant if*

$$(x_1, \dots, x_N)$$

*is  $G_N$ -invariant for each  $N \in \mathbb{N}$ .*

In other words, the condition is that the joint distribution of  $(x_1, \dots, x_N)$  should be invariant under the following coaction map, for each  $N \in \mathbb{N}$ :

$$\alpha_N : \mathbb{C} \langle t_1, \dots, t_N \rangle \rightarrow \mathbb{C} \langle t_1, \dots, t_N \rangle \otimes C(G_N)$$

It is convenient to extend these coactions to a coaction on the algebra of noncommutative polynomials on an infinite number of variables, in the following way:

$$\beta_N : \mathbb{C} \langle t_i | i \in \mathbb{N} \rangle \rightarrow \mathbb{C} \langle t_i | i \in \mathbb{N} \rangle \otimes C(G_N)$$

Indeed, we can define  $\beta_N$  to be the unique unital morphism satisfying:

$$\beta_N(t_j) = \begin{cases} \sum_{i=1}^N t_i \otimes v_{ij} & \text{if } 1 \leq j \leq N \\ t_j \otimes 1 & \text{if } j > N \end{cases}$$

It is clear that  $\beta_N$  as constructed above is a coaction of  $G_N$ . Also, we have the following relations, where  $\iota_N : \mathbb{C} \langle t_1, \dots, t_N \rangle \rightarrow \mathbb{C} \langle t_i | i \in \mathbb{N} \rangle$  is the natural inclusion:

$$(id \otimes \omega_{N,M})\beta_M = \beta_N$$

$$(\iota_N \otimes id)\alpha_N = \beta_N \iota_N$$

By using these compatibility relations, we obtain the following result:

**PROPOSITION 14.24.** *An infinite sequence of random variables  $(x_i)_{i \in \mathbb{N}}$  is  $G$ -invariant if and only if the joint distribution functional*

$$\mu_x : \mathbb{C} \langle t_i | i \in \mathbb{N} \rangle \rightarrow \mathbb{C}$$

$$P \rightarrow tr(P(x))$$

*is invariant under the coaction  $\beta_N$ , for each  $N \in \mathbb{N}$ .*

**PROOF.** This is clear indeed from the above discussion.  $\square$

In what follows  $(x_i)_{i \in \mathbb{N}}$  will be a sequence of self-adjoint random variables in a von Neumann algebra  $(M, tr)$ . We will assume that  $M$  is generated by  $(x_i)_{i \in \mathbb{N}}$ . We denote by  $L^2(M, tr)$  the corresponding GNS Hilbert space, with inner product as follows:

$$\langle m_1, m_2 \rangle = tr(m_1 m_2^*)$$

Also, the strong topology on  $M$ , that we will use in what follows, will be taken by definition with respect to the faithful representation on the space  $L^2(M, tr)$ .

We let  $P_N$  be the fixed point algebra of the action  $\beta_N$ , and we set:

$$B_N = \left\{ p(x) \middle| p \in P_N \right\}''$$

We have then an inclusion  $B_{N+1} \subset B_N$ , for any  $N \geq 1$ , and we can then define the  $G$ -invariant subalgebra as the common intersection of these algebras:

$$B = \bigcap_{N \geq 1} B_N$$

With these conventions, we have the following result, from [17]:

**PROPOSITION 14.25.** *If an infinite sequence of random variables  $(x_i)_{i \in \mathbb{N}}$  is  $G$ -invariant, then for each  $N \in \mathbb{N}$  there is a coaction*

$$\tilde{\beta}_N : M \rightarrow M \otimes L^\infty(G_N)$$

*determined by the following formula, for any  $p \in \mathcal{P}_\infty$ :*

$$\tilde{\beta}_N(p(x)) = (ev_x \otimes \pi_N)\beta_N(p)$$

*The fixed point algebra of  $\tilde{\beta}_N$  is then  $B_N$ .*

PROOF. This is indeed clear from definitions, and from the various compatibility formulae given above, between the coactions  $\alpha_N$  and  $\beta_N$ .  $\square$

We have as well the following result, which is clear as well:

PROPOSITION 14.26. *In the above context, that of an infinite sequence of random variables belonging to an arbitrary von Neumann algebra  $M$  with a trace*

$$(x_i)_{i \in \mathbb{N}}$$

*which is  $G$ -invariant, for each  $N \in \mathbb{N}$  there is a trace-preserving conditional expectation  $E_N : M \rightarrow B_N$  given by integrating the action  $\tilde{\beta}_N$ :*

$$E_N(m) = \left( id \otimes \int_G \right) \tilde{\beta}_N(m)$$

*By taking the limit of these expectations as  $N \rightarrow \infty$ , we obtain a trace-preserving conditional expectation onto the  $G$ -invariant subalgebra.*

PROOF. Once again, this is clear from definitions, and from the various compatibility formulae given above, between the coactions  $\alpha_N$  and  $\beta_N$ .  $\square$

We are now prepared to state and prove the main theorem, from [17], which comes as a complement to the reverse De Finetti theorem that we already established:

THEOREM 14.27. *Let  $(x_i)_{i \in \mathbb{N}}$  be a  $G$ -invariant sequence of self-adjoint random variables in  $(M, tr)$ , and assume that  $M = \langle (x_i)_{i \in \mathbb{N}} \rangle$ . Then there exists a subalgebra  $B \subset M$  and a trace-preserving conditional expectation  $E : M \rightarrow B$  such that:*

- (1) *If  $G = (S_N)$ , then  $(x_i)_{i \in \mathbb{N}}$  are conditionally independent and identically distributed given  $B$ .*
- (2) *If  $G = (S_N^+)$ , then  $(x_i)_{i \in \mathbb{N}}$  are freely independent and identically distributed with amalgamation over  $B$ .*
- (3) *If  $G = (O_N)$ , then  $(x_i)_{i \in \mathbb{N}}$  are conditionally independent, and have Gaussian distributions with mean zero and common variance, given  $B$ .*
- (4) *If  $G = (O_N^+)$ , then  $(x_i)_{i \in \mathbb{N}}$  form a  $B$ -valued free semicircular family with mean zero and common variance.*

PROOF. We use the various partial results and formulae established above. Let  $j_1, \dots, j_k \in \mathbb{N}$  and  $b_0, \dots, b_k \in B$ . We have then the following computation:

$$\begin{aligned} E(b_0 x_{j_1} \dots x_{j_k} b_k) &= \lim_{N \rightarrow \infty} E_N(b_0 x_{j_1} \dots x_{j_k} b_k) \\ &= \lim_{N \rightarrow \infty} \sum_{\sigma \leq \ker j} \sum_{\pi} W_{kN}(\pi, \sigma) \sum_{\pi \leq \ker i} b_0 x_{i_1} \dots x_{i_k} b_k \\ &= \lim_{N \rightarrow \infty} \sum_{\sigma \leq \ker j} \sum_{\pi \leq \sigma} \mu_{D(k)}(\pi, \sigma) N^{-|\pi|} \sum_{\pi \leq \ker i} b_0 x_{i_1} \dots x_{i_k} b_k \end{aligned}$$

Let us recall now from the above that we have the following compatibility formula, where  $\tilde{\iota}_N : W^*(x_1, \dots, x_N) \rightarrow M$  is the canonical inclusion, and  $\tilde{\alpha}_N$  is as before:

$$(\tilde{\iota}_N \otimes id)\tilde{\alpha}_N = \tilde{\beta}_N \tilde{\iota}_N$$

By using this formula, and the above cumulant results, we have:

$$E(b_0 x_{j_1} \dots x_{j_k} b_k) = \lim_{N \rightarrow \infty} \sum_{\sigma \leq \ker j} \sum_{\pi \leq \sigma} \mu_{D(k)}(\pi, \sigma) E_N^{(\pi)}(b_0 x_1 b_1, \dots, x_1 b_k)$$

We therefore obtain the following formula:

$$E(b_0 x_{j_1} \dots x_{j_k} b_k) = \sum_{\sigma \leq \ker j} \sum_{\pi \leq \sigma} \mu_{D(k)}(\pi, \sigma) E^{(\pi)}(b_0 x_1 b_1, \dots, x_1 b_k)$$

We can replace the sum of expectation functionals by cumulants, as to obtain:

$$E(b_0 x_{j_1} \dots x_{j_k} b_k) = \sum_{\sigma \leq \ker j} \xi_E^{(\sigma)}(b_0 x_1 b_1, \dots, x_1 b_k)$$

Here and in what follows  $\xi$  denotes as usual the relevant free or classical cumulants, depending on the quantum group that we are dealing with, free or classical.

Now since the cumulants are determined by the moment-cumulant formulae, we conclude that we have the following formula:

$$\xi_E^{(\sigma)}(b_0 x_{j_1} b_1, \dots, x_{j_k} b_k) = \begin{cases} \xi_E^{(\sigma)}(b_0 x_1 b_1, \dots, x_1 b_k) & \text{if } \sigma \in D(k) \text{ and } \sigma \leq \ker j \\ 0 & \text{otherwise} \end{cases}$$

With this formula in hand, the result then follows from the characterizations of these joint distributions in terms of cumulants, and we are done.  $\square$

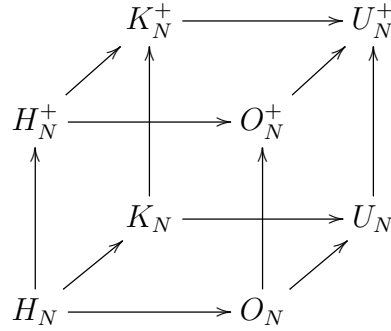
Summarizing, we are done with our first and main objective, namely establishing De Finetti theorems for the main quantum permutation and rotation groups, namely:

$$\begin{array}{ccc} S_N^+ & \longrightarrow & O_N^+ \\ \uparrow & & \uparrow \\ S_N & \longrightarrow & O_N \end{array}$$

The story is of course not over here, and there are many related interesting questions left, which are more technical, in relation with the invariance questions with respect to these quantum groups. We refer here to [17], [38], [39], [61], [62] and related papers.

Regarding now our second objective, which appears as a variation of this, fully in tune with the present book, we would like to understand as well what happens to the invariance

questions with respect to the basic quantum reflection and rotation groups, namely:



Here the answer is more or less known as well from [17], but with the problem however that the paper [17] is extremely general, and in relation with our cube question, more general than needed. In any case, for this and for further aspects of invariance questions, we refer as before to [17], [38], [39], [61], [62] and related papers.

### 14e. Exercises

Things have been quite technical in this chapter, dealing with advanced probability theory, and so will be our exercises here. As a first exercise, we have:

**EXERCISE 14.28.** *Formulate and prove the classical De Finetti theorem, concerning sequences which are invariant under  $S_\infty$ , without using representation theory methods.*

This is something very standard, and is a must-do exercise, the point being that all the Weingarten technology used in this chapter, which is something quite heavy, was motivated by the fact that we want to deal with several quantum groups at the same time, in a “uniform” way. In the case of the symmetric group itself things are in fact much simpler, and the exercise is about understanding how this works.

**EXERCISE 14.29.** *Formulate and prove the free De Finetti theorem, concerning sequences which are invariant under  $(S_N^+)$ , without using representation theory methods.*

The same comments as for the previous exercise apply, the idea being that, once again, the Weingarten function machinery can be avoided in this case.

**EXERCISE 14.30.** *Work out the full proof of the explicit formula for the Weingarten function for  $S_N$ , namely*

$$W_{kN}(\pi, \nu) = \sum_{\tau \leq \pi \wedge \nu} \mu(\tau, \pi) \mu(\tau, \nu) \frac{(N - |\tau|)!}{N!}$$

*then of the main estimate for this function, namely*

$$W_{kN}(\pi, \nu) = N^{-|\pi \wedge \nu|} (\mu(\pi \wedge \nu, \pi) \mu(\pi \wedge \nu, \nu) + O(N^{-1}))$$

*where  $\mu$  is the Möbius function of  $P(k)$ .*

This was something that was already discussed in the above, the idea being that all this comes from the explicit knowledge of the integrals over  $S_N$ , via the Möbius inversion formula, and the problem now is that of working out all the details.

EXERCISE 14.31. *Work out estimates for the integrals of type*

$$\int_{S_N^+} v_{i_1 j_1} v_{i_2 j_2} v_{i_3 j_3} v_{i_4 j_4}$$

*and then for the Weingarten function of  $S_N^+$  at  $k = 4$ .*

Once again, this was something partly discussed in the above, with the comment that things are clear at  $k = 2, 3$ , due to the formula  $P(k) = NC(k)$  valid here. The problem now is that of working out what happens at  $k = 4$ , where things are non-trivial.

EXERCISE 14.32. *Prove directly that the function*

$$d(\pi, \nu) = \frac{|\pi| + |\nu|}{2} - |\pi \vee \nu|$$

*is a distance on  $P(k)$ .*

To be more precise here, this is something that we talked about in the above, with the idea being that this follows from a number of well-known facts regarding the partitions in  $P(k)$ . The problem now is that of proving directly this result.

## CHAPTER 15

### Operator algebras

#### 15a. Operator algebras

We have now a quite complete picture of free probability from a combinatorial point of view, in relation with basic questions from random matrices and quantum groups. In this chapter and the next one we go for the real thing, namely discussing the connections between free probability and selected topics from von Neumann algebra theory.

The von Neumann algebras are the algebras of bounded operators  $A \subset B(H)$  which are weakly closed, and we will see their basic properties in a moment. These algebras are important in the general context of quantum mechanics, somewhat of equal importance as the Lie groups and their generalizations, and with all this going back to the influential work and books of John von Neumann [91] and Hermann Weyl [96].

From a modern perspective, some time has passed since [91], [96], with quantum mechanics having evolved a bit, and finding what exact von Neumann algebra  $A \subset B(H)$  or Lie group  $G \subset GL_N(\mathbb{C})$  is to be studied, and what exactly is to be done with it, is already a quite difficult task. We will slowly make our way here, through these difficult questions, by always orienting our presentation towards things which seem modern.

Needless to say, all this will be quite subjective, and sometimes even controversial. The main problem, in connection with free probability, is that freeness, or rather “true and useful freeness” in von Neumann algebras appears where you would expect it the least, namely in the context of a certain special, innocent and far-from-free looking von Neumann algebra, namely the Murray-von Neumann hyperfinite factor  $R$ .

So, this will be our goal here, explaining what this  $R$  beast is, how freeness appears inside it, why all this is interesting, and why everyone thinking otherwise, with dozens of research directions to be discussed here, both old and more recent, is wrong.

In short, welcome to theoretical physics. Take it easy, and enjoy.

In order to get started now, recall from chapters 5-6 above that we already know a few things about the algebras of operators  $A \subset B(H)$ , which are norm closed. The von Neumann algebras will be by definition such algebras which are weakly closed. In order to discuss this, let us start with a standard result, as follows:

PROPOSITION 15.1. *For an algebra  $A \subset B(H)$ , the following are equivalent:*

- (1)  *$A$  is closed under the weak operator topology, making each of the linear maps  $T \rightarrow \langle Tx, y \rangle$  continuous.*
- (2)  *$A$  is closed under the strong operator topology, making each of the linear maps  $T \rightarrow Tx$  continuous.*

*In the case where these conditions are satisfied,  $A$  is closed under the norm topology.*

PROOF. There are several statements here, the proof being as follows:

(1) It is clear that the norm topology is stronger than the strong operator topology, which is in turn stronger than the weak operator topology. At the level of the subsets  $E \subset B(H)$  which are closed things get reversed, in the sense that weakly closed implies strongly closed, which in turn implies norm closed. Thus, we are left with proving that for any operator algebra  $A \subset B(H)$ , strongly closed implies weakly closed.

(2) But this latter fact is standard, and can be proved by using an amplification trick. Consider the Hilbert space obtained by summing  $k$  times  $H$  with itself:

$$H^+ = H \oplus \dots \oplus H$$

The operators over  $H^+$  can be regarded as being square matrices with entries in  $B(H)$ , and in particular, we have a representation  $\pi : B(H) \rightarrow B(H^+)$ , given by:

$$\pi(T) = \begin{pmatrix} T & & \\ & \ddots & \\ & & T \end{pmatrix}$$

Assume now that we are given an operator  $T \in \bar{A}$ , with the bar denoting the weak closure. We have, by using the Hahn-Banach theorem, for any  $\xi \in H^+$ :

$$\begin{aligned} T \in \bar{A} &\implies \pi(T) \in \overline{\pi(A)} \\ &\implies \pi(T)x \in \overline{\pi(A)\xi} \\ &\implies \pi(T)x \in \overline{\pi(A)\xi}^{\|\cdot\|} \end{aligned}$$

Now observe that the last formula tells us that for any  $\xi = (\xi_1, \dots, \xi_k)$ , and any  $\varepsilon > 0$ , we can find an operator  $S \in A$  such that the following holds, for any  $i$ :

$$\|S\xi_i - T\xi_i\| < \varepsilon$$

It follows that  $T$  belongs to the strong operator closure of  $A$ , as desired.  $\square$

In the above statement the terminology, while quite standard, is a bit confusing, because the norm topology is stronger than the strong operator topology. As a solution to this issue, we agree in what follows to call the norm topology “strong”, and the weak and strong operator topologies “weak”, whenever these two topologies coincide.

With this convention, the operator algebras  $A \subset B(H)$  from Proposition 15.1 are those which are weakly closed, and we can now formulate:

DEFINITION 15.2. *A von Neumann algebra is a  $*$ -algebra of operators*

$$A \subset B(H)$$

*which is closed under the weak topology.*

As basic examples, we have the algebra  $B(H)$  itself, then the singly generated von Neumann algebras,  $A = \langle T \rangle$ , with  $T \in B(H)$ , and then the multiply generated von Neumann algebras, namely  $A = \langle T_i \rangle$ , with  $T_i \in B(H)$ . There are many other examples, and also general methods for constructing examples, and we will discuss this later.

At the level of the general results, we first have the bicommutant theorem of von Neumann, which provides a useful alternative to Definition 15.2 above, as follows:

THEOREM 15.3. *For a  $*$ -algebra  $A \subset B(H)$ , the following are equivalent:*

- (1)  *$A$  is weakly closed, so it is a von Neumann algebra.*
- (2)  *$A$  equals its algebraic bicommutant  $A''$ , taken inside  $B(H)$ .*

PROOF. Since the commutants are weakly closed, it is enough to show that weakly closed implies  $A = A''$ . For this purpose, we will prove something a bit more general, stating that given a  $*$ -algebra of operators  $A \subset B(H)$ , the following holds, with  $A''$  being the bicommutant inside  $B(H)$ , and with  $\bar{A}$  being the weak closure:

$$A'' = \bar{A}$$

We can prove this by double inclusion, as follows:

“ $\supset$ ” Since any operator commutes with the operators that it commutes with, we have an inclusion  $E \subset E''$ , valid for any set  $E \subset B(H)$ . In particular, we have:

$$A \subset A''$$

Our claim now is that the algebra  $A'' \subset B(H)$  is closed, with respect to the strong operator topology. Indeed, assuming that we have  $T_i \rightarrow T$  in this topology, we have:

$$\begin{aligned} T_i \in A'' &\implies ST_i = T_iS, \forall S \in A' \\ &\implies ST = TS, \forall S \in A' \\ &\implies T \in A \end{aligned}$$

Thus our claim is proved, and together with Proposition 15.1, which allows to pass from the strong to the weak operator topology, this gives the desired inclusion, namely:

$$\bar{A} \subset A''$$

“C” Here we must prove that we have the following implication, valid for any operator  $T \in B(H)$ , with the bar denoting as usual the weak operator closure:

$$T \in A'' \implies T \in \bar{A}$$

For this purpose, we use the same amplification trick as in the proof of Proposition 15.1. Consider the Hilbert space obtained by summing  $k$  times  $H$  with itself:

$$H^+ = H \oplus \dots \oplus H$$

The operators over  $H^+$  can be regarded as being square matrices with entries in  $B(H)$ , and in particular, we have a representation  $\pi : B(H) \rightarrow B(H^+)$ , given by:

$$\pi(T) = \begin{pmatrix} T & & \\ & \ddots & \\ & & T \end{pmatrix}$$

The idea will be that of doing the computations in this latter representation. First, in this representation, the image of our algebra  $A \subset B(H)$  is given by:

$$\pi(A) = \left\{ \begin{pmatrix} T & & \\ & \ddots & \\ & & T \end{pmatrix} \mid T \in A \right\}$$

We can now compute the commutant of this image, exactly as in the usual scalar matrix case, and we obtain the following formula:

$$\pi(A)' = \left\{ \begin{pmatrix} S_{11} & \dots & S_{1k} \\ \vdots & & \vdots \\ S_{k1} & \dots & S_{kk} \end{pmatrix} \mid S_{ij} \in A' \right\}$$

We conclude from this that, given  $T \in A''$  as above, we have:

$$\begin{pmatrix} T & & \\ & \ddots & \\ & & T \end{pmatrix} \in \pi(A)''$$

In other words, the conclusion of all this is that we have the following implication:

$$T \in A'' \implies \pi(T) \in \pi(A)''$$

Now given  $\xi \in H^+$ , consider the orthogonal projection  $P \in B(H^+)$  on the norm closure of the vector space  $\pi(A)\xi \subset H^+$ . Since the subspace  $\pi(A)\xi \subset H^+$  is invariant under the action of  $\pi(A)$ , so is its norm closure inside  $H^+$ , and we obtain from this:

$$P \in \pi(A)'$$

By combining this with what we found above, we conclude that:

$$T \in A'' \implies \pi(T)P = P\pi(T)$$

Now since this holds for any  $\xi \in H^+$ , it follows that any  $T \in A''$  belongs to the strong operator closure of  $A$ . By using now Proposition 15.1, which allows us to pass from the strong to the weak operator closure, we conclude that we have  $A'' \subset \bar{A}$ , as desired.  $\square$

In order to develop now some general theory for the von Neumann algebras, let us start by investigating the commutative case. The result here is as follows:

**THEOREM 15.4.** *The commutative von Neumann algebras are the algebras of type*

$$A = L^\infty(X)$$

*with  $X$  being a measured space.*

**PROOF.** We have two assertions to be proved, the idea being as follows:

(1) In one sense, we must prove that given a measured space  $X$ , we can realize the commutative algebra  $A = L^\infty(X)$  as a von Neumann algebra, on a certain Hilbert space  $H$ . But this can be done as follows, using a probability measure on  $X$ :

$$L^\infty(X) \subset B(L^2(X)) \quad , \quad f \rightarrow (g \rightarrow fg)$$

(2) In the other sense, given a commutative von Neumann algebra  $A \subset B(H)$ , any operator  $T \in A$  is normal. So, let us pick a linear space basis  $\{T_i\} \subset A$ , as to have:

$$A = \langle T_i \rangle$$

The generators  $T_i \in B(H)$  are then commuting normal operators, and by using the spectral theorem for such families of operators, we obtain the result.  $\square$

The above result is very interesting, because it shows that an arbitrary von Neumann algebra  $A \subset B(H)$  can be thought of as being of the form  $A = L^\infty(X)$ , with  $X$  being a “quantum measured space”. Thus, we have here a connection with the various quantum group and noncommutative geometry considerations made before.

### 15b. Freeness, factors

Moving ahead now, we will be interested here in the “free” von Neumann algebras. These algebras, traditionally called factors, can be axiomatized as follows:

**DEFINITION 15.5.** *A factor is a von Neumann algebra  $A \subset B(H)$  whose center*

$$Z(A) = A \cap A'$$

*which is a commutative von Neumann algebra, reduces to the scalars,  $Z(A) = \mathbb{C}$ .*

Here the fact that the center is indeed a von Neumann algebra follows from the bicommutant theorem, which shows that the commutant of any  $*$ -algebra is a von Neumann algebra. Thus, the intersection  $Z(A) = A \cap A'$  is indeed a von Neumann algebra.

As already mentioned, Definition 15.5 comes somehow as something opposed to Theorem 15.4, the idea being that of axiomatizing the von Neumann algebras which are “free”,

with this meaning being as far as possible from the commutative ones. In what follows, we will mainly use this point of view on the factors, which fits our purposes.

Before going further, let us mention however that there are some deeper reasons as well for the consideration of the factors, which among others fully justify the term “factor”, coming from the following advanced theorem of von Neumann:

**THEOREM 15.6.** *Given a von Neumann algebra  $A \subset B(H)$ , if we write its center as*

$$Z(A) = L^\infty(X)$$

*then we have a decomposition as follows, with the fibers  $A_x$  being factors:*

$$A = \int_X A_x dx$$

*Moreover, in the case where  $A$  has a trace,  $tr : A \rightarrow \mathbb{C}$ , this trace decomposes as*

$$tr = \int_X tr_x dx$$

*with each  $tr_x : A_x \rightarrow \mathbb{C}$  being the restriction of  $tr$  to the factor  $A_x$ .*

**PROOF.** As a first observation, this is something that we know to hold in finite dimensions, because here the algebra decomposes as follows, with the summands corresponding precisely to the points of the spectrum of the center,  $Z(A) \simeq \mathbb{C}^k$ :

$$A = M_{N_1}(\mathbb{C}) \oplus \dots \oplus M_{N_k}(\mathbb{C})$$

In general, however, this is something quite difficult to prove, requiring a good knowledge of advanced operator theory and advanced functional analysis. We will not need this result in what follows, and we refer here to any good operator algebra book.  $\square$

Moving ahead now, in order to do probability on our factors we will need a trace as well. Leaving aside the somewhat trivial case  $A = M_N(\mathbb{C})$ , we are led in this way to:

**DEFINITION 15.7.** *A  $II_1$  factor is a von Neumann algebra  $A \subset B(H)$  which is infinite dimensional, has trivial center, and has a trace  $tr : A \rightarrow \mathbb{C}$ .*

As a first observation, according to Theorem 15.6, such factors are exactly those appearing in the spectral decomposition of the von Neumann algebras  $A \subset B(H)$  which have traces,  $tr : A \rightarrow \mathbb{C}$ , provided that we add some extra axioms which avoid trivial summands of type  $M_N(\mathbb{C})$ . Moreover, by results of Connes, adding to those of von Neumann, and which are non-trivial as well, the non-tracial case basically reduces to the tracial case, via certain crossed product type operations, and so the conclusion is that “the  $II_1$  factors are the building blocks of the von Neumann algebra theory”.

Summarizing, some heavy things going on here. In what follows we will be mainly interested in concrete mathematics and combinatorics, and we will take Definition 15.7 as it is, as a simple and intuitive definition for the “free von Neumann algebras”.

As a first observation, in practice, and forgetting about Theorem 15.6, it is not even clear that such beasts exist. Fortunately the group von Neumann algebras are there, and we have the following result, providing us with examples of  $\text{II}_1$  factors:

**THEOREM 15.8.** *The center of a group von Neumann algebra  $L(G)$  is*

$$Z(L(G)) = \left\{ \sum_g \lambda_g g \mid \lambda_{gh} = \lambda_{hg} \right\}''$$

and if  $G \neq \{1\}$  has infinite conjugacy classes, in the sense that

$$|\{ghg^{-1} \mid g \in G\}| = \infty \quad , \quad \forall h \neq 1$$

with this being called ICC property, the algebra  $L(G)$  is a  $\text{II}_1$  factor.

**PROOF.** There are two assertions here, the idea being as follows:

(1) Consider a linear combination of group elements, which is in the weak closure of the group algebra  $\mathbb{C}[G]$ , and so defines an element of the von Neumann algebra  $L(G)$ :

$$a = \sum_g \lambda_g g$$

By linearity, this element  $a \in L(G)$  belongs to the center of the algebra  $L(G)$  precisely when it commutes with all the group elements  $h \in G$ , and this gives:

$$\begin{aligned} a \in Z(A) &\iff ah = ha \\ &\iff \sum_g \lambda_g gh = \sum_g \lambda_g hg \\ &\iff \sum_k \lambda_{kh^{-1}} k = \sum_k \lambda_{h^{-1}k} k \\ &\iff \lambda_{kh^{-1}} = \lambda_{h^{-1}k} \end{aligned}$$

Thus, we obtain the formula for the center  $Z(L(G))$  in the statement.

(2) We have to examine here the 3 conditions defining the  $\text{II}_1$  factors. We already know, from chapter 6, that the group algebra  $L(G)$  has a trace, given by:

$$\text{tr}(g) = \delta_{g,1}$$

Regarding now the center, the condition  $\lambda_{gh} = \lambda_{hg}$  that we found above is equivalent to the fact that  $g \rightarrow \lambda_g$  is constant on the conjugacy classes, and we obtain:

$$Z(L(G)) = \mathbb{C} \iff G = \text{ICC}$$

Finally, assuming that this ICC condition is satisfied, and with  $G \neq \{1\}$ , then our group  $G$  is infinite, and so the algebra  $L(G)$  is infinite dimensional, as desired.  $\square$

In order to investigate now the  $\text{II}_1$  factors, the idea, following Murray and von Neumann [69], will be that of looking at the projections. We first have the following result:

PROPOSITION 15.9. *Given two projections  $p, q \in A$  in a  $\text{II}_1$  factor, we have*

$$p \preceq q \quad \text{or} \quad q \preceq p$$

*and so  $\preceq$  is a total order on the equivalence classes of projections  $p \in A$ .*

PROOF. This is something which actually holds for any factor, with the only non-trivial part being that of proving the following implication:

$$p \preceq q, q \preceq p \implies p \simeq q$$

But this is clear in the present  $\text{II}_1$  factor setting, by using the trace.  $\square$

We can now formulate a first main result regarding the  $\text{II}_1$  factors, as follows:

THEOREM 15.10. *Given a  $\text{II}_1$  factor  $A$ , the traces of the projections*

$$\text{tr}(p) \in [0, 1]$$

*can take any values in  $[0, 1]$ .*

PROOF. Given a number  $x \in [0, 1]$ , consider the following set:

$$E = \left\{ p^2 = p = p^* \in A \mid \text{tr}(p) \leq x \right\}$$

This set satisfies the assumptions of the Zorn lemma, and so by this lemma we can find a maximal element  $p \in E$ . Assume by contradiction that:

$$\text{tr}(p) < x$$

Now by using some standard operator theory arguments, we can slightly enlarge the trace of  $p$ , and we obtain a contradiction, as desired. See [69].  $\square$

As a second main result now regarding the  $\text{II}_1$  factors, also from [69], we have:

THEOREM 15.11. *The trace of a  $\text{II}_1$  factor*

$$\text{tr} : A \rightarrow \mathbb{C}$$

*is unique.*

PROOF. This can be proved in many ways, a standard one being that of proving that any two traces agree on the projections, as a consequence of the above results. We refer here to [69], or to our go-to operator algebra book by Blackadar [27].  $\square$

This was for the basic theory of the  $\text{II}_1$  factors, following Murray and von Neumann [69]. From a modern perspective, an interesting question is that of working out the quantum group analogue of Theorem 15.8, and also to work out the reduction theory of the quantum group algebras  $L(\Gamma)$ , and we refer here to [28] and related papers.

In relation to these questions, the conjecture is that if a compact quantum group  $G = \widehat{\Gamma}$  is free, in a suitable sense, then the corresponding von Neumann algebra  $L^\infty(G) = L(\Gamma)$  should be a  $\text{II}_1$  factor. More generally, the conjecture is that the algebras of type  $L^\infty(X)$ , with  $X$  being a free manifold in the sense of chapter 13, are  $\text{II}_1$  factors. These are all difficult questions, and once again we refer here to [28] and related papers.

Getting back now to generalities, from an abstract algebraic perspective, with a touch of functional analysis, a  $\text{II}_1$  factor  $A \subset B(H)$  is not really in need of the ambient Hilbert space  $H$ , and the question of “representing” it appears. We first have:

DEFINITION 15.12. *Given a von Neumann algebra  $A$  with a trace  $\text{tr} : A \rightarrow \mathbb{C}$ , the emdedding*

$$A \subset B(L^2(A))$$

*obtained by GNS construction is called standard form of  $A$ .*

Here we use the GNS construction, explained in chapter 6. As the name indicates, the standard representation is something “standard”, to be compared with any other representation  $A \subset B(H)$ , in order to understand this latter representation.

As already seen in chapter 6, the GNS construction has a number of unique features, that can be exploited. In the present setting, the main result is as follows:

PROPOSITION 15.13. *In the context of the standard representation we have*

$$A' = JAJ$$

*with  $J : L^2(A) \rightarrow L^2(A)$  being the antilinear map given by  $T \rightarrow T^*$ .*

PROOF. Observe first that any  $T \in A$  can be regarded as a vector  $T \in L^2(A)$ , to which we can associate, in an antilinear way, the vector  $T^* \in L^2(A)$ . Thus we have indeed an antilinear map  $J$  as in the statement. In terms of the standard cyclic and separating vector  $\Omega$  for the GNS representation, the formula of this map  $J$  is:

$$J(T\Omega) = T^*\Omega$$

But this gives the equality  $A' = JAJ$  in the statement, via double inclusion, with the proof of both inclusions being routine computations.  $\square$

As a basic illustration for the above result, the commutant of a group algebra  $L(G)$ , which is obtained by definition by using the left regular representation, is the group algebra  $R(G)$ , obtained by using the right regular representation.

We are now in position of constructing the coupling constant, a key notion, due again to Murray and von Neumann [69], which eventually closes the discussion regarding the various Hilbert space representations  $A \subset B(H)$  of a given  $\text{II}_1$  factor  $A$ :

**THEOREM 15.14.** *Given a representation of a  $\text{II}_1$  factor  $A \subset B(H)$ , we can talk about the corresponding coupling constant*

$$\dim_A H \in (0, \infty]$$

*which for the standard form, where  $H = L^2(A)$ , takes the value 1, and which in general measures how far is  $A \subset B(H)$  from the standard form.*

**PROOF.** There are several proofs for this fact, the idea being as follows:

(1) We can amplify the standard representation of  $A$ , on the Hilbert space  $L^2(A)$ , into a representation on  $L^2(A) \otimes l^2(\mathbb{N})$ , and then cut it down with a projection. We obtain in this way a whole family of embeddings  $A \subset B(H)$ , which are quite explicit.

(2) The point now is that of proving, via a technical  $2 \times 2$  matrix trick, that any representation  $A \subset B(H)$  appears in this way. In this picture, the coupling constant appears as the trace of the projection used to cut down  $L^2(A) \otimes l^2(\mathbb{N})$ .

(3) Thus, we are led to the conclusion in the statement. Alternatively, the coupling constant can be defined as follows, with the number on the right being independent of the choice on a nonzero vector  $x \in H$ , and with this being the original definition from [69]:

$$\dim_A H = \frac{\text{tr}_A(P_{A'x})}{\text{tr}_{A'}(P_{Ax})}$$

We refer to [69], or for instance to the book [27], for more details here.  $\square$

As an illustration for the above, given an inclusion of ICC groups  $H \subset G$ , whose group algebras are both  $\text{II}_1$  factors, we have the following formula:

$$\dim_{L(H)} L^2(G) = [G : H]$$

There are many other examples of explicit computations of coupling constants, all leading us into interesting mathematics. We will be back to this.

### 15c. Subfactor theory

Given a  $\text{II}_1$  factor  $A_0$ , let us discuss now the representations  $A_0 \subset A_1$ , with  $A_1$  being another  $\text{II}_1$  factor. This is a quite natural notion too, and perhaps even more natural than the representations  $A_0 \subset B(H)$  studied above, because we have decided in the above that the  $\text{II}_1$  factors  $A_1$ , and not the full operator algebras  $B(H)$ , are the correct infinite dimensional generalization of the usual matrix algebras  $M_N(\mathbb{C})$ .

Given an inclusion of  $\text{II}_1$  factors  $A_0 \subset A_1$ , a first question is that of defining its index, measuring how big is  $A_1$ , when compared to  $A_0$ . This can be done as follows:

THEOREM 15.15. *Given an inclusion of  $\text{II}_1$  factors  $A_0 \subset A_1$ , the number*

$$N = \frac{\dim_{A_0} H}{\dim_{A_1} H}$$

*is independent of the ambient Hilbert space  $H$ , and is called index.*

PROOF. This is standard, with the fact that the index as defined by the above formula is indeed independent of the ambient Hilbert space  $H$  coming from the various basic properties of the coupling constant, from Theorem 15.14 and its proof.  $\square$

There are many examples of subfactors coming from groups, and every time we obtain the intuitive index. We will be back with details in a moment. In general now, following Jones [55], let us start with the following standard result:

PROPOSITION 15.16. *Given a subfactor  $A_0 \subset A_1$ , there is a unique linear map*

$$E : A_1 \rightarrow A_0$$

*which is positive, unital, trace-preserving and which is such that, for any  $a_1, a_2 \in A_0$ :*

$$E(a_1 b a_2) = a_1 E(b) a_2$$

*This map is called conditional expectation from  $A_1$  onto  $A_0$ .*

PROOF. We make use of the standard representation of the  $\text{II}_1$  factor  $A_1$ , with respect to its unique trace  $\text{tr} : A_1 \rightarrow \mathbb{C}$ , as constructed before, namely:

$$A_1 \subset L^2(A_1)$$

If we denote by  $\Omega$  the standard cyclic and separating vector of  $L^2(A_1)$ , we have an identification of vector spaces  $A_0 \Omega = L^2(A_0)$ . Consider now the following projection:

$$e : L^2(A_1) \rightarrow L^2(A_0)$$

It follows from definitions that we have an inclusion  $e(A_1 \Omega) \subset A_0 \Omega$ . Thus the above projection  $e$  induces by restriction a certain linear map, as follows:

$$E : A_1 \rightarrow A_0$$

This linear map  $E$  and the orthogonal projection  $e$  are related by:

$$e x e = E(x) e$$

But this shows that the linear map  $E$  satisfies the various conditions in the statement, namely positivity, unitality, trace preservation and bimodule property. As for the uniqueness assertion, this follows by using the same argument, applied backwards, the idea being that a map  $E$  as in the statement must come from a projection  $e$ .  $\square$

Following Jones [55], we will be interested in what follows in the orthogonal projection  $e : L^2(A_1) \rightarrow L^2(A_0)$  producing the expectation  $E : A_1 \rightarrow A_0$ , rather than in  $E$  itself:

DEFINITION 15.17. *Associated to any subfactor  $A_0 \subset A_1$  is the orthogonal projection*

$$e : L^2(A_1) \rightarrow L^2(A_0)$$

*producing the conditional expectation  $E : A_1 \rightarrow A_0$  via the following formula:*

$$exe = E(x)e$$

*This projection is called Jones projection for the subfactor  $A_0 \subset A_1$ .*

Quite remarkably, the subfactor  $A_0 \subset A_1$ , as well as its commutant, can be recovered from the knowledge of this projection, in the following way:

PROPOSITION 15.18. *Given a subfactor  $A_0 \subset A_1$ , with Jones projection  $e$ , we have*

$$A_0 = A_1 \cap \{e\}' \quad , \quad A_0' = (A_1' \cap \{e\})''$$

*as equalities of von Neumann algebras, acting on the space  $L^2(A_1)$ .*

PROOF. The above two formulae both follow from  $exe = E(x)e$ , via some elementary computations, and for details here, we refer to Jones' paper [55].  $\square$

We are now ready to formulate a key definition, as follows:

DEFINITION 15.19. *Associated to any subfactor  $A_0 \subset A_1$  is the basic construction*

$$A_0 \subset_e A_1 \subset A_2$$

*with  $A_2 = \langle A_1, e \rangle$  being the algebra generated by  $A_1$  and by the Jones projection*

$$e : L^2(A_1) \rightarrow L^2(A_0)$$

*acting on the Hilbert space  $L^2(A_1)$ .*

The idea now, following [55], will be that  $A_1 \subset A_2$  appears as a kind of “reflection” of  $A_0 \subset A_1$ , and also that the basic construction can be iterated, and with all this leading to non-trivial results. Let us start by further studying the basic construction:

THEOREM 15.20. *Given a subfactor  $A_0 \subset A_1$  having finite index,*

$$[A_1 : A_0] < \infty$$

*the basic construction  $A_0 \subset_e A_1 \subset A_2$  has the following properties:*

- (1)  $A_2 = JA_0'J$ .
- (2)  $A_2 = \overline{A_1 + A_1eb}$ .
- (3)  $A_2$  is a  $\text{II}_1$  factor.
- (4)  $[A_2 : A_1] = [A_1 : A_0]$ .
- (5)  $eA_2e = A_0e$ .
- (6)  $\text{tr}(e) = [A_1 : A_0]^{-1}$ .
- (7)  $\text{tr}(xe) = \text{tr}(x)[A_1 : A_0]^{-1}$ , for any  $x \in A_1$ .

PROOF. All this is standard, the idea being as follows:

(1) We know that we have  $JA'_1J = A_1$  and  $JeJ = e$ , and this gives:

$$\begin{aligned} JA'_0J &= J < A'_1, e > J \\ &= < JA'_1J, JeJ > \\ &= < A_1, e > \\ &= A_2 \end{aligned}$$

(2) This simply follows from the fact that  $A_1 + A_1eA_1$  is closed under multiplication, and from the fact that we have  $exe = E(x)e$ .

(3) This follows from the fact, that we know from the above, that our finite index assumption  $[A_1 : A_0] < \infty$  is equivalent to the fact that the algebra  $A'_0$  is a factor. But this in turn is equivalent to the fact that  $A_2 = JA'_0J$  is a factor, as desired.

(4) This follows indeed from the following computation:

$$\begin{aligned} [A_2 : A_1] &= \frac{\dim_{A_1} L^2(A_1)}{\dim_{A_2} L^2(A_1)} \\ &= \frac{1}{\dim_{A_2} L^2(A_1)} \\ &= \frac{1}{\dim_{JA'_0J} L^2(A_1)} \\ &= \frac{1}{\dim_{A'_0} L^2(A_1)} \\ &= \dim_{A_0} L^2(A_1) \\ &= [A_1 : A_0] \end{aligned}$$

(5) This follows from (2) and from the formula  $exe = E(x)e$ .

(6) We have here the following computation:

$$\begin{aligned} 1 &= \dim_{A_0} L^2(A_0) \\ &= \dim_{A_0}(eL^2(A_1)) \\ &= \text{tr}_{A'_0}(e) \dim_{A_0}(L^2(A_1)) \\ &= \text{tr}_{A'_0}(a)[A_1 : A_0] \end{aligned}$$

Now since  $A_2 = JA'_0J$  and  $JeJ = e$ , we obtain from this, as desired:

$$\text{tr}(e) = \text{tr}_{JA'_0J}(JeJ) = \text{tr}_{A'_0}(e) = [A_1 : A_0]^{-1}$$

(7) We already know from (6) above that the formula in the statement holds for  $x = 1$ . In order to discuss the general case, observe first that for  $x, y \in A_0$  we have:

$$\text{tr}(xye) = \text{tr}(yex) = \text{tr}(yxe)$$

Thus the linear map  $x \rightarrow tr(xe)$  is a trace on  $A_0$ , and by uniqueness of the trace on  $A_0$ , we must have  $tr(xe) = c \cdot tr(x)$ , for a certain constant  $c > 0$ . Now by using (6) above we conclude that the constant must be  $c = [A_1 : A_0]^{-1}$ . Thus, we have proved the result for  $x \in A_0$ . The passage to the general case  $x \in A_1$  can be performed as follows:

$$\begin{aligned} tr(xe) &= tr(exe) \\ &= tr(E(x)e) \\ &= tr(E(x))c \\ &= tr(x)c \end{aligned}$$

Thus, we have proved the last formula in the statement, and we are done.  $\square$

The above result is quite interesting, potentially leading to some interesting mathematics, so let us perform now twice the basic construction, and see what we get. The result here, which is something more technical, at least at the first glance, is as follows:

PROPOSITION 15.21. *Associated to  $A_0 \subset A_1$  is the double basic construction*

$$A_0 \subset_e A_1 \subset_f A_2 \subset A_3$$

*with  $e : L^2(A_1) \rightarrow L^2(A_0)$  and  $f : L^2(A_2) \rightarrow L^2(A_1)$  having the following properties:*

$$fef = [A_1 : A_0]^{-1}f, \quad efe = [A_1 : A_0]^{-1}e$$

PROOF. We have two formulae to be proved, the idea being as follows:

(1) The first formula in the statement is clear, because we have:

$$fef = E(e)f = tr(e)f = [A_1 : A_0]^{-1}f$$

(2) Regarding now the second formula, it is enough to check this on the dense subset  $(A_1 + A_1eA_1)\Omega$ . Thus, we must show that for any  $x, y, z \in A_1$ , we have:

$$efe(x + yez)\Omega = [A_1 : A_0]^{-1}e(x + yez)\Omega$$

For this purpose, we will prove that we have, for any  $x, y, z \in A_1$ :

$$efex\Omega = [A_1 : A_0]^{-1}ex\Omega$$

$$efeyez\Omega = [A_1 : A_0]^{-1}eyez\Omega$$

But the first formula can be established as follows:

$$\begin{aligned} efex\Omega &= efexf\Omega \\ &= eE(ex)f\Omega \\ &= eE(e)xf\Omega \\ &= [A_1 : A_0]^{-1}exf\Omega \\ &= [A_1 : A_0]^{-1}ex\Omega \end{aligned}$$

As for the second formula, this can be established as follows:

$$\begin{aligned}
efeyez\Omega &= efeyezf\Omega \\
&= eE(eyez)f\Omega \\
&= eE(eye)zf\Omega \\
&= eE(E(y)e)zf\Omega \\
&= eE(y)E(e)zf\Omega \\
&= [A_1 : A_0]^{-1}eE(y)zf\Omega \\
&= [A_1 : A_0]^{-1}eyezf\Omega \\
&= [A_1 : A_0]^{-1}eyez\Omega
\end{aligned}$$

Thus, we are led to the conclusion in the statement.  $\square$

We can in fact perform the basic construction by recurrence, and we obtain:

**THEOREM 15.22.** *Associated to any subfactor  $A_0 \subset A_1$  is the Jones tower*

$$A_0 \subset_{e_1} A_1 \subset_{e_2} A_2 \subset_{e_3} A_3 \subset \dots$$

*with the Jones projections having the following properties:*

- (1)  $e_i^2 = e_i = e_i^*$ .
- (2)  $e_i e_j = e_j e_i$  for  $|i - j| \geq 2$ .
- (3)  $e_i e_{i \pm 1} e_i = [A_1 : A_0]^{-1} e_i$ .
- (4)  $tr(w e_{n+1}) = [A_1 : A_0]^{-1} tr(w)$ , for any word  $w \in \langle e_1, \dots, e_n \rangle$ .

**PROOF.** This follows from Theorem 15.20 and Proposition 15.21, because the triple basic construction does not need in fact any further study. See [55].  $\square$

The relations found in Theorem 15.22 are in fact well-known, from the standard theory of the Temperley-Lieb algebra. This algebra, discovered by Temperley and Lieb in the context of statistical mechanics [82], has a very simple definition, as follows:

**DEFINITION 15.23.** *The Temperley-Lieb algebra of index  $N \in [1, \infty)$  is defined as*

$$TL_N(k) = \text{span}(NC_2(k, k))$$

*with product given by vertical concatenation, with the rule*

$$\bigcirc = N$$

*for the closed circles that might appear when concatenating.*

In other words, the algebra  $TL_N(k)$ , depending on parameters  $k \in \mathbb{N}$  and  $N \in [1, \infty)$ , is the linear span of the pairings  $\pi \in NC_2(k, k)$ . The product operation is obtained by linearity, for the pairings which span  $TL_N(k)$  this being the usual vertical concatenation, with the conventions that things go “from top to bottom”, and that each circle that might appear when concatenating is replaced by a scalar factor, equal to  $N$ .

As already mentioned, this algebra was discovered by Temperley and Lieb in the context of general statistical mechanics, and we refer here to [82], and subsequent work. In what concerns us, we will just need some elementary results. First, we have:

PROPOSITION 15.24. *The Temperley-Lieb algebra  $TL_N(k)$  is generated by the diagrams*

$$\begin{aligned}\varepsilon_1 &= \bigcup \\ \varepsilon_2 &= \bigcup \\ \varepsilon_3 &= \bigcup \\ &\vdots\end{aligned}$$

*which are all multiples of projections, in the sense that their rescaled versions*

$$e_i = N^{-1}\varepsilon_i$$

*satisfy the abstract projection relations  $e_i^2 = e_i = e_i^*$ .*

PROOF. We have two assertions here, the idea being as follows:

(1) The fact that the Temperley-Lieb algebra  $TL_N(k)$  is indeed generated by the sequence  $\varepsilon_1, \varepsilon_2, \dots$  follows by drawing pictures, and more specifically by decomposing each basis element  $\pi \in NC_2(k, k)$  as a product of such elements  $\varepsilon_i$ .

(2) Regarding now the projection assertion, when composing  $\varepsilon_i$  with itself we obtain  $\varepsilon_i$  itself, times a circle. Thus, according to our multiplication convention, we have:

$$\varepsilon_i^2 = N\varepsilon_i$$

Also, when turning upside-down  $\varepsilon_i$ , we obtain  $\varepsilon_i$  itself. Thus, according to our involution convention for the Temperley-Lieb algebra, we have the following formula:

$$\varepsilon_i^* = \varepsilon_i$$

We conclude that the rescalings  $e_i = N^{-1}\varepsilon_i$  satisfy  $e_i^2 = e_i = e_i^*$ , as desired.  $\square$

As a second result now, making the link with Theorem 15.22, we have:

PROPOSITION 15.25. *The standard generators  $e_i = N^{-1}\varepsilon_i$  of the Temperley-Lieb algebra  $TL_N(k)$  have the following properties, where  $tr$  is the trace obtained by closing:*

- (1)  $e_i e_j = e_j e_i$  for  $|i - j| \geq 2$ .
- (2)  $e_i e_{i \pm 1} e_i = N^{-1} e_i$ .
- (3)  $tr(w e_{n+1}) = N^{-1} tr(w)$ , for any word  $w \in \langle e_1, \dots, e_n \rangle$ .

PROOF. This follows indeed by doing some elementary computations with diagrams, in the spirit of those performed in the proof of Proposition 15.24.  $\square$

With the above results in hand, and still following Jones' paper [55], we can now reformulate Theorem 15.22 into something more conceptual, as follows:

THEOREM 15.26. *Given a subfactor  $A_0 \subset A_1$ , construct its the Jones tower:*

$$A_0 \subset_{e_1} A_1 \subset_{e_2} A_2 \subset_{e_3} A_3 \subset \dots$$

*The rescaled sequence of projections  $e_1, e_2, e_3, \dots \in B(H)$  produces then a representation*

$$TL_N \subset B(H)$$

*of the Temperley-Lieb algebra of index  $N = [A_1 : A_0]$ .*

PROOF. We know from Theorem 15.22 that the rescaled sequence of projections  $e_1, e_2, e_3, \dots \in B(H)$  behaves algebraically exactly as the following  $TL_N$  diagrams:

$$\begin{aligned} \varepsilon_1 &= \bigcup \\ \varepsilon_2 &= \bigcup \\ \varepsilon_3 &= \bigcup \\ &\vdots \end{aligned}$$

But these diagrams generate  $TL_N$ , and so we have an embedding  $TL_N \subset B(H)$ , where  $H$  is the Hilbert space where our subfactor  $A_0 \subset A_1$  lives, as claimed.  $\square$

Before going further, with some examples, more theory, and consequences of Theorem 15.26 above, let us make the following key observation, also from [55]:

THEOREM 15.27. *Given a finite index subfactor  $A_0 \subset A_1$ , the graded algebra  $P = (P_k)$  formed by the sequence of higher relative commutants*

$$P_k = A'_0 \cap A_k$$

*contains the copy of the Temperley-Lieb algebra constructed above,  $TL_N \subset P$ . This graded algebra  $P = (P_k)$  is called “planar algebra” of the subfactor.*

PROOF. As a first observation, since the Jones projection  $e_1 : A_1 \rightarrow A_0$  commutes with  $A_0$ , we have  $e_1 \in P_2$ . By translation we obtain, for any  $k \in \mathbb{N}$ :

$$e_1, \dots, e_{k-1} \in P_k$$

Thus we have indeed an inclusion of graded algebras  $TL_N \subset P$ , as claimed.  $\square$

The point with the above result, which explains among others the terminology at the end, is that, in the context of Theorem 15.26 above, the planar algebra structure of  $TL_N$ , obtained by composing diagrams, extends into an abstract planar algebra structure of  $P$ . See [57]. We will discuss all this, with full details, in the next chapter.

As an interesting consequence of Theorem 15.26, somehow contradicting the “continuous geometry” philosophy that has been going on so far, in relation with the  $\text{II}_1$  factors, we have the following surprising result, also from Jones’ original paper [55]:

THEOREM 15.28. *The index of subfactors  $A_0 \subset A_1$  is “quantized” in the  $[1, 4]$  range,*

$$N \in \left\{ 4 \cos^2 \left( \frac{\pi}{n} \right) \mid n \geq 3 \right\} \cup [4, \infty]$$

*with the obstruction coming from the existence of the representation  $TL_N \subset B(H)$ .*

PROOF. This comes from the combinatorics of  $e_1, e_2, e_3, \dots$ , as follows:

(1) In order to best comment on what happens, when iterating the basic construction, let us record the first few values of the numbers in the statement, namely:

$$\begin{aligned} 4 \cos^2 \left( \frac{\pi}{3} \right) &= 1 \quad , \quad 4 \cos^2 \left( \frac{\pi}{4} \right) = 2 \\ 4 \cos^2 \left( \frac{\pi}{5} \right) &= \frac{3 + \sqrt{5}}{2} \quad , \quad 4 \cos^2 \left( \frac{\pi}{6} \right) = 3 \end{aligned}$$

(2) By using a basic construction, we get, by trace manipulations on  $e_1$ :

$$N \notin (1, 2)$$

With a double basic construction, we get, by trace manipulations on  $\langle e_1, e_2 \rangle$ :

$$N \notin \left( 2, \frac{3 + \sqrt{5}}{2} \right)$$

And so on. Thus, by doing computations, we are led to the conclusion in the statement, by a kind of recurrence, involving a certain family of orthogonal polynomials.

(3) In practice now, following [55], the most elegant way of proving the result is by using the fact, explained in Theorem 15.26, that that sequence of Jones projections  $e_1, e_2, e_3, \dots \subset B(H)$  generate a copy of the Temperley-Lieb algebra of index  $N$ :

$$TL_N \subset B(H)$$

With this result in hand, we must prove that such a representation cannot exist in index  $N < 4$ , unless we are in the following special situation:

$$N = 4 \cos^2 \left( \frac{\pi}{n} \right)$$

But this can be proved by using some suitable trace and positivity manipulations on  $TL_N$ , as in (2) above, and for full details here, we refer to Jones’ paper [55].  $\square$

### 15d. Basic examples

Let us discuss now some basic examples of subfactors, with concrete illustrations for all the above notions, constructions, and general theory. These examples will all come from group actions  $G \curvearrowright Q$ , which are assumed to be minimal, in the sense that:

$$(Q^G)' \cap Q = \mathbb{C}$$

As a starting point, we have the following result, heavily used by Jones [55]:

PROPOSITION 15.29. *Assuming that  $G$  is a compact group, acting minimally on a  $\text{II}_1$  factor  $Q$ , and that  $H \subset G$  is a subgroup of finite index, we have a subfactor*

$$Q^G \subset Q^H$$

*having index  $N = [G : H]$ , called Jones subfactor.*

PROOF. This is something standard, the idea being that the factoriality of  $Q^G, Q^H$  comes from the minimality of the action, and that the index formula is clear.  $\square$

Along the same lines, we have the following result, due to Ocneanu [91]:

PROPOSITION 15.30. *Assuming that  $G$  is a finite group, acting minimally on a  $\text{II}_1$  factor  $Q$ , we have a subfactor as follows,*

$$Q \subset Q \rtimes G$$

*having index  $N = |G|$ , called Ocneanu subfactor.*

PROOF. This is standard as well, the idea being that the factoriality of  $Q \rtimes G$  comes from the minimality of the action, and that the index formula is clear.  $\square$

We have as well a third result of the same type, due to Wassermann [94], namely:

PROPOSITION 15.31. *Assuming that  $G$  is a compact group, acting minimally on a  $\text{II}_1$  factor  $Q$ , and that  $G \rightarrow \text{PU}_n$  is a projective representation, we have a subfactor*

$$Q^G \subset (M_n(\mathbb{C}) \otimes Q)^G$$

*having index  $N = n^2$ , called Wassermann subfactor.*

PROOF. As before, the idea is that the factoriality of  $Q^G, (M_n(\mathbb{C}) \otimes Q)^G$  comes from the minimality of the action, and the index formula is clear.  $\square$

The above subfactors look quite related, and indeed they are, due to:

THEOREM 15.32. *The Jones, Ocneanu and Wassermann subfactors are all of the same nature, and can be written as follows,*

$$\begin{aligned} (Q^G \subset Q^H) &\simeq ((\mathbb{C} \otimes Q)^G \subset (l^\infty(G/H) \otimes Q)^G) \\ (Q \subset Q \rtimes G) &\simeq ((l^\infty(G) \otimes Q)^G \subset (\mathcal{L}(l^2(G)) \otimes Q)^G) \\ (Q^G \subset (M_n(\mathbb{C}) \otimes Q)^G) &\simeq ((\mathbb{C} \otimes Q)^G \subset (M_n(\mathbb{C}) \otimes Q)^G) \end{aligned}$$

*with standard identifications for the various tensor products and fixed point algebras.*

PROOF. This is something standard, from [7], modulo all kinds of standard identifications. We will explain all this more in detail later, after unifying these subfactors.  $\square$

In order to unify now the above constructions of subfactors, following [7], [94], the idea is quite clear. Given a compact group  $G$ , acting minimally on a  $\text{II}_1$  factor  $Q$ , and an inclusion of finite dimensional algebras  $B_0 \subset B_1$ , endowed as well with an action of  $G$ , we would like to construct a kind of generalized Wassermann subfactor, as follows:

$$(B_0 \otimes Q)^G \subset (B_1 \otimes Q)^G$$

In order to do this, we must talk first about the finite dimensional algebras  $B$ , and about inclusions of such algebras  $B_0 \subset B_1$ . Let us start with the following definition:

**DEFINITION 15.33.** *Associated to any finite dimensional algebra  $B$  is its canonical trace, obtained by composing the left regular representation with the trace of  $\mathcal{L}(B)$ :*

$$\text{tr} : B \subset \mathcal{L}(B) \rightarrow \mathbb{C}$$

*We say that an inclusion of finite dimensional algebras  $B_0 \subset B_1$  is Markov if it commutes with the canonical traces of  $B_0, B_1$ .*

In what regards the first notion, that of the canonical trace, this is something that we know well, from chapter 13. Indeed, as explained there, we can formally write  $B = C(X)$ , with  $X$  being a finite quantum space, and the canonical trace  $\text{tr} : B \rightarrow \mathbb{C}$  is then precisely the integration with respect to the “counting measure” on  $X$ .

In what regards the second notion, that of a Markov inclusion, this is something very natural too, and as a first example here, any inclusion of type  $\mathbb{C} \subset B$  is Markov. In general, if we write  $B_0 = C(X_0)$  and  $B_1 = C(X_1)$ , then the inclusion  $B_0 \subset B_1$  must come from a certain fibration  $X_1 \rightarrow X_0$ , and the inclusion  $B_0 \subset B_1$  is Markov precisely when the fibration  $X_1 \rightarrow X_0$  commutes with the respective counting measures.

We will be back to Markov inclusions and their various properties on several occasions, in what follows. For our next purposes here, we just need the following result:

**PROPOSITION 15.34.** *Given a Markov inclusion of finite dimensional algebras  $B_0 \subset B_1$  we can perform to it the basic construction, as to obtain a Jones tower*

$$B_0 \subset_{e_1} B_1 \subset_{e_2} B_2 \subset_{e_3} B_3 \subset \dots$$

*exactly as we did in the above for the inclusions of  $\text{II}_1$  factors.*

**PROOF.** This is something standard, from [55], by following the computations in the above, from the case of the  $\text{II}_1$  factors, and with everything extending well. It is of course possible to do something more general here, unifying the constructions for the inclusions of  $\text{II}_1$  factors  $A_0 \subset A_1$ , and for the inclusions of Markov inclusions of finite dimensional algebras  $B_0 \subset B_1$ , but we will not need this degree of generality, in what follows.  $\square$

With these ingredients in hand, getting back now to the Jones, Ocneanu and Wassermann subfactors, from Theorem 15.32, the point is that these constructions can be unified, and then further studied, the final result on the subject being as follows:

**THEOREM 15.35.** *Let  $G$  be a compact group, and  $G \rightarrow \text{Aut}(Q)$  be a minimal action on a  $\text{II}_1$  factor. Consider a Markov inclusion of finite dimensional algebras*

$$B_0 \subset B_1$$

*and let  $G \rightarrow \text{Aut}(B_1)$  be an action which leaves invariant  $B_0$ , and which is such that its restrictions to the centers of  $B_0$  and  $B_1$  are ergodic. We have then a subfactor*

$$(B_0 \otimes Q)^G \subset (B_1 \otimes Q)^G$$

*of index  $N = [B_1 : B_0]$ , called generalized Wassermann subfactor, whose Jones tower is*

$$(B_1 \otimes Q)^G \subset (B_2 \otimes Q)^G \subset (B_3 \otimes Q)^G \subset \dots$$

*where  $\{B_i\}_{i \geq 1}$  are the algebras in the Jones tower for  $B_0 \subset B_1$ , with the canonical actions of  $G$  coming from the action  $G \rightarrow \text{Aut}(B_1)$ , and whose planar algebra is given by:*

$$P_k = (B'_0 \cap B_k)^G$$

*These subfactors generalize the Jones, Ocneanu and Wassermann subfactors.*

**PROOF.** This is something which is routine, from [7], following Wassermann [94], and we will be back to this in a moment, with details, directly in a more general setting.  $\square$

In addition to the Jones, Ocneanu and Wassermann subfactors, discussed and unified in the above, we have the Popa subfactors, which are constructed as follows:

**PROPOSITION 15.36.** *Given a discrete group  $\Gamma = \langle g_1, \dots, g_n \rangle$ , acting faithfully via outer automorphisms on a  $\text{II}_1$  factor  $P$ , we have the following “diagonal” subfactor*

$$\left\{ \begin{pmatrix} g_1(q) & & \\ & \ddots & \\ & & g_n(q) \end{pmatrix} \middle| q \in P \right\} \subset M_n(P)$$

*having index  $N = n^2$ , called Popa subfactor.*

**PROOF.** This is something standard, a bit as for the Jones, Ocneanu and Wassermann subfactors, with the result basically coming from the work of Popa [72], [73], who was the main user of such subfactors. We will come in a moment with a more general result in this direction, involving discrete quantum groups, along with a complete proof.  $\square$

In order to unify now Theorem 15.35 and Proposition 15.36, observe that the diagonal subfactors can be written in the following way, by using a group dual:

$$(P \rtimes \Gamma)^{\widehat{\Gamma}} \subset (M_n(\mathbb{C}) \otimes (P \rtimes \Gamma))^{\widehat{\Gamma}}$$

Here the group dual  $\widehat{\Gamma}$  acts on  $Q = P \rtimes \Gamma$  via the dual of the action  $\Gamma \subset \text{Aut}(P)$ , and on  $M_n(\mathbb{C})$  via the adjoint action of the following formal representation:

$$\oplus g_i : \widehat{\Gamma} \rightarrow \mathbb{C}^n$$

Summarizing, we are led into quantum groups. So, let us start with:

DEFINITION 15.37. A coaction of a Woronowicz algebra  $A$  on a finite von Neumann algebra  $Q$  is an injective morphism  $\Phi : Q \rightarrow Q \otimes A''$  satisfying the following conditions:

- (1) Coassociativity:  $(\Phi \otimes id)\Phi = (id \otimes \Delta)\Phi$ .
- (2) Trace equivariance:  $(tr \otimes id)\Phi = tr(\cdot)1$ .
- (3) Smoothness:  $\overline{Q}^w = Q$ , where  $Q = \Phi^{-1}(Q \otimes_{alg} A)$ .

The above conditions come from what happens in the commutative case,  $A = C(G)$ , where they correspond to the usual associativity, trace equivariance and smoothness of the corresponding action  $G \curvearrowright Q$ . In general, what we have is a generalization of this.

Along the same lines, but more technically now, we have as well:

DEFINITION 15.38. A coaction  $\Phi : Q \rightarrow Q \otimes A''$  as above is called:

- (1) Ergodic, if the algebra  $Q^\Phi = \{p \in Q \mid \Phi(p) = p \otimes 1\}$  reduces to  $\mathbb{C}$ .
- (2) Faithful, if the span of  $\{(f \otimes id)\Phi(Q) \mid f \in Q_*\}$  is dense in  $A''$ .
- (3) Minimal, if it is faithful, and satisfies  $(Q^\Phi)' \cap Q = \mathbb{C}$ .

Observe that the minimality of the action implies in particular that the fixed point algebra  $Q^\Phi$  is a factor. Thus, we are getting here to the case that we are interested in, actions producing factors, via their fixed point algebras. More on this later.

In order to prove our subfactor results, we need of some general theory regarding the minimal actions. Following Wassermann [94], let us start with the following definition:

DEFINITION 15.39. Let  $\Phi : Q \rightarrow Q \otimes A''$  be a coaction. An eigenmatrix for a corepresentation  $u \in B(H) \otimes A$  is an element  $M \in B(H) \otimes Q$  satisfying:

$$(id \otimes \Phi)M = M_{12}u_{13}$$

A coaction is called semidual if each corepresentation has a unitary eigenmatrix.

As a basic example here, the canonical coaction  $\Delta : A \rightarrow A \otimes A$  is semidual. Following the work of Wassermann in the usual compact group case, we have:

THEOREM 15.40. The minimal coactions are semidual.

PROOF. Let  $S$  be the set of finite dimensional unitary corepresentations of  $A$  which have unitary eigenmatrices. It is then routine to check that  $S$  is stable under:

- (1) Making sums.
- (2) Making tensor products.
- (3) Performing substractions.
- (4) Taking complex conjugates.

By using now the Peter-Weyl theory for the compact quantum groups, due to Woronowicz [98], we conclude that we have  $S = Rep(A)$ , which gives the result. See [7].  $\square$

Following [7], let us construct now the fixed point subfactors. We first have:

PROPOSITION 15.41. *Consider a Woronowicz algebra  $A = (A, \Delta, S)$ , and denote by  $A_\sigma$  the Woronowicz algebra  $(A, \sigma\Delta, S)$ , where  $\sigma$  is the flip. Given two coactions*

$$\beta : B \rightarrow B \otimes A$$

$$\pi : Q \rightarrow Q \otimes A_\sigma$$

*with  $B$  being finite dimensional, the following linear map, while not being multiplicative in general, is coassociative with respect to the comultiplication  $\sigma\Delta$  of  $A_\sigma$ ,*

$$\beta \odot \pi : B \otimes Q \rightarrow B \otimes Q \otimes A_\sigma$$

$$b \otimes p \rightarrow \pi(p)_{23}((id \otimes S)\beta(b))_{13}$$

*and its fixed point space, which is by definition the following linear space,*

$$(B \otimes Q)^{\beta \odot \pi} = \left\{ x \in B \otimes Q \mid (\beta \odot \pi)x = x \otimes 1 \right\}$$

*is then a von Neumann subalgebra of  $B \otimes Q$ .*

PROOF. This is something standard, which follows from a straightforward algebraic verification, explained in [7]. As mentioned in the statement, to be noted is that the tensor product coaction  $\beta \odot \pi$  is not multiplicative in general. See [7].  $\square$

Our first task is to investigate the factoriality of such algebras, and we have here:

THEOREM 15.42. *If  $\beta : B \rightarrow B \otimes A$  is a coaction and  $\pi : Q \rightarrow Q \otimes A_\sigma$  is a minimal coaction, then the following conditions are equivalent:*

- (1) *The von Neumann algebra  $(B \otimes Q)^{\beta \odot \pi}$  is a factor.*
- (2) *The coaction  $\beta$  is centrally ergodic,  $Z(B) \cap B^\beta = \mathbb{C}$ .*

PROOF. This is something standard, from [7], the idea being as follows:

(1) Our first claim, which is something whose proof is a routine verification, explained in [7], based on the semiduality of the minimal coaction  $\pi$ , that we know from Theorem 15.40, is that the following diagram is a non-degenerate commuting square:

$$\begin{array}{ccc} Q & \subset & B \otimes Q \\ \cup & & \cup \\ Q^\pi & \subset & (B \otimes Q)^{\beta \odot \pi} \end{array}$$

(2) In order to prove now the result, it is enough to check the following equality, between subalgebras of the von Neumann algebra  $B \otimes Q$ :

$$Z((B \otimes Q)^{\beta \odot \pi}) = (Z(B) \cap B^\beta) \otimes 1$$

So, let  $x$  be in the algebra on the left. Then  $x$  commutes with  $1 \otimes Q^\pi$ , so it has to be of the form  $b \otimes 1$ . Thus  $x$  commutes with  $1 \otimes Q$ . But  $x$  commutes with  $(B \otimes Q)^{\beta \odot \pi}$ , and from the non-degeneracy of the above square,  $x$  commutes with  $B \otimes Q$ , and in particular with  $B \otimes 1$ . Thus  $b \in Z(B) \cap B^\beta$ . As for the other inclusion, this is obvious.  $\square$

With the above technical results in hand, we can now formulate our main theorem regarding the fixed point subfactors, of the most possible general type, as follows:

**THEOREM 15.43.** *Let  $G$  be a compact quantum group, and  $G \rightarrow \text{Aut}(Q)$  be a minimal action on a  $\text{II}_1$  factor. Consider a Markov inclusion of finite dimensional algebras*

$$B_0 \subset B_1$$

*and let  $G \rightarrow \text{Aut}(B_1)$  be an action which leaves invariant  $B_0$  and which is such that its restrictions to the centers of  $B_0$  and  $B_1$  are ergodic. We have then a subfactor*

$$(B_0 \otimes Q)^G \subset (B_1 \otimes Q)^G$$

*of index  $N = [B_1 : B_0]$ , called generalized Wassermann subfactor, whose Jones tower is*

$$(B_1 \otimes Q)^G \subset (B_2 \otimes Q)^G \subset (B_3 \otimes Q)^G \subset \dots$$

*where  $\{B_i\}_{i \geq 1}$  are the algebras in the Jones tower for  $B_0 \subset B_1$ , with the canonical actions of  $G$  coming from the action  $G \rightarrow \text{Aut}(B_1)$ , and whose planar algebra is given by:*

$$P_k = (B'_0 \cap B_k)^G$$

*These subfactors generalize the Jones, Ocneanu, Wassermann and Popa subfactors.*

**PROOF.** This is something routine, based on the above general theory and results, and for the full story here, and technical details, we refer to [7], [94].  $\square$

The above result is important in connection with probability questions, because our usual character computations for  $G$ , for instance in the case where  $G \subset U_N^+$  is easy, take place in the associated planar algebra  $P_k = (B'_0 \cap B_k)^G$ . More on this later.

## 15e. Exercises

There have been many things in this chapter, and as a first exercise, we have:

**EXERCISE 15.44.** *Learn some further fundamental von Neumann algebra theory, and write a brief account of what you learned.*

This is something quite loose, and things here can vary a lot. Ideally, you should learn about reduction theory, hyperfiniteness, and the combination of the two.

**EXERCISE 15.45.** *Learn some further fundamental Jones subfactor theory, and write a brief account of what you learned.*

As before, this is something quite loose, and things here can vary a lot. We will actually be back to this in the next chapter, with more generalities about subfactors.

## CHAPTER 16

### Quantum algebra

#### 16a. Planar algebras

We have seen in the previous chapter the foundations of von Neumann algebra theory, which naturally leads into factors, and then into subfactors. In this final chapter we discuss what can be done with all this, and hopefully, with some help from free probability.

As already mentioned in the previous chapter, things here will be quite tricky, not to say controversial, or even bitterly controversial, requiring a number of smart choices, based on personal feeling, and knowledge of mathematics and physics. Hang on.

Taking for granted the fact that the factors are the most interesting von Neumann algebras, which is something largely agreed upon, a first crossroads appears when talking about subfactors. Shall we really do this, or stay with the factors? More specifically, we have two types of problems that we can try to solve, as follows:

(1) On one hand we can try to further investigate the  $\text{II}_1$  factors, which are by definition the “free” von Neumann algebras, with tools from free probability. An interesting question here, which actually was the main motivation of Voiculescu, when starting free probability [85], is that of finding probabilistic tools in order to distinguish the group von Neumann algebras  $L(\Gamma)$ . For instance it is unknown whether  $L(F_2)$  is isomorphic to  $L(F_3)$  or not, and Voiculescu’s idea from his later paper [89] was that a clever, entropy type invariant can produce 2 for  $L(F_2)$  and 3 for  $L(F_3)$ , as to prove the non-isomorphism. This is certainly nice, and although a lot of work has gone into such questions [25], [43], [52], [60], [76], [77], [89], the problem is still open, and waiting for volunteers.

(2) On the other hand, we can simply declare that we are overwhelmed by the beauty of Jones’ subfactor discovery, concerning the combinatorics of the Jones projections  $e_1, e_2, e_3, \dots$ , which leads into the Temperley-Lieb algebra  $TL_N$ , so forget about pure mathematics, forget about free probability, forget about everything, and let us just stick with this. And with the remark of course that by doing so, we are not far from free probability, and this because the simplest examples of subfactors are the fixed point subfactors  $(B_0 \otimes Q)^G \subset (B_1 \otimes Q)^G$ , whose simplest instances are those coming from the simplest quantum groups  $G \subset U_N^+$ , which are our old friends the easy quantum groups. And so that no worries, we are certainly into free probability here.

We will choose here (2), isn't quantum mechanics a bit crazy, so let us be a bit crazy too. Now with this choice made, we are facing another crossroads, about what to do with our subfactors. Obviously, everything happens inside the planar algebra  $P_n = A'_0 \cap A_n$ , and with this in mind, we have again two types of questions to be solved:

(3) A first question is that of understanding the exact relation between the planar algebra  $P = (P_n)$  and the original subfactor  $A_0 \subset A_1$ . This was something basically solved by Popa in [73], by using algebras  $A_0, A_1$  of free group factor type, and incorporated by Jones into his planar algebra theory, in [57]. So, no interesting questions here, with the whole thing being quite technical, basically downgrading us to (1) above, although a more modern version of all this was worked out by Guionnet, Jones and Shlyakhtenko in [49]. On the positive side, however, there is a big interesting question, going back to [55], regarding the axiomatization of the planar algebras appearing from hyperfinite subfactors, in continuation of another paper of Popa, namely [72].

(4) A second question is that of forgetting about the subfactor  $A_0 \subset A_1$  itself, and rather focusing on the planar algebra  $P = (P_n)$ . Indeed, in the case of the simplest among the simplest subfactors, namely the fixed point subfactors  $(B_0 \otimes Q)^G \subset (B_1 \otimes Q)^G$  coming from easy quantum groups  $G \subset U_N^+$ , the corresponding planar algebra  $P = (P_n)$  is what is needed for computing laws of characters, doing free probability, Weingarten calculus, and many more things. Thus, we would to understand whether such things can be done or not with an arbitrary planar algebra  $P = (P_n)$ . As for the subfactor  $A_0 \subset A_1$  itself, we can conjecturally assume that it is a hyperfinite subfactor, and patiently wait for an improvement of the above-mentioned paper of Popa [72], before getting into it.

Summarizing, what we have here are two closely related directions of research, with everything gravitating around the notion of hyperfinite subfactor, and Popa's paper [72]. However, passed this key common problem, which is extremely difficult, with no valuable idea in sight, at least so far, (3,4) above appear quite different, with (3) rather bringing us back to (1), and with (4) being most likely the future. So, we will choose here (4).

This was for our general discussion, and as a conclusion, we would like first to talk about planar algebras  $P = (P_n)$ , then about the "laws of characters" associated to such planar algebras, making of course the link with free probability, and even extending free probability to this broad, virgin territory which is subfactor theory, and finally talking about Weingarten integration and other advanced things, again in the subfactor context. And with all this being promised for some further discussion, later, involving von Neumann algebras and functional analysis, with the relevant von Neumann algebra being, perhaps a bit surprisingly, the good old Murray-von Neumann hyperfinite factor  $R$ .

So, many things to be explained. Let us start with a discussion regarding planar algebras. As a starting point here, we have the following result, from chapter 15:

THEOREM 16.1. *Given an inclusion of  $\text{II}_1$  factors  $A_0 \subset A_1$ , with Jones tower*

$$A_0 \subset_{e_1} A_1 \subset_{e_2} A_2 \subset_{e_3} A_3 \subset \dots$$

*the sequence of projections  $e_1, e_2, e_3, \dots \in B(H)$  produces a representation*

$$TL_N \subset B(H)$$

*of the Temperley-Lieb algebra of index  $N = [A_1 : A_0]$ . Moreover, we have*

$$TL_N \subset P$$

*where  $P = (P_n)$  is the graded algebra formed by the commutants  $P_n = A'_0 \cap A_n$ .*

PROOF. This is the main result in [55], that we know well from chapter 15, coming from a detailed study of the iterated basic construction, which leads to the conclusion that the corresponding sequence of Jones projections  $e_1, e_2, e_3, \dots \in B(H)$  behaves algebraically exactly as the sequence of standard generators  $\varepsilon_1, \varepsilon_2, \varepsilon_3, \dots \in TL_N$ .  $\square$

Quite remarkably, the planar algebra structure of  $TL_N$ , taken in an intuitive sense, of composing diagrams, extends to a planar algebra structure on  $P$ . In order to discuss this, let us start with axioms for the planar algebras. Following Jones [57], we have:

DEFINITION 16.2. *The planar algebras are defined as follows:*

- (1) *We consider rectangles in the plane, with the sides parallel to the coordinate axes, and taken up to planar isotopy, and we call such rectangles boxes.*
- (2) *A labelled box is a box with  $2n$  marked points on its boundary,  $n$  on its upper side, and  $n$  on its lower side, for some integer  $n \in \mathbb{N}$ .*
- (3) *A tangle is labelled box, containing a number of labelled boxes, with all marked points, on the big and small boxes, being connected by noncrossing strings.*
- (4) *A planar algebra is a sequence of finite dimensional vector spaces  $P = (P_n)$ , together with linear maps  $P_{n_1} \otimes \dots \otimes P_{n_k} \rightarrow P_n$ , one for each tangle, such that the gluing of tangles corresponds to the composition of linear maps.*

In this definition we are using rectangles, but everything being up to isotopy, we could have used instead circles with marked points, as in [57]. Our choice for using rectangles comes from the main examples that we have in mind, to be discussed below, where the planar algebra structure is best viewed by using rectangles, as above.

Let us also mention that Definition 16.2 is something quite simplified, based on [57]. As explained in [57], in order for subfactors to produce planar algebras and vice versa, there are quite a number of supplementary axioms that must be added, and in view of this, it is perhaps better to start with something stronger than Definition 16.2, as basic axioms. However, as before with rectangles vs circles, our axiomatic choices here are mainly motivated by the concrete examples that we have in mind. More on this later.

As a basic example of a planar algebra, we have the Temperley-Lieb algebra:

THEOREM 16.3. *The Temperley-Lieb algebra  $TL_N$ , viewed as graded algebra*

$$TL_N = (TL_N(n))_{n \in \mathbb{N}}$$

*is a planar algebra, with the corresponding linear maps associated to the planar tangles*

$$TL_N(n_1) \otimes \dots \otimes TL_N(n_k) \rightarrow TL_N(n)$$

*appearing by putting the various  $TL_N(n_i)$  diagrams into the small boxes of the given tangle, which produces a  $TL_N(n)$  diagram.*

PROOF. This is something trivial, which follows from definitions:

(1) Assume indeed that we are given a planar tangle  $\pi$ , as in Definition 16.2, consisting of a box having  $2n$  marked points on its boundary, and containing  $k$  small boxes, having respectively  $2n_1, \dots, 2n_k$  marked points on their boundaries, and then a total of  $n + \sum n_i$  noncrossing strings, connecting the various  $2n + \sum 2n_i$  marked points.

(2) We want to associate to this tangle  $\pi$  a linear map as follows:

$$T_\pi : TL_N(n_1) \otimes \dots \otimes TL_N(n_k) \rightarrow TL_N(n)$$

For this purpose, by linearity, it is enough to construct elements as follows, for any choice of Temperley-Lieb diagrams  $\sigma_i \in TL_N(n_i)$ , with  $i = 1, \dots, k$ :

$$T_\pi(\sigma_1 \otimes \dots \otimes \sigma_k) \in TL_N(n)$$

(3) But constructing such an element is obvious, just by putting the various diagrams  $\sigma_i \in TL_N(n_i)$  into the small boxes the given tangle  $\pi$ . Indeed, this procedure produces a certain diagram in  $TL_N(n)$ , that we can call  $T_\pi(\sigma_1 \otimes \dots \otimes \sigma_k)$ , as above.

(4) Finally, we have to check that everything is well-defined up to planar isotopy, and that the gluing of tangles corresponds to the composition of linear maps. But both these checks are trivial, coming from the definition of  $TL_N$ , and we are done.  $\square$

As a conclusion to all this,  $P = TL_N$  is indeed a planar algebra, but of somewhat “trivial” type, with the triviality coming from the fact that, in this case, the elements of  $P$  are planar diagrams themselves, and so the planar structure appears trivially.

The Temperley-Lieb planar algebra  $TL_N$  is however an important planar algebra, because it is the “smallest” one, appearing inside the planar algebra of any subfactor. But more on this later, when talking about planar algebras and subfactors.

Moving ahead now, here is our second basic example of a planar algebra, which is also “trivial” in the above sense, with the elements of the planar algebra being planar diagrams themselves, but which appears in a bit more complicated way:

THEOREM 16.4. *The Fuss-Catalan algebra  $FC_{N,M}$ , which appears by coloring the Temperley-Lieb diagrams with black/white colors, clockwise, as follows*

$$\circ \bullet \bullet \circ \circ \bullet \bullet \circ \dots \circ \bullet \bullet \circ$$

*and keeping those diagrams whose strings connect either  $\circ - \circ$  or  $\bullet - \bullet$ , is a planar algebra, with again the corresponding linear maps associated to the planar tangles*

$$FC_{N,M}(n_1) \otimes \dots \otimes FC_{N,M}(n_k) \rightarrow FC_{N,M}(n)$$

*appearing by putting the various  $FC_{N,M}(n_i)$  diagrams into the small boxes of the given tangle, which produces a  $FC_{N,M}(n)$  diagram.*

PROOF. The proof here is nearly identical to the proof of Theorem 16.3, with the only change appearing at the level of the colors. To be more precise:

(1) Forgetting about upper and lower sequences of points, which must be joined by strings, a Temperley-Lieb diagram can be thought of as being a collection of strings, say black strings, which compose in the obvious way, with the rule that the value of the circle, which is now a black circle, is  $N$ . And it is this obvious composition rule that gives the planar algebra structure, as explained in the proof of Theorem 16.3.

(2) Similarly, forgetting about points, a Fuss-Catalan diagram can be thought of as being a collection of strings, which come now in two colors, black and white. These Fuss-Catalan diagrams compose then in the obvious way, with the rule that the value of the black circle is  $N$ , and the value of the white circle is  $M$ . And it is this obvious composition rule that gives the planar algebra structure, as before for  $TL_N$ .  $\square$

Getting back now to generalities, and to Definition 16.2, that of a general planar algebra, we have so far two illustrations for it, which, while both important, are both “trivial”, with the planar structure simply coming from the fact that, in both these cases, the elements of the planar algebra are planar diagrams themselves.

In general, the planar algebras can be more complicated than this, and we will see some further examples in a moment. However, the idea is very simple, namely “the elements of a planar algebra are not necessarily diagrams, but they behave like diagrams”.

In relation now with subfactors, the result, which extends Theorem 16.1, and which was found by Jones in [57], almost 20 years after [55], is as follows:

THEOREM 16.5. *Given a subfactor  $A_0 \subset A_1$ , the collection  $P = (P_n)$  of linear spaces*

$$P_n = A'_0 \cap A_n$$

*has a planar algebra structure, extending the planar algebra structure of  $TL_N$ .*

PROOF. We know from Theorem 16.1 that we have an inclusion as follows, coming from the basic construction, and with  $TL_N$  itself being a planar algebra:

$$TL_N \subset P$$

Thus, the whole point is that of proving that the trivial planar algebra structure of  $TL_N$  extends into a planar algebra structure of  $P$ . But this can be done via a long algebraic study, and for the full computation here, we refer to Jones' paper [57].  $\square$

As a first illustration for the above result, we have:

THEOREM 16.6. *We have the following universality results:*

- (1) *The Temperley-Lieb algebra  $TL_N$  appears inside the planar algebra of any subfactor  $A_0 \subset A_1$  having index  $N$ .*
- (2) *The Fuss-Catalan algebra  $FC_{N,M}$  appears inside the planar algebra of any subfactor  $A_0 \subset A_1$ , in the presence of an intermediate subfactor  $A_0 \subset B \subset A_1$ .*

PROOF. Here the first assertion is something that we already know, from Theorem 16.1, and the second assertion is something quite standard as well, by carefully working out the basic construction for  $A_0 \subset A_1$ , in the presence of an intermediate subfactor  $A_0 \subset B \subset A_1$ . For details here, we refer to the paper of Bisch and Jones [26].  $\square$

As a free probability comment here, the Temperley-Lieb algebra, which appears by definition as the span of  $NC_2$ , is certainly a free probability object, and one way of being more concrete here is by saying that suitable fixed point subfactors associated to  $S_N^+, O_N^+, U_N^+$  have planar algebra equal to  $TL_N$ . See [6], [6], [7]. As in what regards the Fuss-Catalan algebra, this is related to the bicolored partitions appearing in the study of  $H_N^+$ , and more generally of  $H_N^{s+}$ , and again, the precise subfactor statement about this concerns fixed point subfactors associated to the quantum groups  $H_N^{s+}$ . See [8], [80].

The above results raise the question on whether any planar algebra produces a subfactor. The answer here is yes, but with many subtleties, and in order to talk about this, we first need to introduce a certain distinguished  $II_1$  factor  $R$ , as follows:

DEFINITION 16.7. *The Murray-von Neumann hyperfinite  $II_1$  factor is*

$$R = \overline{\bigcup_i M_{n_i}(\mathbb{C})}^w$$

*independently of the choice of the algebras  $M_{n_i}(\mathbb{C})$ , and of the embeddings between them.*

To be more precise, all this is based on two theorems of Murray and von Neumann [69], stating on one hand that when performing the above inductive limit construction we obtain, after taking the weak closure, a certain  $II_1$  factor, and on the other hand, that the factor that we obtain is independent on the choice of the algebras  $M_{n_i}(\mathbb{C})$ , and of the embeddings between them. All this is certainly non-trivial, and even less trivial is the following theorem, coming as a continuation of the work in [69], due to Connes [36]:

**THEOREM 16.8.** *The Murray-von Neumann  $\text{II}_1$  factor  $R$  is the unique  $\text{II}_1$  factor which is amenable, in the sense that we have a conditional expectation as follows:*

$$E : B(H) \rightarrow R$$

*In particular, for a discrete group  $\Gamma$  we have  $L(\Gamma) = R$  precisely when  $\Gamma \neq \{1\}$  has the ICC property, and is amenable in the usual group theory sense.*

**PROOF.** This is something terribly complicated, to the point of causing troubles not only to mathematicians, and no surprise here, but to physicists as well. In case you know a good physicist, best is to ask that physicist, but there is no guarantee here, guy might well be clueless on all this. So, read from time to time operator algebras, say from [27], and once ready go through [36]. And in the meantime do not hesitate to ask around, this being a good test for distinguishing good physicists from first-class physicists.  $\square$

Jokes left aside now, what is difficult in the above is the proof of “amenability implies hyperfiniteness”. Indeed, the converse can only be something standard, namely proving that a certain concrete algebra,  $R$  from Definition 16.7, has a certain concrete property. As for the last assertion, this cannot be complicated either, because one of the possible definitions of the amenability of  $\Gamma$  is in terms of an invariant mean  $m : l^\infty(\Gamma) \rightarrow \mathbb{C}$ , and this makes the connection with the expectation  $E : B(l^2(\Gamma)) \rightarrow L(\Gamma)$ . See [36].

Getting back now to subfactors, and to our questions regarding the correspondence between subfactors and planar algebras, these are difficult questions too, and the various answers to these questions can be summarized, a bit informally, as follows:

**THEOREM 16.9.** *We have the following results:*

- (1) *Any planar algebra with positivity produces a subfactor.*
- (2) *In particular, we have  $TL$  and  $FC$  type subfactors.*
- (3) *In the amenable case, and with  $A_1 = R$ , the correspondence is bijective.*
- (4) *In general, we must take  $A_1 = L(F_\infty)$ , and we do not have bijectivity.*
- (5) *The axiomatization of  $P$ , in the case  $A_1 = R$ , is not known.*

**PROOF.** All this is quite heavy, basically coming from the work of Popa in the 90s, using heavy functional analysis, the idea being as follows:

(1) As already mentioned in the comments after Definition 16.2, our planar algebra axioms here are something quite simplified, based on [57]. However, when getting back to Theorem 16.6, the conclusion is that the subfactor planar algebras there satisfy a number of supplementary “positivity” conditions, basically coming from the positivity of the  $\text{II}_1$  factor trace. And the point is that, with these positivity conditions axiomatized, we reach to something which is equivalent to Popa’s axiomatization of the lattice of higher relative commutants  $A'_i \cap A_j$  of the finite index subfactors [73], obtained in the 90s via heavy functional analysis. For the full story here, and details, we refer to Jones’ paper [57].

(2) The existence of the  $TL_N$  subfactors, also known as “ $A_\infty$  subfactors”, is something which was known for some time, since some early work of Popa on the subject. As for the existence of the  $FC_{N,M}$  subfactors, this can be shown by using the intermediate subfactor picture,  $A_0 \subset B \subset A_1$ , by composing two  $A_\infty$  subfactors of suitable indices,  $A_0 \subset B$  and  $B \subset A_1$ . For the full story here, we refer to [26], [57].

(3) This is something fairly heavy, as it is always the case with operator algebra results regarding hyperfiniteness and amenability, due to Popa [72], [73].

(4) This is something a bit more recent, obtained by further building on the above-mentioned constructions of Popa, and we refer here to [49] and related work.

(5) This is the big open question in subfactors. The story here goes back to Jones’ original paper [55], which contains at the end the question, due to Connes, of finding the possible values of the index for the irreducible subfactors of  $R$ . This question, which certainly looks much easier than (5) in the statement, is in fact still open, now 40 years after its formulation, and with on one having any valuable idea in dealing with it.  $\square$

In relation with question (5) above, let us mention that, even in the case of the simplest subfactors, namely the fixed point subfactors  $(B_0 \otimes Q)^G \subset (B_1 \otimes P)^Q$ , things are far from being understood. Passed some algebra, to be explained in what follows, the whole thing comes down to the question on whether, given a closed subgroup  $G \subset U_N^+$ , we can find or not a  $\text{II}_1$  factor  $Q$ , with a minimal action  $G \curvearrowright Q$ , such that the following happens:

$$Q^G \simeq R$$

This is something non-trivial, and many efforts, starting with Ocneanu [71], have focused on constructing examples of type  $Q = R \rtimes \Gamma$ , with  $\Gamma = \widehat{G}$ , based on:

**PROPOSITION 16.10.** *Assuming that a discrete group or quantum group  $\Gamma$  has an action  $\Gamma \curvearrowright R$  which is outer, we can form the crossed product*

$$Q = R \rtimes \Gamma$$

*and then we have a dual action  $G \curvearrowright P$  of the compact quantum group  $G = \widehat{\Gamma}$ , which is minimal, and with the corresponding fixed point algebra being  $Q^G = R$ .*

**PROOF.** This is indeed something trivial, coming straight from the definitions of the various quantum groups and actions involved, explained in chapter 15.  $\square$

Summarizing, we are led in this way to the construction of outer actions  $\Gamma \curvearrowright R$ . And the subject here is quite interesting, related to a wide array of questions from mathematics and physics, with the central notion being that of a matrix model. All this is quite technical, and for more on all this, we refer to [7], [28], [71], [84] and related papers.

Let us mention however that the conjectural answer to these quantum group action questions is “yes”, and so from the perspective of the associated fixed point subfactors,

which can be chosen to be hyperfinite, we are led to the conclusion that the best algebra for doing free probability is, guess who, the good old hyperfinite  $\text{II}_1$  factor  $R$ .

Needless to say, you may agree or not with this. Up to you here, and in any case always remember that we have only one life given in this world. Whether we should use this life for doing physics inside  $R$ , or mathematics inside  $L(F_\infty)$ , is up to you.

### 16b. Bipartite graphs

Let us discuss now, as a continuation of the above, what exactly happens for the fixed point subfactors. We recall from chapter 15 that the main examples of subfactors are all of integer index, appearing as particular cases of the following construction:

**THEOREM 16.11.** *Given a compact quantum group  $G$ , acting on a Markov inclusion of finite dimensional algebras  $B_0 \subset B_1$ , and acting minimally on a  $\text{II}_1$  factor  $Q$ ,*

$$(B_0 \otimes Q)^G \subset (B_1 \otimes Q)^G$$

*is a subfactor, of same index as  $B_0 \subset B_1$ , whose planar algebra is*

$$P_n = (B'_0 \cap B_n)^G$$

*where  $\{B_n\}$  are the algebras in the Jones tower for  $B_0 \subset B_1$ .*

**PROOF.** This is something that we know from chapter 15, with the key step in the proof being that of proving that  $\{(B_n \otimes Q)^G\}$  is the Jones tower for the subfactor.  $\square$

In order to further advance, the idea will be that of associating to the original inclusion  $B_0 \subset B_1$  a certain combinatorial planar algebra  $P(B_0 \subset B_1)$ , as for the planar algebra associated to the fixed point subfactor itself to appear as follows:

$$P = P(B_0 \subset B_1)^G$$

In practice, we will need for all this the notion of planar algebra of a bipartite graph, constructed by Jones in [58]. So, let  $\Gamma$  be a bipartite graph, with vertex set  $\Gamma_a \cup \Gamma_b$ . It is useful to think of  $\Gamma$  as being the Bratteli diagram of an inclusion  $A \subset B$ .

Our first task is to define the graded vector space  $P$ . Since the elements of  $P$  will be subject to a planar calculus, it is convenient to introduce them in boxes, as follows:

**DEFINITION 16.12.** *Associated to  $\Gamma$  is the abstract vector space  $P_n$  spanned by the  $2n$ -loops based at points of  $\Gamma_a$ . The basis elements of  $P_n$  will be denoted*

$$x = \begin{pmatrix} e_1 & e_2 & \cdots & e_n \\ e_{2n} & e_{2n-1} & \cdots & e_{n+1} \end{pmatrix}$$

*where  $e_1, e_2, \dots, e_{2n}$  is the sequence of edges of the corresponding  $2n$ -loop.*

Consider now the adjacency matrix of  $\Gamma$ , which is a matrix of the following type:

$$M = \begin{pmatrix} 0 & m \\ m^t & 0 \end{pmatrix}$$

We pick first an  $M$ -eigenvalue  $\gamma \neq 0$ , and then a  $\gamma$ -eigenvector, as follows:

$$\eta : \Gamma_a \cup \Gamma_b \rightarrow \mathbb{C} - \{0\}$$

With this data in hand, we have the following construction, of Jones [58]:

DEFINITION 16.13. *Associated to any tangle is the multilinear map*

$$T(x_1 \otimes \dots \otimes x_k) = \gamma^c \sum_x \delta(x_1, \dots, x_k, x) \prod_m \mu(e_m)^{\pm 1} x$$

where the objects on the right are as follows:

- (1) The sum is over the basis of  $P_n$ , and  $c$  is the number of circles of  $T$ .
- (2)  $\delta = 1$  if all strings of  $T$  join pairs of identical edges, and  $\delta = 0$  if not.
- (3) The product is over all local maxima and minima of the strings of  $T$ .
- (4)  $e_m$  is the edge of  $\Gamma$  labelling the string passing through  $m$  (when  $\delta = 1$ ).
- (5)  $\mu(e) = \sqrt{\eta(e_f)/\eta(e_i)}$ , where  $e_i, e_f$  are the initial and final vertex of  $e$ .
- (6) The  $\pm$  sign is  $+$  for a local maximum, and  $-$  for a local minimum.

In other words, we plug the loops  $x_1, \dots, x_k$  into the input boxes of  $T$ , then we construct the “output”, as being the sum of all loops  $x$  satisfying the compatibility condition  $\delta = 1$ , altered by certain normalization factors, coming from the eigenvector  $\eta$ .

Let us work out now the precise formula of the action, for 6 carefully chosen tangles, which are of key importance for the considerations to follow. This will be useful as well as an introduction to Jones’ result in [58], stating that  $P$  is a planar algebra:

DEFINITION 16.14. *We have the following examples of tangles:*

- (1) Identity  $1_n$ : the  $(n, n)$ -tangle having  $2n$  vertical strings.
- (2) Multiplication  $M_n$ : the  $(n, n, n)$ -tangle having  $3n$  vertical strings.
- (3) Inclusion  $I_n$ : the  $(n, n+1)$ -tangle like  $1_n$ , with an extra string at right.
- (4) Shift  $J_n$ : the  $(n, n+2)$ -tangle like  $1_n$ , with two extra strings at left.
- (5) Expectation  $U_n$ : the  $(n+1, n)$ -tangle like  $1_n$ , with a curved string at right.
- (6) Jones projection  $E_n$ : the  $(n+2)$ -tangle having two semicircles at right.

Let us look first at the identity tangle  $1_n$ . Since the solutions of  $\delta(x, y) = 1$  are the pairs of the form  $(x, x)$ , this tangle acts by the identity:

$$1_n \begin{pmatrix} f_1 & \dots & f_n \\ e_1 & \dots & e_n \end{pmatrix} = \begin{pmatrix} f_1 & \dots & f_n \\ e_1 & \dots & e_n \end{pmatrix}$$

A similar argument applies to the multiplication tangle  $M_n$ , which acts as follows:

$$M_n \left( \begin{pmatrix} f_1 & \cdots & f_n \\ e_1 & \cdots & e_n \end{pmatrix} \otimes \begin{pmatrix} h_1 & \cdots & h_n \\ g_1 & \cdots & g_n \end{pmatrix} \right) = \delta_{f_1 g_1} \cdots \delta_{f_n g_n} \begin{pmatrix} h_1 & \cdots & h_n \\ e_1 & \cdots & e_n \end{pmatrix}$$

Regarding now the inclusion tangle  $I_n$ , the solutions of  $\delta(x_0, x) = 1$  being the elements  $x$  obtained from  $x_0$  by adding to the right a vector of the form  $\begin{pmatrix} g \\ g \end{pmatrix}$ , we have:

$$I_n \begin{pmatrix} f_1 & \cdots & f_n \\ e_1 & \cdots & e_n \end{pmatrix} = \sum_g \begin{pmatrix} f_1 & \cdots & f_n & g \\ e_1 & \cdots & e_n & g \end{pmatrix}$$

The same method applies to the shift tangle  $J_n$ , whose action is given by:

$$J_n \begin{pmatrix} f_1 & \cdots & f_n \\ e_1 & \cdots & e_n \end{pmatrix} = \sum_{gh} \begin{pmatrix} g & h & f_1 & \cdots & f_n \\ g & h & e_1 & \cdots & e_n \end{pmatrix}$$

Let us record now some partial conclusions, coming from the above formulae:

**PROPOSITION 16.15.** *The graded vector space  $P = (P_n)$  constructed above becomes a graded algebra, with the multiplication map given by*

$$xy = M_n(x \otimes y)$$

*on each vector space  $P_n$ , and with the above inclusion maps  $I_n$ . Also, the shift tangle  $J_n$  acts as an injective morphism of algebras  $P_n \rightarrow P_{n+2}$ .*

**PROOF.** The fact that the multiplication is associative follows from its formula above, which is nothing but a generalization of the usual matrix multiplication. The assertions about the inclusions and shifts follow as well by using their above explicit formula.  $\square$

Let us go back now to the remaining tangles in Definition 16.14. The usual method applies to the expectation  $U_n$ , which acts with a spin factor, as follows:

$$U_n \begin{pmatrix} f_1 & \cdots & f_n & h \\ e_1 & \cdots & e_n & g \end{pmatrix} = \delta_{gh} \mu(g)^2 \begin{pmatrix} f_1 & \cdots & f_n \\ e_1 & \cdots & e_n \end{pmatrix}$$

As for the Jones projection tangle  $E_n$ , this has no input box, so we can only apply it to the unit of  $\mathbb{C}$ . And when doing so, we obtain the following element:

$$E_n(1) = \sum_{egh} \mu(g) \mu(h) \begin{pmatrix} e_1 & \cdots & e_n & h & h \\ e_1 & \cdots & e_n & g & g \end{pmatrix}$$

Once again, let us record some partial conclusions, coming from these formulae:

**PROPOSITION 16.16.** *The elements  $e_k = \gamma^{-1} E_n(1)$  are projections, and define a representation of the Temperley-Lieb algebra, as follows:*

$$TL(\gamma) \rightarrow P$$

*The maps  $U_n$  are bimodule morphisms with respect to  $I_n$ , and their composition is the canonical trace on the image of  $TL(\gamma)$ .*

PROOF. The proof of all these assertions is standard, by using the fact that  $\eta$  is a  $\gamma$ -eigenvector of the adjacency matrix. Note that the statement itself is just a generalization of the usual Temperley-Lieb algebra representation on tensors, from [57].  $\square$

In fact, more generally, we have the following result, from Jones' paper [58]:

THEOREM 16.17. *The graded linear space  $P = (P_n)$ , together with the action of the planar tangles given above, is a planar algebra.*

PROOF. This is something which is quite routine, starting from the above study of the main planar algebra tangles, which are known to generate the whole set of planar tangles. For full details on all this, we refer to Jones' paper [58].  $\square$

In order to formulate our main result, regarding the subfactors associated to the compact quantum groups  $G$ , we will need a few abstract notions. Let us start with:

DEFINITION 16.18. *Let  $P_1, P_2$  be two finite dimensional algebras, coming with coactions  $\alpha_i : P_i \rightarrow P_i \otimes L^\infty(G)$ , and let  $T : P_1 \rightarrow P_2$  be a linear map.*

- (1) *We say that  $T$  is  $G$ -equivariant if  $(T \otimes id)\alpha_1 = \alpha_2 T$ .*
- (2) *We say that  $T$  is weakly  $G$ -equivariant if  $T(P_1^G) \subset P_2^G$ .*

Consider now a planar algebra  $P = (P_k)$ . The annular category over  $P$  is the collection of maps  $T : P_k \rightarrow P_l$  coming from the annular tangles, having at most one input box. These maps form sets  $Hom(k, l)$ , and these sets form a category [57]. We have:

DEFINITION 16.19. *A coaction of  $L^\infty(G)$  on  $P$  is a graded algebra coaction*

$$\alpha : P \rightarrow P \otimes L^\infty(G)$$

*such that the annular tangles are weakly  $G$ -equivariant.*

This definition is something a bit technical, but this is what comes out of the known examples that we have, all coming from the fixed point subfactors. In fact, as we will show below, the examples are basically those coming from actions of quantum groups on Markov inclusions  $B_0 \subset B_1$ , under the following abelianity assumption:

$$[B_0, B_1] = 0$$

For the moment, at the generality level of Definition 16.19, we have:

PROPOSITION 16.20. *If  $G$  acts on a planar algebra  $P$ , then  $P^G$  is a planar algebra.*

PROOF. The weak equivariance condition tells us that the annular category is contained in the suboperad  $\mathcal{P}' \subset \mathcal{P}$  consisting of tangles which leave invariant  $P^G$ . On the other hand the multiplicativity of  $\alpha$  gives  $M_n \in \mathcal{P}'$ , for any  $n$ . Now since  $\mathcal{P}$  is generated by multiplications and annular tangles, we obtain  $\mathcal{P}' = \mathcal{P}$ , and we are done.  $\square$

Let us go back now to the abelian inclusions. We have the following result:

PROPOSITION 16.21. *If  $G$  acts on an abelian inclusion  $A \subset B$ , the canonical extension of this coaction to the Jones tower is a coaction of  $G$  on the planar algebra  $P(A \subset B)$ .*

PROOF. We know from the above that, as a graded algebra,  $P = P(A \subset B)$  coincides with the Jones tower for our inclusion, that we can denote as follows:

$$A \subset B \subset A_1 \subset B_1 \subset \dots$$

Thus the coaction in the statement is a graded coaction, as follows:

$$\alpha : P \rightarrow P \otimes L^\infty(G)$$

In order to finish, we have to prove that the annular tangles are weakly equivariant, in the sense of Definition 16.18, and this can be done as follows:

(1) First, since the annular category is generated by the tangles  $I_n, E_n, U_n, J_n$ , we just have to prove that these 4 particular tangles are weakly equivariant. Now since  $I_n, E_n, U_n$  are plainly equivariant, by construction of the coaction of  $G$  on the Jones tower, it remains to prove that the shift  $J_n$  is weakly equivariant.

(2) We know that the image of the fixed point subfactor shift  $J'_n$  is formed by the  $G$ -invariant elements of the commutant  $A'_1 \cap P_{n+2} = P_n$ . Now since this commutant is the image of the planar shift  $J_n$ , we have  $Im(J_n) = Im(J'_n)$ , and this gives the result.  $\square$

With the above result in hand, we can now prove:

PROPOSITION 16.22. *Assume that  $G$  acts on an abelian inclusion  $A \subset B$ . Then the graded vector space of fixed points  $P(A \subset B)^G$  is a planar subalgebra of  $P(A \subset B)$ .*

PROOF. This follows indeed from Proposition 16.20 and Proposition 16.21.  $\square$

We are now in position of stating and proving our main result:

THEOREM 16.23. *In the abelian case, the planar algebra of a fixed point subfactor*

$$(B_0 \otimes Q)^G \subset (B_1 \otimes Q)^G$$

*is the fixed point algebra  $P(B_0 \subset B_1)^G$  of the bipartite graph algebra  $P(B_0 \subset B_1)$ .*

PROOF. This basically follows from what we have, as follows:

(1) Let  $P = P(B_0 \subset B_1)$ , and let  $R$  be the planar algebra of the fixed point subfactor. We know that we have an equality of graded algebras  $R = P^G$ . Thus, it remains to prove that the planar algebra structure on  $R$  coming from the fixed point subfactor agrees with the planar algebra structure of  $P$ , the one from Theorem 16.17.

(2) Since  $\mathcal{P}$  is generated by the annular category  $\mathcal{A}$  and by the multiplications  $M_n$ , we just have to check that the annular tangles agree on  $R, P$ . Moreover, since  $\mathcal{A}$  is generated by  $I_n, E_n, U_n, J_n$ , we just have to check that these tangles agree on  $R, P$ .

(3) We know that  $R \subset P$  is an inclusion of graded algebras, that all Jones projections for  $P$  are contained in  $R$ , and that the conditional expectations agree. Thus the tangles  $I_n, E_n, U_n$  agree on  $R, P$ , and the only verification left is that for the shift  $J_n$ .

(4) Now by using either the axioms of Popa in [73], or the construction of Jones in [55], it is enough to show that the image of the subfactor shift  $J'_n$  coincides with that of the planar shift  $J_n$ . But this follows as in the proof of Proposition 16.22.  $\square$

This was for the basic theory of the bipartite planar algebras, and their subalgebras. For further details on all the above, and for more, we refer to [7], [58], [80].

### 16c. Spectral measures

In what follows we discuss various structure and classification questions for the subfactors, all interesting questions, related to physics, regarded from a probabilistic viewpoint. In order to get started, we need invariants for our subfactors. We have the choice here between algebraic and analytic invariants, the situation being as follows:

DEFINITION 16.24. *Associated to any finite index subfactor  $A_0 \subset A_1$ , having planar algebra  $P = (P_n)$ , are the following invariants:*

- (1) *Its principal graph  $\Gamma$ , which describes the inclusions  $P_0 \subset P_1 \subset P_2 \subset \dots$ , with the reflections coming from basic constructions removed.*
- (2) *Its fusion algebra  $F$ , which describes the fusion rules for the various types of bimodules that can appear, namely  $A_0 - A_0$ ,  $A_0 - A_1$ ,  $A_1 - A_0$ ,  $A_1 - A_1$ .*
- (3) *Its Poincaré series  $f$ , which is the generating series of the graded components of the planar algebra,  $f(z) = \sum_n \dim(P_n)z^n$ .*
- (4) *Its spectral measure  $\mu$ , which is the probability measure having as moments the dimensions of the planar algebra components,  $\int x^n d\mu(x) = \dim(P_n)$ .*

This definition is of course something a bit informal, and there is certainly some work to be done, in order to fully define all these invariants  $\Gamma, F, f, \mu$ , and to work out the precise relation between them. We will be back to this later, but for the moment, let us keep in mind the fact that associated to a given subfactor  $A \subset B$  are several invariants, which are not exactly equivalent, but are definitely versions of the same thing, the “combinatorics of the subfactor”, and which come in algebraic or analytic flavors.

More in detail, let us begin by explaining how the principal graph  $\Gamma$  is constructed. Consider a finite index irreducible subfactor  $A_0 \subset A_1$ , with associated planar algebra  $P_n = A'_0 \cap A_n$ , and let us look at the following system of inclusions:

$$P_0 \subset P_1 \subset P_2 \subset \dots$$

By taking the Bratelli diagram of this system of inclusions, and then deleting the reflections coming from basic constructions, which appear at each step, according to the

various results from chapter 15, we obtain a certain graph  $\Gamma$ , called principal graph of  $A_0 \subset A_1$ . The main properties of  $\Gamma$  can be summarized as follows:

**PROPOSITION 16.25.** *The principal graph  $\Gamma$  has the following properties:*

- (1) *The higher relative commutant  $P_n = A'_0 \cap A_n$  is isomorphic to the abstract vector space spanned by the  $2n$ -loops on  $\Gamma$  based at the root.*
- (2) *In the amenable case, where  $A_1 = R$  and when the subfactor is “amenable”, the index of  $A_0 \subset A_1$  is given by  $N = \|\Gamma\|^2$ .*

**PROOF.** This is something standard, the idea being as follows:

(1) The statement here, which explains among others the relation between the principal graph  $\Gamma$ , and the other subfactor invariants, from Definition 16.24, comes from the definition of the principal graph, as a Bratelli diagram, with the reflections removed.

(2) This is actually a quite subtle statement, but for our purposes here, we can take the equality  $N = \|\Gamma\|^2$ , which reminds the Kesten amenability condition for discrete groups, as a definition for the amenability of the subfactor. With the remark that for the Popa diagonal subfactors what we have here is precisely the Kesten amenability condition for the underlying discrete group  $G$ , and that, more generally, for the arbitrary generalized Popa or Wassermann subfactors, what we have here is precisely the Kesten type amenability condition for the underlying discrete quantum group  $G$ .  $\square$

Regarding now the Poincaré series, following [12] and related papers, it is convenient to stay, at least for the beginning, at a rather elementary level, and associate such series to any rooted bipartite graph. We have the following definition, which is something straightforward, inspired by Definition 16.24 and Proposition 16.25:

**DEFINITION 16.26.** *The Poincaré series of a rooted bipartite graph  $\Gamma$  is*

$$f(z) = \sum_{n=0}^{\infty} \text{loop}_{\Gamma}(2n) z^n$$

where  $\text{loop}_{\Gamma}(2n)$  is the number of  $2n$ -loops based at the root.

In the case where  $\Gamma$  is the principal graph of a subfactor  $A_0 \subset A_1$ , this series  $f$  is the Poincaré series of the subfactor, in the usual sense, namely:

$$f(z) = \sum_{n=0}^{\infty} \dim(A'_0 \cap A_n) z^n$$

In general, the Poincaré series should be thought of as being a basic representation theory invariant of the underlying group-like object. For instance for the Wassermann subfactor associated to a compact Lie group  $G \subset U_N$ , the Poincaré series is:

$$f(z) = \int_G \frac{1}{1 - \text{Tr}(g)z} dg$$

Again following [12], let us discuss now the measure-theoretic versions of the above invariants. Once again, we start with an arbitrary rooted bipartite graph  $\Gamma$ . We can first introduce a real measure  $\mu$ , whose Stieltjes transform is  $f$ , as follows:

DEFINITION 16.27. *The real measure  $\mu$  of a rooted bipartite graph  $\Gamma$  is given by*

$$f(z) = \int_0^\infty \frac{1}{1-xz} d\mu(x)$$

where  $f$  is the Poincaré series.

In the case where  $\Gamma$  is the principal graph of a subfactor  $A_0 \subset A_1$ , we recover in this way the spectral measure of the subfactor, as introduced in Definition 16.24, with the remark however that the existence of such a measure  $\mu$  was not discussed there. In general, and so also in the particular subfactor case, clarifying the things here, the fact that a measure  $\mu$  as above exists indeed comes from the following simple fact:

PROPOSITION 16.28. *The real measure  $\mu$  of a rooted bipartite graph  $\Gamma$  is given by the following formula, where  $L = EE^t$ , with  $E$  being the adjacency matrix of the graph,*

$$\mu = \text{law}(L)$$

and with the probabilistic computation being with respect to the expectation

$$A \rightarrow \langle A \rangle$$

with  $\langle A \rangle$  being the  $(*,*)$ -entry of a matrix  $A$ , where  $*$  is the root.

PROOF. With the conventions in the statement, namely  $L = EE^t$ , with  $E$  being the adjacency matrix, and  $\langle A \rangle$  being the  $(*,*)$ -entry of a matrix  $A$ , we have:

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} \text{loop}_\Gamma(2n) z^n \\ &= \sum_{n=0}^{\infty} \langle L^n \rangle z^n \\ &= \left\langle \frac{1}{1-Lz} \right\rangle \end{aligned}$$

But this shows that we have indeed the formula  $\mu = \text{law}(L)$ , as desired.  $\square$

In the subfactor case some further interpretations are available as well. For instance in the case of the fixed point subfactors coming from of a compact Lie group  $G \subset U_N$ , discussed after Definition 16.26,  $\mu$  is the spectral measure of the main character:

$$\mu = \text{law}(\chi)$$

Before getting into computations, again following [12], let us introduce as well some alternative invariants, inspired by the fundamental work of Jones in [59], in relation with the annular structure of the subfactors. The main result in [59] is as follows:

THEOREM 16.29. *The theta series of a subfactor of index  $N > 4$ , which is given by*

$$\Theta(q) = q + \frac{1-q}{1+q} f\left(\frac{q}{(1+q)^2}\right)$$

with  $f = \sum_n \dim(P_n)z^n$  being the Poincaré series, has positive coefficients.

PROOF. This is something advanced, the idea being that  $\Theta$  is the generating series of a certain series of multiplicities associated to the subfactor, and more specifically associated to the canonical inclusion  $TL_N \subset P$ . We refer here to Jones' paper [59].  $\square$

Regarding now the theta series for the graphs, this can be introduced as a version of the Poincaré series, via the change of variables  $z^{-1/2} = q^{1/2} + q^{-1/2}$ , as follows:

DEFINITION 16.30. *The theta series of a rooted bipartite graph  $\Gamma$  is*

$$\Theta(q) = q + \frac{1-q}{1+q} f\left(\frac{q}{(1+q)^2}\right)$$

where  $f$  is the Poincaré series.

The theta series can be written as  $\Theta(q) = \sum_s a_s q^s$ , and it follows from the above formula, via some simple manipulations, that its coefficients  $a_s$  are integers. In fact, we have here the following explicit formula from [59], relating the coefficients of the theta series  $\Theta(q) = \sum_s a_s q^s$  to those of the Poincaré series  $f(z) = \sum_n c_n z^n$ :

$$a_s = \sum_{n=0}^s (-1)^{s-n} \frac{2s}{s+n} \binom{s+n}{s-n} c_n$$

In the case where  $\Gamma$  is the principal graph of a subfactor  $A_0 \subset A_1$  of index  $N > 4$ , it is known from [59] that the numbers  $a_s$  are certain multiplicities associated to the planar algebra inclusion  $TL_N \subset P$ , as explained in Theorem 16.29 and its proof. In particular, the coefficients of the theta series are in this case positive integers:

$$a_s \in \mathbb{N}$$

Still following [12], let us introduce as well the following notion:

DEFINITION 16.31. *The circular measure  $\varepsilon$  of a rooted bipartite graph  $\Gamma$  is given by*

$$d\varepsilon(q) = d\mu((q + q^{-1})^2)$$

where  $\mu$  is the associated real measure.

In other words, the circular measure  $\varepsilon$  appears as the pullback of the usual real measure  $\mu$  via the following map, coming from the theory of the theta series in [59]:

$$\begin{aligned} \mathbb{R} \cup \mathbb{T} &\rightarrow \mathbb{R}_+ \\ q &\rightarrow (q + q^{-1})^2 \end{aligned}$$

As a basic example for this, assume that  $\mu$  is a discrete measure, supported by  $k$  positive numbers  $x_1 < \dots < x_k$ , with corresponding densities  $p_1, \dots, p_k$ :

$$\mu = \sum_{i=1}^k p_i \delta_{x_i}$$

For each  $i \in \{1, \dots, k\}$  the equation  $(q + q^{-1})^2 = x_i$  has then four solutions, that we can denote  $q_i, q_i^{-1}, -q_i, -q_i^{-1}$ , and with this notation, we have:

$$\varepsilon = \frac{1}{4} \sum_{i=1}^k p_i \left( \delta_{q_i} + \delta_{q_i^{-1}} + \delta_{-q_i} + \delta_{-q_i^{-1}} \right)$$

In general, the basic properties of the circular measure  $\varepsilon$  can be summarized as follows:

**PROPOSITION 16.32.** *The circular measure has the following properties:*

- (1)  $\varepsilon$  has equal density at  $q, q^{-1}, -q, -q^{-1}$ .
- (2) The odd moments of  $\varepsilon$  are 0.
- (3) The even moments of  $\varepsilon$  are half-integers.
- (4) When  $\Gamma$  has norm  $\leq 2$ ,  $\varepsilon$  is supported by the unit circle.
- (5) When  $\Gamma$  is finite,  $\varepsilon$  is discrete.
- (6) If  $K$  is a solution of  $L = (K + K^{-1})^2$ , then  $\varepsilon = \text{law}(K)$ .

**PROOF.** These results can be deduced from definitions, the idea being that (1-5) are trivial, and that (6) follows from the formula of  $\mu$  from Proposition 16.28.  $\square$

In addition to the above result, we have the following key formula, which gives the even moments of  $\varepsilon$ , and makes the connection with the Jones theta series:

**THEOREM 16.33.** *We have the Stieltjes transform type formula*

$$2 \int \frac{1}{1 - qu^2} d\varepsilon(u) = 1 + T(q)(1 - q)$$

where the  $T$  series of a rooted bipartite graph  $X$  is by definition given by

$$T(q) = \frac{\Theta(q) - q}{1 - q}$$

with  $\Theta$  being the associated theta series.

**PROOF.** This follows by applying the change of variables  $q \rightarrow (q + q^{-1})^2$  to the fact that  $f$  is the Stieltjes transform of  $\mu$ . Indeed, we obtain:

$$\begin{aligned} 2 \int \frac{1}{1 - qu^2} d\varepsilon(u) &= 1 + \frac{1 - q}{1 + q} f\left(\frac{q}{(1 + q)^2}\right) \\ &= 1 + \Theta(q) - q \\ &= 1 + T(q)(1 - q) \end{aligned}$$

Thus, we are led to the conclusion in the statement.  $\square$

As a final theoretical result about all these invariants, which is this time something non-trivial, in the subfactor case, we have the following result, due to Jones [59]:

**THEOREM 16.34.** *In the case where  $\Gamma$  is the principal graph of an irreducible subfactor of index  $> 4$ , the moments of  $\varepsilon$  are positive numbers.*

**PROOF.** This follows indeed from the result in [59] stating that the coefficients of  $\Theta$  are positive numbers, from Theorem 16.29, via the formula in Theorem 16.33.  $\square$

As an illustration for all this, let us first discuss the case of the small index subfactors,  $N \in [1, 4]$ . Following Jones [55] and related work, we first have the following result:

**THEOREM 16.35.** *The index of the subfactors is subject to the condition*

$$N \in \left\{ 4 \cos^2 \left( \frac{\pi}{n} \right) \mid n \geq 3 \right\} \cup [4, \infty]$$

*and at  $N \leq 4$ , the principal graph must be one of the Coxeter-Dynkin ADE graphs.*

**PROOF.** Here the first assertion is something that we already know, from chapter 15. As for the second assertion, this comes via a refinement of all this, the key ingredient being the fact that in index  $N \leq 4$ , and in fact more generally in the amenable case, as discussed before, we must have  $N = \|\Gamma\|^2$ . See [55].  $\square$

More in detail now, the usual Coxeter-Dynkin ADE graphs are as follows:

$$\begin{array}{ll} A_k = \bullet - \circ - \circ \cdots \circ - \circ - \circ & A_\infty = \bullet - \circ - \circ - \circ \cdots \\ & \quad \quad \quad \circ \\ & \quad \quad \quad | \\ D_k = \bullet - \circ - \circ \cdots \circ - \circ - \circ & \\ \\ \tilde{A}_{2k} = \begin{array}{c} \circ - \circ - \circ \cdots \circ - \circ - \circ \\ | \qquad \qquad \qquad | \\ \bullet - \circ - \circ - \circ - \circ - \circ \end{array} & A_{-\infty, \infty} = \begin{array}{c} \circ - \circ - \circ - \circ \cdots \\ | \\ \bullet - \circ - \circ - \circ \cdots \end{array} \\ \\ \tilde{D}_k = \begin{array}{c} \circ \qquad \qquad \qquad \circ \\ | \qquad \qquad \qquad | \\ \bullet - \circ - \circ - \circ \cdots \circ - \circ - \circ \end{array} & D_\infty = \begin{array}{c} \circ \\ | \\ \bullet - \circ - \circ - \circ \cdots \end{array} \end{array}$$

In the above, the graphs  $A_k$  with  $k \geq 2$  and  $D_k$  with  $k \geq 3$  have by definition  $k$  vertices each,  $\tilde{A}_{2k}$  with  $k \geq 1$  has  $2k$  vertices, and  $\tilde{D}_k$  with  $k \geq 4$  has  $k+1$  vertices. Thus, the first graph in each series is by definition as follows:

$$\begin{array}{llll} A_2 = \bullet - \circ & D_3 = \begin{array}{c} \circ \\ | \\ \bullet - \circ \end{array} & \tilde{A}_2 = \begin{array}{c} \circ \\ || \\ \bullet \end{array} & \tilde{D}_4 = \begin{array}{c} \circ \quad \circ \\ \backslash \quad / \\ \bullet - \circ - \circ \end{array} \end{array}$$

There are as well a number of exceptional Coxeter-Dynkin graphs. First we have:

$$\begin{aligned}
 E_6 &= \bullet - \circ - \overset{\circ}{\underset{|}{\circ}} - \circ - \circ \\
 E_7 &= \bullet - \circ - \circ - \overset{\circ}{\underset{|}{\circ}} - \circ - \circ \\
 E_8 &= \bullet - \circ - \circ - \circ - \circ - \overset{\circ}{\underset{|}{\circ}} - \circ
 \end{aligned}$$

Finally, we have index 4 versions of the above exceptional graphs, as follows:

$$\begin{aligned}
 \tilde{E}_6 &= \bullet - \circ - \overset{\circ}{\underset{\circ}{\underset{|}{\circ}}} - \circ - \circ \\
 \tilde{E}_7 &= \bullet - \circ - \circ - \overset{\circ}{\underset{|}{\circ}} - \circ - \circ - \circ \\
 \tilde{E}_8 &= \bullet - \circ - \circ - \circ - \circ - \circ - \overset{\circ}{\underset{|}{\circ}} - \circ
 \end{aligned}$$

Getting back now to Theorem 16.35, with this list in hand, the story is not over, because we still have to understand which of these graphs can really appear as principal graphs of subfactors. And, for those graphs which can appear, we must understand the structure and classification of the subfactors of  $R$ , having them as principal graphs.

In short, still a lot of work to be done, as a continuation of Theorem 16.35. The subfactors of index  $\leq 4$  were intensively studied in the 80s and early 90s, and about 10 years after Jones' foundational paper [55], a complete classification result was found, with contributions by many authors. A simplified form of this result is as follows:

**THEOREM 16.36.** *The principal graphs of subfactors of index  $\leq 4$  are:*

- (1) *Index  $< 4$  graphs:*  $A_k$ ,  $D_{\text{even}}$ ,  $E_6$ ,  $E_8$ .
- (2) *Index 4 finite graphs:*  $\tilde{A}_{2k}$ ,  $\tilde{D}_k$ ,  $\tilde{E}_6$ ,  $\tilde{E}_7$ ,  $\tilde{E}_8$ .
- (3) *Index 4 infinite graphs:*  $A_\infty$ ,  $A_{-\infty, \infty}$ ,  $D_\infty$ .

**PROOF.** As already mentioned, this is something quite heavy, with contributions by many authors, and notably Ocneanu [71]. Observe that the graphs  $D_{\text{odd}}$  and  $E_7$  don't appear in the above list, a subtlety of subfactor theory. For a discussion, see [72].  $\square$

Getting back to probability, our claim is that the circular measure  $\varepsilon$  is the “best” invariant. As a first illustration, let us compute  $\varepsilon$  for the simplest graph in the index range  $N \in [1, 4]$ , namely the graph  $\tilde{A}_{2k}$ . We obtain here something nice, as follows:

THEOREM 16.37. *The circular measure of the basic index 4 graph, namely*

$$\tilde{A}_{2k} = \begin{array}{cccccccc} \circ & - & \circ & - & \circ & \cdots & \circ & - & \circ & - & \circ \\ | & & & & & & & & & & | \\ \bullet & - & \circ & - & \circ & - & \circ & - & \circ & - & \circ \end{array}$$

*is the uniform measure on the  $2k$ -roots of unity.*

PROOF. Let us identify indeed the vertices of  $\Gamma = \tilde{A}_{2k}$  with the group  $\{w^s\}$  formed by the  $2k$ -th roots of unity in the complex plane, where  $w = e^{\pi i/k}$ . The adjacency matrix of  $\Gamma$  acts then on the functions  $f \in C(X)$  in the following way:

$$Mf(w^s) = f(w^{s-1}) + f(w^{s+1})$$

But this formula shows that we have  $M = K + K^{-1}$ , where  $K$  is given by:

$$Kf(w^s) = f(w^{s+1})$$

Thus we can use the last assertion in Proposition 16.32, and we obtain  $\varepsilon = \text{law}(K)$ , which is the uniform measure on the  $2k$ -roots of unity. See [12].  $\square$

In order to discuss all this more systematically, and for all ADE graphs, the idea will be that of looking at the combinatorics of the roots of unity. As before following [12] and related papers, let us introduce the following technical notion:

DEFINITION 16.38. *The series of the form*

$$\xi(n_1, \dots, n_s : m_1, \dots, m_t) = \frac{(1 - q^{n_1}) \cdots (1 - q^{n_s})}{(1 - q^{m_1}) \cdots (1 - q^{m_t})}$$

*with  $n_i, m_i \in \mathbb{N}$  are called cyclotomic.*

The point is that the Poincaré series of the ADE graphs are given by quite complicated formulae, but the corresponding  $T$  series are all cyclotomic. In order to explain this, and formulate our final results, we will need some more theory. Let us introduce as well:

DEFINITION 16.39. *A cyclotomic measure is a probability measure  $\varepsilon$  on the unit circle, having the following properties:*

- (1)  $\varepsilon$  is supported by the  $2k$ -roots of unity, for some  $k \in \mathbb{N}$ .
- (2)  $\varepsilon$  has equal density at  $q, q^{-1}, -q, -q^{-1}$ .

With all these ingredients in hand, following [12] and follow-up papers, we are now ready to discuss the circular measures of the various ADE graphs.

The idea is that all these measures are all cyclotomic, of level  $\leq 3$ , and can be expressed in terms of the basic polynomial densities of degree  $\leq 6$ , namely:

$$\alpha = \operatorname{Re}(1 - q^2)$$

$$\beta = \operatorname{Re}(1 - q^4)$$

$$\gamma = \operatorname{Re}(1 - q^6)$$

To be more precise, we have the following result, with the densities  $\alpha, \beta, \gamma$  being as above, with  $d_k$  being the uniform measure on the  $2k$ -th roots of unity, and with  $d'_k = 2d_{2k} - d_k$  being the uniform measure on the odd  $4k$ -roots of unity:

**THEOREM 16.40.** *The circular measures of the ADE graphs are given by:*

- (1)  $A_{k-1} \rightarrow \alpha_k.$
- (2)  $\tilde{A}_{2k} \rightarrow d_k.$
- (3)  $D_{k+1} \rightarrow \alpha'_k.$
- (4)  $\tilde{D}_{k+2} \rightarrow (d_k + d'_1)/2.$
- (5)  $E_6 \rightarrow \alpha_{12} + (d_{12} - d_6 - d_4 + d_3)/2.$
- (6)  $E_7 \rightarrow \beta'_9 + (d'_1 - d'_3)/2.$
- (7)  $E_8 \rightarrow \alpha'_{15} + \gamma'_{15} - (d'_5 + d'_3)/2.$
- (8)  $\tilde{E}_{k+3} \rightarrow (d_k + d_3 + d_2 - d_1)/2.$

**PROOF.** This is something that we already know for  $\tilde{A}_{2k}$ , from Theorem 16.37. In general, this follows by a similar method, namely counting loops, then computing the corresponding  $T$  series, and finally converting the  $T$  series formulae into measure-theoretic results, as in the statement. We refer here to [12] and follow-up papers.  $\square$

It is possible to further build along the above lines, with a combinatorial refinement of the formulae in Theorem 16.40, making appear a certain connection with the Deligne work on the exceptional series of Lie groups, which is not understood yet.

All the above, which was something nice, was about index  $N \in [1, 4]$ , where the Jones annular theory result from [59] does not apply. In higher index now,  $N \in (4, \infty)$ , where the Jones result does apply, the precise correct “blowup” manipulation on the spectral measure is not known yet. The known results here are as follows:

- (1) On one hand, there is a computation for certain basic Hadamard subfactors, with nice blowup, on a certain noncommutative manifold [9].
- (2) On the other hand, there are many computations by Evans-Pugh, with quite technical blowup results, on suitable real algebraic manifolds [44].

We will briefly discuss in what follows (1), and to be more precise the computation of the spectral measure, and then the blowup problem, for the subfactors coming from the deformed Fourier matrices. Following [9], let us start with:

DEFINITION 16.41. *Given two finite abelian groups  $A, B$ , we consider the corresponding deformed Fourier matrix, given by the following formula*

$$(F_A \otimes_Q F_B)_{ia,jb} = Q_{ib}(F_A)_{ij}(F_B)_{ab}$$

*and we factorize the associated representation  $\pi_Q$  of the algebra  $C(S_{A \times B}^+)$ ,*

$$\begin{array}{ccc} C(S_{A \times B}^+) & \xrightarrow{\pi_Q} & M_{A \times B}(\mathbb{C}) \\ & \searrow & \nearrow \pi \\ & C(G_Q) & \end{array}$$

*with  $C(G_Q)$  being the Hopf image of this representation  $\pi_Q$ .*

All this is perhaps a bit heavy, but the idea is that, passed some trivial examples, the above complex Hadamard matrices are the most basic ones. Following [9], we have:

THEOREM 16.42. *When  $Q$  is generic, the minimal factorization for  $\pi_Q$  is*

$$\begin{array}{ccc} C(S_{A \times B}^+) & \xrightarrow{\pi_Q} & M_{A \times B}(\mathbb{C}) \\ & \searrow & \nearrow \pi \\ & C^*(\Gamma_{A,B}) \rtimes C(A) & \end{array}$$

*where the group on the bottom is  $\Gamma_{A,B} \simeq \mathbb{Z}^{(|A|-1)(|B|-1)} \rtimes B$ .*

PROOF. This is something quite technical, obtained by doing a lot of technical abstract algebra, and for full details on this computation, we refer here to [9].  $\square$

In what regards now the law of the main character, the result here is as follows:

THEOREM 16.43. *We have the moment formula*

$$\int \chi^p = \frac{1}{|A| \cdot |B|} \# \left\{ \begin{array}{l} i_1, \dots, i_p \in A \mid [(i_1, d_1), (i_2, d_2), \dots, (i_p, d_p)] \\ d_1, \dots, d_p \in B \mid = [(i_1, d_p), (i_2, d_1), \dots, (i_p, d_{p-1})] \end{array} \right\}$$

*where the sets between square brackets are by definition sets with repetition.*

PROOF. This is a straightforward consequence of the result from Theorem 16.42, and for full details regarding this computation, we refer to [9].  $\square$

The point now is that the above formula can be interpreted as follows:

THEOREM 16.44. *With  $M = |A|$ ,  $N = |B|$  we have the formula*

$$\text{law}(\chi) = \left(1 - \frac{1}{N}\right) \delta_0 + \frac{1}{N} \text{law}(G)$$

where the matrix

$$G \in C(\mathbb{T}^{MN}, M_M(\mathbb{C}))$$

is given by  $G(q) = \text{Gram matrix of the rows of } q$ .

PROOF. This follows indeed from a standard computation, based on the formula from Theorem 16.43. As before, for details on all this, we refer to [9].  $\square$

In relation with free probability, again following [9], the point is that in a suitable  $M, N \rightarrow \infty$  regime we obtain as limiting measure a free Poisson law:

THEOREM 16.45. *With  $M = \alpha K$ ,  $N = \beta K$ ,  $K \rightarrow \infty$  we have*

$$\text{law}(\chi) = \left(1 - \frac{1}{\alpha\beta K^2}\right) \delta_0 + \frac{1}{\alpha\beta K^2} D_{\frac{1}{\beta K}}(\pi_{\alpha/\beta})$$

where  $D$  is the dilation operation, given by  $D_r(\text{law}(X)) = \text{law}(rX)$ .

PROOF. This follows from the formula in Theorem 16.44, by doing some combinatorics, and as before, for details on all this, we refer to [9].  $\square$

Getting back now to our blowup questions, the conclusion, coming from Theorem 16.44, is that the correct blowup most likely appears on some suitable noncommutative manifold. However, this finding, which is not exactly in tune with the computations in [44], rather using complicated real algebraic manifolds, remains to be confirmed.

### 16d. Further questions

Beyond what has been said above, the world remains large. We have meander determinant problematics. Weingarten function problematics. De Finetti theorem problematics. Matrix model problematics. And so on. All these questions are interesting for subfactors, and they are related as well to the block-modified Wishart matrices.

### 16e. Exercises

Congratulations for having read this book, and no exercises here. However, if looking for a difficult exercise, you can try to solve the hyperfiniteness questions mentioned in the beginning of this chapter. But these are again, at least in the quantum group case, related to matrix model questions. So, in the end, our conclusion would be that doing some further free probability theory in the subfactor context is a good idea.

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