



**HAL**  
open science

# Prospective primary school teachers' recognition of proportional reasoning in pupils' solution to probability comparison tasks

María Burgos, Verónica Albanese, María del Mar López-Martín

## ► To cite this version:

María Burgos, Verónica Albanese, María del Mar López-Martín. Prospective primary school teachers' recognition of proportional reasoning in pupils' solution to probability comparison tasks. Twelfth Congress of the European Society for Research in Mathematics Education (CERME12), Feb 2022, Bozen-Bolzano, Italy. hal-03751836

**HAL Id: hal-03751836**

**<https://hal.science/hal-03751836>**

Submitted on 15 Aug 2022

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Prospective primary school teachers' recognition of proportional reasoning in pupils' solution to probability comparison tasks

María Burgos<sup>1</sup>, Verónica Albanese<sup>1</sup> and María del Mar López-Martín<sup>2</sup>

<sup>1</sup>Universidad de Granada, Facultad de Ciencias de la Educación, España; [mariaburgos@ugr.es](mailto:mariaburgos@ugr.es); [vealbanese@ugr.es](mailto:vealbanese@ugr.es)

<sup>2</sup>Universidad de Almería, Facultad de Ciencias de la Educación, España; [mdm.lopez@ual.es](mailto:mdm.lopez@ual.es)

*In order to foster the learning of mathematics, the teacher must be able to interpret and analyse the students' mathematical activity. This cognitive analysis competence allows the teacher to understand the processes of mathematical learning, to foresee conflicts of meanings and to establish different possibilities for institutionalising the mathematical knowledge involved. In this paper, we are concerned with assessing the initial knowledge and competence of prospective primary school teachers in order to analyse responses of primary school pupils when solving urn probability comparison tasks. Specifically, we are interested in analysing what degrees of proportional reasoning the prospective teachers identify in the pupils' answers. The results reveal the prospective teachers' limitations for correctly identify proportional reasoning, specially to discriminate additive and multiplicative comparison.*

*Keywords: Probability, proportional reasoning, specialized probabilistic knowledge, teacher's education, urn tasks.*

## Introduction

A critical issue in mathematics education research is to clarify the type of didactic-mathematical knowledge that mathematics teachers should have in order to develop their teaching work in an appropriate manner (Chapman, 2014; Mason, 2016). The mathematics education research community accepts that teachers should have a certain level of mathematical competence, that is, they should be able to perform the mathematical practices necessary to solve the problems that the curriculum proposes and to articulate them with the subsequent mathematical contents. Teachers should also have a specialized knowledge of the content itself, of the transformations that have to be applied to it in teaching and learning processes, and of the psychological, sociological and pedagogical factors, among others, that condition these processes. Although the analysis of pupils' thinking is considered one of the main tasks of mathematics teaching, identifying the mathematical ideas inherent to the strategies that a pupil uses during mathematical problem solving could be difficult for the teacher (Fernández et al., 2013). In this sense, several researches indicate that both pre-service and in-service teachers have difficulties to interpret the responses of primary education students when solving mathematical tasks involving proportional reasoning, as well as making action decisions based on how pupils seems to understand proportionality (Buforn et al., 2020; Fernández et al., 2013). These investigations concluded that further research is needed on pre-service teachers' didactic and mathematical knowledge related to proportional reasoning.

The aim of this work is to assess the knowledge and skills of prospective primary school teachers to interpret pupils' responses to probability comparison tasks, identify incorrect strategies and recognise proportional reasoning in their mathematical activity. We address the following research questions:

*What do prospective teachers understand by proportional reasoning in the context of probability comparison tasks? How do they identify and use it to justify their assessment of pupils' solutions?*

Proportional and probabilistic reasoning are strongly linked; both involve quantitative and qualitative analysis, establishing relationships, making inferences and predicting outcomes. The results will allow us to design, implement and evaluate training actions to develop these didactic-mathematical knowledge and skills in prospective mathematics teachers.

## **Previous research**

Proportional reasoning, understood as the ability to establish multiplicative relationships between two quantities and to extend this relationship to another pair of quantities (Lamon, 2007), is an objective present in the Primary Education curriculum, which integrates both the various interpretations of the rational number (ratio, operator, part-whole, measure and quotient) and the ways of reasoning with these meanings (up and down reasoning, relational thinking, covariance, etc.). Proportional reasoning is “a form of mathematical reasoning that involves a sense of co-variation and of multiple comparisons, and the ability to mentally store and process several pieces of information. Proportional reasoning is very much concerned with inference and prediction and involves both qualitative and quantitative methods of thought” (Lesh et al., 1988, p. 93).

Extensive research has focused on investigating pupils' strategies and levels of proportional reasoning in the context of probability, mainly in the urns probabilities comparison setting (Bryant & Nunes, 2012; Cañizares & Batanero, 1998; Langrall & Mooney, 2005; Van Dooren, 2014; Watson, 2005). According to Falk et al. (1980), probability can be thought of as being composed of two sub-constructs: chance and proportion. Various researches in mathematics education reveal that both students and teachers have difficulties in applying proportional reasoning in statistical and probabilistic contexts (Gal, 2002). Specifically, the lack of proportional reasoning to solve elementary probability comparison problems is found not only in students, but also in prospective primary school teachers (Contreras et al., 2011; Gómez et al., 2016; Vásquez & Alsina, 2015). Begolli et al. (2021) suggest that “prior knowledge of proportional reasoning reveals deeper insights into students' potential for learning about probabilistic reasoning, than prior knowledge of the concept of probability itself” (p. 462). For this reason, it is essential that prospective teachers be aware of the different components of proportional reasoning and how they appear in probability.

## **Didactic-mathematical knowledge and competence model**

The study of the type of didactic and mathematical knowledge and competences that teachers should have in order to manage the pupils' learning process has generated several models that intend to characterize such teachers' knowledge and competences (Chapman, 2014; Hill et al., 2008). In this research we adopt the teacher's Didactic-Mathematical Knowledge and Competence (DMKC) model (Godino et al., 2017) developed within the Onto-Semiotic Approach (OSA). The DMKC model considers that the teacher should have a *common mathematical knowledge* regarding a certain educational level where he/she teaches, as well as an *expanded mathematical content knowledge* that allows him/her to articulate the content with higher educational levels. In addition, as some mathematical content is put at stake, the teacher should have a didactic-mathematical or *specialized knowledge* of the different facets involved in the educational process: *epistemic* (institutional content

meanings), *ecological* (aligning tasks according to institutional mandatory curriculum), *cognitive* (understanding student's thinking), *affective* (reacting to anguish, indifference, anger, etc., manifested by students), *interactional* (identifying and answering to students' conflicts and interactions), and *mediational* (choosing the best suitable resources for teaching). In the epistemic facet, the specialized knowledge allows the teacher to recognize the diversity of meanings involved, be able to solve the task using different strategies and justify the accuracy of the procedures. In the cognitive facet, the specialized knowledge guarantees the teacher being able to understand the ways of reasoning, difficulties and personal meanings that students may present when working with the specific mathematical situation. It makes the teacher competent to identifying possible different solution strategies in a probability problem, assessing students' responses and recognizing the mathematical objects involved. In particular, analyzing the proportional reasoning put at stake in the mathematical practices involved in their resolutions. This will provide teachers appropriate responses to real classroom situations.

## Method

We performed a content and descriptive analysis of the written solutions to the problem in order to classify the responses into different categories building on previous research and refining these categories through a cyclical and inductive process; this is typical of qualitative research. This research was conducted with 116 prospective primary school teachers (PPT in the following) at a Spanish university. During their undergraduate studies, these prospective teachers received specific preparation on the epistemic, cognitive, instructional and curricular aspects of teaching statistics and probability. Specifically, the intervention was carried out once the training process of the PPTs on the main contents of Data Processing, Chance and Probability had been completed. This deals with the fundamentals of the Didactics of Mathematics in terms of the main concepts, properties and procedures that form the primary school mathematics curriculum, mathematical learning, errors and difficulties and instructional aspects (tasks, materials and resources) related to this content. The task (see Figure 1) was proposed to the PPTs to be solved individually and voluntarily. The written answers of the PPTs to this task were analysed using content and descriptive analysis methods.

Below you find a problem and some solutions to it developed by primary school pupils.

In box A two white balls and two black balls have been placed. In box B there are 4 white balls and 4 black balls. In which box is there a greater chance of getting a black ball?

Alba: "In box B because it has 2 more black balls than box A".

Daniel: "The same, because in box B there are 2 more white balls, but there are also two more black balls".

Lucía: "The same, because in both the white balls are half as many as the black balls".

Salva: "The same because in box B the number of white and black balls has been multiplied by 2 compared to A".

Justify whether each pupil's answer seems correct or incorrect and identify proportional reasoning in the pupils' answers.

**Figure 1: Task proposed to PPTs to assess the analysis of pupils' solutions to a probability comparison problem**

To assess the cognitive facet of the PPTs' didactical-mathematical knowledge and competence, we proposed to the participants to analyse the correctness degree of different pupils' solutions to a probability comparison problem in urns, identifying the proportional reasoning involved or not, as a relevant mathematical element of pupils' mathematical thinking when solving this type of task.

Notice that Alba’s answer is incorrect, as she only compares the favourable cases, which leads her to decide that in box B the probability of drawing the black ball is higher. To Daniel, the probability is the same, “because in box B there are two more white balls, but there are also two more black balls”. Daniel compares in an additive way the unfavourable cases (white balls) and the favourable cases (black balls) of both boxes: as the difference between the unfavourable cases of both boxes and the favourable cases of both boxes is the same, two, he concludes that the probability of drawing a black ball in both boxes is the same. This argument is not correct, since it only leads to a right answer in the case where the number of favourable cases matches the number of unfavourable cases. In Lucía’s solution, it is true that “if the number of unfavourable cases is the number of favourable cases multiplied by a scalar (in our case 2) the probability remains constant”. However, it is not true that “in both cases the white balls are half of the black balls”. It seems that Lucía confuses “unfavourable cases” with “possible cases”. We note that Lucía uses a correspondence strategy (proportional reasoning) but the multiplicative relationship that she establishes in one urn to be extended to the other is not correct. Finally, to justify that the probability of drawing a black ball in both boxes is the same, Salva uses proportional reasoning: the probability, as the ratio between the number of favourable and possible cases (favourable plus unfavourable) remains constant if both the number of favourable and unfavourable cases is multiplied by a scalar.

## Results

Of the 116 participants, 100 (i.e., 86.21%) considered Daniel’s answer to be correct, but only 73 (62.93%) provided some conclusive justification. Table 1 shows that more than half of PPTs (50.86%) considered Lucía’s solution to be right, however, just 24 (20.69%) of them gave a clear description of why the argument used by this student is appropriate. Of the 84 (i.e. 72.41%) PPTs who explained why they considered the solution given by Salva to be correct or not, 78 (67.24%) considered it to be correct and only 6 (5.17%) considered his argument to be inadequate (they believe that it is only valid in this particular case or that he should have relied on the use of Laplace's rule). In addition, two PPTs considered all pupils’ answers to be correct, without justifying their assessment. These two PPTs were the only ones who implicitly considered Alba’s solution being right, for which 106 (91.38%) PPTs justified her error.

**Table 1: Frequencies (percentages) in the assessment of the correctness degree of pupils' responses**

	Alba	Lucía	Daniel	Salva
No answer/Not conclusive evaluation	10 (8.62)	35 (30.17)	34 (29.31)	32 (27.59)
Pupil’s solution correct	0 (0)	24 (20.69)	73 (62.93)	78 (67.24)
Pupil’s solution incorrect	106 (91.38)	57 (49.14)	19 (16.38)	6 (5.17)
Total	116 (100)	116 (100)	116 (100)	116 (100)

In view of the interest of this paper, of the arguments used by PPTs to justify their assessment of primary school pupils’ answers as correct or incorrect, we focus our attention on those evaluations that refer to the presence or absence of proportional reasoning (see Table 1). This will give us useful information about what prospective teachers understand by proportional reasoning in the context of probability comparison problems and when and how they identify it in the pupils’ responses.

As we see in Table 2, eleven PPTs consider that comparing only favourable cases in an additive way shows an absence of proportional reasoning in Alba's strategy, which they interpret in the context of probability as the ability to establish a proportion, or to establish a multiplicative relation/comparison. For example, PPT41's answer:

PPT41: I have been able to identify proportional reasoning in pupils who think that there is the same probability due to the fact that the number of balls in one box is proportional to the number of balls in the other box. Error that Alba may have made is assuming that because there are more black balls in one box than in the other, there is a much higher probability of getting a black ball, regardless of the number of white balls.

**Table 2: Reference to proportional reasoning in the PPTs' assessment of pupils' solutions**

Student	Reference to proportional reasoning	Frequency
Alba	Lack of proportional reasoning (multiplicative comparison, covariation) as a cause of error	11
	Evidence of proportional reasoning understood as a "more, more" type comparison	7
	Evidence of proportional reasoning understood as an additive comparison	3
Lucía	Evidence of proportional reasoning (multiplicative relationship) leading to the conclusion that there is the same probability	31
	Inadequate or incomplete proportional reasoning (multiplicative relationship) as a cause for her error	10
Daniel	Proportional reasoning (proportion, equivalence of fractions) as a guarantee of successful response	19
Salva	Proportional reasoning (proportion, equivalence of fractions) as a guarantee of successful response	13
	Proportional reasoning (multiplicative relation, variation) guarantee of successful answer	26

Furthermore, we note that seven PPTs of the 106 that identify and justify Alba's answer as incorrect, identify proportional reasoning like a "more, more" type comparison. See for example, PPT52's answer:

PPT52: [...] In Alba's solution, a proportional reasoning can be identified, since the student observes more balls in box B and therefore thinks that there is a higher probability of drawing one in this box.

Three other PPTs also identify proportional reasoning in Alba's incorrect answer, which they interpret as an additive relationship. For example:

PPT110: In Alba's answer we can notice some proportional reasoning, she has taken into account the increase of two balls in the black colour [...] That would explain her mistake, to have thought proportionally in only one colour.

Likewise, other participants contemplate that Lucía has an error due to the use of proportional reasoning when it was not appropriate ("the main error refers to using proportional reasoning when it was not appropriate to do so", PPT24). They consider that the most suitable strategy is to establish

an additive comparison. In this sense, the analysis of the PPTs' reports shows that several participants show inadequate knowledge of proportional reasoning because they consider it in terms of additive comparisons. That is the case of PPT18:

PPT18: We can say that proportional reasoning is found in the following three answers [referred to Daniel, Lucía and Salva], in all three the same amount is added or taken equally in the two boxes. There is the same probability in box A as in box B.

Furthermore, 22 PPTs points out that pupils correctly apply proportional reasoning when they understand and use relevantly the equivalence of fractions and employ it to calculate and compare probabilities. They consider this leads to success for Daniel and Salva in their response. For instance, PPT38:

PPT38: Regarding proportional reasoning, I identify it in two cases, Daniel and Salva, since both have taken into account the equivalence of fractions. Thus, they have observed that in box A there are  $\frac{2}{4}$  white and  $\frac{2}{4}$  black, while in box B there are  $\frac{4}{8}$  white and  $\frac{4}{8}$  black, which means that the probability in both cases is the same.

To 50 (i.e., 43.10%) of the PPTs, proportional reasoning is involved in those pupils' answers in which a multiplicative comparison is established (see Table 2). This is observed in the evaluations of Lucía or Salva:

PPT11: Lucía has reasoned that in box A, there are half as many black balls and half as many white balls as in box B. She aimed to say, in some way, that the quantity in one box and the other changes proportionally.

PPT45: Salva uses proportional reasoning because he says that in box B both black and white balls have been multiplied by two with respect to box A.

As we can see, PPT11 and PPT45 consider the multiplicative relation between the favorable cases (and unfavorable cases) of both boxes. Other participants look at the correspondence within each of the boxes to ensure the same probability. For instance, PPT38 considers "there is the same number of both white (2) and black (2) balls in box A and, also the same number of white (4) and black (4) balls in box B, so the ratio is maintained and the probability is the same".

In summary, we have noticed that, a high-rise percentage of prospective teachers consider proportional reasoning to be based on "more ..., more ..." relationships (these descriptions appear in 23.91% of the occasions in which PPTs identify proportional reasoning). Besides, even when some prospective teachers discriminate between additive comparisons and multiplicative comparisons, they do not always properly describe the multiplicative relationship, or the magnitudes involved in the proportionality correspondence. That is, they do not always correctly establish a multiplicative comparison between the ratios of favorable and unfavorable cases (or of favorable and possible cases) within both boxes, or between the ratios of favorable cases between the boxes and of unfavorable or possible cases between the boxes. Furthermore, some participants, consider that the establishment of a multiplicative relationship only between favorable cases or (also solely) between possible cases is sufficient to identify proportional reasoning and respond successfully to the task.

## Implications for teaching and research

“The identification of the relevant mathematical elements in a problem and the interpretation of how they are present in the students’ answers allow prospective teachers to be in better conditions to make relevant instructional decisions and help students develop their proportional reasoning” (Llinares, 2013, p. 81). Hence to interpret different pupils’ solution to a probability comparison task by recognising how proportional reasoning is involved in their answers, can help to enhance prospective teachers’ didactic-mathematical knowledge and competences (especially in the cognitive facet) regarding this topic.

Our results show the need to strengthen teachers’ education in relation to the connection between proportional and probabilistic reasoning. Prospective teachers find limitations in identifying and justify possible erroneous strategies behind pupils' incorrect answers. A biased or insufficient knowledge of proportional reasoning could explain why PPTs do not identify it in the incorrect answers and when they do, they show errors when interpreting the proportionality relationship and the properties that characterise it (Burgos & Godino, 2021).

We think that our results provide additional valuable information for the design of materials in teacher education programs that consider the characteristics of prospective teachers’ understanding of proportional reasoning in probability tasks. First, to guarantee that prospective teachers are able to recognize and respond to students' errors, teacher education programs should develop a deep understanding of the conceptual, propositional and argumentative components of proportional reasoning involved in probability setting. On the one hand, proportional reasoning is an integral part of probabilistic reasoning. But on the other hand, probability is an enabling environment for future teachers to overcome a limited view of proportional reasoning linked to solving missing-value problems. “Balancing the amount of probability instruction with proportional reasoning instruction may be more successful than teaching only about probabilities” (Begolli et al., 2021, p 463). Therefore, specific actions should be designed in teacher training to reinforce the structural components of proportional reasoning by integrating it with probabilistic reasoning.

## References

- Begolli, K. N., Dai, T., McGinn, K. M., & Booth, J. L. (2021) Could probability be out of proportion? Self-explanation and example-based practice help students with lower proportional reasoning skills learn probability. *Instructional Science*, 49, 441–473. <https://doi.org/10.1007/s11251-021-09550-9>
- Bryant, P., & Nunes, T. (2012). *Children’s understanding of probability: a literature review*. Nuffield Foundation.
- Buform, A., Llinares, S., Fernández, C., Coles, A., & Brown, L. (2020) Pre-service teachers’ knowledge of the unitizing process in recognizing students’ reasoning to propose teaching decisions, *International Journal of Mathematical Education in Science and Technology*. <https://doi.org/10.1080/0020739X.2020.1777333>
- Burgos, M., & Godino, J. D. (2021). Prospective Primary School Teachers’ Competence for the Cognitive Analysis of Students’ Solutions to Proportionality Tasks. *J Math Didakt.* <https://doi.org/10.1007/s13138-021-00193-4>



- Cañizares, M. J., & Batanero, C. (1998). Influencia del razonamiento proporcional y de las creencias subjetivas en la comparación de probabilidades [Influence of proportional reasoning and subjective beliefs on probability comparison]. *UNO*, *14*, 99–114.
- Chapman, O. (2014). Overall commentary: understanding and changing mathematics teachers. In J. J. Lo, K. R. Leatham, & L. R. Van Zoest (Eds.), *Research Trends in Mathematics Teacher Education* (pp. 295–309). Springer International Publishing. [https://doi.org/10.1007/978-3-319-02562-9\\_16](https://doi.org/10.1007/978-3-319-02562-9_16)
- Contreras, J. M., Batanero, C., Díaz, C., & Fernandes, J. A. (2011). Prospective teachers' common and specialized knowledge in a probability task. *Proceedings of CERME 7*, Rzeszow, Polonia, February, 2011.
- Falk, R., Falk, R., & Levin, I. (1980). A potential for learning probability in young children. *Educational Studies in Mathematics*, *11*(2), 181–204. <https://doi.org/10.1007/BF00304355>
- Fernández, C., Llinares, S., & Valls, J. (2013). Primary school teacher's noticing of students' mathematical thinking in problem solving. *The Mathematics Enthusiast*, *10*(1), 441–468. <https://doi.org/10.54870/1551-3440.1274>
- Gómez, E., Batanero, C., Díaz, C., & Contreras, J. M. (2016). Developing a questionnaire to assess the probability content knowledge of prospective primary school teachers. *Statistics Education Research Journal*, *15*(2), 197–215. <https://doi.org/10.52041/serj.v15i2.248>
- Hill, H. C., Ball, D. L., & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education*, *39*, 372–400. <https://doi.org/10.5951/jresmetheduc.39.4.0372>
- Langrall, C. W., & Mooney, E. S. (2005). Characteristics of elementary school students' probabilistic reasoning. In G. Jones (Ed.), *Exploring probability in school* (pp. 95–119). Springer. [https://doi.org/10.1007/0-387-24530-8\\_5](https://doi.org/10.1007/0-387-24530-8_5)
- Lesh, R., Post, T., & Behr, M. (1988). Proportional reasoning. In J. Hiebert, & M. Behr (Eds.), *Number concepts and operations for the middle grades* (pp. 93–118). NCTM.
- Llinares, S. (2013). Professional noticing: a component of the mathematics teacher's professional practice. *Sisyphus — Journal of Education*, *1*(3), 76–93. <https://doi.org/10.25749/sis.3707>
- Mason, J. (2016). Perception, interpretation and decision making: Understanding gaps between competence and performance—a commentary. *ZDM*, *48*(1-2), 219–226. <https://doi.org/10.1007/s11858-016-0764-1>
- Van Dooren, W. (2014). Probabilistic thinking: analyses from a psychological perspective. In *Probabilistic Thinking* (pp. 123–126). Springer. [https://doi.org/10.1007/978-94-007-7155-0\\_7](https://doi.org/10.1007/978-94-007-7155-0_7)
- Vásquez, C., & Alsina, A. (2015). El conocimiento del profesorado para enseñar probabilidad: Un análisis global desde el modelo del Conocimiento Didáctico-Matemático [Teachers' knowledge to teach probability: A global analysis from the model of Didactic-Mathematical Knowledge]. *Avances de Investigación en Educación Matemática, AIEM*, *7*, 27–48.
- Watson, J. (2005). The probabilistic reasoning of middle school students. In G. A. Jones (Ed.), *Exploring probability in school: Challenges for teaching and learning* (pp. 145–169). Springer. [https://doi.org/10.1007/0-387-24530-8\\_7](https://doi.org/10.1007/0-387-24530-8_7)