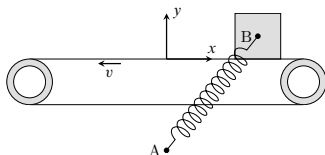


Compact weighted residual formulation for periodic solutions of systems  
undergoing unilateral contact and frictional occurrences  
ENOC 2022

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- Dynamics [with  $\gamma(x, y) = (\sqrt{x^2 + (L + y)^2} - L) / \sqrt{x^2 + (L + y)^2}$ ]

$$\Psi_x(r_T, x, y, \ddot{x}) = m\ddot{x} + k\gamma(x, y)x - r_T - f = 0 \quad (1)$$

$$\Psi_y(r_N, x, y, \ddot{y}) = m\ddot{y} + k\gamma(x, y)(y + L) - r_N + mg = 0 \quad (2)$$

- Unilateral contact<sup>1</sup>

- ▶ displacement level

$$r_N \geq 0, \quad y \geq 0, \quad r_N \cdot y = 0 \quad (3)$$

- ▶ velocity level

$$\begin{cases} y < 0 & \implies r_N \geq 0, \quad \dot{y}^+ + e\dot{y}^- \geq 0, \quad r_N \cdot (\dot{y}^+ + e\dot{y}^-) = 0 \\ y \geq 0 & \implies r_N = 0 \end{cases} \quad (4)$$

- Coulomb's friction [with  $\dot{u}_T(t) = -v + \dot{x}(t)$ ]

$$\begin{cases} \dot{u}_T = 0 & \implies |r_T| \leq \mu r_N \\ \dot{u}_T \neq 0 & \implies |r_T| = \mu |r_N| \text{ and } \exists \alpha \geq 0 \mid r_T = -\alpha \dot{u}_T \end{cases} \quad (5)$$

<sup>1</sup>Acary and Brogliato. *Numerical methods for nonsmooth dynamical systems: Applications in mechanics and electronics*. Lecture Notes in Applied and Computational Mechanics. Springer Verlag, 2008. doi: 10.1007/978-3-540-75392-6.

- Unilateral contact

- ▶ Displacement level<sup>2</sup>

$$\begin{aligned}
 r_N \geq 0, \quad y \geq 0, \quad r_N \cdot y = 0 &\Leftrightarrow \forall \rho_N > 0, \quad \Psi_N(r_N, y) = r_N - \max(r_N - \rho_N y, 0) = 0 \\
 &\Leftrightarrow \Psi_N(r_N, y) = \min(r_N, y) = 0
 \end{aligned} \tag{6}$$

- ▶ Velocity level

$$\begin{cases}
 y < 0 &\implies r_N \geq 0, \quad \dot{y}^+ + e\dot{y}^- \geq 0, \quad r_N \cdot (\dot{y}^+ + e\dot{y}^-) = 0 \\
 y \geq 0 &\implies r_N = 0
 \end{cases}$$

$$\Leftrightarrow \forall \rho_N, \Psi_N(r_N, y, \dot{y}) = (\text{sign } y - 1)(r_N - \max(r_N - \rho_N(\dot{y}^+ + e\dot{y}^-), 0)) + (\text{sign } y + 1)r_N = 0 \tag{7}$$

with convention:  $\text{sign } y = 1$  if  $y \geq 0$  and  $\text{sign } y = -1$  otherwise

- Frictional contact<sup>3,4,5</sup>

$$\begin{cases}
 \dot{u}_T = 0 &\implies |r_T| \leq \mu r_N \\
 \dot{u}_T \neq 0 &\implies |r_T| = \mu |r_N| \text{ and } \exists \alpha \geq 0 \mid r_T = -\alpha \dot{u}_T
 \end{cases}$$

$$\Leftrightarrow \forall \rho_T > 0, \Psi_T(r_T, \dot{u}_T) = r_T \max(\mu r_N, |r_T - \rho_T \dot{u}_T|) - \mu r_N (r_T - \rho_T \dot{u}_T) = 0$$

$$\Leftrightarrow \forall \rho_T > 0, \Psi_T(r_T, \dot{u}_T) = \dot{u}_T + \min(0, \rho_T(r_T + \mu r_N) - \dot{u}_T) + \max(0, \rho_T(r_T - \mu r_N) - \dot{u}_T) = 0 \tag{8}$$

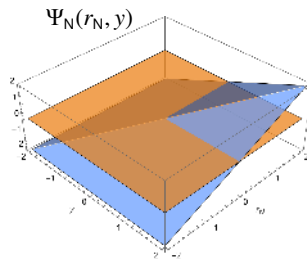
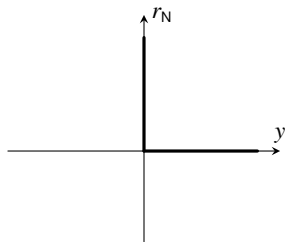
<sup>2</sup>Acary and Brogliato. *Numerical methods for nonsmooth dynamical systems: Applications in mechanics and electronics*. Lecture Notes in Applied and Computational Mechanics. Springer Verlag, 2008. DOI: 10.1007/978-3-540-75392-6.

<sup>3</sup>Stadler. "Semismooth Newton and augmented Lagrangian methods for a simplified friction problem". *SIAM Journal on Optimization* (2004). [hal-01371364].

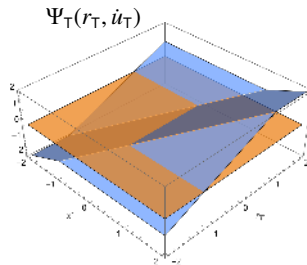
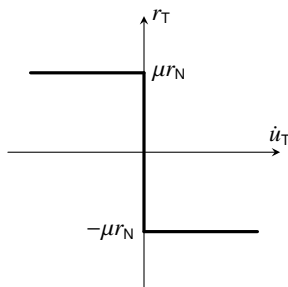
<sup>4</sup>Acary, Brémond, and Huber. "On solving contact problems with Coulomb friction: formulations and numerical comparisons". *Transactions of the European Network for Nonsmooth Dynamics*. Springer, 2018. [hal-01878539].

<sup>5</sup>Hüeber, Stadler, and Wohlmuth. "A primal-dual active set algorithm for three-dimensional contact problems with Coulomb friction". *SIAM Journal on Scientific Computing* (2008). [hal-01371357].

- Unilateral contact (displacement level)



- Frictional contact



- Discretization of the unknowns

$$\begin{aligned}
 x(t) \approx x_h(t) &= \sum_k x_k \phi_k(t), & y(t) \approx y_h(t) &= \sum_k y_k \phi_k(t) \\
 r_N(t) \approx r_{Nh}(t) &= \sum_k N_k \phi_k(t), & r_T(t) \approx r_{Th}(t) &= \sum_k T_k \phi_k(t)
 \end{aligned} \tag{9}$$

- ▶  $\phi_k(t)$ : family of pre-selected shape functions (for instance, Fourier functions)
- ▶  $x_k, y_k, N_k, T_k$ : to be found participations of the shape functions in the solution

- Differentiation in time

$$\ddot{x}(t) \approx \ddot{x}_h(t) = \sum_k x_k \ddot{\phi}_k(t), \quad \ddot{y}(t) \approx \ddot{y}_h(t) = \sum_k y_k \ddot{\phi}_k(t) \tag{10}$$

- Ritz-Galerkin projection

$$\begin{aligned}
 \forall k, \int_t \phi_k(t) \Psi_x(r_{Th}, x_h, y_h, \ddot{x}_h) dt &= 0, & \int_t \phi_k(t) \Psi_y(r_{Th}, x_h, y_h, \ddot{y}_h) dt &= 0 & \text{(dynamics)} \\
 \forall k, \int_t \phi_k(t) \Psi_N(r_{Nh}, y_h) dt &= 0, & \int_t \phi_k(t) \Psi_T(r_{Th}, x_h) dt &= 0 & \text{(contact)}
 \end{aligned} \tag{11}$$

- Solution method
  - ▶ With a range  $k \in [1, \dots, N_h]$ ,  $4N_h$  nonlinear/nonsmooth equations for  $4N_h$  unknowns
  - ▶ Integral computation  $\rightarrow$  classical quadrature schemes
  - ▶ Nonlinear solver  $\rightarrow$  Hybrid Powell with line search (Matlab solver)
  - ▶ Solution accuracy solely dictated by the selected  $\{\phi_k\}$  family
  - ▶ Parameters  $\rho_N$  and  $\rho_T$  can play a role in the convergence rate
- Comparison with the DLFT method<sup>6,7</sup>  $\rightarrow$  similar, however:
  - ▶ Not limited to Fourier functions for the  $\{\phi_k\}$  family
  - ▶ No need of the Alternating Frequency Time procedure<sup>8</sup> to express the Signorini and Coulomb conditions in the time domain using FFT
  - ▶ DLFT algorithm  $\rightarrow$  energy dissipation in the normal direction (plastic impact)
- No penalization needed
- No regularization/smoothing procedure involved
- Possibility to express a piecewise Jacobian
- Other equivalent techniques: shooting<sup>9</sup>

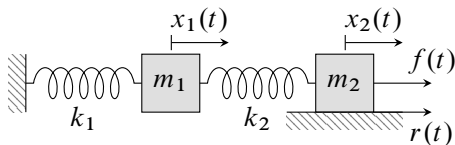
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<sup>6</sup>Nacivet, Pierre, Thouverez, and Jézéquel. "A dynamic Lagrangian frequency–time method for the vibration of dry-friction-damped systems". *Journal of Sound and Vibration* (2003). [hal-011635272].

<sup>7</sup>Vadcard, Batailly, and Thouverez. "On Harmonic Balance Method-based Lagrangian contact formulations for vibro-impact problems". *Journal of Sound and Vibration* (2022). [hal-03665624].

<sup>8</sup>Cameron and Griffin. "An Alternating Frequency/Time Domain Method for Calculating the Steady-State Response of Nonlinear Dynamic Systems". *Journal of Applied Mechanics* (1989). [hal-01333697].

<sup>9</sup>Schreyer and Leine. "A mixed shooting – harmonic balance method for unilaterally constrained mechanical systems". *Archive of Mechanical Engineering* (2016). [hal-01356796].



- Dynamics

$$m_1 \ddot{x}_1 + (k_1 + k_2)x_2 - k_1 x_1 = 0 \quad (12)$$

$$m_2 \ddot{x}_2 + k_2 x_2 - k_2 x_1 = f \cos \omega t + r \quad (13)$$

$$\dot{x}_2 + \min(0, \rho(r + \mu N) - \dot{x}_2) + \max(0, \rho(r - \mu N) - \dot{x}_2) = 0 \quad (14)$$

- Approximated solution: Fourier functions (odd harmonics only)

$$x_{1h}(t) = \frac{a_1}{2} + \sum_{k=1}^{N_h} [a_{2k} \cos(2k-1)\omega t + a_{2k+1} \sin(2k-1)\omega t]$$

$$x_{2h}(t) = \frac{b_1}{2} + \sum_{k=1}^{N_h} [b_{2k} \cos(2k-1)\omega t + b_{2k+1} \sin(2k-1)\omega t] \quad (15)$$

$$r_h(t) = \frac{c_1}{2} + \sum_{k=1}^{N_h} [c_{2k} \cos(2k-1)\omega t + c_{2k+1} \sin(2k-1)\omega t]$$

$m_1 = 1; k_1 = 1; m_2 = 1; k_2 = 10; \mu = 0.9; N = 1; \omega = 0.2; f = 10$

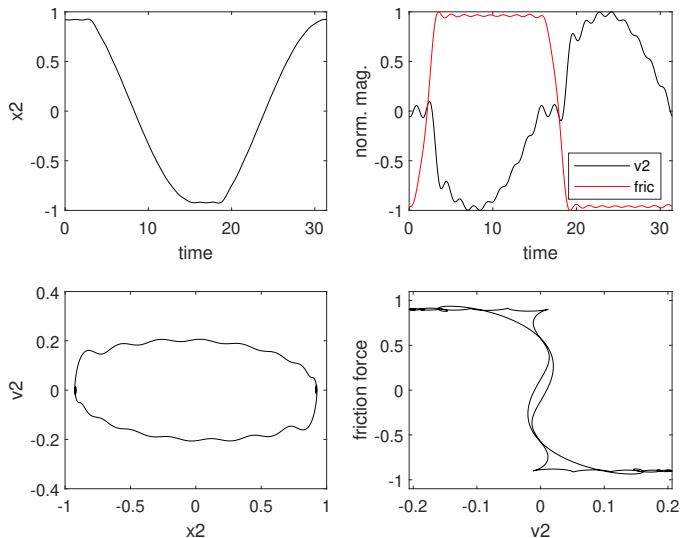


Figure 1:  $N_h = 10$



$m_1 = 1; k_1 = 1; m_2 = 1; k_2 = 10; \mu = 0.9; N = 1; \omega = 0.2; f = 10$

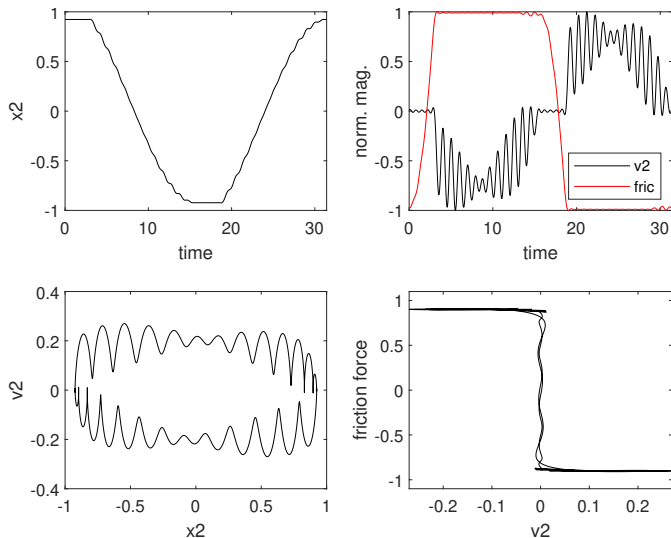


Figure 1:  $N_h = 20$

$m_1 = 1; k_1 = 1; m_2 = 1; k_2 = 10; \mu = 0.9; N = 1; \omega = 0.2; f = 10$

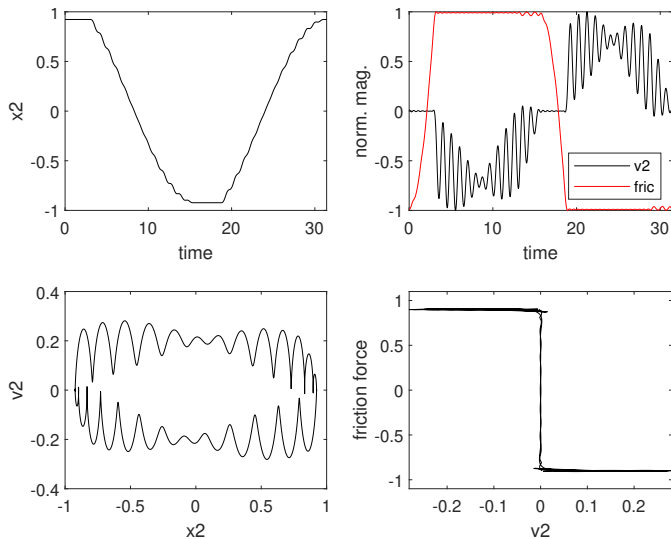


Figure 1:  $N_h = 30$

$m_1 = 1; k_1 = 1; m_2 = 1; k_2 = 10; \mu = 0.9; N = 1; \omega = 0.2; f = 10$

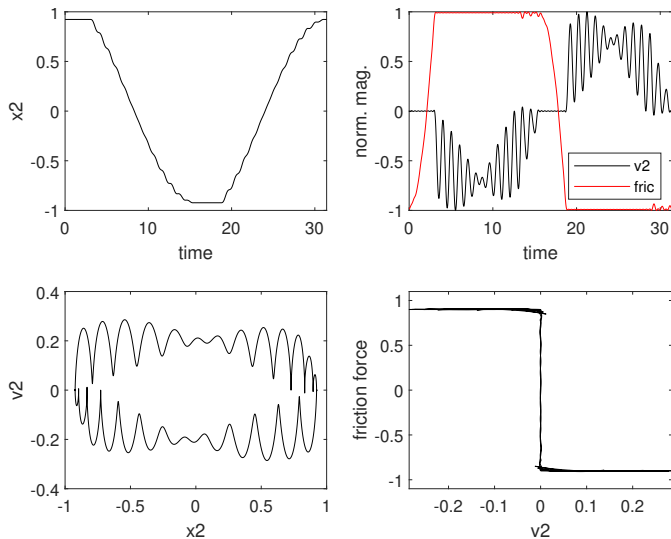
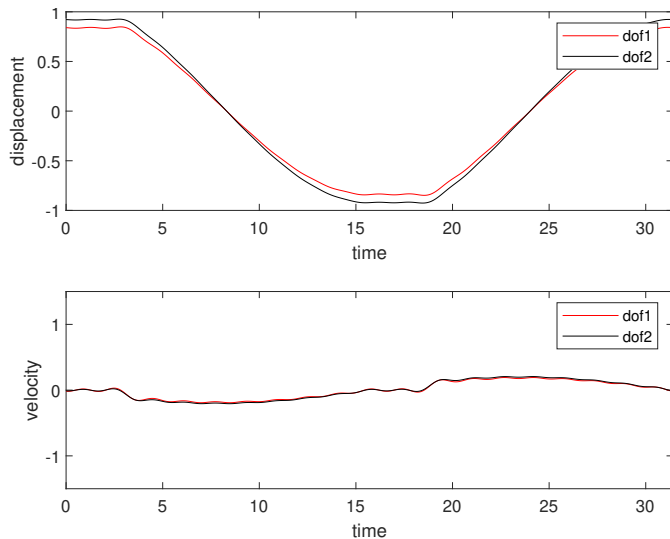
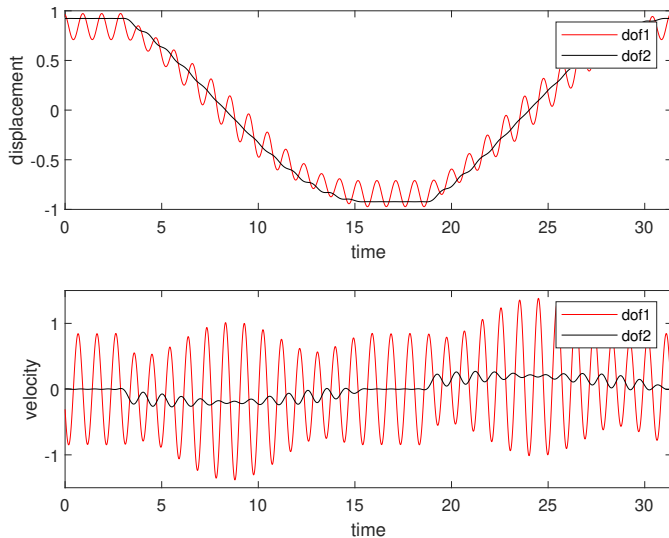
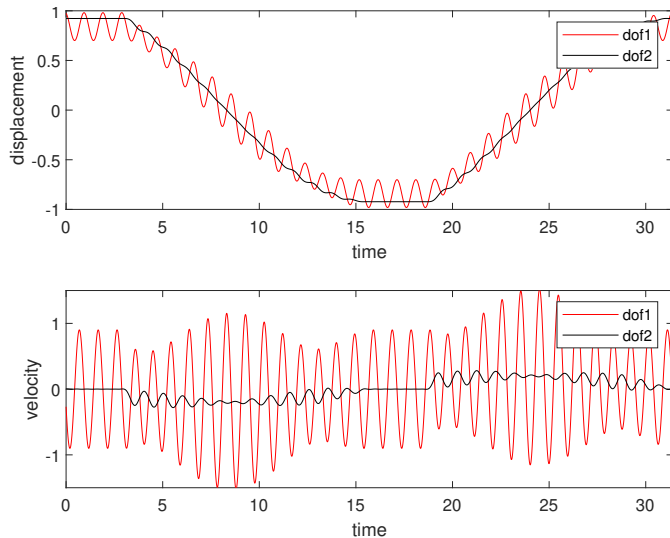


Figure 1:  $N_h = 40$

$m_1 = 1; k_1 = 1; m_2 = 1; k_2 = 10; \mu = 0.9; N = 1; \omega = 0.2; f = 10$ Figure 1:  $N_h = 10$

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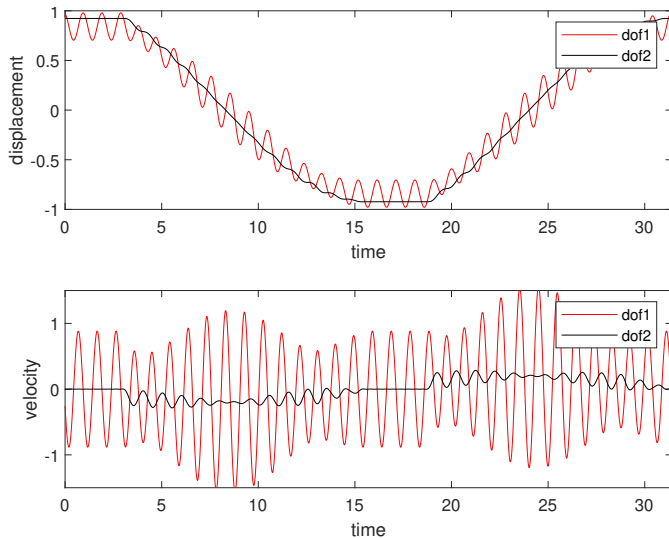


Figure 1:  $N_h = 40$

$m_1 = 1; k_1 = 1; m_2 = 1; k_2 = 10; \mu = 0.9; N = 8; \omega = 4.5; f = 10$

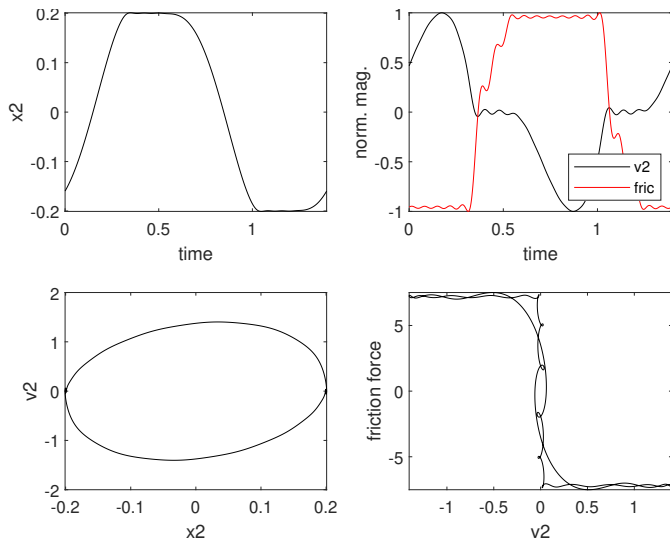


Figure 2:  $N_h = 10$



$m_1 = 1; k_1 = 1; m_2 = 1; k_2 = 10; \mu = 0.9; N = 8; \omega = 4.5; f = 10$

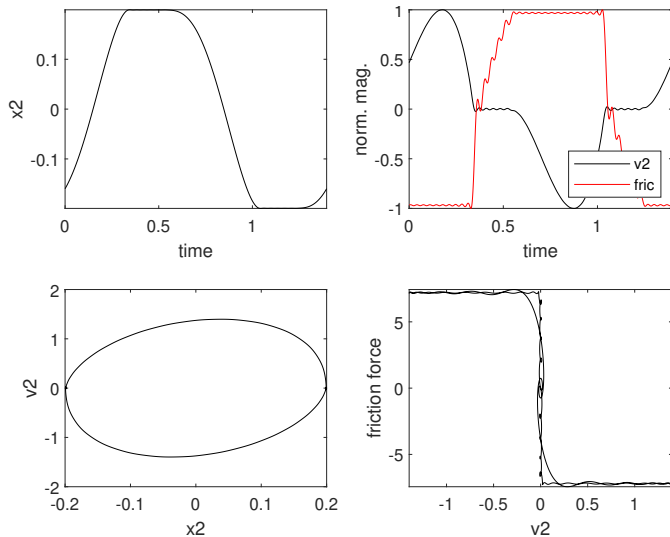


Figure 2:  $N_h = 20$

$m_1 = 1; k_1 = 1; m_2 = 1; k_2 = 10; \mu = 0.9; N = 8; \omega = 4.5; f = 10$

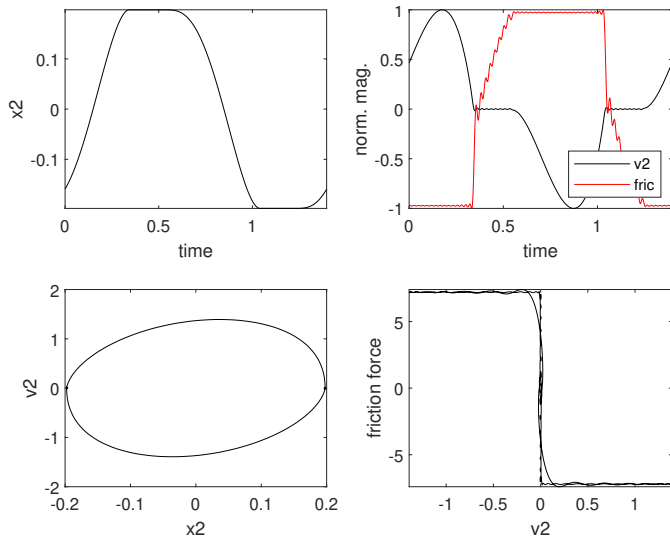


Figure 2:  $N_h = 30$

$m_1 = 1; k_1 = 1; m_2 = 1; k_2 = 10; \mu = 0.9; N = 8; \omega = 4.5; f = 10$

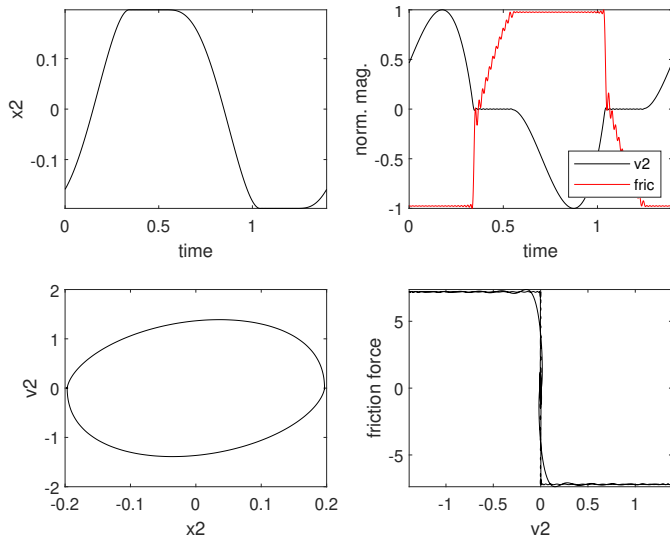


Figure 2:  $N_h = 40$

$m_1 = 1; k_1 = 1; m_2 = 1; k_2 = 10; \mu = 0.9; N = 8; \omega = 4.5; f = 10$

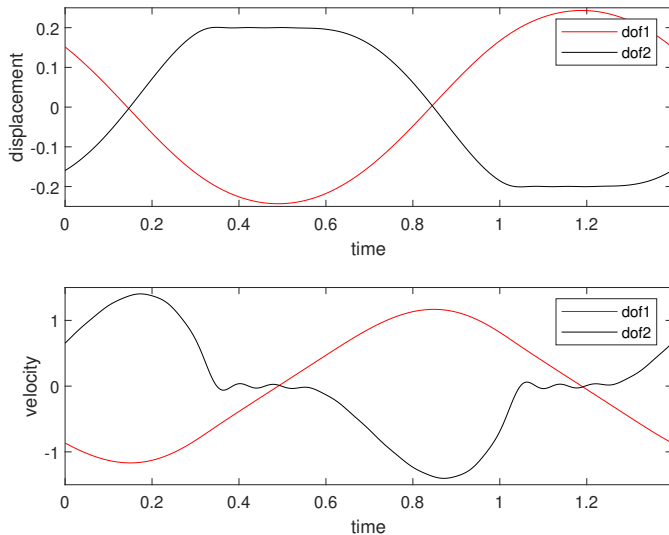
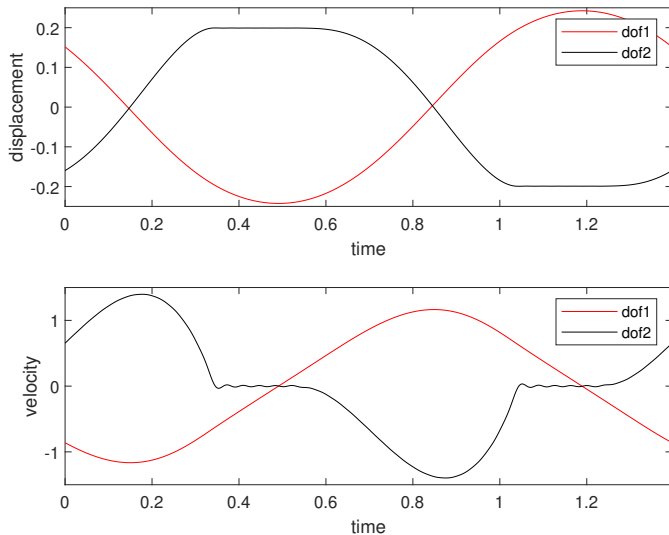


Figure 2:  $N_h = 10$

$m_1 = 1; k_1 = 1; m_2 = 1; k_2 = 10; \mu = 0.9; N = 8; \omega = 4.5; f = 10$ Figure 2:  $N_h = 20$

$m_1 = 1; k_1 = 1; m_2 = 1; k_2 = 10; \mu = 0.9; N = 8; \omega = 4.5; f = 10$

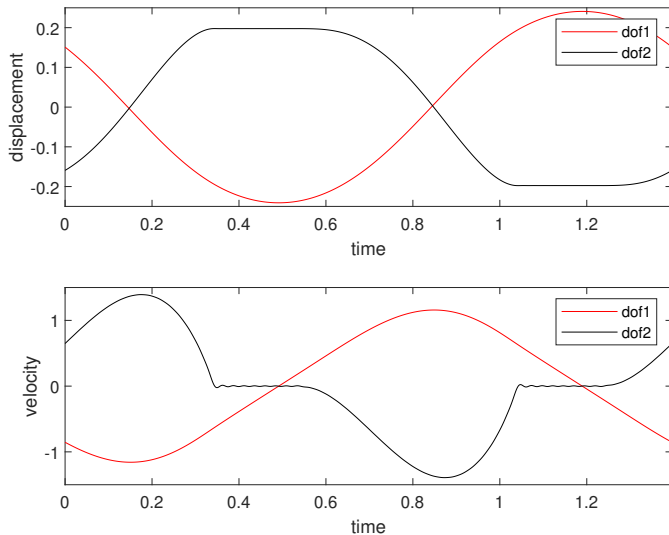
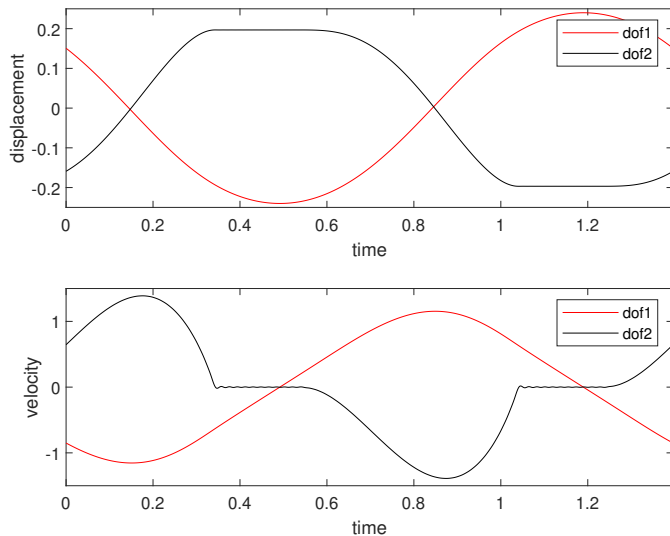


Figure 2:  $N_h = 30$

$m_1 = 1; k_1 = 1; m_2 = 1; k_2 = 10; \mu = 0.9; N = 8; \omega = 4.5; f = 10$ Figure 2:  $N_h = 40$

- Summary
  - ▶ Very simple and general formulation in terms of nonsmooth equalities with Ritz-Galerkin projection
  - ▶ Successful implementation using basic Matlab commands: `fsolve`, `integral`
- Future work
  - ▶ Systematic convergence analysis on  $\rho_T, \rho_N$
  - ▶ Systematic comparison to time-domain solution techniques
  - ▶ Investigation with other shape functions  $\{\phi_k(t)\}$ : wavekets, finite elements in time...
  - ▶ Investigation with other domains of integration: quasi-periodic solutions (square), damped solutions (on  $[0, \infty[$ )
  - ▶ Investigation of forced response: energy-frequency curves
  - ▶ Extension to 2D frictional framework
  - ▶ Extension to nonlinear/nonsmooth modal analysis within the invariant manifold framework
- Pending question: stability analysis?