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Numerical analysis for backward Euler Spectral discretization for Stokes equations with boundary conditions involving the pressure: Part I

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Abstract

In this paper, we address the study of the time-dependent Stokes system with boundary conditions involving the pressure. We obtain existence and uniqueness for a class of Lipschitz-continuous domains. Next, a spectral discretizations of the problem is proposed combined with the backward Euler scheme. The discrete spaces are defined in a way to give exactly divergence-free discrete approximations for the velocity. Then, we prove the associated discrete inf-sup condition and derive a priori error estimates. Finally, some numerical experiments are presented.

Keywords. Stokes equations, pressure boundary conditions, spectral discretization, divergence-free approximations.

1. Introduction

The Stokes and Navier-Stokes equations have been traditionally joined with Dirichlet boundary conditions (BC) due to their simple treatment (see for instance [15, 17, 21, 20, 27]). However, these BCs may not be appropriate from a modeling point of view. This is the case if the physical domain has several inlet/outlet interfaces. In the pioneering paper [4], the authors considered non standard BCs where the pressure and the tangential trace of the velocity are prescribed on a part of the boundary. Since that many researchers have been interested in the study of this problem. We refer to [3, 5, 6, 13, 9, 18, 24] for the analysis of the continuous problem. While most works uses the rotational formulation of the Laplace operator, the symmetric stress tensor was considered in [13]. This actually leads to a well-posedness result in the same $H^1 \times L^2$ functional spaces as for the usual Dirichlet BCs. As regards the numerical analysis, a priori and a posteriori error analysis were performed in [9] for the finite element method. Hood-Taylor spaces were used in [13] in order to test the efficiency of a double saddle point formulation with an application to physiological flows. Concerning the spectral methods, several spectral discretizations were proposed in [8] in the case of the unit square. The first method presented in that paper gives rise to exactly divergence-free discrete velocities. This was done by identifying all the spurious modes and then define the space approximating the pressure ensuring this property and which gives a "good" rate of convergence for the pressure. This method is usually used when one wants to approximate the velocity by exactly divergence-free discrete velocities (see for instance [11, Sec. 24], [7] and [1]). However, the main inconvenient of this method is the complexity of the discrete pressure space.

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In this paper we consider the time-dependent Stokes equations in a domain $\Omega \subset \mathbb{R}^3$:

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + \nabla p &= \mathbf{f} & \text{in } \Omega \times]0, T[\\ \operatorname{div} \mathbf{u} &= 0 & \text{in } \Omega \times]0, T[, \\ \mathbf{u}(\cdot, 0) &= \mathbf{u}_0 & \text{in } \Omega \end{aligned} \quad (1.1)$$

where T is the final simulation time and ν is the kinematic viscosity coefficient. The homogeneous BCs on the velocity associated to these equations are given as follows (see [6] for the non homogeneous case) :

$$\begin{aligned} \mathbf{u} &= \mathbf{0} & \text{on } \Gamma_D \times]0, T[, \\ \mathbf{u} \times \mathbf{n} &= \mathbf{0} & \text{on } \Gamma_m \times]0, T[, \\ p &= p_m & \text{on } \Gamma_m \times]0, T[. \end{aligned} \quad (1.2)$$

Where Γ_D and Γ_m constitute a partition of the boundary $\partial\Omega$. We give a velocity-pressure variational formulation for problem (1.1)-(1.2). Similarly to all the previous works, the pressure BC is treated in a natural way. Then, we prove the well-posedness result in $H^1 \times L^2$. Next, we focus on spectral approximations with *pointwise* divergence-free velocities. The properties and advantages of such schemes are well-known for the finite element methods. The review paper [22] discusses in depth the effects of the divergence constraint on simulations. We propose a spectral discretization in the unit cube. It is a classical method and was considered in [8] for the stationary problem in the unit square.

The rest of this paper is organized as follows: In section 2 we consider the variational formulation in the primitive variables of the Stokes problem and we prove its well-posedness. Next, the stability analysis and error estimates of the time-discrete problem, which is based on the backward Euler's scheme is shown in section 3. Then, the fully discrete formulation is considered in section 4 where the discrete inf-sup condition is established. The Stability of the scheme and the optimal a priori errors analysis are obtained in Section 5. Numerical results and examples are presented in section 6.

2. Variational formulation

Let $\Omega \subset \mathbb{R}^3$ be a bounded connected Lipschitz-continuous domain. We assume that the domain Ω is such that :

$$\partial\Omega = \bar{\Gamma}_D \cup \bar{\Gamma}_m, \quad \Gamma_D \cap \Gamma_m = \emptyset,$$

where both Γ_D and Γ_m have a finite number of connected components and a positive measure in $\partial\Omega$. We also assume that

$$\text{each connected component of } \Gamma_m \text{ is flat with a convex neighborhood in } \Omega. \quad (2.1)$$

In all what follows, we denote by (\cdot, \cdot) (resp. by $(\cdot, \cdot)_\Gamma$) the L^2 -scalar product defined on Ω (resp. on a part Γ of the boundary). We use the whole scale Sobolev spaces $H^s(\Omega)$, $s \geq 0$, equipped with the norm $\|\cdot\|_{H^s(\Omega)}$. Let X be a separable Hilbert space equipped with the norm $\|\cdot\|_X$. For any positive integer k , we denote by $H^k(0, T; X)$ the space of functions v defined from $]0, T[$ with values in X such that the functions $t \mapsto \|u^{(i)}\|_X$, $0 \leq i \leq k$, are square integrable, where $u^{(i)}$ is the i -th derivative of u in the sense of distributions. We also introduce the well known spaces

$$H(\operatorname{div}; \Omega) = \{\mathbf{v} \in L^2(\Omega)^3; \operatorname{div} \mathbf{v} \in L^2(\Omega)\}$$

and

$$H(\operatorname{curl}; \Omega) = \{\mathbf{v} \in L^2(\Omega)^3; \operatorname{curl} \mathbf{v} \in L^2(\Omega)^3\},$$

which are Hilbert spaces for the associated graph norms

$$\|\mathbf{v}\|_{H(\operatorname{div}, \Omega)} = \left(\|\mathbf{v}\|_{L^2(\Omega)}^2 + \|\operatorname{div} \mathbf{v}\|_{L^2(\Omega)}^2 \right)^{1/2}, \quad \|\mathbf{v}\|_{H(\operatorname{curl}, \Omega)} = \left(\|\mathbf{v}\|_{L^2(\Omega)}^2 + \|\operatorname{curl} \mathbf{v}\|_{L^2(\Omega)}^2 \right)^{1/2}.$$

We recall from [20, Sec. I.2.2 and I.2.3] that the normal trace operator $\mathbf{v} \rightarrow \mathbf{v} \cdot \mathbf{n}$ is continuous from $H(\operatorname{div}, \Omega)$ onto $H^{-1/2}(\partial\Omega)^3$ and that the tangential trace operator $\mathbf{v} \rightarrow \mathbf{v} \times \mathbf{n}$ is continuous from $H(\operatorname{curl}, \Omega)$ into $H^{-1/2}(\Omega)^3$. We denote by $H_0(\operatorname{div}, \Omega)$ and $H_0(\operatorname{curl}, \Omega)$ the kernels of the normal trace and tangential trace operators respectively. So, we introduce our variational space, which its aim is to take into account the boundary conditions on the velocity

$$\mathbf{X} = \{\mathbf{v} \in H^1(\Omega)^3, \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma_D \text{ and } \mathbf{v} \times \mathbf{n} = 0 \text{ on } \partial\Omega\} \quad (2.2)$$

equipped with the norm

$$\|\mathbf{v}\|_{\mathbf{X}} = \left(\|\mathbf{v}\|_{L^2(\Omega)}^2 + \|\operatorname{curl} \mathbf{v}\|_{L^2(\Omega)}^2 + \|\operatorname{div} \mathbf{v}\|_{L^2(\Omega)}^2 \right)^{1/2}.$$

Proposition 2.1. *The space \mathbf{X} equipped with the norm $\|\cdot\|_{\mathbf{X}}$ is a Hilbert space.*

Proof. We use similar arguments to [18, proof of Thm A.1]. It suffices to show that the space

$$\mathbf{U} = \{\mathbf{v} \in H(\operatorname{div}, \Omega) \cap H(\operatorname{curl}, \Omega), \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma_D \text{ and } \mathbf{v} \times \mathbf{n} = 0 \text{ on } \partial\Omega\}$$

is continuously embedded in $H^1(\Omega)^3$. We consider an covering $\{\theta_i\}_{i=0}^n$ of Ω , i.e. $\bar{\Omega} = \bigcup_{i=0}^n \theta_i$, with $\theta_0 = \Omega$. For every $i = 1, \dots, n$, we denote by θ_i^* the intersection of θ_i and Ω . Since Γ_m satisfies (2.1), we can assume that if θ_i^* meets both Γ_m and Γ_D or if $\partial\theta_i^* \cap \partial\Omega$ is contained in Γ_m , then θ_i^* is convex. Now, we introduce the associated partition of unity $\{\alpha_i\}_{i=0}^n$:

$$\begin{aligned} \alpha_i \in \mathcal{D}(\theta_i) \quad \text{and} \quad 0 \leq \alpha_i(\mathbf{x}) \leq 1, \quad \forall i = 0, \dots, n, \quad \forall \mathbf{x} \in \bar{\Omega}, \\ \sum_{i=0}^n \alpha_i(\mathbf{x}) = 1 \quad \forall \mathbf{x} \in \bar{\Omega}. \end{aligned}$$

Thus, any function $\mathbf{v} \in \mathbf{U}$ can be written in the form $\mathbf{v} = \sum_{i=0}^n \alpha_i \mathbf{v}$. Obviously, we have

$$\alpha_i \mathbf{v} \in H(\operatorname{div}, \theta_i^*) \cap H_0(\operatorname{curl}, \theta_i^*) \quad \text{and} \quad \theta_i^* \text{ is convex}$$

or

$$\alpha_i \mathbf{v} \in H_0(\operatorname{div}, \theta_i^*) \cap H_0(\operatorname{curl}, \theta_i^*)$$

which implies thanks to Theorems 2.5 and 2.12 in [2] that $\alpha_i \mathbf{v} \in H^1(\Omega)^3$. The estimates $\|\alpha_i \mathbf{v}\|_{H^1(\Omega)} \leq c \|\mathbf{v}\|_{\mathbf{X}}$ are ensured by the theorems already mentioned. \square

To write the variational formulation of our problem, we first define the following bilinear forms

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) &= \int_{\Omega} \operatorname{curl} \mathbf{u}(t, \mathbf{x}) \cdot \operatorname{curl} \mathbf{v}(\mathbf{x}) \, d\mathbf{x}, \\ b(\mathbf{v}, q) &= - \int_{\Omega} (\operatorname{div} \mathbf{v})(\mathbf{x}) q(\mathbf{x}) \, d\mathbf{x}. \end{aligned} \quad (2.3)$$

We now assume that the source function \mathbf{f} belongs to $L^2(0, T; L^2(\Omega)^3)$ and p_m belongs to the space $L^2(0, T; H_{00}^{\frac{1}{2}}(\Gamma_m))$, where $H_{00}^{\frac{1}{2}}(\Gamma_m)$ denotes the space of functions in $H^{\frac{1}{2}}(\partial\Omega)$ with zero value on Γ_D . The variational formulation of the problem can be written as follows:

Find $(\mathbf{u}, p) \in L^2(0, T; \mathbf{X}) \cap H^1(0, T; L^2(\Omega)^3) \times L^2(0, T; L^2(\Omega))$ such that for almost every $t \in [0, T]$ and for all $(\mathbf{v}, q) \in \mathbf{X} \times L^2(\Omega)$

$$\begin{aligned} \left(\frac{\partial \mathbf{u}}{\partial t}, \mathbf{v} \right) + \nu a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) &= (\mathbf{f}, \mathbf{v}) - (p_m, \mathbf{v} \cdot \mathbf{n})_{\Gamma_m}, \\ b(\mathbf{u}, q) &= 0 \end{aligned} \quad (2.4)$$

with

$$\mathbf{u}(\cdot, 0) = \mathbf{u}_0 \text{ in } \Omega. \quad (2.5)$$

We have the following equivalence result.

Proposition 2.2. *Any pair (\mathbf{u}, p) in $H^1(0, T; L^2(\Omega)^3) \cap L^2(0, T; \mathbf{X})$ such that $\mathbf{curl curl} \mathbf{u}(\cdot, t)$ belongs to $L^2(\Omega)^3$ for almost every $t \in [0, T]$ is a solution of problem (2.4)-(2.5) if and only if it is a solution of problem (1.1)-(1.2).*

Proof. Let (\mathbf{u}, p) be a solution of problem (2.4)-(2.5). First, the velocity boundary conditions in (1.2) are satisfied since $\mathbf{u}(\cdot, t)$ belongs to \mathbf{X} . In addition, the divergence-free constraint is obviously equivalent to the second equation in (2.4). Now, taking $\mathbf{v} \in \mathcal{D}(\Omega)$ and differentiating in the sense of distributions gives the momentum equation (1.1), which implies that $p(\cdot, t) \in H^1(\Omega)$ for almost every $t \in [0, T]$.

We return to equation (2.4) and we perform an integration by parts. This gives

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) &= \int_{\Omega} \mathbf{curl curl} \mathbf{u}(\mathbf{x}, t) \cdot \mathbf{v}(\mathbf{x}) \, d\mathbf{x}, \\ b(\mathbf{v}, p) &= \int_{\Omega} \mathbf{v}(\mathbf{x}) \cdot \nabla p(\mathbf{x}, t) \, d\mathbf{x} - (p, \mathbf{v} \cdot \mathbf{n})_{\Gamma_m}. \end{aligned}$$

We deduce that

$$\forall \mathbf{v} \in \mathbf{X}, \quad (p - p_m, \mathbf{v} \cdot \mathbf{n})_{\Gamma_m} = 0.$$

This implies the pressure boundary condition in (1.2). The converse can be proved using the same arguments by counter-integrating by parts. \square

In order to study the well-posedness of the weak problem (2.4)-(2.5), we will prove the following inf-sup condition, which all arguments are those of [8, 9]

Lemma 2.3. *There exists a constant $\beta > 0$ such that the following inf-sup condition holds*

$$\forall q \in L^2(\Omega), \quad \sup_{\mathbf{v} \in \mathbf{X}} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_{\mathbf{X}}} \geq \beta \|q\|_{L^2(\Omega)}. \quad (2.6)$$

Proof. Any function $q \in L^2(\Omega)$ can be split into $q = \tilde{q} + \bar{q}$, where

$$\tilde{q} \in L_0^2(\Omega) \quad \text{and} \quad \bar{q} = \frac{1}{|\Omega|} \int_{\Omega} q(x) \, dx.$$

Thanks to [20, Cor. I.2.4], for every $\tilde{q} \in L_0^2(\Omega)$, there exists $\tilde{\mathbf{v}} \in H_0^1(\Omega)^3$ such that

$$\operatorname{div} \tilde{\mathbf{v}} = -\tilde{q} \quad \text{and} \quad \|\tilde{\mathbf{v}}\|_{H^1(\Omega)} \leq c \|\tilde{q}\|_{L^2(\Omega)}. \quad (2.7)$$

On the other hand, we observe that for every $\mathbf{v} \in \mathbf{X}$,

$$b(\mathbf{v}, \bar{q}) := - \int_{\Omega} \bar{q} \operatorname{div} \mathbf{v} \, dx = \bar{q} \int_{\Gamma_m} \mathbf{v} \cdot \mathbf{n} \, d\sigma.$$

So, let $\boldsymbol{\varphi} \in \mathbf{X}$ be a function with positive integral $\int_{\Gamma_m} \boldsymbol{\varphi}(\mathbf{x}) \cdot \mathbf{n} \, d\sigma > 0$. Setting $\bar{\mathbf{v}} = -\bar{q}\boldsymbol{\varphi}$, we can easily check that

$$\bar{\mathbf{v}} \in \mathbf{X}, \quad b(\bar{\mathbf{v}}, \bar{q}) \geq c \|\bar{q}\|_{L^2(\Omega)}^2 \quad \text{and} \quad \|\bar{\mathbf{v}}\|_{\mathbf{X}} \leq c \|\bar{q}\|_{L^2(\Omega)}. \quad (2.8)$$

Following Boland-Nicolaidis [14], we set $\mathbf{v} = \tilde{\mathbf{v}} + \lambda \bar{\mathbf{v}}$ for a positive constant λ . Since $b(\tilde{\mathbf{v}}, \bar{q}) = 0$, we derive from (2.7)-(2.8) that

$$\begin{aligned} b(\mathbf{v}, q) &= b(\tilde{\mathbf{v}}, \tilde{q}) + \lambda b(\bar{\mathbf{v}}, \bar{q}) + \lambda b(\bar{\mathbf{v}}, \tilde{q}) \\ &\geq \|\tilde{q}\|_{L^2(\Omega)}^2 + \lambda c_1 \|\bar{q}\|_{L^2(\Omega)}^2 - \lambda c_2 \|\tilde{q}\|_{L^2(\Omega)} \|\bar{q}\|_{L^2(\Omega)}. \end{aligned}$$

Using Young's inequality

$$ab \leq \frac{a^2}{2\varepsilon} + \frac{\varepsilon b^2}{2} \quad (2.9)$$

gives

$$b(\mathbf{v}, q) \geq \frac{1}{2} \|\tilde{q}\|_{L^2(\Omega)}^2 + \lambda \left(c_1 - \frac{\lambda c_2^2}{2} \right) \|\bar{q}\|_{L^2(\Omega)}^2.$$

By taking $\lambda = \frac{c_1}{c_2^2}$ and using the orthogonality of \tilde{q} and \bar{q} in $L^2(\Omega)$, we obtain

$$b(\mathbf{v}, q) \geq c' \|q\|_{L^2(\Omega)}^2,$$

The inequality $\|\mathbf{v}\|_{\mathbf{X}} \leq c \|q\|_{L^2(\Omega)}$ is straightforward. \square

As usual, in order to introduce the reduced problem, we define the kernel

$$\mathbf{V} = \{ \mathbf{v} \in \mathbf{X}, b(\mathbf{v}, q) = 0 \quad \forall q \in L^2(\Omega) \}$$

which coincides with the closed subspace of divergence-free functions

$$\mathbf{V} = \{ \mathbf{v} \in \mathbf{X}, \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega \}.$$

We denote by \mathbf{H} the closure of \mathbf{V} in $L^2(\Omega)^3$.

Let us consider the following problem: Find $\mathbf{u} \in L^2(0, T; \mathbf{V}) \cap H^1(0, T; \mathbf{H})$ such that for almost every $t \in [0, T]$

$$\forall \mathbf{v} \in \mathbf{V}, \quad \left(\frac{\partial \mathbf{u}}{\partial t}, \mathbf{v} \right) + \nu a(\mathbf{u}, \mathbf{v}) = (f, \mathbf{v}) - (p_m, \mathbf{v} \cdot \mathbf{n})_{\Gamma_m}, \quad (2.10)$$

$$\mathbf{u}(\cdot, 0) = \mathbf{u}_0 \text{ in } \Omega, \quad (2.11)$$

We now prove the well-posedness result in the following theorem:

Theorem 2.4. *Assume that the data (\mathbf{f}, p_m) belong to $L^2(0, T; L^2(\Omega)^3) \times L^2(0, T; H_{00}^{\frac{1}{2}}(\Gamma_m))$ and the initial velocity \mathbf{u}_0 belongs to \mathbf{V} . Then problem (2.10)-(2.11) has a unique solution $\mathbf{u} \in L^2(0, T; \mathbf{V}) \cap L^\infty(0, T; L^2(\Omega)^3)$. Furthermore, this solution satisfies the following stability estimates:*

$$\|\mathbf{u}(\cdot, t)\|_{L^2(\Omega)} + \|\mathbf{u}\|_{L^2(0, t; \mathbf{X})} \leq c \left(\|\mathbf{f}\|_{L^2(0, t; L^2(\Omega))} + \|p_m\|_{L^2(0, t; H_{00}^{\frac{1}{2}}(\Gamma_m))} + \|\mathbf{u}_0\|_{L^2(\Omega)} \right), \quad (2.12)$$

$$\left\| \frac{\partial \mathbf{u}}{\partial t} \right\|_{L^2(0, t; L^2(\Omega)^3)} \leq c \left(\|\mathbf{f}\|_{L^2(0, t; L^2(\Omega))} + \|p_m\|_{L^2(0, t; H_{00}^{\frac{1}{2}}(\Gamma_m))} + \|\mathbf{u}_0\|_{H(\operatorname{curl}, \Omega)} \right). \quad (2.13)$$

Proof. We will prove the existence of the solution using the time discretization. To do so, let us consider a uniform partition of the interval $[0, T]$ into K sub-intervals. We denote by τ the corresponding time-step and set $t_k = k\tau$, $1 \leq k \leq K$. Now, we introduce the approximate problem as follows:

Find $(\mathbf{u}^k)_{1 \leq k \leq K}$ in $(\mathbf{V})^K$ such that for every k ,

$$\forall \mathbf{v} \in \mathbf{V}, \quad \left(\frac{\mathbf{u}^k - \mathbf{u}^{k-1}}{\tau_k}, \mathbf{v} \right) + \nu a(\mathbf{u}^k, \mathbf{v}) = (\mathbf{f}^k, \mathbf{v}) - (p_m^k, \mathbf{v} \cdot \mathbf{n})_{\Gamma_m}, \quad (2.14)$$

$$\mathbf{u}^0 = \mathbf{u}_0 \quad \text{in } \Omega, \quad (2.15)$$

where

$$\mathbf{f}^k = \frac{1}{\tau} \int_{t_{k-1}}^{t_k} \mathbf{f}(\cdot, s) ds \quad \text{and} \quad p_m^k = \frac{1}{\tau} \int_{t_{k-1}}^{t_k} p_m(\cdot, s) ds.$$

We associate to the elements $\mathbf{u}^1, \dots, \mathbf{u}^K$, the following functions:

$$\mathbf{u}_\tau : [0, T] \longrightarrow \mathbf{V}, \quad \mathbf{u}_\tau(t) = \mathbf{u}^k \text{ for every } t \in [t_{k-1}, t_k]$$

and $\mathbf{w}_\tau : [0, T] \longrightarrow \mathbf{V}$ continuous, affine on each interval $[t_{k-1}, t_k]$ such that $\mathbf{w}_\tau(t_k) = \mathbf{u}^k$. Clearly, for every k , problem (2.14)-(2.15) has a unique solution (see [20, Cor. I.4.1]). Now, we derive a priori estimates. Let $\mathbf{v} = \mathbf{u}^k$ in (2.14). Applying Young's inequality (2.9) gives

$$\begin{aligned} \|\mathbf{u}^k\|_{L^2(\Omega)}^2 + \nu \tau \|\mathbf{curl} \mathbf{u}^k\|_{L^2(\Omega)}^2 &\leq \frac{1}{2}(1 + \tau) \|\mathbf{u}^k\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\mathbf{u}^{k-1}\|_{L^2(\Omega)}^2 \\ &\quad + \tau \left(\|\mathbf{f}^k\|_{L^2(\Omega)}^2 + \tau_k \|p_m^k\|_{H_{00}^{\frac{1}{2}}(\Gamma_m)}^2 \right). \end{aligned}$$

Rearranging the terms of the last inequality, we get using induction

$$\|\mathbf{u}^k\|_{L^2(\Omega)}^2 + 2\nu \sum_{i=1}^k \frac{\tau}{\gamma_i^k} \|\mathbf{curl} \mathbf{u}^i\|_{L^2(\Omega)}^2 \leq \frac{1}{\gamma_k^k} \|\mathbf{u}_0\|_{L^2(\Omega)}^2 + 2 \sum_{i=1}^k \frac{\tau}{\gamma_i^k} \left(\|\mathbf{f}^i\|_{L^2(\Omega)}^2 + \|p_m^i\|_{H_{00}^{\frac{1}{2}}(\Gamma_m)}^2 \right).$$

Where we have set for every i , $1 \leq i \leq k$, $\gamma_i^k = (1 - \tau)^{k-i+1}$. Now, observe that

$$(\gamma_i^k)^{-1} \leq (1 - \tau)^{-K} =: c(\tau)$$

and $c(\tau)$ tends to e^{-T} when τ goes to 0. This yields that $c(\tau)$ is uniformly bounded by a constant c . All this leads to the following estimate

$$\|\mathbf{u}^k\|_{L^2(\Omega)}^2 + 2\nu \sum_{i=1}^k \tau \|\mathbf{curl} \mathbf{u}^i\|_{L^2(\Omega)}^2 \leq c \left(\|\mathbf{u}_0\|_{L^2(\Omega)}^2 + \tau \sum_{i=1}^k \left(\|\mathbf{f}^i\|_{L^2(\Omega)}^2 + \|p_m^i\|_{H_{00}^{\frac{1}{2}}(\Gamma_m)}^2 \right) \right). \quad (2.16)$$

In addition, taking $\mathbf{v} = \mathbf{u}^k - \mathbf{u}^{k-1}$ in (2.10) and using Young's inequality and the identity

$$(a, b) = \frac{1}{2} (\|a\|^2 + \|a - b\|^2 - \|b\|^2),$$

obtaining

$$\begin{aligned} \|\mathbf{u}^k - \mathbf{u}^{k-1}\|_{L^2(\Omega)}^2 + \frac{\nu \tau}{2} \left(\|\mathbf{curl} \mathbf{u}^k\|_{L^2(\Omega)}^2 + \|\mathbf{curl}(\mathbf{u}^k - \mathbf{u}^{k-1})\|_{L^2(\Omega)}^2 - \|\mathbf{curl} \mathbf{u}^{k-1}\|_{L^2(\Omega)}^2 \right) \\ \leq \tau^2 \left(\|\mathbf{f}^k\|_{L^2(\Omega)}^2 + \|p_m^k\|_{H_{00}^{\frac{1}{2}}(\Gamma_m)}^2 \right) + \frac{1}{2} \|\mathbf{u}^k - \mathbf{u}^{k-1}\|_{L^2(\Omega)}^2. \end{aligned}$$

Dividing both sides by τ and summing over k yield

$$\sum_{i=1}^k \tau_i \left\| \frac{\mathbf{u}^i - \mathbf{u}^{i-1}}{\tau_i} \right\|_{L^2(\Omega)}^2 \leq \nu \|\mathbf{curl} \mathbf{u}_0\|_{L^2(\Omega)}^2 + 2 \sum_{i=1}^k \tau_i \left(\|\mathbf{f}^i\|_{L^2(\Omega)}^2 + \|p_m^i\|_{H_{00}^{\frac{1}{2}}(\Gamma_m)}^2 \right). \quad (2.17)$$

From (2.16), we deduce that the sequences $(\mathbf{u}_\tau)_\tau$ and $(\mathbf{w}_\tau)_\tau$ are uniformly bounded in $L^2(0, T; \mathbf{V}) \cap L^\infty(0, T; L^2(\Omega))$. Moreover, estimate (2.17) shows that $(\partial \mathbf{w}_\tau / \partial t)_\tau$ is bounded in $L^2(0, T; L^2(\Omega)^3)$. Then standard arguments, see for example [27, Subsect. III.4.4], ensure the existence of a solution \mathbf{u} in $H^1(0, T; H) \cap L^2(0, T; \mathbf{V})$. The stability estimates (2.12)-(2.13) are consequences of estimates (2.16)-(2.17) and since problem (2.10)-(2.11) is linear, the uniqueness of the solution follows from the stability estimate (2.12). \square

Remark 2.5. *If the function p_m belongs only to $L^2(0, T; L^2(\Gamma_m))$, the well-posedness theorem holds true when the boundary $\partial\Omega$ is connected. Indeed, applying [2, Cor. 3.19], we arrive at*

$$\begin{aligned} \forall \mathbf{v} \in \mathbf{V}, \quad (p_m^k, \mathbf{v} \cdot \mathbf{n})_{\Gamma_m} &\leq \|p_m^k\|_{L^2(\Gamma_m)} \|\mathbf{v}\|_{L^2(\Gamma_m)} \\ &\leq c \|p_m^k\|_{L^2(\Gamma_m)} \|\mathbf{v}\|_{H^1(\Omega)} \\ &\leq c \|p_m^k\|_{L^2(\Gamma_m)} \|\mathbf{curl} \mathbf{v}\|_{L^2(\Omega)} \\ &\leq c \|p_m^k\|_{L^2(\Gamma_m)}^2 + \frac{\nu}{2} \|\mathbf{curl} \mathbf{v}\|_{L^2(\Omega)}^2. \end{aligned}$$

It suffices to use this estimate to bound the term $(p_m^k, \mathbf{v} \cdot \mathbf{n})_{\Gamma_m}$ in the proof of Theorem 2.4.

Theorem 2.6. *Under the same hypothesis of Theorem 2.4, problem (2.4)-(2.5) has a unique solution (\mathbf{u}, p) such that the velocity \mathbf{u} is the solution of the reduced problem (2.10)-(2.11) and the pressure p satisfies*

$$\|p\|_{L^2(0,t;L^2(\Omega))} \leq c \left(\|\mathbf{f}\|_{L^2(0,t;L^2(\Omega))} + \|p_m\|_{L^2(0,t;H_{00}^{\frac{1}{2}}(\Gamma_m))} + \|\mathbf{u}_0\|_{H(\mathbf{curl},\Omega)} \right).$$

Proof. For every $\mathbf{v} \in \mathbf{X}$ and almost every $t \in [0, T]$, we define the linear form L_t by

$$L_t(\mathbf{v}) = (\mathbf{f}, \mathbf{v}) - (p_m, \mathbf{v} \cdot \mathbf{n})_{\Gamma_m} - \left(\frac{\partial \mathbf{u}}{\partial t}, \mathbf{v} \right) - \nu a(\mathbf{u}, \mathbf{v}). \quad (2.18)$$

Clearly, this is a continuous linear form on \mathbf{X} and according to (2.10), it vanishes on \mathbf{V} . Hence, see [20, Lem. I.4.1], there exists a function $p(\cdot, t) \in L^2(\Omega)$ such that

$$\forall \mathbf{v} \in \mathbf{X}, \quad L_t(\mathbf{v}) = b(\mathbf{v}, p(\cdot, t)) \quad (2.19)$$

and

$$\|p(\cdot, t)\|_{L^2(\Omega)} \leq \frac{1}{\beta} \sup_{\mathbf{v} \in \mathbf{X}} \frac{L_t(\mathbf{v})}{\|\mathbf{v}\|_{\mathbf{X}}}. \quad (2.20)$$

From (2.19)-(2.20) we deduce that $p \in L^2(0, T; L^2(\Omega))$ and (\mathbf{u}, p) is a solution of problem (2.4)-(2.5). Now, assume that (\mathbf{u}_1, p_1) and (\mathbf{u}_2, p_2) are two solutions of problem (2.4)-(2.5) and consider the difference (\mathbf{z}, y) . Then, this pair satisfies

$$\begin{aligned} \forall \mathbf{v} \in \mathbf{X}, \quad & \left(\frac{\partial \mathbf{z}}{\partial t}, \mathbf{v} \right) + a(\mathbf{z}, \mathbf{v}) + b(\mathbf{v}, y) = 0, \\ & \forall q \in L^2(\Omega), \quad (\operatorname{div} \mathbf{z}, q) = 0, \\ & \mathbf{z}(\cdot, 0) = \mathbf{0}. \end{aligned}$$

From the stability estimates (2.12)-(2.20), we get $(\mathbf{z}, y) = \mathbf{0}$. □

3. Semi-discrete problem

In this section, our interests focus on the semi-discrete problem. We consider a sequence of nodes t_k with $k = 0, 1, \dots, K$, which induces a partition of the interval $[0, T]$ into sub-intervals $[t_{k-1}, t_k]$. We denote by τ_k the time-step $t_k - t_{k-1}$ and by τ the maximum of τ_k . For any data (\mathbf{f}, p_m) in $\mathcal{C}^0(0, T; L^2(\Omega)^3) \times \mathcal{C}^0(0, T; H_{00}^{\frac{1}{2}}(\Gamma_m))$, the backward Euler's scheme reads as :

Find $(\mathbf{u}^k)_{1 \leq k \leq K}$ in $(\mathbf{X})^K$ and $(p^k)_{1 \leq k \leq K}$ in $(L^2(\Omega))^K$ such that for all $k = 1, 2, \dots, K$,

$$\forall \mathbf{v} \in \mathbf{X}, \quad \left(\frac{\mathbf{u}^k - \mathbf{u}^{k-1}}{\tau_k}, \mathbf{v} \right) + \nu a(\mathbf{u}^k, \mathbf{v}) + b(\mathbf{v}, p^k) = (\mathbf{f}^k, \mathbf{v}) - (p_m^k, \mathbf{v} \cdot \mathbf{n})_{\Gamma_m}, \quad (3.1)$$

$$\begin{aligned} \forall q \in L^2(\Omega), \quad & b(\mathbf{u}^k, q) = 0, \\ & \mathbf{u}^0 = \mathbf{u}_0 \quad \text{in } \Omega, \end{aligned} \quad (3.2)$$

where $\mathbf{f}^k = \mathbf{f}(\cdot, t_k)$ and $p_m^k = p_m(\cdot, t_k)$.

Now we can establish the well-posedness result for this scheme.

Theorem 3.1. *Assume that the data $(\mathbf{f}, p_m) \in \mathcal{C}^0(0, T; L^2(\Omega)^3) \times \mathcal{C}^0(0, T; H_{00}^{\frac{1}{2}}(\Gamma_m))$ and the initial velocity \mathbf{u}_0 belongs to $H(\mathbf{curl}, \Omega)$. Then, problem (3.1)-(3.2) has a unique solution $(\mathbf{u}^k, p^k)_{1 \leq k \leq K}$. Moreover, the following estimates hold*

$$\|\mathbf{u}^k\|_{L^2(\Omega)}^2 + 2\nu \sum_{i=1}^k \tau_i \|\mathbf{curl} \mathbf{u}^i\|_{L^2(\Omega)}^2 \leq e \left(\|\mathbf{u}_0\|_{L^2(\Omega)}^2 + 4T \sum_{i=1}^k \tau_i \left(\|\mathbf{f}^i\|_{L^2(\Omega)}^2 + \|p_m^i\|_{H_{00}^{\frac{1}{2}}(\Gamma_m)}^2 \right) \right), \quad (3.3)$$

$$\sum_{i=1}^k \tau_i \left\| \frac{\mathbf{u}^i - \mathbf{u}^{i-1}}{\tau_i} \right\|_{L^2(\Omega)}^2 \leq \nu \|\mathbf{curl} \mathbf{u}_0\|_{L^2(\Omega)}^2 + 2 \sum_{i=1}^k \tau_i \left(\|\mathbf{f}^i\|_{L^2(\Omega)}^2 + \|p_m^i\|_{H_{00}^{\frac{1}{2}}(\Gamma_m)}^2 \right). \quad (3.4)$$

Similarly, the sequence of pressures satisfies for all $1 \leq k \leq K$,

$$\sum_{i=1}^k \tau_i \|p^i\|_{L^2(\Omega)}^2 \leq c \left(\|\mathbf{u}_0\|_{L^2(\Omega)}^2 + \nu \|\mathbf{curl} \mathbf{u}_0\|_{L^2(\Omega)}^2 + \sum_{i=1}^k \tau_i \left(\|\mathbf{f}^i\|_{L^2(\Omega)}^2 + \|p_m^i\|_{H_{00}^{\frac{1}{2}}(\Gamma_m)}^2 \right) \right). \quad (3.5)$$

Proof. For every $k = 1, 2, \dots, K$, the coercivity of the bilinear form $a(\cdot, \cdot)$ and the inf-sup condition (2.6) imply existence and uniqueness of the solution of problem (3.1)-(3.2), (see [20, Cor. I.4.1]).

First, to derive estimate (3.3), let $\mathbf{v} = \mathbf{u}^k$ in (3.1). Cauchy-Schwarz and Young's inequalities give

$$\left(\frac{1}{2} - \frac{\tau_k}{4T} \right) \|\mathbf{u}^k\|_{L^2(\Omega)}^2 + \nu \tau_k \|\mathbf{curl} \mathbf{u}^k\|_{L^2(\Omega)}^2 \leq \frac{1}{2} \|\mathbf{u}^{k-1}\|_{L^2(\Omega)}^2 + 2T \tau_k \left(\|\mathbf{f}^k\|_{L^2(\Omega)}^2 + \tau_k \|p_m^k\|_{H^{\frac{1}{2}}_{00}(\Gamma_m)}^2 \right). \quad (3.6)$$

By induction we deduce that

$$\|\mathbf{u}^k\|_{L^2(\Omega)}^2 + 2\nu \sum_{i=1}^k \frac{\tau_i}{\gamma_i^k} \|\mathbf{curl} \mathbf{u}^i\|_{L^2(\Omega)}^2 \leq \frac{1}{\gamma_1^k} \|\mathbf{u}_0\|_{L^2(\Omega)}^2 + 4T \sum_{i=1}^k \frac{\tau_i}{\gamma_i^k} \left(\|\mathbf{f}^i\|_{L^2(\Omega)}^2 + \|p_m^i\|_{H_{00}^{\frac{1}{2}}(\Gamma_m)}^2 \right),$$

where we have set for every i , $1 \leq i \leq K$,

$$\gamma_i^k = \prod_{j=i}^k \left(1 - \frac{\tau_j}{2T} \right).$$

Now, taking into account the inequality $\ln(1 - \frac{x}{2}) \geq -x$ valid for every $x \in [0, 1]$, we deduce that

$$\ln(\gamma_1^k) = \sum_{i=1}^k \ln \left(1 - \frac{\tau_i}{2T} \right) \geq - \sum_{i=1}^k \frac{\tau_i}{T} = -1,$$

which implies that $\gamma_1^k \geq e^{-1}$ and the proof is complete.

Next, to get estimate (3.4), we choose $\mathbf{v} = \mathbf{u}^k - \mathbf{u}^{k-1}$ in (3.1). Once again, using successively Cauchy-Schwarz and Young's inequalities combining with the identity

$$(a, ab) = \frac{1}{2} (\|a\|^2 + \|a - b\|^2 - \|b\|^2),$$

yielding

$$\begin{aligned} \|\mathbf{u}^k - \mathbf{u}^{k-1}\|_{L^2(\Omega)}^2 + \frac{\nu \tau_k}{2} \left(\|\mathbf{curl} \mathbf{u}^k\|_{L^2(\Omega)}^2 + \|\mathbf{curl}(\mathbf{u}^k - \mathbf{u}^{k-1})\|_{L^2(\Omega)}^2 - \|\mathbf{curl} \mathbf{u}^{k-1}\|_{L^2(\Omega)}^2 \right) \\ \leq \tau_k^2 \left(\|\mathbf{f}^k\|_{L^2(\Omega)}^2 + \|p_m^k\|_{H_{00}^{1/2}(\Gamma_m)}^2 \right) + \frac{1}{2} \|\mathbf{u}^k - \mathbf{u}^{k-1}\|_{L^2(\Omega)}^2. \end{aligned}$$

Then, dividing by τ_k and summing over k yield estimate (3.4).

Finally, to prove (3.5), we use the inf-sup condition (2.6), we obtain

$$\begin{aligned} \beta \|p\|_{L^2(\Omega)} &\leq \sup_{\mathbf{v} \in \mathbf{X}} \frac{b(\mathbf{v}, p)}{\|\mathbf{v}\|_{\mathbf{X}}} \\ &\leq \left\| \frac{\mathbf{u}^k - \mathbf{u}^{k-1}}{\tau_k} \right\|_{L^2(\Omega)} + \nu \|\mathbf{curl} \mathbf{u}^k\|_{L^2(\Omega)} + \|\mathbf{f}^k\|_{L^2(\Omega)} + \|p_m^k\|_{H_{00}^{1/2}(\Gamma_m)}. \end{aligned}$$

The desired estimate for the pressure follows by taking the square of both sides, multiplying by τ_k and summing over k . \square

Remark 3.2. The classical way to obtain estimate (3.3) is to use an inequality of the form

$$\forall \mathbf{v} \in \mathbf{V}, \quad \|\mathbf{v}\|_{L^2(\Omega)} \leq c \|\mathbf{curl} \mathbf{v}\|_{L^2(\Omega)}.$$

We avoided following this method because to apply the above inequality we usually assume some restrictions on the domain Ω (see [2, Cor. 3.19] and [18, Appendix A.1]).

Let us now estimate the velocity and pressure errors approximation relatives to scheme (3.1)-(3.2). To do this, we consider the sequence $(\mathbf{e}^k)_{0 \leq k \leq K}$ defined by $\mathbf{e}^k = \mathbf{u}(\cdot, t_k) - \mathbf{u}^k$, where \mathbf{u} is the exact solution of the problem (2.4)-(2.5). For every $1 \leq k \leq K$, this sequence satisfies the following system:

$$\begin{aligned} \forall \mathbf{v} \in \mathbf{X}, \quad (\mathbf{e}^k, \mathbf{v}) + \nu a(\mathbf{e}^k, \mathbf{v}) + b(\mathbf{v}, p^k) &= (\mathbf{e}^{k-1}, \mathbf{v}) + (\varepsilon^k, \mathbf{v}), \\ \forall q \in L^2(\Omega), \quad b(\mathbf{e}^k, q) &= 0, \\ \mathbf{e}^0 &= 0 \text{ in } \Omega, \end{aligned} \quad (3.7)$$

where the consistency error is given by

$$\varepsilon^k(\mathbf{u}) = \frac{\mathbf{u}(\cdot, t_k) - \mathbf{u}(\cdot, t_{k-1})}{\tau_k} - \frac{\partial \mathbf{u}}{\partial t}(\cdot, t_k).$$

Using Taylor's formula with the integral form of the remainder, we obtain

$$\varepsilon^k(\mathbf{u}) = \frac{1}{\tau_k} \int_{t_{k-1}}^{t_k} (s - t_{k-1}) \frac{\partial^2 \mathbf{u}}{\partial t^2}(s) ds,$$

which yields

$$\|\varepsilon^k(\mathbf{u})\|_{L^2(\Omega)} \leq \frac{1}{\sqrt{3}} \tau_k^{1/2} \left\| \frac{\partial^2 \mathbf{u}}{\partial t^2} \right\|_{L^2(t_{k-1}, t_k; L^2(\Omega)^3)}.$$

Proceeding in a similar manner to the stability proof and taking into account the above estimate for $\varepsilon^k(\mathbf{u})$, we easily derive the next convergence result.

Proposition 3.3. Under the hypothesis of the previous theorem, assume in addition that the exact velocity \mathbf{u} , solution of (2.4)-(2.5) belongs to $H^2(0, T; L^2(\Omega)^3)$. Then, the following error estimates hold

$$\begin{aligned} \left(\|\mathbf{u}(\cdot, t_k) - \mathbf{u}^k\|_{L^2(\Omega)}^2 + 2\nu \sum_{i=1}^k \tau_i \|\mathbf{curl} \mathbf{u}(\cdot, t_i) - \mathbf{curl} \mathbf{u}^i\|_{L^2(\Omega)}^2 \right)^{1/2} &\leq \sqrt{\frac{eT}{3}} \tau \left\| \frac{\partial^2 \mathbf{u}}{\partial t^2} \right\|_{L^2(0, t_k; L^2(\Omega)^3)} \\ \left(\sum_{i=1}^k \tau_k \left\| \frac{(\mathbf{u}(\cdot, t_i) - \mathbf{u}^i) - (\mathbf{u}(\cdot, t_{i-1}) - \mathbf{u}^{i-1})}{\tau_i} \right\|_{L^2(\Omega)}^2 \right)^{1/2} &\leq \tau \left\| \frac{\partial^2 \mathbf{u}}{\partial t^2} \right\|_{L^2(0, t_k; L^2(\Omega)^3)} \\ \text{and} \quad \left(\sum_{i=1}^k \tau_i \|p(\cdot, t_i) - p^i\|_{L^2(\Omega)}^2 \right)^{1/2} &\leq c \tau \left\| \frac{\partial^2 \mathbf{u}}{\partial t^2} \right\|_{L^2(0, t_k; L^2(\Omega)^3)}. \end{aligned}$$

4. Analysis of the fully discrete problem

4.1. Fully discrete formulation

In this section, we study the space approximation using Spectral method. We then are interested in the case where Ω is the unit cube $] -1, 1[^3$ and the interface Γ_m is the region $\{(x, y, z) \in \partial\Omega : y = \pm 1\}$. We denote by $\mathbb{P}_{l,m,n}(\Omega)$ the space of restrictions to Ω of polynomials with degree $\leq l$ with respect to x , $\leq m$ with respect to y and $\leq n$ with respect to z . When l, m and n are equal, this space is denoted by $\mathbb{P}_n(\Omega)$.

Let $N \geq 3$. Setting $\xi_0 = -1$ and $\xi_N = 1$, we introduce the nodes ξ_i , $1 \leq i \leq N-1$, and the associated weights ρ_i , $0 \leq i \leq N$, of the Gauss-Lobatto quadrature formula. We denote by \mathcal{I}_N the Lagrange interpolation operator associated to the nodes (ξ_i, ξ_j, ξ_k) , $0 \leq i, j, k \leq N$, with values in $\mathbb{P}_N(\Omega)$. Denoting by $\mathbb{P}_N(-1, 1)$ the space of restrictions to $[-1, 1]$ of polynomials with degree $\leq N$, we recall the following exactness property

$$\forall \varphi \in \mathbb{P}_{2N-1}(-1, 1), \quad \int_{-1}^1 \varphi(x) dx = \sum_{i=1}^N \varphi(\xi_i) \rho_i. \quad (4.1)$$

The following equivalence property will be useful in the sequel

$$\forall \varphi \in \mathbb{P}_N(-1, 1), \quad \|\varphi\|_{L^2(-1,1)}^2 \leq \sum_{i=1}^N \varphi(\xi_i)^2 \rho_i \leq 3 \|\varphi\|_{L^2(-1,1)}^2. \quad (4.2)$$

This formula allows us to define the discrete scalar product on the space of continuous functions u and v as follows

$$(u, v)_N = \sum_{i=1}^N u(\xi_i, \xi_j, \xi_k) v(\xi_i, \xi_j, \xi_k) \rho_i \rho_j \rho_k.$$

The associated discrete norm will be denoted by $\|\cdot\|_N$. We refer to the review books [11, 12, 16] for more details on this topics.

The first spectral discretization of our problem is based on the following velocity discrete space:

$$\mathbf{X}_N = (\mathbb{P}_{N,N-1,N-1}(\Omega) \times \mathbb{P}_{N-1,N,N-1}(\Omega) \times \mathbb{P}_{N-1,N-1,N}(\Omega)) \cap \mathbf{X}.$$

The space \mathbb{M}_N of discrete pressures is a subspace of $\mathbb{P}_{N-1}(\Omega)$ and will be fixed later on. In fact, the pair $(\mathbf{X}_N, \mathbb{P}_{N-1}(\Omega))$ is not stable. Indeed, defining the space of spurious modes

$$Z_N := \left\{ q_N \in \mathbb{P}_{N-1}(\Omega) : \forall \mathbf{v}_N \in \mathbf{X}_N, \quad b_N(\mathbf{v}_N, q_N) = 0 \right\},$$

the following lemma shows that it is not empty (see [11, Rem. 24.1]).

Lemma 4.1. *The space Z_N has dimension $12N - 16$ and is spanned by the polynomials*

$$\begin{aligned} (L'_N \pm L'_{N-1})(x)(L'_N \pm L'_{N-1})(y) \varphi_N(z), & \quad (L'_N \pm L'_{N-1})(x) \varphi_N(y) (L'_N \pm L'_{N-1})(z) \\ \text{and} & \quad \varphi_N(x) (L'_N \pm L'_{N-1})(y) (L'_N \pm L'_{N-1})(z), \end{aligned} \quad (4.3)$$

where φ_N runs through $\mathbb{P}_{N-1}(-1, 1)$.

Now we define the space \mathbb{M}_N following Bernardi et al. [11, Sect. 24] (see also [1]). Let $\lambda \in]0, 1[$ be a fixed real number. Denoting by $[\lambda N]$ the integer part of λN and by $\pi_{[\lambda N]}$ the orthogonal projection operator from $L^2(-1, 1)$ onto $\mathbb{P}_{[\lambda N]}(-1, 1)$, we set

$$\Lambda_N := L'_N - \pi_{[\lambda N]} L'_N, \quad \Lambda_{N-1} := L'_{N-1} - \pi_{[\lambda N]} L'_{N-1}.$$

Définition 4.2. *The space \mathbb{M}_N is the orthogonal in $\mathbb{P}_{N-1}(\Omega)$ for the $L^2(\Omega)$ -scalar product of the space \tilde{Z}_N spanned by the polynomials*

$$\begin{aligned} (\Lambda_N \pm \Lambda_{N-1})(x) (\Lambda_N \pm \Lambda_{N-1})(y) \varphi_N(z), \\ (\Lambda_N \pm \Lambda_{N-1})(x) \varphi_N(y) (\Lambda_N \pm \Lambda_{N-1})(z), \\ \text{and} & \quad \varphi_N(x) (\Lambda_N \pm \Lambda_{N-1})(y) (\Lambda_N \pm \Lambda_{N-1})(z), \end{aligned} \quad (4.4)$$

where φ_N runs through $\mathbb{P}_{N-1}(-1, 1)$.

Since Z_N and \tilde{Z}_N have the same dimension, we can check that

$$\mathbb{P}_{N-1}(\Omega) = \mathbb{M}_N \oplus Z_N.$$

The fully discrete problem is now defined using the spectral Galerkin method with numerical integration as follows:

Find $(\mathbf{u}_N^k)_{1 \leq k \leq K} \in \mathbf{X}_N$ and $(p_N^k)_{1 \leq k \leq K} \in \mathbb{M}_N$ such that for all $k = 1, 2, \dots, K$,

$$\forall \mathbf{v}_N \in \mathbf{X}_N, \quad \left(\frac{\mathbf{u}_N^k - \mathbf{u}_N^{k-1}}{\tau_k}, \mathbf{v}_N \right)_N + \nu a_N(\mathbf{u}_N^k, \mathbf{v}_N) + b(\mathbf{v}_N, p_N^k) = (\mathbf{f}^k, \mathbf{v}_N)_N - (p_m^k, \mathbf{v}_N \cdot \mathbf{n})_{N, \Gamma_m}, \quad (4.5)$$

$$\forall q_N \in \mathbb{M}_N, \quad b(\mathbf{u}_N^k, q_N) = 0, \quad \mathbf{u}_N^0 = \mathcal{I}_N \mathbf{u}_0 \quad \text{in } \Omega, \quad (4.6)$$

where the discrete bilinear form $a_N(\cdot, \cdot)$ is given by

$$a_N(\mathbf{u}_N, \mathbf{v}_N) = (\mathbf{curl} \mathbf{u}_N, \mathbf{curl} \mathbf{v}_N)_N$$

and the boundary term $(p_m^k, \mathbf{v}_N \cdot \mathbf{n})_{N, \Gamma_m}$ is defined using Gauss-Legendre-Lobatto quadrature formula.

Observe that the bilinear form $b(\cdot, \cdot)$ is not replaced by the corresponding discrete one since they coincide in the space $\mathbf{X}_N \times \mathbb{M}_N$ (this is a consequence of the exactness property (4.2)).

Remark 4.3. *The space of discretely divergence-free velocities*

$$\mathbf{V}_N := \{ \mathbf{v}_N \in \mathbf{X}_N : \forall q_N \in \mathbb{M}_N, b(\mathbf{v}_N, q_N) = 0 \}$$

coincides with $\mathbf{X}_N \cap \mathbf{V}$. Indeed, let \mathbf{v}_N be any polynomial in \mathbf{V}_N . Since $\text{div} \mathbf{v}_N$ belongs to $\mathbb{P}_{N-1}(\Omega)$, the identity $\mathbb{P}_{N-1}(\Omega) = \mathbb{M}_N \oplus Z_N$ implies that

$$\text{div} \mathbf{v}_N = q_N^1 + q_N^2,$$

where $q_N^1 \in \mathbb{M}_N$ and $q_N^2 \in Z_N$. According to the definition of \mathbf{V}_N and Z_N , we know that

$$b(\mathbf{v}_N, q_N^1) = b(\mathbf{v}_N, q_N^2) = 0.$$

Thus,

$$0 = b(\mathbf{v}_N, \text{div} \mathbf{v}_N) = \|\text{div} \mathbf{v}_N\|_{L^2(\Omega)}^2.$$

4.2. Discrete inf-sup condition

In order to get an analysis of (4.5), (4.6) and establish the priori error estimates, we need to check the discrete inf-sup condition. To do so, its proof is devoted to the demonstration of two rather technical results. We adopt the notation of the previous section. Using the inclusion $A^\perp \subset (A \oplus B)^\perp \oplus B$ valid for any subspaces A and B of a Hilbert space, we may write

$$\mathbb{M}_N \subset (\mathbb{M}_N^1 \oplus \mathbb{M}_N^2) \cap L_0^2(\Omega) \oplus \mathbb{R},$$

where \mathbb{M}_N^1 is the orthogonal of the space \tilde{Z}_N and of the polynomials

$$\begin{aligned} & L_{N-1}(x), L_{N-1}(y), L_{N-1}(z); \\ & L_{N-1}(x) L_{N-1}(y), L_{N-1}(x) L_{N-1}(z), L_{N-1}(y) L_{N-1}(z) \\ & \text{and } L_{N-1}(x) L_{N-1}(y) L_{N-1}(z). \end{aligned} \quad (4.7)$$

\mathbb{M}_N^2 is the space spanned by the polynomials in (4.7) and \mathbb{R} is the space of constant polynomials.

Lemma 4.4. For any $q_N^1 \in \mathbb{M}_N^1 \cap L_0^2(\Omega)$, there exists a function $\mathbf{v}_N^1 \in \mathbb{P}_{N-1}(\Omega) \cap H_0^1(\Omega)^3$ such that

$$b(\mathbf{v}_N^1, q_N^1) = \|q_N^1\|_{L^2(\Omega)}^2 \quad \text{and} \quad \|\mathbf{v}_N^1\|_{H^1(\Omega)} \leq cN \|q_N^1\|_{L^2(\Omega)}.$$

Furthermore, we have

$$\forall q_N^2 \in \mathbb{M}_N^2 \quad b(\mathbf{v}_N^1, q_N^2) = 0 \quad \text{and} \quad \forall q \in \mathbb{R}, \quad b(\mathbf{v}_N^1, q) = 0$$

Proof. This is a consequence of [11, Thms 24.7 and 24.9]. For the first orthogonality property, see [11, Rem. 24.1]. \square

Lemma 4.5. For any $q_N^2 \in \mathbb{M}_N^2 \cap L_0^2(\Omega)$, there exists a function $\mathbf{v}_N^2 \in \mathbb{X}_N \cap H_0^1(\Omega)^3$ such that

$$b(\mathbf{v}_N^2, q_N^2) = \|q_N^2\|_{L^2(\Omega)}^2 \quad \text{and} \quad \|\mathbf{v}_N^2\|_{H^1(\Omega)} \leq c \|q_N^2\|_{L^2(\Omega)}.$$

Furthermore, we have

$$\forall q \in \mathbb{R}, \quad b(\mathbf{v}_N^2, q) = 0$$

Proof. We use similar argument to that of [1, proof of Lem. a.4]. Let q_N^2 be any element of \mathbb{M}_N^2 . Then, if we denote by q_i^* , $i = 1, 7$, the seven polynomials in (5.1), we may write q_N^2 as a linear combination of the q_i^* 's. For simplicity, we set $q_N^2 = \sum_{i=1}^7 q_i^*$. Let us consider the polynomial

$$\mathbf{v}_1^* = \frac{1}{64} \left(\left(\frac{L_N - L_{N-2}}{2N-1} \right) (x)(L_2 - L_0)(y)(L_2 - L_0)(z), 0, 0 \right).$$

We have

$$b(\mathbf{v}_1^*, q_1^*) = b(\mathbf{v}_1^*, L_{N-1}(x)) = \|q_1^*\|_{L^2(\Omega)}^2.$$

Similarly, setting

$$\mathbf{v}_4^* = \frac{-1}{8} \left(\left(\frac{L_N - L_{N-2}}{2N-1} \right) (x)(L_{N-1} - L_{N-3})(y)(L_2 - L_0)(z), 0, 0 \right),$$

we obtain

$$b(\mathbf{v}_4^*, q_4^*) = b(\mathbf{v}_4^*, L_{N-1}(x)L_{N-1}(y)) = \|q_4^*\|_{L^2(\Omega)}^2.$$

Finally, let

$$\mathbf{v}_7^* = \left(\left(\frac{L_N - L_{N-2}}{2N-1} \right) (x)(L_{N-1} - L_{N-3})(y)(L_{N-1} - L_{N-3})(z), 0, 0 \right).$$

Again, we have $b(\mathbf{v}_7^*, q_7^*) = \|q_7^*\|_{L^2(\Omega)}^2$. We can repeat the same arguments to control the other q_i^* 's. Setting

$\mathbf{v}_N^2 = \sum_{i=1}^7 \mathbf{v}_i^*$ and using orthogonality properties, lead to $b(\mathbf{v}_N^2, q_N^2) = \|q_N^2\|_{L^2(\Omega)}^2$. The estimate $\|\mathbf{v}_N^2\|_{H^1(\Omega)} \leq c \|q_N^2\|_{L^2(\Omega)}$ can be easily checked. \square

Lemma 4.6. There exists a constant $c > 0$ independent of N such that for any $\bar{q} \in \mathbb{R}$, there exists $\bar{\mathbf{v}} \in \mathbf{X}_N$ that satisfies

$$b(\bar{\mathbf{v}}, \bar{q}) \geq c \|\bar{q}\|_{L^2(\Omega)}^2 \quad \text{and} \quad \|\bar{\mathbf{v}}\|_{\mathbf{X}} \leq c_2 \|\bar{q}\|_{L^2(\Omega)}.$$

Proof. Consider the function $\varphi \in \mathbb{P}_2(\Omega)$ defined by $\varphi(x, y, z) = (1 - x^2)(1 - z^2)$. Clearly, this function vanishes on Γ_D and satisfies

$$\int_{\Gamma_m} \varphi(\sigma) d\sigma > 0.$$

According to [12, Thm. III.3.1], there exists a polynomial $\mathbf{v}_N \in \mathbf{X}_N$ such that

$$\mathbf{v}_N|_{\Gamma_m} = -q \varphi \mathbf{n}.$$

To conclude we apply the divergence theorem and we obtain

$$b(\mathbf{v}_N, \bar{q}) = \bar{q}^2 \int_{\Gamma_m} \varphi(\sigma) d\sigma \geq c \|\bar{q}\|_{L^2(\Omega)}^2.$$

□

Now, we conclude using Boland and Nicolaides argument [14].

Lemma 4.7. *There exists a constant $\beta > 0$ independent of N such that the following inf-sup condition holds*

$$\forall q_N \in \mathbb{M}_N, \quad \sup_{\mathbf{v}_N \in \mathbf{X}_N} \frac{b_N(\mathbf{v}_N, q_N)}{\|\mathbf{v}_N\|_{\mathbf{X}}} \geq \beta N^{-1} \|q_N\|_{L^2(\Omega)}. \quad (4.8)$$

Proof. Let q_N be any element of \mathbb{M}_N . We already know that $q_N = q_N^1 + q_N^2 + q_N^3$, where $q_N^1 \in \mathbb{M}_N^1$, $q_N^2 \in \mathbb{M}_N^2$ and $q_N^3 \in \mathbb{R}$. Let $\lambda, \mu > 0$ be two real numbers and consider the polynomial $\mathbf{v}_N = \mathbf{v}_N^1 + \lambda \mathbf{v}_N^2 + \mu \mathbf{v}_N^3$, where \mathbf{v}_N^i is the associated polynomials to q_N^i by one of the above lemmas. We have

$$\begin{aligned} b(\mathbf{v}_N, q_N) &= \|q_N^1\|_{L^2(\Omega)}^2 + \lambda \|q_N^2\|_{L^2(\Omega)}^2 + \mu \|q_N^3\|_{L^2(\Omega)}^2 + \lambda b(\mathbf{v}_N^2, q_N^1) + \mu b(\mathbf{v}_N^3, q_N^1) + \mu b(\mathbf{v}_N^3, q_N^2) \\ &\geq \|q_N^1\|_{L^2(\Omega)}^2 + \lambda \|q_N^2\|_{L^2(\Omega)}^2 + \mu \|q_N^3\|_{L^2(\Omega)}^2 - \lambda c_2 \|q_N^2\|_{L^2(\Omega)} \|q_N^1\|_{L^2(\Omega)} \\ &\quad - \mu c_3 \|q_N^3\|_{L^2(\Omega)} \|q_N^1\|_{L^2(\Omega)} - \mu c_3 \|q_N^3\|_{L^2(\Omega)} \|q_N^2\|_{L^2(\Omega)} \end{aligned}$$

Using Young's inequality, we arrive at

$$b(\mathbf{v}_N, q_N) \geq \frac{1}{2} \|q_N^1\|_{L^2(\Omega)}^2 + (\lambda - 2\lambda^2 c_2^2) \|q_N^2\|_{L^2(\Omega)}^2 + \left(\mu - \mu^2 c_3^2 - \frac{1}{4} \left(\frac{\mu c_3}{\lambda c_2} \right)^2 \right).$$

Thus, if we take $\lambda = (4c_2^2)^{-1}$ and $\mu = (2c_3^2(1 + 4c_2^2))^{-1}$, we obtain

$$b(\mathbf{v}_N, q_N) \geq c \left(\|q_N^1\|_{L^2(\Omega)}^2 + \|q_N^2\|_{L^2(\Omega)}^2 + \|q_N^3\|_{L^2(\Omega)}^2 \right).$$

Taking into account the orthogonality of the q_N^i 's, we deduce that

$$b(\mathbf{v}_N, q_N) \geq c \|q_N\|_{L^2(\Omega)}^2 \quad \text{and} \quad \|\mathbf{v}_N\|_{\mathbf{X}} \leq cN \|q_N\|_{L^2(\Omega)}.$$

□

5. Stability and a priori error estimates

Repeating exactly the same techniques of the semi-discrete case and using the equivalence property (4.2), the following result is straightforward (the boundary term is treated as in Remark 2.5)

Theorem 5.1. *Assume that the data (\mathbf{f}, p_m) belongs to $\mathcal{C}^0(0, T; L^2(\Omega)^3) \times \mathcal{C}^0(0, T; H^\mu(\Gamma_m))$, $\mu > 1$, and the initial velocity \mathbf{u}_0 belongs to $H(\mathbf{curl}, \Omega)$. Then, problem (4.5)-(4.6) has a unique solution $(\mathbf{u}^k, p^k)_{1 \leq k \leq K}$. Moreover, the following estimates hold*

$$\begin{aligned} \|\mathbf{u}_N^k\|_{L^2(\Omega)}^2 + 2\nu \sum_{i=1}^k \tau_i \|\mathbf{curl} \mathbf{u}_N^i\|_{L^2(\Omega)}^2 &\leq c \left(\|\mathcal{I}_N \mathbf{u}_0\|_{L^2(\Omega)}^2 + \sum_{i=1}^k \tau_i \left(\|\mathcal{I}_N \mathbf{f}^i\|_{L^2(\Omega)}^2 + \|\mathcal{I}_N p_m^i\|_{L^2(\Gamma_m)}^2 \right) \right) \\ \sum_{i=1}^k \tau_i \left\| \frac{\mathbf{u}_N^i - \mathbf{u}_N^{i-1}}{\tau_i} \right\|_{L^2(\Omega)}^2 &\leq c \left(\|\mathbf{curl} \mathcal{I}_N \mathbf{u}_0\|_{L^2(\Omega)}^2 + \sum_{i=1}^k \tau_i \left(\|\mathcal{I}_N \mathbf{f}^i\|_{L^2(\Omega)}^2 + \|\mathcal{I}_N p_m^i\|_{L^2(\Gamma_m)}^2 \right) \right). \end{aligned}$$

Remark 5.2. *The space $H^\mu(\Gamma_m)$ in Theorem 5.1 denotes the space of functions which belong to $H^\mu(\gamma)$ for each face γ contained in Γ_m .*

To derive error estimates for the discrete scheme (4.5)-(4.6), we need two preliminary lemmas. The first one gives a required characterization of some interpolation spaces (for the definition of these spaces see [23, Chap. 1]).

Lemma 5.3. *The following interpolation result holds for any real number θ , $0 < \theta < 1$,*

$$[H^3(\Omega)^3 \cap \mathbf{V}, \mathbf{V}]_\theta = H^{3(1-\theta)+\theta}(\Omega)^3 \cap \mathbf{V}.$$

Proof. In view of [23, Thm. I.14.3], it suffices to construct a continuous operator \mathcal{R} defined from $H^s(\Omega)$ into $H^{s+1}(\Omega)^3 \cap \mathbf{X}$, $0 \leq s \leq 2$, such that

$$\forall g \in H^s(\Omega), \quad \operatorname{div}(\mathcal{R}g) = g.$$

Let \mathcal{O} be an open ball containing $\bar{\Omega}$. As in [10, Thm. 3.5], we consider an extension operator \mathcal{E} and modify it so that for any $g \in H^s(\Omega)$, we have $\int_{\mathcal{O}} \mathcal{E}g \, d\mathbf{x} = 0$. Now, we solve the following Stokes problem

$$\begin{aligned} -\nu \Delta \mathbf{v} + \nabla q &= \mathbf{0} & \text{in } \mathcal{O}, \\ \operatorname{div} \mathbf{v} &= \mathcal{E}g & \text{in } \mathcal{O}, \\ \mathbf{v} &= \mathbf{0} & \text{in } \partial\mathcal{O}. \end{aligned}$$

This problem has a unique solution $(\mathbf{v}, q) \in H_0^1(\mathcal{O})^3 \times L_0^2(\mathcal{O})$. In addition, the mapping $g \mapsto \mathbf{v}|_{\Omega}$ is continuous from $H^s(\Omega)$ into $H^{s+1}(\Omega)^3$. On the other hand, a divergence-free lifting $\tilde{\mathbf{v}} \in H^{s+1}(\Omega)^3$ for the trace of $\mathbf{v}|_{\Omega}$ on the boundary $\partial\Omega$ is constructed in [10, Thm. 4.11] and satisfies

$$\|\tilde{\mathbf{v}}\|_{H^{s+1}(\Omega)} \leq c \|\mathbf{v}|_{\Omega}\|_{H^{s+1}(\Omega)}.$$

Thus it suffices to take $\mathcal{R}g = \mathbf{v}|_{\Omega} - \tilde{\mathbf{v}}$ to conclude. \square

Lemma 5.4. *Let s be any real number ≥ 1 . The following approximation estimate holds for any function \mathbf{v} in $H^s(\Omega)^3 \cap \mathbf{V}$:*

$$\inf_{\mathbf{v}_N \in \mathbb{P}_N(\Omega)^3 \cap \mathbf{V}} \|\mathbf{v} - \mathbf{v}_N\|_{\mathbf{X}} \leq cN^{1-s} \|\mathbf{v}\|_{H^s(\Omega)}. \quad (5.1)$$

Proof. We know from [26, Thm. 2.3] that the following result holds for any divergence-free function \mathbf{v} in $H^s(\Omega) \cap H_0^1(\Omega)^3$ and $s \geq 3$:

$$\inf_{\mathbf{v}_N \in \mathbb{P}_N(\Omega)^3 \cap \mathbf{V}} \|\mathbf{v} - \mathbf{v}_N\|_{H^1(\Omega)} \leq cN^{1-s} \|\mathbf{v}\|_{H^s(\Omega)}.$$

However, it can be easily checked that this last estimate holds also for functions satisfying the boundary conditions of \mathbf{X} (see [26, Rem. 2.2]). Furthermore, since Ω is convex, the norms $\|\cdot\|_{H^1(\Omega)}$ and $\|\cdot\|_{\mathbf{X}}$ are equivalent in the space \mathbf{V} (see [2, Thm. 2.17]). Thus, estimate (5.1) holds for $s \geq 3$ and obviously, it is also true for $s = 1$. The principal interpolation theorem [23, Thm I.5.1] and Lemma 5.3 give the desired result. \square

Proposition 5.5. *Assume that the initial velocity \mathbf{u}_0 belongs to \mathbf{V} . Moreover, if*

$$\mathbf{u} \in H^2(0, T; H^s(\Omega)^3), \quad \mathbf{u}_0 \in H^s(\Omega)^3, \quad \text{for a real number } s \geq 1,$$

and

$$\mathbf{f} \in \mathcal{C}^0(0, T; H^\sigma(\Omega)^3), \quad p_m \in \mathcal{C}^0(0, T; H^\mu(\Gamma_m)) \quad \text{for } \sigma > \frac{3}{2} \quad \text{and } \mu > 1,$$

then, for every $k = 1, \dots, K$, the following a priori estimate holds

$$\begin{aligned}
& \left(\|\mathbf{u}(\cdot, t_k) - \mathbf{u}_N^k\|_{L^2(\Omega)}^2 + \nu \sum_{i=1}^k \tau_i \|\mathbf{curl} \mathbf{u}(\cdot, t_i) - \mathbf{curl} \mathbf{u}_N^i\|_{L^2(\Omega)}^2 \right)^{1/2} \\
& \leq C \left(N^{1-s} \left(\|\mathbf{u}_0\|_{H^s(\Omega)}^2 + \|\mathbf{u}(\cdot, t_k)\|_{H^s(\Omega)}^2 + \sum_{i=1}^k \tau_i \|\mathbf{u}(\cdot, t_i)\|_{H^s(\Omega)}^2 + \sum_{i=1}^k \tau_i \left\| \frac{\partial \mathbf{u}}{\partial t}(\cdot, t_k) \right\|_{H^s(\Omega)}^2 \right)^{1/2} \right. \\
& \quad \left. + \tau \left\| \frac{\partial^2 \mathbf{u}}{\partial t^2} \right\|_{L^2(0, t_k; H^s(\Omega)^3)} + N^{-\sigma} \left(\sum_{i=1}^k \tau_i \|\mathbf{f}^i\|_{H^\sigma(\Omega)}^2 \right)^{1/2} + N^{-\mu} \left(\sum_{i=1}^k \tau_i \|p_m^i\|_{H^\mu(\Gamma_m)} \right)^{1/2} \right),
\end{aligned} \tag{5.2}$$

The error estimate corresponding to the acceleration term is given as follows

$$\begin{aligned}
& \left(\sum_{i=1}^k \tau_i \left\| \frac{(\mathbf{u}(\cdot, t_i) - \mathbf{u}_N^i) - (\mathbf{u}(\cdot, t_{i-1}) - \mathbf{u}_N^{i-1})}{\tau_i} \right\|_{L^2(\Omega)}^2 \right)^{1/2} \\
& \leq C \left(N^{1-s} \left(\|\mathbf{u}_0\|_{H^s(\Omega)}^2 + \sum_{i=1}^k \tau_i \|\mathbf{u}(\cdot, t_i)\|_{H^s(\Omega)}^2 + \sum_{i=1}^k \tau_i \left\| \frac{\partial \mathbf{u}}{\partial t}(\cdot, t_k) \right\|_{H^s(\Omega)}^2 \right)^{1/2} \right. \\
& \quad \left. \tau \left\| \frac{\partial^2 \mathbf{u}}{\partial t^2} \right\|_{L^2(0, t_k; H^s(\Omega)^3)} + N^{-\sigma} \left(\sum_{i=1}^k \tau_i \|\mathbf{f}^i\|_{H^\sigma(\Omega)}^2 \right)^{1/2} + N^{-\mu} \left(\sum_{i=1}^k \tau_i \|p_m^i\|_{H^\mu(\Gamma_m)} \right)^{1/2} \right).
\end{aligned} \tag{5.3}$$

Proof. Let us consider the function $\tilde{\mathbf{u}}_N(\cdot, t) = \pi_{\mathbf{V}_{N-1}} \mathbf{u}(\cdot, t)$ the projection with respect to the bilinear form $a(\cdot, \cdot)$, i.e.

$$\forall \mathbf{v}_N \in \mathbf{V}_{N-1}, \quad \forall t \in [0, T], \quad a(\mathbf{u}(\cdot, t) - \tilde{\mathbf{u}}_N(\cdot, t), \mathbf{v}_N) = 0.$$

It can be easily checked that this function satisfies for all $\mathbf{v}_N \in \mathbf{V}_{N-1}$

$$\begin{aligned}
\left(\frac{\partial \tilde{\mathbf{u}}_N}{\partial t}(\cdot, t_k), \mathbf{v}_N \right) + \nu a(\tilde{\mathbf{u}}_N(\cdot, t_k), \mathbf{v}_N) &= \left(\frac{\partial}{\partial t}(\tilde{\mathbf{u}}_N - \mathbf{u})(\cdot, t_k), \mathbf{v}_N \right) + \nu a((\tilde{\mathbf{u}}_N - \mathbf{u})(\cdot, t_k), \mathbf{v}_N) \\
&\quad + (\mathbf{f}^k, \mathbf{v}_N) - (p_m^k, \mathbf{v}_N \cdot \mathbf{n})_{\Gamma_m}.
\end{aligned}$$

This last equation is also equivalent to, for all $\mathbf{v}_N \in \mathbf{V}_{N-1}$

$$\begin{aligned}
\left(\frac{\tilde{\mathbf{u}}_N(\cdot, t_k) - \tilde{\mathbf{u}}_N(\cdot, t_{k-1})}{\tau_k}, \mathbf{v}_N \right) + \nu a_N(\tilde{\mathbf{u}}_N(\cdot, t_k), \mathbf{v}_N) \\
= \sum_{i=1}^3 \mathbf{E}_i^k(\mathbf{v}_N) + (\mathbf{f}^k, \mathbf{v}_N) - (p_m^k, \mathbf{v}_N \cdot \mathbf{n})_{\Gamma_m},
\end{aligned} \tag{5.4}$$

where we have set

$$\begin{aligned}
\mathbf{E}_1^k(\mathbf{v}_N) &= \left(\frac{\tilde{\mathbf{u}}_N(\cdot, t_k) - \tilde{\mathbf{u}}_N(\cdot, t_{k-1})}{\tau_k}, \mathbf{v}_N \right) - \left(\frac{\partial \tilde{\mathbf{u}}_N}{\partial t}(\cdot, t_k), \mathbf{v}_N \right), \\
\mathbf{E}_2^k(\mathbf{v}_N) &= \left(\frac{\partial \tilde{\mathbf{u}}_N}{\partial t}(\cdot, t_k) - \frac{\partial \mathbf{u}}{\partial t}(\cdot, t_k), \mathbf{v}_N \right).
\end{aligned}$$

Now, setting $\mathbf{e}_N^k = \tilde{\mathbf{u}}_N(\cdot, t_k) - \mathbf{u}_N^k$ and subtracting equation (4.5) from (5.4), we get the following error equation

$$\forall \mathbf{v}_N \in \mathbf{V}_{N-1}, \quad \left(\frac{\mathbf{e}_N^k - \mathbf{e}_N^{k-1}}{\tau_k}, \mathbf{v}_N \right) + \nu a_N(\mathbf{e}_N^k, \mathbf{v}_N) = \sum_{i=1}^5 \mathbf{E}_i^k(\mathbf{v}_N), \tag{5.5}$$

where

$$\mathbf{E}_4^k(\mathbf{v}_N) = (\mathbf{f}^k, \mathbf{v}_N) - (\mathbf{f}^k, \mathbf{v}_N)_N \quad \text{and} \quad \mathbf{E}_5^k(\mathbf{v}_N) = (p_m^k, \mathbf{v}_N \cdot \mathbf{n})_{N, \Gamma_m} - (p_m^k, \mathbf{v}_N \cdot \mathbf{n})_{\Gamma_m}.$$

Using Taylor's formula with the integral form of the remainder, the first consistency error $\mathbf{E}_1^k(\mathbf{v}_N)$ can be bounded as follows

$$\mathbf{E}_1^k(\mathbf{v}_N) \leq \frac{1}{\sqrt{3}} \tau^{1/2} \left\| \frac{\partial^2 \mathbf{u}}{\partial t^2} \right\|_{L^2(t_k, t_{k-1}; L^2(\Omega)^3)} \|\mathbf{v}_N\|_{L^2(\Omega)}.$$

In order to derive similar bounds for $\mathbf{E}_i^k(\mathbf{v}_N)$, $i = 2, \dots, 5$, we need standard approximation results [12, Chap. III and IV] and Lemma 5.4. For instance, we have

$$\mathbf{E}_2^k(\mathbf{v}_N) \leq CN^{1-s} \left\| \frac{\partial \mathbf{u}}{\partial t}(\cdot, t_k) \right\|_{H^s(\Omega)} \|\mathbf{v}_N\|_{L^2(\Omega)}.$$

Next, taking $\mathbf{v}_N = \mathbf{e}_N^k$ in (5.5) and proceeding in a similar manner to the stability proof, leads to

$$\begin{aligned} \|\mathbf{e}_N^k\|_{L^2(\Omega)}^2 + \nu \sum_{i=1}^k \tau_i \|\mathbf{curl} \mathbf{e}_N^i\|_{L^2(\Omega)}^2 &\leq C \left(N^{2(1-s)} \left(\|\mathbf{u}_0\|_{H^s(\Omega)}^2 + \sum_{i=1}^k \tau_i \left\| \frac{\partial \mathbf{u}}{\partial t}(\cdot, t_k) \right\|_{H^s(\Omega)}^2 \right) \right. \\ &\quad \left. + \tau^2 \left\| \frac{\partial^2 \mathbf{u}}{\partial t^2} \right\|_{L^2(0, T; H^s(\Omega))}^2 + N^{-2\sigma} \left(\sum_{i=1}^k \tau_i \|\mathbf{f}^i\|_{H^\sigma(\Omega)}^2 \right) + N^{-2\mu} \left(\sum_{i=1}^k \tau_i \|p_m^i\|_{H^\mu(\Gamma_m)}^2 \right) \right). \end{aligned}$$

Thus, estimates (5.2) follows by combining estimate (5.1), the identity

$$\mathbf{u}(\cdot, t_k) - \mathbf{u}_N^k = \mathbf{u}(\cdot, t_k) - \tilde{\mathbf{u}}_N(\cdot, t_k) + \tilde{\mathbf{u}}_N(\cdot, t_k) - \mathbf{u}_N^k,$$

and the triangle inequality.

Now we prove the error estimate (5.3). Setting $\mathbf{v}_N = \mathbf{e}_N^k - \mathbf{e}_N^{k-1}$ in (5.5), using Young's inequality and the standard approximation results mentioned above, we obtain

$$\begin{aligned} \|\mathbf{e}_N^k - \mathbf{e}_N^{k-1}\|_N^2 + \frac{\nu \tau_k}{2} \left(\|\mathbf{curl} \mathbf{e}_N^k\|_N^2 + \|\mathbf{curl}(\mathbf{e}_N^k - \mathbf{e}_N^{k-1})\|_N^2 - \|\mathbf{curl} \mathbf{e}_N^{k-1}\|_N^2 \right) \\ \leq C \tau_k^2 \left(N^{2(1-s)} \left(\|\mathbf{u}(\cdot, t_k)\|_{H^s(\Omega)}^2 + \left\| \frac{\partial \mathbf{u}}{\partial t}(\cdot, t_k) \right\|_{H^s(\Omega)}^2 \right) + \tau \left\| \frac{\partial^2 \mathbf{u}}{\partial t^2} \right\|_{L^2(t_{k-1}, t_k; L^2(\Omega)^3)}^2 \right. \\ \left. + N^{-2\sigma} \sum_{i=1}^k \tau_i \|\mathbf{f}^i\|_{H^\sigma(\Omega)}^2 + N^{-2\mu} \sum_{i=1}^k \tau_i \|p_m^i\|_{H^\mu(\Gamma_m)}^2 \right) + \frac{1}{2} \|\mathbf{e}_N^k - \mathbf{e}_N^{k-1}\|_N^2. \end{aligned}$$

Dividing both sides by τ_k , summing over k and using a triangle inequality, give the desired estimate. \square

Proposition 5.6. *Under assumptions of Proposition 5.5, suppose moreover that the exact pressure, solution of (2.4)-(2.5) belongs to $H^1(0, T; H^r(\Omega))$, for $r \geq 1$. Then, the following approximation estimate holds*

$$\begin{aligned} &\left(\sum_{i=1}^k \tau_i \|p(\cdot, t_i) - p_N^i\|_{L^2(\Omega)}^2 \right)^{1/2} \\ &\leq CN \left(N^{1-s} \left(\|\mathbf{u}_0\|_{H^s(\Omega)}^2 + \sum_{i=1}^k \tau_i \|\mathbf{u}\|_{H^s(\Omega)}^2 + \sum_{i=1}^k \tau_i \left\| \frac{\partial \mathbf{u}}{\partial t}(\cdot, t_k) \right\|_{H^s(\Omega)}^2 \right) \right)^{1/2} \\ &\quad + \tau \left\| \frac{\partial^2 \mathbf{u}}{\partial t^2} \right\|_{L^2(0, t_k; H^s(\Omega)^3)} + N^{-r} \left(\sum_{i=1}^k \tau_i \|p(\cdot, t_i)\|_{H^r(\Omega)}^2 \right)^{1/2} \\ &\quad + N^{-\sigma} \left(\sum_{i=1}^k \tau_i \|\mathbf{f}^i\|_{H^\sigma(\Omega)}^2 \right)^{1/2} + N^{-\mu} \left(\sum_{i=1}^k \tau_i \|p_m^i\|_{H^\mu(\Gamma_m)}^2 \right)^{1/2}. \end{aligned} \tag{5.6}$$

Proof. For almost every $t \in [0, T]$, we consider the function $\tilde{p}_N(\cdot, t) = \pi_{[\lambda N]} p(\cdot, t)$, where $\pi_{[\lambda N]}$ is the projection operator defined from $L^2(\Omega)$ onto $\mathbb{P}_{[\lambda N]} N$. For every $k = 1, \dots, K$, we have

$$b(\mathbf{v}_N, \tilde{p}_N(\cdot, t_k) - p_N^k) = b(\mathbf{v}_N, \tilde{p}_N(\cdot, t_k) - p(\cdot, t_k)) + b(\mathbf{v}_N, p(\cdot, t_k)) - b(\mathbf{v}_N, p_N^k).$$

If we replace $b(\mathbf{v}_N, p_N^k)$ and $b(\mathbf{v}_N, p(\cdot, t_k))$ by their equivalents in (4.5) and (2.4) respectively, we find the following error equation

$$\begin{aligned} b(\mathbf{v}_N, \tilde{p}_N(\cdot, t_k) - p_N^k) &= b(\mathbf{v}_N, \tilde{p}_N(\cdot, t_k) - p(\cdot, t_k)) - \left(\frac{\partial \mathbf{u}}{\partial t}(\cdot, t_k), \mathbf{v}_N \right) + \left(\frac{\mathbf{u}_N^k - \mathbf{u}_N^{k-1}}{\tau_k}, \mathbf{v}_N \right)_N \\ &\quad - \nu a(\mathbf{u}(\cdot, t_k), \mathbf{v}_N) + \nu a_N(\mathbf{u}_N^k, \mathbf{v}_N^k) + (\mathbf{f}^k, \mathbf{v}_N) - (\mathbf{f}^k, \mathbf{v}_N)_N - (p_m^k, \mathbf{v}_N \cdot \mathbf{n})_{\Gamma_m} + (p_m^k, \mathbf{v}_N \cdot \mathbf{n})_{N, \Gamma_m}. \end{aligned}$$

Hence, using the inf-sup condition (4.8) and taking into account the error estimates already proved for the velocity and of standard approximation results, we derive the desired estimate. \square

Before we enunciate the conclusive theorem, we introduce the functions \mathbf{u}_{N_τ} and p_{N_τ} defined on $[0, T]$, affines on each subinterval $[t_{k-1}, t_k]$, $k = 1, \dots, K$, and such that

$$\mathbf{u}_{N_\tau}(t_k) = \mathbf{u}_N^k \quad \text{and} \quad p_{N_\tau}(t_k) = p_N^k.$$

Theorem 5.7. *Assume that the initial velocity \mathbf{u}_0 belongs to \mathbf{V} . Moreover if*

$$\mathbf{u} \in H^2(0, T; H^s(\Omega)^3), \quad p \in H^1(0, T; H^r(\Omega)) \quad \mathbf{u}_0 \in H^s(\Omega)^3, \quad \text{for } s, r \geq 1,$$

and

$$\mathbf{f} \in C^0(0, T; H^\sigma(\Omega)^3), \quad p_m \in C^0(0, T; H^\mu(\Omega)), \quad \text{for } \sigma > \frac{3}{2}, \quad \mu > 1.$$

Then, for every $k = 1, \dots, K$, the following a priori estimates hold

$$\begin{aligned} &\|\mathbf{u} - \mathbf{u}_{N_\tau}\|_{L^2(0, T; \mathbf{X})} + \|\mathbf{u} - \mathbf{u}_{N_\tau}\|_{C(0, T; L^2(\Omega))} \\ &\leq C \left((N^{1-s} + \tau) \|\mathbf{u}\|_{H^2(0, T; H^s(\Omega)^3)} + N^{-\sigma} \|\mathbf{f}\|_{C^0(0, T; H^\sigma(\Omega)^3)} + N^{-\mu} \|p_m\|_{C^0(0, T; H^\mu(\Gamma_m))} \right) \end{aligned}$$

and

$$\begin{aligned} \|p - p_{N_\tau}\|_{L^2(0, T; L^2(\Omega))} &\leq C \left((N^{1-s} + \tau) \|\mathbf{u}\|_{H^2(0, T; H^s(\Omega)^3)} + (N^{1-r} + \tau) \|p\|_{H^1(0, T; H^r(\Omega))} \right. \\ &\quad \left. + N^{-\sigma} \|\mathbf{f}\|_{C^0(0, T; H^\sigma(\Omega)^3)} + N^{-\mu} \|p_m\|_{C^0(0, T; H^\mu(\Gamma_m))} \right). \end{aligned}$$

6. Numerical experiments

In this section we present numerical results to examine the performance of the fully discrete scheme analyzed in this manuscript. We first assess the accuracy of the scheme for two test examples with known manufactured solutions for a Stokes problem with physical boundary conditions. The first illustrate the temporal accuracy test and the second concerns the spatial accuracy. For simplicity, we only consider the numerical experiments when the same degree of the polynomial basis are equal in each directions. At each time step, the obtained linear systems of algebraic equations are solved using a preconditioned Generalized Minimal Residual (GMRES) iterative solver [25]. Finally, all the computations are performed using FreeFEM-3D software [19, 28].

6.1. Time accuracy test

In order to show the convergence rate with respect to time, we construct the three-dimensional problem where the exact solution is given in the unit cube $]0,1[^3$ by

$$\begin{aligned} u_1^e(x, y, z) &= (x^2 - y^4 + z^6) \sin(t) - \log(1 + t^2) y^3 z^2 \\ u_2^e(x, y, z) &= \sin(t) (x^3 z^2 + y^5) + x^3 z^2 \sqrt{1 + t^2} - t \\ u_3^e(x, y, z) &= -\sin(t) (2xz + 5y^4 z - x^3 y^3) + (x^2 + y^2) \exp(-t^2) + t^3 \\ p^e(x, y, z) &= (x^3 + y^2 + z) (t^2 + \sin(\pi t)). \end{aligned}$$

Hence, the suitable forcing functions \mathbf{f} and the boundary conditions are obtained using this exact solution and $\nu = 1$. The boundary Γ_m coincides with the upper face $z = 1$ and $\Gamma_d = \partial\Omega/\Gamma_m$. We use the degree $N = 7$ in each direction to insure negligible spatial error for the chosen range of time step. The velocity and pressure errors to the exact solution will be computed at the final time $T = 1$.

We recover a convergence order for velocity and pressure that decreases as the time step tends to 0, with convergence rate equal to 1, which confirms the theoretical estimate obtained in Theorem 5.7, see Figure 3.

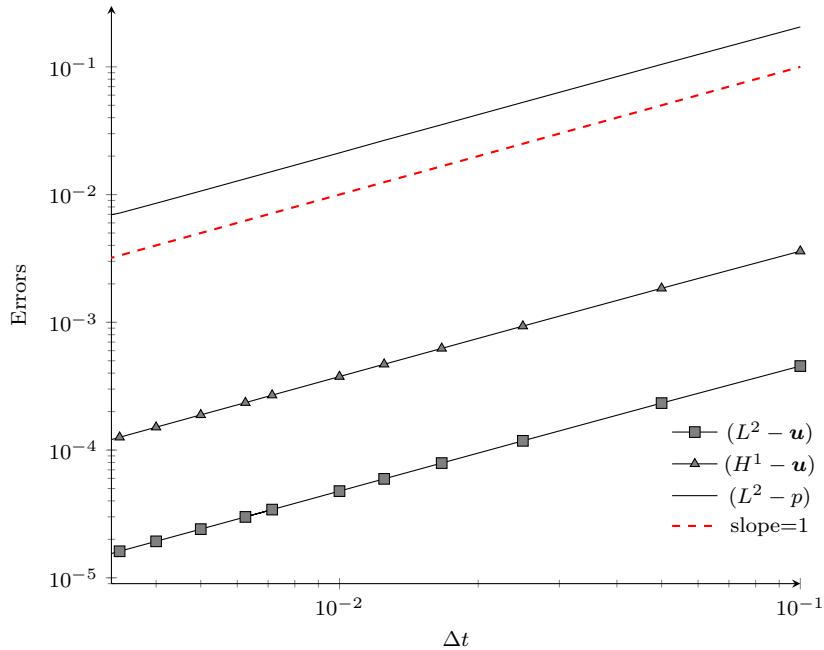


Figure 1: Convergence rate with respect to the time step Δt for velocity in L^2 and H^1 norms and L^2 norm for the pressure.

Figure 2 shows (the slices of) the exact and the computed (velocity, pressure), using $N = 7$ and $\Delta t = \frac{1}{320}$. It can be noted that the top and the bottom levels of this figure have little differences.

6.2. Spatial accuracy test

The aim of the second test is to verify the space accuracy. The computational domain is the square $]0,1[^2$ and the exact solution of the Stokes problem is given as follows by

$$\begin{aligned} u_1^e(x, y) &= \sin(\pi x) \sin(\pi y) t \\ u_2^e(x, y) &= \cos(\pi x) \cos(\pi y) t \\ p^e(x, y) &= \sin(\pi(x - y)) t. \end{aligned}$$

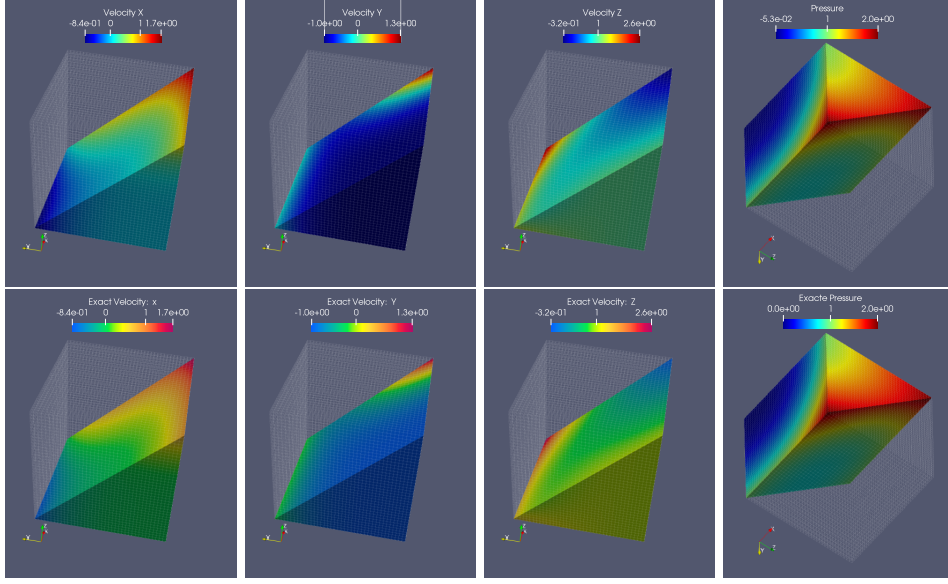


Figure 2: Top: Discrete solution using $(N_1, N_2, N_3) = (7, 7, 7)$ and $\Delta t = 1/320$. From the left to the right: slice of velocity components u_1, u_2, u_3 and pressure p . Bottom: Exact solution: slice of velocity components (u_1^e, u_2^e, u_3^e) and pressure p^e .

We fix the time step Δt equal to 0.1, and we plot the L^2 and H^1 norms error of the velocity and L^2 -norm error of the pressure between the numerical solution and the exact solution at the final time $T = 1$ with a successive polynomial degree from $N = 4$ to $N = 18$. It is clear from Figure 3 that the convergence slopes of both velocity and pressure in all norms coincide with the slope of e^{-x} . Hence the spectral convergence for all unknowns is obtained which is consistent with the error estimate obtained in the present work.

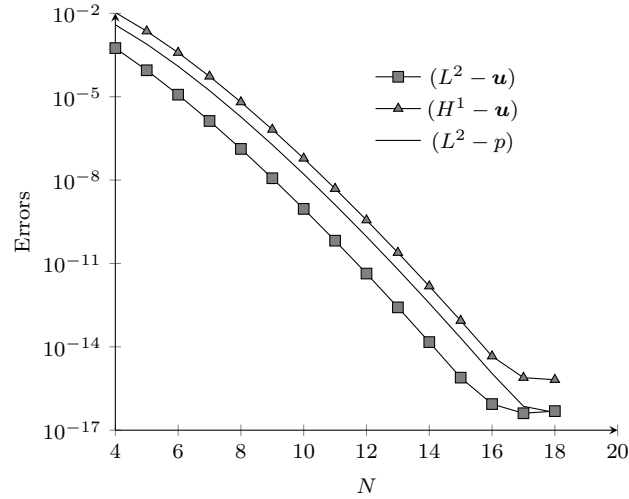


Figure 3: Convergence rate with respect to the polynomials degree N for velocity in L^2 and H^1 norms and L^2 norm for the pressure.

7. Conclusion

In this paper we have dealt with the time-dependent Stokes equations with unusual boundary conditions. We have seen that the associated variational space have different behavior from that of classical ones and this often requires some restrictions on the boundary. In addition, we have proposed a spectral discretization for this problem. Although the numerical scheme gives pointwise divergence-free discrete velocities which is always attractive, the complexity of the pressures space renders this method impractical from a numerical point of view.

There are at least one immediate direction for future work. This is the purpose of the second part of this manuscript. In fact, we will propose another discretization based on a penalty method by "relaxing" the tangential boundary conditions verified by the discrete velocities. This will eliminate all the spurious modes and we will show that the inf-sup constant scales like $\mathcal{O}(N^{-1})$. The main objective of this second method is to give a spectral approximation with the same properties as the first one, i.e. exactly divergence-free discrete velocities and at least with the same rate of convergence, but with a much simpler discrete pressures space.

References

- [1] K. Amoura, C. Bernardi, N. Chorfi, and S. Saadi. Spectral discretization of the stokes problem with mixed boundary conditions. *Progress in Computational Physics*, 2:42–61, 2012.
- [2] C. Amrouche, C. Bernardi, M. Dauge, and V. Girault. Vector potentials in three-dimensional non-smooth domains. *Math. Methods Appl. Sci.*, 21(9):823–864, 1998.
- [3] C. Amrouche and N. Seloula. L^p -theory for vector potentials and Sobolev's inequalities for vector fields: application to the Stokes equations with pressure boundary conditions. *Math. Models Methods Appl. Sci.*, 23(1):37–92, 2013.
- [4] C. Bègue, C. Conca, F. Murat, and O. Pironneau. Les équations de Stokes et de Navier-Stokes avec des conditions aux limites sur la pression. In *Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. IX (Paris, 1985–1986)*, volume 181 of *Pitman Res. Notes Math. Ser.*, pages 179–264. Longman Sci. Tech., Harlow, 1988.
- [5] J. M. Bernard. Non-standard Stokes and Navier-Stokes problems: existence and regularity in stationary case. *Math. Methods Appl. Sci.*, 25(8):627–661, 2002.
- [6] J. M. Bernard. Time-dependent Stokes and Navier-Stokes problems with boundary conditions involving pressure, existence and regularity. *Nonlinear Anal. Real World Appl.*, 4(5):805–839, 2003.
- [7] J. M. Bernard. Spectral discretizations of the Stokes equations with non standard boundary conditions. *J. Sci. Comput.*, 20(3):355–377, 2004.
- [8] C. Bernardi, C. Canuto, and Y. Maday. Spectral approximations of the Stokes equations with boundary conditions on the pressure. *SIAM J. Numer. Anal.*, 28(2):333–362, 1991.
- [9] C. Bernardi, T. Chacón Rebollo, and D. Yakoubi. Finite element discretization of the Stokes and Navier-Stokes equations with boundary conditions on the pressure. *SIAM J. Numer. Anal.*, 53(3):1256–1279, 2015.
- [10] C. Bernardi, M. Dauge, and Y. Maday. Interpolation of nullspaces for polynomial approximation of divergence-free functions in a cube. In *Boundary value problems and integral equations in nonsmooth domains (Luminy, 1993)*, volume 167 of *Lecture Notes in Pure and Appl. Math.*, pages 27–46. Dekker, New York, 1995.
- [11] C. Bernardi and Y. Maday. Spectral methods. In *Handbook of numerical analysis, Vol. V*, Handb. Numer. Anal., V, pages 209–485. North-Holland, Amsterdam, 1997.
- [12] C. Bernardi, Y. Maday, and F. Rapetti. *Discrétisations Variationnelles de Problèmes Aux Limites Elliptiques*. Mathématiques et Applications. Springer, Berlin, Heidelberg, 2004.
- [13] S. Bertoluzza, V. Chabannes, C. Prud'homme, and M. Szopos. Boundary conditions involving pressure for the Stokes problem and applications in computational hemodynamics. *Comput. Methods Appl. Mech. Engrg.*, 322:58–80, 2017.
- [14] J. M. Boland and R. A. Nicolaides. Stability of finite elements under divergence constraints. *SIAM J. Numer. Anal.*, 20(4):722–731, 1983.
- [15] F. Boyer and P. Fabrie. *Mathematical tools for the study of the incompressible Navier-Stokes equations and related models*, volume 183. Springer Science & Business Media, 2012.
- [16] C. Canuto, M.Y. Hussaini, A. Quarteroni, and T.A. Zang. *Spectral Methods*. Scientific Computation. Springer-Verlag, Berlin, 2006.
- [17] T. Chacón Rebollo and R. Lewandowski. *Mathematical and numerical foundations of turbulence models and applications*. Modeling and Simulation in Science, Engineering and Technology. Birkhäuser/Springer, New York, 2014.
- [18] C. Conca, F. Murat, and O. Pironneau. The Stokes and Navier-Stokes equations with boundary conditions involving the pressure. *Japan. J. Math. (N.S.)*, 20(2):279–318, 1994.
- [19] S. Del Pino and O. Pironneau. A fictitious domain based general PDE solver. In *Numerical methods for scientific computing. Variational problems and applications*, pages 20–30. Internat. Center Numer. Methods Eng. (CIMNE), Barcelona, 2003.
- [20] V. Girault and P.-A. Raviart. *Finite Element Methods for Navier-Stokes Equations, Theory and Algorithms*, volume 5 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 1986.

- [21] V. John. *Finite element methods for incompressible flow problems*, volume 51 of *Springer Series in Computational Mathematics*. Springer, Cham, 2016.
- [22] V. John, A. Linke, C. Merdon, M. Neilan, and L G. Rebholz. On the divergence constraint in mixed finite element methods for incompressible flows. *SIAM Rev.*, 59(3):492–544, 2017.
- [23] J.-L. Lions and E. Magenes. *Problèmes aux limites non homogènes et applications, Vol. 1*. Dunod, Paris, 1968.
- [24] S. Marušić. On the Navier-Stokes system with pressure boundary condition. *Ann. Univ. Ferrara Sez. VII Sci. Mat.*, 53(2):319–331, 2007.
- [25] Y. Saad. *Iterative Methods for Sparse Linear Systems*. SIAM, Philadelphia, 2003.
- [26] G. Sacchi Landriani and H. Vandeven. Polynomial approximation of divergence-free functions. *Math. Comp.*, 52(185):103–130, 1989.
- [27] R. Temam. *Navier-Stokes equations. Theory and numerical analysis*. Studies in Mathematics and its Applications, Vol. 2. North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977.
- [28] D. Yakoubi. *Analyse et Mise En œuvre de Nouveaux Algorithmes En Méthodes Spectrales*. PhD Thesis, Université Pierre et Marie Curie, Paris, December 2007.