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Concept Formation of Geometrical Reflection in a Digital Learning Setting

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This paper gives insights into an ongoing study that investigates pairs of students working on geometrical tasks in a cooperative way. The goal of the study is to learn more about students' understanding of geometrical reflection. First results of the study show that «symmetry» and «congruence» are useful categories to describe the students' learning processes on reflections. This paper briefly summarises the theoretical background of the study. Subsequently, it discusses two examples of observed phenomena about these categories of the conceptual understanding.

Keywords: Geometrical reflection, conceptual learning, basic ideas.

Introduction

The formation of conceptual structures is a central aim in the teaching of mathematics, in particular in geometry. The students have to form mental images of objects and their mathematical properties mediating between their real-world experiences on the one hand and the abstract mathematical structures on the other hand. Developing these representations and concepts to capture the meaning of concepts requires a social process (Dörfler, 1995). This includes explanations, drawings, negotiations and discussions in interactive settings (Krummheuer, 1995; Voigt, 1995). During the pandemic these learning-processes had to take place in online learning environments. Schwob and Gudladt (2021) show that online learning environments can support successful interactions and enable the exchange of individual knowledge. Based on this research, the present project uses the dynamic geometry software «GeoGebra» within such online learning environments to gain insights into the students' understanding of the congruence mapping reflection. This contribution discusses very first results of the ongoing project, focusing on the following question: Which conceptual aspects of reflection can be observed?

Theoretical Background

This paper focuses on the teaching and learning of mathematical concepts. More precisely speaking, the aim is on the students' understanding of reflection. From our theoretical point of view, the matter of understanding a concept goes beyond the knowledge about the mathematical definition of concepts (Tall & Vinner, 1981). Roughly speaking, the students have to develop *basic ideas*¹ about what reflections mean within different demands. For example, in figure 1 below we can understand reflection in a statical way: If you see one quadrangle $ABCD$ you can use the characteristics of reflection to check out whether g is a symmetry-axis of the figure. Hence, symmetry is understood as

¹ “basic ideas” is our translation of the term “*Grundvorstellungen*”, which is used by vom Hofe and Blum (2016).

a static basic idea of reflection. If the students ask themselves whether both halves are equal or not while solving the tasks, the category «symmetry» may be prominent. On the other hand, you can also see the triangles ABC and ACD and ask yourself, whether these two triangles are congruent to each other. In this case, you have to activate a more dynamic view on reflections: Can we move one triangle on the other one by reflection at the axis g ? In order to build a comprehensive concept it is necessary for the students to work empirically by operating with the objects and thus gaining insights in their properties (Dörfler, 1995). In terms of a comprehensive concept students would benefit from getting to know static and dynamic aspects of reflection.

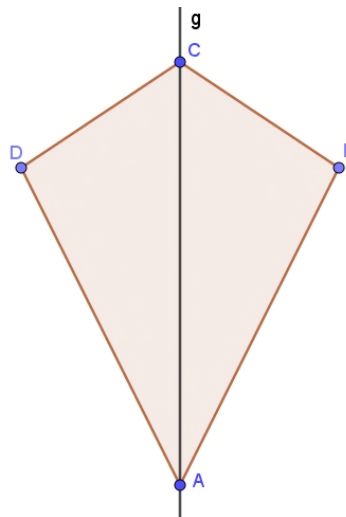


Figure 1: Basic ideas of reflection at quadrangle ABCD

Of course, for the mathematical experts, both meanings of reflection result from the same mathematical definition. But, from our point of view, they are different *basic ideas* that serve as a bridge between individual images and officially and formal definitions.

So, *Grundvorstellungen* can be construed as mediating elements or as objects of transition between the world of mathematics and the individual conceptual world of the learner. GVs thus describe relationships between mathematical structures, individual-psychological processes, and subject-related contexts, or, in short: the relationships between mathematics, the individual, and reality. (vom Hofe & Blum, 2016, p. 231)

These basic ideas can be differentiated into two aspects: The *normative aspect* describes what concepts the students should have (vom Hofe & Blum, 2016). The *descriptive aspect* shows the occurring activated concepts of individual students (vom Hofe & Blum, 2016). The focus of the present project is on the descriptive aspects of basic ideas.

The Benefit of Using Dynamic Geometry Software²

Researchers like Ng and Sinclair (2015) combined analogue (paper-and-pencil work) and digital settings (dynamic computer-based environment) for reasoning about symmetry. The students worked

² Hereafter often abbreviated as DGS.

on the topic of symmetry for three lessons and increasingly used functional relationships to discuss the symmetry of objects e.g. by dragging them to explore the connection of pre-image and image. Starting from a static understanding of reflection, students seem to have extended their knowledge by operating with the respective figures in this manner. In addition, students developed new terms while communicating about symmetry (Ng & Sinclair, 2015). In other studies, the influence of the dynamic geometry software GeoGebra was investigated. Here it was shown that students who had to reflect drawn objects by using GeoGebra performed better in geometric transformations, especially in reflection (Pavethira & Kwan Eu, 2016).

Methodology and Methods

With the aim of gaining insights into descriptive aspects of the explored basic ideas we planned online interviews via the meeting tool «BigBlueButton» (<https://bigbluebutton.org/>) with pairs of students in secondary education (communication *through* technology) (Drijvers et al., 2016). While the interviews feature pairs of same-aged students, every student works on his or her own computer at home without enabling their webcams. One of them gets access to the tasks in «GeoGebra Classroom» (<https://www.geogebra.org/>) and has the authority to manipulate applets and share his screen. Based on the shared screen the students and the interviewer discuss the effects of manipulating reflections in the dynamic geometry software «GeoGebra» (communication *of* technology) (Drijvers et al., 2016). The interviews feature students from sixth to tenth grade. This age range was deliberately chosen because curricular standards on the topics symmetry and congruence are often not properly implemented in German schools. The students in the following episodes are in grade 9 (episode 1) and grade 6 (episode 2). The examples show that an adequate conceptual understanding of reflection doesn't necessarily depend on the grade level. Both student groups have basic knowledge of the DGS «GeoGebra» and its functionality.

Working on the student's understandings, we decided to organise our research as a *qualitative* study. The interviews are being transcribed³ and analysed applying ethnomethodological methods (Krummheuer, 1995; Voigt, 1995) to work out different conceptual understandings of reflection. Furthermore, the transcribed episodes will be analysed by using comparative analysis to connect the theoretical background and empirical results via reconstructive categories (Krummheuer, 2018).

The Learning Environment

Altogether the learning environment includes eight tasks. Based on the work of Senftleben (1996) the tasks contain four phases: First, the researcher asks about the effects of a geometric operation and supports the students with a visualisation of the initial geometric construction of the object (Figure 2) below. Secondly, the students work on the question in a cooperative way. It is important that they do not use any representation within this phase – all operations are performed in their mind only. At the end of this phase, the students make a hypothesis about the effects of the discussed operation. Within the next phase, this hypothesis is validated by performing the operation within «GeoGebra». If the original hypothesis proves to be wrong, the students are then asked to find an explanation for

³ All transcripts in the present paper have been translated to English by the authors.

their observation. While one of the students may manipulate objects on the screen at his free will, the other student is forced to verbalise his ideas and give instructions. Through this interaction both students must *discuss-the-screen* (Drijvers et al., 2010).

Example: Two Episodes

The students in the following episodes worked on six tasks before. The first three tasks encompass the reflection at one axis: The *Pentagon1* was reflected at the straight line g , so that the image *Pentagon2* originates. The students worked on the manipulation of *Pentagon1* (one vertex and position of *Pentagon1*) and the single axis of reflection g and their effects on *Pentagon2*. Afterwards the tasks focus on a double reflection at two parallel axes (figure 2): First the *Pentagon1* was reflected at the straight line g , so that the image *Pentagon2* originates. In addition, *Pentagon2* was reflected at the straight line h , the image is named *Pentagon3*. First, the interviewer explains this double reflection. In the following we provide an analysis of two episodes that focus on the following question:

*“Imagine you move the axis of reflection g towards *Pentagon1*. What happens to the *Pentagon2* and *3*? Describe and explain.”*

In the following episodes the pairs of students work on this question. The students have already discussed their ideas within the first phase of the task, i.e. they built a hypothesis about the expected movements of the pentagons. The episodes start with the (very short) discussion of the accompanying hypothesis.

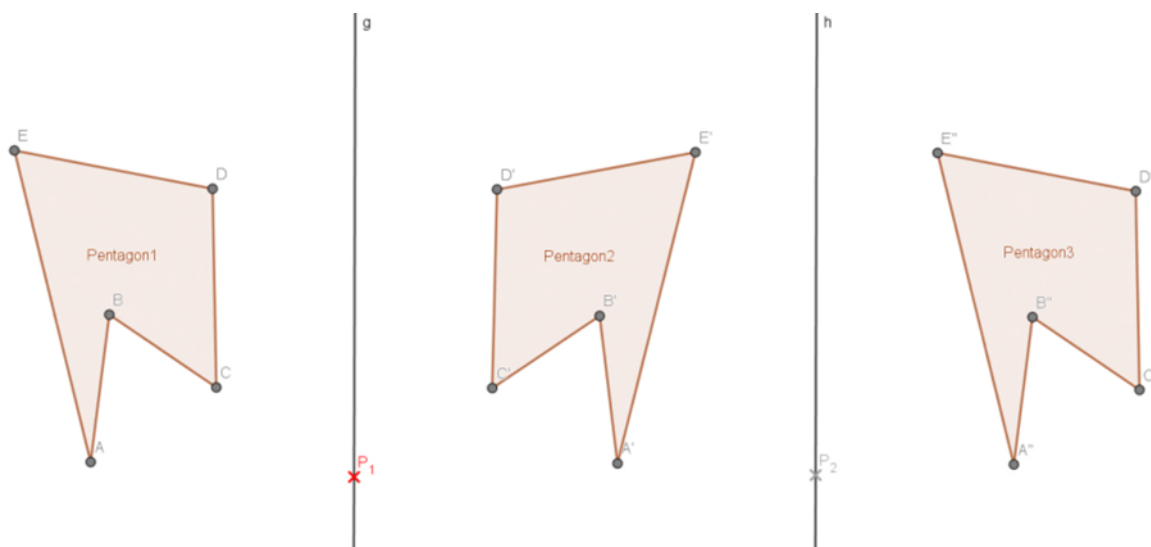


Figure 2: Construction of the task

Episode 1: Symmetry

Both students assumed what happens to the pentagons while moving the axis of reflection g . This episode begins with the verbalisation of the hypothesis and continues with the interviewer’s question on what happens to the pentagons:

- 203 Michael: this is exactly the same as the task before, that 2 stops and that the 1, err
pentagon 1 moves to the left
- 204 Nicolas: yes and 3 stops also, doesn't it?

During line 205 and 208 the students verify their hypothesis.

- 209 Interviewer: what happened now?
- 210 Michael: err pentagon 2 moves towards pentagon 1 and 3 moves to the side
- 211 Interviewer: and why?
- 212 Michael: so to the right (...) do you have an idea why?
- 213 Nicolas: um move the axis a little bit to the left (8 sec.) yes because, yes because the
first reflects err the second which is symmetrical again and then the second
is mirrored again err to the third and the third must also have the same
distance again as the second so they are symmetrical again. you know?
- 214 Michael: oh yes.

Michael and Nicolas relate their hypothesis to the previous task, dealing with the same pentagons and reflections. But, within that task, the reflection axis h was moved. Hence, only pentagon 3 moved. Both assume that pentagons 2 and 3 wouldn't move (turn 204) because the axis of reflection h isn't shifted. They hypothesise, that only pentagon 1 would show an effect by "[moving] to the left" (turn 203). At first sight, this hypothesis sounds strange. But, having in mind a static conceptual understanding of reflection, it seems quite plausible: Assume that the students understand pentagons 1 and 2 as a single figure and the line g as its symmetry-axis. Within this interpretation of the figure, it is very rational (besides alternative hypotheses) to expect that a movement of g would cause a movement of pentagon 1 in order to maintain the symmetry-axis of the figure.

When the students check their hypothesis by manipulating the objects in the «GeoGebra» applet (turn 210 and 212), they are confronted with the correct movements of the pentagons. The interviewer then asks them to find a reasoning for their observation (turn 211). Nicolas comes up with the explanation that pairs of subsequent pentagons are symmetrical (turn 213). One possible interpretation could be that he combines pentagons 1 and 2, which are described as symmetrical to each other, as well as pentagons 2 and 3. In this interpretation, pentagon 2 belongs to both combined symmetrical figures and represents one half of each figure. As Nicolas focuses on pentagons 2 and 3 and their respective distances to h , a static understanding of reflection can be assumed. The role that pentagon 2 has within this interpretation won't be further explained here. Within this episode, the research question can be answered as follows: The students Michael and Nicolas seem to have a static basic idea of reflection, as they apparently combine two pentagons to one symmetrical figure. So, it seems that the category of «symmetry» has been activated in the students in this context.

Episode 2: Congruence

Both students assumed what happens to the figures while moving the axis of reflection g . At the beginning of this episode Niklas and Jonas present their hypothesis:

- 395 Niklas: pentagon 2 pulled itself towards pentagon 1 and pentagon3 moves itself
away from pentagon 2

During line 396 and 400 the students verify their hypothesis.

- 401 Interviewer: very good. okay. now here's the same question again. why is nothing happening with pentagon 1?
- 402 Niklas: um (*moves the line g in the applet back and forth*)
- 403 Jonas: maybe because pentagon 1 is the main figure (*hesitant*) (..)* that was drawn created?
- 404 Niklas: * (*laughs*)
- 405 Interviewer: what do you mean by main figure?
- 406 Jonas: the one that has been drawn first and the others are just the reflections of it
- 407 Interviewer: mhm (*affirmative*)
- 408 Niklas: oh so this is the starting figure of the reflected pentagon 2 and 3 both of which are reflections. so they must be reflected err, towards pentagon 1 don't they? so pentagon 3 to pentagon 2 of course.
- 409 Jonas: yes.
- 410 Interviewer: what is the difference between the main figure or the, the original figure and the pentagons 2 and 3?
- 411 Niklas: um pentagon 2 and 3 (*mumbling*) (6 sec.) so they sort of need this exact reflection to continue being reflections, don't they?
- 412 Jonas: they even need pentagon 1 to exist because the other two are only reflections

Niklas presents a correct hypothesis about the movement of pentagon 2 and 3 when the axis of reflection g is shifted in the direction of pentagon 1. He assumes that they would move according to the respective changes in distance to the axes of reflection g and h (turn 395). As the students verify their hypothesis, they are asked to find a reason why pentagon 1 doesn't move (turn 401). They describe pentagon 1 as the main figure (turn 403) that was drawn first (turn 406). In contrast to this main pentagon, the others are identified as only reflections (turn 408). Since they describe the three pentagons with different terms, one can assume that the students interpret them as distinct geometrical objects, which are congruent to each other. The interviewer then asks them to explain the difference between the main figure and the reflected figures (turn 410). At this point, Jonas stresses the necessity of the main figure (pentagon 1) for the existence of the reflected figures (pentagons 2 and 3; turn 412). Within this episode, the research question can be answered as follows: The students Niklas and Jonas seem to have a dynamic view of reflection, as they apparently compare the pentagon 1 as the main figure with the pentagons 2 and 3 as reflected figures. So, it seems that the category of «congruence» has been activated in the students in this context.

Discussion

A comparison of the two episodes shows very different views on the geometrical objects and thematised operations. These different views depend on the question, whether the students understand the pentagons as different objects or not. An interpretation of the pentagons as halves of a summarised figure, as in the first episode, leads to an understanding of the lines g and h as axes of symmetry. If, on the other hand, the students' concept of reflection is different, as within the second episode, the lines are interpreted as axes of a congruence mapping that leads to a movement, i.e. the basic idea of congruence is activated. Both interpretations of reflection are important for students' understanding of the concept of geometrical reflection. In some situations, the static basic idea «symmetry» may enable students to construct symmetrical figures or identify properties of figures etc. In contrast the dynamic basic idea «congruence» may be important in situations where more figures and the functional relationship between them are relevant. To be clear the tasks in the learning environment

focus from a normative point of view on the dynamic basic idea «congruence». However, both basic ideas can be used to solve the tasks successfully.

The flexibility of switching through previous tasks in «GeoGebra Classroom» may support students in answering questions like "Did figure one move in previous tasks?" or "Which operations have been executed in these tasks?". By dragging objects in the respective applets, students may realise, that figure one only ever moves if it is dragged directly. Based on this observation, technology could potentially support the understanding of the functional relationship between the pre-image and its images. Thus, conducting research on how a connection between image and pre-image is established, could lead to further insights. In this context it might be interesting to consider whether the categories «symmetry» and «congruence» affect the students' understanding of this functional relationship.

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