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# Dialectics of language and Plato's cave of mathematics

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*This is a theoretical article hailing at classroom practice. It was originally intended to be a report on an ongoing study introducing the fundamental theorem of calculus in a STEM freshman course, but the necessary explanation of our epistemological position has taken up most of the available space. Based on Hegel and Lacan, we argue for the reality of language against the implicit realistic positions that dominate mathematics education research. We describe how Cauchy's "one says that" (on dit que) created a bubble of semantic conventions. From inside this bubble, mathematicians only recognize the students by their shadows on the wall, like in Plato's cave. We argue that learning occurs outside the bubble. Instead of dragging the students in, to teach them ready-made notations and methods, our study exercises the dialectics of language in interaction with the students. We stick faithful to our lemma: one teaches by listening and learns by talking.*

*Keywords: Hegel and Lacan, dialectics of language, Cauchy, fundamental theorem of calculus, infinitesimals.*

## Introduction

Initially, we intended to report on our experience with constrained remote teaching during two semesters as teacher-researchers (Tabach, 2006). We introduced the fundamental theorem of calculus (FTC) by working out the graphs of Covid-19 as displayed in the media. Once the students could meaningfully state the FTC about the discrete bar graphs, we moved on to the continuous case by a sudden change of scale (Ellis et al., 2020), thereby changing the referent of discourse. Ideas of limits and infinitesimals emerged from the students. This didactic strategy is justified in the ensuing theoretical development. Some of the worksheets used are available in <https://cabraldinos.mat.br/>

In the present analysis, in order to bring the dialectics of language and the subject of speaking to the foreground, we engage in a critical dialogue with a certain mathematicians' perspective about teaching (Tall, 2009; Tall & Katz, 2014; Thompson, 1994, 2019; Ely, 2017). We focus on three signifiers used by Cauchy (1821): "one says that" (*on dit que*), "becomes an infinitesimal" (*devient un infiniment petit*), and "the neighborhood" (*le voisinage*). Referring to Hegel (Hippolyte, 1977) and Lacan (1973), we show how these signifiers have created a bubble of semantic conventions that can be called *mathematics of the twentieth century* (M20); this bubble is the epistemological habitat of mathematicians. We argue that, due to the inherent semantic limitation, mathematicians are unable to leave the bubble to meet the students on their path to the bubble. They can only drag the students in and produce models of students' ways of thinking by looking at their shadows on the wall. Important concepts like *mathematics*, *number*, and *infinity*, remain outside the semantic reach from the bubble. Mathematicians tend to mix up teaching, understanding and learning.

## Language and the subject come first

To ground our study, we sought to discern events that could be placed upstream of the epistemological flux that led to the FTC. Marx's maxim advised us: the anatomy of man is the key to the anatomy of the ape. We asked ourselves: do we know what we are looking for? Yes, in our classes we expect the students to express in current language: *the integral of the derivative is the variation of the function*, and *the derivative of the integral is the function itself*. Guided by these current language statements, we were able to locate one of the origins of the FTC in Barrow's theorem (Barrow, 1976, p. 78, Figure 109). It is well-known that this theorem became the cornerstone of a discussion that lasted two hundred years. We will take this theorem as an example of a *signifier*, a core concept for us.

Barrow did not write the theorem for us; he was addressing his contemporaries that shared a certain common epistemological spirit. We say that Barrow *took that theorem as a signifier to represent himself as a subject to another signifier* (Lacan, 1973, p. 188). More than three centuries later, it happens that this second signifier is available to us, so that not only can we understand the theorem and its proof, but we can *represent ourselves as subjects* by this second signifier to a third signifier, belonging to the reader, etc. This movement of language and subject from one signifier to the next we call the *dialectics of language*.

Language is not only a system of signs alien to the signified, it is also the existing universe of sense, and this universe is the interiorization of the world as well as the exteriorization of the "I".

Language is a double movement that must be understood in its unity. (Hippolyte, 1977, p. 24)

Due to this subtle but ubiquitous double movement, words acquire their meaning and we become their speakers. We were able to locate Barrow's theorem because what we were looking for was already specified in language. *Insofar as we 'make sense of the world' it is the world that posits its sense across us*. A double movement constitutes both, *human subjects* and *language*. The movement is prior to the opposing poles that it generates. Accordingly, there is neither a universal "I", owner of an inner meaning to be expressed, nor a world out there of which one can speak and make sense by "using" language. There is no "dialectics between", no "dialectic relation" (Pais, 2016).

We apologize for demanding of the reader the effort to understand ideas so strange to commonsense. They are necessary to elicit the realistic<sup>1</sup> positions prevailing in mathematics education research. For instance, the belief in "underlying mechanisms that shape human thought, building from the fundamental level of human perception (...)" (Tall & Katz, 2014, p. 100) leads to the postulate that "the underlying brain activity is more fundamental" (p. 102). The next step in this line of reasoning is to state that "our brains make sense of the world by assembling neuronal information" (Tall, 2009, p. 482). The apex of this chain of psychological materialism is the search for the "number neuron" (Dehaene, 1997, p. 57). A criticism of this chain of thought may be found in Pais (2019) and Webel & Stigliano (2004), as well as in Baldino (2019), a parody about the number neuron.

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<sup>1</sup> [https://en.wikipedia.org/wiki/Scientific\\_realism](https://en.wikipedia.org/wiki/Scientific_realism)

## Dialectics vs understanding

By the time of Barrow, mathematics and philosophy had formed a single body of knowledge whose esthetic was modeled on Euclid. Newton followed this model, while Leibniz broke apart from it, and the two vied for being credited with the "discovery" of calculus. Berkely criticized them both. The story of this imbroglio is well known.

In the beginning of the nineteenth century, Hegel criticized the mathematicians for not been able to provide rational foundation for their science (1985, p. 236-309). By the end of the eighteenth century, Kant's philosophy had become prevalent. Hegel called it *understanding* (*Menschenverstand*) and referred to his own system as *speculative philosophy*. Hegel's went beyond Kant's philosophy. Understanding establishes clear cut distinctions between thought and being, subject and object, form and matter, discourse and referent, us and the world, etc. Due to these distinctions, understanding was prone to being well received in the imbroglio of philosophy and mathematics. Understanding was a reference for the movement for rigor that ran throughout the nineteenth century and is becoming stronger in STEM today. For instance, Kant's philosophy has been explicitly evoked to support the hegemony of mathematics over mathematics education; an "apprenticeship model" attributed to Dawkins and Weber (2017) is proposed by Rittberg et al. (2020) and criticized in Baldino and Cabral (2021).

Understanding postulates the existence of a beyond that cannot be reached by knowledge nor, consequently, by *language*: the so-called *thing in itself* is not cognoscible. This philosophical position favors the polysemy of the signifier "mathematics" discussed in Cabral and Baldino (2021): it favors inconsequent research in mathematics education, while the classroom resists. We argue that language must be brought to the fore. This endeavor naturally points to Lacan who goes beyond Hegel and states plainly: "there is no being outside language"<sup>2</sup>. We cannot survey the meaning of the word "dialectics" from Plato to the present. We will retain the meaning it has in a formula that may condense the whole of Lacan's work: "dialectics of the subject and the Other" (Lacan, 1973, p. 205, 239). Accordingly, by "dialectics" we will refer to both, the dialectics of language, as in Lacan, and to Hegel's speculative philosophy which Žižek has elicited as the background of Lacan's work.

However, *dialectics is not a substitute for understanding*; this would be an assumption proper to understanding. *Dialectics exerts itself on understanding*, leading it to recognize the contradictions that unavoidably stem from its black-and-white divisions. Dialectics does not fight for victory; it is guided by the political necessity of the moment; it fights to continue fighting. If understanding dies, speculative philosophy dies too "and in this night of mere reflection and of the calculating intellect, in this night which is the noonday of life, commonsense and speculation can meet one another" (Hegel, 1977, p. 103).

The disentanglement of mathematics and philosophy is taking place today, under the arrogance of science over humanities. In his paper on quantum mechanics, Gauthier (2010, p. 2) aims to "make explicit the concept of probability in order to extract the mathematical content from its mystical

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<sup>2</sup> " Il n'y a d'être que dans le langage." [https://www.youtube.com/watch?v=njA-1a4N\\_iw](https://www.youtube.com/watch?v=njA-1a4N_iw)

(philosophical) gangue [sic]”. What is it that is being separated from its “philosophical gangue”? Strangely enough, this is left to philosophers to explain, since mathematicians do not say what mathematics is. Scott (2012) shows what happens when a physicist (Stephen Hawking) draws on his professional status to gather audience to speak about what he does not know. We must be aware of such antagonisms because understanding and dialectics face off when they meet the student in the common ground of the classroom, in the “noonday of life”.

### **Cauchy and the bubble: *on dit que***

Concerned with the polysemy of the signifier, Cabral and Baldino (2021) characterize “mathematics” as special form of discourse that was born in Ancient Greece. This discourse, which they call *quilted speech*, aims to stop the slide of the signified under the signifier: each signifier would have a single precise meaning:

The understanding would like “a fixity and an exactitude that is not found in existing language; the idea of creating a pure language, a system of symbols which remain absolutely invariant over the course of the diverse combinations they undergo comes from this. (Hippolyte, 1977, p. 46)

The whole imbroglia following Newton and Leibniz may be looked at as an attempt to quilt speeches; people refer to it as “rigor”, “formalization”, “definition of the concept” etc. In 1821, Cauchy introduced a precision into quilted speeches that eventually completed the separation of mathematics and philosophy. In his *Cours d’analyse de l’École Royale Polytechnique*, he wrote: “*One says that a variable quantity becomes infinitely small when its numerical value decreases indefinitely so as to converge towards its limit zero*” (Cauchy, 1821, p. 26, added emphasis).<sup>3</sup>

On the one hand, he cut the gordian knot of misunderstanding with a *semantic convention*: “one says that” (*on dit que*). However, on the other hand he conserved the movement of dialectics of language by saying “becomes” (*devient*).<sup>4</sup> Something ceases to be what it is and becomes something else. Hegel would appreciate this proposition as a perfectly dialectical one: its very utterance imposes a movement of language that preserves what it denies; this is expressed by the German verb “to transcend” (*aufheben*). Hegel (1985, p. 69) calls this movement of language “becoming” (*das Werden*).

Cauchy’s semantic convention was followed by Weierstrass, Hilbert and many others. They created a language bubble that we call M20. This bubble became the habitat of a *community of speech* whose members are the mathematicians (Cabral & Baldino, 2021). This community decides what counts as a valid inference; for instance, it informally accepts the use of the axiom of choice and the continuous hypothesis.

The membrane that delimits the bubble is opaque for those inside it. Like in Plato’s cave, mathematicians only see shadows of the external world. The infinite movement of the dialectics of

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<sup>3</sup> On dit qu’une quantité variable devient *infinitement petite*, lors que sa valeur numérique décroît indéfiniment de manière à converger vers la limite zéro.

<sup>4</sup> In Portuguese, *devenir*, to become, is a reflexive verb: *tornar-se*: to change oneself is a perfect expression for learning, a movement that occurs in language through which a subject changes herself. It is up to the native speakers to compare the relative weight of these verbs in their respective languages.

language is left outside, including important concepts, like “mathematics”, “number” and “infinity”, that find no “definition” inside the bubble. The “repetition of a process with an underlying pattern of successive states” (Tall, 2009, p. 483) is what Hegel calls the bad (*schlechten*) infinity. The passage from one member to the next in a sequence is an indefinite repetition of sameness, with no risk of producing changes either in the subject or in the object in the dialectics of language. Only this finitized infinity is admitted into the bubble; the apex of finitization of M20 is Weierstrass epsilonics.

When mathematicians face the necessity of teaching, they have an inkling that something is wrong. They do not realize that the student is outside the bubble, plunged into the infinite movement of language. They try to pull her in to teach her the concept by “defining” it. Looking at her shadow on the wall of the cave, they offer us this precious confirmation of our criticism: “Students’ mathematics is the mathematical reality they experience, which is wholly theirs and is unknowable to us in the same way dark matter is unknowable to us” (Thompson, 2019, p. 39).

It is transparent that Kant’s philosophy supports this ideology: the thing in itself is not cognoscible. Mathematicians are aware of the failure of their teaching methods. For instance, even after four semesters of calculus, students are not only unable to use the FTC, but they do not recognize its use when it is presented to them (Thompson, 1994, p. 256, ex. 7.10). Limited by the opacity of the membrane, mathematicians tend to explain this failure by the tautology of lacking: the difficulties with the FTC “stem from impoverished concepts of rate of change and from poorly developed and poorly coordinated images of functional covariation and multiplicatively-constructed quantities” (Thompson, 1994, p. 229). That is, the difficulty lies precisely in what was taught to these students.

Tall (2009, p. 484) refers to the “notion of a generic limit” to account for the difficulties of students with the limit of sequences and the concept of infinitesimal. However, as justification of this concept we only find the expedients: “a *natural human belief* that the limiting object is endowed with the same properties as the individual terms” and “infinitesimal concepts are *natural products* of human imagination” (p. 483, both emphases added). We finally collect the mathematicians’ recognition of the impossibility: “Like with dark matter, the best we can do is make models that fit observations and are consistent with other models” (Thompson, 2019, p. 39).

### **Cauchy and infinitesimals: *devient...le voisinage***

A full account of the polemics around Cauchy can be found in Katz and Katz, (2011) and Tall and Katz (2014); these articles include an impressive list of references only available in universities of the so-called First World. By requiring “mastery of the field” as an academic prerequisite for anyone who has something to say on the subject, one risks blocking out the new and foment cultural imperialism.

We read the second part of Cauchy’s “becomes”, together with this other excerpt: “Besides, one also says that the function  $f(x)$  is, in *the neighborhood* of a *particular value* assigned to the variable  $x$ , a continuous function of this variable (...)” (Cauchy, 1821, p. 35, added emphasis).<sup>5</sup> The French

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<sup>5</sup> On dit encore que la fonction  $f(x)$  est, dans le voisinage d’une valeur particulière attribué à la variable  $x$ , fonction continue de cette variable (...).

singular “*le*” stresses that a numerical “value”, has only one “neighborhood”. Consequently, when this value is zero, we may assume that this neighborhood consists of the “infinitely small variables” (*variables infiniment petites*, p. 65). These signifiers do not belong to the quilted speech of the bubble. We agree with Katz and Katz (2011, p. 426) that we should “jettison the automatic translation-to-limits” in reading Cauchy. Instead, we propose a reading-to-infinitesimals as in Sad et al. (2001). Cauchy is saying that a variable that converges to zero becomes an element of the *monad of zero*,  $\mathfrak{o}$ , a whole universe of infinitesimals that is incommensurable with ours.

From the dialectical perspective, Robinson is not a “consequence” of Cauchy. Marx’s aphorism about the man and the primate must be evoked once more. Cauchy’s discourse on infinitesimals has actually been quilted and finitized by Robinson. Only from this perspective can we look back and ask where this quilting came from and what its trajectory to the present has been. To the myriad things that have been said about Cauchy, we dare to add: the mature Cauchy of 1853 was *waiting for rigor to catch up to him* when he restated his 1821 theorem (Sad et al., 2001). This was the trajectory of the dialectics of language that led from Leibniz to Robinson.

The completion of  $\mathbb{Q}$  with the monads stemming from his sequences was already an embryo in Cauchy’s “intuition”, with no need to invent classes of equivalence of such sequences to fill in the gaps of  $\mathbb{Q}$ . Indeed, consider sequences  $(a_k)$  of rational numbers that satisfy Cauchy’s condition:

$$(\forall \varepsilon \in \mathbb{R}^+)(\exists N \in \mathbb{N})(\forall m, n \in \mathbb{N})(m, n > N \rightarrow |a_n - a_m| < \varepsilon)$$

We can take adequate equivalence classes of such sequences and form what we now call the field of finite hyperrational numbers  $\mathfrak{O}$ . If we introduce the operations of addition and multiplication as well as the order relation elementwise among the monads, we get a complete ordered field (isomorphic to  $\mathbb{R}$ ), namely, the quotient ring  $\mathfrak{O}/\mathfrak{o}$  (Stroyan & Luxemburg, 1976, p. 9, Katz & Katz, 2011, p. 448). We say that a single movement of the dialectics of language that is still taking place posits these signifiers under the form of an identity quilted speech inside the bubble and also *posits ourselves as subjects who utter these speeches*.

## Consequences for mathematics education

The teaching experience that we initially intended to report consists in working on the COVID-19 graphs until the students are able to formally express and meaningfully utter that 1) the variation of the running total is the area under the daily deaths, 2) the moving average of the daily deaths is the variation rate of the running total and 3) the one-day-based moving average is the number of deaths of the day before. These, of course, are the statements of FTC. Next, we replaced the discrete graphs of a few days in the pandemic with continuous graphs of an endemic disease that lasted for many years (Ellis et al., 2020). We insisted upon the expression and formula for the one-day-based moving average. The whole discussion occurred in the realm of the *dialectics of language*; we only tried to collimate the students’ discourses towards the signifiers in the bubble. For instance, only when they were trying to express one day in sixty years did we suggest  $dt$ , “a little bit of”, according to Tompson (1914). For the areas under the graph, the students suggested  $A(a, b)$ ; neither Leibniz notation for the integral nor Riemann sums were necessary.

We did not drag the students into the bubble to teach them how to interpret the M20 readymade symbols (Ely, 2017). We believe that the more one tries to do so, the greater the difficulty for the students. Asunder from the dialectics of language and modeling shadows of students with finitary processes, mathematicians can do no more than teach finite processes. Learning depends on the infinity of variables in the dialectics of language, insofar as the students start representing themselves as subjects in *current language* by new signifiers. In this respect, the possibility of open-camera, eye-to-eye communication and video recording was a blessing to our long-standing methodology (Cabral, Pais, & Baldino, 2019) based on the aphorism *one teaches by listening and learns by talking*.

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