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1 **QUASI-REVERSIBILITY METHODS OF OPTIMAL CONTROL FOR ILL-POSED**  
2 **FINAL VALUE DIFFUSION EQUATIONS**

3 GISÈLE MOPHOU AND MAHAMADI WARMA

ABSTRACT. We consider optimal control problems associated to generally non-well posed Cauchy problems in a general framework. Firstly, we approximate the ill-posed problem with a family of well-posed one and show that solutions of the latter one converge to solutions of the former one. Secondly, we investigate the minimization problem associated with the approximated state equation. We prove the existence and uniqueness of minimizers that we characterize with the optimality systems. Finally, we show that minimizers of the approximated problems converge to the minimizers of the optimal control subjected to the ill-posed state equation that we characterize with a singular optimality system. This characterization is obtained as the limit of the optimality systems of the approximated minimization problem. We use the techniques of quasi-reversibility developed by Lattès and Lions in 1969. Our general framework includes classical elliptic second order operators with Dirichlet and Robin conditions, as well as the fractional Laplace operator with the Dirichlet exterior condition.

4 1. INTRODUCTION AND PROBLEM FORMULATION

5 Let  $\Omega \subset \mathbb{R}^N$  ( $N \geq 1$ ) be an arbitrary open set with boundary  $\partial\Omega$ . We are interested in the optimal  
6 control associated to the following ill-posed diffusion equation:

$$\begin{cases} \rho_t + A\rho = f & \text{in } Q := \Omega \times (0, T), \\ \rho(\cdot, T) = \rho^T & \text{in } \Omega, \end{cases} \quad (1.1)$$

7 where  $T > 0$  is a real number,  $f$  is the control function, and  $\rho^T \in L^2(\Omega)$  is a given function. The  
8 (unbounded) linear operator  $A$  with domain  $D(A)$  in  $L^2(\Omega)$  satisfies the following assumptions:

9 **Assumption 1.1.** *We assume the following.*

- 10 (a) *The operator  $A$  is non-negative, selfadjoint, invertible, and has a compact resolvent.*  
(b) *The operator  $-A$  generates a strongly continuous, analytic, and compact semigroup  $S = (S(t))_{t \geq 0}$  on  $L^2(\Omega)$  which is also submarkovian in the sense that each operator  $S(t)$ ,  $t \geq 0$ , is positive, and also  $L^\infty$ -contractive, that is, for all  $t \geq 0$ ,*

$$\|S(t)u\|_{L^\infty(\Omega)} \leq \|u\|_{L^\infty(\Omega)}, \quad \forall u \in L^2(\Omega) \cap L^\infty(\Omega).$$

Of concern is the following optimization problem:

$$\min_{(f, \rho) \in \mathcal{A}} \frac{1}{2} \left( \left\| \rho(\cdot, 0) - \rho^d \right\|_{L^2(\Omega)}^2 + \xi \|f\|_{L^2(Q)}^2 \right) \quad (1.2)$$

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1 subject to the constraints that  $\rho$  solves the ill-posed state equation (1.1),  $\rho^d \in L^2(\Omega)$  is a given target,  
 2 the constant penalty parameter  $\xi > 0$  is given, the control  $f \in \mathbb{U}_{ad}$ , where  $\mathbb{U}_{ad}$  is a given nonempty  
 3 closed and convex subset of  $L^2(Q)$ , and finally

$$\mathcal{A} = \{(f, \rho) : f \in \mathbb{U}_{ad} \text{ and } \rho \text{ is the associated strong solution of (1.1)}\}. \quad (1.3)$$

4 The function  $\rho^T$  is given such that for every  $f \in \mathbb{U}_{ad}$ , the state equation (1.1) has a strong solution  $\rho$ .  
 5 The Banach space where  $\rho$  belongs shall be specified later.

One comes across such a model (1.1) while dealing for instance of environmental phenomena. Actually in such a situation, one may not know when the phenomenon began or have information on the boundary. Such models with missing data are therefore ill-posed in the sense of Hadamard. In particular, the uniqueness of the solution as well as the continuous dependence of solutions on the given data  $f$  and  $\rho^T$  are not always satisfied. Therefore, even if we can prove using minimizing sequences that the optimization problem (1.2)-(1.1) has a unique solution  $(\bar{f}, \bar{\rho})$ , it will be difficult to characterize this solution since the increase of the state and the control are linked as shown the optimality condition:

$$\int_{\Omega} (\rho(x, 0) - \bar{\rho}(x, 0))(\bar{\rho}(x, 0) - \rho^d(x))dx + \int_Q \xi \bar{f}(f - \bar{f}) \geq 0 \quad \forall (f, \rho) \in \mathcal{A}.$$

6 We refer to Section 3 for more details. This is mostly due to the fact that the map  $(f, \rho^T) \mapsto \rho$   
 7 not in general bijective. So, to obtain an optimality system in which the state and the control are  
 8 not related because they belong to  $\mathcal{A}$ , we approached the ill-posed problem by a family of well-posed  
 9 problems using the quasi-reversibility method developed by Lattès and Lions [13]. More precisely, for  
 10 any  $\varepsilon > 0$ , we approximate the problem (1.1) with the well-posed problem:

$$\begin{cases} \rho_t^\varepsilon + A\rho^\varepsilon = f & \text{in } Q, \\ \varepsilon\rho^\varepsilon(\cdot, 0) + \rho^\varepsilon(\cdot, T) = \rho^T & \text{in } \Omega. \end{cases} \quad (1.4)$$

We show that solutions of the approximated problem (1.4) converge to the solutions of our ill-posed problem (1.1). Then, we consider the associated approximated minimization problem:

$$\min_{(f, \rho^\varepsilon)} \frac{1}{2} \left( \|\rho^\varepsilon(\cdot, 0) - \rho^d\|_{L^2(\Omega)}^2 + \xi \|f\|_{L^2(Q)}^2 \right) \quad (1.5)$$

11 subject to the constraints that  $\rho^\varepsilon$  solves the approximated state equation (1.4) and the control  $f \in \mathbb{U}_{ad}$   
 12 where  $\mathbb{U}_{ad}$  is a nonempty closed and convex subset of  $L^2(Q)$ .

13 The quasi-reversibility method has already been used by some authors for ill-posed problems. In  
 14 [13], Lattès and Lions used the following family of well-posed problems:

$$\begin{cases} v_\varepsilon'(t) + Av_\varepsilon(t) - \varepsilon A^2 v_\varepsilon(t) & = 0, & 0 < t < T \\ u_\varepsilon'(t) + Av_\varepsilon(t) & = 0, & 0 < t < T \\ u_\varepsilon(0) & = v_\varepsilon(0), \\ v_\varepsilon(T) & = g \end{cases} . \quad (1.6)$$

15 to approach the ill-posed problem

$$\begin{cases} u'(t) + Au(t) = 0, & 0 < t < T \\ u(T) = g \end{cases} \quad (1.7)$$

16 where  $g \in H$  is given, and  $A$  is a non-negative selfadjoint operator on a Hilbert space  $H$  such that  
 17  $-A$  generates a strongly continuous, analytic, and compact semigroup of contractions. They have  
 18 shown that if  $u_\varepsilon$  is a solutions of (1.6), then  $u_\varepsilon(T)$  converges to  $g$  in  $H$ , as  $\varepsilon \downarrow 0$ . They did not get a  
 19 convergence of  $u_\varepsilon(t)$ ,  $0 \leq t < T$ , as  $\varepsilon \downarrow 0$ .

1 Showalter [20] has approximated (1.7) with

$$\begin{cases} v'_\varepsilon(t) + \varepsilon Av'_\varepsilon(t) + Av_\varepsilon(t) & = 0, & 0 < t < T \\ u'_\varepsilon(t) + Au_\varepsilon(t) & = 0, & 0 < t < T \\ u_\varepsilon(0) & = v_\varepsilon(0), \\ v_\varepsilon(T) & = g. \end{cases} \quad (1.8)$$

2 The author has proved that  $u_\varepsilon(T)$  converges to  $g$  in  $H$ , as  $\varepsilon \downarrow 0$ , and that  $u_\varepsilon(t)$  converges to the  
3 solution  $u(t)$  of (1.7) in  $H$ , uniformly in  $t \in [0, T]$ , as  $\varepsilon \downarrow 0$ , if and only if such a solution exists. In all  
4 the above convergences, the norm of the function carrying  $g$  to  $v_\varepsilon(0)$  is quite large for small values of  
5  $\varepsilon$ .

Miller [15] has addressed the problem of large norm by finding an optimal perturbation of the  
operator  $A$ . The author stated that it should be possible to make the norm in the order of  $c/\varepsilon$  rather  
than  $e^{c/\varepsilon}$  and derive conditions on the perturbation to achieve the best possible results. As above the  
author approximated (1.7) with

$$\begin{cases} v'(t) + g(A)v(t) = 0, & 0 < t < T \\ v(T) = g, \end{cases}$$

6 and again solved the problem forward using  $v(0)$  as an initial datum. Miller called this method,  
7 stabilized quasi-reversibility.

8 Clark and Oppenheimer [5] have approximated (1.7) with

$$\begin{cases} u'(t) + Au(t) = 0, & 0 < t < T \\ \varepsilon u(0) + u(T) = g, \end{cases} \quad (1.9)$$

9 and they have shown that  $u_\varepsilon(T)$  converges to  $g$  in  $H$ , as  $\varepsilon \downarrow 0$ , and that  $u_\varepsilon(t)$  converges to the solution  
10  $u(t)$  in  $H$ , uniformly in  $t \in [0, T]$ , as  $\varepsilon \downarrow 0$ . They have also obtained a better polynomial convergence  
11 rate.

12 Finally, Denche and Bessila [8] have approximated (1.7) with the following problem:

$$\begin{cases} u'(t) + Au(t) = 0, & 0 < t < T \\ u(T) - \varepsilon u'(0) = g. \end{cases} \quad (1.10)$$

13 They have obtained nice convergence results as in [5] but here with a logarithmic convergence rate.

14 Concerning optimal control problems associated to ill-posed problems using quasi-reversibility type  
15 methods, some few results are available in the literature. In [17], Nakoulima used a regularization-  
16 penalization method to solve an optimal control problem associated to ill-posed elliptic equations.  
17 The result is achieved by considering the control problem as a limit of sequence of well-posed control  
18 problems. In [16], Mophou and Nakoulima studied an optimal control problem subjected to an ill-  
19 posed elliptic equation. Their analysis consisted in viewing the ill-posed problem as the limit of a  
20 family of well-posed equations. Then, under the Slater condition on the admissible set, they gave a  
21 singular optimality system that characterizes the optimal control.

22 As one can notice, none of the above works using quasi-reversibility methods has considered a general  
23 non-homogeneous parabolic problem as in (1.1) where we have some extract terms in the solutions  
24 that can complicate the computations. Our main goal are twofold.

- 25 • Firstly, we would like to investigate this more general problem by considering the approximation  
26 (1.9) proposed in the reference [5] which is a particular case of the system (1.4). We would like  
27 to show the existence of minimizers of the control problem (1.5)-(1.4) and give the optimality  
28 systems that characterizes this sequence of minimizers.

- 1 • Secondly, we would like to prove that the sequence of minimizers of the optimal control problem  
 2 (1.5)-(1.4) converges to minimizer of the control problem (1.2)-(1.1) and give the singular  
 3 optimality system that characterizes this minimizer.

4 Let us notice that as a direct consequence of Assumption 1.1 we have the following.

5 **Remark 1.2.** It follows from Assumption 1.1 that the following assertions hold.

- (a) The operator  $A$  is given by a bilinear, symmetric, continuous, closed, and coercive form  $\mathcal{E}$  with domain  $D(\mathcal{E}) = D(\sqrt{A})$ . This means that  $\mathcal{E}(u, v) = \mathcal{E}(v, u)$  for all  $u, v \in D(\mathcal{E})$ , and

$$\|u\|_{D(\mathcal{E})} := (\mathcal{E}(u, u))^{1/2} \quad (1.11)$$

6 is the norm on the space  $D(\mathcal{E})$ .

There is a constant  $C > 0$  such that for every  $u, v \in D(\mathcal{E})$  we have

$$|\mathcal{E}(u, v)| \leq \|u\|_{D(\mathcal{E})} \|v\|_{D(\mathcal{E})}. \quad (1.12)$$

There is a constant  $C > 0$  such that for every  $u \in D(\mathcal{E})$  we have

$$\|u\|_{D(\mathcal{E})} \geq C \|u\|_{L^2(\Omega)}. \quad (1.13)$$

For every  $u \in D(A)$  and  $v \in D(\mathcal{E})$  we have

$$(Au, v)_{L^2(\Omega)} = \mathcal{E}(u, v). \quad (1.14)$$

- (b) The spectrum of  $A$  is formed with eigenvalues  $\lambda_n$  ( $n \in \mathbb{N}$ ) satisfying

$$0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \leq \dots \quad \text{and} \quad \lim_{n \rightarrow \infty} \lambda_n = +\infty.$$

We shall denote by  $\phi_n$  the orthonormal basis of eigenfunctions associated with the eigenvalues  $\lambda_n$ . Then the eigenvalues of the operator  $S(t)$  are  $e^{-t\lambda_n}$ , and possible zero (see e.g. [18] for more details on abstract results on semigroups). For each  $\varepsilon > 0$  and  $t \geq 0$ , the operator  $\varepsilon I + S(t)$  is invertible. Also if  $u \in L^2(\Omega)$  has the expansion  $u = \sum_{k=1}^{\infty} a_k \phi_k$ , then

$$S(t)u = \sum_{k=1}^{\infty} e^{-t\lambda_k} a_k \phi_k \quad \text{and} \quad (S(t)u, u)_{L^2(\Omega)} = \sum_{k=1}^{\infty} e^{-t\lambda_k} a_k^2 \geq 0, \quad (1.15)$$

where  $a_k = (u, \phi_k)_{L^2(\Omega)}$ . It follows from this accretivity type condition that for every  $t \geq 0$  and  $\varepsilon > 0$  we have

$$\|(\varepsilon I + S(t))^{-1}\| \leq \frac{1}{\varepsilon}. \quad (1.16)$$

- (c) If  $u \in L^2(\Omega)$  has the expansion  $u = \sum_{k=1}^{\infty} a_k \phi_k$ , then for every  $\varepsilon > 0$  we have that

$$(\varepsilon I + S(T))u = \sum_{k=1}^{\infty} (\varepsilon + e^{-T\lambda_k}) a_k \phi_k \quad \text{and} \quad (\varepsilon I + S(T))^{-1}u = \sum_{k=1}^{\infty} \frac{a_k}{\varepsilon + e^{-T\lambda_k}} \phi_k. \quad (1.17)$$

- (d) Let  $(D(\mathcal{E}))^*$  denote the dual of  $D(\mathcal{E})$  with respect to the pivot space  $L^2(\Omega)$  so that we have the following continuous and dense embeddings:  $D(\mathcal{E}) \hookrightarrow L^2(\Omega) \hookrightarrow (D(\mathcal{E}))^*$ . Then  $A$  can be also viewed as an operator from  $D(\mathcal{E})$  into  $(D(\mathcal{E}))^*$  given for every  $u, v \in D(\mathcal{E})$  by

$$\langle Au, v \rangle_{(D(\mathcal{E}))^*, D(\mathcal{E})} = \mathcal{E}(u, v). \quad (1.18)$$

7 In the following if there is no confusion, we shall not make any distinction between the two  
 8 operators.

Throughout the remainder of the paper, without any mention,  $\Omega \subset \mathbb{R}^N$  ( $N \geq 1$ ) is an arbitrary open set with boundary  $\partial\Omega$ . The regularity needed on  $\Omega$  will depend on the operator and the boundary or exterior conditions that we will specify in our examples in Section 4.

We shall let

$$\mathbb{V} := D(\mathcal{E}) \tag{1.19}$$

and  $\mathbb{V}^* = (D(\mathcal{E}))^*$  the dual of  $\mathbb{V}$  with respect to the pivot space  $L^2(\Omega)$ . Then  $\mathbb{V}$  endowed with the norm given in (1.11) is a Hilbert space.

If we set

$$W(0, T; \mathbb{V}) := \left\{ \zeta \in L^2((0, T); \mathbb{V}) : \zeta_t \in L^2((0, T); \mathbb{V}^*) \right\}, \tag{1.20}$$

then  $W(0, T; \mathbb{V})$  endowed with the norm given by

$$\|\psi\|_{W(0, T; \mathbb{V})}^2 = \|\psi\|_{L^2((0, T); \mathbb{V})}^2 + \|\psi_t\|_{L^2((0, T); \mathbb{V}^*)}^2, \quad \forall \psi \in W(0, T; \mathbb{V}), \tag{1.21}$$

is a Hilbert space. Moreover, by [14, Chapter III, Theorem 1.1] we have the following continuous embedding:

$$W(0, T; \mathbb{V}) \hookrightarrow C([0, T]; L^2(\Omega)). \tag{1.22}$$

In the present paper we have obtained the following interesting and specific results:

- Our first main result (Theorem 2.7) shows that the approximated problem (1.4) has a unique strong solution which depends continuously on the given data  $f$  and  $\rho^T$ . In addition, we have obtained in Theorem 2.10 that any strong solution of (1.4) belongs to the space  $L^2((0, T); \mathbb{V}) \cap H^1((0, T); \mathbb{V}^*)$ . Thus, according to the embedding (1.22), we can deduce that any strong solution of (1.4) belongs to  $C([0, T]; L^2(\Omega))$ .
- Theorems 2.15 and 2.16 show that solutions of the approximated problem (1.4) converge (uniformly in  $t \in [0, T]$ ) to solutions of (1.1) and we have obtained the precise rate of convergence. This latter result is very useful for numerical analysis and approximations.
- In Theorems 3.2 and 3.3 we show the existence of a unique minimizer of the minimization problem (1.5)-(1.4) and we characterize the associated optimality conditions and systems.
- In Theorem 3.4 we prove that minimizers and solutions of the optimality systems of the minimization problem (1.5)-(1.4) converge to minimizers and associated optimality systems of the control problem (1.2)-(1.1).

The rest of the paper is structured as follows. In Section 2 we give some results concerning the existence, regularity, and the representation of solutions to the ill-posed equation (1.1), the approximated state equation (1.4) and the associated dual system. We also discuss the convergence of solutions of the approximated problem (1.4) to solutions of the problem (1.1) and obtain the precise rate of convergence. Section 3 concerns the optimal control problems. We first show the existence and uniqueness of the sequence of minimizers to the minimization problem (1.5)-(1.4) and give the optimality systems that characterize these minimizers. Then, we prove that the sequence of minimizers converge to the minimizer of the problem (1.2)-(1.1) that we characterizes with a singular optimality system. Finally in Section 4 we give several examples of concrete operators that enter in our framework.

## 2. EXISTENCE RESULTS

Throughout the remainder of the paper, without any mention, we assume that the operator  $A$  satisfies Assumption 1.1. In addition,  $\{\phi_n\}$  denotes the orthonormal basis of eigenfunctions of  $A$  associated with the eigenvalues  $\{\lambda_n\}$ .

In this section we investigate the well-posedness and regularity of solutions to the non-well posed and the approximated state equations. We also give some necessary and sufficient conditions on the given data such that the ill-posed problem has a solution. In that case, some convergence results of

1 solutions of the approximated problem to solutions of the ill-posed problem have been obtained. We  
2 start by studying the ill-posed problem.

3 **2.1. Existence results of the non-well posed problem.** Here is our notion of strong solutions to  
4 the ill-posed problem.

5 **Definition 2.1.** Let  $f \in L^2(Q)$  and  $\rho^T \in L^2(\Omega)$ . A function  $\rho \in C([0, T]; L^2(\Omega))$  is said to be a strong  
6 solution of (1.1) if the following assertions hold:

- *Regularity:*

$$\rho_t(\cdot, t) \in L^2(\Omega) \text{ and } \rho(\cdot, t) \in D(A) \text{ for a.e. } t \in (0, T)$$

7 and the first equation in (1.1) is satisfied for a.e.  $t \in (0, T)$ .

- *Final condition:*

$$\rho(\cdot, T) = \rho^T \text{ a.e. in } \Omega.$$

8 The following result gives a necessary and sufficient condition in terms of the given data  $f$  and  $\rho^T$   
9 such that the non well-posed system (1.1) has a strong solution.

10 **Lemma 2.2.** Let  $f \in L^2(Q)$  and  $\rho^T \in L^2(\Omega)$  have the expansions  $f(\cdot, t) = \sum_{k=1}^{\infty} f_k(t)\phi_k$  and  $\rho^T =$   
11  $\sum_{k=1}^{\infty} b_k\phi_k$  where  $f_k(t) = (f(\cdot, t), \phi_k)_{L^2(\Omega)}$  and  $b_k = (\rho^T, \phi_k)_{L^2(\Omega)}$ . Then the following assertions are  
12 equivalent.

- 13 (a) The system (1.1) has a strong solution  $\rho$ .  
14 (b) The following two series

$$\sum_{k=1}^{\infty} b_k^2 e^{2T\lambda_k} \text{ and } \sum_{k=1}^{\infty} \left( \int_0^T e^{t\lambda_k} |f_k(t)| dt \right)^2. \quad (2.1)$$

14 converge.

15 *Proof.* Let  $f(\cdot, t) = \sum_{k=1}^{\infty} f_k(t)\phi_k$  and  $\rho^T = \sum_{k=1}^{\infty} b_k\phi_k$ .

(b)  $\Rightarrow$  (a): Assume that the two series in (2.1) converge. Define

$$\rho(\cdot, t) := \sum_{k=1}^{\infty} e^{(T-t)\lambda_k} b_k \phi_k - \int_t^T \sum_{k=1}^{\infty} e^{(\tau-t)\lambda_k} f_k(\tau) \phi_k d\tau, \quad t \in [0, T].$$

16 In that case, it is easy to verify that  $\rho$  is a strong solution of (1.1).

(a)  $\Rightarrow$  (b): Firstly, let  $\rho$  be a solution of (1.1) with  $f = 0$ . Since  $\rho(\cdot, 0) \in L^2(\Omega)$ , it follows that it  
has an eigenfunction expansion of the form  $\rho(\cdot, 0) = \sum_{k=1}^{\infty} a_k \phi_k$ . Thus, calculating and using (1.15)  
we get that

$$S(T)\rho(\cdot, 0) = \sum_{k=1}^{\infty} e^{-T\lambda_k} a_k \phi_k = \rho^T = \sum_{k=1}^{\infty} b_k \phi_k. \quad (2.2)$$

17 The identity (2.2) implies that  $e^{-T\lambda_k} a_k = b_k$ . Therefore,  $a_k = b_k e^{T\lambda_k}$ . Since  $\rho(\cdot, 0) \in L^2(\Omega)$ , we have  
18 that  $\sum_{k=1}^{\infty} |a_k|^2 < \infty$ . Thus, we can deduce that  $\sum_{k=1}^{\infty} b_k^2 e^{2T\lambda_k} < \infty$ .

Secondly, let  $\rho$  be a solution of (1.1) with  $\rho^T = 0$ . Here also,  $\rho(\cdot, 0) \in L^2(\Omega)$  has an eigenfunction  
expansion of the form  $\rho(\cdot, 0) = \sum_{k=1}^{\infty} a_k \phi_k$ . Therefore, using (1.15) again we obtain

$$S(T)\rho(\cdot, 0) = \sum_{k=1}^{\infty} e^{-T\lambda_k} a_k \phi_k = - \sum_{k=1}^{\infty} \int_0^T e^{-(T-\tau)\lambda_k} f_k(\tau) \phi_k d\tau.$$

This implies that

$$e^{-T\lambda_k} a_k = - \int_0^T e^{-(T-\tau)\lambda_k} f_k(\tau) d\tau.$$

Thus,

$$a_k = - \int_0^T e^{t\lambda_k} f_k(t) dt.$$

1 Since  $\sum_{k=1}^{\infty} |a_k|^2 < \infty$ , we can deduce that the series  $\sum_{k=1}^{\infty} \left( \int_0^T e^{t\lambda_k} f_k(t) dt \right)^2$  converges. The proof  
2 is finished.  $\square$

3 **Remark 2.3.** We observe the following fact that will be useful in the next sections. Let  $f \in L^2(Q)$   
4 and  $\rho^T \in L^2(\Omega)$  satisfy (2.1) so that (1.1) has a strong solution.

(a) Using semigroups theory we can show that  $\rho$  is a strong solution of (1.1) if and only if  $\rho \in L^2((0, T); \mathbb{V}) \cap C([0, T]; L^2(\Omega))$  and the identity

$$- \int_Q \rho \phi_t dx dt + \int_0^T \mathcal{E}(\rho, \phi) dx = - \int_{\Omega} \rho^T(x) \phi(x, T) dx + \int_Q f \phi dx dt \quad (2.3)$$

5 holds for every  $\phi \in \{\phi \in L^2((0, T); \mathbb{V}) : \phi_t \in L^2(Q) \text{ and } \phi(\cdot, 0) = 0 \text{ in } \Omega\}$ .

6 Moreover, we have the following estimates: There is a constant  $C > 0$  such that

$$\|\rho^T\|_{L^2(\Omega)}^2 + \|\rho\|_{L^2((0, T); \mathbb{V})}^2 \leq C \|f\|_{L^2(Q)}^2 + \|\rho(0)\|_{L^2(\Omega)}^2. \quad (2.4)$$

(b) The function  $\rho$  is a strong solution of (1.1) if and only if  $\rho \in L^2((0, T); \mathbb{V}) \cap H^1((0, T); \mathbb{V}^*)$ ,  $\rho(\cdot, T) = \rho^T$  in  $\Omega$ , and the equality

$$\langle \rho_t, \phi \rangle_{\mathbb{V}^*, \mathbb{V}} + \mathcal{E}(\rho, \phi) = \int_{\Omega} f \phi dx \quad (2.5)$$

7 holds for every  $\phi \in \mathbb{V}$  and a.e.  $t \in (0, T)$ .

8 **2.2. Existence and regularity of solutions to the approximated state equation.** In this section  
9 we give several results of existence and regularity of solutions to the approximated state equation (1.4)  
10 and the associated dual equation.

11 Recall that we have approximated (1.1) with (1.4). Here is our notion of solutions of (1.4).

12 **Definition 2.4.** Let  $\varepsilon > 0$ ,  $f \in L^2(Q)$ , and  $\rho^T \in L^2(\Omega)$ . A function  $\rho^\varepsilon \in C([0, T]; L^2(\Omega))$  is said to  
13 be a strong solution of (1.4) if the following assertions hold:

- *Regularity:*

$$\rho_t^\varepsilon(\cdot, t) \in L^2(\Omega) \text{ and } \rho^\varepsilon(\cdot, t) \in D(A) \text{ for a.e. } t \in (0, T)$$

14 and the first equation in (1.4) is satisfied for a.e.  $t \in (0, T)$ .

- *Initial condition:*

$$\varepsilon \rho^\varepsilon(\cdot, 0) + \rho^\varepsilon(\cdot, T) = \rho^T \text{ a.e. in } \Omega. \quad (2.6)$$

15 Next, we introduce the natural candidate for a solution of the approximated problem (1.4). Recall  
16 that  $S = (S(t))_{t \geq 0}$  denotes the strongly continuous, analytic, and submarkovian semigroup on  $L^2(\Omega)$   
17 generated by the operator  $-A$ .



**Definition 2.5.** Let  $f \in L^2(Q)$  and  $\rho^T \in L^2(\Omega)$ . For  $\varepsilon > 0$  and  $t \in [0, T]$ , we let

$$\begin{aligned} \rho^\varepsilon(\cdot, t) := & S(t)(\varepsilon I + S(T))^{-1} \rho^T - \int_0^T S(T+t-\tau)(\varepsilon I + S(T))^{-1} f(\cdot, \tau) d\tau \\ & + \int_0^t S(t-\tau) f(\cdot, \tau) d\tau. \end{aligned} \quad (2.7)$$

**Remark 2.6.** Using the semigroup property we easily get the following identity that will be useful. For  $\varepsilon > 0$  and  $t \in [0, T]$ , let  $\rho^\varepsilon$  be given by (2.7). Then,

$$\begin{aligned} \rho^\varepsilon(\cdot, t) = & S(t)(\varepsilon I + S(T))^{-1} \rho^T - \int_0^t S(t-\tau) \left[ I - S(T)(\varepsilon I + S(T))^{-1} \right] f(\cdot, \tau) d\tau \\ & - \int_t^T S(T+t-\tau)(\varepsilon I + S(T))^{-1} f(\cdot, \tau) d\tau. \end{aligned} \quad (2.8)$$

1 We have the following theorem which is our first main result.

2 **Theorem 2.7.** Let  $f \in L^2(Q)$  and  $\rho^T \in L^2(\Omega)$ . Then, the function  $\rho^\varepsilon$  given in (2.7) is the unique  
3 strong solution of the system (1.4) and it depends continuously on the given data  $\rho^T$  and  $f$ .

4 *Proof.* We proceed in three steps.

5 **Step 1:** Since the semigroup  $S$  is analytic, we have that  $S(t)u \in D(A)$  for all  $u \in L^2(\Omega)$  and  $t > 0$ .  
6 In addition,  $\frac{d}{dt} S(t)u = -AS(t)u$  for all  $t > 0$ , and  $(\varepsilon I + S(T))^{-1} \rho^T \in D(A)$  for every  $\varepsilon > 0$ . Thus,  
7 from semigroups theory, it is well-known that  $\rho^\varepsilon$  has the regularity given in Definition 2.4.

Next, we show that  $\rho^\varepsilon$  satisfies the first equation in (1.4) for a.e.  $t \in (0, T)$ . Taking the time derivative of (2.7) we get that for a.e.  $t \in (0, T)$ ,

$$\begin{aligned} \rho_t^\varepsilon(\cdot, t) = & -AS(t)(\varepsilon I + S(T))^{-1} \rho^T + A \int_0^T S(T+t-\tau)(\varepsilon I + S(T))^{-1} f(\cdot, \tau) d\tau \\ & - A \int_0^t S(t-\tau) f(\cdot, \tau) d\tau + f(\cdot, t) \\ = & -A\rho^\varepsilon(\cdot, t) + f(\cdot, t). \end{aligned}$$

8 Thus,  $\rho^\varepsilon$  satisfies the first equation in (1.4) for a.e.  $t \in (0, T)$ .

Next, we verify that  $\rho^\varepsilon$  satisfies the identity (2.6). Since  $\rho^\varepsilon \in C([0, T]; L^2(\Omega))$  and  $S(0) = I$ , it follows from (2.7) that

$$\varepsilon \rho^\varepsilon(\cdot, 0) = \varepsilon(\varepsilon I + S(T))^{-1} \rho^T - \varepsilon \int_0^T S(T-\tau)(\varepsilon I + S(T))^{-1} f(\cdot, \tau) d\tau. \quad (2.9)$$

Using (2.7) and the semigroup property again, we also get

$$\begin{aligned}
\rho^\varepsilon(\cdot, T) &= S(T)(\varepsilon I + S(T))^{-1} \rho^T - \int_0^T S(2T - \tau)(\varepsilon I + S(T))^{-1} f(\cdot, \tau) d\tau + \int_0^T S(T - \tau) f(\cdot, \tau) d\tau \\
&= S(T)(\varepsilon I + S(T))^{-1} \rho^T - \int_0^T S(T - \tau) S(T)(\varepsilon I + S(T))^{-1} f(\tau) d\tau + \int_0^T S(T - \tau) f(\tau) d\tau \\
&= S(T)(\varepsilon I + S(T))^{-1} \rho^T - \int_0^T S(T - \tau)(S(T) + \varepsilon I - \varepsilon I)(\varepsilon I + S(T))^{-1} f(\cdot, \tau) d\tau \\
&\quad + \int_0^T S(T - \tau) f(\cdot, \tau) d\tau \\
&= S(T)(\varepsilon I + S(T))^{-1} \rho^T - \int_0^T S(T - \tau) f(\tau) d\tau + \varepsilon \int_0^T S(T - \tau)(\varepsilon I + S(T))^{-1} f(\cdot, \tau) d\tau \\
&\quad + \int_0^T S(T - \tau) f(\cdot, \tau) d\tau \\
&= S(T)(\varepsilon I + S(T))^{-1} \rho^T + \varepsilon \int_0^T S(T - \tau)(\varepsilon I + S(T))^{-1} f(\cdot, \tau) d\tau. \tag{2.10}
\end{aligned}$$

Combining (2.9)-(2.10), we get

$$\begin{aligned}
\varepsilon \rho^\varepsilon(\cdot, 0) + \rho^\varepsilon(\cdot, T) &= \varepsilon(\varepsilon I + S(T))^{-1} \rho^T + S(T)(\varepsilon I + S(T))^{-1} \rho^T \\
&= (\varepsilon I + S(T))(\varepsilon I + S(T))^{-1} \rho^T = \rho^T.
\end{aligned}$$

1 Thus,  $\rho^\varepsilon$  satisfies (2.6) and is a strong solution of (1.4) in the sense of Definition 2.4.

**Step 2:** Next, we show the continuous dependence of  $\rho^\varepsilon$  on the data  $\rho^T$  and  $f$ . Using (1.16) and the fact that  $S$  is a semigroup of contraction, we get from (2.7) the following estimates for every  $t \in [0, T]$ :

$$\begin{aligned}
\|\rho^\varepsilon(\cdot, t)\|_{L^2(\Omega)} &\leq \|(\varepsilon I + S(T))^{-1} \rho^T\|_{L^2(\Omega)} + \int_0^T \|(\varepsilon I + S(T))^{-1} f(\cdot, \tau)\|_{L^2(\Omega)} d\tau \\
&\quad + \int_0^t \|f(\cdot, \tau)\|_{L^2(\Omega)} d\tau \\
&\leq \frac{1}{\varepsilon} \|\rho^T\|_{L^2(\Omega)} + \frac{1}{\varepsilon} \|f\|_{L^2(Q)} + \|f\|_{L^2(Q)}. \tag{2.11}
\end{aligned}$$

2 Thus,  $\rho^\varepsilon$  depends continuously on the given data  $f$  and  $\rho^T$ .

3 **Step 3:** Finally we show uniqueness. Since the system is linear, it follows from the continuous  
4 dependence on the data given in (2.11) that, if  $f = 0$  in  $Q$  and  $\rho^T = 0$  in  $\Omega$ , then  $\rho^\varepsilon = 0$  in  $Q$ . We  
5 have shown the uniqueness of solutions and the proof is finished.  $\square$

6 **Remark 2.8.** We observe the following.

(a) Since  $\rho^\varepsilon \in C([0, T]; L^2(\Omega))$ , it follows from (2.11) that

$$\|\rho^\varepsilon\|_{C([0, T]; L^2(\Omega))} \leq \frac{1}{\varepsilon} \|\rho^T\|_{L^2(\Omega)} + \left(\frac{1}{\varepsilon} + 1\right) \|f\|_{L^2(Q)}. \tag{2.12}$$

(b) It follows from (2.12) that for every  $0 < \varepsilon \leq 1$ , we have

$$\|\varepsilon \rho^\varepsilon(\cdot, 0)\|_{L^2(\Omega)} \leq \|\rho^T\|_{L^2(\Omega)} + 2\|f\|_{L^2(Q)}. \tag{2.13}$$

(c) It follows from (2.6) and (2.13) that for every  $0 < \varepsilon \leq 1$ ,

$$\|\rho^T - \rho^\varepsilon(\cdot, T)\|_{L^2(\Omega)} = \|\varepsilon \rho^\varepsilon(\cdot, 0)\|_{L^2(\Omega)} \leq \|\rho^T\|_{L^2(\Omega)} + 2\|f\|_{L^2(Q)}. \quad (2.14)$$

1 Next, we give some fine estimates of solutions.

**Proposition 2.9.** *Let  $f \in L^2(Q)$  and  $\rho^T \in L^2(\Omega)$ . Then for every  $\varepsilon > 0$  and  $t \in [0, T]$ , we have that*

$$\begin{aligned} \|\rho^\varepsilon(\cdot, t)\|_{L^2(\Omega)}^2 &\leq \varepsilon^{\frac{2(t-T)}{T}} \|\rho^T\|_{L^2(\Omega)}^2 + \|I - S(T)(\varepsilon I + S(T))^{-1}\|^2 \|f\|_{L^2(Q)}^2 \\ &\quad + \int_t^T \|S(T+t-\tau)(\varepsilon I + S(T))^{-1}\|^2 \|f(\cdot, \tau)\|_{L^2(\Omega)}^2 d\tau. \end{aligned} \quad (2.15)$$

*Proof.* Since  $\rho^T \in L^2(\Omega)$ , it follows that  $\rho^T$  has the expansion  $\rho^T = \sum_{k=1}^{\infty} b_k \phi_k$  and  $\|\rho^T\|_{L^2(\Omega)}^2 = \sum_{k=1}^{\infty} b_k^2$  with  $b_k = (\rho^T, \phi_k)_{L^2(\Omega)}$ . Using (2.8) and making some calculations we get that

$$\begin{aligned} \|\rho^\varepsilon(\cdot, t)\|_{L^2(\Omega)}^2 &\leq \sum_{k=1}^{\infty} e^{-2t\lambda_k} b_k^2 (\varepsilon + e^{-T\lambda_k})^{-2} + \|I - S(T)(\varepsilon I + S(T))^{-1}\|^2 \|f\|_{L^2(Q)}^2 \\ &\quad + \int_t^T \|S(T+t-\tau)(\varepsilon I + S(T))^{-1}\|^2 \|f(\cdot, \tau)\|_{L^2(\Omega)}^2 d\tau \\ &\leq \sum_{k=1}^{\infty} e^{-2t\lambda_k} b_k^2 \left[ (\varepsilon + e^{-T\lambda_k})^{\frac{t}{T}} (\varepsilon + e^{-T\lambda_k})^{1-\frac{t}{T}} \right]^{-2} \\ &\quad + \|I - S(T)(\varepsilon I + S(T))^{-1}\|^2 \|f\|_{L^2((0,T) \times \Omega)}^2 \\ &\quad + \int_t^T \|S(T+t-\tau)(\varepsilon I + S(T))^{-1}\|^2 \|f(\cdot, \tau)\|_{L^2(\Omega)}^2 d\tau \\ &\leq \sum_{k=1}^{\infty} b_k^2 \left( \varepsilon^{\frac{t-T}{T}} \right)^2 + \|I - S(T)(\varepsilon I + S(T))^{-1}\|^2 \|f\|_{L^2(Q)}^2 \\ &\quad + \int_t^T \|S(T+t-\tau)(\varepsilon I + S(T))^{-1}\|^2 \|f(\cdot, \tau)\|_{L^2(\Omega)}^2 d\tau \\ &\leq \left( \varepsilon^{\frac{t-T}{T}} \right)^2 \sum_{k=1}^{\infty} b_k^2 + \|I - S(T)(\varepsilon I + S(T))^{-1}\|^2 \|f\|_{L^2(Q)}^2 \\ &\quad + \int_t^T \|S(T+t-\tau)(\varepsilon I + S(T))^{-1}\|^2 \|f(\cdot, \tau)\|_{L^2(\Omega)}^2 d\tau. \end{aligned}$$

2 We have shown (2.15) and the proof is finished.  $\square$

3 Next, we show further regularity results and fines estimates of solutions that will be useful in the  
4 next sections.

5 **Theorem 2.10.** *Let  $f \in L^2(Q)$ ,  $\rho^T \in L^2(\Omega)$ , and let  $\rho^\varepsilon$  be the unique strong solution of (1.4). Then  
6  $\rho^\varepsilon \in L^2((0, T); \mathbb{V}) \cap C([0, T]; L^2(\Omega))$  and there is a constant  $C > 0$  independent of  $\varepsilon$  such that*

$$\|\rho^\varepsilon\|_{C([0,T]; L^2(\Omega))}^2 + \|\rho^\varepsilon\|_{L^2((0,T); \mathbb{V})}^2 \leq C \left( \|f\|_{L^2(Q)}^2 + \frac{1}{\varepsilon^2} \|\rho^T\|_{L^2(\Omega)}^2 + \frac{1}{\varepsilon^2} \|f\|_{L^2(Q)}^2 \right). \quad (2.16)$$

*Proof.* Notice that we already know that  $\rho^\varepsilon \in C([0, T]; L^2(\Omega))$ ,  $\rho_t^\varepsilon(\cdot, t) \in L^2(\Omega)$ , and  $\rho^\varepsilon(\cdot, t) \in D(A) \subset \mathbb{V}$  for a.e.  $t \in (0, T)$ . Multiplying the first equation in (1.4) by  $\rho^\varepsilon$ , integrating over  $\Omega$ , taking into

account (1.14) and (1.13), and using the Hölder and Young inequalities, we obtain that there is a constant  $C_0 > 0$  (independent of  $\varepsilon$ ) such that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\rho^\varepsilon(\cdot, t)\|_{L^2(\Omega)}^2 + \mathcal{E}(\rho^\varepsilon(\cdot, t), \rho^\varepsilon(\cdot, t)) &= \int_{\Omega} f \rho^\varepsilon \, dx \\ &\leq \frac{1}{2\delta} \|f(\cdot, t)\|_{L^2(\Omega)}^2 + \frac{\delta}{2} \|\rho^\varepsilon(t, \cdot)\|_{L^2(\Omega)}^2 \\ &\leq \frac{1}{2\delta} \|f(\cdot, t)\|_{L^2(\Omega)}^2 + \frac{C_0 \delta}{2} \|\rho^\varepsilon(t, \cdot)\|_{\mathbb{V}}^2, \end{aligned}$$

- 1 for every  $\delta > 0$ . Noticing that  $\mathcal{E}(\rho^\varepsilon(\cdot, t), \rho^\varepsilon(\cdot, t)) = \|\rho^\varepsilon(\cdot, t)\|_{\mathbb{V}}^2$  and choosing  $\delta = \frac{1}{C_0}$  in the preceding  
2 identity, we can deduce that

$$\frac{1}{2} \frac{d}{dt} \|\rho^\varepsilon(\cdot, t)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\rho^\varepsilon(\cdot, t)\|_{\mathbb{V}}^2 \leq \frac{C_0}{2} \|f(\cdot, t)\|_{L^2(\Omega)}^2. \quad (2.17)$$

Now, integrating (2.17) over  $(0, s)$  with  $s \in [0, T]$ , we obtain that there is a constant  $C_0 > 0$  independent of  $\varepsilon$  such that

$$\frac{1}{2} \|\rho^\varepsilon(\cdot, s)\|_{L^2(\Omega)}^2 + \frac{1}{2} \int_0^s \|\rho^\varepsilon(\cdot, t)\|_{\mathbb{V}}^2 \, dt \leq \frac{C_0}{2} \|f\|_{L^2(Q)}^2 + \frac{1}{2} \|\rho^\varepsilon(\cdot, 0)\|_{L^2(\Omega)}^2, \quad (2.18)$$

which in view of (2.13) implies that there exists a constant  $C > 0$  independent of  $\varepsilon$  such that

$$\|\rho^\varepsilon(\cdot, s)\|_{L^2(\Omega)}^2 + \int_0^s \|\rho^\varepsilon(\cdot, t)\|_{\mathbb{V}}^2 \, dt \leq C \left( \|f\|_{L^2(Q)}^2 + \frac{1}{\varepsilon^2} \|\rho^T\|_{L^2(\Omega)}^2 + \frac{1}{\varepsilon^2} \|f\|_{L^2(Q)}^2 \right),$$

for every  $s \in [0, T]$ . Hence, we can deduce that

$$\sup_{s \in [0, T]} \|\rho^\varepsilon(\cdot, s)\|_{L^2(\Omega)}^2 + \int_0^T \|\rho^\varepsilon(\cdot, t)\|_{\mathbb{V}}^2 \, dt \leq C \left( \|f\|_{L^2(Q)}^2 + \frac{1}{\varepsilon^2} \|\rho^T\|_{L^2(\Omega)}^2 + \frac{1}{\varepsilon^2} \|f\|_{L^2(Q)}^2 \right).$$

- 3 We have shown (2.16) and the proof is finished.  $\square$

4 **Remark 2.11.** We observe that using semigroups theory, we can show the following.

- (a) A function  $\rho^\varepsilon$  is a strong solution of (1.4) in the sense of Definition 2.4 if and only if  $\rho^\varepsilon \in L^2((0, T); \mathbb{V}) \cap C([0, T]; L^2(\Omega))$  and the identity

$$\begin{aligned} - \int_Q \rho^\varepsilon \phi_t \, dxdt + \int_0^T \mathcal{E}(\rho^\varepsilon, \phi) \, dt + \int_{\Omega} \rho^\varepsilon(x, T) \phi(x, T) \, dx \\ = \int_Q f \phi \, dxdt + \int_{\Omega} \rho^\varepsilon(x, 0) \phi(x, 0) \, dx \end{aligned} \quad (2.19)$$

holds for every  $\phi \in H(Q) := \{z \in L^2((0, T); \mathbb{V}) : z_t \in L^2(Q)\}$ , or equivalently

$$\begin{aligned} - \int_Q \rho^\varepsilon \phi_t \, dxdt + \int_0^T \mathcal{E}(\rho^\varepsilon, \phi) \, dt + \int_{\Omega} \rho^\varepsilon(x, T) \phi(x, T) \, dx \\ = \int_Q f \phi \, dxdt + \frac{1}{\varepsilon} \int_{\Omega} (\rho^T - \rho^\varepsilon(x, T)) \phi(x, 0) \, dx \end{aligned} \quad (2.20)$$

- 5 holds for every  $\phi \in H(Q)$ . Moreover, if we set

$$\rho^\varepsilon = \mathcal{J}_\varepsilon(f, \rho^T), \quad (2.21)$$

- 6 then, it follows from Theorem 2.10 that the map:  $(f, \rho^T) \mapsto \mathcal{J}_\varepsilon(f, \rho^T)$  is bilinear and continuous  
7 from  $L^2(Q) \times L^2(\Omega)$  into  $L^2((0, T); \mathbb{V}) \cap C([0, T]; L^2(\Omega))$ .

(b) A function  $\rho^\varepsilon$  is a strong solution of (1.4) in the sense of Definition 2.4 if and only if  $\rho^\varepsilon \in L^2((0, T); \mathbb{V}) \cap H^1((0, T); \mathbb{V}^*)$  and the identity

$$\langle \rho_t^\varepsilon, \phi \rangle_{\mathbb{V}^*, \mathbb{V}} + \mathcal{E}(\rho^\varepsilon, \phi) = \int_{\Omega} f \phi \, dx \quad (2.22)$$

holds for every  $\phi \in \mathbb{V}$  and a.e.  $t \in [0, T]$ .

The study of the following system

$$\begin{cases} -p_t^\varepsilon + Ap^\varepsilon = 0 & \text{in } Q, \\ \varepsilon p^\varepsilon(\cdot, T) + p^\varepsilon(\cdot, 0) = p^T & \text{in } \Omega. \end{cases} \quad (2.23)$$

which can be viewed as the dual system associated with the approximated problem (1.4) is useful for the approximated optimal control.

**Definition 2.12.** Let  $p^T \in L^2(\Omega)$ . A function  $p^\varepsilon \in C([0, T]; L^2(\Omega)) \cap C^1((0, T); L^2(\Omega)) \cap C((0, T); D(A))$  is said to be a strong solution of (2.23) if the following assertions hold:

- *Regularity:*

$$p^\varepsilon \in C([0, T]; L^2(\Omega)) \cap C^1((0, T); L^2(\Omega)) \cap C((0, T); D(A))$$

and the first equation in (2.23) holds for a.e.  $t \in (0, T)$ .

- *Initial condition:*

$$\varepsilon p^\varepsilon(\cdot, T) + p^\varepsilon(\cdot, 0) = p^T \text{ a.e. in } \Omega.$$

We have the following existence result.

**Proposition 2.13.** Let  $p^T \in L^2(\Omega)$ . Then, there exists a unique strong solution  $p^\varepsilon$  to (2.23) in the sense of Definition 2.12. In addition,  $p^\varepsilon \in L^2((0, T); \mathbb{V}) \cap C([0, T]; L^2(\Omega))$  and there is a constant  $C > 0$  independent of  $\varepsilon$  such that

$$\|p^\varepsilon\|_{C([0, T]; L^2(\Omega))}^2 + \|p^\varepsilon\|_{L^2((0, T); \mathbb{V})}^2 \leq \frac{C}{\varepsilon^2} \|p^T\|_{L^2(\Omega)}^2. \quad (2.24)$$

*Proof.* Making the change of variable  $t \mapsto T - t$  in (2.23), we obtain that  $\psi^\varepsilon(T - t, x) := p^\varepsilon(t, x)$  solves

$$\begin{cases} \psi_t^\varepsilon + A\psi^\varepsilon = 0 & \text{in } Q, \\ \varepsilon \psi^\varepsilon(\cdot, 0) + \psi^\varepsilon(\cdot, T) = p^T & \text{in } \Omega. \end{cases}$$

That is,  $\psi^\varepsilon$  is a strong solution of (1.4) with  $f = 0$  and  $\rho^T = p^T$ . That is, for all  $t \in [0, T]$  we have

$$p^\varepsilon(\cdot, t) = \psi^\varepsilon(T - t, \cdot) = S(T - t)(\varepsilon I + S(T))^{-1} p^T.$$

Using the analyticity of the semigroup  $S$  we can easily verify that  $p^\varepsilon$  has the required regularity. Proceeding as in Theorem 2.7 we can easily deduce that  $p^\varepsilon$  satisfies the first equation in (2.23) for a.e.  $t \in (0, T)$ . In addition, we have that

$$\begin{aligned} \varepsilon p^\varepsilon(\cdot, T) + p^\varepsilon(\cdot, 0) &= \varepsilon \psi^\varepsilon(\cdot, 0) + \psi^\varepsilon(\cdot, T) \\ &= \varepsilon S(T)(\varepsilon I + S(T))^{-1} p^T + (\varepsilon I + S(T))^{-1} p^T = p^T. \end{aligned}$$

We can also deduce from Theorems 2.7 and 2.10 that  $p^\varepsilon \in L^2((0, T); \mathbb{V}) \cap C([0, T]; L^2(\Omega))$ . From the estimates (2.12) and (2.16) we get that there is a constant  $C > 0$  (independent of  $\varepsilon$ ) such that

$$\sup_{t \in [0, T]} \|p^\varepsilon(\cdot, t)\|_{L^2(\Omega)}^2 + \|p^\varepsilon\|_{L^2((0, T); \mathbb{V})}^2 \leq \frac{C}{\varepsilon^2} \|p^T\|_{L^2(\Omega)}^2.$$

We have shown (2.24) and the proof is finished.  $\square$

1 **2.3. Convergence of solutions.** The main concern of this section is to show that solutions of the  
 2 approximated problem (1.4) converge to solutions of the original problem (1.1). We also want to derive  
 3 the convergence rate.

4 The following convergence result is our second main result.

**Theorem 2.14.** *Let  $f \in L^2(Q)$ ,  $\rho^T \in L^2(\Omega)$ , and  $\rho^\varepsilon$  be given by (2.7). Then*

$$\lim_{\varepsilon \downarrow 0} \|\rho^\varepsilon(\cdot, T) - \rho^T\|_{L^2(\Omega)} = 0. \quad (2.25)$$

*Proof.* Let  $\rho^T = \sum_{k=1}^{\infty} b_k \phi_k \in L^2(\Omega)$  and  $f \in L^2(Q)$ . Using (2.7) we get the following estimates:

$$\begin{aligned} \|\rho^\varepsilon(\cdot, T) - \rho^T\|_{L^2(\Omega)}^2 &\leq \|S(T)(\varepsilon I + S(T))^{-1} \rho^T - \rho^T\|_{L^2(\Omega)}^2 \\ &\quad + \int_0^T \left\| S(T-\tau) \left[ I - S(T)(\varepsilon I + S(T))^{-1} \right] \right\|^2 \|f(\cdot, \tau)\|_{L^2(\Omega)}^2 d\tau \\ &\leq \varepsilon^2 \|(\varepsilon I + S(T))^{-1} \rho^T\|_{L^2(\Omega)}^2 + \left\| I - S(T)(\varepsilon I + S(T))^{-1} \right\|^2 \|f\|_{L^2(Q)}^2 \\ &= \sum_{k=1}^{\infty} \varepsilon^2 b_k^2 (\varepsilon + e^{-T\lambda_k})^{-2} + \left\| I - S(T)(\varepsilon I + S(T))^{-1} \right\|^2 \|f\|_{L^2(Q)}^2. \end{aligned} \quad (2.26)$$

Using the dominated convergence theorem, it is easily seen that

$$\lim_{\varepsilon \downarrow 0} \left\| I - S(T)(\varepsilon I + S(T))^{-1} \right\| = 0. \quad (2.27)$$

Next, fix  $\delta > 0$  but otherwise arbitrary. Choose  $N \in \mathbb{N}$  large enough such that  $\sum_{k=N+1}^{\infty} b_k^2 < \frac{\delta}{2}$ . Then,

$$\sum_{k=1}^{\infty} \varepsilon^2 b_k^2 (\varepsilon + e^{-T\lambda_k})^{-2} \leq \sum_{k=1}^N \varepsilon^2 b_k^2 (\varepsilon + e^{-T\lambda_k})^{-2} + \frac{\delta}{2} \leq \varepsilon^2 \sum_{k=1}^N b_k^2 e^{2T\lambda_k} + \frac{\delta}{2}.$$

Now, let  $\varepsilon > 0$  be such that

$$\varepsilon^2 < \frac{\delta}{4} \left( \sum_{k=1}^N b_k^2 e^{2T\lambda_k} \right)^{-1}.$$

This implies that

$$\sum_{k=1}^{\infty} \varepsilon^2 b_k^2 (\varepsilon + e^{-T\lambda_k})^{-2} < \delta.$$

We have shown that

$$\lim_{\varepsilon \downarrow 0} \sum_{k=1}^{\infty} \varepsilon^2 b_k^2 (\varepsilon + e^{-T\lambda_k})^{-2} = 0. \quad (2.28)$$

5 Combining (2.27)-(2.28) and using (2.26), we get (2.25) and the proof is finished.  $\square$

6 Next, we have the following finer convergence result. In particular, it gives a necessary and sufficient  
 7 condition in terms of solutions of the approximated problem (1.4) such that the non well-posed problem  
 8 (1.1) has a strong solution.

**Theorem 2.15.** *Let  $f \in L^2(Q)$ ,  $\rho^T \in L^2(\Omega)$ , and  $\rho^\varepsilon$  be given by (2.7). Then, the system (1.1) has a strong solution  $\rho$  if and only if the sequence  $\{\rho^\varepsilon(\cdot, 0)\}$  converges in  $L^2(\Omega)$ , as  $\varepsilon \downarrow 0$ . In addition, as  $\varepsilon \downarrow 0$ , we have that*

$$\rho^\varepsilon(\cdot, t) \text{ converges strongly to } \rho \text{ in } L^2(\Omega), \text{ uniformly in } t \in [0, T], \quad (2.29a)$$

$$\rho^\varepsilon \text{ converges strongly to } \rho \text{ in } L^2((0, T), \mathbb{V}). \quad (2.29b)$$

*Proof.* If the system (1.1) has a strong solution  $\rho$ , then  $\rho \in C([0, T]; L^2(\Omega))$ . Hence  $\rho(\cdot, 0)$  exists and belongs to  $L^2(\Omega)$ . Assume that  $\lim_{\varepsilon \downarrow 0} \rho^\varepsilon(\cdot, 0) := \rho^0 = \rho(\cdot, 0)$ . Define

$$\rho(\cdot, t) := S(t)\rho^0 + \int_0^t S(t-\tau)f(\cdot, \tau) d\tau. \quad (2.30)$$

1 Recall that by (2.7) or (2.8) we have that

$$\rho^\varepsilon(\cdot, 0) = (\varepsilon I + S(T))^{-1} \rho^T - \int_0^T S(T-\tau)(\varepsilon I + S(T))^{-1} f(\cdot, \tau) d\tau. \quad (2.31)$$

Using (2.8) and the semigroup property we get

$$\begin{aligned} \rho^\varepsilon(\cdot, t) - \rho(\cdot, t) &= S(t) (\varepsilon I + S(T))^{-1} \rho^T + \int_0^t S(t-\tau) \left[ I - S(T) (\varepsilon I + S(T))^{-1} \right] f(\cdot, \tau) d\tau \\ &\quad - \int_\tau^T S(T+t-\tau) (\varepsilon I + S(T))^{-1} f(\cdot, \tau) d\tau - S(t)\rho^0 \\ &\quad - \int_0^t S(T-\tau)f(\cdot, \tau) d\tau \\ &= S(t) (\varepsilon I + S(T))^{-1} \rho^T - S(t)\rho^0 \\ &\quad - \int_0^T S(T+t-\tau) (\varepsilon I + S(T))^{-1} f(\cdot, \tau) d\tau \\ &= S(t) \left[ \left( (\varepsilon I + S(T))^{-1} - \int_0^T S(T-\tau) (\varepsilon I + S(T))^{-1} f(\cdot, \tau) d\tau \right) - \rho^0 \right], \end{aligned}$$

which in view of (2.31) gives

$$\rho^\varepsilon(\cdot, t) - \rho(\cdot, t) = S(t) \left[ \rho^\varepsilon(\cdot, 0) - \rho^0 \right].$$

Since  $\lim_{\varepsilon \downarrow 0} \|\rho^\varepsilon(\cdot, T) - \rho^T\|_{L^2(\Omega)} = 0$  (by (2.25)) and

$$\lim_{\varepsilon \downarrow 0} \|\rho^\varepsilon(\cdot, t) - \rho(\cdot, t)\|_{L^2(\Omega)} = \lim_{\varepsilon \downarrow 0} \|S(t)(\rho^0 - \rho^\varepsilon(\cdot, 0))\|_{L^2(\Omega)} \leq \lim_{\varepsilon \downarrow 0} \|\rho^0 - \rho^\varepsilon(\cdot, 0)\|_{L^2(\Omega)} = 0,$$

2 we can deduce that  $\rho(\cdot, T) = \rho^T$  in  $\Omega$  and the function  $\rho$  given by (2.30) solves the problem (1.1). In  
3 addition, we have already shown that  $\rho^\varepsilon(\cdot, t)$  converges to  $\rho(\cdot, t)$  uniformly in  $t \in [0, T]$ .

Conversely, assume that  $\rho$  is a solution of (1.1). Let  $f \in L^2(Q)$  and  $\rho^T \in L^2(\Omega)$  have the expansions  $f(\cdot, t) = \sum_{k=1}^{\infty} f_k(t)\phi_k$  and  $\rho^T = \sum_{k=1}^{\infty} b_k\phi_k$ . It follows from Lemma 2.2 that

$$\|\rho(\cdot, 0)\|_{L^2(\Omega)}^2 \leq \sum_{k=1}^{\infty} b_k^2 e^{2T\lambda_k} + \sum_{k=1}^{\infty} \left( \int_0^T e^{\lambda_k \tau} f_k(\tau) d\tau \right)^2.$$

Let  $\delta > 0$  be given. Choose  $N \in \mathbb{N}$  large enough such that

$$\sum_{k=N+1}^{\infty} e^{2T\lambda_k} b_k^2 + \int_0^T \sum_{k=N+1}^{\infty} e^{2\tau\lambda_k} |f_k(\tau)|^2 d\tau < \frac{\delta}{2}. \quad (2.32)$$

Let  $\varepsilon, \eta > 0$ . Calculating, we have that

$$\begin{aligned} \rho^\varepsilon(\cdot, 0) - \rho^\eta(\cdot, 0) &= [(\varepsilon I + S(T))^{-1} - (\eta + S(T))^{-1}] \rho^T \\ &\quad + \int_0^T S(T - \tau) [(\varepsilon I + S(T))^{-1} - (\eta + S(T))^{-1}] f(\cdot, \tau) d\tau. \end{aligned}$$

This identity implies that we have the following estimates:

$$\begin{aligned} \|\rho^\varepsilon(\cdot, 0) - \rho^\eta(\cdot, 0)\|_{L^2(\Omega)}^2 &\leq \left\| [(\varepsilon I + S(T))^{-1} - (\eta + S(T))^{-1}] \rho^T \right\|_{L^2(\Omega)}^2 \\ &\quad + \int_0^T \left\| [(\varepsilon I + S(T))^{-1} - (\eta + S(T))^{-1}] f(\cdot, \tau) \right\|_{L^2(\Omega)}^2 d\tau \\ &\leq \sum_{k=1}^{\infty} (\eta - \varepsilon)^2 \left( \varepsilon\eta + (\varepsilon + \eta)e^{-T\lambda_k} + e^{-2T\lambda_k} \right)^{-2} b_k^2 \\ &\quad + \int_0^T \sum_{k=1}^{\infty} (\eta - \varepsilon)^2 \left( \varepsilon\eta + (\varepsilon + \eta)e^{-T\lambda_k} + e^{-2T\lambda_k} \right)^{-2} |f_k(\tau)|^2 d\tau \\ &= \sum_{k=1}^N (\eta - \varepsilon)^2 \left( \varepsilon\eta + (\varepsilon + \eta)e^{-T\lambda_k} + e^{-2T\lambda_k} \right)^{-2} b_k^2 \\ &\quad + \sum_{k=N+1}^{\infty} (\eta - \varepsilon)^2 \left( \varepsilon\eta + (\varepsilon + \eta)e^{-T\lambda_k} + e^{-2T\lambda_k} \right)^{-2} b_k^2 \\ &\quad + \int_0^T \sum_{k=1}^N (\eta - \varepsilon)^2 \left( \varepsilon\eta + (\varepsilon + \eta)e^{-T\lambda_k} + e^{-2T\lambda_k} \right)^{-2} |f_k(\tau)|^2 d\tau \\ &\quad + \int_0^T \sum_{k=N+1}^{\infty} (\eta - \varepsilon)^2 \left( \varepsilon\eta + (\varepsilon + \eta)e^{-T\lambda_k} + e^{-2T\lambda_k} \right)^{-2} |f_k(\tau)|^2 d\tau \\ &\leq \sum_{k=1}^N (\eta - \varepsilon)^2 e^{4T\lambda_k} b_k^2 + \sum_{k=N+1}^{\infty} \left( \frac{\eta - \varepsilon}{\varepsilon + \eta} \right)^2 e^{2T\lambda_k} b_k^2 \\ &\quad + \int_0^T \sum_{k=1}^N (\eta - \varepsilon)^2 e^{4T\lambda_k} f_k(\tau)^2 d\tau + \int_0^T \sum_{k=N+1}^{\infty} \left( \frac{\eta - \varepsilon}{\varepsilon + \eta} \right)^2 e^{2T\lambda_k} |f_k(\tau)|^2 d\tau \\ &\leq \sum_{k=1}^N (\eta - \varepsilon)^2 e^{4T\lambda_k} b_k^2 + \int_0^T \sum_{k=1}^N (\eta - \varepsilon)^2 e^{4T\lambda_k} |f_k(\tau)|^2 d\tau + \frac{\delta}{2}. \quad (2.33) \end{aligned}$$

Now choosing  $\gamma > 0$  such that

$$\gamma^2 < \delta \left( \sum_{k=1}^N (\eta - \varepsilon)^2 e^{4T\lambda_k} b_k^2 + \int_0^T \sum_{k=1}^N (\eta - \varepsilon)^2 e^{4T\lambda_k} |f_k(\tau)|^2 d\tau \right)^{-1} \quad (2.34)$$



and requiring that  $\varepsilon$  and  $\eta$  are less than  $\gamma$ , we can deduce from (2.32), (2.33) and (2.34) that

$$\|\rho^\varepsilon(\cdot, 0) - \rho^\eta(\cdot, 0)\|_{L^2(\Omega)}^2 < \delta.$$

1 We have shown that  $\{\rho^\varepsilon(\cdot, 0)\}_\varepsilon$  is a Cauchy sequence in  $L^2(\Omega)$ , hence it converges.

2 Now set  $z^\varepsilon = \rho^\varepsilon - \rho$ , where  $\rho^\varepsilon$  and  $\rho$  are respectively solutions of (1.4) and (1.1). Then,  $z^\varepsilon$  is a  
3 solution of

$$\begin{cases} z_t^\varepsilon + Az^\varepsilon = 0 & \text{in } Q := (0, T) \times \Omega, \\ \varepsilon \rho^\varepsilon(\cdot, 0) + z^\varepsilon(\cdot, T) = 0 & \text{in } \Omega. \end{cases} \quad (2.35)$$

Since  $\rho^\varepsilon, \rho \in L^2((0, T); \mathbb{V}) \cap C([0, T]; L^2(\Omega))$ , if we multiply the first equation in (2.35) by  $z^\varepsilon$ , integrate by part over  $Q$  and take into account (1.13), (1.14) (1.11) and (1.19), we have that

$$\frac{1}{2} \|z^\varepsilon(\cdot, T)\|_{L^2(\Omega)}^2 + \|z^\varepsilon\|_{L^2((0, T); \mathbb{V})}^2 = \frac{1}{2} \|z^\varepsilon(\cdot, 0)\|_{L^2(\Omega)}^2,$$

which in view of the fact that  $z^\varepsilon = \rho^\varepsilon - \rho$  and  $z^\varepsilon(\cdot, T) = \varepsilon \rho^\varepsilon(\cdot, 0)$  is equivalent to

$$\frac{\varepsilon^2}{2} \|\rho^\varepsilon(\cdot, 0)\|_{L^2(\Omega)}^2 + \|\rho^\varepsilon - \rho\|_{L^2((0, T); \mathbb{V})}^2 = \frac{1}{2} \|\rho^\varepsilon(\cdot, 0) - \rho(\cdot, 0)\|_{L^2(\Omega)}^2.$$

Thus

$$\|\rho^\varepsilon - \rho\|_{L^2((0, T); \mathbb{V})}^2 \leq \frac{1}{2} \|\rho^\varepsilon(\cdot, 0) - \rho(\cdot, 0)\|_{L^2(\Omega)}^2. \quad (2.36)$$

4 Since  $\rho^\varepsilon(\cdot, t)$  converges to  $\rho$ , as  $\varepsilon \downarrow 0$ , uniformly in  $t \in [0, T]$ , we can deduce from (2.36) that  $\rho^\varepsilon$   
5 converges to  $\rho$  in  $L^2((0, T), \mathbb{V})$ , as  $\varepsilon \downarrow 0$ . The proof is finished.  $\square$

6 Next, we give the rate of convergence.

7 **Theorem 2.16.** *Let  $f \in L^2(Q)$  and  $\rho^T \in L^2(\Omega)$  have the expansions*

$$f(\cdot, t) = \sum_{k=1}^{\infty} f_k(t) \phi_k \quad \text{and} \quad \rho^T = \sum_{k=1}^{\infty} b_k \phi_k. \quad (2.37)$$

*If there exists  $\gamma > 0$  such that the series*

$$\sum_{k=1}^{\infty} b_k^2 e^{\gamma T \lambda_k} \quad \text{and} \quad \sum_{k=1}^{\infty} \int_0^T e^{\gamma t \lambda_k} |f_k(t)|^2 dt \quad (2.38)$$

8 *converge, then  $\|\rho^\varepsilon(\cdot, T) - \rho^T\|_{L^2(\Omega)}$  converges to zero with rate  $\varepsilon^\gamma \gamma^{-2}$ .*

9 *Proof.* Firstly, we notice that if (2.38) holds, then (2.1) holds so that (1.1) has a strong solution.

Secondly, let  $\gamma \in (0, 2)$  be such that (2.38) holds. Let  $j \in (0, 2)$ . Fix  $n \in \mathbb{N}$  and define the function  $g_n(\varepsilon) := \frac{\varepsilon^j}{(\varepsilon + e^{-T\lambda_n})^2}$ . Then,  $g'_n(\varepsilon) = \varepsilon^{j-1} \frac{(j-2)\varepsilon + j e^{-T\lambda_n}}{(\varepsilon + e^{-T\lambda_n})^3}$ . Thus,  $g'_n(\varepsilon) = 0$  when  $\varepsilon = 0$  or  $\varepsilon = \frac{j}{j-2} e^{-T\lambda_n}$ . Since

$$g_n(\varepsilon) > 0, \quad g_n(0) = 0, \quad \text{and} \quad \lim_{\varepsilon \rightarrow \infty} g_n(\varepsilon) = 0,$$

it follows that  $g_n$  achieves its maximum at the point  $\varepsilon_0 := \frac{j}{j-2}e^{-T\lambda_n}$ . Thus, we can deduce that  $g_n(\varepsilon) \leq \frac{\left(\frac{j}{2-j}\right)^j e^{-jT\lambda_n}}{(\varepsilon_0 + e^{-T\lambda_n})^2}$ . Calculating, we get the following estimates:

$$\begin{aligned}
\|\rho^\varepsilon(\cdot, T) - \rho^T\|_{L^2(\Omega)}^2 &\leq 2 \sum_{n=1}^{\infty} b_n^2 \varepsilon^2 (\varepsilon + e^{-T\lambda_n})^{-2} + 2 \sum_{n=1}^{\infty} \int_0^T e^{-2(T-\tau)\lambda_n} \left(1 - \frac{e^{-T\lambda_n}}{\varepsilon + e^{-T\lambda_n}}\right)^{-2} |f_k(\tau)|^2 d\tau \\
&\leq 2 \sum_{n=1}^{\infty} b_n^2 \varepsilon^2 (\varepsilon + e^{-T\lambda_n})^{-2} + 2 \sum_{n=1}^{\infty} \int_0^T e^{-2(T-\tau)\lambda_n} \left(\frac{\varepsilon}{\varepsilon + e^{-T\lambda_n}}\right)^2 |f_k(\tau)|^2 d\tau \\
&\leq 2\varepsilon^{2-j} \sum_{n=1}^{\infty} b_n^2 \left(\frac{j}{2-j}\right)^j e^{-jT\lambda_n} (\varepsilon_0 + e^{-T\lambda_n})^{-2} \\
&\quad + 2 \sum_{n=1}^{\infty} \varepsilon^{2-j} \int_0^T \left(\frac{j}{2-j}\right)^j e^{-jT\lambda_n} e^{-2(T-\tau)\lambda_n} (\varepsilon_0 + e^{-T\lambda_n})^{-2} |f_k(\tau)|^2 d\tau \\
&\leq 2\varepsilon^{2-j} \sum_{n=1}^{\infty} b_n^2 \left(\frac{j}{2-j}\right)^j e^{(2-j)T\lambda_n} (\varepsilon_0^2 + 2\varepsilon_0 e^{T\lambda_n})^{-1} \\
&\quad + 2 \sum_{n=1}^{\infty} \varepsilon^{2-j} \int_0^T \left(\frac{j}{2-j}\right)^j e^{(2-j)T\lambda_n} e^{-2(T-\tau)\lambda_n} (\varepsilon_0 + 2\varepsilon_0 e^{T\lambda_n})^{-1} |f_k(\tau)|^2 d\tau \\
&\leq \varepsilon^{2-j} \sum_{n=1}^{\infty} b_n^2 \left(\frac{j}{2-j}\right)^j e^{(2-j)T\lambda_n} \\
&\quad + \varepsilon^{2-j} \sum_{n=1}^{\infty} \int_0^T \left(\frac{j}{2-j}\right)^j e^{(2-j)T\lambda_n} e^{-2(T-\tau)\lambda_n} |f_k(\tau)|^2 d\tau \\
&\leq \varepsilon^{2-j} \left(\frac{j}{2-j}\right)^j \left[ \sum_{n=1}^{\infty} b_n^2 e^{(2-j)T\lambda_n} + \sum_{n=1}^{\infty} \int_0^T e^{(2-j)T\lambda_n} |f_k(\tau)|^2 d\tau \right].
\end{aligned}$$

Letting  $j = 2 - \gamma$  we obtain

$$\|\rho^\varepsilon(\cdot, T) - \rho^T\|_{L^2(\Omega)}^2 \leq \left(\frac{2}{\gamma}\right)^2 \varepsilon^\gamma \left[ \sum_{n=1}^{\infty} b_n^2 e^{\gamma T\lambda_n} + \sum_{n=1}^{\infty} \int_0^T e^{\gamma T\lambda_n} |f_k(\tau)|^2 d\tau \right] = C \varepsilon^\gamma \gamma^{-2},$$

where

$$C := 4 \left( \sum_{n=1}^{\infty} b_n^2 e^{\gamma T\lambda_n} + \sum_{n=1}^{\infty} \int_0^T e^{\gamma T\lambda_n} |f_k(\tau)|^2 d\tau \right).$$

1 The proof is finished. □

2 We have the following result as a direct consequence of Theorem 2.16. Its proof follows the lines of  
3 the proof of Theorem 2.16 with the obvious modifications.

**Corollary 2.17.** *Let  $f \in L^2(Q)$  and  $\rho^T \in L^2(\Omega)$  have the expansions (2.37). If there exists  $\gamma > 0$  such that the series*

$$\sum_{k=1}^{\infty} b_k^2 e^{(2+\gamma)T\lambda_k} \text{ and } \sum_{k=1}^{\infty} \int_0^T e^{(2+\gamma)t\lambda_k} |f_k(t)|^2 dt$$

1 converge, then  $\|\rho^\varepsilon(\cdot, t) - \rho(\cdot, t)\|_{L^2(\Omega)}$  converges to zero, uniformly in  $t \in [0, T]$ , as  $\varepsilon \downarrow 0$ , with conver-  
 2 gence rate  $\varepsilon^\gamma \gamma^{-2}$ .

### 3. THE OPTIMAL CONTROL PROBLEM

4 Throughout this section we assume that  $f \in L^2(Q)$  and  $\rho^T \in L^2(\Omega)$  satisfies (2.1).

We are interested in the optimal control problem associated to the non well-posed state equation (1.1), that is, we want to solve the optimization problem (1.2)-(1.1). Let  $\mathbb{U}_{ad}$  be a nonempty closed and convex subset of  $L^2(Q)$  and  $\mathcal{A}$  be defined as in (1.3). For example one may take  $\mathbb{U}_{ad}$  to be the following nonempty closed and convex subset of  $L^2(Q)$ :

$$\mathbb{U}_{ad} := \left\{ f \in L^2(Q) \text{ such that } \sum_{k=1}^{\infty} \left( \int_0^T e^{t\lambda_k} |f_k(t)| dt \right)^2 < \infty \right\},$$

5 where  $\lambda_k$  and  $\phi_k$  ( $k \in \mathbb{N}$ ) are defined as in Remark 1.2, and  $f_k(t) := (f(\cdot, t), \phi_k)_{L^2(\Omega)}$ .

6 **Remark 3.1.** We make the following observations.

- 7 (a) From Lemma 2.2, we have that the system (1.1) has a strong solution  $\rho$  in the sense of Definition  
 8 2.1. This shows that  $\mathcal{A} \neq \emptyset$ . Moreover, it follows from Remark 2.3 that  $\rho \in L^2((0, T); \mathbb{V}) \cap$   
 9  $C([0, T]; L^2(\Omega))$  and (2.3) holds true.  
 10 (b) Since  $\rho \in L^2((0, T); \mathbb{V}) \cap C([0, T]; L^2(\Omega))$ , we know that  $\rho(\cdot, 0) \in L^2(\Omega)$ . We can thus define  
 11 the cost function

$$J(f, \rho) = \frac{1}{2} \left( \|\rho(\cdot, 0) - \rho^d\|_{L^2(\Omega)}^2 + \xi \|f\|_{L^2(Q)}^2 \right), \quad (3.1)$$

12 where  $\rho^d \in L^2(\Omega)$  and  $\xi > 0$  is a real parameter.

- 13 (c) Using minimizing sequences, the structure of the cost function  $J$  given in (3.1), and the estimate  
 14 (2.4), we can prove that there exists a unique optimal pair  $(\bar{f}, \bar{\rho}) \in \mathcal{A}$  which is the solution to the  
 15 optimization problem (1.2)-(1.1). Moreover, writing the first order Euler-Lagrange optimality  
 16 condition that characterises the optimal pair  $(\bar{f}, \bar{\rho})$ , we have that

$$\int_{\Omega} (\rho(x, 0) - \bar{\rho}(x, 0))(\bar{\rho}(x, 0) - \rho^d(x)) dx + \int_Q \xi \bar{f}(f - \bar{f}) dx dt \geq 0 \quad \forall (f, \rho) \in \mathcal{A}. \quad (3.2)$$

17 As mentioned in the introduction, the increase of the state and the control in (3.2) are linked because  
 18 the system is not well-posed in the sense of Hadamard (in particular, the solution whenever it exists)  
 19 may not be unique, and the continuous dependence of solutions on the given data may be lost).

20 **3.1. The approximated optimal control problem.** Here, we study the existence and uniqueness  
 21 of minimizers of the optimization problems (1.5)-(1.4), and we give a characterization of the associated  
 22 optimality conditions. Let us notice that in this section we do not assume that  $f$  and  $\rho^T$  satisfy (2.1).

23 We let

$$J^\varepsilon(f) = \frac{1}{2} \left( \|\mathcal{T}_\varepsilon(f, \rho^T)(\cdot, 0) - \rho^d\|_{L^2(\Omega)}^2 + \xi \|f\|_{L^2(Q)}^2 \right), \quad (3.3)$$

24 where in view of (2.21),  $\rho^\varepsilon = \mathcal{T}_\varepsilon(f, \rho^T)$  solves (1.4). Recall that  $\mathbb{U}_{ad}$  is a nonempty closed and convex  
 25 subset of  $L^2(Q)$ .

26 We have the following existence and uniqueness result of optimal controls.

27 **Theorem 3.2.** For every  $\varepsilon > 0$ , there exists a unique  $f^\varepsilon \in \mathbb{U}_{ad}$  solution of the minimization problem  
 28 (1.5)-(1.4). The associated state  $\rho^\varepsilon$  is the unique strong solution of (1.4) with  $f$  replaced by  $f^\varepsilon$ .

1 *Proof.* Since the functional  $J^\varepsilon$  given in (3.3) is bounded from below by zero, it is possible to construct  
 2 a minimizing sequence  $(f^{\varepsilon n})_{n \in \mathbb{N}}$  such that

$$\lim_{n \rightarrow \infty} J^\varepsilon(f^{\varepsilon n}) = \inf_{f \in \mathcal{U}_{\text{ad}}} J^\varepsilon(f). \quad (3.4)$$

3 We denote by  $\psi^{\varepsilon n} := \mathcal{T}_\varepsilon(f^{\varepsilon n}, \rho^T)$  the state associated with the control  $f^{\varepsilon n}$ . For each  $n \in \mathbb{N}$ , by Remark  
 4 2.11, we have that  $\psi^{\varepsilon n} \in L^2((0, T); \mathbb{V}) \cap H^1((0, T); \mathbb{V}^*)$  is the unique strong solution of

$$\begin{cases} \psi_t^{\varepsilon n} + A\psi^{\varepsilon n} & = f^{\varepsilon n} & \text{in } Q, \\ \varepsilon\psi^{\varepsilon n}(\cdot, 0) + \psi^{\varepsilon n}(\cdot, T) & = \rho^T & \text{in } \Omega. \end{cases} \quad (3.5)$$

It follows from (3.4) and the structure of the cost function  $J^\varepsilon$  that, there is a constant  $C > 0$  (independent of  $n$  and  $\varepsilon$ ) such that,

$$\|f^{\varepsilon n}\|_{L^2(Q)} \leq C, \quad (3.6)$$

$$\|\psi^{\varepsilon n}(\cdot, 0)\|_{L^2(\Omega)} \leq C. \quad (3.7)$$

5 It follows from (3.7) and the second equation in (3.5) that there is a constant  $C > 0$  independent of  $n$   
 6 and  $\varepsilon$  such that

$$\|\psi^{\varepsilon n}(\cdot, T)\|_{L^2(\Omega)} = \|\rho^T - \varepsilon\psi^{\varepsilon n}(\cdot, 0)\|_{L^2(\Omega)} \leq C\varepsilon + \|\rho^T\|_{L^2(\Omega)}. \quad (3.8)$$

7 Since  $\psi^{\varepsilon n} \in L^2((0, T); \mathbb{V}) \cap C([0, T]; L^2(\Omega))$  is the unique strong solution of (3.5), it follows from (2.12),  
 8 (2.16), and (3.6) that there is a constant  $C > 0$  independent of  $n$  and  $\varepsilon$  such that

$$\|\psi^{\varepsilon n}\|_{C([0, T]; L^2(\Omega))}^2 + \|\psi^{\varepsilon n}\|_{L^2((0, T); \mathbb{V})}^2 \leq C \left[ \left( \frac{1}{\varepsilon^2} + 1 \right) + \frac{1}{\varepsilon^2} \|\rho^T\|_{L^2(\Omega)}^2 \right]. \quad (3.9)$$

It follows from (3.6), (3.7), (3.8), and (3.9) that there exist  $f^\varepsilon \in L^2(Q)$ ,  $\psi_T^\varepsilon, \psi_0^\varepsilon \in L^2(\Omega)$ , and  $\rho^\varepsilon \in L^2((0, T); \mathbb{V})$  such that, as  $n \rightarrow \infty$ , we have the following convergences:

$$f^{\varepsilon n} \rightarrow f^\varepsilon \quad \text{weakly in } L^2(Q), \quad (3.10a)$$

$$\psi^{\varepsilon n}(\cdot, 0) \rightarrow \psi_0^\varepsilon \quad \text{weakly in } L^2(\Omega), \quad (3.10b)$$

$$\psi^{\varepsilon n}(\cdot, T) \rightarrow \psi_T^\varepsilon \quad \text{weakly in } L^2(\Omega), \quad (3.10c)$$

$$\psi^{\varepsilon n} \rightarrow \rho^\varepsilon \quad \text{weakly in } L^2((0, T); \mathbb{V}). \quad (3.10d)$$

9 Since  $f^{\varepsilon n} \in \mathcal{U}_{\text{ad}}$  which is a closed subset of  $L^2(Q)$ , we can deduce that

$$f^\varepsilon \in \mathcal{U}_{\text{ad}}. \quad (3.11)$$

Next, we show that  $\rho^\varepsilon$  belongs to  $H^1((0, T); \mathbb{V}^*)$ . It follows from (2.22) in Remark 2.11 that

$$\langle \psi_t^{\varepsilon n}, \phi \rangle_{\mathbb{V}^*, \mathbb{V}} + \mathcal{E}(\psi^{\varepsilon n}, \phi) dt = \int_{\Omega} f^{\varepsilon n} \phi dx dt$$

for every  $\phi \in \mathbb{V}$  and almost every  $t \in (0, T)$ . Using the continuity of the bilinear form  $\mathcal{E}$  (see (1.12)), we get from the previous identity that there is a constant  $C > 0$  independent of  $n$  and  $\varepsilon$  such that

$$\begin{aligned} |\langle \psi_t^{\varepsilon n}, \phi \rangle_{\mathbb{V}^*, \mathbb{V}}| &\leq \|\psi^{\varepsilon n}(\cdot, t)\|_{\mathbb{V}} \|\phi\|_{\mathbb{V}} + \|f^{\varepsilon n}(\cdot, t)\|_{L^2(\Omega)} \|\phi\|_{L^2(\Omega)} \\ &\leq C \left( \|\psi^{\varepsilon n}(\cdot, t)\|_{\mathbb{V}} + \|f^{\varepsilon n}(\cdot, t)\|_{L^2(\Omega)} \right) \|\phi\|_{\mathbb{V}}. \end{aligned}$$

This shows that

$$\|\psi_t^{\varepsilon n}\|_{\mathbb{V}^*} \leq C \left( \|\psi^{\varepsilon n}(\cdot, t)\|_{\mathbb{V}} + \|f^{\varepsilon n}(\cdot, t)\|_{L^2(\Omega)} \right).$$

Integrating the latter inequality over  $(0, T)$ , we get that

$$\int_0^T \|\psi_t^{\varepsilon n}\|_{\mathbb{V}^*}^2 \leq C \int_0^T \left( \|\psi^{\varepsilon n}(\cdot, t)\|_{\mathbb{V}}^2 + \|f^{\varepsilon n}(\cdot, t)\|_{L^2(\Omega)}^2 \right) dt.$$

It follows from (3.6), (3.9), and the previous estimate that there is a constant  $C > 0$  independent of  $n$  and  $\varepsilon$  such that

$$\|\psi_t^{\varepsilon n}\|_{L^2((0, T); \mathbb{V}^*)}^2 \leq C \left[ \left( \frac{1}{\varepsilon^2} + 1 \right) + \frac{1}{\varepsilon^2} \|\rho^T\|_{L^2(\Omega)}^2 \right].$$

- 1 This shows that  $\psi_t^{\varepsilon n}$  converges weakly to some function  $\psi^\varepsilon \in L^2((0, T); \mathbb{V}^*)$  as  $n \rightarrow \infty$ . The uniqueness  
 2 of the limit implies that  $\psi^\varepsilon = \rho_t^\varepsilon$ . This fact combined with (3.10d) implies that  $\rho^\varepsilon \in H^1((0, T); \mathbb{V}^*)$ .  
 3 Since  $\rho^\varepsilon \in L^2((0, T); \mathbb{V}) \cap H^1((0, T); \mathbb{V}^*)$ , we have that  $\rho^\varepsilon \in C([0, T]; L^2(\Omega))$  by (1.22).

On the other hand, observing that  $\rho^\varepsilon \in C([0, T]; L^2(\Omega))$ , and using the uniqueness of the limit, we can deduce from (3.9), (3.10d), (3.10b) and (3.10c) that  $\psi_0^\varepsilon = \rho^\varepsilon(\cdot, 0)$  and  $\psi_T^\varepsilon = \rho^\varepsilon(\cdot, T)$  in  $\Omega$ . Consequently, as  $n \rightarrow \infty$ , we have that

$$\psi^{\varepsilon n}(\cdot, 0) \rightarrow \rho^\varepsilon(\cdot, 0) \text{ weakly in } L^2(\Omega), \quad (3.12a)$$

$$\psi^{\varepsilon n}(\cdot, T) \rightarrow \rho^\varepsilon(\cdot, T) \text{ weakly in } L^2(\Omega). \quad (3.12b)$$

Recall that by Remark 2.11, for every  $\phi \in H(Q) = \{z \in L^2((0, T); \mathbb{V}) : z_t \in L^2(Q)\}$ , we have that

$$\begin{aligned} & - \int_Q \phi_t \psi^{\varepsilon n} dx dt + \int_0^T \mathcal{E}(\psi^{\varepsilon n}, \phi) dt + \int_\Omega \psi^{\varepsilon n}(x, T) \phi(x, T) dx \\ & = \int_Q f^{\varepsilon n} \phi dx dt + \int_\Omega \psi^{\varepsilon n}(x, 0) \phi(x, 0) dx \\ & = \int_Q f^{\varepsilon n} \phi dx dt + \frac{1}{\varepsilon} \int_\Omega (\psi^{\varepsilon n}(x, T) - \rho^T) \phi(x, 0) dx. \end{aligned} \quad (3.13)$$

Using (3.10a), (3.10d), (3.12a), and (3.12b) while taking the limit of (3.13) as  $n \rightarrow \infty$ , we get that

$$\begin{aligned} & - \int_Q \phi_t \rho^\varepsilon dx dt + \int_0^T \mathcal{E}(\rho^\varepsilon, \phi) dt + \int_\Omega \rho^\varepsilon(x, T) \phi(x, T) dx \\ & = \int_Q f^\varepsilon \phi dx dt + \int_\Omega \rho^\varepsilon(x, 0) \phi(x, 0) dx \\ & = \int_Q f^\varepsilon \phi dx dt + \frac{1}{\varepsilon} \int_\Omega (\rho^\varepsilon(x, T) - \rho^T) \phi(x, 0) dx \end{aligned} \quad (3.14)$$

for every  $\phi \in H(Q)$ . By Remark 2.11, (3.14) means that  $\rho^\varepsilon \in L^2((0, T); \mathbb{V}) \cap C([0, T]; L^2(\Omega))$  is the unique strong solution of the system

$$\begin{cases} \rho_t^\varepsilon + A\rho^\varepsilon = f^\varepsilon & \text{in } Q, \\ \varepsilon \rho^\varepsilon(\cdot, 0) + \rho^\varepsilon(\cdot, T) = \rho^T & \text{in } \Omega. \end{cases}$$

Next, since the functional  $J^\varepsilon$  is convex and lower semi-continuous, using (3.10a), (3.12a) and (3.11) we can deduce that

$$J^\varepsilon(f^\varepsilon) \leq \liminf_{n \rightarrow +\infty} J^\varepsilon(f^{\varepsilon n}) \leq \lim_{n \rightarrow +\infty} J^\varepsilon(f^{\varepsilon n}) \leq \inf_{f \in \mathcal{U}_{\text{ad}}} J^\varepsilon(f) \leq J^\varepsilon(f^\varepsilon).$$

- 4 We have shown that  $(f^\varepsilon, \rho^\varepsilon)$  is the optimal solution of (1.5)-(1.4). The uniqueness is straightforward  
 5 and follows directly from the strict convexity of the functional  $J^\varepsilon$ . The proof is finished.  $\square$

1 Next, we characterize the associated optimality conditions.

2 **Theorem 3.3.** *Let  $(\rho^\varepsilon, f^\varepsilon)$  be the solution of the minimization problem (1.5)-(1.4). Then, there exists*  
 3  *$q^\varepsilon \in L^2((0, T); \mathbb{V}) \cap H^1((0, T); \mathbb{V}^*)$  such that we have the following optimality systems:*

$$\begin{cases} \rho_t^\varepsilon + A\rho^\varepsilon = f^\varepsilon & \text{in } Q, \\ \varepsilon\rho^\varepsilon(\cdot, 0) + \rho^\varepsilon(\cdot, T) = \rho^T & \text{in } \Omega, \end{cases} \quad (3.15)$$

4 and

$$\begin{cases} -q_t^\varepsilon + Aq^\varepsilon = 0 & \text{in } Q, \\ \varepsilon q^\varepsilon(\cdot, T) + q^\varepsilon(\cdot, 0) = \rho^\varepsilon(\cdot, 0) - \rho^d & \text{in } \Omega, \end{cases} \quad (3.16)$$

5 and

$$\int_Q (\xi f^\varepsilon - q^\varepsilon)(f - f^\varepsilon) \, dx \, dt \geq 0, \quad \forall f \in \mathcal{U}_{ad}. \quad (3.17)$$

6 *Proof.* We have already shown (3.15) in the proof of Theorem 3.2. It is also clear that (3.16) is the dual  
 7 system associated with (3.15). To complete the proof of the Theorem, we write the Euler-Lagrange  
 8 first order optimality condition that characterizes the optimal control  $f^\varepsilon$ . That is,

$$\lim_{\lambda \downarrow 0} \frac{J^\varepsilon(f^\varepsilon + \lambda(f - f^\varepsilon)) - J^\varepsilon(f^\varepsilon)}{\lambda} \geq 0, \quad \forall f \in \mathcal{U}_{ad}. \quad (3.18)$$

9 After some straightforward calculations we have that

$$\int_\Omega z^\varepsilon(x, 0)(\rho^\varepsilon(x, 0) - \rho^d) \, dx + \xi \int_Q f^\varepsilon(f - f^\varepsilon) \, dx \, dt \geq 0, \quad \forall f \in \mathcal{U}_{ad}, \quad (3.19)$$

10 where  $z^\varepsilon \in L^2((0, T); \mathbb{V}) \cap H^1((0, T); \mathbb{V}^*)$  is the unique strong solution of

$$\begin{cases} z_t^\varepsilon + Az^\varepsilon = f - f^\varepsilon & \text{in } Q, \\ \varepsilon z^\varepsilon(\cdot, 0) + z^\varepsilon(\cdot, T) = 0 & \text{in } \Omega. \end{cases} \quad (3.20)$$

11 Notice that the existence of such  $z^\varepsilon$  is given by Theorems 2.7 and 2.10.

12 To interpret (3.19), we use the adjoint state given by (3.16). Note that in view of Proposition 2.13,  
 13 we have  $q^\varepsilon \in L^2((0, T); \mathbb{V}) \cap H^1((0, T); \mathbb{V}^*)$ . So, if we multiply the first equation in (3.20) by  $q^\varepsilon$  solution  
 14 of (3.16), and integrate by parts over  $Q$ , we get

$$\int_\Omega z^\varepsilon(x, 0)(\rho^\varepsilon(x, 0) - \rho^d(x)) \, dx + \int_Q (f - f^\varepsilon)q^\varepsilon \, dx \, dt = 0. \quad (3.21)$$

15 Combining (3.19)-(3.21) we get (3.17). This completes the proof.  $\square$

16 **3.2. Convergence of the approximated optimal control problem.** The goal of this section is  
 17 to show that solutions of the optimal control problem (1.5)-(1.4) converge to solutions of the optimal  
 18 control problem (1.2)-(1.1), as  $\varepsilon \downarrow 0$ . Throughout the following  $\mathcal{D}((0, T); \mathbb{V}) := C_c^\infty((0, T); \mathbb{V})$ . Recall  
 19 that we have assumed that  $f$  and  $\rho^T$  satisfy (2.1). We have the following result.

**Theorem 3.4.** *Let  $(\bar{f}, \bar{\rho})$  be a solution of the minimization problem (1.2)-(1.1), and let  $(f^\varepsilon, \rho^\varepsilon, q^\varepsilon)$  be  
 as in Theorems 3.2 and 3.3. Assume that  $f^\varepsilon \in \mathcal{U}_{ad}$  and that the interior of  $\mathcal{U}_{ad}$  is not empty, that is,*

$\text{Int}(\mathbb{U}_{ad}) \neq \emptyset$ . Then, there exists  $\bar{q} \in L^2(Q)$  such that, as  $\varepsilon \downarrow 0$ , we have the following convergences:

$$f^\varepsilon \text{ converges strongly to } \bar{f} \in L^2(Q) \text{ and } \bar{f} \in \mathbb{U}_{ad}, \quad (3.22)$$

$$\rho^\varepsilon \text{ converges weakly to } \bar{\rho} \in L^2((0, T); \mathbb{V}) \cap H^1((0, T); \mathbb{V}^*), \quad (3.23)$$

$$\rho^\varepsilon(\cdot, T) \text{ converges strongly to } \rho^T \in L^2(\Omega), \quad (3.24)$$

$$\rho^\varepsilon(\cdot, 0) \text{ converges strongly to } \bar{\rho}(\cdot, 0) \in L^2(\Omega), \quad (3.25)$$

$$q^\varepsilon \text{ converges weakly to } \bar{q} \in L^2(Q). \quad (3.26)$$

*Proof.* We proceed in three steps.

**Step 1.** From (3.6) and the convergence in (3.10a), we can deduce that there is a constant  $C > 0$  independent of  $\varepsilon$  such that

$$\|f^\varepsilon\|_{L^2(Q)} \leq C. \quad (3.27)$$

It follows from (3.27) that there exists  $f^* \in L^2(Q)$  such that

$$f^\varepsilon \rightarrow f^* \text{ weakly in } L^2(Q), \text{ as } \varepsilon \downarrow 0. \quad (3.28)$$

Since  $f^\varepsilon \in \mathbb{U}_{ad}$  which is a closed subset of  $L^2(Q)$ , we can deduce that  $f^* \in \mathbb{U}_{ad}$ .

From (3.7) and (3.12a), we have that there is a constant  $C > 0$  independent of  $\varepsilon$  such that

$$\|\rho^\varepsilon(\cdot, 0)\|_{L^2(\Omega)} \leq C. \quad (3.29)$$

It follows from (3.29) and (3.15) that there is a constant  $C > 0$  independent of  $\varepsilon$  such that

$$\|\rho^T - \rho^\varepsilon(\cdot, T)\|_{L^2(\Omega)} \leq C\varepsilon. \quad (3.30)$$

Since  $\rho^\varepsilon$  is the unique strong solution of (3.15), using (3.30), (3.27) and (2.18), we have that there is a constant  $C > 0$  independent of  $\varepsilon$  such that

$$\sup_{t \in [0, T]} \|\rho^\varepsilon(\cdot, t)\|_{L^2(\Omega)}^2 \leq C\|f^\varepsilon\|_{L^2(Q)}^2 + \frac{1}{\varepsilon^2} \|\rho^T - \rho^\varepsilon(\cdot, T)\|_{L^2(\Omega)}^2 \leq C^2 \quad (3.31)$$

and

$$\int_0^T \|\rho^\varepsilon(\cdot, t)\|_{\mathbb{V}}^2 dt \leq C\|f^\varepsilon\|_{L^2(Q)}^2 + \frac{2}{\varepsilon^2} \|\rho^T - \rho^\varepsilon(\cdot, T)\|_{L^2(\Omega)}^2 \leq C^2. \quad (3.32)$$

From (3.32), we have that there exists  $\rho^* \in L^2((0, T); \mathbb{V})$  such that, as  $\varepsilon \downarrow 0$ ,

$$\rho^\varepsilon \rightarrow \rho^* \text{ weakly in } L^2((0, T); \mathbb{V}). \quad (3.33)$$

Proceeding as in the proof of Theorem 3.2 we can also obtain that, as  $\varepsilon \downarrow 0$ ,

$$\rho^\varepsilon \rightarrow \rho^* \text{ weakly in } H^1((0, T); \mathbb{V}^*). \quad (3.34)$$

It follows from (3.33) and (3.34) that  $\rho^* \in L^2((0, T); \mathbb{V}) \cap H^1((0, T); \mathbb{V}^*) \subset C([0, T]; L^2(\Omega))$ , and

$$\rho^\varepsilon \rightarrow \rho^* \text{ converges weakly in } L^2((0, T); \mathbb{V}) \cap H^1((0, T); \mathbb{V}^*), \text{ as } \varepsilon \downarrow 0. \quad (3.35)$$

From (3.30), we have that, as  $\varepsilon \downarrow 0$ ,

$$\rho^\varepsilon(\cdot, T) \rightarrow \rho^T \text{ strongly in } L^2(\Omega). \quad (3.36)$$

and we have shown (3.24).

Using (3.31), (3.29), the fact that  $\rho^* \in C([0, T]; L^2(\Omega))$ , and the uniqueness of the limit, we can deduce that, as  $\varepsilon \downarrow 0$ ,

$$\rho^\varepsilon(\cdot, 0) = \frac{1}{\varepsilon}(\rho^T - \rho^\varepsilon(\cdot, T)) \rightarrow \rho^*(\cdot, 0) \text{ weakly in } L^2(\Omega). \quad (3.37)$$

Recall that it follows from (2.19) in Remark 2.11 that for every  $\phi \in H(Q)$ , we have that

$$\begin{aligned} & - \int_Q \phi_t \rho^\varepsilon dx dt + \int_0^T \mathcal{E}(\rho^\varepsilon, \phi) dt + \int_\Omega \rho^\varepsilon(x, T) \phi(x, T) dx \\ & = \int_Q f^\varepsilon \phi dx dt + \int_\Omega \rho^\varepsilon(x, 0) \phi(0) dx. \end{aligned} \quad (3.38)$$

1 Using (3.28), (3.23), (3.24) and (3.37), while taking the limit of (3.38) as  $\varepsilon \downarrow 0$ , we get

$$- \int_Q \phi_t \rho^* dx dt + \int_0^T \mathcal{E}(\rho^*, \phi) dt + \int_\Omega \rho^T(x) \phi(x, T) dx = \int_Q f^* \phi dx dt + \int_\Omega \rho^*(x, 0) \phi(x, 0) dx \quad (3.39)$$

2 for every  $\phi \in H(Q)$ . Thus,  $\rho^* \in L^2((0, T); \mathbb{V}) \cap C([0, T]; L^2(\Omega))$  and for every  $\phi \in H(Q)$  such that  
3  $\phi(\cdot, 0) = 0$  in  $\Omega$ , we have

$$- \int_Q \phi_t \rho^* dx dt + \int_0^T \mathcal{E}(\rho^*, \phi) dt + \int_\Omega \rho^T(x) \phi(x, T) dx = \int_Q f^* \phi dx dt. \quad (3.40)$$

4 Since  $f^* \in \mathbb{U}_{ad}$  and  $\rho^T$  satisfy (2.1), it follows from Remark 2.3 that  $\rho^*$  is a strong solution of

$$\begin{cases} \rho_t^* + A\rho^* = f^* & \text{in } Q, \\ \rho^*(\cdot, T) = \rho^T & \text{in } \Omega. \end{cases} \quad (3.41)$$

5 Now, since (3.41) has a strong solution  $\rho^*$ , it follows from Theorem 2.15 that  $\rho^\varepsilon(\cdot, 0)$  converges strongly  
6 in  $L^2(\Omega)$ , as  $\varepsilon \downarrow 0$ . The uniqueness of the limit shows that

$$\rho^\varepsilon(\cdot, 0) \rightarrow \rho^*(\cdot, 0) \text{ strongly in } L^2(\Omega) \text{ as } \varepsilon \downarrow 0. \quad (3.42)$$

7 **Step 2.** Let  $(\bar{f}, \bar{\rho}) \in \mathcal{A}$  be the solution of the optimization problem (1.2)-(1.1). We claim that  
8  $(\bar{f}, \bar{\rho}) = (f^*, \rho^*)$ . Since  $(f^*, \rho^*) \in \mathcal{A}$ , we have that

$$J(\bar{f}, \bar{\rho}) \leq J(f^*, \rho^*). \quad (3.43)$$

Observing that on the one hand  $f^\varepsilon$  is the optimal control solution of (1.5)-(1.4), and on the other  
hand that  $\bar{f} \in \mathbb{U}_{ad}$ , we can write

$$\begin{aligned} \|\mathcal{J}_\varepsilon(f^\varepsilon, \rho^T)(\cdot, 0) - \rho^d\|_{L^2(\Omega)} + \xi \|f^\varepsilon\|_{L^2(Q)}^2 &= J^\varepsilon(f^\varepsilon) \leq J^\varepsilon(\bar{f}) \\ &= \|\mathcal{J}_\varepsilon(\bar{f}, \rho^T)(\cdot, 0) - \rho^d\|_{L^2(\Omega)} + \xi \|\bar{f}\|_{L^2(Q)}^2. \end{aligned} \quad (3.44)$$

In view of (2.21),  $\rho^\varepsilon = \mathcal{J}_\varepsilon(f^\varepsilon, \rho^T)$ . Using (3.42), (3.28) and (2.29a), we can deduce from (3.44) that

$$\|\rho^*(\cdot, 0) - \rho^d\|_{L^2(\Omega)} + \xi \|f^*\|_{L^2(Q)}^2 = J(f^*, \rho^*) \leq J(\bar{f}, \bar{\rho}) = \|\bar{\rho}(\cdot, 0) - \rho^d\|_{L^2(\Omega)} + \xi \|\bar{f}\|_{L^2(Q)}^2. \quad (3.45)$$

9 Thus,

$$J(f^*, \rho^*) \leq J(\bar{f}, \bar{\rho}). \quad (3.46)$$

Combining (3.43)-(3.46), we obtain that

$$J(\bar{f}, \bar{\rho}) \leq J(f^*, \rho^*) \leq J(\bar{f}, \bar{\rho}).$$

10 This implies that

$$(\bar{f}, \bar{\rho}) = (f^*, \rho^*) \quad (3.47)$$

11 and the claim is proved.

Since

$$J(f^*, \rho^*) \leq \lim_{\varepsilon \downarrow 0} J^\varepsilon(f^\varepsilon) \leq J(f^*, \rho^*),$$



1 we can also deduce that

$$J^\varepsilon \rightarrow J, \text{ as } \varepsilon \downarrow 0. \quad (3.48)$$

2 Using (3.47) in (3.35) and (3.42) we obtain that (3.23) and (3.25) hold true.

3 Using again (3.47) in (3.37) and in (3.28), we get that, as  $\varepsilon \downarrow 0$ ,

$$\rho^\varepsilon(\cdot, 0) \rightarrow \bar{\rho}(\cdot, 0) \text{ weakly in } L^2(\Omega) \quad (3.49)$$

4 and

$$f^\varepsilon \rightarrow \bar{f} \text{ weakly in } L^2(Q). \quad (3.50)$$

5 Next, we show that  $f^\varepsilon \rightarrow \bar{f}$  strongly in  $L^2(Q)$ , as  $\varepsilon \downarrow 0$ . Notice that (3.48) can be rewritten as

$$\lim_{\varepsilon \downarrow 0} \left( \|\rho^\varepsilon(\cdot, 0) - \rho^d\|_{L^2(\Omega)} + \xi \|f^\varepsilon\|_{L^2(Q)}^2 \right) = \|\bar{\rho}(\cdot, 0) - \rho^d\|_{L^2(\Omega)} + \xi \|\bar{f}\|_{L^2(Q)}^2. \quad (3.51)$$

Using (3.50) and (3.49), we obtain that

$$\|\bar{\rho}(\cdot, 0) - \rho^d\|_{L^2(\Omega)} \leq \liminf_{\varepsilon \downarrow 0} \|\rho^\varepsilon(\cdot, 0) - \rho^d\|_{L^2(\Omega)},$$

and

$$\|\bar{f}\|_{L^2(Q)}^2 \leq \liminf_{\varepsilon \downarrow 0} \|f^\varepsilon\|_{L^2(Q)}^2.$$

Combining the above two inequalities with (3.51) we get

$$\|\bar{\rho}(\cdot, 0) - \rho^d\|_{L^2(\Omega)} = \lim_{\varepsilon \downarrow 0} \|\rho^\varepsilon(\cdot, 0) - \rho^d\|_{L^2(\Omega)}, \quad (3.52)$$

$$\|\bar{f}\|_{L^2(Q)}^2 = \lim_{\varepsilon \downarrow 0} \|f^\varepsilon\|_{L^2(Q)}^2. \quad (3.53)$$

Observe that

$$\|f^\varepsilon - \bar{f}\|_{L^2(Q)}^2 = \|f^\varepsilon\|_{L^2(Q)}^2 + \|\bar{f}\|_{L^2(Q)}^2 - 2 \int_Q f^\varepsilon \bar{f} dx dt.$$

Taking the limit of the latter identity, as  $\varepsilon \downarrow 0$ , while using (3.50) and (3.53), we get

$$\lim_{\varepsilon \downarrow 0} \|f^\varepsilon - \bar{f}\|_{L^2(Q)}^2 = \|\bar{f}\|_{L^2(Q)}^2 + \|\bar{f}\|_{L^2(Q)}^2 - 2 \int_Q (\bar{f})^2 dx dt = 0.$$

This implies that

$$f^\varepsilon \rightarrow \bar{f} \text{ strongly in } L^2(Q) \text{ as } \varepsilon \downarrow 0,$$

6 and we have shown (3.22).

**Step 3.** Since  $\text{Int}(\mathbb{U}_{ad}) \neq \emptyset$ , it follows that there are  $w \in \text{Int}(\mathbb{U}_{ad})$  and  $r > 0$  such that

$$\|v - w\|_{L^2(Q)} < r \text{ implies } v \in \mathbb{U}_{ad}.$$

7 Since  $\mathbb{U}_{ad}$  is a closed and convex subset of  $L^2(Q)$  with nonempty interior, we have that

$$\mathbb{U}_{ad} = \overline{\text{Int}(\mathbb{U}_{ad})}. \quad (3.54)$$

8 Hence, there exists a minimizing sequence  $\{v_n^\varepsilon\}_{n \in \mathbb{N}} \subset \text{Int}(\mathbb{U}_{ad})$  of  $f^\varepsilon$  such that

$$\|v_n^\varepsilon - f^\varepsilon\|_{L^2(Q)} \leq r. \quad (3.55)$$

9 Let  $\phi^{\varepsilon n}$  be the state associated to the control  $v_n^\varepsilon$ . Since  $v_n^\varepsilon \in \mathbb{U}_{ad}$ , we have that  $\phi^{\varepsilon n}$  satisfies

$$\begin{cases} \phi_t^{\varepsilon n} + A\phi^{\varepsilon n} = v_n^\varepsilon & \text{in } Q, \\ \varepsilon \phi^{\varepsilon n}(\cdot, 0) + \phi^{\varepsilon n}(\cdot, T) = \rho^T & \text{in } \Omega. \end{cases} \quad (3.56)$$

This shows that the estimates (3.6), (3.7), and (3.9) hold for  $(v_n^\varepsilon, \phi^{\varepsilon n})$ . That is, there is a constant  $C > 0$  independent of  $n$  and  $\varepsilon$  such that

$$\|v_n^\varepsilon\|_{L^2(Q)} \leq C, \quad (3.57a)$$

$$\|\phi^{\varepsilon n}(\cdot, 0)\|_{L^2(\Omega)} \leq C, \quad (3.57b)$$

$$\|\phi^{\varepsilon n}\|_{C([0,T];L^2(\Omega))}^2 + \|\phi^{\varepsilon n}\|_{L^2((0,T);V)}^2 \leq C \left[ \left( \frac{1}{\varepsilon^2} + 1 \right) + \frac{1}{\varepsilon^2} \|\rho^T\|_{L^2(\Omega)}^2 \right]. \quad (3.57c)$$

- 1 Set  $z^{\varepsilon n} := \rho^\varepsilon - \phi^{\varepsilon n}$  where  $\rho^\varepsilon$  is the state associated to the optimal control  $f^\varepsilon$ . In view of (3.56) and  
 2 (3.15),  $z^{\varepsilon n}$  is a solution of the system

$$\begin{cases} z_t^{\varepsilon n} + Az^{\varepsilon n} = f^\varepsilon - v_n^\varepsilon & \text{in } Q, \\ \varepsilon z^{\varepsilon n}(\cdot, 0) + z^{\varepsilon n}(\cdot, T) = 0 & \text{in } \Omega. \end{cases} \quad (3.58)$$

- 3 Since  $\phi^{\varepsilon n}$  and  $\rho^\varepsilon$  satisfy (3.57b) and (3.29), respectively, we have that

$$\|z^{\varepsilon n}(\cdot, 0)\|_{L^2(\Omega)} = \|\rho^\varepsilon(\cdot, 0) - \phi^{\varepsilon n}(\cdot, 0)\|_{L^2(\Omega)} \leq C, \quad (3.59)$$

for some constant  $C > 0$  independent of  $n$  and  $\varepsilon$ . If we multiply the first equation in (3.58) by  $q^\varepsilon$ , solution of (3.16), and integrate by parts, we obtain that

$$\begin{aligned} - \int_0^T \langle q_t^\varepsilon, z^{\varepsilon n} \rangle_{V^*, V} dt + \int_0^T \mathcal{E}(z^{\varepsilon n}, q^\varepsilon) dt + \int_\Omega z^{\varepsilon n}(x, T) q^\varepsilon(x, T) dx \\ - \int_\Omega z^{\varepsilon n}(x, 0) q^\varepsilon(x, 0) dx = \int_Q (f^\varepsilon - v_n^\varepsilon) q^\varepsilon dx dt. \end{aligned}$$

Since  $q^\varepsilon$  is a solution of (3.16), it follows that

$$\int_\Omega z^{\varepsilon n}(x, 0) (\rho^d(x) - \rho^\varepsilon(x, 0)) dx = \int_Q (f^\varepsilon - v_n^\varepsilon) q^\varepsilon dx dt.$$

- 4 Using the Cauchy Schwarz inequality, (3.59), and (3.29), we obtain that there is a constant  $C > 0$   
 5 independent of  $n$  and  $\varepsilon$  such that

$$\left| \int_Q (f^\varepsilon - v_n^\varepsilon) q^\varepsilon dx dt \right| \leq C. \quad (3.60)$$

Let  $v \in \mathbb{U}_{ad}$  be such that  $\|v - v_n^\varepsilon\|_{L^2(Q)} \leq r$ . Then

$$\begin{aligned} \int_Q (\xi f^\varepsilon - q^\varepsilon)(f - f^\varepsilon) dx dt &= \int_Q (\xi f^\varepsilon - q^\varepsilon)(f - v) dx dt + \int_Q (\xi f^\varepsilon - q^\varepsilon)(v - f^\varepsilon) dx dt \\ &= \int_Q \xi f^\varepsilon (f - v) dx dt - \int_Q q^\varepsilon (f - v) dx dt \\ &\quad + \int_Q (\xi f^\varepsilon - q^\varepsilon)(v - f^\varepsilon) dx dt. \end{aligned} \quad (3.61)$$

Setting

$$X^\varepsilon := \int_Q \xi f^\varepsilon (f - v) dx dt + \int_Q (\xi f^\varepsilon - q^\varepsilon)(v - f^\varepsilon) dx dt,$$

- 6 we have that

$$\int_Q (\xi f^\varepsilon - q^\varepsilon)(f - f^\varepsilon) dx dt = X^\varepsilon - \int_Q q^\varepsilon (f - v) dx dt. \quad (3.62)$$

1 Choosing  $v = v_n^\varepsilon$ , we obtain that

$$\int_Q (\xi f^\varepsilon - q^\varepsilon)(f - f^\varepsilon) dx dt = X_1^\varepsilon - \int_Q q^\varepsilon (f - v_n^\varepsilon) dx dt, \quad (3.63)$$

where

$$\begin{aligned} X_1^\varepsilon &= \int_Q \xi f^\varepsilon (f - v_n^\varepsilon) dx dt + \int_Q (\xi f^\varepsilon - q^\varepsilon)(v_n^\varepsilon - f^\varepsilon) dx dt \\ &= \int_Q \xi f^\varepsilon (f - v_n^\varepsilon) dx dt + \int_Q \xi f^\varepsilon (v_n^\varepsilon - f^\varepsilon) dx dt - \int_Q q^\varepsilon (v_n^\varepsilon - f^\varepsilon) dx dt. \end{aligned}$$

Using (3.55), (3.60), and (3.27), we can deduce that there is a constant  $C > 0$  independent of  $n$  and  $\varepsilon$  such that

$$|X_1^\varepsilon| \leq 2\xi C r + r = C(\xi, r).$$

It follows from (3.17) that

$$\left| \int_Q q^\varepsilon (f - v_n^\varepsilon) dx dt \right| \leq C(\xi, r) \quad \forall f \in \mathbb{U}_{ad} \text{ with } \|f - v_n^\varepsilon\|_{L^2(Q)} \leq r.$$

Consequently, we have that  $\|q^\varepsilon\|_{L^2(Q)} \leq C$  for some constant  $C > 0$  independent of  $\varepsilon$ . Thus, there exists  $\bar{q} \in L^2(Q)$  such that

$$q^\varepsilon \rightarrow \bar{q} \text{ weakly in } L^2(Q), \text{ as } \varepsilon \downarrow 0.$$

2 The proof is concluded. □

3 **Theorem 3.5.** Assume that  $\text{Int}(\mathbb{U}_{ad}) \neq \emptyset$ . Then,  $(\bar{f}, \bar{\rho})$  is the solution of the minimization problem  
4 (1.2)-(1.1) if and only if there exists  $\bar{q} \in L^2(Q)$  such that the triple  $(\bar{f}, \bar{\rho}, \bar{q})$  satisfies the following  
5 singular optimality systems:  $\bar{\rho}$  is a strong solution of

$$\begin{cases} \bar{\rho}_t + A\bar{\rho} = \bar{f} & \text{in } Q, \\ \bar{\rho}(\cdot, T) = \rho^T & \text{in } \Omega, \end{cases} \quad (3.64)$$

6 and

$$\int_Q \bar{q} (\phi_t + A\phi) dx dt = 0 \quad (3.65)$$

7 for every  $\phi \in \mathcal{D}((0, T), D(A))$ , and finally

$$\int_Q (\xi \bar{f} - \bar{q})(f - \bar{f}) dx dt \geq 0, \quad \forall f \in \mathbb{U}_{ad}. \quad (3.66)$$

8 *Proof.* From (3.47) and (3.41), we can deduce (3.64). Using (3.22) and (3.26) while passing to the  
9 limit, as  $\varepsilon \downarrow 0$ , we obtain (3.66). To complete the proof of the theorem, we need to prove (3.65).

10 Indeed, since  $q^\varepsilon \in L^2((0, T); \mathbb{V}) \cap H^1((0, T); \mathbb{V}^*)$  is a strong solution to (3.16), we have that

$$\int_Q q^\varepsilon \phi_t dx dt + \int_0^T \mathcal{E}(q^\varepsilon(t), \phi(t)) dt = 0, \quad \forall \phi \in \mathcal{D}((0, T); \mathbb{V}). \quad (3.67)$$

11 Integrating by parts, we obtain from (3.67) that

$$\int_Q q^\varepsilon (\phi_t + A\phi) dx dt = 0, \quad \forall \phi \in \mathcal{D}((0, T); D(A)). \quad (3.68)$$

Using (3.26) and taking the limit of (3.68) as  $\varepsilon \downarrow 0$ , we can deduce that

$$\int_Q \bar{q} (\phi_t + A\phi) dx dt = 0, \quad \forall \phi \in \mathcal{D}((0, T); D(A)).$$

1 We have shown (3.65) and the proof is finished.  $\square$

2 We conclude this section with the following observation.

**Remark 3.6.** In Theorem 3.5, we can in addition show that  $\bar{q}(\cdot, t) \in \mathbb{V}$  for a.e.  $t \in (0, T)$ . We are not able to prove that  $\bar{q} \in L^2((0, T); \mathbb{V}) \cap H^1((0, T); \mathbb{V}^*)$  so that  $\bar{q}$  will be a strong solution of the ill-posed system

$$\begin{cases} -\bar{q}_t + A\bar{q} = 0 & \text{in } Q \\ \bar{q}(\cdot, 0) = \rho^T & \text{in } \Omega, \end{cases}$$

3 which can be viewed as the dual system associated with (1.1). We do not know if such a result holds  
4 true. In the classical case of well-posed problems this is always true, but in the present setting we do  
5 not know. This is an interesting open problem that deserves to be investigated.

#### 6 4. SOME EXAMPLES OF OPERATORS

7 In this section we give some examples of operators satisfying Assumption 1.1. We start with the  
8 fractional Laplace operator.

9 **4.1. The fractional Laplace operator with Dirichlet exterior conditions.** In order to introduce  
10 the fractional Laplace operator with the zero Dirichlet exterior condition, we need some preparations.  
11 We start by introducing the function spaces.

Let  $\Omega \subset \mathbb{R}^N$  be an arbitrary open set. Given  $0 < s < 1$  a real number, we define the fractional order Sobolev space

$$H^s(\Omega) := \left\{ w \in L^2(\Omega) : \int_{\Omega} \int_{\Omega} \frac{|w(x) - w(y)|^2}{|x - y|^{N+2s}} dx dy < \infty \right\},$$

and we endow it with the norm given by

$$\|w\|_{H^s(\Omega)} := \left( \int_{\Omega} |w(x)|^2 dx + \int_{\Omega} \int_{\Omega} \frac{|w(x) - w(y)|^2}{|x - y|^{N+2s}} dx dy \right)^{\frac{1}{2}}.$$

We set

$$H_0^s(\Omega) := \left\{ w \in H^s(\mathbb{R}^N) : w = 0 \text{ in } \mathbb{R}^N \setminus \Omega \right\}.$$

12 Then,  $H_0^s(\Omega)$  endowed with the norm

$$\|w\|_{H_0^s(\Omega)} = \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|w(x) - w(y)|^2}{|x - y|^{N+2s}} dx dy \right)^{1/2}, \quad (4.1)$$

13 is a Hilbert space (see e.g. [19, Lemma 7] and [9, 11]). We let  $H^{-s}(\Omega) := (H_0^s(\Omega))^*$  denote the dual  
14 space of  $H_0^s(\Omega)$  with respect to the pivot space  $L^2(\Omega)$ , so that we have the following continuous and  
15 dense embeddings (see e.g. [2, 9, 11]):

$$H_0^s(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-s}(\Omega). \quad (4.2)$$

Under the assumption that  $\Omega$  is bounded and has a Lipschitz continuous boundary, we have that  $\mathcal{D}(\Omega)$  is dense in  $H_0^s(\Omega)$  for every  $0 < s < 1$  (see e.g. [10]). In addition, by [11, Chapter 1], if  $0 < s \neq 1/2 < 1$ , then

$$H_0^s(\Omega) = \overline{\mathcal{D}(\Omega)}^{H^s(\Omega)},$$

1 with equivalent norms, where  $\mathcal{D}(\Omega)$  denotes the space of all continuously infinitely differentiable func-  
 2 tions with compact support in  $\Omega$ . But if  $s = 1/2$ , then  $H_0^s(\Omega)$  is a proper subspace of  $\overline{\mathcal{D}(\Omega)}^{H^s(\Omega)}$ .

3 **Remark 4.1.** We observe the following facts. Let  $\Omega \subset \mathbb{R}^N$  be an arbitrary bounded open set.

4 (a) If  $N > 2s$ , then there exists a constant  $C_0 = C(N, s) > 0$  (depending only on  $\Omega$  and  $s$ ) such that  
 5 for every  $u \in H_0^s(\Omega)$ ,

$$\|u\|_{L^r(\Omega)} \leq C_0 \|u\|_{H_0^s(\Omega)}, \quad (4.3)$$

6 where  $r = \frac{2N}{N-2s}$  (see e.g. [19, Lemma 6 a]) and [9, 11]).

7 (b) It is well-known that the continuous injection

$$H_0^s(\Omega) \hookrightarrow L^2(\Omega), \quad (4.4)$$

8 is also compact, see [9, 11].

9 For more information on fractional order Sobolev spaces, we refer to [9, 11, 22] and their references.  
 Next, we give a rigorous definition of the fractional Laplacian. Given  $0 < s < 1$ , we let

$$\mathcal{L}_s^1(\mathbb{R}^N) := \left\{ w : \mathbb{R}^N \rightarrow \mathbb{R} \text{ measurable and } \int_{\mathbb{R}^N} \frac{|w(x)|}{(1+|x|)^{N+2s}} dx < \infty \right\}.$$

For  $w \in \mathcal{L}_s^1(\mathbb{R}^N)$  and  $\varepsilon > 0$ , we set

$$(-\Delta)_\varepsilon^s w(x) := C_{N,s} \int_{\{y \in \mathbb{R}^N : |x-y| > \varepsilon\}} \frac{w(x) - w(y)}{|x-y|^{N+2s}} dy, \quad x \in \mathbb{R}^N,$$

where  $C_{N,s}$  is a normalization constant given by

$$C_{N,s} := \frac{s 2^{2s} \Gamma\left(\frac{2s+N}{2}\right)}{\pi^{\frac{N}{2}} \Gamma(1-s)}, \quad (4.5)$$

10 and  $\Gamma$  denotes the usual Euler Gamma function.

The fractional Laplacian  $(-\Delta)^s$  is defined by the following singular integral:

$$(-\Delta)^s w(x) := C_{N,s} \text{P.V.} \int_{\mathbb{R}^N} \frac{w(x) - w(y)}{|x-y|^{N+2s}} dy = \lim_{\varepsilon \downarrow 0} (-\Delta)_\varepsilon^s w(x), \quad x \in \mathbb{R}^N, \quad (4.6)$$

11 provided that the limit exists for a.e.  $x \in \mathbb{R}^N$ . We refer to [9] and their references regarding the class  
 12 of functions for which the limit in (4.6) exists for a.e.  $x \in \mathbb{R}^N$ .

If the function  $w$  is smooth enough, say, in the Schwartz space  $\mathcal{S}(\mathbb{R}^N)$ , then  $(-\Delta)^s$  can be also defined as

$$(-\Delta)^s w(x) = -\frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \frac{w(x+y) + w(x-y) - w(x)}{|x-y|^{N+2s}} dy.$$

13 For more information on the fractional Laplace operator, we refer to [9, 12, 22] and their references.

14 Now we are ready to give our first example.

**Example 4.2 (The Dirichlet exterior condition).** Let  $\Omega \subset \mathbb{R}^N$  ( $N \geq 1$ ) be an arbitrary bounded open set with boundary  $\partial\Omega$  and  $(-\Delta)^s$  the fractional Laplace operator defined in (4.6). Let  $(-\Delta)_D^s$  be the realization in  $L^2(\Omega)$  of  $(-\Delta)^s$  with the zero Dirichlet exterior condition. More precisely,

$$D((-\Delta)_D^s) = \left\{ u \in H_0^s(\Omega) : (-\Delta)^s u \in L^2(\Omega) \right\}, \quad (-\Delta)_D^s u := (-\Delta)^s u \text{ in } L^2(\Omega).$$

It is well known that the operator  $-(-\Delta)_D^s$  generates a strongly continuous, compact, and analytic semigroup  $S = (S(t))_{t \geq 0}$  on  $L^2(\Omega)$  which is also submarkovian (see e.g. [6]). In addition, we have that the operator  $(-\Delta)_D^s$  has a compact resolvent and its spectrum is formed with eigenvalues  $\lambda_n$  satisfying

$$0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \leq \dots \quad \text{and} \quad \lim_{n \rightarrow \infty} \lambda_n = +\infty.$$

Thus, we can conclude that the operator  $(-\Delta)_D^s$  satisfies all the conditions in Assumption 1.1. We also mention that the bilinear form  $\mathcal{E}$  associated with  $(-\Delta)_D^s$  is given by  $D(\mathcal{E}) = H_0^s(\Omega)$  and

$$\mathcal{E}(u, v) = \frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dx dy.$$

1 The form  $\mathcal{E}$  is closed, continuous, and coercive.

2 **4.2. Second order elliptic operators with classical boundary conditions.** Let us consider the  
3 second order elliptic operator in divergence form

$$Lu = - \sum_{i,j=1}^N D_j(a_{ij}(x)D_j u),$$

where we assume that the coefficients  $a_{ij}$  are symmetric, that is,  $a_{ij} = a_{j,i}$  ( $i, j = 1, 2, \dots, N$ ), belong to  $L^\infty(\Omega)$ , and satisfy the ellipticity condition

$$\sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j \geq \mu|\xi|^2$$

4 for some constant  $\mu > 0$ .

5 Throughout the following,  $H_0^1(\Omega)$  and  $H^1(\Omega)$  shall denote the classical well-known first order Sobolev  
6 spaces.

**Example 4.3 (The Dirichlet boundary condition).** Let  $\Omega \subset \mathbb{R}^N$  be an arbitrary bounded open set. Let  $\mathcal{E}_D : H_0^1(\Omega) \times H_0^1(\Omega) \rightarrow \mathbb{R}$  be the bilinear, symmetric, continuous, and coercive form given by

$$\mathcal{E}_D(u, v) = \sum_{i,j=1}^N \int_{\Omega} a_{ij} D_i u D_j v dx.$$

Let  $A_D$  be the selfadjoint operator on  $L^2(\Omega)$  associated with  $(\mathcal{E}_D, H_0^1(\Omega))$  in the sense that

$$\begin{cases} D(A_D) = \{u \in H_0^1(\Omega) : \exists f \in L^2(\Omega) \text{ such that } \mathcal{E}_D(u, v) = (f, v)_{L^2(\Omega)} \forall v \in H_0^1(\Omega)\} \\ A_D u = f \end{cases}$$

7 It is well-known that  $A_D$  satisfies all the conditions in Assumption 1.1. We refer to [1, 7] and their  
8 references for more details.

**Example 4.4 (The Robin boundary condition).** Let  $\Omega \subset \mathbb{R}^N$  be a bounded open set with a Lipschitz continuous boundary  $\partial\Omega$ . Let  $\beta \in L^\infty(\partial\Omega)$  be such that  $\beta(x) \geq \beta_0 > 0$  for  $\sigma$ -a.e.  $x \in \partial\Omega$ , where  $\beta_0$  is a constant and  $\sigma$  denotes the usual Lebesgue surface measure. Let  $\mathcal{E}_R : H^1(\Omega) \times H^1(\Omega) \rightarrow \mathbb{R}$  be the bilinear, symmetric, continuous, and coercive form given by

$$\mathcal{E}_R(u, v) = \sum_{i,j=1}^N \int_{\Omega} a_{ij} D_i u D_j v dx + \int_{\partial\Omega} \beta uv d\sigma.$$

Let  $A_R$  be the selfadjoint operator on  $L^2(\Omega)$  associated with  $(\mathcal{E}_R, H^1(\Omega))$  in the sense that

$$\begin{cases} D(A_R) = \{u \in H^1(\Omega) : \exists f \in L^2(\Omega) \text{ such that } \mathcal{E}_R(u, v) = (f, v)_{L^2(\Omega)} \forall v \in H^1(\Omega)\} \\ A_R u = f \end{cases}$$

1 It is also well-known that  $A_R$  satisfies all the conditions in Assumption 1.1. We refer to [3, 4, 21] for  
2 more details.

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