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# A framework for analysing drawings as tools for mathematical reasoning

Astrid Hågensen Kleven

Norwegian University of Science and Technology [astrid.h.kleven@ntnu.no](mailto:astrid.h.kleven@ntnu.no)

*The current paper presents a way of explaining and understanding children's drawings as tools for mathematical reasoning. The paper has two aims; to present a framework for analysing drawings as tools for problem-solving and describe how different drawings support children's mathematical reasoning in problem-solving. The study shows that drawings can offer different support for children's reasoning. Some drawings offer explicit tools to be manipulated for solving a problem. Some drawings are structuring tools, supporting systematic checking and re-checking of solutions, while others do not help solve a problem, but serve as tools for communicating with others.*

*Keywords: reasoning, collaboration, drawing, representation, early years mathematics*

## **Introduction**

Reasoning has acquired a more prominent role in several nations' curriculum, including Norway (Utdanningsdirektoratet, 2019). The Norwegian curriculum states that children should learn to 'follow, assess and understand mathematical chains of thought' and 'formulate their own reasoning to understand and to solve problems' (Utdanningsdirektoratet, 2019). Even though teachers are expected to facilitate reasoning on all school levels, some find it challenging to identify reasoning in young pupils' actions or words and, therefore, challenging to facilitate (Bragg et al., 2016). They also claim that the development of mathematical reasoning requires 'appropriate encouragement and feedback from [their] teacher who can only do this if they recognise mathematical reasoning in children's actions and words' (p. 523). According to Battista (2016), mathematical reasoning is about making conclusions based on evidence or assumptions after having manipulated and analysed objects, representations, and statements (p.1), and Bragg et al. (2016) claim mathematical reasoning consists of 'following a line of enquiry, conjecturing relationships and generalisations, and developing an argument, justification or proof using mathematical language'. In this study, reasoning is understood as children making conclusions or assumptions regarding problems after manipulating the task using drawing, and using drawing to follow lines of inquiry while solving tasks.

One way of developing reasoning skills is through paying attention to the different ways children create meaning and their representations. Mathematical representations capture the process behind mathematical concepts or relationships (Woleck, 2001) and play a critical role in children's conceptual and mathematical development (MacDonald, 2013). Mathematical representations can be symbols, physical objects, verbal language, or drawings used to understand mathematical concepts and relations and are a part of construction knowledge (Bobis & Way, 2018). Despite a large body of research on representations, little research focuses on one of the most important representations for the youngest mathematicians; drawings (Woleck, 2001). Drawings can serve as mediation or re-presentation of meaning that allows children's inner pictures and reasoning to become available to others. Thus, children's drawings play a role in meaning-making, problem-solving and early symbolism (Thom & McGarvey, 2015). Drawing is one of children's earliest (mathematical)

representations (Papandreou, 2014; Woleck, 2001). Drawings and other written signs have a degree of permanence, allowing them to be re-examined, revised and argued for (Papandreou, 2019), which is why some claim that drawing play a central role in fostering mathematical reasoning for children (Saundry & Nicol, 2006).

By utilising children's representations of their understandings, like drawings or narratives, we are offered a window into their minds (MacDonald, 2013; Woleck, 2001). Children use drawings to bring their ideas or thoughts to the surface and to communicate them with others (Papandreou, 2014; Woleck, 2001). Learning to draw and drawing naturally occurs for most children, but the difference between drawing for fun and drawing with a purpose is big, and the transition from drawing out-of-school and drawing in-school contexts is a topic where several researchers claim teachers lack knowledge (Bakar, 2017; Woleck, 2001). Given the lack of research on drawings as representations (Woleck, 2001) and their central role in fostering children's mathematical reasoning (Saundry & Nicol, 2006), drawings should be offered more attention when researching reasoning and young children.

The project applies a sociocultural perspective on learning and development and considers knowledge to be developed and shared between people (Vygotsky, 1978). Although Vygotsky is mainly known for the relation between speech and reasoning, we can also relate his work to the drawing and reasoning. According to Vygotskian sociocultural theory, higher mental functions and human actions, like communication, are mediated by tools and signs (e.g. language and drawings) (Dahl et al., 2017). Tools are, within sociocultural theory, defined as the resources – both language-based and physical, available to us that we use to understand the world around us (Säljö, 2009, p. 21, my translation). Applying this perspective, viewing drawings as tools used to communicate and to understand the world around us, we can see the connection between drawing and reasoning.

This paper has two aims; the first is to present a theoretically anchored framework for analysing children's mathematical drawings as tools for mathematical reasoning. Which can be used for further research on drawing as a representation, both for researchers wanting to contribute or further develop the framework and teachers wanting to develop their pupils' competencies of using drawing as a meaningful representation and tool for reasoning. The second aim is to explain how different drawings support children's mathematical reasoning through some illustrative examples from two first-grade classrooms.

## **Framework for analysing drawings as tools in mathematical problem solving**

The framework that is the starting point for what is presented below was initially developed by me for my master thesis (Kleven, 2019). It considers two aspects of drawings in mathematics: the use of the drawing, and visual and multimodal aspects of drawings, and is a synthesis of other frameworks for examining drawings as a mathematical representation.

The first aspect concerns if and how drawings are used. It includes categories like used or did not use drawing, and different ways drawings that can be used for and of problem-solving (Saundry & Nicol, 2006). If used *for* problem-solving, the drawing is used for solving the problem, and drawing is both a process and a product (p. 57). In drawing for problem-solving, the drawing and problem-solving happen simultaneously. The drawing represents thoughts or internal pictures linked to children's

mathematical reasoning. A drawing *of* problem-solving is produced after the task is solved, representing a solution (rather than a process). It functions as a tool for communicating with others (rather than solving the problem). Even though drawings of problem-solving are considered to be produced after solving the problem, Saundry and Nicol (2006) claim that pupils' artefacts when working on mathematical problems are a part of their reasoning and cannot be separated from it.

The second included approach to analysing drawings are the *visual and multimodal aspects* of drawings. The distinction between pictographic and iconic drawings is commonly used to describe visual aspects of drawings which could be useful for identifying and describing mathematical reasoning. Both pictographic and iconic drawings can support children's reasoning and problem-solving in mathematics, but at different times and in different ways. A pictographic (situational) drawing can, according to Rellensmann et al. (2017), help pupils better understand a problem by providing a way of organising the information provided in the task. However, simultaneously, a pictographic drawing can also include irrelevant details, getting in the way of an effective problem-solving process. In contrast, an iconic, mathematical drawing often only includes relevant elements, but given its abstraction, it often requires a higher level of mathematical skills to utilise. Further categories are based on Papandreou (2009), and concern whether the pupils include numbers, letters or words in their solution, and if the children use gestures or verbal language as a supplementary mode of communication. Although the table below (table 1) presents the whole framework, this short paper will only focus on a few categories and aspects. The colour-coding in the table is as follows: use of drawings in blue, and visual and multimodal aspects of drawings in purple.

**Table 1 – A framework for analysing drawings as tools for mathematical problem solving**

Category	Description and indicators of use
Used drawing	The pupil found one or several solutions using drawing.
Did not use drawing	The pupil only used numbers, letters or did not draw at all. Including the use of other concrete manipulatives (like counting on the fingers or using counters). Both correct and incorrect answers included.
Manipulative	Movement, like circles or lines, represent calculations or operations, similarly to physical manipulatives (Saundry & Nicol, 2006; Woleck, 2001). Drawings are used to organise and count the elements needed to solve the problem (Woleck, 2001) and serves as a placeholder for thoughts.
System support	A passive drawing without movement. Drawings are used in an elimination process by systematically sorting elements and creating a structure (Saundry & Nicol, 2006). The drawing is crucial for solving the task, and pupils often use it to count and re-count their solution.
Narrative	Pupils create a story with elements in the task or their surrounding life (Soundy & Drucker, 2009), then act out the story. Creativity, previous knowledge, and the ability to differentiate relevant and irrelevant information become explicit.
Dramatic representation	The pupil draws themselves as part of the problem (Woleck, 2001). E.g., a pupil draws himself pointing at a number line, showing the physical drama of solving the problem.

Imagery/ Visualisation	The problem is solved internally. The process is played out in their mind, and then they draw to <i>communicate</i> a solution (Saundry & Nicol, 2006). Information is still processed visually, even though it is not always visible on paper.
Pictographic	Is recognised by its realism compared to the elements in the task. E.g., If the task asks the pupils to put flowers into vases, the pupils will draw either flowers, vases, or both. Also called situational drawings (Rellensmann et al., 2017). Depict the surface of the problem have a low level of abstraction.
Iconic	Simple lines and shapes created to imitate the elements in the problem. It lacks the realism or affiliation to reality found in pictographic drawings. Also called mathematical drawings (Rellensmann et al., 2017). It depicts a mathematical structure and has a high level of abstraction.
Symbolic	Pupils include numbers or letters/words. Defined in this framework as the conventional numerals 0,1,2,3,4,5,6,7,8,9, all letters/words and non-conventional number symbols (Papandreou, 2009).
Gestures and verbal language	The pupils supplement their solution with gestures or use verbal language to support their communication.

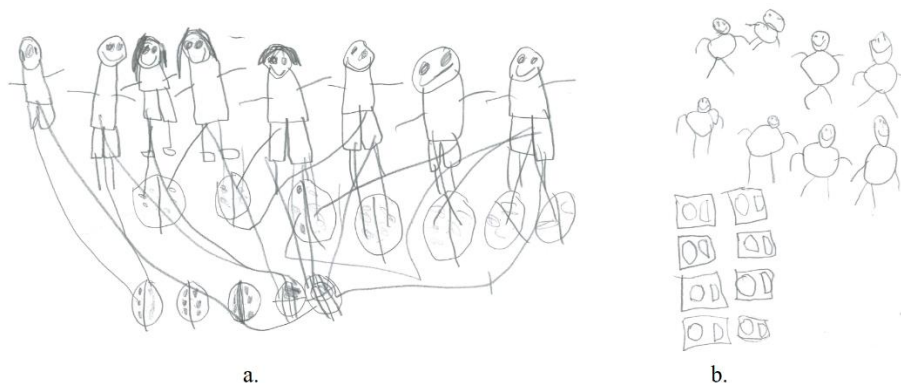
The framework above shows different aspects concerning drawing, that one might consider when researching childrens mathematical drawings. Several other aspects were also considered in putting together the framework, and there is a possibility to build further on the framework, including categories and aspects relevant for the particular study one is conducting. The framework is meant to be a flexible framework, where one can make adaptations based on context and needs.

The examples below are from an Educational Design Research study investigating how teachers can facilitate productive conversations, where young pupils are provided with opportunities to develop their reasoning competencies. Video recordings of twenty-seven first-grade pupils (age 5-7) and three teachers have been collected to understand how pupils use drawings to reason and communicate in mathematics. Pupils worked in groups (of 2-4) collaboratively solving problem-solving tasks in mathematics. Participation was based on informed consent from pupils and parents, and the names included below are pseudonyms. For analysing how drawings were used, collected pupil work and video recordings were used. Video recordings were helpful in addition to the drawings because video recordings could help determine whether the children drew while solving the problem or afterwards. All video recordings were transcribed and analysed to supplement the analysis of the drawings. After categorising the ways of using drawings, I aimed to identify reasoning in the drawings, using the video recordings and transcriptions as supplementary information about the process. The examples below are from different sessions, where pupils work on the following problems: (1) 8 children share 12 cookies equally. How many cookies does each child get? And (2) A farmer has some animals that altogether have 14 feet. How many animals, and what animals does the farmer have?

### **Reasoning in drawings for problem-solving**

The examples illustrate drawings used as manipulative and system support. In Figure 1a, drawing is used as manipulative to give one cookie to each child physically. We can see lines representing the process of sharing twelve cookies equally between eight children, resulting in each child getting one and a half cookies each. Based on the drawing, we can see that the pupils gave each child one cookie

and then split the leftover cookies to give the same amount. Based on the drawing and the video recording of the two pupils collaborating, the drawing appears to be closely linked to their reasoning process of giving as many whole cookies as possible first and then splitting the rest equally as well.



**Figure 1a - Manipulative and 1b - System support**

In comparison, Figure 1b shows a drawing used as system support for the same problem. The drawing provides an overview and organisation of the elements, but no operations or calculations are visible. In this drawing, we could need to turn to the video recording alone to identify whether the pupil put one and a half cookies in the lunch boxes at once, or if they put one cookie in each of the lunch boxes first, and then put half in each when it was not possible to give a whole cookie more to each child. Identifying the children's reasoning processes can be more difficult in drawings as system support, but the drawing still depicts some of the children's mental processes of sharing cookies equally.

### **Reasoning in drawings of problem-solving**

For identifying imagery, one needs to be present during the process or have recordings (audio or video); this is because one requirement is that the problem is already *solved* when the drawing is produced. As an example of imagery, I would like to highlight Leo and Sam working on problem 2, the farmer problem (figure 2a below). Initially, Sam suggests five sheep, five cows and four hens while holding up five, five and then four fingers. This adds up to a total of 14 animals. After a while, Sam realised that his answer was wrong because it was 14 animals and not legs. Sam starts counting using his fingers and stops while holding up eight fingers. He says, "Two cows", and then Leo draws. Sam continues using his fingers to count and ends up at one sheep and one hen. The boys count all the feet and end up at 14 feet and are satisfied. Sam solved the problem before they made the drawing, and the drawing was used to communicate with others, not to solve the problem. The boys did, however, use the drawing to check their solution.



**Figure 2a - Imagery and 2b – Visual aspects**

### **Visual aspects and multimodality**

As a short example of an analysis of visual aspects of a drawing, I would like to present a drawing made by a group of four pupils working on problem one, the cookie problem. Malin, Leo, Eric and Martin made an iconic drawing of cookies and children (figure 2b), in addition to a number line of the even numbers from 2-12. The video recordings show the four pupils discussing and trying to agree on which of the circles are children and which ones are cookies because they used the twelve circles to count both children and cookies. This sparks a discussion about if a circle is a cookie or a child, making the equal sharing of the cookies difficult. They are not able to solve the problem using drawing. In addition to using an iconic drawing, we can see that the group used both conventional number symbols and the words for children and cookies in Norwegian. The abstract nature of the drawing made it difficult for the children to reason both individually and in collaboration. A more pictographic drawing with a clear difference between children and cookies could have made this easier.

### **Discussion**

This study aimed at presenting a theoretical framework for describing and analysing how children use drawings in mathematical problem-solving and show some examples of how different drawings can support mathematical reasoning. The framework was presented as a whole, although I only focused on small aspects of the framework in this study. However, the framework is a synthesis of different theoretical frameworks for describing different aspects of drawing in mathematics, enabling research on many different areas or aspects connected to mathematical drawings.

This study further shows that different drawings can support differently in reasoning processes. In drawings as manipulatives (Figure 1a), we can see that a manipulative-drawing allows children to use the drawing to solve the problem physically. A drawing as a manipulative can support the pupils reasoning process by giving the pupil something concrete (external) to solve the problem, thereby allowing the internal processes to become external. All problem-solving processes are visible on paper and are useful for teachers to identify how the pupils solved the problem without being present during the process. Looking back at Battista (2016), defining reasoning to concern manipulating a problem to make decisions or conclusions, a drawing used as a manipulative explicitly enables the use of a drawing to solve the problem, making the drawing a tool for the child's reasoning. As we can see in figure 1a, the lines enable the children to give one and one cookie, keeping tabs on the division

happening in the problem. In drawings as system support, more extensive parts of the mathematical reasoning happen internally and therefore "beyond reach" of the drawing or those witnessing the process. Drawings as system support can support children's reasoning by providing them with a tool for organising elements systematically, making it easier to check multiple solutions, and keeping tabs of all elements of the problem without being overwhelmed by the problem itself (Woleck, 2001). As figure 1b shows, the lunch boxes with cookies allow the pupil to count and check his solution, both whilst solving the problem, and afterwards. Reasoning can also be identified in the process and products of pupils using drawings as *imagery*. A requirement for imagery is that the problem is already solved when drawing occurs. The drawing is then used to communicate or show the process or product to others. The drawing serves as mediation or re-presentation of meaning, allowing pupils inner pictures and reasoning to become available to others. Applying a sociocultural perspective allows us to think about the process (and product) of drawing as a direct re-presentation of mental pictures, and therefore as reasoning made available to others. To identify reasoning in drawings of problem-solving, we need to turn to children's verbal utterances and descriptions of their drawings, as illustrated above in the discussion in figure 2a.

Regardless of how the children use drawings, whether as manipulatives, system support, narratives or dramatic representations, all drawings have a degree of permanence to them, allowing them to be discussed, revised and argued for in collaboration with others or for oneself (Papandreou, 2019). By gaining knowledge on children's drawings as mathematical representations, we are offered a window into their minds and understanding of mathematical problems and concepts (MacDonald, 2013; Woleck, 2001). Though examining the children's drawings in the cookie-sharing context, one can gain insight into how young children treat the concept of dividing equally, which again can tell us something about their understanding of division more generally. Facilitating reasoning and sense-making should be the primary goal of mathematics instruction (Battista, 2016). It is, therefore, interesting to investigate how teachers can achieve this, particularly in the early years, where the amount of empirical research is limited.

A possible further analysis of data is needed, in order to see if there is possible to identify a connection between the different ways of using drawing (as a manipulative, system support and so on) and whether the students are able to present understandable chains of reasoning and arguments for their statements when communicating with others. Further studies into the use of more than one modality simultaneously and the quality of young children reasoning is another aspect which it will be both exiting and relevant to look at in the future.

The study had its limitations. What is presented above are empirical examples from one study, and should for that reason, not be generalised without considering all contributing factors. In addition, the framework could benefit from studies in other contexts with more participants with other prerequisites. Given the limitations in scope, several of the framework categories were not discussed, and more studies utilising and further developing the framework is both welcomed and needed to improve the quality of the proposed framework.

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