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1.1 Introduction

There is a vast number of recent studies concerning the fluid-structure interaction problems involving moving interfaces. We can classify these types of models broadly into two types: either the structure is moving inside the fluid or the structure is located at the boundary of the fluid domain. Since in this chapter we are interested in studying the motion of ball inside the compressible fluid domain, below we mention related works from the literature concerning this case only.

In [5], Desjardins and Esteban studied the global-in-time existence (up to contact) of weak solutions for the system of several rigid bodies interacting with a compressible viscous flow (for $\gamma \geq 2$) in a bounded domain of $\mathbb{R}^3$. Feireisl improved the global existence result (for $\gamma > 3/2$) in [8] and allowed the possible collisions of the rigid bodies or a contact of the rigid bodies with the exterior boundary. Regarding strong solutions, the existence and uniqueness of global solutions for small initial data have been achieved in [3] in the Hilbert space framework by Boulakia and Guerrero as long as no collisions occur. Their work is based on a method proposed by Matsumura and Nishida in [16] for a viscous compressible fluid (without structure). In a L^p-L^q setting, the authors in [10] proved the existence and uniqueness of local-in-time strong solutions for the system composed by rigid bodies immersed into a viscous compressible fluid and in [9], the authors establish the global in time existence up to contact.

Let us mention some works related to a control supported on the rigid body only. In the 1d case for a Burgers-particle system, the authors in [4] proved that the particle reaches a point arbitrarily close to a given target, whereas the velocities

of the fluid and of the particle are driven exactly to zero. In [17], the same result has been established but with a uniform time for all initial data. In [20], the authors consider the 3d case for a rigid ball moving into viscous incompressible fluid and obtain a stabilization result by using a control of spring and damper type connecting the center of the ball to a fixed point h_1. Recently the same problem has been tackled for the compressible viscous fluid in [18] and the main difference is that we need to deal with more regular solutions in this case than the incompressible one.

In [18], the stabilization result is obtained under a smallness condition on the initial velocities and on the distance between the initial position of the center of the ball and h_1. In the present work we would like to have a stabilization result similar to [18] but without the requirement that the initial position of the center of the ball is close to h_1. We achieve this by imposing a more complicated control law, in which the anchor point of the spring and damper is not fixed at h_1, but instead it jumps between a finite number of possible points. In [18], we work in the Hilbert space setting and we need to verify several compatibility conditions for the initial conditions. The main hindrance to impose such a switching control law in [18] is that we need to verify the compatibility conditions in each junction of such switching control. We find a remedy to such a problem by working in the L^p-L^q set up where we do not need to verify several compatibility conditions for the initial conditions.

1.1.1 Mathematical description of the model

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with C^4 boundary occupied by a fluid and a rigid ball. We denote by $\mathcal{S}(t) \subset \Omega$, the domain of the rigid ball at instant $t \geqslant 0$. We suppose that the fluid domain $\mathcal{F}(t) := \Omega \setminus \overline{\mathcal{S}(t)}$ is connected is for all $t \geqslant 0$. Furthermore, let us assume that initially the ball is away from boundary:

$$\operatorname{dist}(\mathcal{S}(0), \partial\Omega) \geqslant \nu > 0. \tag{1.1}$$

The fluid is modeled by the compressible Navier-Stokes system whereas the motion of the rigid ball is governed by the balance equations for linear and angular momentum. At time $t \geqslant 0$, let $\widetilde{h}(t) \in \mathbb{R}^3$, $Q(t) \in SO_3(\mathbb{R})$ and $\widetilde{\omega}(t) \in \mathbb{R}^3$ denote the position of the center of mass, the orthogonal matrix giving the orientation of the solid and the angular velocity of the rigid ball. Therefore we have,

$$\dot{Q}(t)Q(t)^{-1}y = \mathbb{A}(\widetilde{\omega}(t))y = \widetilde{\omega}(t) \times y \text{ for all } y \in \mathbb{R}^3,$$

where $\mathbb{A}(\omega)$ is the skew-symmetric matrix:

$$\mathbb{A}(\omega) = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}.$$

Without loss of generality we can assume that $Q(0) = I_3$. Thus the domain occupied by the structure $\mathcal{S}(t)$ at time $t \geqslant 0$ is given by

$$\mathcal{S}(t) = \left\{ \widetilde{h}(t) + Q(t)(y - h_0),\ y \in \mathcal{S}(0) \right\}, \quad t \geqslant 0 \tag{1.2}$$

where $h_0 \in \Omega$ is the initial position of the centre of the mass of the rigid ball.

The full system of equations modelling the rigid ball inside the fluid can be written as

$$\begin{cases} \dfrac{\partial \widetilde{\rho}}{\partial t} + \operatorname{div}(\widetilde{\rho}\widetilde{u}) = 0 & t > 0,\ x \in \mathcal{F}(t), \\ \widetilde{\rho}\left(\dfrac{\partial \widetilde{u}}{\partial t} + (\widetilde{u} \cdot \nabla)\, \widetilde{u} \right) - \operatorname{div}\sigma(\widetilde{u}, \widetilde{p}) = 0 & t > 0,\ x \in \mathcal{F}(t), \\ \widetilde{u}(t,x) = 0 & t > 0,\ x \in \partial\Omega, \\ \widetilde{u}(t,x) = \widetilde{h}'(t) + \widetilde{\omega}(t) \times (x - \widetilde{h}(t)) \quad t > 0,\ x \in \partial\mathcal{S}(t), & \\ m \dfrac{d^2}{dt^2}\widetilde{h} = - \displaystyle\int_{\partial\mathcal{S}(t)} \sigma(\widetilde{u}, \widetilde{p}) n \, \mathrm{d}\Gamma + w & t > 0, \\ J \dfrac{d}{dt}\widetilde{\omega} = (J\widetilde{\omega}) \times \widetilde{\omega} - \displaystyle\int_{\partial\mathcal{S}(t)} (x - h(t)) \times \sigma(\widetilde{u}, \widetilde{p})\, \mathrm{d}\Gamma & t > 0, \\ \widetilde{h}(0) = h_0, \quad \widetilde{h}'(0) = \ell_0, \quad \widetilde{\omega}(0) = \omega_0, & \\ \widetilde{\rho}(0,x) = \widetilde{\rho}_0(x), \quad \widetilde{u}(0,x) = u_0(x) & x \in \mathcal{F}(0). \end{cases} \tag{1.3}$$

In the above equations, $\widetilde{\rho} = \widetilde{\rho}(t,x)$ and $\widetilde{u} = \widetilde{u}(t,x)$ represent respectively the density and the velocity of the fluid and the pressure of the fluid is denoted by $\widetilde{p}$. We assume that the flow is in the barotropic regime and we focus on the isentropic case where the relation between $\widetilde{p}$ and $\widetilde{\rho}$ is given by the constitutive law:

$$\widetilde{p} = a\widetilde{\rho}^{\gamma},$$

with $a > 0$ and the adiabatic constant $\gamma \geqslant 1$. The Cauchy stress tensor is defined as:

$$\sigma(u,p) = 2\mu\mathbb{D}(u) + \alpha \operatorname{div} u I_3 - pI_3,$$

where $\mathbb{D}(u) = \frac{1}{2}\left(\nabla u + \nabla u^{\top}\right)$ denotes the symmetric part of the velocity gradient ($\nabla u^{\top}$ is the transpose of the matrix ∇u) and α, μ are the viscosity coefficients satisfying

$$\mu > 0, \quad \alpha + \frac{2}{3}\mu \geqslant 0. \tag{1.4}$$

We have denoted by $\partial\mathcal{S}(t)$ the boundary of the rigid structure at time t and by $n(t,x)$ the unit normal to $\partial\mathcal{S}(t)$ at the point x directed towards the interior of the rigid ball. Let m, J be the mass and the moment of inertia of the rigid ball

respectively. If $\rho_{\mathcal{S}}$ is the mass density of the ball, then the formulae for m and J are

$$m = \frac{4}{3}\pi\rho_{\mathcal{S}}, \quad J = \frac{2m}{5}I_3.$$

Finally, w in $(1.3)_5$ is our control that we take as a feedback control:

$$w(t) = k_p(h_1 - \widetilde{h}(t)) - k_d\widetilde{h}'(t), \tag{1.5}$$

where k_d and k_p are well-chosen so that

$$\lim_{t\to\infty} \widetilde{h}(t) = h_1,$$

whereas the velocities of the fluid and of the rigid ball go to 0:

$$\lim_{t\to\infty} \widetilde{u}(t) = 0, \quad \lim_{t\to\infty} \widetilde{h}'(t) = 0, \quad \lim_{t\to\infty} \widetilde{\omega}(t) = 0.$$

In literature, this type of control is known as Proportional-Derivative (PD) controller generated by a spring and a damper. The spring-damper is connected from the center of mass of the ball to the fixed anchor point h_1 and it is attracting the ball towards the point h_1.

Now we want to mention precisely the stabilization result obtained in [18] so that we can state clearly the main difference between [18] and our present work. Let us introduce

$$\Omega^0 := \{x \in \Omega \ ; \ \mathrm{dist}(x, \partial\Omega) > 1\}. \tag{1.6}$$

We assume that the set Ω^0 is connected.

For $0 \le T_1 < T_2 \le \infty$, we introduce the following space:

$$\begin{aligned}\widehat{\mathcal{S}}_{T_1,T_2} = \Big\{&(\rho, u, \ell, \omega) \mid \rho \in L^2(T_1, T_2; H^3(\mathcal{F}(t))) \cap C_b^0([T_1, T_2]; H^3(\mathcal{F}(t)))\\ &\cap H^1(T_1, T_2; H^2(\mathcal{F}(t)) \cap \ C_b^1([T_1, T_2]; H^2(\mathcal{F}(t))) \cap H^2(T_1, T_2; L^2(\mathcal{F}(t))),\\ &u \in L^2(T_1, T_2; H^4(\mathcal{F}(t))) \cap C_b^0([T_1, T_2]; H^3(\mathcal{F}(t))) \cap H^1(T_1, T_2; H^2(\mathcal{F}(t))\\ &\cap C_b^1([T_1, T_2]; H^1(\mathcal{F}(t))) \cap \ H^2(T_1, T_2; L^2(\mathcal{F}(t))), \ \ell \in H^2(T_1, T_2), \ \omega \in H^2(T_1, T_2)\Big\},\end{aligned} \tag{1.7}$$

where we denote by C_b^k the set of continuous and bounded functions with derivatives continuous and bounded up to the order k.

In [18], the authors proved the following result:

Theorem 1.1.1. *Assume that Ω^0 is non empty and connected. Let $h_1 \in \Omega^0$ and $\overline{\rho} > 0$. Assume w is given by the feedback law* (1.5)*. There exists $\delta > 0$ such that for any*

$$h_0 \in \Omega^0, \ \rho_0 \in H^3(\mathcal{F}(0)), \ \rho_0 > 0, \ u_0 \in H^3(\mathcal{F}(0)), \ \ell_0, \ \omega_0 \in \mathbb{R}^3, \tag{1.8}$$

satisfying the compatibility conditions

$$u_0(y) = \ell_0 + \omega_0 \times (y - h_0) \text{ for } y \in \partial\mathcal{B}(0), \quad u_0 = 0 \text{ on } \partial\Omega, \tag{1.9}$$

$$-\frac{1}{\rho_0}\operatorname{div}\sigma(u_0,p_0)=0 \text{ on } \partial\Omega, \tag{1.10}$$

$$\begin{aligned}&\Big(\omega_0\times(\omega_0\times(y-h_0))\Big)-\frac{1}{\rho_0}\operatorname{div}\sigma(u_0,p_0)(y)\\&=\frac{1}{m}\left[\int\limits_{\partial\mathcal{B}(0)}\sigma(u_0,p_0)n\,d\Gamma\right]+\left[J^{-1}\int\limits_{\partial\mathcal{B}(0)}(x-h_0)\times\sigma(u_0,p_0)n\,d\Gamma\right]\times(y-h_0),\\&\qquad\qquad\text{for } y\in\partial\mathcal{B}(0),\end{aligned} \tag{1.11}$$

where

$$p_0=a\rho_0^\gamma.$$

and the smallness condition

$$\|(\rho_0,u_0,\ell_0,\omega_0)\|_{\widehat{\mathcal{S}}_{0,0}}+|h_1-h_0|\le\delta, \tag{1.12}$$

the system (1.3) *admits a unique strong solution* $(\rho,u,\ell,\omega)\in\widehat{\mathcal{S}}_{0,\infty}$, $h\in L^\infty(0,\infty)$. *Moreover, the solution* (ρ,u,h,ℓ,ω) *of* (1.3) *satisfies*

$$\begin{gathered}\lim_{t\to\infty}\|\rho(t,\cdot)-\overline{\rho}\|_{H^2(\mathcal{F}(t))}=0,\quad \lim_{t\to\infty}\|u(t,\cdot)\|_{H^2(\mathcal{F}(t))}=0,\\ \lim_{t\to\infty}h(t)=h_1,\quad \lim_{t\to\infty}\ell(t)=0,\quad \lim_{t\to\infty}\omega(t)=0.\end{gathered}$$

The above theorem tells us that we can move the ball from any initial point $h_0\in\Omega^0$ asymptotically to another point $h_1\in\Omega^0$, if h_0 and h_1 are sufficiently close to each other, by connecting a spring and a damper to the ball and pulling the ball towards h_1.

In this work, we would like to have a result that tells us similar to the above theorem, but without the requirement that h_0 is close to h_1 (see the precise statement in Theorem 1.4.1). The main novelties of our work are:

- In [18], we need several compatibility conditions (1.9)–(1.11) to prove the global existence theorem under the presence of feedback control w of the form (1.5). But in the present work, we only need (1.16) (which is same as (1.9)) to prove the global existence result under same feedback.
- We have obtained the stability properties (1.63)–(1.64) for the velocities of the fluid and body and the position of the body without the closeness assumption for initial and final position of the body.
- We have developed a complicated control law in which the anchor point of the spring and damper is not fixed but instead it jumps between a finite number of possible points.

The outline of the chapter is as follows. In Section 1.2, we have introduced the necessary notations required throughout the chapter. The global in time existence and asymptotic analysis of solutions using the feedback (1.5) under a smallness assumption of the initial data has been established in Section 1.3. One of the main difficulties to do so is that the domain of the fluid equation is one of the unknowns. We overcome this difficulty by introducing Lagrangian variables. Apart from allowing to rewrite the coupled system in a fixed cylindrical domain this allows us to tackle the term $u\cdot\nabla\rho$ in the density equation. Next we associate to the original nonlinear problem a linear one, involving source terms and establish the L^p-L^q regularity property for this linear problem by proving that the associated linear operators are $\mathcal{R}$-sectorial in an appropriate Banach spaces. The existence and uniqueness proof is completed by estimating the nonlinear terms and by using Banach fixed point theorem. The exponential stability of the semigroup generated by the part of the fluid-structure operator helps us to analyze the asymptotic behaviour of the solutions. In Section 1.4, we obtain the stability properties (1.63)–(1.64) where we can skip the smallness condition on the distance between the initial and final position of the ball by introducing a switching feedback control.

1.2 Notations

In this section, we fix notations, that will be used throughout the paper. Firstly $W^{s,q}(\mathcal{F})$, with $s\geqslant 0$ and $q>1$, denote the usual Sobolev spaces. Let $k\in\mathbb{N}$. For every $0<s<k$, $1\leqslant p<\infty$, $1\leqslant q<\infty$, we define the Besov spaces by real interpolation of Sobolev spaces

$$B^s_{q,p}(\mathcal{F})=(L^q(\mathcal{F}),W^{k,q}(\mathcal{F}))_{s/k,p}\ .$$

We refer to [1] and [22] for a detailed presentation of Besov spaces. If $T\in(0,\infty]$, we set

$$W^{1,2}_{p,q}((0,T);\mathcal{F})=L^p(0,T;W^{2,q}(\mathcal{F}))\cap W^{1,p}(0,T;L^q(\mathcal{F})).$$

It is well known that the following embedding holds

$$W^{1,2}_{p,q}((0,T);\mathcal{F})\hookrightarrow C^0_b([0,T);B^{2(1-1/p)}_{q,p}(\mathcal{F})),$$

where we denote by C^k_b the set of continuous and bounded functions with derivatives continuous and bounded up to the order k. In view of this observation, we use the following norm form $W^{1,2}_{p,q}((0,T);\mathcal{F})$:

$$\|f\|_{W^{1,2}_{p,q}((0,T);\mathcal{F})}:=\|f\|_{L^p(0,T;W^{2,q}(\mathcal{F}))}+\|f\|_{W^{1,p}(0,T;L^q(\mathcal{F}))}+\|f\|_{C^0_b([0,T);B^{2(1-1/p)}_{q,p}(\mathcal{F}))}.$$

We also introduce the functional spaces with time decay. For any $\beta\in\mathbb{R}$, $p\in[1,\infty]$ and $\mathbb{X}$ a Banach space, we define

$$L^p_\beta(0,\infty;\mathbb{X})=:\left\{f\mid t\to e^{\beta t}f(t)\in L^p(0,\infty;\mathbb{X})\right\},$$

and a similar notation for $W^{1,2}_{p,q,\beta}((0,\infty)\times\mathcal{F})$.

We also need a definition of Sobolev spaces in the time dependent domain $\mathcal{F}(t)$. Let $\Lambda(t,\cdot)$ be a C^1-diffeomorphism from $\mathcal{F}(0)$ onto $\mathcal{F}(t)$ such that all the derivatives up to second order in space variable and all the derivatives up to first order in time variable exist. For all functions $v(t,\cdot):\mathcal{F}(t)\to\mathbb{R}$, we denote $\widehat{v}(t,y)=v(t,\Lambda(t,y))$. Then for any $1<p,q<\infty$ we define

$$
\begin{aligned}
&L^p(0,T;L^q(\mathcal{F}(\cdot))=\{v\mid \widehat{v}\in L^p(0,T;L^q(\mathcal{F}(0)))\},\\
&L^p(0,T;W^{2,q}(\mathcal{F}(\cdot)))=\left\{v\mid \widehat{v}\in L^p(0,T;W^{2,q}(\mathcal{F}(0)))\right\},\\
&W^{1,p}(0,T;L^q(\mathcal{F}(\cdot)))=\left\{v\mid \widehat{v}\in W^{1,p}(0,T;L^q(\mathcal{F}(0)))\right\},\\
&C([0,T];W^{1,q}(\mathcal{F}(\cdot)))=\left\{v\mid \widehat{v}\in C([0,T];W^{1,q}(\mathcal{F}(0)))\right\},\\
&C([0,T];B^{2(1-1/p)}_{q,p}(\mathcal{F}(\cdot)))=\left\{v\mid \widehat{v}\in C([0,T];B^{2(1-1/p)}_{q,p}(\mathcal{F}(0))\right\}.
\end{aligned}
$$

1.3 Local stabilization

In this section, we prove a stabilization result when h_0 and h_1 are sufficiently close to each other.

Let us now give the conditions we require on initial data and on (p,q) for the system (1.3):

$$2<p<\infty,\qquad 3<q<\infty,\qquad \frac{1}{p}+\frac{1}{2q}\neq\frac{1}{2},\tag{1.13}$$

$$h_0\in\Omega^0,\quad h_1\in\Omega^0,\quad \ell_0\in\mathbb{R}^3,\quad \omega_0\in\mathbb{R}^3,,\tag{1.14}$$

$$\widetilde{\rho}_0\in W^{1,q}(\mathcal{F}(0)),\quad u_0\in B^{2(1-1/p)}_{q,p}(\mathcal{F}(0))^3,\quad \min_{\overline{\mathcal{F}(0)}}\widetilde{\rho}_0>0,\tag{1.15}$$

with the compatibility conditions

$$u_0=0 \text{ on } \partial\Omega \text{ and } u_0(y)=\ell_0+\omega_0\times(y-h_0),\ y\in\partial\mathcal{S}(0).\tag{1.16}$$

Our first main result in this section asserts global existence and uniqueness of the system (1.3) under smallness assumption on the initial data.

Theorem 1.3.1. *Let us assume (p,q) satisfy (1.13), and let $\overline{\rho}>0$ be a given constant. Assume that Ω^0 (defined in (1.6)) is non empty, connected and (1.1) is satisfied, and $h_1\in\Omega^0$. Assume w is given by the feedback law (1.5) with $k_p>0$, $k_d>0$. Then there exist $\beta>0$ and $\delta>0$, such that, for any $(\widetilde{\rho}_0,u_0,h_0,\ell_0,\omega_0)$ satisfying (1.14)-(1.16), with*

$$\overline{\rho}=\frac{1}{|\mathcal{F}(0)|}\int_{\mathcal{F}(0)}\widetilde{\rho}_0\ \mathrm{d}x,\tag{1.17}$$

and

$$\|\widetilde{\rho}_0-\overline{\rho}\|_{W^{1,q}(\mathcal{F}(0))}+\|u_0\|_{B^{2(1-1/p)}_{q,p}(\mathcal{F}(0))^3}+\|h_0-h_1\|_{\mathbb{R}^3}+\|\ell_0\|_{\mathbb{R}^3}+\|\omega_0\|_{\mathbb{R}^3}\leqslant\delta,\tag{1.18}$$

the system (1.3) *admits a unique strong solution* $(\widetilde{\rho}, \widetilde{u}, \widetilde{h}, \widetilde{\ell}, \widetilde{\omega})$ *satisfying*

$$\widetilde{\rho} \in C_b^0([0,\infty); W^{1,q}(\mathcal{F}(\cdot))),\ \nabla\widetilde{\rho} \in W^{1,p}_\beta(0,\infty; L^q(\mathcal{F}(\cdot))),$$
$$\widetilde{u} \in W^{1,2}_{p,q,\beta}((0,\infty); \mathcal{F}(\cdot)),$$
$$\widetilde{h} - h_1 \in W^{2,p}_\beta(0,\infty;\mathbb{R}^3), \quad \widetilde{\ell} \in W^{1,p}_\beta(0,\infty;\mathbb{R}^3), \quad \widetilde{\omega} \in W^{1,p}_\beta(0,\infty;\mathbb{R},^3).$$

Moreover, $\widetilde{\rho}(t,x) \geqslant \dfrac{\overline{\rho}}{2}$ *for all* $t \in (0,\infty)$, $x \in \overline{\mathcal{F}(t)}$ *and* $\operatorname{dist}(\mathcal{S}(t), \partial\Omega) \geqslant \nu/2$ *for all* $t \in [0,\infty)$.

As a simple consequence of the above theorem, we have the following local stabilisation result

Corollary 1.3.2. *With the assumptions and notation in Theorem 1.3.1 we have*

$$\|\widetilde{\rho}(t,\cdot) - \overline{\rho}\|_{W^{1,q}(\mathcal{F}(t))} + \|\widetilde{u}(t,\cdot)\|_{B^{2(1-1/p)}_{q,p}(\mathcal{F}(t))^3} + \|\widetilde{h}(t) - h_1\|_{\mathbb{R}^3} + \|\widetilde{\ell}(t)\|_{\mathbb{R}^3} + \|\widetilde{\omega}(t)\|_{\mathbb{R}^3} \leqslant C\delta e^{-\beta t},$$

where the constant C *is independent of* $t > 0$. *In particular,*

$$\lim_{t\to\infty} \|\widetilde{\rho}(t,\cdot) - \overline{\rho}\|_{W^{1,q}(\mathcal{F}(t))} = 0, \quad \lim_{t\to\infty} \|\widetilde{u}(t,\cdot)\|_{B^{2(1-1/p)}_{q,p}(\mathcal{F}(t))^3} = 0,$$
$$\lim_{t\to\infty} \widetilde{h}(t) = h_1, \quad \lim_{t\to\infty} \widetilde{\ell}(t) = 0, \quad \lim_{t\to\infty} \widetilde{\omega}(t) = 0.$$

The rest of this section is devoted towards the proofs of Theorem 1.3.1 and Corollary 1.3.2.

1.3.1 Change of variables

In order to prove Theorem 1.3.1, we first reformulate the problem in a fixed spatial domain. Let us set $\mathcal{F} = \mathcal{F}(0)$ and $\mathcal{S} = \mathcal{S}(0)$. We consider the characteristics X associated with the fluid velocity $\widetilde{u}$:

$$\begin{cases} \partial_t X(t,y) = \widetilde{u}(t, X(t,y)) & t > 0, \\ X(0,y) = y \in \overline{\mathcal{F}}. \end{cases} \tag{1.19}$$

Due to the boundary conditions of $\widetilde{u}$, we have

$$X(t,y) = \begin{cases} h(t) + Q(t)(y - h_0) & \text{if } y \in \partial\mathcal{S}, \\ y & \text{if } y \in \partial\Omega. \end{cases}$$

Assume that $X(t,\cdot)$ is a C^1-diffeomorphism from $\overline{\mathcal{F}}$ onto $\overline{\mathcal{F}(t)}$ for all $t > 0$. For each $t > 0$, we denote by $Y(t,\cdot) = X(t,\cdot)^{-1}$ the inverse of $X(t,\cdot)$. We consider the following change of variables

$$\rho(t,y) = \widetilde{\rho}(t, X(t,y)) - \overline{\rho}, \qquad u(t,y) = Q^\top(t)\widetilde{u}(t, X(t,y)), \tag{1.20}$$
$$h(t) = \widetilde{h}(t) - h_1, \qquad \ell(t) = Q^\top(t)\widetilde{h}'(t), \qquad \omega(t) = Q^\top(t)\widetilde{\omega}(t), \tag{1.21}$$

for $(t,y) \in (0,\infty) \times \mathcal{F}$. In particular,

$$\widetilde{\rho}(t,x) = \overline{\rho} + \rho(t, Y(t,x)), \qquad \widetilde{u}(t,x) = Q(t)u(t, Y(t,x)), \tag{1.22}$$

for $(t,x) \in (0,\infty) \times \mathcal{F}(t)$.

The above change of variable is the slight variation of the usual Lagrangian change of coordinates, which allows us to rewrite the governing equations in a fixed spatial domain and to preserve the linear form of the transmission condition for the velocity field.

The system satisfied by $(\rho, u, h, \ell, \omega)$ reads as follows:

$$\begin{cases} \partial_t \rho + \overline{\rho}\mathrm{div} u = \mathcal{F}_1 & \text{in } (0,\infty) \times \mathcal{F}, \\ \partial_t u - \mathrm{div}\sigma_l(\rho,u) = \mathcal{F}_2 & \text{in } (0,\infty) \times \mathcal{F}, \\ u = 0 & \text{on } (0,\infty) \times \partial\Omega, \\ u = \ell + \omega \times y & \text{on } (0,\infty) \times \partial\mathcal{S}, \\ \dfrac{d}{dt} h = \ell & t \in (0,\infty), \\ \dfrac{d}{dt}\ell + m^{-1}k_p h + m^{-1}k_d \ell = -m^{-1}\displaystyle\int_{\partial\mathcal{S}} \sigma_l(\rho,u)n \, \mathrm{d}\Gamma + \mathcal{G}_1 & t \in (0,\infty), \\ \dfrac{d}{dt}\omega = -J(0)^{-1}\displaystyle\int_{\partial\mathcal{S}} y \times \sigma_l(\rho,u)n \, \mathrm{d}\Gamma + \mathcal{G}_2 & t \in (0,\infty), \\ \rho(0) = \rho_0 := \widetilde{\rho}_0 - \overline{\rho}, \quad u(0) = u_0, & \text{in } \mathcal{F}, \\ h(0) = h_0 - h_1, \quad \ell(0) = \ell_0, \quad \omega(0) = \omega_0, \end{cases} \tag{1.23}$$

where

$$\sigma_l(\rho,u) = \frac{2\mu}{\overline{\rho}}\mathbb{D}u + \frac{1}{\overline{\rho}}\left(\alpha \mathrm{div} u - a\gamma(\overline{\rho})^{\gamma-1}\rho\right) I_3, \quad \mathbb{D}(u) = \frac{1}{2}(\nabla u + \nabla u^\top), \tag{1.24}$$

and

$$X(t,y) = y + \int_0^t Q(s)u(s,y) \, \mathrm{d}s, \tag{1.25}$$

for every $y \in \mathcal{F}$ and $t \geqslant 0$. Using the notation

$$Z(t,y) = (Z_{ij})_{1\leqslant i,j\leqslant 3} = [\nabla X]^{-1}(t,y) \qquad (t \geqslant 0,\ y \in \mathcal{F}), \tag{1.26}$$

the remaining terms in (1.23) are defined by

$$\mathcal{F}_1(\rho,u,h,\ell,\omega) = -\rho \mathrm{div} u - (\rho + \overline{\rho})(Z^\top - I_3) : \nabla u, \tag{1.27}$$

$$
\begin{aligned}
(\mathcal{F}_2)_i(\rho,u,h,\ell,\omega) =& -(\rho+\overline{\rho})(\omega\times Qu)_i+\rho(\partial_t u)_i-(\rho+\overline{\rho})\left[(Q-I_3)\partial_t u\right]_i\\
&+\mu\sum_{l,j,k}\frac{\partial^2(Qu)_i}{\partial y_l\partial y_k}\left(Z_{kj}-\delta_{kj}\right)Z_{l,j}+\mu\sum_{l,k}\frac{\partial^2(Qu)_i}{\partial y_l\partial y_k}\left(Z_{lk}-\delta_{lk}\right)\\
&+\mu\left[(Q-I_3)\Delta u\right]_i+\mu\sum_{l,j,k}Z_{lj}\frac{\partial(Qu)_i}{\partial y_k}\frac{\partial Z_{kj}}{\partial y_l}\\
&+(\mu+\alpha)\sum_{l,j,k}\frac{\partial^2(Qu)_j}{\partial y_l\partial y_k}\left(Z_{kj}-\delta_{kj}\right)Z_{li}+(\mu+\alpha)\sum_{l,j}\frac{\partial^2(Qu)_j}{\partial y_l\partial y_j}\left(Z_{li}-\delta_{li}\right)\\
&+(\alpha+\mu)\frac{\partial}{\partial y_i}\left[\nabla u:(Q^\top-I_3)\right]+\frac{\alpha+\mu}{\overline{\rho}}\sum_{l,j,k}Z_{li}\frac{\partial(Qu)_j}{\partial y_k}\frac{\partial Z_{kj}}{\partial y_l}\\
&-a\gamma\frac{(\widetilde{\rho}+\overline{\rho})^{\gamma-1}}{\overline{\rho}}\sum_{j,l}Q_{ji}\frac{\partial\rho}{\partial y_l}(Z_{jl}-\delta_{j,l}),
\end{aligned}
\tag{1.28}
$$

$$
\begin{aligned}
\mathcal{G}_0(\rho,u,h,\ell,\omega) =& \frac{\mu}{\overline{\rho}}\left[\nabla u(ZQ-I_3)+\left[(ZQ)^\top-I_3\right](\nabla u)^\top\right]\\
&+\frac{\alpha}{\overline{\rho}}\left(([ZQ]^\top-I_3):\nabla u\right)I_3-\frac{a}{\overline{\rho}}(\overline{\rho}+\rho)^\gamma I_3,
\end{aligned}
\tag{1.29}
$$

$$
\mathcal{G}_1(\rho,u,h,\ell,\omega)=-m^{-1}k_p(Q^\top-I_3)h-\frac{m}{\overline{\rho}}(\omega\times\ell)-\int_{\partial\mathcal{S}}\mathcal{G}_0 n\,\mathrm{d}\Gamma,
\tag{1.30}
$$

$$
\mathcal{G}_2(\rho,u,h,\ell,\omega)=\frac{J(0)}{\overline{\rho}}\omega\times\omega-\int_{\partial\mathcal{S}}y\times\mathcal{G}_0 n\,\mathrm{d}\Gamma.
\tag{1.31}
$$

Using the above change of variables, Theorem 1.3.1 can be reformulated as

Theorem 1.3.3. *Let us assume (p,q) satisfy (1.13), and let $\overline{\rho}>0$ be a given constant as in (1.17). Assume that Ω^0 (defined in (1.6)) is non empty, connected and (1.1) is satisfied, $h_1\in\Omega^0$, $k_p>0$, and $k_d>0$. Then there exist $\beta>0$ and $\delta>0$, such that, for any $(\rho_0+\overline{\rho},u_0,h_0,\ell_0,\omega_0)$ satisfying (1.14)-(1.16) and*

$$
\|\rho_0\|_{W^{1,q}(\mathcal{F}(0))}+\|u_0\|_{B^{2(1-1/p)}_{q,p}(\mathcal{F}(0))^3}+\|h_0-h_1\|_{\mathbb{R}^3}+\|\ell_0\|_{\mathbb{R}^3}+\|\omega_0\|_{\mathbb{R}^3}\leqslant\delta,
\tag{1.32}
$$

the system (1.23)–(1.29) admits a unique strong solution (ρ,u,h,ℓ,ω) satisfying

$$
\begin{aligned}
\|\rho\|_{L^\infty(0,\infty;W^{1,q}(\mathcal{F}))}&+\|\nabla\rho\|_{W^{1,p}_\beta(0,\infty;L^q(\mathcal{F}))}+\|\partial_t\rho\|_{L^p_\beta(0,\infty;L^q(\mathcal{F}))}\\
&+\|u\|_{W^{1,2}_{p,q,\beta}((0,\infty);\mathcal{F})^3}+\|h\|_{W^{2,p}_\beta(0,\infty;\mathbb{R}^3)}\\
&+\|\ell\|_{W^{1,p}_\beta(0,\infty;\mathbb{R}^3)}+\|\omega\|_{W^{1,p}_\beta(0,\infty;\mathbb{R}^3)}\leqslant C\delta.
\end{aligned}
\tag{1.33}
$$

Moreover, $X\in L^\infty(0,\infty;W^{2,q}(\mathcal{F}))^3\cap W^{1,\infty}(0,\infty;W^{1,q}(\mathcal{F}))$ and $X(t,\cdot):\mathcal{F}\to\mathcal{F}(t)$ is a C^1-diffeormorphim for all $t\in[0,\infty)$.

The rest of this section is devoted towards the proof of above theorem.

1.3.2 Linearized System

The proof of Theorem 1.3.3 relies on a linearization and a fixed point theorem. Thus as a first step, we linearise the system (1.23)–(1.29) around $(\overline{\rho}, 0, 0, 0, 0)$. More precisely, we consider the following linear system

$$\begin{cases} \partial_t \rho + \overline{\rho}\mathrm{div} u = f_1 & \text{in } (0,\infty)\times\mathcal{F}, \\ \partial_t u - \mathrm{div}\sigma_l(\rho,u) = f_2 & \text{in } (0,\infty)\times\mathcal{F}, \\ u = 0 & \text{on } (0,\infty)\times\partial\Omega, \\ u = \ell + \omega\times y & \text{on } (0,\infty)\times\partial\mathcal{S}, \\ \dfrac{\mathrm{d}}{\mathrm{dt}} h = \ell & \text{in } (0,\infty), \\ \dfrac{\mathrm{d}}{\mathrm{dt}}\ell + m^{-1}k_p h + m^{-1}k_d\ell = -m^{-1}\displaystyle\int_{\partial\mathcal{S}} \sigma_l(\rho,u)n \ \mathrm{d}\Gamma + \mathrm{g}_1 & \text{in } (0,\infty), \\ \dfrac{\mathrm{d}}{\mathrm{dt}}\omega = -J(0)^{-1}\displaystyle\int_{\partial\mathcal{S}} y\times\sigma_l(\rho,u)n \ \mathrm{d}\Gamma + \mathrm{g}_2 & \text{in } (0,\infty), \\ \rho(0) = \rho_0, \quad u(0) = u_0, & \text{in } \mathcal{F}, \\ h(0) = h_0 - h_1, \quad \ell(0) = \ell_0, \quad \omega(0) = \omega_0, \end{cases} \tag{1.34}$$

where $\sigma_l(\rho,u)$ is defined as in (1.24).

The above system is known as the linearized fluid-structure system. We can say the considered system is "monolithic", in the sense that the linear system preserve the fluid-structure coupling. Our aim is to study the regularity and decay properties of the above linear system. To this aim, in the section below, we will introduce the corresponding linear *fluid-structure* operator $\mathcal{A}_{FS}$ in a suitable Banach space $\mathcal{X}$. We are going to show that the operator $\mathcal{A}_{FS}$ is $\mathcal{R}$-sectorial in $\mathcal{X}$ (see Theorem 1.3.8). Then according to [23, Theorem 4.3], the operator $\mathcal{A}_{FS}$ has the maximal L^p regularity property in the Banach space $\mathcal{X}$. This will help us to obtain $L^p - L^q$ type regularity of the linear system (1.34). However, in order to obtain exponential decay we need to consider certain subspace $\mathcal{X}_{\mathrm{m}}$ of $\mathcal{X}$. This is done in Theorem 1.3.9. Finally, the exponential stability of the system, $L^p - L^q$ type regularity and the Banach fixed point theorem, will help us to conclude the proof of Theorem 1.3.3.

Let us mention that, maximal $L^p - L^q$ type regularity of various fluid-structure system has been studied recently by many authors. We refer to the articles [12, 15, 9, 13, 14] and the references therein for a detailed presentation on this topic. In fact, our analysis here is based on the articles [12, 15] and [9].

1.3.3 The fluid-structure operator

In this subsection, we want to rewrite the system (1.34) in a suitable operator. We first introduce the operator

$$\mathcal{D}(A_{\rm u}) = W^{2,q}(\mathcal{F}) \cap W_0^{1,q}(\mathcal{F}), \quad A_{\rm u} = \frac{\mu}{\overline{\rho}}\Delta + \frac{\alpha+\mu}{\overline{\rho}}\nabla{\rm div}. \tag{1.35}$$

It is known that, the operator $A_{\rm u}$ is an isomorphism from $\mathcal{D}(A_{\rm u})$ to $L^q(\mathcal{F})^3$ for any $q \in (1,\infty)$ (see for instance [19]). Given $(\ell,\omega) \in \mathbb{C}^3 \times \mathbb{C}^3$, let us consider the following problem

$$\begin{cases} -\frac{\mu}{\overline{\rho}}\Delta W - \frac{\alpha+\mu}{\overline{\rho}}\nabla{\rm div} W = 0 \text{ in } \mathcal{F}, \\ W = \ell + \omega \times y \text{ on } \partial\mathcal{S}, \quad W = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.36}$$

Using the trace properties, it is easy to verify that the above system admits a unique solution $W \in W^{2,q}(\mathcal{F})^3$. Therefore, we can define the Dirichlet operator

$$D_{\rm s} \in \mathcal{L}(\mathbb{C}^3 \times \mathbb{C}^3; W^{2,q}(\mathcal{F})^3), \qquad D_{\rm s}(\ell,\omega) = W, \tag{1.37}$$

where W solves (1.36). For $q \in (1,\infty)$, let us set

$$\mathcal{X} = W^{1,q}(\mathcal{F}) \times L^q(\mathcal{F})^3 \times \mathbb{C}^3 \times \mathbb{C}^3 \times \mathbb{C}^3. \tag{1.38}$$

Using the above definitions, we introduce the *fluid-structure operator* $\mathcal{A}_{\rm FS}$: $\mathcal{D}(\mathcal{A}_{\rm FS}) \to \mathcal{X}$ defined by

$$\mathcal{D}(\mathcal{A}_{\rm FS}) = \Big\{ (\rho,u,h,\ell,\omega) \in W^{1,q}(\mathcal{F}) \times W^{2,q}(\mathcal{F})^3 \times \mathbb{C}^3 \times \mathbb{C}^3 \times \mathbb{C}^3 \,; \\ u - D_{\rm s}(\ell,\omega) \in \mathcal{D}(A_{\rm u}) \Big\}, \tag{1.39}$$

and

$$\mathcal{A}_{\rm FS} = \mathcal{A}^0_{\rm FS} + \mathcal{B}_{\rm FS},$$

with

$$\mathcal{A}^0_{\rm FS} \begin{bmatrix} \rho \\ u \\ h \\ \ell \\ \omega \end{bmatrix} = \begin{bmatrix} -\overline{\rho}{\rm div}\ u \\ A_{\rm u}\left(u - D_{\rm s}(\ell,\omega)\right) \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{1.40}$$

$$\mathcal{B}_{\rm FS} \begin{bmatrix} \rho \\ u \\ h \\ \ell \\ \omega \end{bmatrix} = \begin{bmatrix} 0 \\ -a\gamma(\overline{\rho})^{\gamma-1} \\ \ell \\ -m^{-1}\displaystyle\int_{\partial\mathcal{S}} \sigma_l(\rho,u)n \ {\rm d}\Gamma - {\rm m}^{-1}{\rm k_p h} - {\rm m}^{-1}{\rm k_d}\ell \\ -J(0)^{-1}\displaystyle\int_{\partial\mathcal{S}} y \times \sigma_l(\rho,u)n \ {\rm d}\Gamma \end{bmatrix}. \tag{1.41}$$

The following equivalence holds:

Theorem 1.3.4. *Let $1 < p < \infty$ and $1 < q < \infty$. Let $h \in W^{2,p}(0,\infty;\mathbb{C}^3)$, $\ell \in W^{1,p}(0,\infty;\mathbb{C}^3)$, $\omega \in W^{1,p}(0,\infty;\mathbb{C}^3)$, $\rho \in W^{1,p}(0,\infty;W^{1,q}(\mathcal{F}))$, $u \in W^{1,2}_{p,q}((0,\infty);\mathcal{F})^3$. Then (ρ,u,h,ℓ,ω) is a solution to the system (1.34) if and only if (ρ,u,h,ℓ,ω) satisfies*

$$\frac{\mathrm{d}}{\mathrm{dt}}\begin{bmatrix}\rho\\u\\h\\\ell\\\omega\end{bmatrix} = \mathcal{A}_{\mathrm{FS}}\begin{bmatrix}\rho\\u\\h\\\ell\\\omega\end{bmatrix} + \begin{bmatrix}f_1\\f_2\\0\\g_1\\g_2\end{bmatrix} \qquad (t\in(0,\infty)), \qquad \begin{bmatrix}\rho(0)\\u(0)\\h(0)\\\ell(0)\\\omega(0)\end{bmatrix} = \begin{bmatrix}\rho_0\\u_0\\h_0\\\ell_0\\\omega_0\end{bmatrix}. \tag{1.42}$$

1.3.4 Some background on $\mathcal{R}$-sectorial operators

In this section, we briefly introduce $\mathcal{R}$-sectorial operators. We refer to the article by Weis [23] for a detailed presentation and properties of $\mathcal{R}$-sectorial operators. First, let us give definition of $\mathcal{R}$-boundedness family of operators.

Definition 1.3.5. *Assume $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces and $\mathcal{E} \subset \mathcal{L}(\mathcal{X},\mathcal{Y})$. We say that $\mathcal{E}$ is $\mathcal{R}$-bounded if there exist $p \in [1,\infty)$ and a constant $C > 0$, such that for any integer $N \geqslant 1$, any $T_1, T_2, \cdots, T_N \in \mathcal{E}$ any independent Rademacher random variables $r_1, r_2, \cdots, r_N$, and any $x_1, x_2, \cdots, x_N \in \mathcal{X}$,*

$$\left(\mathbb{E}\left\|\sum_{j=1}^{N} r_j T_j x_j\right\|_{\mathcal{Y}}^{p}\right)^{1/p} \leqslant C\left(\mathbb{E}\left\|\sum_{j=1}^{N} r_j x_j\right\|_{\mathcal{X}}^{p}\right)^{1/p},$$

where $\mathbb{E}$ denotes the expectation of a random variable.

Note that the above definition is independent of $p \in [1,\infty)$. Also, if $\mathcal{E} \subset \mathcal{L}(\mathcal{X},\mathcal{Y})$ is $\mathcal{R}$-bounded, then it is uniformly bounded. On the other hand if $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert spaces, then every the definition of $\mathcal{R}$-boundedness coincides with uniform boundedness.

For any $\theta \in (0,\pi)$, we define the sector

$$\Sigma_\theta = \{\lambda \in \mathbb{C}\setminus\{0\} \mid |\arg\lambda| < \theta\}. \tag{1.43}$$

We now give definition of $\mathcal{R}$-sectorial operators

Definition 1.3.6. *Let $\mathcal{X}$ be a Banach space, and let $A : \mathcal{D}(A) \to \mathcal{X}$ be a densely defined closed linear operator on $\mathcal{X}$. The operator A is said to be $\mathcal{R}$-sectorial of angle $\theta \in (0,\pi)$ if $\Sigma_\theta \subset \rho(A)$, and if the set $\left\{\lambda(\lambda - A)^{-1} \mid \lambda \in \Sigma_\theta\right\}$ is $\mathcal{R}$-bounded in $\mathcal{L}(\mathcal{X})$.*

If $\mathcal{X}$ is a Hilbert space, then the definition of $\mathcal{R}$-sectorial operators coincides with the definition of sectorial operators. We also recall that, a Banach space $\mathcal{X}$ is said to be an UMD Banach space if the Hilbert transformer is bounded in $L^p(\mathbb{R};\mathcal{X})$ for $p \in (1,\infty)$.

Let us consider the following system:

$$u' = Au + f \quad \text{in} \quad (0,\infty), \quad u(0) = u_0. \tag{1.44}$$

For our purpose, we need the following version of maximal L^p regularity result:

Theorem 1.3.7. *Let $\mathcal{X}$ be a UMD Banach space, $1 < p < \infty$ and let A be a closed, densely defined operator in $\mathcal{X}$ with domain $\mathcal{D}(A)$. Let us assume that A is a $\mathcal{R}$-sectorial operator of angle $\theta > \frac{\pi}{2}$ and that the semigroup generated by A has negative exponential type. Then for every $u_0 \in (\mathcal{X}, \mathcal{D}(A))_{1-1/p,p}$ and for every $f \in L^p(0,\infty;\mathcal{X})$, (1.44) admits a unique solution in $L^p(0,\infty;\mathcal{D}(A)) \cap W^{1,p}(0,\infty;\mathcal{X})$.*

Proof. The proof follows from [23, Theorem 4.2], [6, Theorem 2.4] and [21, Theorem 1.8.2]. □

1.3.5 $\mathcal{R}$-sectoriality of the fluid-structure operator

We recall the definition of the space $\mathcal{X}$ and the operator $\mathcal{A}_{\rm FS}$ from subsection 1.3.3. We prove the following result on $\mathcal{R}$-sectoriality of the fluid-structure operator $\mathcal{A}_{\rm FS}$:

Theorem 1.3.8. *Let $1 < q < \infty$. The space $\mathcal{X}$ is an UMD Banach space, and there exists $\gamma > 0$ such that $\mathcal{A}_{\rm FS} - \gamma$ is an $\mathcal{R}$-sectorial operator in $\mathcal{X}$ of angle $\theta > \pi/2$.*

Proof. First of all, from [7, Theorem 2.5] we have that $\mathcal{A}_{\rm u} - \gamma$ is a $\mathcal{R}$-sectorial operator of angle $\theta > \beta$ on $L^q(\mathcal{F})^3$. By standard calculation, it is easy to verify that for $\lambda \in \gamma + \Sigma_\theta$, we have

$$\lambda(\lambda I_3 - \mathcal{A}^0_{\rm FS})^{-1} = \begin{bmatrix} \mathrm{Id} & -\overline{\rho}\mathrm{div}(\lambda I_3 - \mathcal{A}_{\rm u})^{-1} & \overline{\rho}\mathrm{div}\mathcal{A}_{\rm u}(\lambda I_3 - \mathcal{A}_{\rm u})^{-1}\widetilde{D}_{\rm s}\lambda^{-1} \\ 0 & \lambda(\lambda I_3 - \mathcal{A}_{\rm u})^{-1} & -\mathcal{A}_{\rm u}(\lambda I_3 - \mathcal{A}_{\rm u})^{-1}\widetilde{D}_{\rm s} \\ 0 & 0 & I_{9\times 9} \end{bmatrix}$$

where $\widetilde{D}_{\rm s}(h,\ell,\omega) = D_{\rm s}(\ell,\omega)$. Then using the properties of $\mathcal{R}$-boundedness operators, we can deduce that $\mathcal{A}^0_{\rm FS} - \gamma$ is an $\mathcal{R}$-sectorial operator of angle $\theta > \pi/2$ (see for instance [13, Theorem 4.2], [9, Theorem 3.12], [12] for more details about the proof).

Next, using trace results, for any $1/q < s < 1$, there exists a constant $C > 0$ such that

$$\left\| \mathcal{B}_{\rm FS} \begin{bmatrix} \rho \\ u \\ h \\ \ell \\ \omega \end{bmatrix} \right\|_{\mathcal{X}} \leqslant C \left(\|\rho\|_{W^{s,q}(\mathcal{F})} + \|u\|_{W^{1+s,q}(\mathcal{F})^3} + \|(h,\ell,\omega)\|_{\mathbb{C}^3\times\mathbb{C}^3\times\mathbb{C}^3} \right).$$

Since the embedding $W^{1,q}(\mathcal{F}) \hookrightarrow W^{s,q}(\mathcal{F})$ is compact for $s \in (1/q, 1)$, we obtain

that for any $\delta > 0$ there exists $C(\delta) > 0$ such that

$$\left\| \mathcal{B}_{\mathrm{FS}} \begin{pmatrix} \rho \\ u \\ h \\ \ell \\ \omega \end{pmatrix} \right\|_{\mathcal{X}} \leqslant \delta \left\| \mathcal{A}^0_{\mathrm{FS}} \begin{pmatrix} \rho \\ u \\ h \\ \ell \\ \omega \end{pmatrix} \right\|_{\mathcal{X}} + C(\delta) \left\| \begin{pmatrix} \rho \\ u \\ h \\ \ell \\ \omega \end{pmatrix} \right\|_{\mathcal{X}}.$$

In particular, $\mathcal{B}_{\mathrm{FS}}$ is a $\mathcal{A}^0_{\mathrm{FS}}$-bounded perturbation. Finally, using [11, Corollary 2] (see also [12, Proposition 1.18]) we conclude the proof of the theorem. □

1.3.6 Exponential stability of the fluid-structure operator

The aim of this subsection is to show the operator $\mathcal{A}_{\mathrm{FS}}$ generates an analytic semigroup of negative type in the following subspace of $\mathcal{X}$:

$$\mathcal{X}_{\mathrm{m}} = \left\{ (f_1, f_2, g_1, g_2, g_3) \in \mathcal{X} \mid \int_{\mathcal{F}} f_1 \mathrm{dy} = 0 \right\}.$$

One can check that the space $\mathcal{X}_{\mathrm{m}}$ is invariant under $(e^{t\mathcal{A}_{\mathrm{FS}}})_{t\geqslant 0}$. Therefore we can consider the restriction of $\mathcal{A}_{\mathrm{FS}}$ to the domain $\mathcal{D}(\mathcal{A}_{\mathrm{FS}}) \cap \mathcal{X}_{\mathrm{m}}$. We prove the following theorem

Theorem 1.3.9. *Let $1 < q < \infty$. Then the part of $\mathcal{A}_{\mathrm{FS}}$ in $\mathcal{X}_{\mathrm{m}}$ generates an exponentially stable semigroup $(e^{t\mathcal{A}_{\mathrm{FS}}})_{t\geqslant 0}$ on $\mathcal{X}_{\mathrm{m}}$. In other words, there exist constants $C > 0$ and $\beta_0 > 0$ such that*

$$\|e^{t\mathcal{A}_{\mathrm{FS}}}(\rho_0, u_0, h_0, \ell_0, \omega_0)^\top\|_{\mathcal{X}_{\mathrm{m}}} \leqslant Ce^{-\beta_0 t}\|(\rho_0, u_0, h_0, \ell_0, \omega_0)^\top\|_{\mathcal{X}_{\mathrm{m}}}, \tag{1.45}$$

for all $(\rho_0, u_0, h_0, \ell_0, \omega_0) \in \mathcal{X}_{\mathrm{m}}$.

Remark 1.3.10. *When $\lambda = 0$, integrating the first equation of (1.46) and using the boundary conditions of u we have*

$$\int_{\mathcal{F}} f_1 \mathrm{dy} = 0.$$

Thus to study exponential stability of the fluid-structure semigroup it is necessary to consider the space $\mathcal{X}_{\mathrm{m}}$.

Proof of Theorem 1.3.9. We first note that the standard results on analytic semigroups (see [2, Proposition 2.9, p.120]) imply that the exponential stability of the semigroup generated by the part of $\mathcal{A}_{FS}$ in $\mathcal{X}_{\mathrm{m}}$ is equivalent to establish

$$\{\lambda \in \mathbb{C} \ ; \ \mathrm{Re}\lambda \geq -\beta\} \subset \rho(\mathcal{A}_{\mathrm{FS}}|_{\mathcal{D}(\mathcal{A}_{\mathrm{FS}})\cap\mathcal{X}_{\mathrm{m}}}), \text{ for some } \beta > 0.$$

We divide our proof into two steps.
Step 1. $\{\lambda \in \mathbb{C} \ ; \ \mathrm{Re}\lambda \geq 0\} \subset \rho(\mathcal{A}_{\mathrm{FS}}|_{\mathcal{D}(\mathcal{A}_{\mathrm{FS}})\cap\mathcal{X}_{\mathrm{m}}})$.

Let $\lambda \in \mathbb{C}^+$ and $(f_1, f_2, g_0, g_1, g_2) \in \mathcal{X}_{\rm m}$. We consider the following resolvent problem

$$\begin{cases} \lambda\rho + \overline{\rho}{\rm div}u = f_1 & \text{in } \mathcal{F}, \\ \lambda u - {\rm div}\sigma_l(\rho, u) = f_2 & \text{in } \mathcal{F}, \\ u = \ell + \omega \times y & \text{on } \partial\mathcal{S}, \\ u = 0 & \text{on } \partial\Omega, \\ \lambda h = \ell + g_0, & \\ \lambda\ell + m^{-1}k_p h + m^{-1}k_d\ell + m^{-1}\displaystyle\int_{\partial\mathcal{S}} \sigma_l(\rho, u)n \ {\rm d}\Gamma = {\rm g}_1, & \\ \lambda\omega + J(0)^{-1}\displaystyle\int_{\partial\mathcal{S}} y \times \sigma_l(\rho, u)n \ {\rm d}\Gamma = {\rm g}_2. & \end{cases} \tag{1.46}$$

We are going to show that the system (1.46) admits a unique solution $(\rho, u, h, \ell, \omega) \in \mathcal{D}(\mathcal{A}_{\rm FS}) \cap \mathcal{X}_{\rm m}$. The proof is divided into several parts.

Step 1.1. Existence and uniqueness for $\lambda = 0$. Consider the system (1.46) with $\lambda = 0$. Then $\ell = -g_0$. Let $\chi \in C_c^\infty(\Omega)$ such that $\chi = 1$ on $\overline{\mathcal{S}}$. We define $v = u - \chi\ell$. Then it is easy to see that, (ρ, v, ω) satisfies

$$\begin{cases} -\mu\Delta v + a\gamma\overline{\rho}^{\gamma-1}\nabla\rho = \overline{\rho}f_2 + \dfrac{\alpha+\mu}{\overline{\rho}}\nabla f_1 & \\ \qquad\qquad -\dfrac{\mu}{\overline{\rho}}\Delta(\chi\ell) - \dfrac{\alpha+\mu}{\overline{\rho}}\nabla({\rm div}(\chi\ell)) & \text{in } \mathcal{F}, \\ {\rm div}v = \dfrac{f_1}{\overline{\rho}} - {\rm div}(\chi\ell) & \text{in } \mathcal{F}, \\ v = \omega \times y & \text{on } \partial\mathcal{S}, \\ v = 0 & \text{on } \partial\Omega, \\ J(0)^{-1}\displaystyle\int_{\partial\mathcal{S}} y \times \sigma_0(\rho, v)n \ {\rm d}\gamma = g_2 - \dfrac{\alpha}{\overline{\rho}^2}\displaystyle\int_{\partial\mathcal{S}} y \times f_1 {\rm d}\gamma, & \end{cases} \tag{1.47}$$

where

$$\sigma_0(\rho, v) = \frac{2\mu}{\overline{\rho}}\mathbb{D}u + \frac{1}{\overline{\rho}}\left(-a\gamma(\overline{\rho})^{\gamma-1}\rho\right) I_3.$$

The above system is similar to the Stokes-rigid ball system studied in [15]. In fact following the proof of Theorem 4.2 in [15], we can show that, the system (1.47) admits a unique solution $(\rho, v, \omega) \in W^{1,q}(\mathcal{F}) \times W^{2,q}(\mathcal{F})^3 \times \mathbb{C}^3$. Finally, the existence of h follows from equation $(1.46)_6$.

Step 1.2. Existence for $\lambda \neq 0$, $\lambda \in \mathbb{C}^+$. By setting $\rho = \dfrac{1}{\lambda}(f_1 - \overline{\rho}{\rm div}u)$ and $h =$

$\frac{1}{\lambda}(\ell + g_0)$ the system (1.46) can be rewritten as

$$\begin{cases} \lambda u - \mathrm{div}\widehat{\sigma}_\lambda(u) = \widehat{f}_2 & \text{in } \mathcal{F}, \\ u = \ell + \omega \times y & \text{on } \partial\mathcal{S}, \\ u = 0 & \text{on } \partial\Omega, \\ \left(\lambda + m^{-1}\dfrac{k_p}{\lambda} + m^{-1}k_d\right)\ell = -m^{-1}\displaystyle\int_{\partial\mathcal{S}} \widehat{\sigma}_\lambda(u)\ n\mathrm{d}\Gamma + \widehat{\mathrm{g}}_1 & \\ \lambda\omega = -J(0)^{-1}\displaystyle\int_{\partial\mathcal{S}(t)} y \times \widehat{\sigma}_\lambda(u) n\mathrm{d}\Gamma + \widehat{\mathrm{g}}_2 & \end{cases} \tag{1.48}$$

where

$$\widehat{\sigma}_\lambda(u) = \frac{2\mu}{\overline{\rho}}\mathbb{D}(u) + \left(\frac{1}{\overline{\rho}}\left(\alpha + \frac{a\gamma\overline{\rho}^\gamma}{\lambda}\right)\mathrm{div}u\right) I_3,$$

$$\widehat{f}_2 = f_2 - \frac{a\overline{\rho}^{\gamma-1}}{\lambda\overline{\rho}}\nabla f_1, \quad \widehat{g}_1 = \left(g_1 + m^{-1}\frac{a\overline{\rho}^{\gamma-1}}{\lambda\overline{\rho}}\int_{\partial\mathcal{S}} f_1 n\mathrm{d}\Gamma\right) - \frac{m^{-1}k_p}{\lambda}g_0,$$

$$\widehat{g}_2 = \left(g_2 + J(0)^{-1}\frac{a\overline{\rho}^{\gamma-1}}{\lambda\overline{\rho}}\int_{\partial\mathcal{S}} y \times f_1 n\mathrm{d}\Gamma\right).$$

The above system is similar to the system (3.39) of [9]. Thus following the idea of the proof of Theorem 3.16 in [9], we can conclude the existence for $\lambda \in \mathbb{C}^+, \lambda \neq 0$.

Step 1.3. Uniqueness for $\lambda \neq 0$, $\lambda \in \mathbb{C}^+$. Let us assume that $(\rho, u, h, \ell, \omega) \in \mathcal{D}(\mathcal{A}_{\mathrm{FS}}) \cap \mathcal{X}_{\mathrm{m}}$ solves the system (1.46) with $(f_1, f_2, g_0, g_1, g_2) = 0$. We claim that, $(\rho, u, h, \ell, \omega) \in W^{1,2}(\mathcal{F}) \times W^{2,2}(\mathcal{F}) \times \mathbb{C}^3 \times \mathbb{C}^3 \times \mathbb{C}^3$. We set

$$\rho = \frac{1}{\lambda}(-\overline{\rho}\mathrm{div}u), \quad h = \frac{1}{\lambda}\ell. \tag{1.49}$$

We take $\lambda_0 > 0$ and the system (1.46) with $(f_1, f_2, g_0, g_1, g_2) = 0$ can be rewritten as

$$\begin{cases} \lambda_0 u - \mathrm{div}\widehat{\sigma}_\lambda(u) = (\lambda_0 - \lambda)u & \text{in } \mathcal{F}, \\ u = \ell + \omega \times y & \text{on } \partial\mathcal{S}, \\ u = 0 & \text{on } \partial\Omega, \\ \left(\lambda_0 + m^{-1}\dfrac{k_p}{\lambda} + m^{-1}k_d\right)\ell = -m^{-1}\displaystyle\int_{\partial\mathcal{S}} \widehat{\sigma}_\lambda(u)\ n\mathrm{d}\Gamma + (\lambda_0 - \lambda)\ell & \\ \lambda_0\omega = -J(0)^{-1}\displaystyle\int_{\partial\mathcal{S}(t)} y \times \widehat{\sigma}_\lambda(u) n\mathrm{d}\Gamma + (\lambda_0 - \lambda)\omega. & \end{cases} \tag{1.50}$$

Since $W^{2,q}(\mathcal{F}) \hookrightarrow L^2(\mathcal{F})$, we deduce from (1.50) that $(u, \ell, \omega) \in W^{2,2}(\mathcal{F}) \times \times \mathbb{C}^3 \times \mathbb{C}^3$. Using the expressions of ρ and h in (1.49), we conclude that $\rho \in W^{1,2}(\mathcal{F})$ and $h \in \mathbb{C}^3$.

Multiplying $(1.46)_2$ by $\overline{v}$, $(1.46)_6$ by $\overline{\ell}$ and $(1.46)_7$ by $\overline{\omega}$, after integration by parts and taking the real part, we obtain

$$\mathrm{Re}\lambda \int_{\mathcal{F}} |u|^2 \mathrm{d}y + 2\mu \int_{\mathcal{F}} |\mathbb{D}u|^2 \mathrm{d}y + \alpha \int_{\mathcal{F}} |\mathrm{div} u|^2 \mathrm{d}y \\ + (\mathrm{Re}\lambda) m|\ell|^2 + \mathrm{Re}(\lambda J(0)\omega \cdot \overline{\omega}) + k_p|h|^2 + k_d|\ell|^2 = 0.$$

Since $\mathrm{Re}\lambda \geqslant 0$, using (1.4), $k_p, k_d > 0$ and the boundary conditions, we obtain $\rho = u = h = \ell = \omega = 0$.

Step 2. $\{\lambda \in \mathbb{C}\ ;\ \mathrm{Re}\lambda \geq -\beta\} \subset \rho(\mathcal{A}_{\mathrm{FS}}|_{\mathcal{D}(\mathcal{A}_{\mathrm{FS}})\cap\mathcal{X}_{\mathrm{m}}})$, **for some** $\beta > 0$**.**

We know from Theorem 1.3.8 that there exists $\gamma > 0$ such that $\mathcal{A}_{\mathrm{FS}} - \gamma$ is an $\mathcal{R}$-sectorial operator in $\mathcal{X}$ of angle greater than $\pi/2$. It implies that there exists $C_1 > 0$ such that for any $\lambda \in \gamma + \Sigma_{\pi-\theta}$ (the notation Σ_θ is defined in (1.43)) with $\theta < \pi/2$,

$$\|(\lambda - \mathcal{A}_{FS})^{-1}\|_{\mathcal{L}(\mathcal{X}_{\mathrm{m}})} \leq C_1.$$

Since

$$\{\lambda \in \mathbb{C} \ \mid\ \mathrm{Re}\lambda \geq 0\} \setminus [\gamma + \Sigma_{\pi-\theta}]$$

is a compact set, there exists $C_2 > 0$ such that for any $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda \geq 0$

$$\|(\lambda - \mathcal{A}_{FS})^{-1}\|_{\mathcal{L}(\mathcal{X}_{\mathrm{m}})} \leq C.$$

This yields that for some $\beta > 0$,

$$\{\lambda \in \mathbb{C} \ \mid\ \mathrm{Re}\lambda \geq -\beta\} \subset \rho(\mathcal{A}_{FS}|_{\mathcal{D}(\mathcal{A}_{\mathrm{FS}})\cap\mathcal{X}_{\mathrm{m}}}).$$

This completes the proof of the theorem. □

1.3.7 Maximal $L^p - L^q$ Regularity for the Linearized Fluid-Structure System

In this section, we prove maximal $L^p - L^q$ type regularity result with decay properties of the linear system (1.34). Following [9, 13], we introduce the following standard decomposition: for any $f \in L^1(\mathcal{F})$,

$$f = f_{\mathrm{m}} + f_{\mathrm{avg}}, \quad \text{with} \quad \int_{\mathcal{F}} f_{\mathrm{m}}(y)\ \mathrm{d}y = 0, \quad f_{\mathrm{avg}} = |\mathcal{F}|^{-1} \int_{\mathcal{F}} f(y)\ \mathrm{d}y. \tag{1.51}$$

For $1 < p, q < \infty$, and $\beta \in \mathbb{R}$, the solution space $\mathcal{E}_{p,q,\beta}$ is defined by

$$\mathcal{E}_{p,q,\beta} = \Big\{(\rho, u, h, \ell, \omega) \mid \rho = \rho_{\mathrm{m}} + \rho_{\mathrm{avg}},\ \rho_{\mathrm{m}} \in W^{1,p}_\beta(0,\infty; W^{1,q}(\mathcal{F})), \rho_{\mathrm{avg}} \in L^\infty(0,\infty), \\ \partial_t \rho_{\mathrm{avg}} \in L^p_\beta(0,\infty), \quad u \in W^{1,2}_{p,q,\beta}((0,\infty);\mathcal{F})^3, \\ h \in W^{2,p}_\beta(0,\infty;\mathbb{R}^3), \quad \ell \in W^{1,p}_\beta(0,\infty;\mathbb{R}^3), \quad \omega \in W^{1,p}_\beta(0,\infty;\mathbb{R}^3)\Big\}$$

and

$$\|(\rho,u,h,\ell,\omega)\|_{\mathcal{E}_{p,q,\beta}} = \|\rho_{\rm m}\|_{W^{1,p}_\beta(0,\infty;W^{1,q}(\mathcal{F}))} + \|\rho_{\rm avg}\|_{L^\infty(0,\infty)} + \|\partial_t\rho_{\rm avg}\|_{L^p_\beta(0,\infty)} + \|u\|_{W^{1,2}_{p,q,\beta}((0,\infty);\mathcal{F})^3} + \|h\|_{W^{2,p}_\beta(0,\infty)} + \|\ell\|_{W^{1,p}_\beta(0,\infty)} + \|\omega\|_{W^{1,p}_\beta(0,\infty)}.$$

Similarly, for $1 < p,q < \infty$, and $\beta \in \mathbb{R}$, we define $\mathcal{B}_{p,q,\beta}$, the space for non-homogeneous source terms, as follows

$$\mathcal{B}_{p,q,\beta} = \Big\{(f_1,f_2,g_1,g_2) \mid f_1 = f_{1,\rm m}+f_{1,\rm avg}, f_1 \in L^p_\beta(0,\infty,W^{1,q}(\mathcal{F})), f_{1,\rm avg} \in L^1(0,\infty), f_2 \in L^p_\beta(0,\infty;L^q(\mathcal{F}))^3, g_1 \in L^p_\beta(0,\infty;\mathbb{R}^3), g_2 \in L^p_\beta(0,\infty;\mathbb{R}^3)\Big\}, \quad (1.52)$$

with

$$\|(f_1,f_2,g_1,g_2)\|_{\mathcal{B}_{p,q,\beta}} = \|f_1\|_{L^p_\beta(0,\infty;W^{1,q}(\mathcal{F}))} + \|f_{1,\rm avg}\|_{L^1(0,\infty)} + \|f_2\|_{L^p_\beta(0,\infty;L^q(\mathcal{F}))^3} + \|g_1\|_{L^p_\beta(0,\infty)} + \|g_2\|_{L^p_\beta(0,\infty)}.$$

We set

$$\mathcal{J}_{p,q} = \Big\{(\rho_0,u_0,h_0-h_1,\ell_0,\omega_0) \mid \rho_0 \in W^{1,q}(\mathcal{F}) \cap L^q_m(\mathcal{F}),\ u_0 \in B^{2(1-1/p)}_{q,p}(\mathcal{F})^3,\ h_0-h_1 \in \mathbb{R}^3,\ \ell_0 \in \mathbb{R}^3,\ \omega_0 \in \mathbb{R}^3\Big\}, \quad (1.53)$$

and we introduce the space of initial data

$$\mathcal{J}^{cc}_{p,q} = \mathcal{J}_{p,q} \quad \text{if} \quad \frac{1}{p}+\frac{1}{2q} > 1,$$

$$\mathcal{J}^{cc}_{p,q} = \Big\{(\rho_0,u_0,h_0-h_1,\ell_0,\omega_0) \in \mathcal{J}_{p,q} \mid u_0 = 0 \text{ on } \partial\Omega, \ u_0(y) = \ell_0 + \omega_0 \times y \quad y \in \partial\mathcal{S}\Big\} \quad \text{if} \quad \frac{1}{p}+\frac{1}{2q} < 1.$$

Theorem 1.3.11. *Let $1 < p < \infty$ and $1 < q < \infty$ satisfying $\frac{1}{p}+\frac{1}{2q} \neq 1$. Let β_0 is the constant introduced in Theorem 1.3.9, and $\beta \in [0,\beta_0)$. Then for any*

$$(\rho_0-\overline{\rho},u_0,h_0-h_1,\ell_0,\omega_0) \in \mathcal{J}^{cc}_{p,q}, \quad (f_1,f_2,g_1,g_2) \in \mathcal{B}_{p,q,\beta}$$

the system (1.34) *admits a unique solution $(\rho,u,h,\ell,\omega) \in \mathcal{E}_{p,q,\beta}$ with*

$$\|(\rho,u,h,\ell,\omega)\|_{\mathcal{E}_{p,q,\beta}} \leqslant C_L\Big(\|(\rho_0,u_0,h_0-h_1,\ell_0,\omega_0)\|_{\mathcal{J}_{p,q}} + \|(f_1,f_2,g_1,g_2)\|_{\mathcal{B}_{p,q,\beta}}\Big). \quad (1.54)$$

Proof. Let us first consider the case $\beta = 0$. We integrate equation $(1.34)_1$ satisfied by ρ in $\mathcal{F}$, use the boundary conditions u and the decomposition (1.51) to conclude that ρ_{avg} is a solution of the following system

$$\partial_t \rho_{\rm avg} = f_{1,\rm avg} \quad t \in (0,\infty), \quad \rho_{\rm avg}(0) = 0. \tag{1.55}$$

As $f_{1,\rm avg}$ belongs to $L^1(0,\infty)$ we have $\rho_{\rm avg} \in L^\infty(0,\infty)$. On the other hand, it is easy to see that $(\rho_{\rm m}, u, h, \ell, \omega)$ satisfies the following system

$$\frac{\rm d}{{\rm dt}} \begin{bmatrix} \rho_{\rm m} \\ u \\ h \\ \ell \\ \omega \end{bmatrix} = \mathcal{A}_{\rm FS} \begin{bmatrix} \rho_{\rm m} \\ u \\ h \\ \ell \\ \omega \end{bmatrix} + \begin{bmatrix} f_{1,\rm m} \\ f_2 \\ 0 \\ g_1 \\ g_2 \end{bmatrix} \qquad (t \in (0,\infty)), \qquad \begin{bmatrix} \rho_{\rm m}(0) \\ u(0) \\ h(0) \\ \ell(0) \\ \omega(0) \end{bmatrix} = \begin{bmatrix} \rho_0 \\ u_0 \\ h_0 - h_1 \\ \ell_0 \\ \omega_0 \end{bmatrix}, \tag{1.56}$$

where $\mathcal{A}_{\rm FS}$ is the linear fluid-structure operator introduced in Section 1.3.3. Recall that, from Theorem 1.3.8 and Theorem 1.3.9 we know that $\mathcal{A}_{\rm FS}$ is a $\mathcal{R}$-sectorial operator on $\mathcal{X}_{\rm m}$, and generates an exponentially stable semigroup on $\mathcal{X}_{\rm m}$. Note that, the hypothesis of the theorem yields that

$$(\rho_0, u_0, h_0 - h_1, \ell_0, \omega_0) \in (\mathcal{X}_{\rm m}, \mathcal{D}(\mathcal{A}_{\rm FS}) \cap \mathcal{X}_{\rm m})_{1-1/p,p},$$

and

$$(f_{1,\rm m}, f_2, 0, g_1, g_2) \in L^p(0,\infty; \mathcal{X}_{\rm m}).$$

Therefore, according to Theorem 1.3.7, the system (1.56) admits a unique solution

$$(\rho_{\rm m}, u, h, \ell, \omega) \in L^p(0,\infty; \mathcal{D}(\mathcal{A}_{\rm FS}) \cap \mathcal{X}_{\rm m}) \cap W^{1,p}(0,\infty; \mathcal{X}_{\rm m}) \subset \mathcal{S}_{\infty,p,q}.$$

The case $\beta > 0$ can be reduced to the previous case by multiplying all the functions by $e^{\beta t}$ and using the fact that $\mathcal{A}_{\rm FS} - \gamma$ is a $\mathcal{R}$-sectorial operator with negative type. This completes the proof of the theorem. □

1.3.8 Proof of Theorem 1.3.1 and Corollary 1.3.2

First we give a short proof of Theorem 1.3.3. The proof is similar to the proof of [9, Theorem 3.22]. We just indicate the main steps.

Throughout this subsection, we assume

$$2 < p < \infty \text{ and } 3 < q < \infty.$$

Let us fix $\beta \in (0, \beta_0)$, where β_0 is the constant introduced in Theorem 1.3.9. For $\delta > 0$, we define the ball $\mathcal{E}_{p,q,\beta,\delta}$ in $\mathcal{E}_{p,q,\beta}$ as follows

$$\mathcal{E}_{p,q,\beta,\delta} = \left\{ (\rho, u, h, \ell, \omega) \mid \|(\rho, u, h, \ell, \omega)\|_{\mathcal{E}_{p,q,\beta}} \leqslant \delta \right\}. \tag{1.57}$$

Let us take $(\sigma, v, \zeta, \kappa, \tau) \in \mathcal{E}_{p,q,\beta,\delta}$ and we consider the system

$$\begin{cases} \partial_t \rho + \overline{\rho}\mathrm{div} u = \mathcal{F}_1(\sigma, v, \zeta, \kappa, \tau) & \text{in } (0,\infty)\times\mathcal{F}, \\ \partial_t u - \mathrm{div}\sigma_l(\rho, u) = \mathcal{F}_2(\sigma, v, \zeta, \kappa, \tau) & \text{in } (0,\infty)\times\mathcal{F}, \\ u = 0 & \text{on } (0,\infty)\times\partial\Omega, \\ u = \ell + \omega \times y & \text{on } (0,\infty)\times\partial\mathcal{S}, \\ \dfrac{d}{dt}h = \ell & t \in (0,\infty), \\ \dfrac{d}{dt}\ell + m^{-1}k_p h + m^{-1}k_d \ell & \\ \qquad = -m^{-1}\displaystyle\int_{\partial\mathcal{S}} \sigma_l(\rho, u)n \ \mathrm{d}\gamma + \mathcal{G}_1(\sigma, v, \zeta, \kappa, \tau) & t \in (0,\infty), \\ \dfrac{d}{dt}\omega = -J(0)^{-1}\displaystyle\int_{\partial\mathcal{S}} y \times \sigma_l(\rho, u)n \ \mathrm{d}\gamma + \mathcal{G}_2(\sigma, v, \zeta, \kappa, \tau) & t \in (0,\infty), \\ \rho(0) = \rho_0 := \widetilde{\rho}_0 - \overline{\rho}, \quad u(0) = u_0, & \text{in } \mathcal{F}, \\ h(0) = h_0 - h_1, \quad \ell(0) = \ell_0, \quad \omega(0) = \omega_0. & \end{cases} \tag{1.58}$$

We consider the mapping

$$\mathcal{N} : \mathcal{E}_{p,q,\beta,\delta} \to \mathcal{E}_{p,q,\beta,\delta} \qquad (\sigma, v, \zeta, \kappa, \tau) \mapsto (\rho, u, h, \ell, \omega), \tag{1.59}$$

where $(\rho, u, h, \ell, \omega)$ is the solution to the system (1.58). To prove Theorem 1.3.3, it is enough to show that following result

Proposition 1.3.12. *There exists $\delta > 0$ such that, for any*

$$\|(\rho_0, u_0, h_0 - h_1, \ell_0, \omega_0)\|_{\mathcal{J}_{p,q}} < \frac{\delta}{2C_L}, \tag{1.60}$$

where C_L is the positive constant appearing in Theorem 1.3.11, the mapping $\mathcal{N}$ defined in (1.59) *is a strict contraction in $\mathcal{E}_{p,q,\beta,\delta}$.*

Let us remark that, the nonlinear terms appearing in (1.23)–(1.29) are exactly same as the nonlinear terms in [9]. In fact, following the proof of Proposition 3.20 and Proposition 3.21 in [9], we can show that, there exists $\delta > 0$ such that, for any $(\sigma, v, \zeta, \kappa, \tau) \in \mathcal{E}_{p,q,\beta,\delta}$, and $(\sigma_i, v_i, \zeta_i, \kappa_i, \tau_i) \in \mathcal{E}_{p,q,\beta,\delta}$, $i = 1, 2$, we have

$$\Big\| \Big(\mathcal{F}_1(\sigma, v, \zeta, \kappa, \tau), \mathcal{F}_2(\sigma, v, \zeta, \kappa, \tau), \mathcal{G}_1(\sigma, v, \zeta, \kappa, \tau), \mathcal{G}_2(\sigma, v, \zeta, \kappa, \tau) \Big) \Big\|_{\mathcal{B}_{p,q,\beta}} \leqslant C\delta^2,$$

and

$$\Big\| (\mathcal{F}_1^1, \mathcal{F}_2^1, \mathcal{G}_1^1, \mathcal{G}_2^1) - (\mathcal{F}_1^2, \mathcal{F}_2^2, \mathcal{G}_1^2, \mathcal{G}_2^2) \|_{\mathcal{B}_{p,q,\beta}} \leqslant C\delta \Big\| (\sigma_1, v_1, \zeta_1, \kappa_1, \tau_1) - (\sigma_1, v_1, \zeta_1, \kappa_1, \tau_1) \Big\|_{\mathcal{E}_{p,q,\beta}},$$

for some $C > 0$ independent of δ, where

$$\mathcal{F}_1^i = \mathcal{F}_1(\sigma_i, v_i, \zeta_i, \kappa_i, \tau_i), \quad \mathcal{F}_2^i = \mathcal{F}_2(\sigma_i, v_i, \zeta_i, \kappa_i, \tau_i),$$

$$\mathcal{G}_1^i = \mathcal{G}_1(\sigma_i, v_i, \zeta_i, \kappa_i, \tau_i), \quad \mathcal{G}_2^i = \mathcal{G}_2(\sigma_i, v_i, \zeta_i, \kappa_i, \tau_i).$$

Using the above estimates together with (1.54), we can easily conclude the proof of Proposition 1.3.12, and hence the proof of Theorem 1.3.3.

Finally, using the change of variables we conclude the proof of Theorem 1.3.1 and Corollary 1.3.2. We refer to [9] for more details regarding the proofs.

1.4 Global Stabilizability

In this section, using switching feedback control, we prove a global stabilization result, where we remove the assumption that h_0 and h_1 are sufficiently close. More precisely, we prove the following result

Theorem 1.4.1. *Under the assumptions of Theorem 1.3.3, there exists $\delta > 0$, depending only on Ω, k_p, k_d such that for any $(\widetilde{\rho}_0, u_0, h_0, \ell_0, \omega_0)$ satisfying* (1.14)-(1.16), (1.17) *and*

$$\|\widetilde{\rho}_0 - \overline{\rho}\|_{W^{1,q}(\mathcal{F}(0))} + \|u_0\|_{B^{2(1-1/p)}_{q,p}(\mathcal{F}(0))^3} + \|\ell_0\|_{\mathbb{R}^3} + \|\omega_0\|_{\mathbb{R}^3} \leqslant \frac{\delta}{2}, \tag{1.61}$$

there exists a piecewise constant function $s : [0, \infty) \to \Omega$ satisfying $\operatorname{dist}(s(t), \partial\Omega) > 1$, for all $t \geqslant 0$, such that the strong solution of (1.3) *with switching feedback law*

$$w(t) = k_p(s(t) - \widetilde{h}(t)) - k_d\widetilde{h}'(t), \quad t \geqslant 0, \tag{1.62}$$

satisfies the stability properties

$$\lim_{t\to\infty} \|\widetilde{\rho}(t,\cdot) - \overline{\rho}\|_{W^{1,q}(\mathcal{F}(t))} = 0, \quad \lim_{t\to\infty} \|\widetilde{u}(t,\cdot)\|_{B^{2(1-1/p)}_{q,p}(\mathcal{F}(t))^3} = 0, \tag{1.63}$$

$$\lim_{t\to\infty} \widetilde{h}(t) = h_1, \quad \lim_{t\to\infty} \widetilde{h}'(t) = 0, \quad \lim_{t\to\infty} \widetilde{\omega}(t) = 0. \tag{1.64}$$

Proof of Theorem 1.4.1. Let us recall the set (1.6):

$$\Omega^0 = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > 1\}.$$

Let us consider h_0, h_1 belong to Ω^0. The path-connectedness of Ω^0 follows from the fact that $\Omega^0 \subset \mathbb{R}^3$ is open and connected. Hence, we have a path $\gamma : [0,1] \to \Omega^0$ such that $\gamma(0) = h_0$ and $\gamma(1) = h_1$. Also observe that $\gamma([0,1])$ is compact, it implies that there exists $\nu > 0$ such that

$$\operatorname{dist}(\gamma([0,1]), \partial\Omega) \geq 1 + \nu.$$

According to Theorem 1.3.1, there exists $\delta > 0$ such that for any $h^* \in \gamma([0,1])$, initial data satisfying

$$\Big\|(\widetilde{\rho}_0 - \overline{\rho}, u_0, h_0 - h_*, \ell_0, \omega_0)\Big\|_{\mathcal{J}_{p,q}} \leqslant \delta,$$

under the feedback law

$$w(t) = k_p[h^* - \widetilde{h}(t)] - k_d\widetilde{h}'(t),$$

the system (1.3) admits a unique global in time strong solution. Furthermore, we can use Corollary 1.3.2 to conclude: as $t \to \infty$, we have

$$\|\widetilde{\rho}(t,\cdot) - \overline{\rho}\|_{W^{1,q}(\mathcal{F}(t))} + \|\widetilde{u}(t,\cdot)\|_{B^{2(1-1/p)}_{q,p}(\mathcal{F}(t))^3} + \|h^* - \widetilde{h}(t)\|_{\mathbb{R}^3} + \|\widetilde{h}'(t)\|_{\mathbb{R}^3} + \|\widetilde{\omega}(t)\|_{\mathbb{R}^3} \to 0.$$

Now we make a partition $\{x_r\}$ of the path joining h_0 to h_1 such that

$$x_0 = h_0, \quad x_{k+1} = h_1, \quad \|x_{r+1} - x_r\|_{\mathbb{R}^3} < \frac{\delta}{2}, \text{ for all } r = 0, 1, 2, ..., k.$$

First we explain how we can go from x_0 to x_1 and from x_1 to x_2. After that we state the general strategy. With the help of the partition $\{x_r\}$ and condition (1.61), we obtain

$$\|\widetilde{\rho}_0 - \overline{\rho}\|_{W^{1,q}(\mathcal{F}(0))} + \|u_0\|_{B^{2(1-1/p)}_{q,p}(\mathcal{F}(0))^3} + \|x_1 - h_0\|_{\mathbb{R}^3} + \|\ell_0\|_{\mathbb{R}^3} + \|\omega_0\|_{\mathbb{R}^3} \leqslant \delta.$$

Under the feedback law

$$w_0(t) = k_p(x_1 - \widetilde{h}(t)) - k_d\widetilde{h}'(t), \quad t \geq 0,$$

the system(1.3) admits a unique global in time strong solution along with the asymptotic behaviour:

$$\|\widetilde{\rho}(t,\cdot) - \overline{\rho}\|_{W^{1,q}(\mathcal{F}(t))} + \|\widetilde{u}(t,\cdot)\|_{B^{2(1-1/p)}_{q,p}(\mathcal{F}(t))^3} + \|x_1 - \widetilde{h}(t)\|_{\mathbb{R}^3} + \|\widetilde{h}'(t)\|_{\mathbb{R}^3} + \|\widetilde{\omega}(t)\|_{\mathbb{R}^3} \to 0, \text{ as } t \to \infty.$$

Thus, we choose $T_1 > 0$ large enough such that

$$\|\widetilde{\rho}(T_1,\cdot) - \overline{\rho}\|_{W^{1,q}(\mathcal{F}(T_1))} + \|\widetilde{u}(T_1,\cdot)\|_{B^{2(1-1/p)}_{q,p}(\mathcal{F}(T_1))^3} + \|x_1 - \widetilde{h}(T_1)\|_{\mathbb{R}^3} + \|\widetilde{h}'(T_1)\|_{\mathbb{R}^3} + \|\widetilde{\omega}(T_1)\|_{\mathbb{R}^3} \leqslant \frac{\delta}{2}.$$

As $\|x_1 - x_2\|_{\mathbb{R}^3} < \frac{\delta}{2}$, we can deduce

$$\|\widetilde{\rho}(T_1,\cdot) - \overline{\rho}\|_{W^{1,q}(\mathcal{F}(T_1))} + \|\widetilde{u}(T_1,\cdot)\|_{B^{2(1-1/p)}_{q,p}(\mathcal{F}(T_1))^3} + \|x_2 - \widetilde{h}(T_1)\|_{\mathbb{R}^3} + \|\widetilde{h}'(T_1)\|_{\mathbb{R}^3} + \|\widetilde{\omega}(T_1)\|_{\mathbb{R}^3} \leqslant \delta.$$

To proceed further, use the feedback law:

$$w_1(t) = k_p(x_2 - \widetilde{h}(t)) - k_d\widetilde{h}'(t), \text{ for } t \geqslant T_1.$$

In general, we construct a partition

$$0 = T_0 < T_1 < T_2 < \cdots < T_k < T_{k+1} = +\infty$$

of $[0, \infty]$ in the following way: let $r \in \{1, 2, ..., k\}$ and $T_r \geqslant 0$ be such that

$$\|\widetilde{\rho}(T_{r-1}, \cdot) - \overline{\rho}\|_{W^{1,q}(\mathcal{F}(T_{r-1}))} + \|\widetilde{u}(T_{r-1}, \cdot)\|_{B^{2(1-1/p)}_{q,p}(\mathcal{F}(T_{r-1}))^3} + \|x_{r-1} - \widetilde{h}(T_{r-1})\|_{\mathbb{R}^3} + \|\widetilde{h}'(T_{r-1})\|_{\mathbb{R}^3} + \|\widetilde{\omega}(T_{r-1})\|_{\mathbb{R}^3} \leqslant \frac{\delta}{2}.$$

Due to the partition $\{x_r\}$, we know $\|x_r - x_{r-1}\|_{\mathbb{R}^3} < \frac{\delta}{2}$, which implies

$$\|\widetilde{\rho}(T_{r-1}, \cdot) - \overline{\rho}\|_{W^{1,q}(\mathcal{F}(T_{r-1}))} + \|\widetilde{u}(T_{r-1}, \cdot)\|_{B^{2(1-1/p)}_{q,p}(\mathcal{F}(T_{r-1}))^3} + \|x_r - \widetilde{h}(T_{r-1})\|_{\mathbb{R}^3} + \|\widetilde{h}'(T_{r-1})\|_{\mathbb{R}^3} + \|\widetilde{\omega}(T_{r-1})\|_{\mathbb{R}^3} \leqslant \delta.$$

Now under the feedback law

$$w_{r-1}(t) = k_p(x_r - \widetilde{h}(t)) - k_d\widetilde{h}'(t), \quad \text{for } t \geqslant T_{r-1},$$

the system (1.3) admits a unique global in time strong solution and there exists $T_r > T_{r-1}$ such that

$$\|\widetilde{\rho}(T_r, \cdot) - \overline{\rho}\|_{W^{1,q}(\mathcal{F}(T_r))} + \|\widetilde{u}(T_r, \cdot)\|_{B^{2(1-1/p)}_{q,p}(\mathcal{F}(T_r))^3} + \|x_r - \widetilde{h}(T_r)\|_{\mathbb{R}^3} + \|\widetilde{h}'(T_r)\|_{\mathbb{R}^3} + \|\widetilde{\omega}(T_r)\|_{\mathbb{R}^3} \leqslant \frac{\delta}{2}.$$

As $|x_{r+1} - x_r| < \frac{\delta}{2}$, we can deduce

$$\|\widetilde{\rho}(T_r, \cdot) - \overline{\rho}\|_{W^{1,q}(\mathcal{F}(T_r))} + \|\widetilde{u}(T_r, \cdot)\|_{B^{2(1-1/p)}_{q,p}(\mathcal{F}(T_r))^3} + \|x_{r+1} - \widetilde{h}(T_r)\|_{\mathbb{R}^3} + \|\widetilde{h}'(T_r)\|_{\mathbb{R}^3} + \|\widetilde{\omega}(T_r)\|_{\mathbb{R}^3} \leqslant \delta.$$

We can iterate the above step with the help of the feedback law:

$$w_r(t) = k_p(x_{r+1} - \widetilde{h}(t)) - k_d\widetilde{h}'(t), \text{ for } t \geqslant T_r.$$

Hence, in the presence of the switching feedback law

$$w(t) = k_p(s(t) - \widetilde{h}(t)) - k_d\widetilde{h}'(t), \quad t \geqslant 0,$$

where piecewise constant function $s : [0, \infty) \to \Omega^0$ is given by

$$s(t) = x_j, \quad j \in \{1, 2, ..., (k+1)\}, \quad t \in [T_{j-1}, T_j),$$

we obtain a unique strong solution of (1.3) and the solution enjoys the stability properties (1.63)-(1.64). This completes the proof of the Theorem. □

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