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# Algebraic techniques and perturbation methods to approach frequency response curves

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## Abstract

The Algebra is exploited to study approximated responses of nonlinear dynamical systems leading to tracing solutions of approximated bifurcation diagrams associated with polynomial equations resulting from search of approximated periodic solutions of nonlinear ordinary differential equations. In detail via using the Gröbner basis, a polynomial with the smallest degree in term of the approximated amplitude of the systems, here the  $L^2$  norm of coefficients of truncated Fourier series, is extracted where its coefficients are parameters of the systems such as the frequency. The presented methodology permits to detect maximal number of solutions even those which belong to isola of the frequency response curves of the system.

*Keywords:* Euclidean division, Gröbner basis,  $L^2$  norm, polynomial

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## 1. Introduction

The real dynamical behaviours of mechanical and structural systems are represented by nonlinear differential equations. For system with smooth nonlinearities (very often polynomial nonlinearities) perturbation methods have been developed to trace system responses, from them we can refer to Krilov-Bogoliubov-Mitropolski (KBM) [1, 2, 3, 4, 5, 6, 7, 8, 9], Harmonic Balance Method (HBM) [10, 11, 12], normal form [13, 14, 15, 16, 17, 18, 19, 20, 21, 22] and multiple scale

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methods [6, 23, 24, 25, 26, 27]. Meanwhile, many other methods exist which are omitted here [23]. Some various trends of applications of asymptotic techniques to mechanical problems via presenting methods to be used in different static and dynamics problems are reported in [28, 29]

Using these different techniques, one obtains approximations of periodic solutions by solving systems of algebraic equations with unknown coefficients (Fourier coefficients of truncated Fourier series for examples) and given parameters. Numerical techniques are used to solve these algebraic equations at given parameters values such as Newton-Raphson methods, or continuation approaches if solutions are tracked versus one (or several ones that is not usually the case in practice) parameter(s) [30, 31]. Nevertheless, finding all the solutions, and especially the isolated branches of solutions is always challenging. Algebraic methods exist to obtain parametric representation of solutions of polynomial equations with multi-variables such as Gröbner basis [32]. There are many different algorithms for evaluation of Gröbner bases from them we can name the one which is developed by Buchberger [33]. The Gröbner basis is used in [34, 35] for obtaining parametrised solutions of polynomial system of equations. The current paper deals with solving of a bifurcation problem depending on variables (e.g. components of  $L^2$  norms of the unknown component of the vector variable of displacements) and parameters via solving a scalar polynomial equations with coefficients depending on parameters. Organisation of the paper is as it follows: In Section 2, we recall how to obtain algebraic equations and some definitions and materials about rings of multivariate polynomials and the Gröbner basis are provided. In Section 3 a basic algorithm without endowing the Gröbner basis is described and illustrated on a simple example (Duffing oscillator). In Section 4, the Gröbner basis is exploited and an algorithm is provided for the case of a single-dof system. It is illustrated via the Duffing example. The single unknown can be an approximation of  $L^2$  norm of a periodic solution or of the Fourier coefficients of approximated Fourier series. Finally, to illustrate the method for general cases (N-dof nonlinear system with forcing or parametric nonlinearities) a two-dof case is examined. Finally, the paper is

concluded in Section 5.

## 2. Basic ideas in rings of polynomials

Let us suppose that we have a set of nonlinear dynamical equations (e.g. Duffing), with the variable  $x(t)$ . For seeking periodic solutions in  $V = L^2([0, T])$  with  $T = \frac{2\pi}{\omega}$ ,  $x$  can be represented as:

$$x(t) = \frac{a_0}{2} + \sum_{j=1}^{+\infty} a_j \cos(j\omega t) + b_j \sin(j\omega t) \quad (1)$$

or in the form of truncated series as:

$$x(t) \simeq F(X_0, X_1, X_2, \dots, X_N, t) = \frac{a_0}{2} + \sum_{j=1}^M a_j \cos(j\omega t) + b_j \sin(j\omega t) \quad (2)$$

with  $a_0 \rightarrow X_0$ ,  $a_1 \rightarrow X_1$ ,  $b_1 \rightarrow X_2$ , ... leading to  $E_j(X_0, X_1, \dots) = 0$ ,  $j = 0, 1, 2, \dots, 2M = N$ .

Let us suppose that there is a function space  $\mathcal{V}$  where  $x \in \mathcal{V}$  and we set the norm of  $x$  as  $\|x\|$ . For instance,

$$\|x\|^2 = \|x\|_2^2 = \left(\frac{a_0^2}{2}\right) + \sum_{j=1}^{+\infty} a_j^2 + b_j^2 \quad \text{where } x \in L^2\left([0, T = \frac{2\pi}{\omega}]\right) \quad (3)$$

It should be mentioned that if there is not any constant term in the system, then  $E_0$  and  $X_0$  can be omitted.

### 2.1. The problem

We are interested to perform bifurcation analyses with the parameter  $\lambda$  on a nonlinear dynamical system. From  $x(t) \simeq F(X_0, X_1, X_2, \dots, X_N, t)$  we reach to:

$$\begin{cases} E_0(X_0, X_1, X_2, \dots, X_N, \lambda) = 0 \\ E_1(X_0, X_1, X_2, \dots, X_N, \lambda) = 0 \\ \vdots \\ E_Z(X_0, X_1, X_2, \dots, X_N, \lambda) = 0 \end{cases} \quad (4)$$

We would like to obtain from all  $E_j$  equations, a problem which is represented as

$$\mathbb{P}(\|x\|^2, \lambda) = 0 \quad (5)$$

With

$$\|x\| \simeq \frac{1}{2}X_0^2 + X_1^2 + X_2^2 + \dots + X_N^2 = X \quad (6)$$

Equation 5 reads:

$$\mathbb{P}(\|(X_0, X_1, X_2, \dots, X_N)\|, \lambda) = P(X, \lambda) = 0 \quad (7)$$

where  $P$  is a polynomial. In the following section we will clarify some definitions and properties which are related to the demanded problem.

## 2.2. Some definitions and properties

- Let us suppose that  $\mathbb{K}[X]$  is the polynomial ring of  $X$  over the field  $\mathbb{K}$  and  $\mathbb{I} \triangleleft \mathbb{K}[X]$ , ideals of the ring  $\mathbb{K}[X]$ .
- Given an ideal  $\mathbb{I} \triangleleft \mathbb{K}[X]$ , it is possible to generate quotient ring  $\mathbb{K}[X]/\mathbb{I}$ .
- In the ring  $\mathbb{K}[X]$ , division of the polynomial  $P$  by  $E$  is represented by

$$P = \underbrace{q}_{\text{quotient}} E + \underbrace{r}_{\text{remainder}} \quad (8)$$

with  $\text{degree}(r) < \text{degree}(E)$ .

- Let us suppose that there is not any constant term in the system so,  $E_0$  and  $X_0$  are omitted. Consider again  $\mathbf{E} = \{E_1, E_2, \dots, E_Z\}$  in the ring  $\mathbb{K}[X_1, X_2, \dots, X_N]$ , the ideal produced by  $\mathbf{E}$  is the set of linear combinations of polynomials from  $\mathbf{E}$  such that:

$$\mathbb{I} = \left\{ \sum_{j=1}^Z h_j E_j, h_1, h_2, \dots, h_Z \in \mathbb{K}[X_1, X_2, \dots, X_N] \right\} \quad (9)$$

- In the ring  $\mathbb{K}[X_1, X_2, \dots, X_N]$ , let us find Gröbner bases  $\mathbf{G} = \{g_1, g_2, \dots, g_Z\}$ , for the set  $\mathbf{E} = \{E_1, E_2, \dots, E_Z\}$ .

- When the polynomial  $P$  belongs to the ideal produced by  $\mathbf{E}$ , the remainder of  $P$  with respect to  $\mathbf{G}$  must be zero.
- In the ring  $\mathbb{K}[X_1, X_2, \dots, X_N]$ , we are generalizing division of polynomial  $P$  by polynomials  $\mathbf{E} = \{E_1, E_2, \dots, E_Z\}$  so that:
  - $P = q_1 E_1 + q_2 E_2 + \dots + q_Z E_Z + r$
  - Polynomials  $q_1, q_2, \dots, q_Z$  are quotients of  $\mathbb{K}[X_1, X_2, \dots, X_N]$ .
  - Polynomial  $r$  is the remainder of  $\mathbb{K}[X_1, X_2, \dots, X_N]$ .

In fact

$$\begin{aligned}
 P &= g_1 q_1 + g_2 q_2 + \dots + g_z q_z + \tilde{r} \\
 P \triangleleft [g_1, g_2, \dots, g_z] &= [E_1, E_2, \dots, E_Z] \Rightarrow \tilde{r} = 0
 \end{aligned} \tag{10}$$

### 3. A very basic algorithm without using the Gröbner basis

Let us consider  $N$  polynomial equations of  $N$  variables  $X_1, \dots, X_N$ , represented as  $E_j(X_1, \dots, X_N) = 0$  for  $j = 1, \dots, N$ . Let us consider the degrees of each equation versus one variable (successively  $X_1, \dots, X_N$ ).

- Let us keep the variable with the smallest maximal degree among the  $N$  equations;
- Let us consider the equation with the smallest degree in this variable:
  - If it is 0, let us choose the variable with the next smallest maximal degree.
  - If it is 1, we solve in that variable.
- Let us substitute in the other equations and go back to the beginning after considering numerators of equations and we keep in memory the value of denominator.
- If it is greater than 2, let us make the Euclidean division of the equation (polynomial) with highest degree in the variable versus the equation with the smallest degree in the same variable.

- We obtain a quotient and a remainder:
  - If the remainder is of degree 1, we solve it.
  - If it is greater than 1, we divide again the equation with the smallest degree in the variable by the remainder.
- We iterate the process till we find a remainder of degree 1.
- If the rest is of degree 0, we have eliminated one variable.

### 3.1. Application to the Duffing oscillator

Let us consider the Duffing example with approximation of the periodic solution by one harmonic via HBM (see appendix A). We obtain two polynomials in  $X_1, X_2$  from  $E_j(X_1, X_2) = 0$  for  $j = 1, 2$ . We can see that for  $X_1$  or  $X_2$  variables, the degrees of  $E_1$  and  $E_2$  are:

- $\text{degree}(E_1, X_1) = 3$
- $\text{degree}(E_2, X_1) = 2$
- $\text{degree}(E_1, X_2) = 2$
- $\text{degree}(E_2, X_2) = 3$

So we consider Euclidean division of  $E_1$  seen as a polynomial  $P_1$  of  $X_1$  by  $E_2$  also seen as a polynomial  $P_2$  of  $X_1$ . The result is:

$$E_1(X_1, X_2) = P_1 = Q_1 P_2 + R_1 \quad (11)$$

with

$$R_1 = \frac{1}{3} \frac{(4a^2 w^2 + 3X_2 cf) X_1}{cX_2^2} + \frac{4aw(X_2 w^2 - X_2 + f)}{3cX_2^2} \quad (12)$$

The  $E_1 = P_1 = 0$  is equivalent to  $P_2 = R_1 = 0$ . Solving  $R_1 = 0$  in  $X_1$  provides

$$X_1 = -4 \frac{aw(X_2 w^2 - X_2 + f)}{4a^2 w^2 + 3X_2 cf} \quad (13)$$

and by replacing in  $P_2$ , we obtain a parametrization of the solutions:  $X_1$  is a function of  $X_2$ . And  $X_2$  verifies a scalar equation governed by:

$$\frac{12cX_2 (afw + aw^3X_2 - awX_2)^2}{(4a^2w^2 + 3cfX_2)^2} + \frac{4aw (afw + aw^3X_2 - awX_2)}{4a^2w^2 + 3cfX_2} + \frac{3b_1^3c}{4} - f - w^2X_2 + X_2 = 0 \quad (14)$$

Finding roots of this equation (and managing the denominator) is simple. So since it can be checked that the latter is singular when the numerator is not zero (for example if  $a = 0$ , the denominator is singular for  $X_2 = 0$  that is not a zero of the numerator), it is possible to define the parametrization by a polynomial in one variable ( $X_2$ ) expressed as the simplified first one of the previous equations ( since  $c \neq 0$  and  $X_2 \neq 0$ ):

$$16 X_2 a^4 w^4 + 16 X_2 a^2 w^6 + 24 X_2^2 a^2 c f w^2 + 9 X_2^3 c^2 f^2 - 32 X_2 a^2 w^4 + 16 a^2 f w^4 - 12 X_2 c f^2 w^2 + 16 X_2 a^2 w^2 - 16 a^2 f w^2 + 12 X_2 c f^2 - 12 c f^3 = 0 \quad (15)$$

Let us consider system parameters which are defined in Table 1. We consider two values for  $w$  as  $w = 1.12$  and  $w = 1.5$ . Corresponding numerical values for  $X_2$ , then  $X_1$  and finally  $X = X_1^2 + X_2^2$  are given in Table 1. We name obtained solutions via this method as "indirect solutions".

Table 1: Numerical solutions of polynomial Eqs. 15 and 13 (Indirect solutions).

$a$	$c$	$f$	$w$	$X_1$	$X_2$	$X = X_1^2 + X_2^2$
0	1	$\frac{1}{10}$	1.12	0	0.7235217920	0.523483783
			1.5	0	-0.080310794277493	0.006449823678
				0	-1.24896418571948	1.559911538
				0	1.32927497999697	1.766971972



#### 4. Seeking periodic solutions via exploitation of Gröbner basis

For detailed information about Gröbner basis the reader is referred to reference [36] and to documentation of the Package Gröbner Basis of Maple<sup>©</sup> [37]. We do not intend to describe known results about Gröbner basis. We simply provide the following basic idea:

- From given polynomials with given variables (and parameters), Gröbner basis can be constructed to generate the same ideal of polynomials (in a ring of polynomials with the given number of variables) than the ideal of polynomials generate by initial equations. Grolet [34] used the idea to eliminate all variables except one. From

$$E'_1(X_1, \dots, X_M), \dots, E'_M(X_1, \dots, X_M) \quad (16)$$

One obtains:

$$\begin{cases} X_{\sigma(1)} = e_1(X_{\sigma(M)}) \\ \vdots \\ X_{\sigma(M-1)} = e_{M-1}(X_{\sigma(M)}) \\ \mathcal{E}(X_{\sigma(M)}) = 0 \end{cases} \quad (17)$$

where  $\mathcal{E}$  is a polynomial in one variable only,  $\sigma$  is one-to-one from  $\{1, \dots, M\}$  to itself, and  $e_j$  functions.

- With a chosen Gröbner basis, the following question can be replied: For a given polynomial  $P$  in the ring of polynomial, does  $P$  belong to the Ideal generated by the Gröbner basis?
  - Yes  $\rightarrow$  the normal form of  $P$  (it is similar to th reminder  $r$  in Eq. 8) versus the Gröbner basis is 0.
  - No  $\rightarrow$  a residual normal form collected versus monomials in the variables of the Ring of polynomials is obtained.

The core idea of our method is:

- To look for a polynomial (with unknown coefficients) of an interesting auxiliary variable (typically the approximated  $L^2$  norm depending on variables of the initial ring of polynomials) so that we could determine the unknown coefficient to obtain a null normal form.
- With increasing the degree and the number of unknown coefficients, we stop when finding the lowest degree of the unknown polynomial leads to a null normal form.

This is detailed in a given example which is provided in next section.

#### 4.1. Presentation of the algorithm for a single degree of freedom system

Let us suppose that we have  $N$  polynomial equations of  $N$  variables  $X_1, \dots, X_N$  reading as:

$$E_j(X_1, \dots, X_N) = 0, \quad j = 1, 2, \dots, N \quad (18)$$

Our proposed algorithm is summarized as it follows:

- We consider the Gröbner basis generated by the polynomials  $E_1, \dots, E_N$  (with parameters of the problem appearing in the coefficients of the equations).
- We introduce a new variable that is the approximation of the  $L^2$  norm of the searched periodic solution, reading as

$$X = X_1^2 + \dots + X_N^2 \quad (19)$$

- We introduce successively the polynomials:

$$P_j = X^j + s_{j-1}X^{j-1} + \dots + s_1X + s_0 \quad (20)$$

for  $j = 1, 2, \dots$

- We test to see if  $P_j$  belongs to the ideal generated by the Gröbner basis produced by the  $N$  polynomial equations. The polynomials  $P_j$  are splitted on the Gröbner basis generated by the equations:
  - If  $P_j$  belongs to the generated ideal, the normal form of  $P_j$  should be 0.
- The normal form of  $P_j$  is examined.
  - It is splitted on polynomials of variables  $X_1, \dots, X_N$  with coefficients depending on initial parameters and depending linearly on the unknown parameters  $s_0, s_1, \dots, s_{j-1}$ .
  - Expression of each coefficient of the normal form provides a linear equation in  $s_0, s_1, \dots, s_{j-1}$ .
  - We stop the algorithm when we can solve the system of linear equations so that the normal form of  $P_j$  is zero.

The result provides a polynomial equation with one variable governing the amplitude-frequency response. Parameters of the problem appear in the final equation. This algorithm converges in finite number of iterations.

#### 4.2. A few results for Duffing equation

Using Maple<sup>©</sup> software, we generate the Gröbner basis and we test polynomials  $P_j$  from  $j = 1$  to  $j = 3$ . For  $j = 1$  and  $j = 2$  it is not possible to choose the coefficients  $s_j$  so that  $P_j$  belongs to the ideal generated by the Gröbner basis. To be more accurate, let us give the results.

##### 4.2.1. Polynomial $P_1$

For polynomial  $P_1$ , the normal form via Gröbner basis defined by the two algebraic equations obtained by HBM taking into account only the first harmonic in cosine and sine is:

$$-X_1 f s_1 + a s_0 w = 0 \tag{21}$$

To get rid of the residual monomials  $X_1$  and 1 we have to demand  $s_1 = 0$  and  $s_0 = 0$ . Consequently  $P_1 = 0$  that is not admissible solution.

4.2.2. *Polynomial  $P_2$*

For  $P_2$  the normal form is

$$-4X_1fs_2w^2 + 4afs_2wX_2 - 3cfs_1X_1 + 3acs_0w + 4X_1fs_2 \quad (22)$$

Again to get rid of all the residual monomials in  $X_1$ ,  $X_2$  and 1, we must demand

$$\begin{cases} 3acs_0w = 0 \\ -4fs_2w^2 - 3cfs_1 + 4fs_2 = 0 \\ 4afs_2w = 0 \end{cases} \quad (23)$$

Finally the solution is  $s_0 = s_1 = s_2 = 0$ , and consequently  $P_2 = 0$  that is again not admissible.

4.2.3. *Polynomial  $P_3$*

For  $j = 3$ , the result of the test for  $P_j$  belonging to this ideal provides the following polynomial:

$$\frac{1}{243c^5f^5} e_1X_1 + \frac{1}{243c^5f^5} e_2X_2 + \frac{1}{243c^5f^5} e_3 = 0 \quad (24)$$

with

$$\begin{aligned}
e_1 &= -5120 a^9 w^{11} + 10240 a^7 w^{13} - 1024 a^5 w^{15} + 5120 a^9 w^9 \\
&\quad -30720 a^7 w^{11} + 5120 a^5 w^{13} + 1536 a^7 c f^2 w^7 + 1728 a^5 c^2 f^2 s_1 w^7 \\
&\quad -4608 a^5 c f^2 w^9 - 576 a^3 c^2 f^2 s_1 w^9 + 30720 a^7 w^9 - 10240 a^5 w^{11} \\
&\quad -1728 a^5 c^2 f^2 s_1 w^5 + 9216 a^5 c f^2 w^7 + 1728 a^3 c^2 f^2 s_1 w^7 - 10240 a^7 w^7 \\
&\quad +10240 a^5 w^9 - 432 a^3 c^3 f^4 s_1 w^3 - 4608 a^5 c f^2 w^5 - 1728 a^3 c^2 f^2 s_1 w^5 \\
&\quad -324 a c^4 f^4 s_2 w^3 - 5120 a^5 w^7 + 576 a^3 c^2 f^2 s_1 w^3 + 324 a c^4 f^4 s_2 w \\
&\quad +1024 a^5 w^5 \\
e_2 &= 1024 a^{10} w^{10} - 10240 a^8 w^{12} + 5120 a^6 w^{14} + 20480 a^8 w^{10} \\
&\quad -20480 a^6 w^{12} - 576 a^6 c^2 f^2 s_1 w^6 + 3072 a^6 c f^2 w^8 + \\
&\quad 1728 a^4 c^2 f^2 s_1 w^8 - 10240 a^8 w^8 + 30720 a^6 w^{10} - 3072 a^6 c f^2 w^6 \\
&\quad -3456 a^4 c^2 f^2 s_1 w^6 - 20480 a^6 w^8 + 1728 a^4 c^2 f^2 s_1 w^4 \\
&\quad +324 a^2 c^4 f^4 s_2 w^2 + 5120 a^6 w^6 \\
e_3 &= -4096 a^8 f w^{10} + 4096 a^6 f w^{12} + 4096 a^8 f w^8 - 12288 a^6 f w^{10} \\
&\quad +768 a^6 c f^3 w^6 + 1152 a^4 c^2 f^3 s_1 w^6 + 12288 a^6 f w^8 \\
&\quad -1152 a^4 c^2 f^3 s_1 w^4 - 4096 a^6 f w^6 + 243 c^5 f^5 s_3
\end{aligned} \tag{25}$$

We can see that choosing the  $s_j$  coefficients to be zero, then all the coefficients of the residual monomials are possible. In that case,  $P_j$  belongs to the ideal. The choice of the  $s_j$  coefficients is done by solving the three following equations versus  $s_0, s_1, s_2$ :

$$\begin{cases} e_1 = 0 \\ e_2 = 0 \\ e_3 = 0 \end{cases} \tag{26}$$

The polynomial  $P_3$  becomes:

$$X^3 + \frac{32}{9c^2 f^2} C_2 X^2 + \frac{256}{81c^4 f^4} C_1 X - C_0 \tag{27}$$

with

$$\begin{aligned}
C_2 &= a^2 w^2 (a^2 w^2 - w^4 + 2 w^2 - 1) \\
C_1 &= a^4 w^4 (a^4 w^4 + 2 a^2 w^6 + w^8 - 4 a^2 w^4 - 4 w^6 - 3 c f^2 w^2 \\
&\quad + 2 a^2 w^2 + 6 w^4 + 3 c f^2 - 4 w^2 + 1) \\
C_0 &= \frac{256}{81} \frac{a^6 w^6}{c^4 f^2}
\end{aligned} \tag{28}$$

#### 4.2.4. Polynomials $P_4$ , $P_5$ and higher

It can be proven that if  $K > 3$

$$P_K = Q_{K-3} P_3(X) \tag{29}$$

where  $Q_{K-3}$  is a polynomial depending on undetermined coefficients  $S_4, \dots, s_K$  and  $X$ . For instance,

$$P_4(X) = Q_1(X, s_4) P_3(X) \tag{30}$$

with

$$X^3 + \frac{32}{9c^2 f^2} C_2 X^2 + \frac{256}{81c^4 f^4} C_1 X - C_0 \tag{31}$$

and

$$Q_1(X, s_4) = s_4 \left( X + \frac{8(1-w^2)}{3c} \right) \tag{32}$$

which introduces spurious solutions due to the fact that the polynomials in  $X$  that are becoming 0 belongs to an ideal generated by  $P_3$ .

#### 4.3. A transversal application

One of transversal applications of explained methodology is identification of system parameters, e.g. nonlinearities of the system. The frequency curves of the system can be detected by experimental tests. Then via consideration of general form of Eq. 27, expressed as:

$$A_3 X^3 + A_2 X^2 + A_1 X + A_0 \tag{33}$$

it will be possible to detect and to identify  $A_j(a, c, f, \omega)$  by one of minimisation techniques.

4.4. Numerical examples: direct and indirect solutions

Let us consider the same system parameters which are reported in Table 1: The results obtained by solving directly  $P_3(X) = 0$  in Eq. 27 (direct solution) are reported in see Table 2. Comparing these results with indirect solutions (see Table 1) shows that there are good agreements between them.

Table 2: System parameters and obtained results from "direct solution".

$a$	$c$	$f$	$w$	$X$
0	1	$\frac{1}{10}$	1.12	0.5234837835
			1.5	0.006449823678
				1.559911538
				1.766971972

It should be mentioned that other choices can be made as well; for example rather than looking for  $P(X)$  one could look for  $P(X_j)$  with  $X_j$  to be any of the variables, or successively all the variables. Similar results are obtained: polynomials are obtained by solving a linear system. This fact is explained in detail in Sect. 4.5.

4.5. Duffing oscillator: Polynomials of a single coefficient of Fourier series

In this Section, we show that it is possible to obtain a polynomial equation only in one coefficient of the truncated Fourier series, for example in variable  $X_1$  (or  $X_2$ , or  $X_1/X_2$  corresponding to tangent of the unknown phase). For this, let us consider again the Duffing oscillator which is described in Eq. 57. Let us look for approximated periodic solution as:

$$x(t) = X_1 \cos(wt) + X_2 \sin(wt) \tag{34}$$

The HBM method provides following two algebraic equations:

$$\begin{aligned} \frac{3}{4} X_1^3 c + \frac{3}{4} X_1 X_2^2 c - w^2 X_1 + X_2 a w + X_1 &= 0 \\ \frac{3}{4} X_1^2 X_2 c + \frac{3}{4} X_2^3 c - X_1 a w - X_2 w^2 + X_2 - f &= 0 \end{aligned} \quad (35)$$

Looking for a polynomial in  $X_1$  belonging to the ideal generated by these equations via the approach leaned on Gröbner basis in the form

$$s_0 + s_1 X_1 + s_2 X_1^2 + s_3 X_1^3 \quad (36)$$

we obtain the normal form as

$$\begin{aligned} \frac{1}{9} \frac{1}{c^2 f^2} (-16 a^4 s_3 w^4 + 16 a^2 s_3 w^6 - 32 a^2 s_3 w^4 - 12 a c f s_2 w^3 \\ + 16 a^2 s_3 w^2 + 12 a c f s_2 w + 9 c^2 f^2 s_1) X_1 + \\ \frac{1}{9} \frac{1}{c^2 f^2} (-32 a^3 s_3 w^5 + 32 w^3 s_3 a^3 + 12 a^2 c f s_2 w^2) X_2 + \\ \frac{1}{9} \frac{1}{c^2 f^2} (-16 a^3 f s_3 w^3 + 9 c^2 f^2 s_0) \end{aligned} \quad (37)$$

Then via setting  $s_0 = 1$ , we solve the linear equations obtained for  $s_1, s_2, s_3$  by demanding elimination of the coefficients of the monomials of the normal form ( $1, X_1$  and  $X_2$ ). We obtain:

$$\begin{aligned} s_1 &= \frac{a^2 w^2 + w^4 - 2 w^2 + 1}{a w f} \\ s_2 &= \frac{3 c (w^2 - 1)}{2 a^2 w^2} \\ s_3 &= \frac{9}{16} \frac{c^2 f}{w^3 a^3} \end{aligned} \quad (38)$$

Without giving details, similar methodology provides for  $X_2$ :

$$\begin{aligned} 16 X_2 a^4 w^4 + 16 X_2 a^2 w^6 + 24 c a^2 w^2 X_2^2 f + 9 c^2 f^2 X_2^3 \\ - 32 X_2 a^2 w^4 + 16 a^2 f w^4 - 12 X_2 c f^2 w^2 + 16 a^2 w^2 X_2 \\ - 16 a^2 f w^2 + 12 X_2 c f^2 - 12 c f^3 = 0 \end{aligned} \quad (39)$$



and for phase tangent (in fact in  $Y = X_1/X_2$ ) following polynomial relation is obtained:

$$\begin{aligned} & -4Y^3a^2w^4 + 4Y^2a^3w^3 + 4Y^3a^2w^2 - 4Ya^2w^4 + 3Y^3cf^2 \\ & + 4a^3w^3 + 4Ya^2w^2 = 0 \end{aligned} \quad (40)$$

Let us consider an academic numerical example:  $a = 3/100, f = 1/100, c = 1$ . In Fig. 1 values for  $X_1, X_2$  and  $X_1/X_2$  are plotted versus  $w$  (respectively red, blue and green curve).

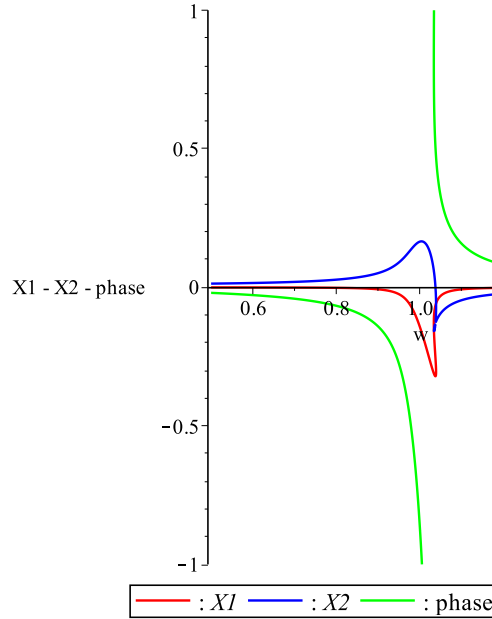


Figure 1: Curves of approximated  $L^2$ -norms of  $x$  (solid line) and  $z$  (dashed line) versus  $w$  for the system with  $\omega_1 = 1, \omega_2 = 1.10$ . Definitions of  $x$  are  $z$  are presented in Eq. 44.

#### 4.6. Two examples of two dof systems

Here, we will two examples of coupled two dof nonlinear systems and we will find equilibrium points of systems via explained methodology in this paper.

4.6.1. Example 1

Let us consider following system:

$$\begin{cases} \frac{d^2x}{dt^2} + \omega_1^2 x + c_1 x^3 + a_1 \frac{dx}{dt} - f_1 \sin(\omega t) + d(x-z)^3 = 0 \\ \frac{d^2z}{dt^2} + \omega_2^2 z + c_2 z^3 + a_2 \frac{dz}{dt} - f_2 \sin(\omega t) - d(x-z)^3 = 0 \end{cases} \quad (41)$$

We consider rough approximations of system variables which are described as:

$$\begin{cases} x = X_1 \cos(\omega t) + X_2 \sin(\omega t) \\ z = Z_1 \cos(\omega t) + Z_2 \sin(\omega t) \end{cases} \quad (42)$$

System parameters which are collected in Table 3. We obtain four equations

Table 3: System parameters of the 2 dof system

$c_1$	$d$	$c_2$	$a_1$	$a_2$	$f_1$	$f_2$
1	1/2	1/3	1/50	1/50	1/100	1/50

generating Gröbner basis:

$$E_1(X_1, X_2, Z_1, Z_2) = 9/2 X_1^3 - 9/2 X_1^2 Z_1 + 9/2 X_1 X_2^2 - 3 X_1 X_2 Z_2 + 9/2 X_1 Z_1^2 + 3/2 X_1 Z_2^2 - 4 X_1 w^2 - 3/2 X_2^2 Z_1 + 3 X_2 Z_1 Z_2 - 3/2 Z_1^3 - 3/2 Z_1 Z_2^2 + \frac{2}{25} X_2 w + 4 X_1 = 0$$

$$E_2(X_1, X_2, Z_1, Z_2) = 9/2 X_1^2 X_2 - 3/2 X_1^2 Z_2 - 3 X_1 X_2 Z_1 + 3 X_1 Z_1 Z_2 + 9/2 X_2^3 - 9/2 X_2^2 Z_2 + 3/2 X_2 Z_1^2 + 9/2 X_2 Z_2^2 - 4 X_2 w^2 - 3/2 Z_1^2 Z_2 - 3/2 Z_2^3 - \frac{2}{25} X_1 w + 4 X_2 = 0$$

$$E_3(X_1, X_2, Z_1, Z_2) = -3/2 X_1^3 + 9/2 X_1^2 Z_1 - 3/2 X_1 X_2^2 + 3 X_1 X_2 Z_2 - 9/2 X_1 Z_1^2 - 3/2 X_1 Z_2^2 + 3/2 X_2^2 Z_1 - 3 X_2 Z_1 Z_2 + 5/2 Z_1^3 + 5/2 Z_1 Z_2^2 - 4 Z_1 w^2 + \frac{2}{25} Z_2 w + 9 Z_1 = 0$$

$$E_4(X_1, X_2, Z_1, Z_2) = -3/2 X_1^2 X_2 + 3/2 X_1^2 Z_2 + 3 X_1 X_2 Z_1 - 3 X_1 Z_1 Z_2 - 3/2 X_2^3 + 9/2 X_2^2 Z_2 - 3/2 X_2 Z_1^2 - 9/2 X_2 Z_2^2 + 5/2 Z_1^2 Z_2 + 5/2 Z_2^3 - 4 Z_2 w^2 - \frac{2}{25} Z_1 w + 9 Z_2 = 0 \quad (43)$$

Let us introduce the approximated  $L^2$ -norms for  $x$  and  $z$  respectively:

$$\begin{aligned} X &= X_1^2 + X_2^2 \simeq \|x\|^2 \\ Z &= Z_1^2 + Z_2^2 \simeq \|z\|^2 \end{aligned} \quad (44)$$

Let us call  $G2$  as the Gröbner basis obtained from the 4 equations  $E_1, \dots, E_4$ .

#### 4.6.2. An example of polynomial in $Z$

We look for polynomial  $P(X)$  (and  $Q(Z)$ ) where  $X$  (and  $Z$ ) is defined in Eq. 44. The process explained for the single dof system runs. By solving linear systems to nullify the remainder of unknown polynomials in  $X$  (respectively  $Z$ ) versus the Gröbner basis  $G2$  we obtain that: the minimum degree in  $X$  (and  $Z$ ) is 15. For example, the polynomial in  $Z$  is defined in Appendix B. Let us detect the frequency-response curves for closely- and not closely-spaced frequencies. As the first example, we consider a system where its frequencies are closely-spaced

frequencies, e.g.  $\omega_1 = 1$  and  $\omega_2 = 1.10$ . Figure 2 collects obtained results for  $L^2$ -norms of  $x$  and  $z$ . The same results for the system with  $\omega_1 = 1$  and  $\omega_2 = \frac{25}{17}$ , i.e. considered as not closely-spaced frequencies, are represents in Fig. 3.

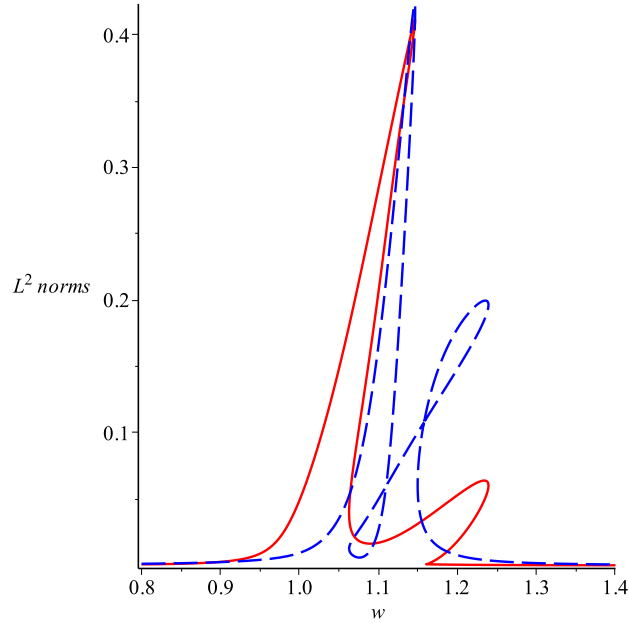


Figure 2: Curves of approximated  $L^2$ -norms of  $x$  (solid line) and  $z$  (dashed line) versus  $w$  for the system with  $\omega_1 = 1, \omega_2 = 1.10$ . Definitions of  $x$  and  $z$  are presented in Eq. 44.

#### 4.6.3. Example 2

Let us consider a model of a linear master system which is coupled to a nonlinear energy sink. We start from

$$\begin{cases} \frac{d^2x}{dt^2} + a \frac{dx}{dt} + x + \epsilon\alpha(x-y)^3 - \epsilon f \sin(wt) = 0 \\ \epsilon \frac{d^2y}{dt^2} + \epsilon\lambda \frac{dy}{dt} + \epsilon\alpha(y-x)^3 = 0 \end{cases} \quad (45)$$

By introducing

$$\begin{aligned} u &= x + \epsilon y \\ v &= x - y \end{aligned} \quad (46)$$

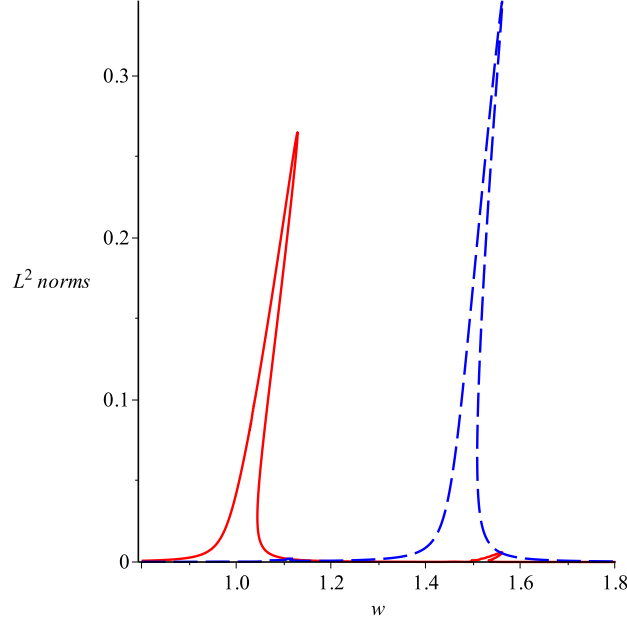


Figure 3: Curves of approximated  $L^2$ -norms of  $x$  (solid line) and  $z$  (dashed line) versus  $w$  for the system with  $\omega_1 = 1, \omega_2 = \frac{25}{17}$ . Definitions of  $x$  and  $z$  are presented in Eq. 44.

which is equivalent to

$$\begin{aligned} x &= \frac{u + \epsilon v}{1 + \epsilon} \\ y &= \frac{u - v}{1 + \epsilon} \end{aligned} \quad (47)$$

and  $\mu = \frac{1}{(1+\epsilon)}, p = (\epsilon\lambda + a)\mu, q = (a\epsilon - \lambda)\mu, r = (a - \lambda)\mu, s = (a\epsilon + \lambda)\mu, g = \epsilon f, \beta = (1 + \epsilon)\alpha$  we obtain the next two differential equations:

$$\begin{aligned} equ &= \frac{d^2u}{dt^2} + p\frac{du}{dt} + q\frac{dv}{dt} + \mu(\epsilon v + u) - g \sin(wt) \\ eqv &= \frac{d^2v}{dt^2} + r\frac{du}{dt} + s\frac{dv}{dt} + \mu(\epsilon v + u) - g \sin(wt) + \beta v^3 \end{aligned} \quad (48)$$

Let us name these two equations as  $equ$  and  $eqv$ . Looking for a rough approximation of periodic solutions for  $u$  and  $v$ , let us introduce:

$$\begin{aligned} u &\simeq X_1 \cos(wt) + X_2 \sin(wt) \\ v &\simeq Z_1 \cos(wt) + Z_2 \sin(wt) \end{aligned} \quad (49)$$

Using Harmonic Balance Method, we obtain four algebraic equations writing

$$\begin{aligned}
\frac{1}{T} \int_0^T equ \cos(wt) dt &= 0 \\
\frac{1}{T} \int_0^T equ \sin(wt) dt &= 0 \\
\frac{1}{T} \int_0^T eqv \cos(wt) &= 0 \\
\frac{1}{T} \int_0^T eqv \sin(wt) &= 0
\end{aligned} \tag{50}$$

Solving the first and second equations,  $X_1, X_2$  are obtained as functions of  $Z_1, Z_2$ :

$$\begin{aligned}
X_1 &= -\frac{-Z_1 \epsilon \mu w^2 + Z_1 pqw^2 - Z_2 \epsilon \mu pw - Z_2 qw^3 + Z_1 \epsilon \mu^2 + Z_2 \mu qw + gpw}{p^2 w^2 + w^4 - 2 \mu w^2 + \mu^2} \\
X_2 &= -\frac{Z_1 \epsilon \mu pw + Z_1 qw^3 - Z_2 \epsilon \mu w^2 + Z_2 pqw^2 - Z_1 \mu qw + Z_2 \epsilon \mu^2 + gw^2 - g\mu}{p^2 w^2 + w^4 - 2 \mu w^2 + \mu^2}
\end{aligned} \tag{51}$$

So two algebraic equations in  $Z_1, Z_2$  are obtained by considering numerators of third and fourth previous equations when replacing  $X_1, X_2$ . These equations can be written as:

$$\begin{aligned}
AZ_1^3 + AZ_1 Z_2^2 + BZ_1 + CZ_2 + E \\
AZ_2 Z_1^2 + AZ_2^3 + BZ_2 - CZ_1 + F
\end{aligned} \tag{52}$$

where  $A, B, C, E, F$  are given in Appendix C.

Let us build a Gröbner basis from this 2 equations in  $Z_1, Z_2$ . We obtain:

$$\begin{aligned}
& [(-AF^3 - BC^2F + C^3E) Z_1 + (AE^2C + ACF^2) Z_2^2 + \\
& (AEF^2 - BC^2E - C^3F) Z_2 - 2C^2EF (AE^2C + ACF^2) Z_1 Z_2 + \\
& (-AEF^2 - BC^2E - C^3F) Z_1 + \\
& (AE^2F + BC^2F - C^3E) Z_2 - C^2E^2 + C^2F^2 \\
& (AE^2C + ACF^2) Z_1^2 + (-AE^2F + BC^2F - C^3E) Z_1 + \\
& (AE^3 + BC^2E + C^3F) Z_2 + 2C^2EF]
\end{aligned} \tag{53}$$

Then we introduce the approximation of square of  $L^2$ -norm of  $v$ :

$$Z = Z_1^2 + Z_2^2 \tag{54}$$

We look for coefficients of a polynomial in  $Z$  of minimal degree so that this polynomial belongs to the ideal generated by the Groebner basis. We obtain a

polynomial of degree 3:

$$s_0 + s_1 Z + s_2 Z^2 + s_3 Z^3 = 0 \quad (55)$$

with:

$$\begin{aligned} s_3 &= 1 \\ s_2 &= 2 \frac{B}{A} \\ s_1 &= \frac{B^2 + C^2}{A^2} \\ s_0 &= -\frac{E^2 + F^2}{A^2} \end{aligned} \quad (56)$$

We obtain approximate periodic solutions. It is simple to analyze the behavior of the frequency - norm curve versus parameters of the problem and occurrence of potential isola.

Let us give an example. We choose  $\epsilon = \frac{1}{100}$ ,  $a = \frac{1}{100}$ ,  $\lambda = 2$ ,  $\alpha = 10$ . We keep two free parameters:  $w$  and  $f$ . Plotting  $Z$  versus  $w$  and  $f$  provides occurrence of multiple solutions and especially potential occurrence of isolas. This is presented in Fig. (4).

## 5. Conclusion

Smooth nonlinear differential equations treated by perturbation methods, e.g. harmonic balance method, yield to a set of polynomial equations. The proposed methodology in this paper, leads to potential solution of a bifurcation problem depending on  $X$ , standing for any component of the approximated  $L^2$  norms (or any components of truncated Fourier series) of the unknown component of the vector variable of generalized displacements and system parameters. It is shown that

- Either computer algebra could be used (if the size is not too large) to keep  $X$  and one or two parameters.
- Or a full numerical method can be designed to solve a polynomial equation in  $X$  with numerical values of parameters and loops to manage the values of parameters.

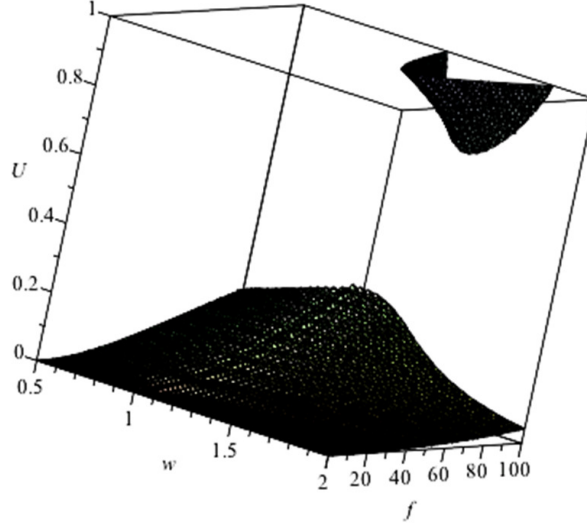


Figure 4: Curves of approximated  $L^2$ -norms of  $v (Z_1^2 + Z_2^2)$  versus  $w$  and  $f$  for  $\epsilon = \frac{1}{100}$ ,  $a = \frac{1}{100}$ ,  $\lambda = 2$ ,  $\alpha = 10$ .

The key point of our developments is to build Gröbner bases from the set of polynomial equations. Then, the remainders should be evaluated which could be splitted via computing the remainder of each monomials  $X^k$ ,  $k = 0, 1, \dots$ . Evaluation of Gröbner bases via softwares such as Maple<sup>©</sup>, Mathematica<sup>©</sup>, etc are straightforward. The explained methodology is general and theoretically can be applied to any (smooth) nonlinear dynamical systems, but creation of Gröbner bases and/or considerations of number of parameters depend on the characteristics of the computer. If the system possesses non polynomial terms, then they can be treated by introducing new variables in polynomials as the MANLAB software [38] does.



## Appendix A

Let us consider the Duffing equation

$$E(x(t)) = \frac{d^2x}{dt^2}(t) + a \frac{dx}{dt}(t) + \omega_0^2 x(t) + cx^3(t) - f \sin(\omega t) = 0 \quad (57)$$

where  $t$  denotes the time, and  $a$ ,  $c$ ,  $f$  are given parameters. We assume (up to a change of time variable)  $\omega_0 = 1$ .  $\omega$  gives external basic frequency for searching periodic solution. We look for approximated periodic solutions of Duffing equation using a truncated Fourier series of the form:

$$x(t) \simeq \frac{a_0}{2} + \sum_{j=1}^N a_j \cos(j\omega t) + b_j \sin(j\omega t) \quad (58)$$

Let us set  $T = \frac{2\pi}{\omega}$ . Algebraic equation are obtained by harmonic balance method as results of the following calculations:

$$\frac{1}{T} \int_0^T E(x(t)) dt = 0 \quad (59)$$

and for  $j = 1, \dots, N$ ,

$$\frac{1}{T} \int_0^T E(x(t)) \cos(j\omega t) dt = 0 \quad (60)$$

and

$$\frac{1}{T} \int_0^T E(x(t)) \sin(j\omega t) dt = 0 \quad (61)$$

Coefficients  $a_0, \dots, b_N$  are the  $2N + 1$  unknowns.

The first equation determines static equilibrium position around which oscillations are examined. In general, it could be omitted via assuming  $a_0 = 0$ . For example, assuming  $a_0 = 0$  and  $N = 1$ , two polynomial equations in  $a_1$  and  $b_1$  are obtained:

$$E_1(a_1, b_1) = \frac{3}{4}a_1^3c + \frac{3}{4}a_1b_1^2c - a_1w^2 + b_1aw + a_1 = 0 \quad (62)$$

$$E_2(a_1, b_1) = \frac{3}{4}a_1^2b_1c + \frac{3}{4}b_1^3c - awa_1 - b_1w^2 + b_1 - f = 0$$

Renaming  $a_j$  by  $X_j$ , for  $j = 1, 2$ , we obtain the corresponding equation to start Euclidean division or Gröbner approach. Let us notice that these equations can be written in the following form

$$E_1 = \frac{3c}{4}a_1N - a_1w^2 + b_1aw + a_1 = 0 \tag{63}$$

$$E_2 = \frac{3c}{4}b_1N - awa_1 - b_1w^2 + b_1 - f = 0$$

where  $N = a_1^2 + b_1^2$ , leading to a linear system in  $a_1, b_1$ . Solving this linear system provides an expression of  $a_1$  and  $b_1$  versus  $N$ . Recalling the relation  $N - a_1^2 - b_1^2 = 0$  leads to a polynomial equation in  $N$  that corresponds to approximation of  $\|x\|_2^2$ . This polynomial is expressed as:

$$\begin{aligned} &9N^3c^2 - 24N^2cw^2 + 16Na^2w^2 + 16Nw^4 + 24N^2c \\ &- 32Nw^2 - 16f^2 + 16N = 0 \end{aligned} \tag{64}$$

It is similar to the result obtained by euclidean division or Gröbner approach.

Appendix B

$$\begin{aligned}
& -\frac{1378169306088554230448128}{3154182529449462890625} w^{16} + \left( \frac{908148539392}{33074001} w^{16} - \frac{1795245925597184}{6890416875} w^{14} + \right. \\
& \quad \frac{925288206189068288}{861302109375} w^{12} - \frac{20303937659080428419072}{8074707275390625} w^{10} + \\
& \quad \frac{1229727953729482810564864}{336446136474609375} w^8 - \frac{3411410189342633673711232}{1009338409423828125} w^6 + \\
& \quad \frac{653616148882221825936344}{336446136474609375} w^4 - \frac{1068477874238692629147146}{1682230682373046875} w^2 + \\
& \left. \frac{1827943346824919744518657}{20186768188476562500} \right) Z^7 + \left( -\frac{1134053163008}{33074001} w^{14} + \frac{5887718468780032}{20671250625} w^{12} - \right. \\
& \quad \frac{4362555462584717312}{4306510546875} w^{10} + \frac{5384114551457063507968}{2691569091796875} w^8 - \frac{10629871028983174135616}{4485948486328125} w^6 + \\
& \quad \frac{113367363707739767193776}{67289227294921875} w^4 - \frac{134473665141617001115444}{201867681884765625} w^2 + \\
& \quad \left. \frac{91308017871954880257533}{807470727539062500} \right) Z^8 + \left( \frac{51252330496}{33074001} w^{12} - \frac{22683650711552}{2296805625} w^{10} + \right. \\
& \quad \frac{38475728778145792}{1435503515625} w^8 - \frac{325127784501945935872}{8074707275390625} w^6 + \frac{32176760582720770288}{897189697265625} w^4 - \\
& \quad \frac{16267166582808973592}{897189697265625} w^2 + \frac{131152668984318128389}{32298829101562500} \left. \right) Z^9 + \left( \frac{6399029248}{1224963} w^{10} - \right. \\
& \quad \frac{4766769442816}{153120375} w^8 + \frac{11748956511902464}{159500390625} w^6 - \frac{344605556432115472}{3987509765625} w^4 + \\
& \quad \frac{120124244975725096}{2392505859375} w^2 - \frac{137993592668846662}{11962529296875} \left. \right) Z^{10} + \left( -\frac{408321}{523905280} w^8 + \right. \\
& \quad \frac{1493761338368}{255200625} w^6 - \frac{1573245655526528}{159500390625} w^4 + \frac{1156202316118244}{159500390625} w^2 - \\
& \quad \left. \frac{310850797475104}{159500390625} \right) Z^{11} + \left( -\frac{78814976}{408321} w^6 + \frac{22315262528}{28355625} w^4 - \frac{757496789528}{708890625} w^2 + \right. \\
& \quad \left. \frac{3080599852667}{6380015625} \right) Z^{12} + \left( \frac{265855600406687329291500272063552}{657121360301971435546875} w^8 - \right. \\
& \quad \frac{87095193364126415138763814569952}{394272816181182861328125} w^6 + \frac{132288258862375468329763133764}{1622521877288818359375} w^4 - \\
& \quad \frac{36005186115362658082043360745472}{1971364080905914306640625} w^2 + \frac{5512741519360}{33074001} w^{24} - \frac{3343050977837056}{1378083375} w^{22} + \\
& \quad \frac{209421832285824483328}{12919531640625} w^{20} - \frac{530952793350881590116352}{8074707275390625} w^{18} + \\
& \quad \frac{910149056368918877845651456}{5046692047119140625} w^{16} - \frac{1111252296156390035946564026368}{3154182529449462890625} w^{14} + \\
& \quad \frac{330301481418404569761234594230272}{657121360301971435546875} w^{12} - \frac{1040290862758469027093870159240192}{1971364080905914306640625} w^{10} + \\
& \left. \frac{3701844637451132472050108936009}{1971364080905914306640625} \right) Z^3 + \left( -\frac{2462811422720}{33074001} w^{22} + \frac{282938851721216}{275616675} w^{20} - \right.
\end{aligned}$$

$$\begin{aligned}
& \frac{83469735214580236288}{12919531640625} w^{18} + \frac{197758922084693295824896}{8074707275390625} w^{16} - \\
& \frac{313435647881896438249078784}{5046692047119140625} w^{14} + \frac{348807083000509485867592216576}{3154182529449462890625} w^{12} - \\
& \frac{3706857255637949525597361077248}{26284854412078857421875} w^{10} + \frac{2030801564587105529363588216128}{15770912647247314453125} w^8 - \\
& \frac{6503741754449884086254164046048}{78854563236236572265625} w^6 + \frac{185490219365537888065035581348}{5256970882415771484375} w^4 - \\
& \frac{78854563236236572265625}{715383594036407656829198137946} w^2 + \frac{334930554784032922352555921327}{315418252944946289062500} )Z^4 + \\
& \left( -\frac{4145241128960}{33074001} w^{20} + \frac{6169415905181696}{4134250125} w^{18} - \frac{4124918675357433856}{516781265625} w^{16} + \right. \\
& \quad \frac{40781055576003251142656}{1614941455078125} w^{14} - \frac{52802737470560161361604608}{1009338409423828125} w^{12} + \\
& \quad \frac{233811739938965398616669913088}{1699130128334449594721506112} w^{10} - \frac{1699130128334449594721506112}{23364315032958984375} w^8 + \\
& \quad \frac{3154182529449462890625}{153757562975591585094666167456} w^6 - \frac{67342870089939159963999879004}{3154182529449462890625} w^4 + \\
& \quad \frac{3154182529449462890625}{5795218416971835320518889324} w^2 - \frac{113043846351105662544097141}{177700424194335937500} )Z^5 + \\
& \left( \frac{3054172635136}{33074001} w^{18} - \frac{6888025348636672}{6890416875} w^{16} + \frac{12448522917675139072}{2583906328125} w^{14} - \right. \\
& \quad \frac{4056662303141916557312}{299063232421875} w^{12} + \frac{124062764458015971484742656}{5046692047119140625} w^{10} - \\
& \quad \frac{3750859833805571136464068096}{126167301177978515625} w^8 + \frac{42055767059326171875}{1008580232405190022172262976} w^6 - \\
& \quad \frac{313847852654861086953748024}{25233460235595703125} w^4 + \frac{31635974345722301381951678}{8411153411865234375} w^2 - \\
& \quad \frac{50968862889420126267899257}{100933840942382812500} )Z^6 - \frac{371735497963208704}{8074707275390625} w^{20} + \\
& \quad \frac{1547811610857618971615166464}{1971364080905914306640625} w^{14} - \frac{1100528098786680305569}{504669204711914062500} - \frac{11341398016}{20671250625} w^{24} - \\
& \quad \frac{16883352779491959714151604224}{16428034007549285888671875} w^{12} + \left( \frac{1870496}{15123} w^4 - \frac{584129312}{1890375} w^2 + \frac{1827186602}{9451875} \right) Z^{13} + \\
& \quad \frac{95966531158016}{12919531640625} w^{22} + \frac{871685003677524819968}{5046692047119140625} w^{18} + \\
& \quad \frac{1216565315255726119030038846464}{1232102550566196441650390625} w^{10} - \frac{851522177212674164034359093248}{1232102550566196441650390625} w^8 + \\
& \quad \left( -\frac{1344}{71} w^2 + \frac{41412}{1775} \right) Z^{14} + \left( \frac{181462368256}{33074001} w^{28} - \frac{67032628330496}{765601875} w^{26} + \right. \\
& \quad \frac{334942201421234176}{516781265625} w^{24} - \frac{23819578878102415081472}{8074707275390625} w^{22} + \\
& \quad \frac{517147007449997697572864}{560743560791015625} w^{20} - \frac{66086231935138517144685248512}{3154182529449462890625} w^{18} + \\
& \quad \frac{7813210067145962903477966012416}{219040453433990478515625} w^{16} - \frac{2278510610280936959236919410982912}{49284102022647857666015625} w^{14} +
\end{aligned}$$

$$\begin{aligned}
& \frac{56482922746171457535692562038431744}{1232102550566196441650390625} w^{12} - \frac{14213757254597023392992218194694144}{410700850188732147216796875} w^{10} + \\
& \frac{24127235897828715075885994451039744}{1232102550566196441650390625} w^8 - \frac{9922114788423397338736265850082496}{1232102550566196441650390625} w^6 + \\
& \frac{37381352907555242865961626841448}{16428034007549285888671875} w^4 - \frac{155915865836905718017344659566}{394272816181182861328125} w^2 + \\
& \frac{402383315056822591565148793}{12616730117797851562500} )Z + ( \frac{8628331889513769041957610278172608}{49284102022647857666015625} w^8 - \\
& \frac{4053557152868564285713095549258656}{49284102022647857666015625} w^6 + \frac{1297687189155867463789643378703548}{49284102022647857666015625} w^4 - \\
& \frac{2029650382567565733103442318956}{394272816181182861328125} w^2 - \frac{1880927240192}{33074001} w^{26} + \frac{17981852455272448}{20671250625} w^{24} - \\
& \frac{79340727352602656768}{12919531640625} w^{22} + \frac{213923019999400703295488}{8074707275390625} w^{20} - \\
& \frac{78646943003392452712988672}{1009338409423828125} w^{18} + \frac{104077022971241060071581679616}{630836505889892578125} w^{16} - \\
& \frac{510021699746424067943739988688896}{1971364080905914306640625} w^{14} + \frac{4997487074654034102224839458234368}{630836505889892578125} w^{12} - \\
& \frac{1971364080905914306640625}{2643494904566063655125361759107072} w^{10} + \frac{16428034007549285888671875}{73198695034948774671932113819} )Z^2 + \\
& \frac{9856820404529571533203125}{369484831146012933221404} w^2 + \frac{423493498330136981072144320256}{1232102550566196441650390625} w^6 - \\
& \frac{15770912647247314453125}{227306693537749813312275616} w^4 + Z^{15} \\
& \frac{1971364080905914306640625}{1971364080905914306640625}
\end{aligned}$$

## Appendix C

$$\begin{aligned}A &= 3\beta p^2 w^2 + 3\beta w^4 - 6\beta \mu w^2 + 3\beta \mu^2 \\B &= 4\epsilon \mu p^2 w^2 - 4\epsilon \mu prw^2 + 4\epsilon \mu w^4 - 4p^2 w^4 - 4qrw^4 - 4w^6 - 4\epsilon \mu^2 w^2 \\&\quad - 4\mu pqw^2 + 4\mu qrw^2 + 8\mu w^4 - 4\mu^2 w^2 \\C &= 4\epsilon \mu rw^3 + 4p^2 sw^3 - 4pqrw^3 + 4sw^5 + 4\epsilon \mu^2 pw - 4\epsilon \mu^2 rw + 4\mu qw^3 \\&\quad - 8\mu sw^3 - 4\mu^2 qw + 4\mu^2 sw \\E &= -4grw^3 - 4g\mu pw + 4g\mu rw \\F &= -4gp^2 w^2 + 4gprw^2 - 4gw^4 + 4g\mu w^2\end{aligned}$$

## References

## References

- [1] N. M. Krylov, N. N. Bogolyubov, Methodes approchees de la mecanique non-lineaire dans leurs application a l'étude de la perturbation des mouvements periodiques de divers phenomenes de resonance s'y rapportant, Académie des Sciences d'Ukraine, Kiev, 1935, 114 pages.
- [2] N. M. Krylov, N. N. Bogolyubov, Introduction to nonlinear mechanics., Izd-vo AN USSR, Kiev, 1937, 367 pages.
- [3] N. Kryloff, N. Bogoliuboff, Introduction to Non-Linear Mechanics. (AM-11), Princeton University Press, 1943, 106 pages.
- [4] N. N. Bogoliubov, Y. A. Mitropolski, Asymptotic Methods in the Theory of Non-Linear Oscillations., Gordon & Breach, New York, 1961, 520 pages.
- [5] J. Hale, Oscillations in Nonlinear Systems, McGraw-Hil, 1963, 180 pages.
- [6] A. H. Nayfeh, Perturbation Methods, John Wiley & Sons Inc, New York, 1973, 448 pages.

- [7] P. Coulet, E. Spiegel, Amplitude equations for systems with competing instabilities, *SIAM Journal on Applied Mathematics* 43 (4) (1983) 776–821. [arXiv:https://doi.org/10.1137/0143052](https://doi.org/10.1137/0143052), [doi:10.1137/0143052](https://doi.org/10.1137/0143052). URL <https://doi.org/10.1137/0143052>
- [8] M. Roseau, *Vibrations in Mechanical Systems*, Springer-Verlag Berlin Heidelberg, 1984, 515 pages.
- [9] D. R. Smith, *Singular-Perturbation Theory: an Introduction with Applications*, Cambridge University Press, Cambridge, 1985, 520 pages.
- [10] C. Hayashi, *Forced oscillations in nonlinear systems*, Nippon Print. and Pub. Co, Osaka, 1953, 164 pages.
- [11] L. Cesari, Functional analysis and galerkin’s method, *The Michigan Mathematical Journal* 11 (4) (1964) 385–414.
- [12] M. Urabe, Galerkin’s procedure for nonlinear periodic systems, *Archive for Rational Mechanics and Analysis* 20 (2) (1965) 120–152.
- [13] H. Poincaré, *Les méthodes nouvelles de la mécanique céleste*, Volume 2, Gauthier-Villars, Paris, 1893.
- [14] G. Birkhoff, *Dynamical systems*, vol. 9, Colloquium publications (American Mathematical Society), New York, 1927.
- [15] C. Elphick, E. Tirapegui, M. Brachet, P. Coulet, G. Iooss, A simple global characterization for normal forms of singular vector fields, *Physica D: Nonlinear Phenomena* 29 (1) (1987) 95 – 127.
- [16] J. Guckenheimer, P. J. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer-Verlag, New York, 1983.
- [17] F. G. Gustavson, On constructing formal integrals of a hamiltonian system near ail equilibrium point, *Astronomical Journal* 71 (8) (1966) 670–686.

- [18] G. Iooss, M. Adelmeyer, *Topics in Bifurcation Theory and Applications*, 2nd Edition, World Scientific, 1999.
- [19] C. L. Siegel, J. K. Moser, *Lectures on Celestial Mechanics*, Springer-Verlag Berlin Heidelberg, 1995.
- [20] R. Rand, D. Armbruster, *Perturbation Methods, Bifurcation Theory and Computer Algebra*, Springer-Verlag New York, 1987.
- [21] V. Zhuravlev, A new algorithm for birkhoff normalization of hamiltonian systems, *Journal of Applied Mathematics and Mechanics* 61 (1) (1997) 9–13.
- [22] V. Zhuravlev, The normal form of perturbations of non-linear oscillatory systems, *Journal of Applied Mathematics and Mechanics* 66 (6) (2002) 881–887.
- [23] A. H. Nayfeh, D. T. Mook, *Nonlinear Oscillations*, Wiley, 1995.
- [24] R. Starosta, G. Sypniewska-Kamińska, J. Awrejcewicz, Parametric and external resonances in kinematically and externally excited nonlinear spring pendulum, *International Journal of Bifurcation and Chaos* 21 (10) (2011) 3013–3021.
- [25] R. Starosta, G. Sypniewska-Kamińska, J. Awrejcewicz, Asymptotic analysis of kinematically excited dynamical systems near resonances, *Nonlinear Dynamics* 68 (4) (2012) 459 – 469.
- [26] A. A. Klimenko, Y. V. Mikhlin, J. Awrejcewicz, Nonlinear normal modes in pendulum systems, *Nonlinear Dynamics* 70 (1) (2012) 797–813.
- [27] Y. Mikhlin, A. Onizhuk, J. Awrejcewicz, Resonance behavior of the system with a limited power supply having the mises girder as absorber, *Nonlinear Dynamics* 99 (1) (2020) 519–536.
- [28] A. Bruno, *Power Geometry in Algebraic and Differential Equations*, Elsevier Science, 2000.



- [29] I. V. Andrianov, J. Awrejcewicz, New Trends in Asymptotic Approaches: Summation and Interpolation Methods, *Applied Mechanics Reviews* 54 (1) (2001) 69–92.
- [30] M. Guskov, J.-J. Sinou, F. Thouverez, Multi-dimensional harmonic balance applied to rotor dynamics, *Mechanics Research Communications* 35 (8) (2008) 537 – 545.
- [31] C. Grenat, S. Baguet, C.-H. Lamarque, R. Dufour, A multi-parametric recursive continuation method for nonlinear dynamical systems, *Mechanical Systems and Signal Processing* 127 (2019) 276 – 289.
- [32] T. Becker, V. Weispfenning, *Gröbner Bases: A Computational Approach to Commutative Algebra*, Springer, New York, 1993, 574 pages.
- [33] B. Buchberger, A theoretical basis for the reduction of polynomials to canonical forms, *SIGSAM Bull.* 10 (3) (1976) 19–29.
- [34] A. Grolet, *Dynamique non-linéaire des structures mécaniques : application aux systèmes à symétrie cyclique*, École doctorale MEGA, Ecole Centrale de Lyon, 2013.
- [35] A. Grolet, F. Thouverez, Computing multiple periodic solutions of nonlinear vibration problems using the harmonic balance method and groebner bases, *Mechanical Systems and Signal Processing* 52-53 (2015) 529 – 547.
- [36] A. Bostan, F. Chyzak, M. Giusti, R. Lebreton, G. Lecerf, B. Salvy, E. Schost, *Algorithmes Efficaces en Calcul Formel*, Frédéric Chyzak (auto-édit.), Palaiseau, 2017, 686 pages. Printed by CreateSpace. Also available in electronic version.  
URL <https://hal.archives-ouvertes.fr/AECF/>
- [37] Maplesoft, a division of Waterloo Maple Inc., Maple.  
URL <https://www.maplesoft.com>
- [38] MANLAB, An interactive path-following and bifurcation analysis software, LMA, CNRS, France, 2019.