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Abstract: The paper presents a dynamic event-triggering mechanism for the constant and the localized damped linear Schrödinger equation. The following results are tackled: the existence of solution to the closed-loop event-triggered control system; the avoidance of the Zeno behavior due to the absence of any accumulation point of the sequence of time instants and the exponential stability based on energy estimate through the observability inequality. A simulation example based on the one-dimensional Schrödinger equation is presented to validate the theoretical results.

Keywords: Schrödinger equation, Dynamic event-triggering mechanism, Exponential stability, Observability inequality.

1. INTRODUCTION

1.1 Control of the linear Schrödinger equation

The Schrödinger equation is a partial differential equation (PDE) modeling the behavior of the wave function of a non-relativistic particle in quantum mechanical systems (Sulem and Sulem (2007)) as laser beam propagation (Andrews and Phillips (2005)) or Bose Einstein condensation. Besides, the solutions to the linear Schrödinger equation also solve a plate equation (Machtyngier and Zuazua (1994)), which models for instance in one-dimension in space, the deflection of a beam due to applied loads. All these applications justify the extensive study of this equation in many aspects, including the well-posedness (Cazenave et al. (1998)), the exact controllability (Machtyngier (1994)), the observability (Phung (2001),) and the stabilization or stability analysis by multiplier techniques and constructing energy functionals (Machtyngier and Zuazua (1994)) or by backstepping approach via the boundary actuation and measurements (Krstic et al. (2011)).

1.2 Event-triggering mechanism

Starting from a stabilized closed-loop system, where the dynamic is described by a Schrödinger equation and a control is applied as a continuous-in-time and localized damping source term, we are interested in this article by the effect of a digital implementation of this control. The time sampling implementation of such a continuous-in-time controller has to be designed efficiently in order to avoid wasting communications, computational and actuating resources. The event-triggering strategy turned out to be a powerful tool for this, since the events are planned to occur only when needed, while allowing stability. Indeed, event-triggering strategies lead to a different paradigm with respect to traditional periodic implementations since the use of event-triggering schemes enables to update control inputs only when specific state-dependent events occur. To the best of our knowledge, there is no exponential stability proof on periodic sampling for the Schrödinger equation. Another important feature of an event-triggering approach is to ensure that there will not be infinitely many updates of the control over a bounded time interval, situation known as the Zeno behavior.

In event-triggered control theory, two approaches can be considered: the *emulation* and the *co-design*. Concerning the emulation method, the control is supposed to be a priori known and only the event-triggering mechanism is being designed, as in Postoyan et al. (2014); Espitia et al. (2017, 2021) respectively for nonlinear finite dimentional systems using hybrid systems tools; linear hyperbolic systems of conservation laws and for constant-parameters reaction-diffusion PDE systems. As far as the co-design approach is concerned, both the controller and the event-triggering mechanism are simultaneously designed. This approach is addressed in some papers, see, for example, Seuret et al. (2016) and reference therein for LQ-stabilization of saturated linear systems. In the current article, as described earlier, we adopt an emulation approach, starting from the results in Machtyngier and Zuazua (1994) and Phung (2001) where a stabilizing controller has been prescribed to stabilize the linear Schrödinger equation.

With the goal of enriching the static event-triggering mechanism (state-based triggering condition) designed in Koudohode et al. (2022), we propose here a dynamic event-triggering rule, similarly to the one introduced for general framework of nonlinear finite-dimentional control system in Girard (2015) and in Zhao et al. (2020); Peralez et al. (2018) using small-gain and high gain methods. This new dynamic rule consists in introducing an additional

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internal dynamic variable to the static law. It is worthwhile to mention that the dynamic event-triggering strategy has already been extended to PDE framework in Espitia (2020) for a coupled 2×2 linear hyperbolic system, in Wang and Krstic (2021) for sandwich hyperbolic PDE systems and in Rathnayake et al. (2021) for a class of reaction-diffusion PDEs with Robin actuation.

Event-based control strategies were also considered for parabolic PDE in Espitia et al. (2021) (with small gain approach), Selivanov and Fridman (2016) (for distributed event-triggered control) and for hyperbolic PDE in Baudouin et al. (2019) (damped wave equation), Espitia et al. (2017) (via Lyapunov-based event triggered sampling and quantization), Davo et al. (2018) (via backstepping method and looped-functionals) and for abstract infinite-dimensional systems in Wakaiki and Sano (2020) and Wakaiki and Sano (2019) (under Lipschitz perturbations).

1.3 Contributions

The contributions of the paper can be summarized as follows:

- We design a novel and dynamic event-triggering mechanism for a locally damped linear Schrödinger equation different from the static rule proposed in Koudohode et al. (2022) and we ensure the avoidance of Zeno phenomenon.
- Thanks to an energy estimate and an observability inequality we prove global exponential stability of the energy of the closed-loop system under the designed dynamic event-triggering rule.

1.4 Outline

The damped linear Schrödinger equation under consideration is introduced in Section 2 and some existing results on well-posedness and stability are presented. Section 3 proposes the design of the dynamic event-triggering mechanism, the proof of the well-posedness and of the avoidance of the Zeno phenomenon. The main event-based stabilization results are provided in Section 4. Finally, Section 5 gives some numerical illustrations of the theoretical results. Concluding remarks and perspectives are given in Section 6.

1.5 Notation

Given an open set $\Omega \subset \mathbb{R}^N$, $L^2(\Omega)$ is the Hilbert space of square integrable scalar functions endowed with norm $\|z\| = (\int_\Omega |z(x)|^2 dx)^{\frac{1}{2}}$. The gradient and Laplacian of z are denoted $\nabla z = (\partial_{x_1} z, \dots \partial_{x_N} z)$ and $\Delta z = \sum_{i=1}^N \partial_{x_i}^2 z$. We define the Sobolev spaces $H_0^1(\Omega) = \{z \in L^2(\Omega), \nabla z \in \left(L^2(\Omega)\right)^N, z|_{\partial\Omega} = 0\}$, with norm $\|z\|_{H_0^1(\Omega)} = \|\nabla z\|$ and $H^2(\Omega) = \{z \in L^2(\Omega), \nabla z \in \left(L^2(\Omega)\right)^N, \partial_{x_j}\partial_{x_i} z \in L^2(\Omega)\}$, the set of all function such that $\int_\Omega \left(|z|^2 + |\nabla z|^2 + |\Delta z|^2\right) dx$ is finite. The dual space of a Sobolev space H is denoted H'. We will often write $\int_\Omega g(t)$ instead of $\int_\Omega g(x,t)dx$ to ease the reading. $Im(z)$ and $Re(z)$ are respectively the imaginary part and real part of $z \in \mathbb{C}$ and its complex conjugate is denoted $\bar{z}$.

2. PRELIMINARY AND PROBLEM DESCRIPTION

This paper deals with the stabilization of a damped linear Schrödinger equation under event-triggered control. Let Ω be an open bounded domain in $\mathbb{R}^N$, $N \in \mathbb{N}^*$ with smooth boundary $\partial\Omega$. Let us consider the following control system

$$\begin{cases} i\partial_t z(x,t) + \Delta z(x,t) = -i\alpha(x) z(x,t) & (x,t) \in \Omega \times \mathbb{R}^+, \\ z(x,t) = 0 & (x,t) \in \partial\Omega \times \mathbb{R}^+ \\ z(x,0) = z_0(x) & x \in \Omega. \end{cases} \tag{1}$$

In system (1), $\alpha \in L^\infty(\Omega; \mathbb{R})$ is the damping coefficient and is such that there exist $\omega \subset \Omega$, α_0 and $\alpha_1 \in \mathbb{R}+$ such that

$$\begin{cases} 0 < \alpha < \alpha_1 = \|\alpha\|_{L^\infty(\Omega)} & \text{a.e. in } \Omega \\ \exists \alpha_0 > 0 : \alpha \geq \alpha_0 & \text{a.e. in } \omega \subset \Omega. \end{cases} \tag{2}$$

This means that the damping will not necessarily act on the whole domain Ω. The well-posedness and exponential stability of system (1) are already documented in the literature. For instance, it has been proved in Cazenave et al. (1998); Machtyngier and Zuazua (1994) the following theorem.

Theorem 2.1. Well-posedness and stabilization (Cazenave et al. (1998); Machtyngier and Zuazua (1994))

(1) For any initial conditions $z_0 \in L^2(\Omega)$, there exists a unique weak solution to (1) satisfying

$$z \in C^0(\mathbb{R}^+; L^2(\Omega)) \cap C^1([0,T]; (H^2(\Omega) \cap H_0^1(\Omega))'). \tag{3}$$

Moreover, for any initial data $z_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, the unique solution to (1) satisfies

$$z \in C^0(\mathbb{R}^+; H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1(\mathbb{R}^+; L^2(\Omega)). \tag{4}$$

(2) For any initial condition in $L^2(\Omega)$, there exist $C > 0$ and $\delta > 0$ such that the weak solution z to (1) verifies for all $t > 0$

$$E(t) := \frac{1}{2}\|z(t)\|^2 \leq CE(0)e^{-2\delta t}. \tag{5}$$

We address the problem of event-triggered implementation of a state-feedback stabilizing law for the control system such that

$$\begin{cases} i\partial_t z + \Delta z = -i\alpha(x) z(t_k), & \text{in } \Omega \times [t_k, t_{k+1}), k \in \mathbb{N} \\ z = 0, & \text{on } \partial\Omega \times \mathbb{R}^+, \\ z(\cdot, 0) = z_0 & \text{in } \Omega \end{cases} \tag{6}$$

where $0 = t_0 < t_1 < \cdots < t_k < t_{k+1}$. In this system, $\{t_k\}_{k\in\mathbb{N}}$, is the sequence of updates according to the event-triggering algorithm presented in Section 3.

Let us define the following time T corresponding to the maximal time under which the event-triggered control system (6) has a unique solution:

$$\begin{cases} T = +\infty & \text{if } (t_k) \text{ is a finite sequence,} \\ T = \limsup_{k\to+\infty} t_k & \text{if not.} \end{cases} \tag{7}$$

Proving that $T = +\infty$ will ensures that there will not be accumulation point of the sequence $\{t_k\}_{k\in\mathbb{N}}$. This will be proved in Theorem 3.1 in order to avoid Zeno behavior.

3. DYNAMIC EVENT-TRIGGERING STRATEGY

We introduce in this section, the event-triggering algorithm, which determines the time instant at which the controller $-\alpha z$ needs to be updated. Some results on the well-posedness and the absence of Zeno behavior will be also presented.

3.1 Definition of the event-triggering mechanism

Inspired by the emulation approach introduced in the context of ordinary differential equation in Tabuada (2007); Postoyan et al. (2014); Girard (2015), the following state-dependent criterion was proposed in Koudohode et al. (2022). Starting from $t_0 = 0$, then $\forall k > 0$

$$t_{k+1} = \inf\left\{t \geq t_k, \|z(t) - z(t_k)\|^2 > \gamma\|z(t)\|^2\right\}. \quad (8)$$

where $\gamma > 0$ is a design parameter. Hence, an event is generated when the error term e_k

$$e_k(x,t) = z(x,t) - z(x,t_k), \quad (9)$$

becomes larger than a proportion of the energy. In this paper, we propose to enrich our event-triggering mechanism (8) by adding an internal scalar dynamic variable m satisfying the following differential equation

$$\dot{m}(t) = -\eta m(t) + 2\gamma E(t) - \|e_k(t)\|^2, \text{ for } t \geq t_k \quad (10)$$

with $m(t_k) = 0$ and $\eta > 0$ a design parameter.

Then, we can describe the event-triggering law under consideration in the paper. Starting from $t_0 = 0$

$$t_{k+1} = \inf\left\{t \geq t_k, \|e_k(t)\|^2 - 2\gamma E(t) > \frac{1}{\theta}m(t)\right\} \quad (11)$$

where $\gamma > 0$ and $\theta > 0$ are design parameters.

Remark 3.1. When the design parameter θ tends to $+\infty$ in the dynamic rule (11), we obtain the static rule (8). Note that the signal $m(t)$ can be considered as a filtered value of $2\gamma E(t) - \|e_k(t)\|^2$.

Similarly to Girard (2015), one gets the following result.

Lemma 3.1. Using the definition of the event-triggering mechanism (11), it follows, for all $t \in [t_k, t_{k+1}), k \geq 0$:

$$m(t) \geq 0 \text{ and } \|e_k(t)\|^2 \leq \frac{1}{\theta}m(t) + 2\gamma E(t) \quad (12)$$

Proof. Indeed, between two triggering instants t_k, t_{k+1}, from (11), we have $\frac{1}{\theta}m(t) + 2\gamma E(t) - \|e_k(t)\|^2 \geq 0$. Combined to (10), this inequality brings

$$\frac{1}{\theta}m(t) + 2\dot{m}(t) + \eta m(t) \geq 0, \text{ i.e } \dot{m}(t) \geq -\left(\frac{1}{\theta} + \eta\right)m(t)$$

for which the comparison principle can be applied to guarantee $m(t) \geq 0$, for all $t \in [t_k, t_{k+1}), k \geq 0$. □

Remark 3.2. The dynamic event-triggering mechanism is usually constructed with $m(t) \leq 0$ as in Espitia et al. (2017); Espitia (2020); Rathnayake et al. (2021); Wang and Krstic (2021) but this paper follows the same approach as in Girard (2015); Karafyllis et al. (2021) where $m(t)$ is positive.

Remark 3.3. For a given state $z(t_k)$ of the event-triggered control system (6), since $m(t) \geq 0$, the next execution time t_{k+1} given by the dynamic rule (11) comes later than the one given by the static rule (8).

3.2 Well-posedness and absence of Zeno behavior

Leveraging on some regularity of the classical solutions to the Schrödinger equation we get the following lemma.

Lemma 3.2. Let Ω be an open bounded domain of class C^2. For any initial conditions $z_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, there exists a unique solution to (6) under the event-triggering mechanism (11), satisfying

$$z \in C^0([0,T]; H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega)). \quad (13)$$

Proof. The proof is constructed by induction and is similar to the one that has been presented in Koudohode et al. (2022) for the static event-triggering law (8). □

Let us now notice that from (9), the event-triggered closed-loop system also reads:

$$\begin{cases} i\partial_t z + \Delta z = -i\alpha z + i\alpha e_k, & \text{in } \Omega \times [t_k, t_{k+1}), \\ z = 0, & \text{on } \partial\Omega \times \mathbb{R}^+, \\ z(\cdot, 0) = z_0, & \text{in } \Omega. \end{cases} \quad (14)$$

Before proving that the Zeno phenomenon cannot occur, let us state the following intermediate result.

Lemma 3.3. Under the event-triggering law (11), for all $t \in [0,T)$ one has:

$$2\gamma\theta E(0)e^{-2Kt} \leq 2\gamma\theta E(t) + m(t), \quad (15)$$

with

$$K = \frac{1}{2}\max\left\{3\alpha_1 + \frac{2}{\theta} + \alpha_1\gamma \;;\; \alpha_1\gamma + \frac{1}{\theta} + \eta\right\}. \quad (16)$$

Proof. Performing the Green formula with $z = 0$ on $\partial\Omega$, the time-derivative of $E(t)$ along the trajectories of system (14) is given by:

$$\dot{E}(t) = \mathcal{R}e\left(\int_\Omega \bar{z}(t)\partial_t z(t)\right) = \mathcal{I}m\left(-\int_\Omega \bar{z}(t)\Delta z(t)\right)$$
$$-\mathcal{I}m\left(\int_\Omega i\alpha(x)|z(t)|^2\right) + \mathcal{I}m\left(i\int_\Omega \alpha(x)e_k(t)\bar{z}(t)\right)$$
$$\dot{E}(t) = -\int_\Omega \alpha(x)|z(t)|^2 + \mathcal{R}e\left(\int_\Omega \alpha(x)\bar{e}_k(t)z(t)\right) \quad (17)$$

Then, we use (2), along with Cauchy-Schwarz and Young's inequalities, to obtain

$$\begin{aligned}\left|\dot{E}(t)\right| &\leq \alpha_1\|z(t)\|^2 + \alpha_1\|e_k(t)\|\|z(t)\| \\ &\leq \frac{3\alpha_1}{2}\|z(t)\|^2 + \frac{\alpha_1}{2}\|e_k(t)\|^2. \end{aligned} \quad (18)$$

Hence, $|2\gamma\theta\dot{E}(t) + \dot{m}(t)| \leq \gamma\theta\left(3\alpha_1\|z(t)\|^2 + \alpha_1\|e_k(t)\|^2\right)$ $+ \eta m(t) + 2\gamma E(t) + \|e_k(t)\|^2$.

From (12) and using $\|z(t)\|^2 = 2E(t)$, we get

$$|2\gamma\theta\dot{E}(t) + \dot{m}(t)| \leq 6\alpha_1\theta\gamma E(t) + \eta m(t) + 2\gamma E(t)$$
$$+ (\alpha_1\theta\gamma + 1)\left(\frac{1}{\theta}m(t) + 2\gamma E(t)\right)$$
$$\leq \left(6\alpha_1\theta\gamma + 4\gamma + 2\alpha_1\theta\gamma^2\right)E(t) + \left(\alpha_1\gamma + \frac{1}{\theta} + \eta\right)m(t),$$

so that with K defined by (16), we can write

$$|2\gamma\theta\dot{E}(t) + \dot{m}(t)| \leq 2K\left(2\gamma\theta E(t) + m(t)\right).$$

From there, $-2K\left(2\gamma\theta E(t) + m(t)\right) \leq 2\gamma\theta\dot{E}(t) + \dot{m}(t)$ and one gets that $F(t) = e^{2Kt}(2\gamma\theta E(t) + m(t))$ satisfies $\dot{F}(t) \leq 0$ so that for any $t \in [t_k, t_{k+1}]$, using $m(t_k) = 0$,

$$2\gamma\theta E(t_k)e^{-2K(t-t_k)} \leq 2\gamma\theta E(t) + m(t).$$

Inferring this inequality for $E(t_{k-1})$ up to $t_0 = 0$, by induction we get (15) for all $t \in [0,T)$ and the lemma is proved. □

We can now provide the main result on the fact that our event-triggering law does not generate some infinite sequence of updates in finite time, proving thus the absence of Zeno behavior.

Theorem 3.1. There is no Zeno phenomenon for the system (6) under the event-triggering mechanism (11). In other words, following (11), there will not be infinitely many updates of the control of system (6) over any bounded time interval.

Proof. The proof is done by contradiction. Let us assume that T defined by (7) is such that $T < +\infty$. Let us also define and study the evolution of the following function :

$$\varphi(t) = \frac{\theta\|e_k(t)\|^2}{2\gamma\theta E(t) + m(t)} \tag{19}$$

As in Girard (2015); Koudohode et al. (2022), the proof is based on the study of φ in the interval $[0, T]$. This function φ is non negative and satisfies, $\forall k \in \mathbb{N}$, $\varphi(t_k^+) = 0$ and jumps from $\varphi(t_{k+1}^+) = 1$ to $\varphi(t_{k+1}^+) = 0$, where $\varphi(t_{k+1}^-)$ is the value of φ before the update in time t_k $\varphi(t_{k+1}^+)$ is the one after the update $k+1$. The time-derivative of φ reads:

$$\dot{\varphi}(t) \;=\; \frac{\theta\frac{d}{dt}\|e_k(t)\|^2}{2\gamma\theta E(t) + m(t)} - \varphi(t)\frac{2\gamma\theta\dot{E} + \dot{m}(t)}{2\gamma\theta E(t) + m(t)} \tag{20}$$

On the one hand, (9) and (6) imply that $i\partial_t e_k(t) = i\partial_t z(t) = -\Delta z(t) - i\alpha(x)z(t) + i\alpha(x)e_k(t)$ so that we have by Cauchy-Schwarz's inequality:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|e_k(t)\|^2 &= \mathcal{I}m\int_\Omega i\bar{e}_k(t)\partial_t e_k(t)\\
&= \mathcal{I}m\int_\Omega \bar{e}_k(t)\left(-\Delta z(t) - i\alpha(x)z(t) + i\alpha(x)e_k\right)\\
&\leq \|\Delta z(t)\|\|e_k(t)\| + \alpha_1\|e_k(t)\|\|z(t)\| + \alpha_1\|e_k(t)\|^2
\end{aligned} \tag{21}$$

From Theorem 2.1, for all $z_0 \in H^2 \cap H_0^1(\Omega)$, the closed-loop system (14) under the event-triggering mechanism (11) has a unique solution $z \in C^0([0, T]; H^2 \cap H_0^1(\Omega))$ so that there exists a constant $C_0 = C_0(T, \|z_0\|_{H^2(\Omega)\cap H^1(\Omega)}) > 0$ such that $\forall t \in [0, T]$,

$$\|\Delta z(t)\| \leq \|\Delta z\|_{L^\infty(0,T;L^2(\Omega))} \leq C_0. \tag{22}$$

By Young's inequality and (2) it follows:

$$\frac{1}{2}\frac{d}{dt}\|e_k(t)\|^2 \leq C_0\|e_k(t)\| + \frac{3\alpha_1}{2}\|e_k(t)\|^2 + \frac{\alpha_1}{2}\|z(t)\|^2.$$

On the other hand, dealing with the numerator of the second term of (20), we have from (18) and (10),

$$\begin{aligned}
|-2\gamma\theta\dot{E} - \dot{m}| \leq 2\theta\gamma&\left(\frac{3\alpha_1}{2}\|z(t)\|^2 + \frac{\alpha_1}{2}\|e_k(t)\|^2\right)\\
&+ \eta m(t) + 2\gamma E(t) + \|e_k(t)\|^2.
\end{aligned} \tag{23}$$

Re-organizing terms in (20), we get

$$\begin{aligned}
(2\gamma\theta E(t) + m(t))\dot{\varphi}(t) &\leq 2C_0\theta\|e_k(t)\| + 3\alpha_1\theta\|e_k(t)\|^2\\
&+ 2\alpha_1\theta E(t) + \varphi(t)\left(6\theta\gamma\alpha_1 + 2\gamma\right)E(t)\\
&+ \varphi(t)(\theta\alpha_1\gamma + 1)\|e_k(t)\|^2 + \varphi(t)\eta m(t).
\end{aligned} \tag{24}$$

In (24), several terms have to be handled. First, from (12) we have $\theta\|e_k(t)\| \leq \sqrt{\theta}\sqrt{(2\theta\gamma E(t) + m(t))}$ so that using Lemma 3.3 we can write, for all $t \in [0, T]$,

$$\begin{aligned}
\frac{\theta\|e_k(t)\|}{2\gamma\theta E(t) + m(t)} &\leq \frac{\sqrt{\theta}}{\sqrt{2\gamma\theta E(t) + m(t)}}\\
&\leq \frac{\sqrt{\theta}}{\sqrt{2\gamma\theta E(0)e^{-2Kt}}} \leq \frac{e^{KT}}{\sqrt{2\gamma E(0)}}.
\end{aligned} \tag{25}$$

Moreover, one should notice that $\frac{aE(t)}{2\gamma\theta E(t)+m(t)} \leq \frac{a}{2\gamma\theta}$ and $\frac{bm(t)}{2\gamma\theta E(t)+m(t)} \leq b$, for any nonnegative scalars a and b. Therefore, back to (24), deviding by $(2\gamma\theta E(t) + m(t))$, recalling $\varphi(t) = \frac{\theta\|e_k(t)\|^2}{2\gamma\theta E(t)+m(t)}$ and using (25), we obtain

$$\begin{aligned}
\dot{\varphi}(t) \leq &\frac{2C_0e^{KT}}{\sqrt{2\gamma E(0)}} + 3\alpha_1\varphi(t) + \frac{\alpha_1}{\gamma}\\
&+ \left(3\alpha_1 + \frac{1}{\theta}\right)\varphi(t) + \frac{\theta\alpha_1\gamma + 1}{\theta}\varphi^2(t) + \eta\varphi(t).
\end{aligned}$$

Finally, denoting by

$$a_0 = \frac{2C_0e^{KT}}{\sqrt{2\gamma E(0)}} + \frac{\alpha_1}{\gamma},\; a_1 = 6\alpha_1 + \eta + \frac{1}{\theta},\; a_2 = \alpha_1\gamma + \frac{1}{\theta}$$

we obtain $\dot{\varphi}(t) \leq a_0 + a_1\varphi(t) + a_2\varphi^2(t)$. Integrating on $[t_k, t_{k+1})$, recalling $\varphi(t_k) = 0$, $\varphi(t_{k+1}) = 1$, one gets

$$1 \leq \frac{1}{A}(t_{k+1} - t_k) \tag{26}$$

where $A = \int_0^1 \frac{ds}{a_0 + a_1 s + a_2 s^2} > 0$ since $a_0, a_1, a_2 > 0$.

Since we assumed that $T < +\infty$, passing to the limit $t_k \to T$ as $k \to +\infty$ in (26) leads to a contradiction. We therefore obtained $T = +\infty$, ensuring the absence of any accumulation points and the avoidance of Zeno behavior. □

Remark 3.4. Differently from the usual literature dealing with event-triggered control for finite-time dimension systems, the proof of Theorem 3.1 is not based on the existence of a dwell-time. Taking another route, the main idea consists in proving that there exisit no accumulation point for the sequence $(t_k)_{k\geq 0}$.

4. EXPONENTIAL STABILITY ANALYSIS

This section addresses the problem of the exponential stability of system (6)-(11).

In order to prove the stability of the closed loop, we consider the Lyapunov candidate function:

$$W(t) = E(t) + m(t), \tag{27}$$

with the energy E defined in (5) and the internal state m defined in (10).

Next, we can first take inspiration from Machtyngier (1994); Machtyngier and Zuazua (1994) in order to upper-bound the functional W, defined in (27). This is reported in the following intermediate Lemma.

Lemma 4.1. Consider the solution z to system (14). For any $\tau > 0$ there exist some constant K_1 and $K_2 > 0$ such that $W(t) = E(t) + m(t)$, satisfies:

$$W(\tau) \leq K_1\int_0^\tau\int_\Omega \alpha(x)|z(t)|^2 dxdt + K_2\int_0^\tau W(t)dt. \tag{28}$$

Proof. Let $\tau > 0$ and let us recall that from the time-derivative of $E(t)$ in (18) we have, since $m(t) \geq 0$,

$$\dot{W}(t) = -\int_\Omega \alpha(x)|z(t)|^2 + Re\left(\int_\Omega \alpha(x)\bar{e}_k(t)z(t)\right) - \eta m(t) + 2\gamma E(t) - \|e_k(t)\|^2 \tag{29}$$
$$\leq Re\left(\int_\Omega \alpha(x)\bar{e}_k(t)z(t)\right) + 2\gamma E(t).$$

From (2), (12), Cauchy-Schwarz and Young's inequalities we get

$$\dot{W}(t) \leq \alpha_1 \left(\frac{\|e_k(t)\|^2}{2\sqrt{\gamma}} + \frac{\sqrt{\gamma}\|z(t)\|^2}{2}\right) + 2\gamma E(t)$$
$$\leq \alpha_1 \frac{1}{2\sqrt{\gamma}}\left(2\gamma E(t) + \frac{1}{\theta}m(t)\right) + \alpha_1\sqrt{\gamma}E(t) + 2\gamma E(t),$$

yielding, $\dot{W}(t) \leq C_1 W(t)$, with

$$C_1 = \max\{2\alpha_1\sqrt{\gamma} + 2\gamma \;;\; \alpha_1/(2\theta\sqrt{\gamma})\} \tag{30}$$

Integrating on $[0,\tau]$, and knowing that $W(0) = E(0)$ since $m(0) = 0$ we get

$$W(\tau) \leq E(0) + C_1 \int_0^\tau W(t)dt. \tag{31}$$

Let us now consider the solution z to (14) as the sum of two variables $y = y(x,t)$ and $\phi = \phi(x,t)$ satisfying

$$\begin{cases} i\partial_t y + \Delta y = -i\alpha z + i\alpha e_k & \text{in } \Omega \times [t_k, t_{k+1}), \\ y = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ y(\cdot, 0) = 0 & \text{in } \Omega, \end{cases} \tag{32}$$

and

$$\begin{cases} i\partial_t \phi + \Delta \phi = 0 & \text{in } \Omega \times \mathbb{R}^+, \\ \phi = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ \phi(\cdot, 0) = z_0 & \text{in } \Omega. \end{cases} \tag{33}$$

As proved, for example, in Machtyngier (1994); Phung (2001), the solution to system (33) satisfies the following property:

Lemma 4.2. (Observability inequality). Let $\omega \subset \Omega$ and $\tau > 0$ be given. There exists $C_{obs} > 0$ such that the solution to (33) satisfies

$$\|\phi(0)\|^2 \leq C_{obs} \int_0^\tau \int_\omega |\phi(x,t)|^2 dx dt.$$

Taking advantage of this important result, from (31), under assumption (2) and the fact that $\phi = z - y$, recalling that for any $a, b \in \mathbb{R}$, $|a-b|^2 \leq 2(a^2+b^2)$, we can write:

$$W(\tau) \leq \frac{1}{2}\|\phi(0)\|^2 + C_1 \int_0^\tau W(t)dt$$
$$\leq \frac{C_{obs}}{2\alpha_0} \int_0^\tau \int_\omega \alpha(x)|\phi(x,t)|^2 dx dt + C_1 \int_0^\tau W(t)dt$$
$$\leq \frac{C_{obs}}{\alpha_0} \int_0^\tau \int_\Omega \alpha(x)|z(t)|^2 dt$$
$$+ \frac{C_{obs}\alpha_1}{\alpha_0}\|y\|^2_{L^\infty(0,\tau;L^2(\omega))} + C_1 \int_0^\tau W(t)dt. \tag{34}$$

Using classical energy estimate (see Cazenave et al. (1998)), on the Schrödinger equation (32), for a $L^2((0,\tau)\times\Omega)$-source term $-i\alpha z + i\alpha e_k$, there exists $C > 0$ such that

$$\|y\|^2_{L^\infty(0,\tau;L^2(\omega)} \leq C\|\alpha(e_k - z)\|^2_{L^2(0,\tau;L^2(\Omega))}$$
$$\leq C\alpha_1^2 \int_0^\tau \|e_k(t)\|^2 dt + C\alpha_1 \int_0^\tau \int_\Omega \alpha(x)|z(t)|^2 dt.$$

From (12), we have $\forall t \in [t_k, t_{k+1}), \|e_k(t)\|^2 \leq C_2 W(t)$ with $C_2 = \max\{2\gamma; \frac{1}{\theta}\}$, thus

$$\|y\|^2_{L^2(0,\tau;L^2(\omega))} \leq C\alpha_1^2 C_2 \int_0^\tau W(t)dt$$
$$+ C\alpha_1 \int_0^\tau \int_\Omega \alpha(x)|z(t)|^2 dt$$

Hence, $W(\tau) \leq \left(\frac{C_{obs}}{\alpha_0} + \frac{C_{obs}C\alpha_1^2}{\alpha_0}\right) \int_0^\tau \int_\Omega \alpha(x)|z(t)|^2 dt$
$$+ \left(C_1 + \frac{C_{obs}C\alpha_1^3 C_2}{\alpha_0}\right) \int_0^\tau W(t)dt.$$

Therefore we get inequality (28) with

$$K_1 = \frac{C_{obs}}{\alpha_0}\left(1 + C\alpha_1^2\right); K_2 = C_1 + \frac{C_{obs}C\alpha_1^3 C_2}{\alpha_0}. \tag{35}$$

□

Then, the main exponential stability result is proven by using the Lyapunov functional candidate W defined in (27) and by studying its time-derivative along the closed-loop system. The result is reported in the theorem below.

Theorem 4.1. There exists $\gamma > 0$ such that for any initial condition $z_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, the closed-loop system (6) under the event-triggering mechanism (11) is exponentially stable with decay rate $\delta > 0$. In other words, there exist $K > 0$ and $\delta > 0$ such that

$$E(t) \leq K E(0) e^{-2\delta t}, \quad \forall t > 0. \tag{36}$$

Proof. We use the following Lyapunov functional candidate $W(t) = E(t) + m(t)$, defined in (27). In the proof we consider two cases depending on the damping.

• *Globally non-vanishing damping.* Let us discuss the case where the damping does not vanish in Ω (corresponding to $\omega = \Omega$). Performing the Cauchy-Schwarz and Young's inequalities and using (2) we get from (12) and (29)

$$\dot{W}(t) \leq -\alpha_0\|z(t)\|^2 + \alpha_1\|z(t)\|\|e_k(t)\|$$
$$-\eta m(t) + 2\gamma E(t) - \|e_k(t)\|^2$$
$$\leq (2\gamma - 2\alpha_0)E(t) + \frac{\alpha_1}{2\varepsilon}\|z(t)\|^2 + \left(\frac{\alpha_1\varepsilon}{2} - 1\right)\|e_k(t)\|^2$$
$$-\eta m(t)$$
$$\leq \left(-2\alpha_0 + \alpha_1\varepsilon\gamma + \frac{\alpha_1}{\varepsilon}\right)E(t) + \left(-\eta + \frac{\alpha_1\varepsilon}{2\theta} - \frac{1}{\theta}\right)m(t).$$

Setting $\delta_1 = \frac{1}{2}\min\left\{2\alpha_0 - \alpha_1\varepsilon\gamma - \frac{\alpha_1}{\varepsilon} \;;\; \eta - \frac{\alpha_1\varepsilon}{2\theta} + \frac{1}{\theta}\right\}$ we obtain

$$\dot{W}(t) \leq -2\delta_1 W(t). \tag{37}$$

Choosing $\varepsilon = 1/\sqrt{\gamma}$, and in order to have $\delta_1 > 0$, easy calculations prove that we can pick the tuning parameters γ, η and θ such that

$$0 < \gamma < \frac{\alpha_0^2}{\alpha_1^2} \quad \text{and} \quad \eta\theta > \frac{\alpha_1^2}{2\alpha_0} - 1. \tag{38}$$

Remarking that (12) gives $E(t) \leq W(t)$ and performing the usual integration calculations, we obtain that for all $t \geq 0$, $E(t) \leq e^{-2\delta_1 t}W(0)$. Finally, since $m(0) = 0$, we get $E(t) \leq e^{-2\delta_1 t}E(0)$ proving that (36) holds with $K = 1$ and $\delta = \delta_1$ in the case of non-vanishing damping in $\omega = \Omega$.

• *Locally non-vanishing damping.* In the general case, one has $\omega \subsetneq \Omega$ and the damping may vanish outside ω. We will thus need to use Lemma 4.1. Let $\tau > 0$. Integrating (29) on $[0, \tau]$, we can write :

$$W(\tau) - W(0) = -\int_0^\tau \int_\Omega \alpha(x)|z(t)|^2 dx + \mathcal{R}e\left(\int_0^\tau \int_\Omega \alpha(x)\bar{e}_k(t)z(t)\right) - \eta\int_0^\tau m(t) + 2\gamma\int_0^\tau E(t) - \int_0^\tau \|e_k(t)\|^2 \quad (39)$$

We can rewrite (28) of Lemma 4.1 as follows

$$-\int_0^\tau \int_\Omega \alpha(x)|z(t)|^2 dxdt \le -\frac{1}{K_1}W(\tau) + \frac{K_2}{K_1}\int_0^\tau W(t)dt.$$

Combining this inequality with (39), and using the usual tricks, we get

$$\left(1+\frac{1}{K_1}\right)W(\tau) \le W(0) + \frac{K_2}{K_1}\int_0^\tau W(t)dt - \eta\int_0^\tau m(t) + \left(\frac{\alpha_1\varepsilon}{2} - 1\right)\int_0^\tau \|e_k(t)\|^2 + \left(2\gamma + \frac{\alpha_1}{\varepsilon}\right)\int_0^\tau E(t)$$

so that using (12),

$$\left(1+\frac{1}{K_1}\right)W(\tau) \le W(0) + \frac{K_2}{K_1}\int_0^\tau W(t) + \left(\alpha_1\gamma\varepsilon + \frac{\alpha_1}{\varepsilon}\right)\int_0^\tau E(t) + \left(-\eta + \frac{\alpha_1\varepsilon}{2\theta} - \frac{1}{\theta}\right)\int_0^\tau m(t).$$

Since $W(0) = E(0)$ and assuming

$$\eta\theta > \frac{\alpha_1}{2\sqrt{\gamma}} - 1$$

we can write :

$$\left(1+\frac{1}{K_1}\right)W(\tau) \le E(0) + \left(\frac{K_2}{K_1} + K_3\right)\int_0^\tau W(t) \quad (40)$$

where we have considered (35), $\varepsilon = 1/\sqrt{\gamma}$ and $K_3 = 2\alpha_1\sqrt{\gamma}$.

It brings by Gronwall's Lemma,

$$W(\tau) \le \frac{K_1}{K_1+1}\exp\left[\frac{K_1}{K_1+1}\left(K_3 + \frac{K_2}{K_1}\right)\tau\right]E(0),$$

that can be written as

$$W(\tau) \le pe^{\mathbf{c}\tau}E(0)$$

with $p = \frac{K_1}{K_1+1}, \mathbf{c} = \frac{K_1}{K_1+1}\left(K_3 + \frac{K_2}{K_1}\right) = \frac{K_1K_3+K_2}{K_1+1}$.

Next, we apply the invariance by translation in time of the linear Schrödinger equation on the interval $[(n-1)\tau, n\tau]$, for $n = 1, 2, \ldots$, to get (denoting $a = pe^{\mathbf{c}\tau}$):

$$W(n\tau) \le aW((n-1)\tau) \le \cdots \le a^n E(0) = e^{-n\tau\kappa}E(0),$$

where we set $a^n = \exp\left(-n\tau\frac{1}{\tau}\ln\left(\frac{1}{a}\right)\right)$ and $\kappa = \frac{1}{\tau}\ln\left(\frac{1}{a}\right)$. Note that $\kappa > 0$ if and only if $a < 1$, so that we must have $pe^{\tau\mathbf{c}} < 1$ which is equivalent to

$$\tau < -\frac{\ln p}{\mathbf{c}} = \frac{(K_1+1)\ln\left(\frac{K_1+1}{K_1}\right)}{(K_1K_3+K_2)}. \quad (41)$$

Now, for every positive time t, there exists $n \in \mathbb{N}^*$ such that $(n-1)\tau < t \le n\tau$. Using (30) and integration on $[(n-1)\tau, t]$ we have:

$$W(t) \le W((n-1)\tau) + C_1\int_{(n-1)\tau}^t W(s)ds \le e^{-n\tau\kappa}e^{\tau\kappa}E(0) + C_1\int_0^t W(s)ds. \quad (42)$$

Since $e^{-n\tau\kappa} \le e^{-\kappa t}$ for $t \le n\tau$, and $e^{\tau\kappa} = 1/a$, we get

$$W(t) \le \frac{1}{a}e^{-\kappa t}E(0) + C_1\int_0^t W(s)ds.$$

Then by Gronwall's Lemma, it follows, for $2\delta = \kappa - C_1$, $E(t) \le W(t) \le \frac{1}{a}e^{-2\delta t}E(0)$, and some calculations prove that we can insure $\delta > 0$ if

$$\frac{1}{\tau}\ln\left(\frac{K_1+1}{K_1}\right) - \frac{K_1K_3+K_2}{K_1+1} > C_1 \quad (43)$$

where K_1 and K_2 given by (35) and K_3 in (40).

The proof of Theorem 4.1 is complete as soon as we can ensure that (43) can be obtained for a good choice of the tuning parameters γ, η and θ of the event-triggering law. Notice first that (41) gives $\frac{1}{\tau}\ln\left(\frac{K_1+1}{K_1}\right) > \frac{(K_1K_3+K_2)}{K_1+1}$ so that (43) becomes true if C_1 can be chosen small enough. Then let us take $\theta > 0$ large enough to have (30) yielding $C_1 = 2\alpha_1\sqrt{\gamma} + 2\gamma$, positive constant that can be as small as needed when choosing $\gamma > 0$ small enough. □

Remark 4.1. Theorem 4.1 is valid not only in the case $\omega = \Omega$ (which corresponds to the fact that the damping does not vanish in Ω), but also in the case $\omega \subsetneq \Omega$ (which corresponds to the fact that the damping may vanish outside ω). Of course the values of K and δ will be different in the two cases.

5. NUMERICAL EXAMPLE

Consider as in Koudohode et al. (2022), the one-dimensional Schrödinger equation (6) under the event-triggering mechanism (11) on $\Omega = (0, \pi)$ with initial condition

$$z_0(x) = \sin(x),\ x \in [0, \pi].$$

We use the divided differences on a uniform grid for the space variable and the discretization with respect to time through Crank Nicolson scheme is performed.

With respect to (2), we select the damping coefficient as follows:

$$\alpha(x) = \begin{cases} 0 & \text{if } x < \pi/6 \text{ or } x > 2\pi/3 \\ 3.14 & \text{if } \pi/6 \le x \le 2\pi/3 \end{cases}$$

Hence, we can take

$$\alpha_0 = \pi/6, \alpha_1 = \pi \text{ and } \omega = (\pi/6, 2\pi/3).$$

We use (Phung, 2001, Theorem 2.2) to select the constants C_{obs} and C as:

$$C_{obs} = 2.8 \text{ and } C = 0.18.$$

From equation (35) one gets

$$K_1 = 9.8343 \text{ and } K_2 = 8.1460.$$

With $\gamma = 0.1$ and $C_1 = 2.1859$, using the proof of Theorem 4.1 two inequalities depending on the damping can be satisfied as reported below:

- In the case of a globally non-vanishing damping (corresponding to $\omega = \Omega$ and the damping does not vanish in Ω), the inequality (38) is verified:

$$0 < \gamma < \frac{\alpha_0^2}{\alpha_1^2} \quad \text{and} \quad \eta\theta > \frac{\alpha_1^2}{2\alpha_0} - 1.$$

- In the case of a locally non-vanishing damping (corresponding to $\omega \subsetneq \Omega$ and the damping may vanish outside ω), the inequality (43) is verified:

$$\frac{1}{\tau}\ln\left(\frac{K_1+1}{K_1}\right)-\frac{K_1K_3+K_2}{K_1+1}>C_1$$

for $\eta = 0.7$, $\theta = 15$, $\tau < 0.020429$, and $K_3 = 1.9859$.

In Figure 1 we compare the imaginary part Imz of the numerical solution z to the continuous-in-time closed-loop systems (1) (top) and the dynamic event-triggered one (6)-(11) (bottom). It also illustrates the guarantee of the exponential stability of the solution as studied in Theorem 4.1.

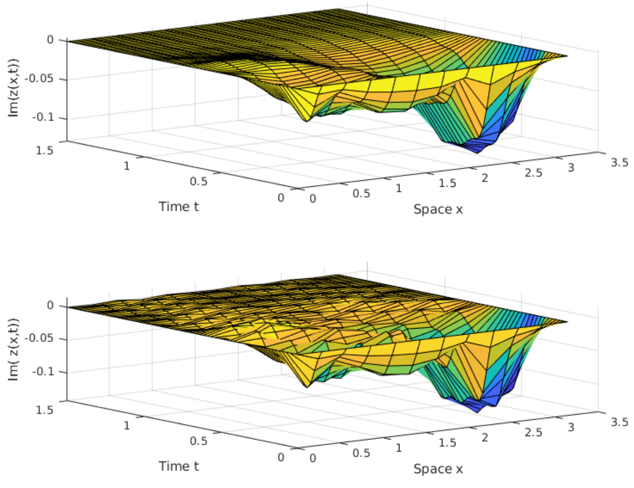

Fig. 1. Imaginary part of the solution to the closed-loop system (6) under the event-triggering mechanism (11) (bottom), and to the continuous-in-time closed-loop system (1) (top).

The guarantee of the exponential stability of the solution is confirmed even more clearly with Figure 2 where we depicted the time-evolution of the energy of the solution to systems (6) under the static (8) and dynamic (11) event-triggering mechanism (ETM).

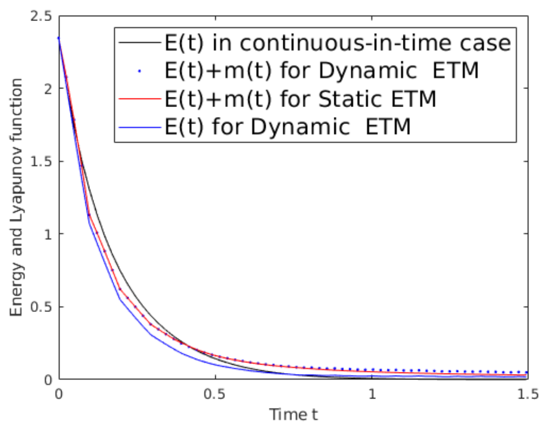

Fig. 2. Time-evolution of the energy $E(t)$ and the Lyapunov function $W(t)$.

Finally, Figure 3 depicts the sampling time t_k and the time-evolution of the dynamic signal $m(t)$.

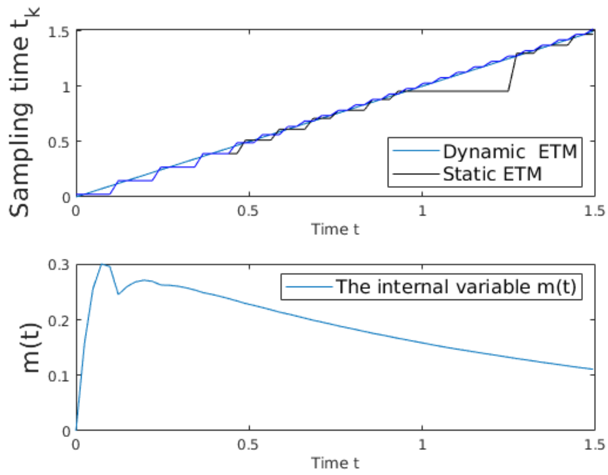

Fig. 3. Time evolution of the internal variable $m(t)$ in the dynamic event-triggering mechanism (bottom), and of the sampling time t_k (top).

6. CONCLUSION AND PERSPECTIVE

A dynamic event-triggering mechanism is proposed to determine when the stabilizing control of the linear Schrödinger equation needs to be updated in digital implementations, while reducing the using of computational resources. The event-triggering condition is such that the exponential stability and well-posedness are maintained while the occurence of Zeno behavior is avoided.

In our future work, we may look into event-triggered boundary control of Schrödinger equation by considered observer-based control law. We may also study the presence of input nonlinearity, as saturation, for example.

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