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WHY DOES PROPOSITIONAL QUANTIFICATION MAKE MODAL AND TEMPORAL LOGICS ON TREES ROBUSTLY HARD?

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ABSTRACT. Adding propositional quantification to the modal logics **K**, **T** or **S4** is known to lead to undecidability but **CTL** with propositional quantification under the tree semantics (\mathbf{QCTL}^t) admits a non-elementary TOWER-complete satisfiability problem. We investigate the complexity of strict fragments of \mathbf{QCTL}^t as well as of the modal logic **K** with propositional quantification under the tree semantics. More specifically, we show that \mathbf{QCTL}^t restricted to the temporal operator **EX** is already TOWER-hard, which is unexpected as **EX** can only enforce local properties. When \mathbf{QCTL}^t restricted to **EX** is interpreted on N -bounded trees for some $N \geq 2$, we prove that the satisfiability problem is $\mathbf{AEXP}_{\text{POL}}$ -complete; $\mathbf{AEXP}_{\text{POL}}$ -hardness is established by reduction from a recently introduced tiling problem, instrumental for studying the model-checking problem for interval temporal logics. As consequences of our proof method, we prove TOWER-hardness of \mathbf{QCTL}^t restricted to **EF** or to **EXEF** and of the well-known modal logics such as **K**, **KD**, **GL**, **K4** and **S4** with propositional quantification under a semantics based on classes of trees.

1. INTRODUCTION

Propositional quantification in modal and temporal logics. A natural way to design logics that dynamically update their models, consists in adding propositional quantification as done for example to define QBF from SAT. Propositional quantification is a very powerful feature to update models but this may have consequences in terms of computability. In the realm of modal logics [12], the paper [16] remains a quite early work adding propositional quantification. The undecidability of the propositional modal logic **K** (resp. **T**, **K4** and **S4**) augmented with propositional quantification is first established in [25], and this is done thanks to a reduction from the second-order arithmetic. By contrast, the decidability of second-order versions of the modal logic **S5** is first proved in [24, Chapter 3] (see also [25, 31]) but two **S5** modalities and propositional quantification already lead to undecidability [1, 30].

Many subsequent works have dealt with second-order modal logics, see *e.g.* [30, 34, 27, 57], but in the realm of temporal logics, **LTL** with propositional quantification (written **QLTL**) is

Key words and phrases: propositional quantifier, modal logic over trees, satisfiability problem, tower-hardness, computational tree logic CTL.

introduced in Sistla’s PhD thesis [54] (see also [53]) and non-elementarity of the satisfiability problem is a consequence of [43]. So, the QLTL satisfiability problem is decidable but with high complexity. By contrast, CTL with propositional quantification (written QCTL) already admits an undecidable satisfiability problem by [25] (as CTL captures the modal logic K) but its variant under the tree semantics (written QCTL^t) admits a non-elementary TOWER-complete satisfiability problem [37, 19] (the complexity class TOWER is introduced in [50]). Having a tree semantics means that the formulae of QCTL^t are interpreted on computation trees obtained from the unfolding of finite (total) Kripke structures, which allows us to regain decidability (see a similar approach in [63] with a quantified version of the modal logic S4). This is a major observation from [37], partly motivated by the design of decision procedures for ATL with strategy contexts [38]. The tree semantics, as far as satisfiability is concerned, amounts to considering Kripke structures that are finite-branching trees in which all the maximal branches are infinite. This is an elegant way to regain decidability. More generally, decidability in the presence of propositional quantification can be regained when tree-like models are involved, see *e.g.* [4, 6, 63, 37], essentially by taking advantage of Rabin’s Theorem [47].

The modal logic K with propositional quantification from [25] is interpreted under the structure semantics, as classified in [37], but many variants of propositional quantification exist in the literature (see *e.g.* [45, 28, 14] and [9, 10] in the context of epistemic reasoning). Sometimes, propositional quantification is syntactically restricted in the temporal language, but its inclusion is motivated by a gain of expressive power while preserving the decidability of the reasoning tasks. By way of example, in [48], an extension of the modal μ -calculus with partial propositional quantification is introduced to perform control synthesis, whereas an extension for model-checking computer systems is also presented in [35].

Interestingly enough, propositional quantification can sometimes have a more hidden presence. For instance, hybrid logics with the down-arrow operator \downarrow_x , see *e.g.* [2, 3, 41, 62, 32], can be understood as a form of propositional quantification since $\downarrow_x \phi$ enforces that the propositional variable x holds true only at the current world (before evaluating the formula ϕ). In such logics, the companion formula $@_x \psi$ expresses that the unique world satisfying x also satisfies ψ ; $@_x$ is a powerful operator to navigate in the structure [41] (obviously, it is related to the universal modality, see *e.g.* [29]).

Our motivations. As recalled above, the modal logic K augmented with propositional quantification is undecidable [25] and a fortiori, undecidability holds for fragments of CTL with propositional quantification. Actually, these results hold under the structure semantics but QCTL^t (tree semantics) admits TOWER-complete satisfiability and model-checking problems [37]. As QCTL^t can express that every tree node has exactly one child, TOWER-hardness for the satisfiability problem for QCTL^t , is a corollary of the TOWER-hardness of the satisfiability problem for QLTL, see *e.g.* [43, 53, 20].

Given the central position of the modal logic K, surprisingly, the complexity of the satisfiability problem for K with propositional quantification under the tree semantics has never been investigated (closely related to QCTL_X^t – *i.e.* QCTL^t restricted to **EX** – as **EX** corresponds to \Diamond in K but with total models). QCTL_X^t is a natural and modest fragment of QCTL^t and we aim at characterising its complexity. Furthermore, the model-checking problem for QCTL^t is TOWER-hard even with input Kripke structures having at most two successor worlds per world and TOWER-hardness of QLTL holds with linear structures of length ω , see *e.g.* [53]. Thus, it is worth understanding what happens with the satisfiability

problem for $\text{QCTL}_{\mathbf{X}}^t$ when the tree models are N -bounded (*i.e.* each node has at most N children) for some fixed $N \geq 2$.

Our contributions. Given QCTL^t , the extension of CTL with propositional quantification under the tree semantics (*i.e.* the models are finite-branching trees where all the maximal branches are infinite), let $\text{QCTL}_{\leq N}^t$ be its variant in which the models are N -bounded, for some $N \geq 2$. We write $\text{QCTL}_{\mathbf{X}}^t$ and $\text{QCTL}_{\mathbf{X}, \leq N}^t$, to denote respectively the restriction to the operator **EX**, and $\text{QCTL}_{\mathbf{F}}^t$ to denote the restriction of QCTL^t to **EF**.

- We first present to the reader the toolkit of local nominals and explain the basic ideas behind the hardness results of this paper by proving, for all $N \geq 2$, that the satisfiability problem for $\text{QCTL}_{\mathbf{X}, \leq N}^t$ is AEXP_{POL} -complete (Section 3). AEXP_{POL} is the class of problems solvable by exponential-time alternating Turing machines with a polynomial number of alternations. By using a small model property and the complexity of model-checking (with upper bound AEXP_{POL}), AEXP_{POL} -easiness is established. As far as AEXP_{POL} -hardness is concerned, the alternating multi-tiling problem introduced in [13] is instrumental to establish that the model-checking problem for the interval temporal logic $B\bar{E}$ with regular expressions is AEXP_{POL} -hard. As a corollary, we get that the modal logic **K** with propositional quantification interpreted on finite trees of branching degree bounded by some fixed $N \geq 2$ is also AEXP_{POL} -complete.
- More generally and despite the modest scope of **EX**, the satisfiability problem for $\text{QCTL}_{\mathbf{X}}^t$ is shown to be TOWER-hard (Theorem 4.8), by a uniform reduction from k -NEXPTIME-complete tiling problems (uniformity is with respect to k). The corresponding upper bound is known from [37] and it is worth noting that all the TOWER upper bound results presented in this paper are based on translations into the satisfiability problem for QCTL^t , sometimes via intermediate decision problems, which eventually uses Rabin's Theorem [47] in some essential way. The hardness proof is one of the main results of the paper and amounts first to showing that one can enforce that a node has a number of children equal to some tower of exponentials of height k with a formula of size exponential in k . By contrast, checking the satisfiability status of CTL^* formulae, requires only to consider tree models with branching degree bounded by the size of the formula, see *e.g.* [21, 20]. Once this complex construction enforcing a very high number of children is performed, the reduction from the tiling problems can be done with the help of other properties on the number of children. Hence, even though $\text{QCTL}_{\mathbf{X}}$ under the structure semantics is undecidable [25] and the variant of $\text{QCTL}_{\mathbf{X}}^t$ under the tree semantics (QCTL^t) is decidable by [37], the problem admits a high complexity despite the local range of **EX**. The TOWER lower bound for QCTL^t crucially depends on the availability of very wide trees, which contrasts with the AEXP_{POL} upper bound for $\text{QCTL}_{\mathbf{X}, \leq N}^t$ for which trees are N -bounded.
- By adapting our proof method, we show that $\text{QCTL}_{\mathbf{F}}^t$ and $\text{QCTL}_{\mathbf{XF}}^t$ (a variant of $\text{QCTL}_{\mathbf{F}}^t$ with the unique operator **EXEF**) are TOWER-hard too (consult Section 5.1).
- As **EX**, **EF** and **EXEF** correspond to the modality \Diamond in several modal logics (*e.g.* **EXEF** corresponds to \Diamond in transitive frames), we are able to establish TOWER-completeness for standard modal logics with propositional quantification when interpreted on tree-like structures (see Section 5.2). For instance, as the provability logic **GL** (after Gödel and Löb) is complete for the class of finite transitive trees, *i.e.* the accessibility relation is the transitive closure of the child-relation in the tree, see *e.g.* [12], we also investigate the

satisfiability problem under the finite tree semantics. We show that $\text{QCTL}_{\mathbf{X}}^{ft}$ and $\text{QCTL}_{\mathbf{XF}}^{ft}$ ('*ft*' stands for 'finite tree semantics') are TOWER-complete too. The satisfiability problem for \mathbf{K} (resp. \mathbf{GL}) with propositional quantification under the finite (resp. transitive) tree semantics is shown to be TOWER-complete. Similar results are shown for \mathbf{KD} , $\mathbf{K4}$ and $\mathbf{S4}$ with propositional quantification but interpreted on appropriate tree-like Kripke structures. See Section 5.2 for the details.

This work is a revised and complete version of our conference paper [7]. To keep the main body of the paper short we delegate all the proofs to the technical appendix.

2. PRELIMINARIES

2.1. Kripke structures and computation trees. Below, we recall standard definitions about Kripke structures. Let $AP = \{p, q, x, y, \dots\}$ be a countably infinite set of propositional variables. A *Kripke structure* \mathcal{K} is a triple $\langle W, R, l \rangle$, where W is a set of *worlds*, $R \subseteq W \times W$ is a transition relation and $l : W \rightarrow 2^{AP}$ is a labelling function. A Kripke structure \mathcal{K} is *total* whenever for all $w \in W$, there is $w' \in W$ such that $(w, w') \in R$. Totality is a standard property for defining classes of models for temporal logics such as CTL. In the sequel, by a 'Kripke structure' we mean a structure according to the above definition, otherwise when arbitrary (or total, finite, etc.) Kripke structures need to be considered, we explicitly specify which classes of structures we have in mind. For instance, a tree model (resp. finite tree model) $\mathfrak{T} = \langle V, E, l \rangle$ is an arbitrary Kripke structure where (V, E) is a (resp. finite) rooted tree. Standard definitions about trees are omitted herein. Such structures play an important role in the paper, as the tree semantics involves specific tree models (those obtained as computation trees of Kripke structures).

Given a Kripke structure $\mathcal{K} = \langle W, R, l \rangle$ and a world $w \in W$, a *finite path* π from w is a finite sequence w_0, \dots, w_n such that $w_0 = w$ and for all $i \in [0, n-1]$, we have $(w_i, w_{i+1}) \in R$. An *infinite path* from w is an infinite sequence w_0, \dots, w_n, \dots such that $w_0 = w$ and for all $i \geq 0$, we have $(w_i, w_{i+1}) \in R$. With $\Pi_{\mathcal{K}, w}$ we denote the set of all finite paths starting from a world $w \in W$ on a Kripke structure \mathcal{K} .

For a Kripke structure $\mathcal{K} = \langle W, R, l \rangle$ and $w \in W$, the *computation tree unfolding* \mathcal{K} from w is the tree model $\mathfrak{T}_{\mathcal{K}, w} = \langle V, E, l' \rangle$ such that the following conditions are satisfied:

- (a) $V \stackrel{\text{def}}{=} \Pi_{\mathcal{K}, w}$,
- (b) $\pi E \pi' \stackrel{\text{def}}{\iff} \pi = w_0, \dots, w_n, \pi' = \pi, w_{n+1}$ for some $w_{n+1} \in W$ and $(w_n, w_{n+1}) \in R$,
- (c) for all $\pi = w_0, \dots, w_n \in V$, we have $l'(\pi) \stackrel{\text{def}}{=} l(w_n)$.

Thus, when \mathcal{K} is finite and total, $\mathfrak{T}_{\mathcal{K}, w}$ is a finite-branching tree model in which all the maximal branches are infinite and it is also a total Kripke structure. Below, unless otherwise stated, the tree semantics involves such finite-branching trees with infinite maximal branches. In the sequel, tree-like Kripke structures are denoted by $\mathfrak{T} = \langle V, E, l \rangle$ in order to emphasize the tree-like nature of such models.

2.2. The logics QCTL^s , QCTL^t , QCTL^{ft} and QCTL^{gt} . In this section, we define the logics QCTL^s , QCTL^t , QCTL^{ft} and QCTL^{gt} whose formulae are interpreted over different classes of Kripke structures. The formulae of QCTL-like logics are defined from the grammar below by

extending the set of formulae from the computation tree logic CTL [18] with propositional quantification:

$$\phi ::= p \mid \neg\phi \mid \phi \wedge \phi \mid \mathbf{EX}\phi \mid \mathbf{E}(\phi\mathbf{U}\phi) \mid \mathbf{A}(\phi\mathbf{U}\phi) \mid \exists p \phi,$$

where $p \in AP$. We use the standard abbreviations $\vee, \rightarrow, \leftrightarrow, \perp, \top$, as well as other operators like $\mathbf{AX}, \mathbf{EF}, \mathbf{AG}$ and \mathbf{AF} :

- $\mathbf{AX}(\phi) \stackrel{\text{def}}{=} \neg\mathbf{EX}(\neg\phi)$,
- $\mathbf{EF}(\phi) \stackrel{\text{def}}{=} \mathbf{E}(\top\mathbf{U}\phi)$, $\mathbf{AG}(\phi) \stackrel{\text{def}}{=} \neg\mathbf{EF}(\neg\phi)$ and $\mathbf{AF}(\phi) \stackrel{\text{def}}{=} \mathbf{A}(\top\mathbf{U}\phi)$.
- The universal quantifier $\forall p \phi$ is used as $\neg\exists p \neg\phi$.

We denote by $|\phi|$ the *length* of the formula ϕ measured in a standard way, *i.e.* as the number of symbols used to write ϕ . The *modal/temporal depth* of a formula ϕ , written $md(\phi)$, is the maximal number of nested temporal operators in ϕ . We stress that $md(\phi)$ is linear in $|\phi|$.

By restricting the set of allowed temporal operators in QCTL to the only temporal operator \mathbf{EX} (resp. \mathbf{EF}) we obtain the logic $\text{QCTL}_{\mathbf{X}}$ (resp. $\text{QCTL}_{\mathbf{F}}$). Note that \mathbf{AF} does not occur in $\text{QCTL}_{\mathbf{F}}$ but \mathbf{AX} (resp. \mathbf{AG}) is allowed in $\text{QCTL}_{\mathbf{X}}$ (resp. in $\text{QCTL}_{\mathbf{F}}$) as it is the dual operator of \mathbf{EX} (resp. \mathbf{EF}). The main object of study in the paper is the logic $\text{QCTL}_{\mathbf{X}}$ under the tree semantics (below, written $\text{QCTL}_{\mathbf{X}}^t$). Moreover, we write $\text{QCTL}_{\mathbf{XF}}$ to denote the restriction of QCTL to the (combined) temporal operator \mathbf{EXEF} , which provides a strict version of the future-time temporal operator \mathbf{EF} . In the rest of the paper, we refer to QCTL (or to some of its fragments) to denote a set of formulae and the notation QCTL^\bullet with a superscript ‘ \bullet ’ refers to a logic based on QCTL (or on some of its fragments) under a specific semantics and for which the symbol ‘ \bullet ’ is just a reminder of the semantics.

To define formally the semantics for propositional quantifiers, we introduce the notion of X -equivalence between two Kripke structures, with X being a set of propositional variables. Intuitively, two Kripke structures are X -equivalent, whenever the only differences (if any) between them are restricted to the interpretation of the propositional variables not in X . Formally, we say that two Kripke structures $\mathcal{K} = \langle W, R, l \rangle$ and $\mathcal{K}' = \langle W', R', l' \rangle$ are X -equivalent (written $\mathcal{K} \approx_X \mathcal{K}'$), iff $W = W'$, $R = R'$ and $l(w) \cap X = l'(w) \cap X$ for all $w \in W$.

Given a Kripke structure $\mathcal{K} = \langle W, R, l \rangle$, a world $w \in W$ and a formula ϕ in QCTL, the satisfaction relation $\mathcal{K}, w \models \phi$ is defined as follows:

$$\begin{array}{ll} \mathcal{K}, w \models p & \text{iff } p \in l(w) \\ \mathcal{K}, w \models \neg\phi & \text{iff } \mathcal{K}, w \not\models \phi \\ \mathcal{K}, w \models \phi \wedge \psi & \text{iff } \mathcal{K}, w \models \phi \text{ and } \mathcal{K}, w \models \psi \\ \mathcal{K}, w \models \mathbf{EX}\phi & \text{iff there is } w' \text{ such that } (w, w') \in R \text{ and } \mathcal{K}, w' \models \phi \\ \mathcal{K}, w \models \mathbf{E}(\phi\mathbf{U}\psi) & \text{iff there is a path } w_0, \dots, w_n \text{ such that } w_0 = w, \\ & \mathcal{K}, w_n \models \psi \text{ and for all } i \in [0, n-1], \text{ we have } \mathcal{K}, w_i \models \phi \\ \mathcal{K}, w \models \mathbf{A}(\phi\mathbf{U}\psi) & \text{iff for all infinite paths } w_0, \dots, w_n, \dots \text{ such that } w_0 = w, \text{ there is} \\ & j \geq 0 \text{ such that } \mathcal{K}, w_j \models \psi \text{ and for all } i \in [0, j-1], \mathcal{K}, w_i \models \phi \\ \mathcal{K}, w \models \exists p \phi & \text{iff there is } \mathcal{K}' \text{ such that } \mathcal{K} \approx_{AP \setminus \{p\}} \mathcal{K}' \text{ and } \mathcal{K}', w \models \phi \end{array}$$

Note that $\mathcal{K}, w \models \mathbf{EF}\phi$ iff there is $w' \in R^*(w)$ (where R^* is the reflexive transitive closure of R) such that $\mathcal{K}, w' \models \phi$. Stating that there is a unique successor of w satisfying the propositional variable x can be expressed by the formula $\mathbf{EX} x \wedge \neg(\exists p \mathbf{EX}(x \wedge p) \wedge \mathbf{EX}(x \wedge \neg p))$, where p is distinct from x .

The satisfiability problem for the logic QCTL^s (under the *structure semantics*) takes as input a formula ϕ in QCTL and asks whether there is a finite and total Kripke structure \mathcal{K} and a world w such that $\mathcal{K}, w \models \phi$.

The *tree semantics* introduced in [37] is obtained by considering as only admissible models the computation trees of finite and total Kripke structures. As noted in [37, Remark 5.7], an equivalent formulation can be provided: the satisfiability problem for the logic QCTL^t (under the *tree semantics*) takes as input a formula in QCTL and asks whether there is a finite-branching tree model \mathfrak{T} in which all the maximal branches are infinite such that $\mathfrak{T}, \varepsilon \models \phi$ and ε is the root of \mathfrak{T} . This is the definition we adopt along the paper.

We write $\text{SAT}(\mathfrak{L})$ to denote the satisfiability problem for the logic \mathfrak{L} . The distinction between the tree semantics and the structure semantics is crucial and affects the computational properties of the satisfiability problems.

Proposition 2.1. (I) $\text{SAT}(\text{QCTL}^s)$ (under the *structure semantics*) is undecidable [37, Theorem 5.1]. (II) $\text{SAT}(\text{QCTL}^t)$ (under the *tree semantics*) is decidable and TOWER-complete [37, Theorem 5.3]. (III) The satisfiability problem for QCTL^t is TOWER-hard even if restricted to ω -sequences [43, 53].

All our forthcoming TOWER upper bound results are based on translations into the satisfiability problem for QCTL^t , sometimes via intermediate decision problems, which eventually invokes Rabin's Theorem [47] in some essential way. This is not surprising, as considering tree-like models and propositional quantification naturally leads to invoking the decidability of $\text{S}\omega\text{S}$ [47] or its linear version, the second-order logic of one successor S1S [15].

Let us recapitulate the different versions of quantified CTL we have seen so far.

- QCTL^s is interpreted over finite and total Kripke structures and $\text{SAT}(\text{QCTL}^s)$ is undecidable, see *e.g.* [37].
- QCTL^t is interpreted over finite-branching trees in which all the maximal branches are infinite and $\text{SAT}(\text{QCTL}^t)$ is TOWER-complete, see *e.g.* [37].

Let us introduce two additional versions that are closely related to modal logics with propositional quantification on tree-like models introduced in the forthcoming Section 5.

- QCTL^{ft} is interpreted over finite trees, $\text{SAT}(\text{QCTL}^{ft})$ can be shown to be in TOWER by a logspace reduction into $\text{SAT}(\text{QCTL}^t)$ and its restriction to **EX** will be shown to admit a TOWER-hard satisfiability problem (see Section 5).
- QCTL^{gt} is interpreted over finite-branching trees (maximal branches may be finite), $\text{SAT}(\text{QCTL}^{gt})$ can be shown to be in TOWER by logspace reduction into $\text{SAT}(\text{QCTL}^t)$ and its restriction to **EXEF** will be shown to admit a TOWER-hard satisfiability problem (see Section 5).

As a side remark, the equivalence between the two formulations of QCTL^t , is mainly due to the fact that QCTL^t can be translated into Monadic Second-Order Logic (MSO) over tree models with arbitrary finite branching (getting decidability by Rabin's Theorem [47]). Indeed, as MSO over tree models with arbitrary finite branching is decidable by Rabin's Theorem [47], the satisfiability problem for QCTL^t is decidable too, by a standard translation internalising the tree semantics for QCTL^t . As explained in [58, Section 6.3], the existence of a tree model implies also the existence of a *regular tree model*, that can be precisely originated by a finite Kripke structure. Hence, QCTL^t has the finite model property. So, not only can every finite and total Kripke structure be unfolded as a finite-branching tree in which all the maximal branches are infinite (a direct consequence of the definition for computation trees)

but the existence of a tree model with the above-mentioned properties and satisfying ϕ implies the existence of the computation tree of a finite and total Kripke structure satisfying ϕ . Hence, satisfiability in the computation tree of a finite Kripke structure is equivalent to satisfiability in a finite-branching tree in which all the maximal branches are infinite, and in the sequel, we shall operate with the latter definition.

Apart from CTL-like logics (see *e.g.* [37] for a wealth of bibliographical references), other logics with propositional quantification have been shown to be decidable by translation into $\text{S}\omega\text{S}$, see *e.g.* [4, 6, 63]. Besides, in [4] a fragment of GL with propositional quantification is shown to be decidable by translation into the weak monadic second-order logic of one successor WS1S [15], and a version of Gödel logic LC with propositional quantification is shown to be decidable by translation into S1S [15, 6], see also [63] solving partially an open problem from [33, §9].

2.3. Complexity classes and tiling problems. In this section, we introduce tiling problems that are mainly used in Sections 3.3 and 4.4.

Let $\mathfrak{t} : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ be a tetration function defined for integers $n, k \geq 0$, inductively as $\mathfrak{t}(0, n) = n$ and $\mathfrak{t}(k+1, n) = 2^{\mathfrak{t}(k, n)}$. Intuitively the function \mathfrak{t} defines the tower of exponentials of height k , *e.g.* we have $\mathfrak{t}(1, n) = 2^n$, $\mathfrak{t}(2, n) = 2^{2^n}$, and so on. By $k\text{-NEXPTIME}$ we denote the class of all problems decidable with a nondeterministic Turing machines of working time in $\mathcal{O}(\mathfrak{t}(k, p(n)))$ for some polynomial $p(\cdot)$, on each input of length n . We define TOWER as the class of all problems of time complexity bounded by a tower of exponentials, whose height is an elementary function [50]. Thus, to show TOWER -hardness (using elementary reductions [50]), it is sufficient to prove $k\text{-NEXPTIME}$ -hardness for all k using uniform reductions [50, Section 3.1]. It is worth recalling that TOWER -hardness is defined with the class of elementary reductions (*i.e.* those with time-complexity bounded by a tower of exponentials of fixed height) [50]. Building a reduction from instances of $k\text{-NEXPTIME}$ -hard problems with time-complexity $\mathfrak{f}(n, k)$ where the size n of the input leads to an elementary reduction when $\mathfrak{f}(n, k)$ itself is elementary in n and k . This is what we mean by a uniform reduction in k to establish TOWER -hardness.

For proving hardness results, we make extensive use of tiling problems, see *e.g.* [60].

Definition 2.2. The *tiling problem* Tiling_k takes as inputs a triple $\langle \mathcal{T}, \mathcal{H}, \mathcal{V} \rangle$ and $c \in \mathcal{T}^n$ for some $n \geq 1$ such that \mathcal{T} is a finite set of *tile types*, $\mathcal{H} \subseteq \mathcal{T} \times \mathcal{T}$ (*resp.* $\mathcal{V} \subseteq \mathcal{T} \times \mathcal{T}$) represents the *horizontal (resp. vertical) matching relation*, and $c = t_0, t_1, \dots, t_{n-1} \in \mathcal{T}^n$ is the *initial condition*. A *solution* for the instance $\langle \mathcal{T}, \mathcal{H}, \mathcal{V} \rangle, c$ is a mapping $\tau : [0, \mathfrak{t}(k, n) - 1] \times [0, \mathfrak{t}(k, n) - 1] \rightarrow \mathcal{T}$ such that:

(init): For all $i \in [0, n - 1]$, $\tau(0, i) = t_i$.

(hori): For all $i, j \in [0, \mathfrak{t}(k, n) - 1]$, $i < \mathfrak{t}(k, n) - 1$ implies $(\tau(i, j), \tau(i+1, j)) \in \mathcal{H}$.

(verti): For all $i, j \in [0, \mathfrak{t}(k, n) - 1]$, $j < \mathfrak{t}(k, n) - 1$ implies $(\tau(i, j), \tau(i, j+1)) \in \mathcal{V}$.

A mapping τ satisfying (hori) and (verti) is called a *tiling*.

The problem of checking if an instance of Tiling_k has a solution (note that k does not appear in the instance and it governs the size of the grid) is $k\text{-NEXPTIME}$ -complete [60].

Given $N \geq 2$, let us consider the satisfiability problem for $\text{QCTL}_{\mathbf{X}, \leq N}^t$ in which the structures are tree models where all the maximal branches are infinite but each node has at most N children (and at least one child). To characterise the complexity of $\text{SAT}(\text{QCTL}_{\mathbf{X}, \leq N}^t)$, we consider the complexity class AEXP_{POL} that consists of all problems decidable by an

alternating Turing machine (ATM) [17] working in exponential-time and using only polynomially many alternations [13, 44]. We stress here that allowing an unbounded number of alternations would give us the class EXPSPACE , and classes similar to AEXP_{POL} have been considered in [11], typically $\text{STA}(\mathbf{f}(n), \mathbf{g}(n), \mathbf{h}(n))$, where $\mathbf{f}(n)$ refers to the restriction on the Space, $\mathbf{g}(n)$ refers to the restriction on the Time, and $\mathbf{h}(n)$ refers to the restriction on the number of Alternations. Consequently, AEXP_{POL} is the union of classes $\text{STA}(\cdot, 2^{\mathbf{g}(n)}, \mathbf{h}(n))$ with polynomials $\mathbf{g}(n)$, $\mathbf{h}(n)$. The complexity of several logical problems has been captured by the class AEXP_{POL} , see *e.g.* [23, 14, 13].

For proving AEXP_{POL} -hardness, we use an elegant modification of Tiling_1 , introduced in [13, 44]. The extension amounts to considering a stack of n tilings, with a matching relation between two consecutive tile types on the same position of the grid, and quantifications over the tile types on the first row (initial conditions). Details follow below.

Definition 2.3. The *alternating multi-tiling problem* AMTP takes as inputs an even number n (in unary), $\langle \mathcal{T}, \mathcal{H}, \mathcal{V} \rangle$ (as for defining Tiling_1), $\mathcal{T}_0 \subseteq \mathcal{T}$, $\mathcal{T}_{\text{acc}} \subseteq \mathcal{T}$ and $\mathcal{T}_{\text{multi}} \subseteq \mathcal{T} \times \mathcal{T}$. Given an initial condition $c = (w_1, \dots, w_n) \in (\mathcal{T}_0^{2^n})^n$, a *solution* for c is a multi-tiling (τ_1, \dots, τ_n) on the grid $[0, 2^n - 1] \times [0, 2^n - 1]$ such that:

(m-init): For all $\alpha \in [1, n]$, for all $j \in [0, 2^n - 1]$, $\tau_\alpha(0, j) = w_\alpha(j)$ (*i.e.* the first row of τ_α is w_α).

(m-tiling): For $\alpha \in [1, n]$, τ_α satisfies (hori) and (verti).

(m-multi): For $\alpha \in [1, n - 1]$, for all $i, j \in [0, 2^n - 1]$, $(\tau_\alpha(i, j), \tau_{\alpha+1}(i, j)) \in \mathcal{T}_{\text{multi}}$.

(m-accept): For some $j \in [0, 2^n - 1]$, $\tau_n(2^n - 1, j) \in \mathcal{T}_{\text{acc}}$.

An instance \mathcal{I} for AMTP made of n , $\langle \mathcal{T}, \mathcal{H}, \mathcal{V} \rangle$, $\mathcal{T}_0, \mathcal{T}_{\text{acc}} \subseteq \mathcal{T}$, $\mathcal{T}_{\text{multi}} \subseteq \mathcal{T} \times \mathcal{T}$ is positive iff for all $w_1 \in \mathcal{T}_0^{2^n}$, there is $w_2 \in \mathcal{T}_0^{2^n}$ such that ... for all $w_{n-1} \in \mathcal{T}_0^{2^n}$, there is $w_n \in \mathcal{T}_0^{2^n}$ such that there is a solution (τ_1, \dots, τ_n) for (w_1, \dots, w_n) . Note that this sequence involves $n-1$ quantifier alternations.

The alternating multi-tiling problem is shown to be AEXP_{POL} -complete in [13, 44].

3. WHAT HAPPENS WHEN TREES ARE BOUNDED?

In this section, we study the satisfiability problem for $\text{QCTL}_{\mathbf{x}, \leq N}^t$ with $N \geq 1$, *i.e.* $\text{QCTL}_{\mathbf{x}}^t$ over trees, where the degree of each node is bounded by a fixed natural number $N \geq 1$. As we have already mentioned in the introduction, the goal of this section is two-fold. First, we would like to make the reader familiar with our proof techniques applied to a simplified scenario. Second, our results show that to get TOWER-hardness of $\text{QCTL}_{\mathbf{x}}^t$ we must focus on trees of arbitrarily large branching.

3.1. A toolkit for introducing local nominals. Below, we introduce formulae to simulate partially the use of nominals from hybrid modal logics [3]. A nominal x is usually understood as a propositional variable true at exactly one world of the model (a global property). In $\text{QCTL}_{\mathbf{x}}^t$, such a property cannot be enforced but it can be done with respect to nodes at a bounded depth from the evaluation node, whence the adjective ‘local’ for the nominals. The use of local nominals is essential in all our hardness proofs, as it allows us to simulate first-order quantification on a given set of nodes of bounded depth.

Proofs of all lemmas from this section are rather straightforward and are shown by careful inspection of the semantics. Hence, we delegate them to Appendix A.1–A.3.

Definition 3.1. Given a tree model \mathfrak{T} and a node v , we say that the propositional variable x is a *nominal for the depth $k \geq 1$ from v* iff there is $v' \in V$ with $vE^k v'$ such that $\mathfrak{T}, v' \models x$, and for all worlds $v'' \neq v'$ s.t. $vE^k v''$, we have $\mathfrak{T}, v'' \not\models x$ (E^k is the k -fold composition of E).

The formula $\text{nom}(x, k)$ defined as $\mathbf{EX}^k x \wedge \neg \exists p (\mathbf{EX}^k (x \wedge p) \wedge \mathbf{EX}^k (x \wedge \neg p))$, where p is distinct from x , states that x is a nominal for the depth k (\mathbf{EX}^k denotes k copies of \mathbf{EX}).

Lemma 3.2. x is a nominal for the depth $k \geq 1$ from v iff $\mathfrak{T}, v \models \text{nom}(x, k)$.

Let us next define $@_x^k \phi$ as the formula $\mathbf{EX}^k (x \wedge \phi)$ (usually assuming that $\text{nom}(x, k)$ holds).

Lemma 3.3. Assuming that x is a nominal for the depth $k \geq 0$ from v such that $vE^k v'$ and $\mathfrak{T}, v' \models x$, we have $\mathfrak{T}, v \models @_x^k \phi$ iff $\mathfrak{T}, v' \models \phi$.

Given $d \geq 1$ and propositional variables x_1, \dots, x_d (that play the role of nominals), we often write $@_{x_1, \dots, x_d} \phi$ to denote the formula $@_{x_1}^1 @_{x_2}^1 \dots @_{x_d}^1 \phi$ (usually assuming that $\text{nom}(x_1, 1)$ holds and for all $i \in [2, d]$, $@_{x_1}^1 @_{x_2}^1 \dots @_{x_{i-1}}^1 \text{nom}(x_i, 1)$ holds true). We also use $@_{\bar{x}} \phi$ instead of $@_{x_1, \dots, x_d} \phi$ when \bar{x} is understood as x_1, \dots, x_d . Given a node v_0 such that

$$\mathfrak{T}, v_0 \models \text{nom}(x_1, 1) \wedge \bigwedge_{i \in [2, d]} @_{x_1}^1 @_{x_2}^1 \dots @_{x_{i-1}}^1 \text{nom}(x_i, 1),$$

we write v_1, \dots, v_d to denote the unique sequence of nodes such that for all $i \in [1, d]$, we have both $v_{i-1} E v_i$ and $\mathfrak{T}, v_i \models x_i$. The existence and uniqueness of the nodes v_1, \dots, v_d follow from Lemma 3.2 and Lemma 3.3. Here is another useful lemma justifying the use of the introduced abbreviations.

Lemma 3.4. Assume that $\mathfrak{T}, v_0 \models \text{nom}(x_1, 1) \wedge \bigwedge_{i \in [2, d]} @_{x_1}^1 @_{x_2}^1 \dots @_{x_{i-1}}^1 \text{nom}(x_i, 1)$ and the sequence v_1, \dots, v_d is associated to $\bar{x} = x_1, \dots, x_d$. Then, $\mathfrak{T}, v_0 \models @_{\bar{x}} \phi$ iff $\mathfrak{T}, v_d \models \phi$.

Let $\text{diff-nom}(x_1, \dots, x_\alpha, k)$ be an abbreviation of $\bigwedge_{i \in [1, \alpha]} \text{nom}(x_i, k) \wedge \bigwedge_{j \in [1, \alpha]} \bigwedge_{i < j} \neg @_{x_i}^k x_j$. It allows us to name α distinct nodes at the depth k . Hence, the respective nodes interpreting the nominals x_1, \dots, x_α are pairwise distinct. It is summarised with the lemma below, which proof is a slight variant the proof of Lemma 3.2.

Lemma 3.5. Given a tree model \mathfrak{T} and a node v , we have $\mathfrak{T}, v \models \text{diff-nom}(x_1, \dots, x_\alpha, k)$ iff there are α distinct nodes v_1, \dots, v_α such that for all $i \in [1, \alpha]$, x_i is a nominal for the depth $k \geq 1$ from v .

Let us illustrate the use of local nominals and $\text{diff-nom}(x_1, \dots, x_\alpha, k)$ to specify that a node has at most 2^n children, with a formula of polynomial size in n . This is exactly the type of properties that can be expressed in graded modal logics [26, 22, 59]. Given a finite set X of propositional variables, we design a formula stating that no pair of distinct children agree on all propositional variables from X , as done in [19]. It is given below:

$$\text{Uni}(X) \stackrel{\text{def}}{=} \forall x, y \text{ diff-nom}(x, y, 1) \rightarrow \neg \left(\bigwedge_{p \in X} (@_x^1 p \leftrightarrow @_y^1 p) \right)$$

Thus, the formula $\Diamond_{\leq 2^n} \top$ from graded modal logics can be expressed in $\text{QCTL}_{\mathbf{X}}^t$ with

$$\Diamond_{\leq 2^n} \top \stackrel{\text{def}}{=} \exists p_0, \dots, p_{n-1} \text{ Uni}(\{p_0, \dots, p_{n-1}\}).$$

In Section 4, we show how to succinctly express hyperexponential bounds.

3.2. Beyond the ExpSpace upper bound: AExp_{pol}. In order to solve $\text{SAT}(\text{QCTL}_{\mathbf{x}, \leq N}^t)$, little is needed if the EXPSPACE upper bound is aimed. Indeed, given a formula ϕ in $\text{QCTL}_{\mathbf{x}, \leq N}^t$, it is clear that for an N -bounded tree model \mathfrak{T} satisfying ϕ at its root node ε , it is irrelevant what happens at nodes of depth strictly more than $md(\phi)$. Hence, the formula ϕ is satisfiable iff there is a finite N -bounded tree structure \mathfrak{T} with all the branches of length exactly $md(\phi)$ satisfying ϕ at its root ε (as the branches of tree models are infinite, we need to consider branches of length exactly $md(\phi)$). Thus, \mathfrak{T} has at most $|\phi|N^{|\phi|}$ nodes. To get an algorithm working in NEXPSPACE, guess such an exponential-size finite tree structure, and perform model-checking on it with an algorithm inherently in PSPACE (as model-checking finite structures with MSO is PSPACE-complete [56, 61] and ϕ can be translated to MSO in the standard way), which leads to NEXPSPACE. By Savitch's Theorem [49], we get the EXPSPACE upper bound.

This bound is not completely satisfactory as it does not use much of $\text{QCTL}_{\mathbf{x}, \leq N}^t$ and more importantly, Section 3.3 proves AEXP_{POL}-hardness of $\text{SAT}(\text{QCTL}_{\mathbf{x}, \leq N}^t)$ as long as $N \geq 2$. Hence, the goal of this section is to establish an AEXP_{POL} upper bound. The tight upper bound for $\text{SAT}(\text{QCTL}_{\mathbf{x}, \leq N}^t)$ relies on the following ingredients.

- (i) Every formula ϕ of $\text{QCTL}_{\mathbf{x}}^t$ is logically equivalent to a $\text{QCTL}_{\mathbf{x}}^t$ formula ϕ' in prenex normal form (PNF) such that ϕ' can be computed in polynomial-time in $|\phi|$. Formulae in PNF are of the form $\mathcal{Q}_1 p_1 \cdots \mathcal{Q}_\beta p_\beta \psi$ where $\{\mathcal{Q}_1, \dots, \mathcal{Q}_\beta\} \subseteq \{\exists, \forall\}$ and ψ is quantifier-free.
- (ii) Existence of an N -bounded tree model for ϕ is equivalent to the existence of an N -bounded finite tree structure such that all branches are of length $md(\phi)$. Then, we simply guess a finite tree of a small (exponential) size with the help of the *shallow model property* - such a tree will be later unravelled to become an infinite tree model.
- (iii) Checking whether $\mathfrak{T}, \varepsilon \models \mathcal{Q}_1 p_1 \cdots \mathcal{Q}_\beta p_\beta \psi$ (involving an N -bounded finite tree with branches of length $md(\psi)$ and the input formula in PNF) can be done with an alternating Turing machine in time $\mathcal{O}((|\psi| + \beta)|\mathfrak{T}|)$ and with at most β alternations.

To establish (i), we cannot rely directly on PNF for QCTL from [37, Prop. 3.1] as the translation in [37, Prop. 3.1] involves temporal operators beyond the language of $\text{QCTL}_{\mathbf{x}}^t$.

Lemma 3.6. *For every formula ϕ in $\text{QCTL}_{\mathbf{x}}^t$, one can compute in polynomial-time in $|\phi|$ a logically equivalent formula in PNF $\mathcal{Q}_1 p_1 \cdots \mathcal{Q}_\beta p_\beta \psi$ with $\beta \leq |\phi|$.*

Proof. On tree models, the following formulae are tautologies, assuming that p does not occur in ψ' (where \mathcal{Q} is either \exists or \forall):

$$\text{EX } \mathcal{Q} p \psi \leftrightarrow \mathcal{Q} p \text{EX } \psi \quad (\mathcal{Q} p \psi) \wedge \psi' \leftrightarrow \mathcal{Q} p (\psi \wedge \psi') \quad \neg \exists p \psi \leftrightarrow \forall p \neg \psi \quad \neg \forall p \psi \leftrightarrow \exists p \neg \psi$$

Hence, by employing the above formulae and rewriting the input, we conclude the lemma. For a more detailed explanation consult Appendix A.4 \square

Now we proceed with the second property. Let us be a bit more precise. Given a tree model \mathfrak{T} , we write $\mathfrak{T}|_n$ to denote its subtree obtained by taking only nodes on the depth at most n from the root. A *completion* of a finite tree \mathfrak{T}' of maximal depth n is an infinite tree \mathfrak{T} (finite-branching and all the maximal branches are infinite) such that $\mathfrak{T}' = \mathfrak{T}|_n$. By the *naive completion* of \mathfrak{T}' of maximal depth n , we refer to the unique completion achieved by replacing each node v at depth n from \mathfrak{T} by an infinite chain of copies of itself.

A *shallow model property* states that what matters for a $\text{QCTL}_{\mathbf{x}}^t$ formula ϕ in its infinite tree model is a relatively small finite part, with paths bounded by the modal depth of ϕ .

Lemma 3.7 (Shallow Model Property). *Let $\mathfrak{T}, \varepsilon$ be a model for the $\text{QCTL}_{\mathbf{X}}^t$ -formula ϕ . Then, any completion of $\mathfrak{T}|_{\text{md}(\phi)}, \varepsilon$ is also a model for ϕ .*

Proof. The construction is standard and goes in exactly the same way as in the literature, e.g. [12, Theorem 2.3]. \square

Since we are interested in the satisfiability problem over N -bounded trees, the overall size of a structure $\mathfrak{T}|_{\text{md}(\phi)}$ can be easily bounded. A simple estimation can be obtained by counting the number of nodes with a certain distance from the root, namely:

$$|\mathfrak{T}|_{\text{md}(\phi)}| \leq 1 + N + N^2 + \dots + N^{\text{md}(\phi)} < |\phi| \cdot N^{|\phi|}$$

As a direct consequence of the above estimation and Lemma 3.7, we obtain:

Lemma 3.8. *For any formula ϕ , ϕ is satisfiable for $\text{QCTL}_{\mathbf{X}, \leq N}^t$ iff ϕ is satisfiable in a finite N -bounded tree structure of size bounded by $|\phi|N^{|\phi|}$ and each branch is of length $\text{md}(\phi)$.*

In order to establish (iii), the details are omitted but we apply the naive model-checking algorithm for MSO with an ATM: existential (resp. universal) quantification $\exists p$ (resp. $\forall p$) requires time $\mathcal{O}(|\mathfrak{T}|)$ and the machine enters a sequence of existential (resp. universal) states. The quantifier-free formula ψ is evaluated as a first-order formula by the standard translation for modal logic. Note also that checking $\mathfrak{T}', v \models \psi$ can be done in polynomial time in $|\psi| + |\mathfrak{T}'|$ (see [18, 51]). By combining (i)-(iii) we establish an improved upper bound.

Theorem 3.9. *For any $N \geq 1$, the satisfiability problem for $\text{QCTL}_{\mathbf{X}, \leq N}^t$ is in AEXP_{POL} .*

When $N = 1$, the upper bound can be improved as the number of alternations is linear and the size of the finite witness “tree” is polynomial in $|\phi|$, and therefore the whole procedure can be implemented with a polynomial-time alternating Turing machine (thus in PSPACE [17]). The matching lower bound is inherited from quantified propositional logic QBF, see e.g. [42].

Corollary 3.10. *The satisfiability problem for $\text{QCTL}_{\mathbf{X}, \leq 1}^t$ is PSPACE -complete.*

3.3. A reduction from AMTP (with fixed $N \geq 2$). Let $N \geq 2$ and let us consider the satisfiability problem for $\text{QCTL}_{\mathbf{X}, \leq N}^t$ in which the structures are tree models where all the maximal branches are infinite and each node has at most N children (and at least one child). In order to show that the problem is AEXP_{POL} -hard, we define below a reduction from the alternating multi-line tiling problem **AMTP** presented in Section 2 and introduced in [13].

To define a grid $[0, 2^n - 1] \times [0, 2^n - 1]$, a major part in the solution of an instance of **AMTP**, we specify a tree such that every node at a distance less than $2n$ from the root ε has exactly two children, implying that there are exactly 2^{2n} nodes at a distance $2n$ from ε . Moreover, each node at a distance $2n$ encodes a position $(\mathfrak{H}, \mathfrak{V})$ in $[0, 2^n - 1] \times [0, 2^n - 1]$, by making the propositional variables $\mathbf{h}_{n-1}, \dots, \mathbf{h}_0$ and $\mathbf{v}_{n-1}, \dots, \mathbf{v}_0$ to be responsible, respectively, for the horizontal and vertical axes. The i -th bit of \mathfrak{H} (resp. \mathfrak{V}) is taken care of by the truth value of \mathbf{h}_i (resp. \mathbf{v}_i) and by convention the least significant bit is encoded by \mathbf{h}_0 (resp. \mathbf{v}_0).

The forthcoming formula $\text{grid}(2n)$ is dedicated to encoding such a grid.

$$\begin{aligned} \text{grid}(2n) \stackrel{\text{def}}{=} & \left(\bigwedge_{i \in [0, 2n-1]} \mathbf{A}\mathbf{X}^i \mathbf{E}\mathbf{X}_{=2}\top \right) \wedge \\ & \forall x, y \text{ diff-nom}(x, y, 2n) \rightarrow \left(\bigvee_{j \in [0, n-1]} \neg(@_x^{2n} \mathbf{h}_j \leftrightarrow @_y^{2n} \mathbf{h}_j) \vee \neg(@_x^{2n} \mathbf{v}_j \leftrightarrow @_y^{2n} \mathbf{v}_j) \right), \end{aligned}$$

where $\mathbf{E}\mathbf{X}_{=2}\top \stackrel{\text{def}}{=} \exists x_1, x_2 \text{ diff-nom}(x_1, x_2, 1) \wedge \mathbf{A}\mathbf{X}(x_1 \vee x_2)$ states that there are exactly two children. Moreover, $\mathbf{A}\mathbf{X}^0 \psi \stackrel{\text{def}}{=} \psi$ and $\mathbf{A}\mathbf{X}^{i+1} \psi \stackrel{\text{def}}{=} \mathbf{A}\mathbf{X}\mathbf{A}\mathbf{X}^i \psi$.

Note that the upper part of $\text{grid}(2n)$ enforces that there are exactly 2^{2n} descendants at a distance $2n$ from the root, while the lower part imposes that any two such descendants differ by at least one propositional variable from $\mathbf{h}_{n-1}, \dots, \mathbf{h}_0, \mathbf{v}_{n-1}, \dots, \mathbf{v}_0$. Hence, the full grid $[0, 2^n - 1] \times [0, 2^n - 1]$ is encoded with $\text{grid}(2n)$. The correctness of $\text{grid}(2n)$ follows from Lemma 3.3 and Lemma 3.5.

Corollary 3.11. *$\mathfrak{T}, v \models \text{grid}(2n)$ iff $\mathfrak{T}|_{2n}$ is a binary tree in which there are exactly 2^{2n} nodes v' satisfying $E^{2n}(v, v')$ and each of such distinct nodes v', v'' is labelled by a different subset of atomic propositions from $\{\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_{n-1}, \mathbf{h}_0, \mathbf{h}_1, \dots, \mathbf{h}_{n-1}\}$.*

Let $\langle \mathcal{T}, \mathcal{H}, \mathcal{V} \rangle$ be a triple from an instance of **AMTP** and let $j \in \mathbb{N}$. Each tile type $t \in \mathcal{T}$ will be represented by a fresh propositional variable t^j . Hence, $\{t^j : t \in \mathcal{T}\}$ (written below \mathcal{T}^j) is a set of propositional variables used to provide a tile type for each position of the grid $[0, 2^n - 1] \times [0, 2^n - 1]$, while the superscript ‘ j ’ is handy to remember that this concerns the j -th tiling (as several tilings are involved in **AMTP** instances).

We first define the formulae ϕ_{cov}^j , $\phi_{\mathcal{H}}^j$ and $\phi_{\mathcal{V}}^j$ whose conjunction states that every position of the grid has a unique tile type in \mathcal{T}^j , and the horizontal and vertical matching conditions are satisfied. Hence, we have a valid tiling of the grid made from \mathcal{T}^j .

The formula ϕ_{cov}^j expresses that every position of the grid has a unique tile type:

$$\phi_{\text{cov}}^j \stackrel{\text{def}}{=} \forall x \text{ nom}(x, 2n) \rightarrow @_x^{2n} \left(\bigvee_{t \in \mathcal{T}} t^j \wedge \bigwedge_{t \neq t' \in \mathcal{T}} \neg(t^j \wedge t'^j) \right).$$

For the horizontal matching constraints, we need to express when two nodes at a distance $2n$ interpreted respectively by x and y and representing respectively the position $(\mathfrak{H}, \mathfrak{V})$ and $(\mathfrak{H}', \mathfrak{V}')$, satisfy $\mathfrak{V} = \mathfrak{V}'$ and $\mathfrak{H}' = \mathfrak{H} + 1$. The formula $\text{HN}(x, y)$ (‘HN’ for ‘horizontal neighbours’) does the job using a standard arithmetical reasoning on binary numbers.

The intuition is that we treat \mathbf{h}_i propositions as bits and to verify that the number encoded on y is equal to the number encoded on x plus 1, we need to (i) find an index i on which the i -th bit is switched on for y but switched off for x , (ii) check that all bits on more significant positions after i for x and y are equal and (iii) ensure that all less significant bits

are switched on for x while switched off for y . This idea is formalised as follows:

$$\text{HN}(x, y) \stackrel{\text{def}}{=} \left(\bigwedge_{\alpha \in [0, n-1]} @_x^{2n} \mathbf{v}_\alpha \leftrightarrow @_y^{2n} \mathbf{v}_\alpha \right) \wedge \bigvee_{i \in [0, n-1]} \left(@_x^{2n} \neg \mathbf{h}_i \wedge @_y^{2n} \mathbf{h}_i \wedge \right. \\ \left. \wedge \bigwedge_{\alpha \in [0, i-1]} (@_x^{2n} \mathbf{h}_\alpha \wedge @_y^{2n} \neg \mathbf{h}_\alpha) \wedge \left(\bigwedge_{\alpha \in [i+1, n]} (@_x^{2n} \mathbf{h}_\alpha \leftrightarrow @_y^{2n} \mathbf{h}_\alpha) \right) \right).$$

Employing $\text{HN}(x, y)$ we can provide a formula $\phi_{\mathcal{H}}^j$ that encodes horizontal matching constraints:

$$\phi_{\mathcal{H}}^j \stackrel{\text{def}}{=} \forall x, y \left(\mathbf{nom}(x, 2n) \wedge \mathbf{nom}(y, 2n) \wedge \text{HN}(x, y) \right) \rightarrow \bigvee_{(t, t') \in \mathcal{H}} (@_x^{2n} t^j \wedge @_y^{2n} t'^j).$$

Let $\text{VN}(x, y)$ (where ‘VN’ stands for ‘vertical neighbours’) be the formula obtained from $\text{HN}(x, y)$ by replacing each occurrence of \mathbf{h}_α (resp. \mathbf{v}_α) by \mathbf{v}_α (resp. \mathbf{h}_α).

The following formula $\phi_{\mathcal{V}}^j$ encodes the vertical matching constraints:

$$\phi_{\mathcal{V}}^j \stackrel{\text{def}}{=} \forall x, y \left(\mathbf{nom}(x, 2n) \wedge \mathbf{nom}(y, 2n) \wedge \text{VN}(x, y) \right) \rightarrow \bigvee_{(t, t') \in \mathcal{V}} (@_x^{2n} t^j \wedge @_y^{2n} t'^j).$$

To state that a root satisfying $\text{grid}(2n)$ encodes a tiling with respect to \mathcal{T}^j , we consider the formula

$$\phi_{\text{tiling}}^j \stackrel{\text{def}}{=} \phi_{\text{cov}}^j \wedge \phi_{\mathcal{H}}^j \wedge \phi_{\mathcal{V}}^j.$$

Lemma 3.12. *Assume that $\mathfrak{T}, v \models \text{grid}(2n)$ holds. Then:*

- *If $\mathfrak{T}, v \models \phi_{\text{tiling}}^j$ then $\tau : [0, 2^n - 1] \times [0, 2^n - 1] \rightarrow \mathcal{T}^j$, defined as $\tau(x, y) = t^j$ for $\mathfrak{T}, v_{(x, y)} \models t^j$, where $v_{(x, y)}$ is the unique encoding of the position (x, y) satisfying $E^{2n}(v, v_{(x, y)})$, is a tiling.*
- *If $\tau : [0, 2^n - 1] \times [0, 2^n - 1] \rightarrow \mathcal{T}^j$ is a tiling, then there exists a tree \mathfrak{T}', v' satisfying $\phi_{\text{tiling}}^j \wedge \text{grid}(2n)$ and being a \mathcal{T}^j -variant of \mathfrak{T} .*

Proof. By careful inspection of the semantics and of presented formulae, cf. Appendix A.5. \square

In order to encode an instance of **AMTP**, there are still properties that need to be expressed. Let us assume that the root node ε satisfies $\text{grid}(2n)$.

- Given the set of initial tile types $\mathcal{T}_0 \subseteq \mathcal{T}$, let us express that for each position of the first row of the grid, exactly one tile type in \mathcal{T}_0^j holds.

$$\phi_{\text{init}}^j \stackrel{\text{def}}{=} \forall x \left(\mathbf{nom}(x, 2n) \wedge @_x^{2n} \left(\bigwedge_{\alpha \in [0, n-1]} \neg \mathbf{h}_\alpha \right) \right) \rightarrow @_x^{2n} \left(\bigvee_{t \in \mathcal{T}_0} t^j \wedge \bigwedge_{t \neq t' \in \mathcal{T}_0} \neg(t^j \wedge t'^j) \right)$$

- Assuming that ε satisfies $\phi_{\text{tiling}}^j \wedge \phi_{\text{init}}^j$, we express that for each position of the first row of the grid, the tile type in \mathcal{T}_0^j coincides with the tile type in $\mathcal{T}_0^{j'}$ (corresponding to (m-init)):

$$\phi_{\text{coinci}}^{j, j'} \stackrel{\text{def}}{=} \forall x \left(\left(\mathbf{nom}(x, 2n) \wedge @_x^{2n} \left(\bigwedge_{\alpha \in [0, n-1]} \neg \mathbf{h}_\alpha \right) \right) \rightarrow @_x^{2n} \left(\bigvee_{t \in \mathcal{T}_0} (t^j \wedge t^{j'}) \right) \right).$$

- Given the set of accepting tile types $\mathcal{T}_{acc} \subseteq \mathcal{T}$ and assuming that ε satisfies ϕ_{tiling}^j , we state that there is a position on the last row with a tile type in \mathcal{T}_{acc} (satisfying (m-accept)):

$$\phi_{acc}^j \stackrel{\text{def}}{=} \exists x \text{ nom}(x, 2n) \wedge @_x^{2n} \left(\left(\bigwedge_{\alpha \in [0, n-1]} h_\alpha \right) \wedge \bigvee_{t \in \mathcal{T}_{acc}} t^j \right).$$

- Given the multi-matching tiling relation $\mathcal{T}_{multi} \subseteq \mathcal{T} \times \mathcal{T}$, and assuming ε satisfies $\phi_{\text{tiling}}^j \wedge \phi_{\text{tiling}}^{j+1}$, on every position, the tile type in \mathcal{T}^j and the tile type in \mathcal{T}^{j+1} are in the relation \mathcal{T}_{multi} (fulfilling the requirements of (m-multi)):

$$\phi_{multi}^j \stackrel{\text{def}}{=} \forall x \text{ nom}(x, 2n) \rightarrow @_x^{2n} \left(\bigvee_{(t, t') \in \mathcal{T}_{multi}} (t^j \wedge t'^{j+1}) \right).$$

It is time to wrap up. Given a finite set of propositional variables $X = \{r_1, \dots, r_\beta\}$, we write $\exists X \psi$ to denote the formula $\exists r_1 \exists r_2 \dots \exists r_\beta \psi$. $\forall X \psi$ is defined similarly. Given an instance \mathcal{I} of **AMTP** made of $n, \langle \mathcal{T}, \mathcal{H}, \mathcal{V} \rangle, \mathcal{T}_0, \mathcal{T}_{acc}, \mathcal{T}_{multi}$, let us define the formula $\phi_{\mathcal{I}}$ below:

$$\begin{aligned} \phi_{\mathcal{I}} \stackrel{\text{def}}{=} & \text{grid}(2n) \wedge \forall \mathcal{T}_0^1 \exists \mathcal{T}_0^2 \forall \mathcal{T}_0^3 \dots \forall \mathcal{T}_0^{n-1} \exists \mathcal{T}_0^n \bigwedge_{j \in [1, n]} \phi_{\text{init}}^j \rightarrow \left(\exists \{t^j : t \in \mathcal{T}, j \in [n+1, 2n]\} \right. \\ & \left. \left(\bigwedge_{j \in [n+1, 2n]} \phi_{\text{tiling}}^j \wedge \phi_{\text{coinci}}^{j, (j-n)} \right) \wedge \left(\bigwedge_{j \in [n+1, 2n-1]} \phi_{\text{multi}}^j \right) \wedge \phi_{acc}^{2n} \right). \end{aligned}$$

Now, we can state the correctness of the reduction.

Lemma 3.13. *\mathcal{I} is a positive instance of **AMTP** iff $\phi_{\mathcal{I}}$ is satisfiable in $\text{QCTL}_{\mathbf{X}, \leq N}^t$.*

Proof. The proof is a bit tedious but has no serious difficulties, as all the conditions for being a solution of \mathcal{I} can be easily expressed, as soon as the grid $[0, 2^n - 1] \times [0, 2^n - 1]$ is encoded. Moreover, the quantifications involved in **AMTP** are straightforwardly taken care of in $\text{QCTL}_{\mathbf{X}, \leq N}^t$ thanks to the presence of propositional quantification. Consult Appendix A.6. \square

The above lemma leads us to one of the main results of the paper.

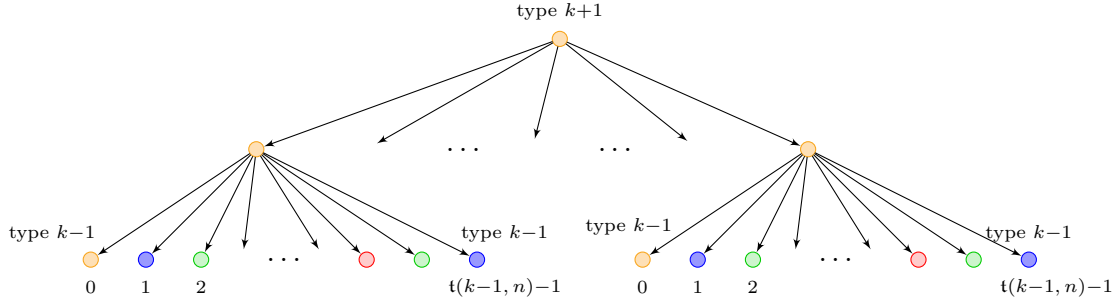
Theorem 3.14. *For all $N \geq 2$, the satisfiability problem for $\text{QCTL}_{\mathbf{X}, \leq N}^t$ is AEXP_{POL} -hard.*

4. TOWER-HARDNESS OF THE SATISFIABILITY PROBLEM $\text{QCTL}_{\mathbf{X}}^t$

We are back to the (general) satisfiability problem for $\text{QCTL}_{\mathbf{X}}^t$, i.e. with no further restrictions on the number of children per node.

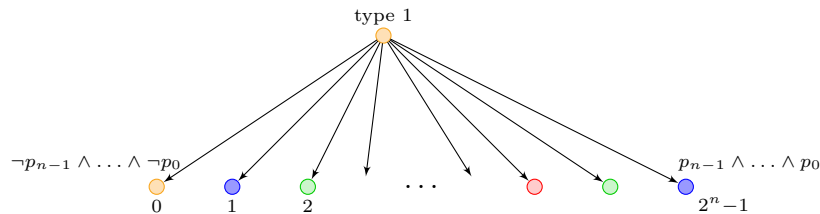
4.1. Overview of the method. In order to show TOWER-hardness, we shall reduce the k -NEXPTIME-complete tiling problem Tiling_k introduced in Section 2.3 to $\text{SAT}(\text{QCTL}_{\mathbf{x}}^t)$ and this should be done in a uniform way so that TOWER-hardness can be concluded (see the discussion in [50, Section 3.1.2] and in Section 2.3). Hence, we need to encode concisely a grid $\mathbf{t}(k, n) \times \mathbf{t}(k, n)$ and to do so, the main task consists in enforcing that a node has $\mathbf{t}(k, n)$ children, using a formula of elementary size in $k + n$ (bounded by a tower of exponentials of fixed height). Actually, our method produces a formula of exponential size in $k + n$. Of course, this is not the end of the story as we need to encode the grid $\mathbf{t}(k, n) \times \mathbf{t}(k, n)$ and to express on it constraints about the tiling τ . First, let us explain how to enforce that a node has exactly $\mathbf{t}(k, n)$ children, by partly taking advantage of the proof technique of local nominals (see Section 3.1).

We recall that $\mathbf{t}(1, n) = 2^n$ and $\mathbf{t}(k+1, n) = 2^{\mathbf{t}(k, n)}$ for $k > 0$. For the forthcoming subsections, we assume that n is fixed. Below, we classify the nodes of a tree model by their *type* (a value in \mathbb{N}) such that any node is of type 0, and if a node is of type $k > 0$, then it has exactly $\mathbf{t}(k, n)$ children and all the children are of type $k-1$. To be more precise, a node may have two types (as one of them is always zero). So, a node of type 1 has exactly 2^n children, a node of type 2 has exactly 2^{2^n} children, etc. Therefore if a node is of type $k > 0$, then the value k is unique. Additional conditions apply for being of type $k > 0$ but we can already notice that a node v of type k implicitly defines a balanced subtree of depth k with root v .



In order to enforce that a node is of type $k \geq 1$ (this is a trivial property for $k = 0$), and therefore has exactly $\mathbf{t}(k, n)$ children, with each node v of type $k \geq 0$ is associated a *number* in $[0, \mathbf{t}(k+1, n) - 1]$. Such a number is written $\text{nb}_{\mathfrak{T}}(v)$. The subscript “ \mathfrak{T} ” may be omitted when the context is clear. Similarly, when $\mathfrak{T}, v_0 \models \text{nom}(x_1, 1) \wedge \bigwedge_{i \in [2, d]} @_{x_1}^1 @_{x_2}^1 \dots @_{x_{i-1}}^1 \text{nom}(x_i, 1)$ and the nodes v_1, \dots, v_d are associated with x_1, \dots, x_d , we write $\text{nb}_{\mathfrak{T}}(x_i)$ instead of $\text{nb}_{\mathfrak{T}}(v_i)$ for all $i \in [1, d]$ (in general, by a slight abuse of notation, we may refer to a node by its local nominal when it exists).

When the type of the node v is zero, its number is defined as the unique m such that the number represented by the truth values of $p_{n-1}, p_{n-2}, \dots, p_0$ is equal to m . As usual, the propositional variable p_i is responsible for the i th bit of the number m and by convention, the least significant bit is encoded by the truth value of p_0 . We illustrate the encoding below:



Otherwise, when the type of v is equal to some $k > 0$, its number is represented by the binary encoding of the propositional variable val on its children assuming that there are $\mathfrak{t}(k, n)$ children whose respective (bit) numbers span all over $[0, \mathfrak{t}(k, n) - 1]$ and therefore all the children are implicitly ordered. This principle makes sense conceptually but it remains to express it in $\text{QCTL}_{\mathbf{X}}^t$, similarly to the TOWER-hardness proof in [46] for the fluted fragment in which counters with high values have to be enforced within a restricted language (see also [56]). That is why, in Table 1, we present a list of formulae to be defined. All of them are interpreted on a node v_0 of type $k \geq 0$, $1 \leq d \leq k$.

Formulae to be defined	Intuitive meaning
$\text{type}(k)$	v_0 is of type k
$\text{first}(k)$	$\mathbf{nb}(v_0) = 0$
$\text{last}(k)$	$\mathbf{nb}(v_0) = \mathfrak{t}(k + 1, n) - 1$
$\text{uniq}(k)$	$\forall v, v' ((v_0 E v) \ \& \ (v_0 E v') \ \& \ v \neq v') \rightarrow \mathbf{nb}(v) \neq \mathbf{nb}(v')$
$\text{compl}(k)$	$\forall v ((v_0 E v) \ \& \ \mathbf{nb}(v) < \mathfrak{t}(k, n) - 1) \rightarrow \exists v' (v_0 E v') \ \wedge \ \mathbf{nb}(v') = \mathbf{nb}(v) + 1$
$\mathbf{nb}(\bar{x}) =_k \mathbf{nb}(\bar{y})$	$\mathbf{nb}(v_d) = \mathbf{nb}(v'_d)$
$\mathbf{nb}(\bar{y}) =_k \mathbf{nb}(\bar{x}) + 1$	$\mathbf{nb}(v'_d) = 1 + \mathbf{nb}(v_d)$
$\mathbf{nb}(\bar{x}) <_k \mathbf{nb}(\bar{y})$	$\mathbf{nb}(v_d) < \mathbf{nb}(v'_d)$

Table 1: Family of auxiliary formulae.

In the last 3 lines of the table, the subscript ‘ k ’ below ‘ $=$ ’ and ‘ $<$ ’ allows us to remember that the formula is evaluated at a node of type k ; as \bar{x} and \bar{y} are of length d , the number comparison is done on nodes of type $k - d$ and the numbers can take values in $[0, \mathfrak{t}(k - d + 1, n) - 1]$. Though most of the intuitive meanings are straightforward, let us notice that $\text{uniq}(k)$ is intended to express that two distinct children of v_0 have distinct numbers. Similarly, $\text{compl}(k)$ is intended to express that if a child of v_0 has a number n strictly less than $\mathfrak{t}(k, n) - 1$, then v_0 has also another child with number equal to $n + 1$ (‘ compl ’ in $\text{compl}(k)$ stands for ‘complete’).

In what follows we will also employ $\widehat{\bar{x}, \bar{y}}$ to denote the formula

$$\mathbf{nom}(x_1, 1) \wedge \mathbf{nom}(y_1, 1) \wedge \bigwedge_{i \in [2, d]} (\@_{x_1}^1 \@_{x_2}^1 \dots \@_{x_{i-1}}^1 \mathbf{nom}(x_i, 1) \wedge \@_{y_1}^1 \@_{y_2}^1 \dots \@_{y_{i-1}}^1 \mathbf{nom}(y_i, 1)).$$

Assuming that the nodes v_1, \dots, v_d are associated with $\bar{x} = x_1, \dots, x_d$ (resp. v'_1, \dots, v'_d are associated with $\bar{y} = y_1, \dots, y_d$), v_0, v_1, \dots, v_d and v_0, v'_1, \dots, v'_d can be understood as two branches rooted at v_0 ending at the node v_d and at the node v'_d respectively. The formula $\widehat{\bar{x}, \bar{y}}$ uses subformulae introduced in Section 3.1 and the wide hat symbol in $\widehat{\bar{x}, \bar{y}}$ above \bar{x} and \bar{y} is a graphical reminder of these two branches. By contrast, the specific formula \bar{x}, \bar{x} states the existence of a single branch with nodes v_0, \dots, v_d .

In order to define $\text{type}(k)$ ($k \geq 1$), we specify that every child is of type $k - 1$, there is a child with number equal to zero, and if a child has number $m < \mathfrak{t}(k, n) - 1$, then there is a child with number equal to $m + 1$. Moreover, two distinct children have distinct numbers in $[0, \mathfrak{t}(k, n) - 1]$. Satisfying these conditions guarantees that the number for the children span all over $[0, \mathfrak{t}(k, n) - 1]$. The formula $\text{type}(k)$ (for $k \geq 1$) is defined as

$$\text{type}(k) \stackrel{\text{def}}{=} \mathbf{AX}(\text{type}(k - 1)) \wedge \mathbf{EX}(\text{first}(k - 1)) \wedge \text{uniq}(k) \wedge \text{compl}(k).$$

Note that the above formula is intended to be built over the propositional variables p_0, \dots, p_{n-1}, val (only).

Let us first explain how we proceed to define all the mentioned formulae. For successive values $N \in \mathbb{N}$, we define inductively the formulae:

- $\text{type}(N)$, $\text{first}(N)$ and $\text{last}(N)$,
- $\text{nb}(x_1, \dots, x_d) =_k \text{nb}(y_1, \dots, y_d)$,
- $\text{nb}(x_1, \dots, x_d) <_k \text{nb}(y_1, \dots, y_d)$,
- $\text{nb}(y_1, \dots, y_d) =_k \text{nb}(x_1, \dots, x_d) + 1$

for all $k - d = N - 1$. For $N = 0$, only the formulae $\text{type}(0)$, $\text{first}(0)$ and $\text{last}(0)$ make sense. The case $N = 1$ is not yet an instance of the general case. We first treat the cases for $N \in \{0, 1\}$ and then we proceed with the general case $N \geq 2$.

4.2. Formulae for types zero and one. When $k = 0$ (thus $k = N = 0$), only the intended properties for the formulae $\text{type}(0)$, $\text{first}(0)$ and $\text{last}(0)$ are meaningful. Let us define them, in the simplest way.

$$\text{type}(0) \stackrel{\text{def}}{=} \top \quad \text{first}(0) \stackrel{\text{def}}{=} \neg p_{n-1} \wedge \dots \wedge \neg p_0 \quad \text{last}(0) \stackrel{\text{def}}{=} p_{n-1} \wedge \dots \wedge p_0.$$

It can be rapidly checked that $\mathfrak{T}, v \models \text{type}(0)$ iff v is of type 0. Moreover, assuming that v is understood as a node of type 0, we have $\mathfrak{T}, v \models \text{first}(0)$ iff $\text{nb}_{\mathfrak{T}}(v) = 0$ and $\mathfrak{T}, v \models \text{last}(0)$ iff $\text{nb}_{\mathfrak{T}}(v) = t(1, n) - 1 = 2^n - 1$.

We next focus on the case when $N = 1$ and we define the formulae $\text{type}(k)$, $\text{first}(k)$ and $\text{last}(k)$ with $k=1$ (*i.e.* when $k=N=1$) as well as $\text{nb}(x_1, \dots, x_d) <_k \text{nb}(y_1, \dots, y_d)$ and $\text{nb}(y_1, \dots, y_d) =_k \text{nb}(x_1, \dots, x_d) + 1$ with $k - d = 0$ (that is when $k-d = N-1$ with $N=1$). We stress that k and d can be arbitrarily large as long as $k = d$.

To start with the formula $\text{nb}(x_1, \dots, x_d) =_k \text{nb}(y_1, \dots, y_d)$, it can be easily defined in terms of $\text{nb}(x_1, \dots, x_d) <_k \text{nb}(y_1, \dots, y_d)$ as:

$$\neg (\text{nb}(x_1, \dots, x_d) <_k \text{nb}(y_1, \dots, y_d)) \wedge \neg (\text{nb}(y_1, \dots, y_d) <_k \text{nb}(x_1, \dots, x_d)).$$

Second, we turn our attention to the formula $\text{type}(1)$. It states that there is a child with number equal to zero, if a child has number $m < 2^n - 1$, then there is a child with number equal to $m + 1$, all the children are of type 0, and two distinct children have distinct numbers in $[0, 2^n - 1]$. Remember that the number of each child (of type 0) is computed from the propositional variables in $\{p_{n-1}, \dots, p_0\}$. The arithmetical reasoning between children, leading to the fact that there are exactly 2^n children whose numbers span all over $[0, 2^n - 1]$ takes advantage of standard arithmetical properties on numbers encoded by n bits. Here is the formula $\text{type}(1)$:

$$\text{type}(1) \stackrel{\text{def}}{=} \mathbf{AX}(\text{type}(0)) \wedge \mathbf{EX}(\text{first}(0)) \wedge \text{uniq}(1) \wedge \text{compl}(1).$$

It remains to specify what exactly the formulae $\text{uniq}(1)$ and $\text{compl}(1)$ are. In order to define $\text{uniq}(1)$, responsible for enforcing the uniqueness among the children's numbering, we simply state that there are no two distinct children (of type 0) having the same number:

$$\forall x, y \text{ diff-nom}(x, y, 1) \rightarrow \neg (\text{nb}(x) =_1 \text{nb}(y)).$$

Note that the formula $\text{diff-nom}(x, y, 1)$ guarantees that we pick two distinct children and the nominals x and y allow us to access them (and check the values of the propositional variables in $\{p_{n-1}, \dots, p_0\}$). Consult Lemma 3.5 for the correctness.

The formula $\text{compl}(1)$ below states that for each child v (of type 0) that is not the last one (*i.e.* does not have the highest possible number among all other nodes), there is also a child v' (also of type 0) such that $\text{nb}_{\mathfrak{T}}(v') = \text{nb}_{\mathfrak{T}}(v) + 1$. Here is the formula $\text{compl}(1)$:

$$\forall x (\text{nom}(x, 1) \wedge @_x^1(\neg \text{last}(0))) \rightarrow \exists y \text{nom}(y, 1) \wedge \text{nb}(y) =_1 \text{nb}(x) + 1.$$

Finally, it remains to define the formulae $\text{nb}(x) =_1 \text{nb}(y)$ and $\text{nb}(y) =_1 \text{nb}(x) + 1$ used respectively in $\text{uniq}(1)$ and in $\text{compl}(1)$. Below, we treat the more general situation with $k = d$ (k is not necessarily equal to 1), and $\text{nb}(x) =_1 \text{nb}(y)$ and $\text{nb}(y) =_1 \text{nb}(x) + 1$ are specific instances with $k = d = 1$. Let assume that $\bar{x} = x_1, \dots, x_k$ and $\bar{y} = y_1, \dots, y_k$ (thus $d = k$). The forthcoming definitions are standard and rely on elementary operations on binary encoding of natural numbers with n bits (again, the least significant bit is represented by the truth value of p_0):

- $\text{nb}(y_1, \dots, y_k) =_k \text{nb}(x_1, \dots, x_k) + 1$ is defined as

$$\bigvee_{i=0}^{n-1} \left(\underbrace{@_{\bar{x}} \left(\neg p_i \wedge \bigwedge_{j=0}^{i-1} p_j \right)}_{\text{look for the first zero bit}} \wedge \underbrace{@_{\bar{y}} \left(\bigwedge_{j=0}^{i-1} \neg p_j \wedge p_i \right)}_{\text{reset previous bits, set } p_i} \wedge \underbrace{\left(\bigwedge_{j=i+1}^{n-1} @_{\bar{x}} p_j \Leftrightarrow @_{\bar{y}} p_j \right)}_{\text{rewrite other bits}} \right)$$

- $\text{nb}(x_1, \dots, x_k) <_k \text{nb}(y_1, \dots, y_k)$ is defined as

$$\bigvee_{i=0}^{n-1} \left(\underbrace{@_{\bar{y}} p_i \wedge @_{\bar{x}} \neg p_i}_{\text{find the first bit that differs}} \wedge \underbrace{\left(\bigwedge_{j=i+1}^{n-1} @_{\bar{x}} p_j \Leftrightarrow @_{\bar{y}} p_j \right)}_{\text{equality of more significant bits}} \right)$$

For the sake of completeness, we define $\text{first}(1) \stackrel{\text{def}}{=} \mathbf{AX}(\neg \text{val})$ and $\text{last}(1) \stackrel{\text{def}}{=} \mathbf{AX}(\text{val})$.

The lemma below states that we have properly proceeded for the binary encoding of numbers with the variables in p_{n-1}, \dots, p_0 (and Lemmas 3.4–3.5 need to be used).

Lemma 4.1. *Let \mathfrak{T} be a tree model and v be one of its nodes such that v satisfies $\widehat{\bar{x}, \bar{y}}$ (\bar{x} and \bar{y} are both of length k) and, v_k and v'_k are understood as nodes of type 0.*

- (I) : $\mathfrak{T}, v \models \text{nb}(\bar{y}) =_k \text{nb}(\bar{x}) + 1$ iff $\text{nb}_{\mathfrak{T}}(v'_k) = 1 + \text{nb}_{\mathfrak{T}}(v_k)$.
- (II) : $\mathfrak{T}, v \models \text{nb}(\bar{x}) <_k \text{nb}(\bar{y})$ iff $\text{nb}_{\mathfrak{T}}(v_k) < \text{nb}_{\mathfrak{T}}(v'_k)$.
- (III) : $\mathfrak{T}, v \models \text{nb}(\bar{x}) =_k \text{nb}(\bar{y})$ iff $\text{nb}_{\mathfrak{T}}(v_k) = \text{nb}_{\mathfrak{T}}(v'_k)$.

Proof. By careful inspection of the presented formulae, cf. Appendix B.1. □

We conclude by presenting the main lemma that gathers all established formulae.

Lemma 4.2. *Let \mathfrak{T} be a tree model and let v be any of its nodes. The following hold:*

- (I) : $\mathfrak{T}, v \models \text{type}(1)$ iff v is of type 1,
- (II) : Assuming v satisfies $\text{type}(1)$, we have $\mathfrak{T}, v \models \text{first}(1)$ iff $\text{nb}_{\mathfrak{T}}(v) = 0$.
- (III) : Assuming v satisfies $\text{type}(1)$, we have $\mathfrak{T}, v \models \text{last}(1)$ iff $\text{nb}_{\mathfrak{T}}(v) = t(2, n) - 1$.

Proof. The properties (II)–(III) follow immediately from the way we encode numbers. Check Appendix B.2 for a more detailed explanation. □

4.3. Formulae with arbitrary $N \geq 2$. Let us consider the arbitrary case $N \geq 2$. Below, we define the formulae $\text{type}(N)$, $\text{first}(N)$ and $\text{last}(N)$ as well as $\text{nb}(x_1, \dots, x_d) <_k \text{nb}(y_1, \dots, y_d)$, and $\text{nb}(y_1, \dots, y_d) =_k \text{nb}(x_1, \dots, x_d) + 1$ with $k-d = N-1$. We also consider the formula $\text{nb}(x_1, \dots, x_d) =_k \text{nb}(y_1, \dots, y_d)$, that is defined as follows:

$$\neg(\text{nb}(x_1, \dots, x_d) <_k \text{nb}(y_1, \dots, y_d)) \wedge \neg(\text{nb}(y_1, \dots, y_d) <_k \text{nb}(x_1, \dots, x_d)).$$

We assume that for all $k < N$, the formulae $\text{type}(k)$, $\text{last}(k)$ and $\text{first}(k)$ are already defined and for $k-d \leq N-2$, the formulae $\text{nb}(\bar{x}) <_k \text{nb}(\bar{y})$, $\text{nb}(\bar{x}) =_k \text{nb}(\bar{y})$ and $\text{nb}(\bar{y}) =_k \text{nb}(\bar{x}) + 1$ are already defined too (\bar{x} and \bar{y} are of length d). This can be understood as an implicit induction hypothesis when proving the correctness of the formulae built for N .

As for the case $N = 1$, the formula $\text{type}(N)$ follows the general schema: it states that there is a child with number equal to zero, if a child has number $m < \mathfrak{t}(N, n) - 1$, then there is a child with number equal to $m + 1$, and two distinct children have distinct numbers in $[0, \mathfrak{t}(N, n) - 1]$. Of course, all the children are enforced to be of type $N - 1$. We present the claimed formula $\text{type}(N)$ below.

$$\text{type}(N) \stackrel{\text{def}}{=} \mathbf{AX}(\text{type}(N-1)) \wedge \mathbf{EX}(\text{first}(N-1)) \wedge \text{uniq}(N) \wedge \text{compl}(N).$$

Again, it remains to specify what $\text{uniq}(N)$ and $\text{compl}(N)$ are. In order to define $\text{uniq}(N)$, we simply state that there are no two distinct children (of type $N - 1$) with the same number:

$$\text{uniq}(N) \stackrel{\text{def}}{=} \forall x, y \text{ diff-nom}(x, y, 1) \rightarrow \neg(\text{nb}(x) =_N \text{nb}(y)).$$

Again, the formula $\text{diff-nom}(x, y, 1)$ allows us to select two distinct children (of type $N - 1$). The formula $\text{compl}(N)$ below states that for each child v (of type $N - 1$) that is not the last one, there is also a child v' (of type $N - 1$ too) such that $\text{nb}_{\mathfrak{T}}(v') = \text{nb}_{\mathfrak{T}}(v) + 1$. Here it is:

$$\text{compl}(N) \stackrel{\text{def}}{=} \forall x (\text{nom}(x, 1) \wedge @_x^1(\neg \text{last}(N-1))) \rightarrow \exists y \text{ nom}(y, 1) \wedge \text{nb}(y) =_N \text{nb}(x) + 1.$$

It remains to define $\text{nb}(x) =_N \text{nb}(y)$ and $\text{nb}(y) =_N \text{nb}(x) + 1$ used respectively in $\text{uniq}(N)$ and in $\text{compl}(N)$. This time, this requires much lengthier developments, apart from using the properties of the formulae constructed for $N - 1$ and for smaller values (implicit induction hypothesis). Below, we treat the more general situation with $k - d = N - 1$, and $\text{nb}(x) =_N \text{nb}(y)$ and $\text{nb}(y) =_N \text{nb}(x) + 1$ are just particular instances with $k = N$ and $d = 1$. Thus, let $\bar{x} = x_1, \dots, x_d$ and $\bar{y} = y_1, \dots, y_d$.

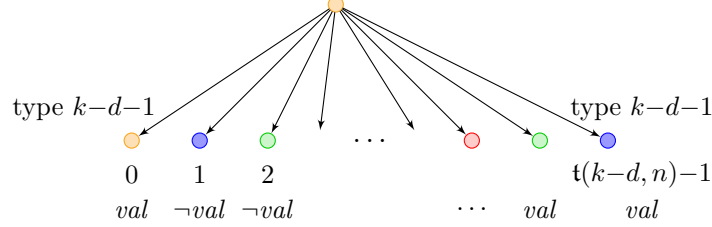
For defining $\text{nb}(y_1, \dots, y_k) =_k \text{nb}(x_1, \dots, x_k) + 1$ (see Section 4.2), we have compared the respective truth values of the propositional variables p_{n-1}, \dots, p_0 for the node v_k (the interpretation of x_k) and for the node v'_k (the interpretation of y_k). The same principle applied for defining $\text{nb}(x_1, \dots, x_k) =_k \text{nb}(y_1, \dots, y_k)$. Typically, $\text{nb}(y_1, \dots, y_k) =_k \text{nb}(x_1, \dots, x_k) + 1$ holds iff there is an index $i \in [0, n-1]$, such that

- for every $j \in [i+1, n-1]$, v_k and v'_k agree on p_j ,
- v_k does not satisfy p_i and v'_k satisfies p_i ,
- for $j \in [0, i-1]$, v_k satisfies p_j , v'_k does not satisfy p_j .

Thus, we needed to define a partition $\{[i+1, n-1], \{i\}, [0, i-1]\}$ of $[0, n-1]$ (understood as the set of bit numbers to encode a value in $[0, 2^n - 1]$). The same principle applies when the bit numbers are among $[0, \mathfrak{t}(k-d, n) - 1]$ to encode a value in $[0, \mathfrak{t}(k-d+1, n) - 1]$. This needs to be done concisely as we cannot go through all the $\mathfrak{t}(k-d, n)$ bit numbers because the whole reduction has to be of elementary complexity.

Now, when attempting to define $\text{nb}(y_1, \dots, y_d) =_k \text{nb}(x_1, \dots, x_d) + 1$, the nodes v_d and v'_d are of type $k - d > 0$ with $\mathfrak{t}(k-d, n)$ children each. The truth values of val on their

respective children determine precisely the numbers $\mathbf{nb}(v_d)$ and $\mathbf{nb}(v'_d)$. Below, we describe what v_d 's children look like.



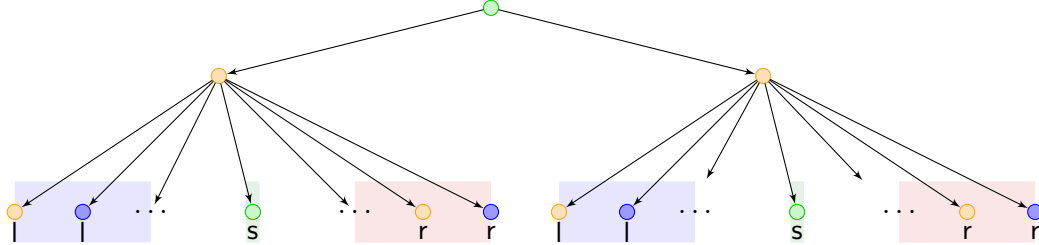
Let $u_{t(k-d, n)-1}, \dots, u_0$ be the children of v_d such that $\mathbf{nb}(u_j) = j$ for all j . Similarly, let $u'_{t(k-d, n)-1}, \dots, u'_0$ be the children of v'_d such that $\mathbf{nb}(u'_j) = j$ for all j . Hence, $\mathbf{nb}(y_1, \dots, y_d) =_k \mathbf{nb}(x_1, \dots, x_d) + 1$ holds iff there is a position $i \in [0, t(k-d, n) - 1]$ satisfying

- for $j \in [i+1, t(k-d, n) - 1]$, u_j and u'_j agree on *val*,
- u_i does not satisfy *val* and u'_i satisfies *val*,
- for every $j \in [0, i-1]$, u_j satisfies *val* and u'_j does not satisfy *val*.

We have to define a partition $\{u_{i+1}, \dots, u_{t(k-d, n)-1}\}, \{u_i\}, \{u_0, \dots, u_{i-1}\}$ of $\{u_{t(k-d, n)-1}, \dots, u_0\}$ (and similarly for the children of v'_d). To do so, we employ the existential quantification on the fresh propositional variables *l* (left), *s* (selected bit), *r* (right) such that

- (a) for every child of v_d (resp. v'_d), exactly one propositional variable among $\{l, s, r\}$ holds true,
- (b) exactly one child of v_d (resp. v'_d) satisfies *s*,
- (c) if v is a child of v_d satisfying *l* (resp. *s*) and v' is child of v_d satisfying *s* (resp. *r*), then $\mathbf{nb}_{\mathcal{T}}(v) < \mathbf{nb}_{\mathcal{T}}(v')$. The same condition holds with v'_d .

Below, we illustrate how the propositional variables *l*, *s* and *r* are distributed.



Additional arithmetical constraints are needed to relate the partition of \bar{x} with the partition of \bar{y} (see below the details) but in a way, it is independent of the partition itself. For instance, the unique child of v_d satisfying *s* and the unique child of v'_d satisfying *s* should have the same (bit) number. Nevertheless, it is clear that we need, at least, to be able to state in $\mathbf{QCTL}_{\mathbf{X}}^t$ the existence of a partition satisfying the conditions (a), (b) and (c). In the sequel, such partitions are called *lsr-partitions*. The forthcoming formula $\text{LSR}_{\bar{x}}(k)$ does the job for v_d (then use $\text{LSR}_{\bar{y}}(k)$ for v'_d).

Take $\bar{x} = x_1, \dots, x_d$. In the context of the definition of $\text{LSR}_{\bar{x}}(k)$, we allow the limit case $d = 0$, with empty sequence ε , assuming that $@_{\varepsilon}\psi \stackrel{\text{def}}{=} \psi$ and $\widehat{\varepsilon, \varepsilon} \stackrel{\text{def}}{=} \top$. Below $0 \leq d < k$ and the formula $\text{LSR}_{\bar{x}}(k)$ is defined as the conjunction $\text{LSR}_{\bar{x}}^1(k) \wedge \text{LSR}_{\bar{x}}^2(k) \wedge \text{LSR}_{\bar{x}}^3(k)$ and is interpreted on a node v_0 of type k satisfying $\widehat{\bar{x}}, \bar{x}$, and therefore this satisfaction is witnessed by the branch v_0, \dots, v_d (notations for developments below).

First, $\text{LSR}_{\bar{x}}^2(k) \stackrel{\text{def}}{=} @_{\bar{x}}(\mathbf{EX}_{=1}(\mathbf{s}))$, with $\mathbf{EX}_{=1}(\psi)$ defined as $\mathbf{EX}\psi \wedge \neg \exists p (\mathbf{EX}(\psi \wedge p) \wedge \mathbf{EX}(\psi \wedge \neg p))$ for a fresh p . Note that the formula $\text{LSR}_{\bar{x}}^2(k)$ simply states that there is a unique child of v_d satisfying \mathbf{s} . Next, let $\text{LSR}_{\bar{x}}^1(k)$ be defined below, stating that for every child of v_d , exactly one propositional variable among $\{\mathbf{l}, \mathbf{s}, \mathbf{r}\}$ holds true:

$$\text{LSR}_{\bar{x}}^1(k) \stackrel{\text{def}}{=} @_{\bar{x}}(\mathbf{AX}((\mathbf{s} \vee \mathbf{l} \vee \mathbf{r}) \wedge \neg(\mathbf{s} \wedge \mathbf{l}) \wedge \neg(\mathbf{s} \wedge \mathbf{r}) \wedge \neg(\mathbf{l} \wedge \mathbf{r}))).$$

Finally, $\text{LSR}_{\bar{x}}^3(k)$ is defined as follows.

$$@_{\bar{x}}(\forall w \forall w' \text{ diff-nom}(w, w', 1) \wedge ((@_w^1(\mathbf{s}) \wedge @_{w'}^1(\mathbf{r})) \vee (@_w^1(\mathbf{l}) \wedge @_{w'}^1(\mathbf{s}))) \rightarrow \mathbf{nb}(w') <_{k-d} \mathbf{nb}(w)).$$

The formula $\text{LSR}_{\bar{x}}^3(k)$ states if v is a child of v_d satisfying \mathbf{l} (resp. \mathbf{s}) and v' is another child of v_d satisfying \mathbf{s} (resp. \mathbf{r}), then $\mathbf{nb}_{\bar{x}}(v) < \mathbf{nb}_{\bar{x}}(v')$. The nodes v and v' are obviously of type $k-d-1$ and their respective numbers belong to $[0, \mathbf{t}(k-d, n)-1]$. It is important to observe that $\mathbf{nb}(w') <_{k-d} \mathbf{nb}(w)$ is well-defined recursively as soon as $k-d \leq N-2$.

Lemma 4.3. *Let \mathfrak{T} be a tree model, v_0 be a node of type $k \geq 0$, and \bar{x} be a (possibly empty) sequence of nominals x_1, \dots, x_d for some $d \in [0, k-1]$ such that $k-d \leq N-1$, $\mathfrak{T}, v_0 \models \bar{x}, \bar{x}$, and its witness branch is v_0, \dots, v_d . Then $\mathfrak{T}, v_0 \models \text{LSR}_{\bar{x}}(k)$ iff the conditions below hold:*

- (a) *For every child of v_d , exactly one propositional variable among $\{\mathbf{l}, \mathbf{s}, \mathbf{r}\}$ holds true.*
- (b) *Exactly one child of v_d satisfies \mathbf{s} .*
- (c) *If v is a child of v_d satisfying \mathbf{l} (resp. \mathbf{s}) and v' is a child of v_d satisfying \mathbf{s} (resp. \mathbf{r}), then $\mathbf{nb}_{\bar{x}}(v) < \mathbf{nb}_{\bar{x}}(v')$.*

Proof. By careful inspection of the presented formulae, cf. Appendix B.3. \square

We come back to the question of defining formulae expressing number comparisons. The formula $\mathbf{nb}(\bar{y}) =_k \mathbf{nb}(\bar{x}) + 1$ (remember $k-d = N-1$) is defined as the expression

$$\exists \mathbf{l}, \mathbf{s}, \mathbf{r} \text{ LSR}_{\bar{x}}(k) \wedge \text{LSR}_{\bar{y}}(k) \wedge \phi_{\text{left}}(k) \wedge \phi_{\text{select}}(k) \wedge \phi_{\text{right}}(k).$$

The conjunction $\phi_{\text{left}}(k) \wedge \phi_{\text{select}}(k) \wedge \phi_{\text{right}}(k)$ takes care of the arithmetical constraints. The formula $\phi_{\text{select}}(k)$ states that the v_d 's unique child satisfying \mathbf{s} (whose number is the pivot bit i) does not satisfy val , and the v'_d 's unique child satisfying \mathbf{s} (whose number is also the pivot bit i) satisfies val :

$$\phi_{\text{select}}(k) \stackrel{\text{def}}{=} @_{\bar{x}}(\mathbf{AX}(\mathbf{s} \rightarrow \neg \text{val})) \wedge @_{\bar{y}}(\mathbf{AX}(\mathbf{s} \rightarrow \text{val})).$$

The formula $\phi_{\text{right}}(k)$ states that for all the children of v_d satisfying \mathbf{r} (and therefore with bit number strictly smaller than i), the bit value is 1, and for all the children of v'_d satisfying \mathbf{r} (and therefore with bit number strictly smaller than i), the bit value is 0.

$$\phi_{\text{right}}(k) \stackrel{\text{def}}{=} @_{\bar{x}}(\mathbf{AX}(\mathbf{r} \rightarrow \text{val})) \wedge @_{\bar{y}}(\mathbf{AX}(\mathbf{r} \rightarrow \neg \text{val})).$$

The formula $\phi_{\text{left}}(k)$ states that the children of v_d satisfying \mathbf{l} induce a set of bit numbers equal to the set of bit numbers induced by the children of v'_d satisfying \mathbf{l} . This entails also that the unique respective children of v_d and v'_d satisfying \mathbf{s} have the same (bit) number. Moreover, we require that children with the same bit number satisfying \mathbf{l} (taken from v_d and from v'_d) have the same bit value witnessed by the truth value of val . The formula $\phi_{\text{left}}(k)$ is equal to $\phi_{\text{left}}^{\bar{x}, \bar{y}}(k) \wedge \phi_{\text{left}}^{\bar{y}, \bar{x}}(k)$ with $\phi_{\text{left}}^{\bar{x}, \bar{y}}(k)$ defined below:

$$\begin{aligned} \phi_{\text{left}}^{\bar{x}, \bar{y}}(k) &\stackrel{\text{def}}{=} \forall w @_{\bar{x}}(\text{nom}(w, 1) \wedge @_w^1(\mathbf{l})) \rightarrow \\ &(\exists w' @_{\bar{y}}(\text{nom}(w', 1) \wedge @_{w'}^1(\mathbf{l})) \wedge \mathbf{nb}(\bar{x}, w) =_k \mathbf{nb}(\bar{y}, w') \wedge (@_{\bar{x}, w} \text{val} \leftrightarrow @_{\bar{y}, w'} \text{val})). \end{aligned}$$

Note that $\mathbf{nb}(\bar{x}, w) =_k \mathbf{nb}(\bar{y}, w')$ is well-defined as $k - (d + 1) \leq N - 2$. Below, we define the formula $\mathbf{nb}(\bar{x}) <_k \mathbf{nb}(\bar{y})$ with $k - d = N - 1$, $\bar{x} = x_1, \dots, x_d$ and $\bar{y} = y_1, \dots, y_d$. Based on previous developments and on standard arithmetical properties of numbers encoded in binary with $k - d$ bits, we define the formula $\mathbf{nb}(\bar{x}) <_k \mathbf{nb}(\bar{y})$ as the expression

$$\mathbf{nb}(\bar{x}) <_k \mathbf{nb}(\bar{y}) \stackrel{\text{def}}{=} \exists s, l, r \text{ LSR}_{\bar{x}}(k) \wedge \text{LSR}_{\bar{y}}(k) \wedge \phi_{\text{left}}(k) \wedge \phi_{\text{select}}(k).$$

As previously, the formula $\mathbf{nb}(\bar{x}) =_k \mathbf{nb}(\bar{y})$ is defined as follows:

$$\mathbf{nb}(\bar{x}) =_k \mathbf{nb}(\bar{y}) \stackrel{\text{def}}{=} \neg(\mathbf{nb}(\bar{x}) <_k \mathbf{nb}(\bar{y})) \wedge \neg(\mathbf{nb}(\bar{y}) <_k \mathbf{nb}(\bar{x})).$$

Lemma 4.4. *Let \mathfrak{T} be a tree model, v be a node satisfying $\widehat{\bar{x}}, \widehat{\bar{y}}$ and, v_d, v'_d are of type $k - d$.*

(I) : *We have $\mathfrak{T}, v \models \mathbf{nb}(\bar{y}) =_k \mathbf{nb}(\bar{x}) + 1$ iff $\mathbf{nb}_{\mathfrak{T}}(v'_d) = 1 + \mathbf{nb}_{\mathfrak{T}}(v_d)$.*

(II) : *We have $\mathfrak{T}, v \models \mathbf{nb}(\bar{x}) <_k \mathbf{nb}(\bar{y})$ iff $\mathbf{nb}_{\mathfrak{T}}(v_d) < \mathbf{nb}_{\mathfrak{T}}(v'_d)$.*

(III) : *We have $\mathfrak{T}, v \models \mathbf{nb}(\bar{x}) =_k \mathbf{nb}(\bar{y})$ iff $\mathbf{nb}_{\mathfrak{T}}(v_d) = \mathbf{nb}_{\mathfrak{T}}(v'_d)$.*

Proof. By careful inspection of the presented formulae, cf. Appendix B.4. \square

Mainly, this allows us to prove Lemma 4.5(I).

Lemma 4.5. *Let $N \geq 2$, \mathfrak{T} be a tree model and v be one of its nodes.*

(I) : *$\mathfrak{T}, v \models \text{type}(N)$ iff v is of type N ,*

(II) : *Assuming that v satisfies $\text{type}(N)$, we have $\mathfrak{T}, v \models \text{first}(N)$ iff $\mathbf{nb}_{\mathfrak{T}}(v) = 0$.*

(III) : *Assuming that v satisfies $\text{type}(N)$, we have $\mathfrak{T}, v \models \text{last}(N)$ iff $\mathbf{nb}_{\mathfrak{T}}(v) = \mathfrak{t}(N + 1, n) - 1$.*

Proof. Similar to the proof of Lemma 4.2, see Appendix B.5 for more details. \square

Consequently, for all $k \geq 0$, $\text{type}(k)$, $\text{first}(k)$ and $\text{last}(k)$ characterise exactly the discussed properties, and similarly for the formulae of form $\mathbf{nb}(\bar{y}) =_k \mathbf{nb}(\bar{x}) + 1$, $\mathbf{nb}(\bar{x}) <_k \mathbf{nb}(\bar{y})$, $\mathbf{nb}(\bar{x}) =_k \mathbf{nb}(\bar{y})$ where \bar{x} and \bar{y} are of length d in $[1, k]$ and $k \geq 1$. It is natural to wonder what is the size of $\text{type}(k)$, using a reasonably succinct encoding for formulae. As the definition of $\text{type}(k)$ requires the subformulae $\text{type}(k - 1)$, $\mathbf{nb}(\bar{x}) =_k \mathbf{nb}(\bar{y})$ and $\mathbf{nb}(\bar{y}) =_k \mathbf{nb}(\bar{x}) + 1$ (the other subformulae are of constant size), and the formula $\text{type}(1)$ is quadratic in n , one can show that $\text{type}(k)$ is of size $2^{\mathcal{O}(k+n)}$. This is sufficient for our purposes.

4.4. Uniform reduction leading to Tower-hardness. Let (\mathcal{P}, c) , where $\mathcal{P} = (\mathcal{T}, \mathcal{H}, \mathcal{V})$ and $c = t_0, t_1, \dots, t_{n-1}$ be an instance of Tiling_k known to be $k\text{-NEXPTime}$ -complete (see also [20, Chapter 11]). We reduce the existence of a tiling $\tau : [0, \mathfrak{t}(k, n) - 1] \times [0, \mathfrak{t}(k, n) - 1] \rightarrow \mathcal{T}$, respecting the initial condition and the horizontal and vertical matching conditions, to the satisfiability of a formula ϕ in $\text{QCTL}_{\mathbf{X}}^t$.

To encode the grid $[0, \mathfrak{t}(k, n) - 1] \times [0, \mathfrak{t}(k, n) - 1]$, we consider a root node ε of type $k + 1$, and we distinguish $\mathfrak{t}(k, n)$ children among all of its $\mathfrak{t}(k + 1, n)$ children. Each child of ε has itself exactly $\mathfrak{t}(k, n)$ children as it is a node of type k . In order to identify the $\mathfrak{t}(k, n)$ first children of ε , we use an lsr -partition so that the unique child satisfying s has precisely the number $\mathfrak{t}(k, n)$. This guarantees that exactly the children of ε whose numbers are in $[0, \mathfrak{t}(k, n) - 1]$ satisfy r . So, the lsr -partition is used in a new context.

Below, we define the new formula $\mathbf{nb} =_k \mathfrak{t}(k, n)$ that expresses that a node of type k has number $\mathfrak{t}(k, n)$ ($k \geq 1$). We recall that a node of type k takes its values in $[0, \mathfrak{t}(k + 1, n) - 1]$. Let us provide an inductive definition for $\mathbf{nb} =_k \mathfrak{t}(k, n)$ with the base case $k = 1$.

- $\mathbf{nb} =_0 \mathfrak{t}(0, n) \stackrel{\text{def}}{=} \neg p_{n-1} \wedge \dots \wedge \neg p_0$.

- $\mathbf{nb} =_1 \mathbf{t}(1, n) \stackrel{\text{def}}{=} \mathbf{AX}(val \leftrightarrow X_n = n)$, where $X_n = n$ is an abbreviation for the formula stating that the truth values for p_{n-1}, \dots, p_0 encode n in binary.
- For $k \geq 2$, $(\mathbf{nb} =_k \mathbf{t}(k, n)) \stackrel{\text{def}}{=} \mathbf{AX}(val \leftrightarrow (\mathbf{nb} =_{k-1} \mathbf{t}(k-1, n)))$.

Lemma 4.6. *Assume that $\mathfrak{T}, v \models \text{type}(k)$. Then $\mathbf{nb}_{\mathfrak{T}}(v) = \mathbf{t}(k, n)$ iff $\mathfrak{T}, v \models (\mathbf{nb} =_k \mathbf{t}(k, n))$.*

Proof. By careful inspection of the semantics, cf. Appendix B.6. \square

Let $\phi_{\mathcal{P}}$ be the formula built from the instance $(\mathcal{P} = (\mathcal{T}, \mathcal{H}, \mathcal{V}), c)$ as follows:

$$\text{type}(k+1) \wedge \exists l, s, r \text{ LSR}_{\varepsilon}(k+1) \wedge \mathbf{EX}(s \wedge (\mathbf{nb} =_k \mathbf{t}(k, n))) \wedge \phi_{\text{cov}} \wedge \phi_{\text{init}} \wedge \phi_{\mathcal{H}} \wedge \phi_{\mathcal{V}}.$$

Definitions and explanations for ϕ_{cov} , ϕ_{init} , $\phi_{\mathcal{H}}$ and $\phi_{\mathcal{V}}$ follow but observe that an lsr -partition is performed for a node of type $k+1$ and exactly $\mathbf{t}(k, n)$ children satisfy r thanks to the satisfaction of the subformula $\mathbf{EX}(s \wedge (\mathbf{nb} =_k \mathbf{t}(k, n)))$. The formula ϕ_{cov} states that every position in $[0, \mathbf{t}(k, n) - 1] \times [0, \mathbf{t}(k, n) - 1]$ has a unique tile:

$$\forall x, y \text{ nom}(x, 1) \wedge @_{x,1}^1 r \wedge @_{x,1}^1 \text{nom}(y, 1) \rightarrow @_{x,y}(\bigvee_{t \in \mathcal{T}} t \wedge \bigwedge_{t' \neq t \in \mathcal{T}} \neg(t \wedge t')).$$

The tile types in \mathcal{T} are understood as propositional variables. In order to access the root node ε to a node encoding a position of the grid, one needs first to access a child v of ε satisfying r (and this is done with the help of the local nominal x) and then to access any child v' of v (done with the local nominal y). Then, to reason propositionally on v' , it is sufficient to consider subformulae of the form $@_{x,y}\psi$. This principle is applied to all the formulae below. The formula $\phi_{\mathcal{H}}$ defined below encodes the horizontal matching constraints.

$$\begin{aligned} \forall x, x', y, y' (\text{nom}(x, 1) \wedge @_{x,1}^1 r \wedge \text{nom}(x', 1) \wedge @_{x',1}^1 r \wedge @_{x,1}^1 \text{nom}(y, 1) \wedge @_{x',1}^1 \text{nom}(y', 1) \wedge \\ \mathbf{nb}(x') =_{k+1} \mathbf{nb}(x) + 1 \wedge \mathbf{nb}(x, y) =_{k+1} \mathbf{nb}(x, y')) \rightarrow \bigvee_{(t,t') \in \mathcal{H}} @_{x,y} t \wedge @_{x',y'} t' \end{aligned}$$

Similarly, the following formula $\phi_{\mathcal{V}}$ encodes the vertical matching constraints.

$$\begin{aligned} \forall x, x', y, y' \text{ nom}(x, 1) \wedge @_{x,1}^1 r \wedge \text{nom}(x', 1) \wedge @_{x',1}^1 r \wedge @_{x,1}^1 \text{nom}(y, 1) \wedge @_{x',1}^1 \text{nom}(y', 1) \wedge \\ \mathbf{nb}(x) =_{k+1} \mathbf{nb}(x') \wedge \mathbf{nb}(x, y') =_{k+1} \mathbf{nb}(x, y) + 1 \rightarrow \bigvee_{(t,t') \in \mathcal{V}} @_{x,y} t \wedge @_{x',y'} t' \end{aligned}$$

It remains to express the initial conditions. It is sufficient to identify the n first children of the first child of ε (identified by the satisfaction of $\text{first}(k)$). For example, to express that the j th child of the first child of ε (say v is this first child of ε) satisfies t_j , perform an lsr -partition on v , enforce that the unique child satisfying s also satisfies t_j and express that there are exactly $j-1$ children of v satisfying r . This is a condition from graded modal logic that is easy to express. Let $\mathbf{EX}_{=i} \psi$ be the formula below stating that exactly $i \geq 1$ children satisfy ψ :

$$\exists q_1, \dots, q_i \text{ diff-nom}(q_1, \dots, q_i, 1) \wedge \mathbf{AX}((q_1 \vee \dots \vee q_i) \leftrightarrow \psi),$$

where q_1, \dots, q_i are fresh propositional variables. By convention $\mathbf{EX}_{=0} \psi$ is defined as $\mathbf{AX}\neg\psi$. The formula ϕ_{init} is defined as

$$\forall x (\text{nom}(x, 1) \wedge @_{x,1}^1 (\text{first}(k))) \rightarrow @_{x,1}^1 (\bigwedge_{i \in [0, n-1]} \exists l, s, r \text{ LSR}_{\varepsilon}(k) \wedge \mathbf{EX}_{=i} r \wedge \mathbf{EX}(s \wedge t_i)).$$

The correctness of the reduction is stated below.

Lemma 4.7. $\mathcal{P} = (\mathcal{T}, \mathcal{H}, \mathcal{V})$, $c = t_0, t_1, \dots, t_{n-1}$ is a positive instance of Tiling_k iff $\phi_{\mathcal{P}}$ is satisfiable in $\text{QCTL}_{\mathbf{X}}^t$.

Proof. Given an instance $\mathcal{P} = (\mathcal{T}, \mathcal{H}, \mathcal{V})$, $c = t_0, t_1, \dots, t_{n-1}$ of the tiling problem Tiling_k , let $\tau : [0, t(k, n) - 1] \times [0, t(k, n) - 1] \rightarrow \mathcal{T}$ be a tiling respecting all the constraints (init), (hori) and (verti). Below, we build a tree model $\mathfrak{T} = \langle V, E, l \rangle$ such that $\mathfrak{T}, \varepsilon \models \phi_{\mathcal{P}}$ where ε is the root node of \mathfrak{T} . Let V be the following subset of \mathbb{N}^* (set of finite sequences over \mathbb{N}):

- ε is the empty string and it belongs to V ,
- For all $j \in [1, k + 1]$, $[0, t(k + 1, n) - 1] \times \dots \times [0, t(j, n) - 1] \subseteq V$.
- $[0, t(k + 1, n) - 1] \times \dots \times [0, t(1, n) - 1] \times 0^+ \subseteq V$.

The binary relation E is simply defined as: $vEv' \stackrel{\text{def}}{\iff} v$ is a prefix of v' , and $v \cdot \alpha = v'$ for some $\alpha \in \mathbb{N}$. So, (V, E) is a finite-branching tree such that all the maximal branches are infinite. The labelling map l is defined in a way so that $\mathfrak{T}, \varepsilon \models \text{type}(k + 1)$. For instance, any node $m_{k+1}, \dots, m_2 \in V \cap \mathbb{N}^{k-1}$ has 2^n children, and their numbers should span over $[0, 2^n - 1]$. This is easy to realise by setting properly the truth values for p_{n-1}, \dots, p_0 . Similarly, any node $m_{k+1}, \dots, m_3 \in V \cap \mathbb{N}^{k-2}$ has $t(2, n)$ children, and their numbers should span over $[0, t(2, n) - 1]$. Again, this is easy to realise by setting properly the truth values of val on the nodes in $[0, t(k + 1, n) - 1] \times \dots \times [0, t(1, n) - 1]$. So, it remains to take care of the propositional variables dedicated to the tile types.

- For all $(i, j) \in [0, t(k, n) - 1] \times [0, t(k, n) - 1]$, $l((i, j)) \cap \mathcal{T}$ is equal to $\{\tau(i, j)\}$ by definition. In particular, this means that for all $(i, j) \in [0, t(k, n) - 1] \times [0, t(k, n) - 1]$, there is exactly one tile type (understood as a propositional variable) satisfied by (i, j) .
- For all $v \in V \setminus [0, t(k, n) - 1] \times [0, t(k, n) - 1]$, the value of the set $l(v) \cap \mathcal{T}$ is irrelevant.

It remains to check that

$$\mathfrak{T}, \varepsilon \models \text{type}(k + 1) \wedge \exists l, s, r \text{LSR}_{\varepsilon}(k + 1) \wedge \mathbf{EX}(s \wedge \mathbf{nb} =_k t(k, n)) \wedge \phi_{\text{cov}} \wedge \phi_{\text{init}} \wedge \phi_{\mathcal{H}} \wedge \phi_{\mathcal{V}},$$

which is routine and done in Appendix B.7. \square

We are ready to conclude the main theorem of this paper.

Theorem 4.8. $\text{SAT}(\text{QCTL}_{\mathbf{X}}^t)$ is TOWER-complete.

Theorem 4.8 significantly improves the TOWER lower bound from [37, Cor. 5.6] by considering as only temporal operator, the (local) modality \mathbf{EX} . TOWER-hardness can be also obtained with arbitrary countable trees. In Section 5 below, we show that this entails more TOWER-hardness results for other fragments of $\text{QCTL}_{\mathbf{X}}^t$ and for modal logics with propositional quantification under appropriate tree semantics.

5. A HARVEST OF TOWER-COMPLETE MODAL AND TEMPORAL LOGICS

We capitalise on the TOWER-hardness of the satisfiability problem for $\text{QCTL}_{\mathbf{X}}^t$, by showing TOWER-hardness of other fragments of QCTL^t that involve only \mathbf{EF} or its strict variant \mathbf{EXEF} (Section 5.1). TOWER-hardness is obtained by reduction from $\text{SAT}(\text{QCTL}_{\mathbf{X}}^t)$ by introducing propositional variables that enforce layers from the root in the tree model and therefore this allows us to simulate \mathbf{EX} . In Section 5.2, we consider well-known modal logics that are complete for classes of tree-like Kripke structures, and we show that their extension with propositional quantification for such classes of tree-like Kripke structures is decidable in TOWER, but more importantly TOWER-hard. Some of such classes involve finite trees

and therefore, we also take the opportunity to study $\text{QCTL}_{\mathbf{X}}^{ft}$ and $\text{QCTL}_{\mathbf{XF}}^{ft}$ that happen, for instance, to be closely related to the modal logics QK^t and QGL^t , respectively.

5.1. The satisfiability problems for $\text{QCTL}_{\mathbf{F}}^t$ and $\text{QCTL}_{\mathbf{XF}}^t$ are Tower-hard! The fragment $\text{QCTL}_{\mathbf{F}}^t$ of QCTL^t is defined according to the following grammar

$$\phi ::= p \mid \neg\phi \mid \phi \wedge \phi \mid \mathbf{EF}\phi \mid \exists p \phi.$$

We recall the standard semantics for \mathbf{EF} -formulae: $\mathfrak{T}, v \models \mathbf{EF}\phi \stackrel{\text{def}}{\iff}$ there is $j \geq 0$ such that $vE^j v'$ and $\mathfrak{T}, v' \models \phi$, and as usual, $\mathbf{AG}\phi \stackrel{\text{def}}{=} \neg\mathbf{EF}\neg\phi$.

In order to show that $\text{SAT}(\text{QCTL}_{\mathbf{F}}^t)$ is TOWER-hard, we design a logarithmic-space many-one reduction from $\text{SAT}(\text{QCTL}_{\mathbf{X}}^t)$. A more sophisticated analysis is also possible to establish TOWER-hardness for even smaller fragments, see the recent work [39].

Let ϕ be a formula in $\text{QCTL}_{\mathbf{X}}^t$ with modal depth $md(\phi) = k$. Without loss of generality, we assume that ϕ may contain occurrences of \mathbf{EX} and no occurrences of \mathbf{AX} . Let us define the formula $\phi' = \text{trans}(k, \phi) \wedge \text{shape}(k)$ in $\text{QCTL}_{\mathbf{F}}^t$, where the formula $\text{shape}(k)$ enforces a discipline for layers (explained below) and $\text{trans}(k, \phi)$ admits a recursive definition, by relativising the occurrences of \mathbf{EX} . We consider the set of propositional variables $Y_k = \{\text{layer}_{-1}, \text{layer}_0, \dots, \text{layer}_k\}$ with the intended meaning that a node satisfying layer_i is of “layer i ”, the root node being of layer k . Indeed, there is a need for such propositional variables, as unlike with the formulae in $\text{QCTL}_{\mathbf{X}}^t$, we have to enforce that moving with \mathbf{EF} leads to a lower layer.

The formula $\text{shape}(k)$ is defined as the conjunction of the following formulae.

- Every node satisfies exactly one propositional variable from Y_k (layer unicity):

$$\mathbf{AG} \left((\text{layer}_{-1} \vee \text{layer}_0 \vee \dots \vee \text{layer}_k) \wedge \bigwedge_{-1 \leq i \neq j \leq k} \neg(\text{layer}_i \wedge \text{layer}_j) \right)$$

- When a node satisfies layer_i with $-1 \leq i \leq k$, none of its descendants satisfies layer_j with $j > i$ (monotonicity of layer numbers):

$$\bigwedge_{-1 \leq i \leq k} \mathbf{AG}(\text{layer}_i \rightarrow \mathbf{AG}(\text{layer}_{-1} \vee \text{layer}_0 \vee \dots \vee \text{layer}_i))$$

- When a node satisfies layer_i with $0 \leq i \leq k$, there is a descendant satisfying layer_{i-1} (weak progress):

$$\bigwedge_{0 \leq i \leq k} \mathbf{AG}(\text{layer}_i \rightarrow \mathbf{EF} \text{layer}_{i-1})$$

- When a node satisfies layer_i with $0 \leq i \leq k$, it has no (strict) descendant satisfying layer_i (no stuttering). This type of constraints does not apply to layer_{-1} .

$$\bigwedge_{0 \leq i \leq k} \mathbf{AG}(\text{layer}_i \rightarrow \neg \exists p (p \wedge \mathbf{EF}(\text{layer}_i \wedge \neg p)))$$

- The root node is at layer k : layer_k .

A tree model $\mathfrak{T} = \langle V, E, l \rangle$ for QCTL^t with root ε is k -layered iff the conditions below hold:

- For every node $v \in V$, $\text{card}(l(v) \cap Y_k) = 1$.
- For all $v \in V$ such that $\text{layer}_j \in l(v)$ for some $j \in [-1, k]$,
 - if $j \geq 0$, then there is v' such that vEv' and $\text{layer}_{j-1} \in l(v')$ and,

- for all v' such that vE^+v' and $layer_{j'} \in l(v')$, we have $j' \leq j$.
- (c) For all $j \in [0, k]$, there are no distinct nodes v and v' such that vE^+v' and $layer_j \in l(v) \cap l(v')$.
- (d) $layer_k \in l(\varepsilon)$, where ε is the root of the tree model.

This means that the only propositional variable from Y_k satisfied by a node reachable in $j \in [1, k]$ steps from ε is $layer_m$ for some $m \leq k - j$, and the only propositional variable from Y_k satisfied by a node reachable in strictly more than k steps from ε is $layer_{-1}$. Moreover, once $layer_{-1}$ holds true, it holds for all its descendants. Actually, the formula $shape(k)$ characterises k -layered structures.

Lemma 5.1. *Let $\mathfrak{T} = \langle V, E, l \rangle$ be a tree model for QCTL^t with the root node ε . We have $\mathfrak{T}, \varepsilon \models shape(k)$ holds if and only if \mathfrak{T} is k -layered.*

Proof. By careful inspection of the semantics, cf. Appendix C.1. □

To define $trans(k, \phi)$, we define inductively $trans(i, \psi)$ where ψ is a subformula of ϕ and $md(\psi) \leq i$.

- $trans(i, p) \stackrel{\text{def}}{=} p$ for all propositional variables p ,
- $trans$ is homomorphic for Boolean connectives and $trans(i, \exists p \psi) \stackrel{\text{def}}{=} \exists p \ trans(i, \psi)$,
- $trans(i, \mathbf{EX}\psi) \stackrel{\text{def}}{=} \mathbf{EF}(layer_{i-1} \wedge trans(i-1, \psi))$.

Note that $trans(k, \phi)$ has no occurrence of $layer_{-1}$ since $md(\phi) = k$ and that translating an \mathbf{EX} -formula decreases the index of the layer by exactly one. The correctness of the reduction can be now stated as follows.

Lemma 5.2. *A formula ϕ is satisfiable for $\text{QCTL}_{\mathbf{X}}^t$ iff $trans(k, \phi) \wedge shape(k)$ is satisfiable for $\text{QCTL}_{\mathbf{F}}^t$.*

Proof. By induction. Consult Appendix C.2. □

Hence we conclude yet another important result.

Theorem 5.3. *The satisfiability problem for $\text{QCTL}_{\mathbf{F}}^t$ is TOWER-complete.*

The TOWER upper bound is established for the full logic QCTL^t in [37] and in particular for $\text{QCTL}_{\mathbf{F}}^t$. Theorem 5.3 also admits a variant in which we only allow to move to proper descendants. It amounts to replacing \mathbf{EF} by \mathbf{EXEF} (treated here as a single modality) in $\text{QCTL}_{\mathbf{F}}^t$, leading to the variant $\text{QCTL}_{\mathbf{XF}}^t$, with formulae obtained from

$$\phi ::= p \mid \neg\phi \mid \phi \wedge \phi \mid \mathbf{EXEF}\phi \mid \exists p \phi$$

As usual, we write $\mathbf{AXAG} \psi$ as an abbreviation of $\neg\mathbf{EXEF}\neg\psi$.

Theorem 5.4. *The satisfiability problem for $\text{QCTL}_{\mathbf{XF}}^t$ is TOWER-complete.*

As above, the TOWER upper bound for $\text{SAT}(\text{QCTL}_{\mathbf{XF}}^t)$ is inherited from $\text{SAT}(\text{QCTL}^t)$ [37]. Note that in QCTL^t , the formula $\mathbf{EF}p$ is logically equivalent to $p \vee \mathbf{EXEF}p$. Thus, we get a (possibly exponential, which is sufficient for us) reduction from $\text{SAT}(\text{QCTL}_{\mathbf{F}}^t)$ to $\text{SAT}(\text{QCTL}_{\mathbf{XF}}^t)$, whence the TOWER-hardness of $\text{SAT}(\text{QCTL}_{\mathbf{XF}}^t)$. Recall that TOWER-hardness is defined with respect to elementary reductions and therefore an exponential-time reduction is fine to establish TOWER-hardness, see e.g. [50] and Section 2.3.

5.2. Modal logics with propositional quantification on trees. Numerous well-known modal logics are complete (a.k.a. determined) for classes of tree-like structures. A modal logic \mathcal{L} (defined from the Hilbert-style system $\mathcal{H}\mathcal{L}$) is complete for a class of Kripke structures \mathcal{C} iff the theoremhood of ϕ in $\mathcal{H}\mathcal{L}$ is equivalent to the validity of ϕ in \mathcal{C} (i.e. for all $\mathcal{K} \in \mathcal{C}$, for all w , we have $\mathcal{K}, w \models \phi$). For instance, the (propositional) modal logic **K** is complete for the class of finite trees [52, 12]. It is worth noting that a given modal logic can be complete for different classes of Kripke models (*e.g.* **K** is complete for the class of all the Kripke models, but also complete for the class of finite Kripke models) and their extension to propositional quantification may lead to distinct logics. Typically, **K** with propositional quantification under the structure semantics is undecidable [25] whereas it is shown below to be TOWER-complete under the finite tree semantics.

Below, for the propositional modal logics \mathcal{L} in **K**, **KD**, **GL**, **K4** and **S4** we define an extension $\mathbf{Q}\mathcal{L}^t$ with propositional quantification under a class of tree-like models that is complete for the logic \mathcal{L} .

In order to avoid too many notations, the modalities for each logic $\mathbf{Q}\mathcal{L}^t$ are **EX** and **AX** (instead of the more standard modal operators \Diamond and \Box) and therefore $\mathbf{Q}\mathcal{L}^t$ formulae are built from the grammar below: $\phi ::= p \mid \neg\phi \mid \phi \wedge \phi \mid \mathbf{EX}\phi \mid \mathbf{AX}\phi \mid \exists p \phi$.

- The propositional modal logic **K** is complete for the class of finite trees and we define \mathbf{QK}^t as the modal logic with propositional quantification over the class of finite trees.
- The propositional modal logic **KD** (**K** with seriality, a.k.a. totality) is known to be complete for the class of finite-branching trees for which all the maximal branches are infinite. Indeed, **KD** is complete for the class of finite total Kripke models. Therefore by using the unfolding construction, completeness applies also for the class of finite-branching trees for which all the maximal branches are infinite, *i.e.* the models for \mathbf{QCTL}^t . Let \mathbf{QKD}^t be the modal logic with propositional quantification over the class of finite-branching trees for which all the maximal branches are infinite. The satisfiability problem for \mathbf{QKD}^t is exactly the problem for $\mathbf{QCTL}_{\mathbf{X}}^t$.
- The modal logic **GL** is known to be complete for the class of finite transitive trees (**GL** is complete with respect to finite irreflexive transitive Kripke models [55]), *i.e.* the class of Kripke structures $\langle V, E^+, l \rangle$ such that $\langle V, E, l \rangle$ is a finite tree model, see *e.g.* [12]. Let \mathbf{QGL}^t be the modal logic with propositional quantification over the class of finite transitive trees, which is precisely $\mathbf{QCTL}_{\mathbf{XF}}^{ft}$. It is worth noting that adding propositional quantification to **GL** is studied in [4], where a fragment is shown to be decidable by translation into the weak monadic second-order logic of one successor **WS1S** [15].
- The modal logic **K4** is complete for the class of Kripke structures $\langle V, E^+, l \rangle$ such that $\langle V, E, l \rangle$ is a finite-branching tree model (some branches may be infinite, some others not). Let $\mathbf{QK4}^t$ be the modal logic with propositional quantification over the class of finite-branching trees.
- The modal logic **S4** is complete for the class of finite Kripke structures such that the accessibility relation is reflexive and transitive, and therefore complete for the class of structures $\langle V, E^*, l \rangle$ such that $\langle V, E, l \rangle$ is a finite-branching tree model in which all the branches are infinite (by unfolding). Let $\mathbf{QS4}^t$ be the modal logic with propositional quantification over the class of finite-branching trees in which all the branches are infinite (precisely the class of models for \mathbf{QCTL}^t). The satisfiability problem for $\mathbf{QS4}^t$ happens to be exactly the problem for $\mathbf{QCTL}_{\mathbf{F}}^t$, modulo the fact that **EX** in $\mathbf{QS4}^t$ corresponds to **EF** in $\mathbf{QCTL}_{\mathbf{F}}^t$.

In this section, we show that the satisfiability problem for the logics QK^t , QKD^t , QGL^t , QK4^t and QS4^t and whose models are tree-like Kripke structures is TOWER-complete. For instance, QK^t corresponds to the modal logic K interpreted on finite trees with propositional quantification, which is precisely $\text{QCTL}_{\mathbf{X}}^{ft}$, i.e. QCTL^{ft} restricted to the **EX** operator.

Theorem 5.5. *$\text{SAT}(\text{QK}^t)$ is TOWER-complete.*

Proof. The satisfiability problem in Theorem 5.5 is exactly $\text{SAT}(\text{QCTL}_{\mathbf{X}}^{ft})$. An easy translation can be found in Appendix C.3. \square

As a corollary of Theorem 3.14, for all $N \geq 2$, the satisfiability problem for $\text{QK}_{\leq N}^t$ is AEXP_{POL} -complete too where $\text{QK}_{\leq N}^t$ is interpreted on finite trees whose branching degree is at most N . The AEXP_{POL} lower bound can be obtained using the same reduction as for $\text{QCTL}_{\mathbf{X}, \leq N}^t$ whereas the AEXP_{POL} upper bound uses also the same arguments as for $\text{QCTL}_{\mathbf{X}, \leq N}^t$.

We have seen that QKD^t is actually defined as $\text{QCTL}_{\mathbf{X}}^t$ (KD is characterised by total and finite Kripke structures whose unfoldings generate finite-branching trees in which all the branches are infinite). Consequently:

Theorem 5.6. *$\text{SAT}(\text{QKD}^t)$ is TOWER-complete.*

Recall that the modal logic GL is known to be complete for the class of finite transitive trees. The logic QGL^t extends it with propositional quantification. We obtain the following:

Theorem 5.7. *$\text{SAT}(\text{QGL}^t)$ is TOWER-complete.*

Proof. The logic can be shown to be precisely $\text{QCTL}_{\mathbf{XF}}^{ft}$ when **EX** is translated into **EXEF**, whence we get a TOWER upper bound. For the TOWER-hardness proof, it is very similar to the one for $\text{QCTL}_{\mathbf{F}}^t$ (actually, it is a bit simpler). Consult Appendix C.4. \square

QK4^t is defined as the modal logic with propositional quantification over the class of finite-branching trees. Our next theorem is as follows.

Theorem 5.8. *$\text{SAT}(\text{QK4}^t)$ is TOWER-complete.*

Proof. Full proof is in Appendix C.5. We first show $\text{SAT}(\text{QK4}^t)$ and $\text{SAT}(\text{QCTL}_{\mathbf{XF}}^{gt})$ are identical modulo the rewriting of **EX** into **EXEF**. This yields the TOWER upper bound. As far as TOWER-hardness is concerned, for any formula ϕ in $\text{QCTL}_{\mathbf{XF}}^t$, one can show that ϕ is satisfiable for $\text{QCTL}_{\mathbf{XF}}^t$ iff $\phi \wedge \mathbf{EX} \mathbf{EF} \top \wedge \mathbf{AXAG} \mathbf{EXEF} \top$ is satisfiable in $\text{QCTL}_{\mathbf{XF}}^{gt}$. \square

Finally, by noting that QS4^t is equal to $\text{QCTL}_{\mathbf{F}}^t$ modulo that **EX** is rewritten into **EF**, using Theorem 5.3, we get the following complexity characterisation.

Theorem 5.9. *$\text{SAT}(\text{QS4}^t)$ is TOWER-complete.*

6. CONCLUSION

In the paper, we have developed a relatively simple proof method to show that the satisfiability problems for $\text{QCTL}_{\mathbf{X}}^t$, $\text{QCTL}_{\mathbf{F}}^t$ and $\text{QCTL}_{\mathbf{XF}}^t$ are TOWER-complete, see also similar methods in [56, 46]. Our contribution is to establish TOWER-hardness, which could be also shown for several modal logics with propositional quantification whose respective classes of models are tree-like structures. Moreover, in the case of fixed degree, we have shown that for all

$N \geq 2$, the satisfiability problem for the variant $\text{QCTL}_{\mathbf{X}, \leq N}^t$ is AEXP_{POL} -complete. Whereas AEXP_{POL} -hardness is established by reducing the alternating multi-tiling problem recently introduced in [13], the TOWER-hardness of $\text{SAT}(\text{QCTL}_{\mathbf{X}}^t)$ is essentially based on the fact that one can enforce concisely that a node has a number of children equal to some tower of exponentials.

Section 5 deals with the TOWER-completeness of $\text{SAT}(\text{QCTL}_{\mathbf{X}}^{ft})$ and $\text{SAT}(\text{QCTL}_{\mathbf{XF}}^{ft})$, as well as TOWER-completeness for the well-known modal logics K, KD, GL, K4 and S4 extended with propositional quantification but with adequate classes of tree-like structures. Though the TOWER upper bound for decision problems on trees should not come as a real surprise, all our TOWER-hardness results significantly improve the current state-of-the-art regarding the fragments of QCTL^t and for the above-mentioned modal logics. In particular, our proof technique for TOWER-hardness of $\text{SAT}(\text{QCTL}_{\mathbf{X}}^t)$ (and therefore for QK^t on finite trees) is simple enough so that it could be further reused or adapted, see *e.g.* a recent refinement of the proof in [8].

This work can be continued in several directions. For instance, TOWER-hardness of $\text{SAT}(\text{QCTL}_{\mathbf{F}}^t)$ is recently refined in [40, 39] by establishing that already $\text{QCTL}_{\mathbf{F}}^t$ restricted to formulae of modal/temporal depth two is also TOWER-hard. Among the several directions, one of them would be to characterise the expressiveness of $\text{QCTL}_{\mathbf{X}}^t$ or $\text{QCTL}_{\mathbf{F}}^t$ along the lines of [19] or [36], see also [5]. More generally, we believe that standard modal logics with propositional quantification, but under the tree semantics, deserve to be much better understood.

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APPENDIX A. PROOFS FROM SECTION 3

A.1. Proof of Lemma 3.2.

Proof. First, suppose that x is a nominal for the depth $k \geq 0$ from v . By definition, this means that there is $v' \in V$ satisfying $vE^k v'$ such that $\mathfrak{T}, v' \models x$ and for all $v'' \neq v'$ satisfying $vE^k v''$ we have $\mathfrak{T}, v'' \not\models x$. Obviously, we have $\mathfrak{T}, v \models \mathbf{EX}^k x$. *Ad absurdum*, suppose that $\mathfrak{T}, v \models \exists p (\mathbf{EX}^k(x \wedge p) \wedge \mathbf{EX}^k(x \wedge \neg p))$ with p being a fresh propositional variable. So, there is a $(AP \setminus \{p\})$ -variant \mathfrak{T}' of \mathfrak{T} (i.e. $\mathfrak{T} \approx_{AP \setminus \{p\}} \mathfrak{T}'$) such that $\mathfrak{T}', v \models \mathbf{EX}^k(x \wedge p) \wedge \mathbf{EX}^k(x \wedge \neg p)$. Therefore, there is v' such that $vE^k v'$ and $\mathfrak{T}', v' \models x \wedge p$. Similarly, there is v'' such that $vE^k v''$ and $\mathfrak{T}', v'' \models x \wedge \neg p$. Because of the constraint on the satisfaction of the propositional variable p , the nodes v' and v'' are distinct, which leads to a contradiction.

Conversely, suppose that $\mathfrak{T}, v \models \text{nom}(x, k)$. As $\mathfrak{T}, v \models \mathbf{EX}^k x$, there exists v' such that $vE^k v'$ and $\mathfrak{T}, v' \models x$. The uniqueness of v' can be concluded from the satisfaction $\neg \exists p (\mathbf{EX}^k(x \wedge p) \wedge \mathbf{EX}^k(x \wedge \neg p))$ (as above), since that formula characterises exactly the property that there are no two distinct nodes reachable in k steps from v satisfying x . \square

A.2. Proof of Lemma 3.3.

Proof. By assumption, there is a node $v' \in V$ satisfying $vE^k v'$ such that $\mathfrak{T}, v' \models x$, and for all $v'' \neq v'$ satisfying $vE^k v''$, we have $\mathfrak{T}, v'' \not\models x$. First, suppose that $\mathfrak{T}, v \models @_x^k \phi$, i.e. $\mathfrak{T}, v \models \mathbf{EX}^k(x \wedge \phi)$. Hence, there exists v'' such that $vE^k v''$ and $\mathfrak{T}, v'' \models x \wedge \phi$. As $\mathfrak{T}, v'' \models x$, the node v'' is necessarily equal to v' and therefore $\mathfrak{T}, v' \models \phi$. For the opposite direction, suppose that $\mathfrak{T}, v' \models \phi$. As $vE^k v'$ and $\mathfrak{T}, v' \models x$, by the definition of the satisfaction relation \models , we conclude that $\mathfrak{T}, v \models \mathbf{EX}^k(x \wedge \phi)$ and $\mathbf{EX}^k(x \wedge \phi)$ is equal to $@_x^k \phi$. \square

A.3. Proof of Lemma 3.4.

Proof. The proof is by induction on d . For the base case, suppose that $d = 1$. Thus, we have that $\mathfrak{T}, v_0 \models \text{nom}(x_1, 1)$, and that v_1 is the unique child of v_0 such that $v_0 E v_1$ and $\mathfrak{T}, v_1 \models x_1$. By Lemma 3.3, we have $\mathfrak{T}, v_0 \models @_{x_1}^1 \phi$ iff $\mathfrak{T}, v_1 \models \phi$ and $@_{x_1}^1 \phi$ is equal to $@_{x_1}^1 \phi$, so we are done. For the induction step with $d \geq 2$, let us assume that

$$\mathfrak{T}, v_0 \models \text{nom}(x_1, 1) \wedge \bigwedge_{i \in [2, d]} @_{x_1}^1 @_{x_2}^1 \cdots @_{x_{i-1}}^1 \text{nom}(x_i, 1),$$

and v_1, \dots, v_d is associated to x_1, \dots, x_d . The propositions below are equivalent

- $\mathfrak{T}, v_0 \models @_{\bar{x}} \phi$,
- $\mathfrak{T}, v_0 \models @_{x_1}^1 @_{x_2}^1 \cdots @_{x_d}^1 \phi$ (by definition of $@_{\bar{x}}$),
- $\mathfrak{T}, v_{d-1} \models @_{x_d}^1 \phi$ (by the induction hypothesis),
- $\mathfrak{T}, v_d \models \phi$ (by Lemma 3.3).

Hence, $\mathfrak{T}, v_0 \models @_{\bar{x}} \phi$ holds iff $\mathfrak{T}, v_d \models \phi$ holds, finishing the proof. \square

A.4. Proof of Lemma 3.6.

Proof. By way of example, we prove that $\mathbf{EX} \forall p \psi \leftrightarrow \forall p \mathbf{EX} \psi$ is valid. (Other formulae, e.g. $\mathbf{EX} \exists p \psi \leftrightarrow \exists p \mathbf{EX} \psi$ can be proved to be valid in an even simpler way). Before doing so, note that then the valid equivalences below provide a rewriting system (by reading the equivalences from left to right) that pushes the propositional quantification outside (in the usual way), leading to formulae in PNF in polynomial-time ($\mathcal{Q} \in \{\exists, \forall\}$).

$$\mathbf{EX} \mathcal{Q} p \psi \leftrightarrow \mathcal{Q} p \mathbf{EX} \psi \quad (\mathcal{Q} p \psi) \wedge \psi' \leftrightarrow \mathcal{Q} p (\psi \wedge \psi') \quad \neg \exists p \psi \leftrightarrow \forall p \neg \psi \quad \neg \forall p \psi \leftrightarrow \exists p \neg \psi$$

assuming that p does not occur in ψ' (otherwise, rename the quantified variable).

First, assume that $\mathfrak{T}, v \models \mathbf{EX} \forall p \psi$ with $\mathfrak{T} = \langle V, E, l \rangle$. Thus, there exists v' such that vEv' and $\mathfrak{T}, v' \models \forall p \psi$. Hence, for all $\mathfrak{T}' \approx_{AP \setminus \{p\}} \mathfrak{T}$, we have $\mathfrak{T}', v' \models \psi$. As v' remains a child of v for all such variants \mathfrak{T}' , we have that for all $\mathfrak{T}' \approx_{AP \setminus \{p\}} \mathfrak{T}$, $vE'v'$ and $\mathfrak{T}', v' \models \psi$. Thus, for all $\mathfrak{T}' \approx_{AP \setminus \{p\}} \mathfrak{T}$ we have $\mathfrak{T}', v \models \mathbf{EX} \psi$. We conclude $\mathfrak{T}, v \models \forall p \mathbf{EX} \psi$.

Conversely (and this is the place where the tree structure is essential), assume $\mathfrak{T}, v \models \forall p \mathbf{EX} \psi$. It means that for all $\mathfrak{T}' \approx_{AP \setminus \{p\}} \mathfrak{T}$, we have $\mathfrak{T}', v \models \mathbf{EX} \psi$. *Ad absurdum*, suppose $\mathfrak{T}, v \not\models \mathbf{EX} \forall p \psi$. Equivalently, we have $\mathfrak{T}, v \models \mathbf{AX} \exists p \neg \psi$. Thus, for all v' such that vEv' , there is $\mathfrak{T}_{v'} \approx_{AP \setminus \{p\}} \mathfrak{T}$ (say $\mathfrak{T}_{v'} = \langle V, E, l_{v'} \rangle$) such that $\mathfrak{T}_{v'}, v' \not\models \psi$. Let $\mathfrak{T}'_{v'} = \langle V_{v'}, E_{v'}, l'_{v'} \rangle$ be the restriction of $\mathfrak{T}_{v'}$ to the subtree whose root is v' . Obviously $\mathfrak{T}'_{v'}, v' \not\models \psi$ too. Two distinct children v_1 and v_2 of v in \mathfrak{T} , lead to tree models $\mathfrak{T}'_{v_1} = \langle V_{v_1}, E_{v_1}, l'_{v_1} \rangle$ and $\mathfrak{T}'_{v_2} = \langle V_{v_2}, E_{v_2}, l'_{v_2} \rangle$ with $V_{v_1} \cap V_{v_2} = \emptyset$. Indeed, \mathfrak{T} is a tree. Let \mathfrak{T}^* be defined as $\langle V, E, (\uplus_{v' \in E(v)} l'_{v'}) \uplus \{v \mapsto l(v)\} \rangle$, where \uplus denotes the disjoint sum. One can show that $\mathfrak{T}^* \approx_{AP \setminus \{p\}} \mathfrak{T}$ and for all v' such that vEv' , $\mathfrak{T}^*, v' \not\models \psi$, which leads to a contradiction. \square

A.5. Proof of Lemma 3.12.

Proof. We start with the first item of the lemma. The definition of τ is correct, since:

- for each position $(x, y) \in [0, 2^n - 1] \times [0, 2^n - 1]$ there is a unique node v' in the distance $2n$ from v encoding the position (x, y) (follows from Corollary 3.11),
- for each v' , as defined above, there is a unique tile proposition t^j such that $\mathfrak{T}, v' \models t^j$ holds (it is a direct consequence of the satisfaction of ϕ_{cov}^j at v).

The satisfaction of the conditions (hori) and (verti) follows from the satisfaction of $\phi_{\mathcal{H}}^j \wedge \phi_{\mathcal{V}}^j$. Indeed, let us discuss the condition (hori) only, since (verti) is analogous. Take any two consecutive positions (a, b) and $(a, b + 1)$ such that $\tau(a, b) = t$ and $\tau(a, b + 1) = t'$. Then, let $v_t, v_{t'}$ be nodes at the distance $2n$ from v , representing the positions (a, b) and $(a, b + 1)$. Set the local nominals x and y at v_t and $v_{t'}$. Then, note that the formula $\text{HN}(x, y)$ is satisfied at v by elementary operations on binary encodings of numbers. Hence, by the right-hand side of the implication in $\phi_{\mathcal{H}}^j$ we conclude that $(t, t') \in \mathcal{H}$.

To show the second item, we take $\mathfrak{T}' = \langle V, E, l' \rangle$ obtained from $\mathfrak{T} = \langle V, E, l \rangle$ by setting $l'(v_{(a,b)}) = (l(v_{(a,b)}) \setminus \mathcal{T}^j) \cup \{\tau(a, b)\}$ for the unique node $v_{(a,b)}$ at the distance $2n$ from v corresponding to (a, b) in the grid. Otherwise l and l' coincide. Since the formula $\text{grid}(2n)$ does not employ propositions from \mathcal{T}^j we conclude $\mathfrak{T}', v \models \text{grid}(2n)$. By the definition of τ we know that each node at the distance $2n$ from v is labelled with exactly one tile proposition from \mathcal{T}^j and hence, $\mathfrak{T}', v \models \phi_{\text{cov}}^j$. Checking that \mathfrak{T}', v satisfies $\phi_{\mathcal{H}}^j \wedge \phi_{\mathcal{V}}^j$ is routine and follows from the fact that τ is a tiling (so it satisfies (hori) and (verti)). \square

A.6. Proof of Lemma 3.13.

Proof. Let \mathcal{I} be the instance $n, \langle \mathcal{T}, \mathcal{H}, \mathcal{V} \rangle, \mathcal{T}_0, \mathcal{T}_{acc}, \mathcal{T}_{multi}$ of AMTP. Before showing that the reduction is correct, we need to state preliminary properties. Moreover, in the proof below, we repeat several formula definitions to follow more smoothly the technical developments.

(GRID) We recall that $\text{grid}(2n)$ is defined as the formula below

$$\begin{aligned} & \left(\bigwedge_{i \in [0, 2n-1]} \mathbf{AX}^i \mathbf{EX}_{=2} \top \right) \wedge \forall x, y \text{ diff-nom}(x, y, 2n) \rightarrow \\ & \left(\bigvee_{j \in [0, n-1]} \neg(@_x^{2n} \mathbf{h}_j \leftrightarrow @_y^{2n} \mathbf{h}_j) \vee \neg(@_x^{2n} \mathbf{v}_j \leftrightarrow @_y^{2n} \mathbf{v}_j) \right). \end{aligned}$$

Given a tree model \mathfrak{T} and a root node ε , one can show that $\mathfrak{T}, \varepsilon \models \text{grid}(2n)$ iff the properties below hold:

- (a) For all $j \in [0, 2n-1]$, we have that $\varepsilon E^j v$ implies that v has exactly two children.
- (b) For all distinct nodes v, v' such that $\varepsilon E^{2n} v$ and $\varepsilon E^{2n} v'$, there is some propositional variable r in $\{\mathbf{h}_{n-1}, \dots, \mathbf{h}_0, \mathbf{v}_{n-1}, \dots, \mathbf{v}_0\}$, such that v satisfies r iff v' does not satisfy r .

Satisfaction of (a) is essentially due to the fact that $\mathbf{EX}_{=2} \top$ is defined as

$$\exists x_1, x_2 \text{ diff-nom}(x_1, x_2, 1) \wedge \mathbf{AX}(x_1 \vee x_2)$$

and one can check that it holds true on nodes having exactly two children. For the satisfaction of (b), we need to invoke Lemma 3.2, Lemma 3.3 and Lemma 3.5. Assuming that $\text{diff-nom}(x, y, 2n)$ holds, (b) is equivalent to have two distinct nodes that are the respective interpretations of the nominals x and y for the depth $2n$. More generally, Lemma 3.12 states the main properties that are used about $\text{grid}(2n)$.

(TILING) Let us state a few properties about the conjunction $\phi_{\text{cov}}^j \wedge \phi_{\mathcal{H}}^j \wedge \phi_{\mathcal{V}}^j$, assuming that $\mathfrak{T}, \varepsilon \models \text{grid}(2n)$ (and therefore the set of nodes of distance $2n$ from the root encodes the grid $[0, 2^n - 1] \times [0, 2^n - 1]$). The formula ϕ_{cov}^j defined as

$$\forall x \text{ nom}(x, 2n) \rightarrow @_x^{2n} \left(\bigvee_{t \in \mathcal{T}} t^j \wedge \bigwedge_{t \neq t' \in \mathcal{T}} \neg(t^j \wedge t'^j) \right)$$

states that all the nodes at distance $2n$ from the root satisfy exactly one tile type from \mathcal{T}^j . This is again a consequence of Lemma 3.2 and Lemma 3.3. The formula $\phi_{\mathcal{H}}^j$ is defined as:

$$\forall x, y (\text{nom}(x, 2n) \wedge \text{nom}(y, 2n) \wedge \text{HN}(x, y)) \rightarrow \bigvee_{(t, t') \in \mathcal{H}} @_x^{2n} t^j \wedge @_y^{2n} t'^j,$$

where $\text{HN}(x, y)$ is the formula below:

$$\begin{aligned} \text{HN}(x, y) \stackrel{\text{def}}{=} & \left(\bigwedge_{\alpha \in [0, n-1]} @_x^{2n} \mathbf{v}_\alpha \leftrightarrow @_y^{2n} \mathbf{v}_\alpha \right) \wedge \bigvee_{i \in [0, n-1]} \left(@_x^{2n} \neg \mathbf{h}_i \wedge @_y^{2n} \mathbf{h}_i \wedge \right. \\ & \left. \wedge \bigwedge_{\alpha \in [0, i-1]} (@_x^{2n} \mathbf{h}_\alpha \wedge @_y^{2n} \neg \mathbf{h}_\alpha) \wedge \left(\bigwedge_{\alpha \in [i+1, n]} (@_x^{2n} \mathbf{h}_\alpha \leftrightarrow @_y^{2n} \mathbf{h}_\alpha) \right) \right). \end{aligned}$$

The formula $\text{HN}(x, y)$ expresses that, assuming that x and y for the depth $2n$, the two nodes at distance $2n$ interpreted respectively by x and y and representing respectively the positions $(\mathfrak{H}, \mathfrak{V})$ and $(\mathfrak{H}', \mathfrak{V}')$ of the grid, satisfies $\mathfrak{V} = \mathfrak{V}'$ and $\mathfrak{H}' = \mathfrak{H} + 1$. So, $\phi_{\mathcal{H}}^j$ encodes the horizontal constraints for the set of tile types \mathcal{T}^j . Similarly, $\phi_{\mathcal{V}}^j$ encodes the vertical

constraints for the set of tile types \mathcal{T}^j . Indeed, in $\text{VN}(x, y)$ we swap the variable \mathbf{h}_α with the variable \mathbf{v}_α (with respect to the definition for $\text{HN}(x, y)$), which amount to stating that assuming that x and y are nominals for the depth $2n$ from ε , the two nodes at distance $2n$ interpreted respectively by x and y and representing respectively the positions $(\mathfrak{H}, \mathfrak{V})$ and $(\mathfrak{H}', \mathfrak{V}')$ of the grid, satisfies $\mathfrak{V}' = \mathfrak{V} + 1$ and $\mathfrak{H}' = \mathfrak{H}$. So, assuming that $\mathfrak{T}, \varepsilon \models \text{grid}(2n)$, the formula $\phi_{\text{cov}}^j \wedge \phi_{\mathcal{H}}^j \wedge \phi_{\mathcal{V}}^j$, expresses that the way the tile types from \mathcal{T}^j holds on the nodes at distance $2n$ from the root defines a proper tiling.

(INIT) Let ϕ_{init}^j be the formula below:

$$\forall x (\text{nom}(x, 2n) \wedge @_x^{2n} (\bigwedge_{\alpha \in [0, n-1]} \neg \mathbf{h}_\alpha)) \rightarrow @_x^{2n} (\bigvee_{t \in \mathcal{T}_0} t^j \wedge \bigwedge_{t' \neq t \in \mathcal{T}_0} \neg(t^j \wedge t'^j)).$$

Note that $(\bigwedge_{\alpha \in [0, n-1]} \neg \mathbf{h}_\alpha)$ states on a node v at distance $2n$ from the root, that all the propositional variables \mathbf{h}_α are false and therefore this is a node on the row zero of the grid. ϕ_{init}^j therefore states (using again Lemma 3.2 and Lemma 3.3) that all the nodes of the row zero have a unique tile type from \mathcal{T}_0 .

(COINCI) Let $\phi_{\text{coinci}}^{j, j'}$ be the formula below (with $j, j' \in \mathbb{N}$):

$$\forall x (\text{nom}(x, 2n) \wedge @_x^{2n} (\bigwedge_{\alpha \in [0, n-1]} \neg \mathbf{h}_\alpha)) \rightarrow @_x^{2n} (\bigvee_{t \in \mathcal{T}_0} t^j \wedge t'^{j'}).$$

Assuming that $\mathfrak{T}, \varepsilon \models \phi_{\text{tiling}}^j \wedge \phi_{\text{init}}^{j'}$, the formula $\phi_{\text{coinci}}^{j, j'}$ states that for every node of the row zero, the tile type from \mathcal{T}^j is the same as the tile type from $\mathcal{T}_0^{j'}$. The reasoning is exactly the same as for **(INIT)**.

(ACCEPT) Let ϕ_{acc}^j be the formula below

$$\exists x \text{ nom}(x, 2n) \wedge @_x^{2n} ((\bigwedge_{\alpha \in [0, n-1]} \mathbf{h}_\alpha) \wedge \bigvee_{t \in \mathcal{T}_{\text{acc}}} t^j).$$

Note that $(\bigwedge_{\alpha \in [0, n-1]} \mathbf{h}_\alpha)$ states on a node v at distance $2n$ from the root, that all the propositional variables \mathbf{h}_α are true and therefore this is a node on the (last) row $2^n - 1$ of the grid.

Assuming that $\mathfrak{T}, \varepsilon \models \phi_{\text{tiling}}^j$, the formula ϕ_{acc}^j therefore states that there is a node encoding a position of the grid on the last row such that the tile type is in $\mathcal{T}_{\text{acc}}^j$.

(MULTI) Finally, let ϕ_{multi}^j be the formula below:

$$\forall x \text{ nom}(x, 2n) \rightarrow @_x^{2n} (\bigvee_{(t, t') \in \mathcal{T}_{\text{multi}}} t^j \wedge t'^{j+1}).$$

As done in **(INIT)**, ϕ_{multi}^j (assuming that $\mathfrak{T}, \varepsilon \models \phi_{\text{tiling}}^j \wedge \phi_{\text{tiling}}^{j+1}$) states that for all nodes at distance $2n$ from the root, the tile type from \mathcal{T}^j and the tile type from \mathcal{T}^{j+1} are in the relation $\mathcal{T}_{\text{multi}}$. Consequently,

$$\mathfrak{T}, \varepsilon \models \text{grid}(2n) \wedge (\bigwedge_{j \in [1, n]} \phi_{\text{init}}^j) \wedge (\bigwedge_{j \in [n+1, 2n]} \phi_{\text{tiling}}^j \wedge \phi_{\text{coinci}}^{j, (j-n)}) \wedge (\bigwedge_{j \in [n+1, 2n-1]} \phi_{\text{multi}}^j) \wedge \phi_{\text{acc}}^{2n})$$

if and only if the initial condition induced from the satisfaction of $(\bigwedge_{j \in [1, n]} \phi_{\text{init}}^j)$ (see the definition of AMTP in Section 2.3) of the form $(w_1, \dots, w_n) \in (\mathcal{T}_0^{2^n})^n$, and the multi-tiling (τ_1, \dots, τ_n) induced by the satisfaction of $(\bigwedge_{j \in [n+1, 2n]} \phi_{\text{tiling}}^j)$, entails that (τ_1, \dots, τ_n) is a solution and satisfies the condition (m-init), (m-tiling), (m-multi) and (m-accept).

Indeed, satisfying $\mathfrak{T}, \varepsilon \models \text{grid}(2n) \wedge (\bigwedge_{j \in [n+1, 2n]} \phi_{\text{tiling}}^j)$ defines a multi-tiling (τ_1, \dots, τ_n) (and reciprocally). Similarly, satisfying $\mathfrak{T}, \varepsilon \models \text{grid}(2n) \wedge (\bigwedge_{j \in [1, n]} \phi_{\text{init}}^j)$ defines an initial condition $(w_1, \dots, w_n) \in (\mathcal{T}_0^{2^n})^n$ (and reciprocally). The details are omitted but it does not pose any difficulty. So given an initial condition $c = (w_1, \dots, w_n) \in (\mathcal{T}_0^{2^n})^n$, we write \mathfrak{T}_c to denote a tree model such that $\mathfrak{T}_c, \varepsilon \models \text{grid}(2n) \wedge (\bigwedge_{j \in [1, n]} \phi_{\text{init}}^j)$ and on the grid induced by \mathfrak{T}_c , we have precisely the initial condition c . Similarly, given a multi-tiling $M = (\tau_1, \dots, \tau_n)$, we write \mathfrak{T}_M to denote a tree model such that $\mathfrak{T}_M, \varepsilon \models \text{grid}(2n) \wedge (\bigwedge_{j \in [n+1, 2n]} \phi_{\text{tiling}}^j)$ and on the grid induced by \mathfrak{T}_M , we have precisely the multi-tiling (τ_1, \dots, τ_n) . More generally, given c and M , we write $\mathfrak{T}_{c, M}$ to denote a tree model such that $\mathfrak{T}_{c, M}, \varepsilon \models \text{grid}(2n) \wedge (\bigwedge_{j \in [1, n]} \phi_{\text{init}}^j) \wedge (\bigwedge_{j \in [n+1, 2n]} \phi_{\text{tiling}}^j)$ and on the grid induced by $\mathfrak{T}_{c, M}$, we have precisely the initial condition c and the multi-tiling M . Reciprocally, assuming a tree model \mathfrak{T} such that $\mathfrak{T}, \varepsilon \models \text{grid}(2n) \wedge (\bigwedge_{j \in [1, n]} \phi_{\text{init}}^j)$, we write $c_{\mathfrak{T}} = (w_1, \dots, w_n)$ to denote the initial condition from the grid defined by \mathfrak{T} . Similarly, assuming a tree model \mathfrak{T} such that $\mathfrak{T}, \varepsilon \models \text{grid}(2n) \wedge (\bigwedge_{j \in [n+1, 2n]} \phi_{\text{tiling}}^j)$, we write $M_{\mathfrak{T}} = (\tau_1, \dots, \tau_n)$ to denote the multi-tiling from the grid defined by \mathfrak{T} .

It remains to conclude by explaining how to handle the quantifications over the tuples $(w_1, \dots, w_n) \in (\mathcal{T}_0^{2^n})^n$. The instance \mathcal{I} is positive iff

- (by definition) for all $w_1 \in \mathcal{T}_0^{2^n}$, there is $w_2 \in \mathcal{T}_0^{2^n}$ such that \dots for all $w_{n-1} \in \mathcal{T}_0^{2^n}$, there is $w_n \in \mathcal{T}_0^{2^n}$ such that there is a solution (τ_1, \dots, τ_n) for (w_1, \dots, w_n) , iff
- (by the above correspondences) there is a tree model \mathfrak{T}^0 such that $\mathfrak{T}^0, \varepsilon \models \text{grid}(2n)$ such that for all \mathfrak{T}^1 such that $(\mathfrak{T}^1 \approx_{AP \setminus \mathcal{T}_0^1} \mathfrak{T}^0)$, $\mathfrak{T}^1, \varepsilon \models \phi_{\text{init}}^1$ and $w_1^{\mathfrak{T}^1} = w_1$ (where $w_1^{\mathfrak{T}^1}$ is the new word induced by \mathfrak{T}^1) there is \mathfrak{T}^2 such that $(\mathfrak{T}^2 \approx_{AP \setminus \mathcal{T}_0^2} \mathfrak{T}^1)$, $\mathfrak{T}^2, \varepsilon \models \phi_{\text{init}}^2$ and $w_2^{\mathfrak{T}^2} = w_2$ \dots for all \mathfrak{T}^{n-1} such that $(\mathfrak{T}^{n-1} \approx_{AP \setminus \mathcal{T}_0^{n-1}} \mathfrak{T}^{n-2})$, $\mathfrak{T}^{n-1}, \varepsilon \models \phi_{\text{init}}^{n-1}$ and $w_{n-1}^{\mathfrak{T}^{n-1}} = w_{n-1}$ there is \mathfrak{T}^n such that $(\mathfrak{T}^n \approx_{AP \setminus \mathcal{T}_0^n} \mathfrak{T}^{n-1})$, $\mathfrak{T}^n, \varepsilon \models \phi_{\text{init}}^n$ and $w_n^{\mathfrak{T}^n} = w_n$, there is \mathfrak{T}^{n+1} such that $(\mathfrak{T}^{n+1} \approx_{AP \setminus \{tj : t \in \mathcal{T}, j \in [n+1, 2n]\}} \mathfrak{T}^n)$ such that $\mathfrak{T}^{n+1}, \varepsilon \models (\bigwedge_{j \in [n+1, 2n]} \phi_{\text{tiling}}^j)$ and $M_{\mathfrak{T}^{n+1}}$ is a solution for $c_{\mathfrak{T}^{n+1}}$, iff
- (by first-order reasoning) there is a tree model \mathfrak{T}^0 such that $\mathfrak{T}^0, \varepsilon \models \text{grid}(2n)$ such that for all \mathfrak{T}^1 such that $(\mathfrak{T}^1 \approx_{AP \setminus \mathcal{T}_0^1} \mathfrak{T}^0)$ there is \mathfrak{T}^2 such that $(\mathfrak{T}^2 \approx_{AP \setminus \mathcal{T}_0^2} \mathfrak{T}^1)$ \dots for all \mathfrak{T}^{n-1} such that $(\mathfrak{T}^{n-1} \approx_{AP \setminus \mathcal{T}_0^{n-1}} \mathfrak{T}^{n-2})$ there is \mathfrak{T}^n such that $(\mathfrak{T}^n \approx_{AP \setminus \mathcal{T}_0^n} \mathfrak{T}^{n-1})$, if $\mathfrak{T}^n, \varepsilon \models (\bigwedge_{j \in [1, n]} \phi_{\text{init}}^j)$, then there is \mathfrak{T}^{n+1} such that $(\mathfrak{T}^{n+1} \approx_{AP \setminus \{tj : t \in \mathcal{T}, j \in [n+1, 2n]\}} \mathfrak{T}^n)$ such that $\mathfrak{T}^{n+1}, \varepsilon \models (\bigwedge_{j \in [n+1, 2n]} \phi_{\text{tiling}}^j)$ and $M_{\mathfrak{T}^{n+1}}$ is a solution for $c_{\mathfrak{T}^{n+1}}$, iff
- (by the encodings of (m-init), (m-coinci), (m-multi), (m-accept)) there is a tree model \mathfrak{T}^0 such that $\mathfrak{T}^0, \varepsilon \models \text{grid}(2n)$ such that for all \mathfrak{T}^1 such that $(\mathfrak{T}^1 \approx_{AP \setminus \mathcal{T}_0^1} \mathfrak{T}^0)$ there is \mathfrak{T}^2 such that $(\mathfrak{T}^2 \approx_{AP \setminus \mathcal{T}_0^2} \mathfrak{T}^1)$ \dots for all \mathfrak{T}^{n-1} such that $(\mathfrak{T}^{n-1} \approx_{AP \setminus \mathcal{T}_0^{n-1}} \mathfrak{T}^{n-2})$ there is \mathfrak{T}^n such that $(\mathfrak{T}^n \approx_{AP \setminus \mathcal{T}_0^n} \mathfrak{T}^{n-1})$, if $\mathfrak{T}^n, \varepsilon \models (\bigwedge_{j \in [1, n]} \phi_{\text{init}}^j)$, then there is \mathfrak{T}^{n+1} such that $(\mathfrak{T}^{n+1} \approx_{AP \setminus \{tj : t \in \mathcal{T}, j \in [n+1, 2n]\}} \mathfrak{T}^n)$ such that $\mathfrak{T}^{n+1}, \varepsilon \models (\bigwedge_{j \in [n+1, 2n]} \phi_{\text{tiling}}^j)$ and $\mathfrak{T}^{n+1}, \varepsilon \models (\bigwedge_{j \in [n+1, 2n]} \phi_{\text{coinci}}^{j, (j-n)}) \wedge (\bigwedge_{j \in [n+1, 2n-1]} \phi_{\text{multi}}^j) \wedge \phi_{\text{acc}}^{2n}$ iff
- (by definition of \models) the formula below is satisfiable:

$$\text{grid}(2n) \wedge \forall \mathcal{T}_0^1 \exists \mathcal{T}_0^2 \forall \mathcal{T}_0^3 \dots \exists \mathcal{T}_0^n \bigwedge_{j \in [1, n]} \phi_{\text{init}}^j \rightarrow$$

$$(\exists \{t^j : t \in \mathcal{T}, j \in [n+1, 2n]\}) \left(\bigwedge_{j \in [n+1, 2n]} \phi_{\text{tiling}}^j \wedge \phi_{\text{coinci}}^{j, (j-n)} \right) \wedge \left(\bigwedge_{j \in [n+1, 2n-1]} \phi_{\text{multi}}^j \right) \wedge \phi_{\text{acc}}^{2n}.$$

□

APPENDIX B. PROOFS FROM SECTION 4

B.1. Proof of Lemma 4.1.

Proof. The property (III) is a direct consequence of (II). Since the proof of (II) is similar to (I), we focus on showing (I) only. In order to define $\mathbf{nb}(y_1, \dots, y_k) =_k \mathbf{nb}(x_1, \dots, x_k) + 1$, and therefore to express that $\mathbf{nb}_{\mathfrak{T}}(v'_k) = \mathbf{nb}_{\mathfrak{T}}(v_k) + 1$ with numbers computed with the propositional variables p_{n-1}, \dots, p_0 , we perform a standard comparison of the respective truth values of p_{n-1}, \dots, p_0 for the node v_k (the interpretation of x_k) and for the node v'_k (the interpretation of y_k). Typically, $\mathbf{nb}_{\mathfrak{T}}(v'_k) = \mathbf{nb}_{\mathfrak{T}}(v_k) + 1$ iff there is $i \in [0, n-1]$, such that

- for every $j \in [i+1, n-1]$, v_k and v'_k agree on p_j ($\mathbf{nb}_{\mathfrak{T}}(v'_k)$ and $\mathbf{nb}_{\mathfrak{T}}(v_k)$ agree on their j th bit),
- v_k does not satisfy p_i and v'_k satisfies p_i ($\mathbf{nb}_{\mathfrak{T}}(v'_k)$ and $\mathbf{nb}_{\mathfrak{T}}(v_k)$ disagree on their i th bit and the i th bit of $\mathbf{nb}_{\mathfrak{T}}(v_k)$ is equal to zero),
- for every $j \in [0, i-1]$, v_k satisfies p_j and v'_k does not satisfy p_j .

The formula $\mathbf{nb}(y_1, \dots, y_k) =_k \mathbf{nb}(x_1, \dots, x_k) + 1$ indeed quantifies existentially on i via a generalised disjunction and the three conditions are checked by three conjuncts in the standard manner. □

B.2. Proof of Lemma 4.2.

Proof. We focus on proving (I), the other properties can be shown analogously.

A node v is of type 1 iff it has exactly 2^n children, say v_0, \dots, v_{2^n-1} , and for all $j \in [0, 2^n-1]$, the number associated to v_j is precisely j when encoded with the truth values of the propositional variables p_{n-1}, \dots, p_0 . The latter proposition amounts to having the following conditions:

- The node v has a child whose number is zero.
- If v' is a child of v with number $m < 2^n - 1$, then v has also a child with number $m + 1$.
- Two distinct children of v have distinct numbers.

Let us recall that $\text{type}(1) = \mathbf{AX}(\text{type}(0)) \wedge \mathbf{EX}(\text{first}(0)) \wedge \text{uniq}(1) \wedge \text{compl}(1)$. Obviously, $\mathbf{AX}(\text{type}(0))$ always holds and $\mathbf{EX}(\text{first}(0))$ expresses exactly the condition (a). It remains to show that $\text{uniq}(1)$ (resp. $\text{compl}(1)$) characterises the conditions (c) (resp. the condition (b)). The formula $\text{uniq}(1)$ is equal to $\forall x, y \text{ diff-nom}(x, y, 1) \rightarrow \neg(\mathbf{nb}(x) =_1 \mathbf{nb}(y))$. By Lemma 4.1(III), and by Lemma 3.5, when interpreted on a node of type 1, $\text{uniq}(1)$ states that for any two distinct children, their respective numbers are different, which is precisely the condition (c). Finally, let us recall the definition of the formula $\text{compl}(1)$:

$$\forall x (\text{nom}(x, 1) \wedge @_x^1(\neg \text{last}(0))) \rightarrow \exists y \text{ nom}(y, 1) \wedge \mathbf{nb}(y) =_1 \mathbf{nb}(x) + 1.$$

By Lemma 4.1(I), by the fact that $\text{last}(0)$ already characterises the nodes of type 0 whose number is $2^n - 1$. By Lemma 3.2 and Lemma 3.3, the formula $\text{compl}(1)$, states that for all children whose number m is different from $2^n - 1$, there is a child with number $m + 1$,

which is precisely the condition (b). This ends the proof as all the formulae $\mathbf{AX}(\text{type}(0))$, $\mathbf{EX}(\text{first}(0))$, $\text{uniq}(1)$, $\text{compl}(1)$ capture exactly the properties specified above. \square

B.3. Proof of Lemma 4.3.

Proof. Assume that v_0 is of type k , $\mathfrak{T}, v_0 \models \widehat{\bar{x}, \bar{x}}$ and the witness branch is v_0, \dots, v_d .

First, suppose that $\mathfrak{T}, v \models \text{LSR}_{\bar{x}}(k)$. As $\mathfrak{T}, v_0 \models @_{\bar{x}}(\mathbf{EX}_{=1}(\mathbf{s}))$, by Lemma 3.4, $\mathfrak{T}, v_d \models \mathbf{EX}_{=1}(\mathbf{s})$. It is easy to show that $\mathbf{EX}_{=1} \psi$ holds whenever there is a unique child satisfying ψ . Consequently, there is a unique child of v_d satisfying \mathbf{s} , which corresponds to the satisfaction of (b). As $\mathfrak{T}, v_0 \models @_{\bar{x}}(\mathbf{AX}((\mathbf{s} \vee \mathbf{l} \vee \mathbf{r}) \wedge \neg(\mathbf{s} \wedge \mathbf{l}) \wedge \neg(\mathbf{s} \wedge \mathbf{r}) \wedge \neg(\mathbf{l} \wedge \mathbf{r})))$, by Lemma 3.4, we have $\mathfrak{T}, v_d \models \mathbf{AX}((\mathbf{s} \vee \mathbf{l} \vee \mathbf{r}) \wedge \neg(\mathbf{s} \wedge \mathbf{l}) \wedge \neg(\mathbf{s} \wedge \mathbf{r}) \wedge \neg(\mathbf{l} \wedge \mathbf{r}))$ and therefore for all children v' of v_d , we have $\mathfrak{T}, v' \models (\mathbf{s} \vee \mathbf{l} \vee \mathbf{r}) \wedge \neg(\mathbf{s} \wedge \mathbf{l}) \wedge \neg(\mathbf{s} \wedge \mathbf{r}) \wedge \neg(\mathbf{l} \wedge \mathbf{r})$. As the formula $(\mathbf{s} \vee \mathbf{l} \vee \mathbf{r}) \wedge \neg(\mathbf{s} \wedge \mathbf{l}) \wedge \neg(\mathbf{s} \wedge \mathbf{r}) \wedge \neg(\mathbf{l} \wedge \mathbf{r})$ precisely states that exactly one propositional variable among $\{\mathbf{l}, \mathbf{s}, \mathbf{r}\}$ holds true, we can conclude that (a) is satisfied. Moreover, $\mathfrak{T}, v_0 \models \text{LSR}_{\bar{x}}^3(k)$ and by Lemma 3.4, we have that $\mathfrak{T}, v_d \models \forall w \forall w' \text{diff-nom}(w, w', 1) \wedge ((@_w^1(\mathbf{s}) \wedge @_{w'}^1(\mathbf{r})) \vee (@_w^1(\mathbf{l}) \wedge @_{w'}^1(\mathbf{s}))) \rightarrow \mathbf{nb}(w') <_{k-d} \mathbf{nb}(w)$.

As $k - d - 1 \leq N - 2$ by assumption, the satisfaction of the formula above on v_d straightforwardly states that for all children v and v' of v_d such that (v satisfies \mathbf{s} and v' satisfies \mathbf{r}) or (v satisfies \mathbf{l} and v' satisfies \mathbf{s}), we have $\mathbf{nb}_{\mathfrak{T}}(v) < \mathbf{nb}_{\mathfrak{T}}(v')$ (here we use the induction hypothesis), which corresponds precisely to the satisfaction of (c).

The proof in the other direction is quite similar as there are equivalences between the formulae $\text{LSR}_{\bar{x}}^1(k)$, $\text{LSR}_{\bar{x}}^2(k)$, and $\text{LSR}_{\bar{x}}^3(k)$, and the conditions (a), (b) and (c). \square

B.4. Proof of Lemma 4.4.

Proof. Let \mathfrak{T} be a tree model and v be such that v satisfies $\widehat{\bar{x}, \bar{y}}$, $k - d = N - 1$ and v_d and v'_d are of type $k - d$. We will focus on the proof of (I) only. The proof of (II) is similar to the proof of (I) and (III) is a direct consequence of (II). Hence, we omit the details.

As v_d and v'_d are of type $k - d$, both have $\mathbf{t}(k - d, n)$ children. Those children are ordered, let $u_0, \dots, u_{\mathbf{t}(k-d, n)-1}$ be the children of v_k such that $\mathbf{nb}(u_j) = j$ for all j . Similarly, let $u'_0, \dots, u'_{\mathbf{t}(k-d, n)-1}$ be the children of v'_k such that $\mathbf{nb}(u'_j) = j$ for all j . By arithmetical reasoning, we have $\mathbf{nb}_{\mathfrak{T}}(v'_d) = \mathbf{nb}_{\mathfrak{T}}(v_d) + 1$ iff there is $i \in [0, \mathbf{t}(k - d, n) - 1]$, such that

- (A) for every $j \in [i + 1, \mathbf{t}(k - d, n) - 1]$, $\mathbf{nb}_{\mathfrak{T}}(v'_d)$ and $\mathbf{nb}_{\mathfrak{T}}(v_d)$ agree on the j th bit, which is equivalent to u_j and u'_j agree on val ,
- (B) the i th bit of $\mathbf{nb}_{\mathfrak{T}}(v_d)$ is equal to 0 and the i th bit of $\mathbf{nb}_{\mathfrak{T}}(v'_d)$ is equal to 1, which is equivalent to u_i does not satisfy val and u'_i satisfies val ,
- (C) for every $j \in [0, i - 1]$, the j th bit of $\mathbf{nb}_{\mathfrak{T}}(v_d)$ is equal to 1 and the j th bit of $\mathbf{nb}_{\mathfrak{T}}(v'_d)$ is equal to 0, which is equivalent to u_j satisfies val and u'_j does not satisfy val .

By using Lemma 3.4, it is easy to check that the condition (A) (resp. (B), (C)) is taken care by $\phi_{\text{left}}(k)$ (resp. $\phi_{\text{select}}(k)$, $\phi_{\text{right}}(k)$). This is quite immediate for $\phi_{\text{select}}(k)$ and $\phi_{\text{right}}(k)$, as no induction hypothesis is used, in particular no comparison between numbers is performed. Concerning the satisfaction of $\phi_{\text{left}}(k)$, we have to be more careful. Let us recall the definition of the formula $\phi_{\text{left}}^{\bar{x}, \bar{y}}(k)$ below:

$$\begin{aligned} & \forall w @_{\bar{x}}(\text{nom}(w, 1) \wedge @_{\bar{w}}^1(\mathbf{l})) \rightarrow \\ & (\exists w' @_{\bar{y}}(\text{nom}(w', 1) \wedge @_{\bar{w}'}^1(\mathbf{l})) \wedge \mathbf{nb}(\bar{x}, w) =_k \mathbf{nb}(\bar{y}, w') \wedge (@_{\bar{x}, w} val \Leftrightarrow @_{\bar{y}, w'} val)). \end{aligned}$$

We have defined $\phi_{\text{left}}(k)$ as $\phi_{\text{left}}^{\bar{x},\bar{y}}(k) \wedge \phi_{\text{left}}^{\bar{y},\bar{x}}(k)$. By Lemma 3.4, and by the induction hypothesis (as $(k - (d + 1)) \leq N - 2$), $\phi_{\text{left}}^{\bar{x},\bar{y}}(k)$ enforces that for all children v of v_d satisfying l , there is a child v' of v'_d satisfying l , such that v and v' agree on val . Moreover, $\phi_{\text{left}}^{\bar{y},\bar{x}}(k)$ enforces that for all children v of v'_d satisfying l , there is a child v' of v_d satisfying l , such that v and v' agree on val . So, the set of numbers of the children of v_d satisfying l is equal to the set of numbers of the children of v'_d satisfying l . This implies also that this property applies for the children satisfying r , and the number of the unique child of v_d satisfying s is equal to the number of the unique child of v'_d . This is a direct consequence of the properties of **lsr**-partitions (see Lemma 4.3). Consequently, $\text{nb}_{\bar{x}}(v'_d) = \text{nb}_{\bar{x}}(v_d) + 1$ implies $\text{nb}(\bar{y}) =_k \text{nb}(\bar{x}) + 1$. The proof for the other direction is similar, as $\phi_{\text{left}}(k)$ is equivalent to (A), $\phi_{\text{select}}(k)$ is equivalent to (B) and $\phi_{\text{right}}(k)$ is equivalent to (C). Moreover, the existential quantification over i corresponds to the existential quantification leading to an **lsr**-partition. \square

B.5. Proof of Lemma 4.5.

Proof. The properties (II) and (III) are easy to verify, hence let us focus on (I). The proof is actually very similar to the proof of Lemma 4.2. A node v is of type N for some $N \geq 2$, iff the conditions below hold: (a) it has exactly $\mathfrak{t}(N, n)$ children, (b) all its children are of type $N - 1$ and (c) $\{\text{nb}_{\bar{x}}(v') : vEv'\} = [0, \mathfrak{t}(N, n) - 1]$. The mentioned conditions can be reformulated as follows so, by induction, (a')+(b')+(c') is equivalent to (a)+(b)+(c):

- (a') The node v has a child whose number is zero.
- (b') If v' is a child of v with number $m < \mathfrak{t}(N, n) - 1$, then v has also a child with number $m + 1$.
- (c') Two distinct children of v have distinct numbers.

Let us recall below the definition of the formula $\text{type}(N)$:

$$\mathbf{AX}(\text{type}(N - 1)) \wedge \mathbf{EX}(\text{first}(N - 1)) \wedge \text{uniq}(N) \wedge \text{compl}(N).$$

Obviously, the formula $\mathbf{AX}(\text{type}(N - 1))$ expresses exactly the condition (b), assuming that $\text{type}(N - 1)$ already characterised the nodes of type $N - 1$. Similarly, the formula $\mathbf{EX}(\text{first}(N - 1))$ expresses exactly the condition (a'), assuming that $\text{first}(N - 1)$ already characterises the nodes of type $N - 1$ whose number is zero. It remains to show that $\text{uniq}(N)$ (resp. $\text{compl}(N)$) characterises the conditions (c') (resp. the condition (b')).

Let us recall below the definition of the formula $\text{uniq}(N)$:

$$\forall x, y \text{ diff-nom}(x, y, 1) \rightarrow \neg(\text{nb}(x) =_N \text{nb}(y)).$$

By Lemma 4.4(III), and by the fact that $\text{diff-nom}(x, y, 1)$ enforces that x and y are interpreted by two distinct children (see Lemma 3.5), when interpreted on a node of type N , the formula $\text{uniq}(N)$ states that for any two distinct children, their respective numbers are different, which is precisely the condition (c'). Finally, let us recall the formula $\text{compl}(N)$:

$$\forall x (\text{nom}(x, 1) \wedge @_x^1(\neg \text{last}(N - 1))) \rightarrow \exists y \text{ nom}(y, 1) \wedge \text{nb}(y) =_N \text{nb}(x) + 1.$$

By Lemma 4.4(I), by the fact that $\text{last}(N - 1)$ already characterises the nodes of type $N - 1$ whose number is $\mathfrak{t}(N, n) - 1$ and by the properties of the subformulae $\text{nom}(x, 1)$ and $\text{nom}(y, 1)$ enforcing local nominals x and y , the formula $\text{compl}(N)$, when interpreted on a node of type N , states that for all children whose number m is different from $\mathfrak{t}(N, n) - 1$, there is a child with number $m + 1$, which is precisely the condition (b'). This ends the proof as the formulae $\mathbf{AX}(\text{type}(N - 1))$, $\mathbf{EX}(\text{first}(N - 1))$, $\text{uniq}(N)$, $\text{compl}(N)$ capture (a')+(b')+(c'). \square

B.6. Proof of Lemma 4.6.

Proof. The case of $k = 0$ is trivial, so take for the case case $k = 1$. Assuming that $\mathfrak{T}, v \models \text{type}(1)$, the node v is of type 1, and therefore v has exactly $\mathfrak{t}(1, n) = 2^n$ children. The number $\mathbf{nb}_{\mathfrak{T}}(v)$ is determined by the truth values of val on its children, and the children have themselves (bit) numbers spanning all over $[0, 2^n - 1]$ and the numbers are encoded by the truth values of p_{n-1}, \dots, p_0 . Consequently, $\mathbf{nb}_{\mathfrak{T}}(v) = \mathfrak{t}(1, n) = 2^n$ iff the unique child v' of v such that $\mathbf{nb}_{\mathfrak{T}}(v') = n$ satisfies val and all the other children do not satisfy val . If the value n encoded with n bits is represented by the sequence of bits $b_{n-1}b_{n-2} \dots b_0$, then $X_n = n$ is a shortcut for $\bigwedge_{i \in [0, n-1]} \ell_i$ where $\ell_i = p_i$ if $b_i = 1$, otherwise $\ell_i = \neg p_i$. Hence, we get $\mathbf{nb}_{\mathfrak{T}}(v) = \mathfrak{t}(k, n)$ iff $\mathfrak{T}, v \models \mathbf{AX}(val \leftrightarrow X_n = n)$.

For the induction step, we reason in a similar way. Assume that for all $1 \leq k' < k$, if $\mathfrak{T}', v' \models \text{type}(k')$, we have $\mathbf{nb}_{\mathfrak{T}'}(v') = \mathfrak{t}(k', n)$ iff $\mathfrak{T}', v' \models \mathbf{nb} =_{k'} \mathfrak{t}(k', n)$. Now, assume that $\mathfrak{T}, v \models \text{type}(k)$, the node v is of type k , and therefore v has exactly $\mathfrak{t}(k, n)$ children. The number $\mathbf{nb}_{\mathfrak{T}}(v)$ is determined by the truth values of val on its children, and the children have themselves (bit) numbers spanning all over $[0, \mathfrak{t}(k, n) - 1]$. Consequently, $\mathbf{nb}_{\mathfrak{T}}(v) = \mathfrak{t}(k, n)$ iff the unique child v' of v such that $\mathbf{nb}_{\mathfrak{T}}(v') = \mathfrak{t}(k-1, n)$ satisfies val and all the other children does not satisfy val . Indeed, if only the $\mathfrak{t}(k-1, n)$ th bit is equal to 1, $\mathbf{nb}_{\mathfrak{T}}(v)$ is equal to $2^{\mathfrak{t}(k-1, n)}$, which is precisely $\mathfrak{t}(k, n)$. Now, checking whether a node of type $k-1$, has value $\mathfrak{t}(k-1, n)$ can be expressed by $\mathbf{nb} =_{k-1} \mathfrak{t}(k-1, n)$ invoking the induction hypothesis. Putting all together, we get $\mathbf{nb}_{\mathfrak{T}}(v) = \mathfrak{t}(k, n)$ iff $\mathfrak{T}, v \models \mathbf{AX}(val \leftrightarrow (\mathbf{nb} =_{k-1} \mathfrak{t}(k-1, n)))$. \square

B.7. More details on the proof of Lemma 4.7.

Proof. We have seen that the satisfaction of $\text{type}(k+1)$ is guaranteed by the way propositional variables hold on the nodes. Observe that for the nodes in $[0, \mathfrak{t}(k+1, n) - 1] \times \dots \times [0, \mathfrak{t}(1, n) - 1] \times 0^+$, the truth values of the propositional variables is irrelevant. Moreover, for all $j \in [1, k+1]$, for all $m_{k+1}, \dots, m_j \in [0, \mathfrak{t}(k+1, n) - 1] \times \dots \times [0, \mathfrak{t}(j, n) - 1]$, we have $\mathbf{nb}_{\mathfrak{T}}(m_{k+1}, \dots, m_j) = m_j$. In order to check the satisfaction of the existentially quantified subformula, we consider the labelling l' variant of l only for the propositional variables l , s , and r such that l holds on $[\mathfrak{t}(k, n) + 1, \mathfrak{t}(k+1, n) - 1]$, s holds on $\mathfrak{t}(k, n)$, and r holds on $[0, \mathfrak{t}(k, n) - 1]$. By Lemma 4.3, $\langle V, E, l' \rangle, \varepsilon \models \text{LSR}_{\varepsilon}(k+1)$ and by Lemma 4.6, we have $\langle V, E, l' \rangle, \varepsilon \models \mathbf{EX}(s \wedge \mathbf{nb} =_k \mathfrak{t}(k, n))$. The satisfaction of the formulae ϕ_{cov} , $\phi_{\mathcal{H}}$ and $\phi_{\mathcal{V}}$ is inherited from the fact that for all $(i, j) \in [0, \mathfrak{t}(k, n) - 1] \times [0, \mathfrak{t}(k, n) - 1]$, there is exactly one tile type satisfied by (i, j) and the mapping τ satisfies horizontal and vertical matching conditions. Here, we use the properties of the formulae of the form $\mathbf{nom}(x, 1)$, $@_x^1 \psi$, $\mathbf{nb}(x) =_{k+1} \mathbf{nb}(x')$ and $\mathbf{nb}(x, y') =_{k+1} \mathbf{nb}(x, y) + 1$ (see e.g. Lemma 4.4), apart from the fact that r holds exactly on the nodes in $[0, \mathfrak{t}(k, n) - 1]$. Concerning the satisfaction of ϕ_{init} , first observe that $\mathbf{EX}_{=j} \psi$ holds true exactly when there are j children of the node satisfying the formula ψ . Hence, the formula below

$$\forall x (\mathbf{nom}(x, 1) \wedge @_x^1(\text{first}(k))) \rightarrow @_x^1 \left(\bigwedge_{j \in [0, n-1]} \exists l, s, r \text{LSR}_{\varepsilon}(k) \wedge \mathbf{EX}_{=j} r \wedge \mathbf{EX}(s \wedge t_j) \right)$$

states for all $(i, j) \in \{0\} \times [0, n-1]$, t_j holds on it. In order to access the j th child of $\{0\}$, an lsr -partition on the children of $\{0\}$ is performed and s holds true exactly on (i, j) by counting how many children satisfies r . The proof for the other direction uses similar principles and is omitted herein. The main idea is to build τ so that assuming that $\mathfrak{T}, \varepsilon \models \phi_{\mathcal{P}}$, for all v, v'

such that $\varepsilon E v E v'$, and $(\mathbf{nb}(v), \mathbf{nb}(v')) \in [0, \mathbf{t}(k, n) - 1] \times [0, \mathbf{t}(k, n) - 1]$, $\tau(\mathbf{nb}(v), \mathbf{nb}(v'))$ takes the value of the unique t in \mathcal{T} satisfied on the node v' . \square

APPENDIX C. PROOFS FROM SECTION 5

C.1. Proof of Lemma 5.1.

Proof. First, let us suppose that \mathfrak{T} is k -layered and we show that $\mathfrak{T}, \varepsilon \models \text{shape}(k)$.

- By the condition (a), we have that:

$$\mathfrak{T}, \varepsilon \models \mathbf{AG} \left((layer_{-1} \vee layer_0 \vee \dots \vee layer_k) \wedge \bigwedge_{-1 \leq i \neq j \leq k} \neg(layer_i \wedge layer_j) \right)$$

as the formula inside \mathbf{AG} states that there is exactly one proposition from $Y_k = \{layer_{-1}, layer_0, \dots, layer_k\}$ holds.

- By the condition (b), namely its second point, for all $i \in [0, k]$, we have

$$\mathfrak{T}, \varepsilon \models \bigwedge_{-1 \leq i \leq k} \mathbf{AG}(layer_i \rightarrow \mathbf{AG}(layer_{-1} \vee layer_0 \vee \dots \vee layer_i)).$$

- By the first point of the condition (b), for all $i \in [0, k]$, we have $\mathfrak{T}, \varepsilon \models \mathbf{AG}(layer_i \rightarrow \mathbf{EF} layer_{i-1})$. Actually, when $\mathfrak{T}, v \models layer_i$ the witness descendant satisfying $layer_{i-1}$ is a child of v by (b).
- By the condition (d), we have $\mathfrak{T}, \varepsilon \models layer_k$.
- By the condition (c), for every node v satisfying $layer_j$ for some $j \in [0, k]$, there is no proper descendant of v satisfying $layer_j$. Observe that the formula $\neg \exists p (p \wedge \mathbf{EF}(layer_j \wedge \neg p))$ holds exactly on the nodes such that there is no proper descendant satisfying $layer_j$. Hence

$$\mathfrak{T}, \varepsilon \models \bigwedge_{0 \leq i \leq k} \mathbf{AG}(layer_i \rightarrow \neg \exists p (p \wedge \mathbf{EF}(layer_i \wedge \neg p))).$$

Conversely, suppose that $\mathfrak{T}, \varepsilon \models \text{shape}(k)$ holds. The satisfaction of (a), (c) and (d) holds thanks to the corresponding formulae in $\text{shape}(k)$ (see above). Let us check that (c) holds true. As

$$\mathfrak{T}, \varepsilon \models \bigwedge_{0 \leq i \leq k} \mathbf{AG}(layer_i \rightarrow \neg \exists p (p \wedge \mathbf{EF}(layer_i \wedge \neg p)))$$

holds and for all $i \in [0, k]$, we have

$$\mathfrak{T}, \varepsilon \models \mathbf{AG}(layer_i \rightarrow \mathbf{EF} layer_{i-1}),$$

on the same branch two distinct nodes cannot satisfy $layer_i$ for some $i \in [0, k]$. Moreover, the satisfaction of $layer_i$ implies that some (proper) descendant satisfies $layer_{i-1}$. Due to the monotonicity of the layer numbers and no stuttering, $layer_i$ implies that a child satisfies $layer_{i-1}$, which corresponds to the first point of (b). The second point of (b) is a consequence of the monotonicity of the layer numbers. \square

C.2. Proof of Lemma 5.2.

Proof. First, let us assume that ϕ ($md(\phi) = k \geq 0$) is satisfiable for $\text{QCTL}_{\mathbf{X}}^t$, i.e. $\mathfrak{T}, \varepsilon \models \phi$ holds with the tree model $\mathfrak{T} = \langle V, E, l \rangle$. Let $\mathfrak{T}' = \langle V, E, l' \rangle$ be the tree model obtained from \mathfrak{T} by providing truth values for the propositional variables in Y_k . More precisely, for all $v \in V$ with $\varepsilon E^j v$, we have that $l'(v) \stackrel{\text{def}}{=} (l(v) \setminus Y_k) \cup \{\text{layer}_{\max(-1, k-j)}\}$. As \mathfrak{T} is a tree model, $\varepsilon E^j v$ implies that j is the unique number of steps to reach v from ε . Obviously, \mathfrak{T}' is k -layered and hence, by Lemma 5.1, $\mathfrak{T}', \varepsilon \models \text{shape}(k)$ holds. Moreover, by structural induction, one can show that for all $j \in [0, k]$, for all $v \in V$ with $\varepsilon E^{(k-j)} v$, and for all subformulae ψ of ϕ of modal depth less than j , $\mathfrak{T}, v \models \psi$ if and only if $\mathfrak{T}', v \models \text{trans}(j, \psi)$. This leads to the satisfaction of $\mathfrak{T}', \varepsilon \models \text{trans}(k, \phi)$.

- For the base case $j = 0$, for all formulae ψ of modal degree 0, we have $\mathfrak{T}, v \models \psi$ iff $\mathfrak{T}', v \models \text{trans}(0, \psi)$ due to the fact that $\text{trans}(0, \psi) = \psi$ and, that \mathfrak{T} and \mathfrak{T}' agree on the propositional variables occurring in ϕ .
- For the induction step, the proof for the cases with Boolean connectives is immediate.
- We now consider the case with propositional quantification. Suppose that $\mathfrak{T}, v \models \exists p \psi$. Hence, there is $\mathfrak{T}^* = \langle V, E, l^* \rangle$ such that $\mathfrak{T}^* \approx_{AP \setminus \{p\}} \mathfrak{T}$ and $\mathfrak{T}^*, v \models \psi$. Let $\mathfrak{T}^{**} = \langle V, E, l^{**} \rangle$ be the variant obtained from \mathfrak{T}^* such that for all $v' \in V$ with $\varepsilon E^j v'$, we have $l^{**}(v') \stackrel{\text{def}}{=} (l^*(v') \setminus Y_k) \cup \{\text{layer}_{\max(-1, k-j)}\}$. By the induction hypothesis, $\mathfrak{T}^{**}, v \models \text{trans}(j, \psi)$. It is easy to check that $\mathfrak{T}^{**} \approx_{AP \setminus \{p\}} \mathfrak{T}'$ and therefore $\mathfrak{T}', v \models \exists p \text{trans}(j, \psi)$. The proof for the other direction is analogous.
- Finally, we consider the case with **EX**. First, let us suppose that $\mathfrak{T}, v \models \mathbf{EX}\psi$ with $\varepsilon E^{(k-j)} v$ and the modal depth of ψ is less than j . Hence, there is v' such that $v E v'$ and $\mathfrak{T}, v' \models \psi$. Thus, $\varepsilon E^{(k-(j-1))} v'$ and therefore v' satisfies layer_{j-1} in \mathfrak{T}' . By the induction hypothesis (ψ is also of modal depth less than $j-1$), we conclude $\mathfrak{T}', v' \models \text{layer}_{j-1} \wedge \text{trans}(j-1, \psi)$. As v' is also a child of v in \mathfrak{T}' (and therefore a descendant), we obtain $\mathfrak{T}', v \models \mathbf{EF}(\text{layer}_{j-1} \wedge \text{trans}(j-1, \psi))$. Conversely, suppose that $\mathfrak{T}', v \models \mathbf{EF}(\text{layer}_{j-1} \wedge \text{trans}(j-1, \psi))$. Thus, there is a descendant v' such that $v E^* v'$ and $\mathfrak{T}', v' \models \text{layer}_{j-1} \wedge \text{trans}(j-1, \psi)$. By definition of l' , we have $\varepsilon E^{k-j+1} v'$ and therefore $v E v'$. By the induction hypothesis, we obtain $\mathfrak{T}, v' \models \psi$ (again, ψ is also of modal depth less than $j-1$), which implies $\mathfrak{T}, v \models \mathbf{EX}\psi$.

For the other implication, we assume that $\text{trans}(k, \phi) \wedge \text{shape}(k)$ is satisfiable for $\text{QCTL}_{\mathbf{F}}^t$, that is $\mathfrak{T}, \varepsilon \models \text{trans}(k, \phi) \wedge \text{shape}(k)$ holds with the tree model $\mathfrak{T} = \langle V, E, l \rangle$ and $md(\phi) = k \geq 0$. By Lemma 5.1, the tree model \mathfrak{T} is k -layered and therefore satisfying layer_i and jumping to a node with the help of $\mathbf{EF}(\text{layer}_{i-1} \wedge \dots)$ leads to a child node (assuming that $i \in [0, k]$). Let $\mathfrak{T}' = \langle V', E', l' \rangle$ be the tree model defined as follows:

- V' is the least subset of V satisfying the conditions below:
 - $\varepsilon \in V'$,
 - if $v \in V'$ and $\text{layer}_j \in l(v)$ for some $j \in [0, k]$, then for all $v' \in V$ such that $\text{layer}_{j-1} \in l(v')$ and $v E v'$, then $v' \in V'$. The children of v that do not satisfy layer_{j-1} are ignored in \mathfrak{T}' .
- l' is the restriction of l to V' .
- For all $v, v' \in V'$, $v E' v' \stackrel{\text{def}}{=} \text{one of the conditions below holds}$:
 - $\text{layer}_{-1} \in l(v) \cap l(v')$ and $v E v'$.
 - For some $j \in [0, k]$, $v E v'$, $\text{layer}_j \in l(v)$ and $\text{layer}_{j-1} \in l(v')$.

It is not difficult to check that \mathfrak{T}' is a tree model (finite-branching tree and all the maximal branches are infinite), as \mathfrak{T} satisfies the formula below (due to the satisfaction of $\text{shape}(k)$):

$$\bigwedge_{i \in [0, k]} \mathbf{AG}(\text{layer}_i \rightarrow \mathbf{EF} \text{layer}_{i-1}).$$

Similarly to what we did above, by structural induction, one can show that for all $j \in [0, k]$, for all $v \in V'$ such that $\text{layer}_j \in l(v)$, and for all subformulae ψ of ϕ of modal depth less than j , we have $\mathfrak{T}, v \models \text{trans}(j, \psi)$ iff $\mathfrak{T}', v \models \psi$. This leads to the satisfaction of $\mathfrak{T}', \varepsilon \models \phi$ as $\mathfrak{T}, \varepsilon \models \text{layer}_k$ holds by the satisfaction of $\text{shape}(k)$.

- For the base case $j = 0$, for all formulae ψ of modal degree 0, we have $\mathfrak{T}, v \models \text{trans}(0, \psi)$ iff $\mathfrak{T}', v \models \psi$ due to the fact that $\text{trans}(0, \psi) = \psi$ and, \mathfrak{T} and \mathfrak{T}' agree on the propositional variables occurring in ϕ .
- For the induction step, the proof for the cases with Boolean connectives and propositional quantification is by easy verification (*cf.* the proof in the other direction).
- Let us treat in depth the case with $\mathbf{EX}\psi$, with $\text{layer}_j \in l(v)$ and $\mathbf{EX}\psi$ is of modal depth less than j . Suppose that $\mathfrak{T}, v \models \text{trans}(j, \mathbf{EX}\psi)$. So, this means that $\mathfrak{T}, v \models \mathbf{EF}(\text{layer}_{j-1} \wedge \text{trans}(j-1, \psi))$. There is $v' \in V$ such that vE^*v' and $\mathfrak{T}, v' \models \text{layer}_{j-1} \wedge \text{trans}(j-1, \psi)$. As \mathfrak{T} is a k -layered tree model, necessarily vEv' (otherwise there are two distinct nodes on the branch from v such that either both satisfy layer_j or both satisfy layer_{j-1} , which leads to a contradiction). By definition of E' , we get $vE'v'$ and by the induction hypothesis (ψ is also of modal depth less than $j-1$), we get $\mathfrak{T}', v' \models \psi$. Hence, we obtain $\mathfrak{T}', v \models \mathbf{EX}\psi$.

Conversely, assume that $\mathfrak{T}', v \models \mathbf{EX}\psi$. Thus, there exists $v' \in V'$ such that $vE'v'$ and $\mathfrak{T}', v' \models \psi$. By definition of E' , $\mathfrak{T}, v' \models \text{layer}_{j-1}$ and hence, by the induction hypothesis, we get $\mathfrak{T}, v' \models \text{layer}_{j-1} \wedge \text{trans}(j-1, \psi)$. As one can check that $E' \subseteq E$ and hence, we conclude that $\mathfrak{T}, v \models \mathbf{EF}(\text{layer}_{j-1} \wedge \text{trans}(j-1, \psi))$. \square

C.3. Proof of Lemma 5.5.

Proof. As far as TOWER-hardness is concerned, in order to enforce finite tree models, it is sufficient to consider the reduction defined for QCTL_X^t in Section 4 but to modify the definition of the formula type(0) so that type(0) is now equal to $\neg \mathbf{EX} \top$. In that way, the finite grids of the form $[0, t(k, n) - 1] \times [0, t(k, n) - 1]$ can still be encoded but with finite tree models.

In order to get the TOWER upper bound, let us define a reduction to the satisfiability problem for QCTL^t by simply identifying finite trees within tree models for QCTL^t (known to be in TOWER by [37]). Let ϕ be a formula in QK^t . Without loss of generality, we assume that ϕ may contain occurrences of \mathbf{EX} and no occurrences of \mathbf{AX} . We introduce the formula $\text{trans}(\phi) \wedge \phi_{\text{fin}}$ in QCTL^t , where ϕ_{fin} enforces that the fresh propositional variable *in* holds true only finitely on each branch and $\text{trans}(\phi)$ admits a recursive definition, by relativising the occurrences of \mathbf{EX} with respect to *in*. Let ϕ_{fin} be the formula $\text{in} \wedge \mathbf{AF} \neg \text{in} \wedge \mathbf{AG}(\neg \text{in} \rightarrow \mathbf{AG} \neg \text{in})$. The satisfiability of ϕ_{fin} at the root node ε implies that *in* holds exactly on a subtree from ε where all the branches are finite. It remains to define $\text{trans}(\phi)$:

- $\text{trans}(p) \stackrel{\text{def}}{=} p$ for all propositional variables p , and trans is homomorphic for Boolean connectives and propositional quantification,
- $\text{trans}(\mathbf{EX}\psi) \stackrel{\text{def}}{=} \mathbf{EX}(\text{in} \wedge \text{trans}(\psi))$.

We recall that the satisfaction of ϕ_{fin} at the root node ε implies *in* holds exactly on a subtree from ε where all the branches are finite.

One can show that ϕ is satisfiable for QK^t iff $\text{trans}(\phi) \wedge \phi_{\text{fin}}$ is satisfiable in QCTL^t (note that *in* does not need to be part of the conjunction as the root is always part of the model). Moreover, as the models for QCTL^t are finite-branching trees (see *e.g.* [37, Remark 5.7]), ϕ is satisfiable in a finite tree model iff $\text{trans}(\phi) \wedge \phi_{\text{fin}}$ is satisfiable in a finite-branching tree model, which leads to the desired upper bound TOWER. \square

C.4. Proof of Lemma 5.7.

Proof. First, let us show that $\text{SAT}(\text{QGL}^t)$ and $\text{SAT}(\text{QCTL}_{\mathbf{XF}}^{ft})$ are identical problems modulo the rewriting of **EX** into **EXEF**. Herein, $\text{QCTL}_{\mathbf{XF}}^{ft}$ is defined as QCTL^{ft} restricted to the combined temporal operator **EXEF**. According to Section 2, the models for QCTL^{ft} are finite trees. In a second part of the proof, we show that $\text{SAT}(\text{QCTL}_{\mathbf{XF}}^{ft})$ is TOWER-complete.

(I) Let t be the map from QGL^t formulae into $\text{QCTL}_{\mathbf{XF}}^{ft}$ formulae such that $t(\phi)$ is defined from ϕ by replacing every occurrence of **EX** by **EXEF**. Similarly, the inverse map t^{-1} is defined so that $t^{-1}(\psi)$ with a $\text{QCTL}_{\mathbf{XF}}^{ft}$ formula ψ , is defined from ψ by replacing every occurrence of **EXEF** by **EX**.

Let us show that (a) ϕ is QGL^t satisfiable iff $t(\phi)$ is $\text{QCTL}_{\mathbf{XF}}^{ft}$ satisfiable and (b) ψ is $\text{QCTL}_{\mathbf{XF}}^{ft}$ satisfiable iff $t^{-1}(\psi)$ is QGL^t satisfiable. Since t is bijective, it is sufficient to show (a). Let $\mathcal{K} = \langle W, R, l \rangle$ be a QGL^t model and $w \in W$ such that $\mathcal{K}, w \models \phi$. As \mathcal{K} is a QGL^t model, $(W, R) = (W, E^+)$ for some finite tree (W, E) with root w . Consequently for all $w_1, w_2 \in W$, we have $(w_1, w_2) \in R$ iff (w_1, w_2) belongs to the transitive closure of E . By structural induction, one can easily show that for all $w' \in W$ and for all subformulae ϕ' of ϕ , we have $\mathcal{K}, w' \models \phi'$ iff $\langle W, E, l \rangle, w' \models t(\phi')$. Consequently, ϕ is QGL^t satisfiable implies $t(\phi)$ is $\text{QCTL}_{\mathbf{XF}}^{ft}$ satisfiable as $\langle W, E, l \rangle$ is a model for $\text{QCTL}_{\mathbf{XF}}^{ft}$ since it is a finite tree.

Conversely, let $\mathfrak{T} = \langle V, E, l \rangle$ be a $\text{QCTL}_{\mathbf{XF}}^{ft}$ model (finite tree) with root ε such that $\mathfrak{T}, \varepsilon \models t(\phi)$. Let \mathcal{K} be the Kripke structure $\langle V, E^+, l \rangle$ defined from \mathfrak{T} and by definition, \mathcal{K} is an QGL^t model. Again, by structural induction, one can easily show that for all $w' \in V$ and for all subformulae ϕ' of ϕ , we have $\langle V, E, l \rangle, w' \models t(\phi')$ iff $\mathcal{K}, w' \models \phi'$. Consequently, ϕ is QGL^t satisfiable as we have assumed that $\mathfrak{T}, \varepsilon \models t(\phi)$ and therefore $\mathcal{K}, \varepsilon \models \phi$.

(II) Let us show that the satisfiability problem for $\text{QCTL}_{\mathbf{XF}}^{ft}$ is TOWER-complete. We need to take care of both the lower bound and of the upper bound. As far as TOWER-hardness is concerned, we define a reduction from the satisfiability for $\text{QCTL}_{\mathbf{X}}^{ft}$ (see Theorem 5.5 as $\text{QCTL}_{\mathbf{X}}^{ft}$ and QK^t are identical), and the proof is very similar to the one for $\text{QCTL}_{\mathbf{F}}^t$ (actually, it is a bit simpler). The main steps are summarised below. Let ϕ be a formula in $\text{QCTL}_{\mathbf{X}}^{ft}$ of the modal depth $md(\phi) = k$. Similarly to what was done before, let us consider the set of fresh propositional variables $Y_k = \{\text{layer}_0, \dots, \text{layer}_k\}$ with the intended meaning that a node satisfying layer_i is of “layer i ”, the root node being of layer k . Let us define the formula $\text{trans}(k, \phi) \wedge \text{shape}(k)$ in $\text{QCTL}_{\mathbf{XF}}^t$, where $\text{shape}(k)$ is the conjunction of the following formulae:

- Every node satisfies exactly one propositional variable from Y_k (layer unicity) and the root satisfies $layer_k$.

$$(layer_k \wedge \bigwedge_{i \neq k} \neg layer_i) \wedge \mathbf{AXAG}((layer_0 \vee \dots \vee layer_k) \wedge \bigwedge_{0 \leq i \neq j \leq k} \neg(layer_i \wedge layer_j)).$$

- When a node satisfies $layer_i$ with $i \leq k$, none of its descendants satisfies some $layer_j$ with $j > i$ (monotonicity of layer numbers).

$$\mathbf{AXAG}(layer_0 \vee \dots \vee layer_{k-1}) \wedge \bigwedge_{i \leq k-1} \mathbf{AXAG}(layer_i \rightarrow \mathbf{AXAG}(layer_0 \vee \dots \vee layer_{i-1})).$$

- When a node satisfies $layer_i$ with $1 \leq i \leq k-1$, there is a descendant satisfying $layer_{i-1}$ (weak progress).

$$\mathbf{EXEF} layer_{k-1} \wedge \mathbf{AXAG}(layer_i \rightarrow \mathbf{EXEF} layer_{i-1}).$$

- The nodes satisfying $layer_0$ have no successor: $\mathbf{AXAG}(layer_0 \rightarrow \neg \mathbf{EXEF} \top)$.

The formula $trans(k, \phi)$ is defined as in the reduction from $\mathbf{QCTL}_{\mathbf{X}}^t$ to $\mathbf{QCTL}_{\mathbf{XF}}^t$ (see Section 5.1). One can show that ϕ is satisfiable in $\mathbf{QCTL}_{\mathbf{X}}^{ft}$ iff $trans(k, \phi) \wedge \text{shape}(k)$ is satisfiable in $\mathbf{QCTL}_{\mathbf{XF}}^{ft}$.

To get the TOWER upper bound, let us define a reduction to the satisfiability problem for \mathbf{QCTL}^t (known to be in TOWER by [37]). Let ϕ be a formula in $\mathbf{QCTL}_{\mathbf{XF}}^{ft}$. We introduce the formula $trans(\phi) \wedge \phi_{\text{fin}}$ in \mathbf{QCTL}^t , where $trans(\phi)$ is recursively defined as follows:

- $trans(p) \stackrel{\text{def}}{=} p$ for all propositional variables p , and $trans$ is homomorphic for Boolean connectives and propositional quantification,
- $trans(\mathbf{EXEF} \psi) \stackrel{\text{def}}{=} \mathbf{EXEF}(in \wedge trans(\psi))$.

As in the proof of Theorem 5.5, ϕ_{fin} is equal to $in \wedge \mathbf{AF} \mathbf{AG} \neg in \wedge \mathbf{AG}(\neg in \rightarrow \mathbf{AG} \neg in)$. One can show that ϕ is satisfiable in a tree model without infinite branches iff $trans(\phi) \wedge \phi_{\text{fin}}$ is satisfiable in \mathbf{QCTL}^t . Again, as the models for \mathbf{QCTL}^t are finite-branching, ϕ is satisfiable in a finite tree model iff $trans(\phi) \wedge \phi_{\text{fin}}$ is satisfiable in a finite-branching tree models, which leads to TOWER-easiness. \square

C.5. Proof of Lemma 5.8.

Proof. First, let us show that $\text{SAT}(\mathbf{QK4}^t)$ and $\text{SAT}(\mathbf{QCTL}_{\mathbf{XF}}^{gt})$ are identical problems modulo the rewriting of \mathbf{EX} into \mathbf{EXEF} . Herein, $\mathbf{QCTL}_{\mathbf{XF}}^{gt}$ is defined as \mathbf{QCTL}^{gt} restricted to the temporal operator \mathbf{EXEF} . According to Section 2, the models for \mathbf{QCTL}^{gt} are finite-branching trees. In a second part of the proof, we show that $\text{SAT}(\mathbf{QCTL}_{\mathbf{XF}}^{gt})$ is TOWER-complete.

(I) Let t be the map from $\mathbf{QK4}^t$ formulae into $\mathbf{QCTL}_{\mathbf{XF}}^{gt}$ formulae such that $t(\phi)$ is defined from ϕ by replacing every occurrence of \mathbf{EX} by \mathbf{EXEF} . Similarly, the inverse map t^{-1} is defined so that $t^{-1}(\psi)$ with a $\mathbf{QCTL}_{\mathbf{XF}}^{ft}$ formula ψ , is defined from ψ by replacing every occurrence of \mathbf{EXEF} by \mathbf{EX} . This is similar to what is done in the proof of Theorem 5.7.

Let us show that (a) ϕ is $\mathbf{QK4}^t$ satisfiable iff $t(\phi)$ is $\mathbf{QCTL}_{\mathbf{XF}}^{gt}$ satisfiable and (b) ψ is $\mathbf{QCTL}_{\mathbf{XF}}^{gt}$ satisfiable iff $t^{-1}(\psi)$ is $\mathbf{QK4}^t$ satisfiable. It is sufficient to show (a). Let $\mathcal{K} = \langle W, R, l \rangle$ be an $\mathbf{QK4}^t$ model and $w \in W$ such that $\mathcal{K}, w \models \phi$. As \mathcal{K} is a $\mathbf{QK4}^t$ model $(W, R) = (W, E^+)$ for some finite-branching tree (W, E) with root w . Consequently for all $w_1, w_2 \in W$, we have

$(w_1, w_2) \in R$ iff (w_1, w_2) belongs to the transitive closure of E . By structural induction, one can easily show that for all $w' \in W$ and for all subformulae ϕ' of ϕ , we have $\mathcal{K}, w' \models \phi'$ iff $\langle W, E, l \rangle, w' \models t(\phi')$. Consequently, ϕ is QK4^t satisfiable implies $t(\phi)$ is $\text{QCTL}_{\mathbf{XF}}^{gt}$ satisfiable as $\langle W, E, l \rangle$ is a model for $\text{QCTL}_{\mathbf{XF}}^{gt}$.

Conversely, let $\mathfrak{T} = \langle V, E, l \rangle$ be a $\text{QCTL}_{\mathbf{XF}}^{gt}$ model (finite-branching tree) with root ε such that $\mathfrak{T}, \varepsilon \models t(\phi)$. Let \mathcal{K} be the Kripke structure $\langle V, E^+, l \rangle$ defined from \mathfrak{T} and by definition, \mathcal{K} is an QK4^t model. Again, by structural induction, one can easily show that $\mathcal{K}, \varepsilon \models \phi$.

(II) In order to establish the upper bound TOWER, let us provide a reduction from the satisfiability for $\text{QCTL}_{\mathbf{XF}}^{gt}$ to the satisfiability problem for $\text{QCTL}_{\mathbf{XF}}^t$. Let ϕ be a formula in $\text{QCTL}_{\mathbf{XF}}^{gt}$. We introduce the formula $\text{trans}(\phi) \wedge \phi'_{\text{fin}}$ in $\text{QCTL}_{\mathbf{XF}}^t$, where ϕ'_{fin} enforces that the propositional variable *in* holds false on all descendants, as soon as it does not hold on a node and that $\text{trans}(\phi)$ admits a recursive definition, by relativising the occurrences of **EXEF** with respect to *in*. Let ϕ'_{fin} be the formula $\text{in} \wedge \mathbf{AXAG}(\neg \text{in} \rightarrow \mathbf{AXAG} \neg \text{in})$.

It remains to define $\text{trans}(\phi)$:

- $\text{trans}(p) \stackrel{\text{def}}{=} p$ for all propositional variables p , and trans is homomorphic for Boolean connectives and propositional quantification,
- $\text{trans}(\mathbf{EXEF}\psi) \stackrel{\text{def}}{=} \mathbf{EXEF}(\text{in} \wedge \text{trans}(\psi))$.

It is easy to see that ϕ is satisfiable for $\text{QCTL}_{\mathbf{XF}}^{gt}$ iff $\text{trans}(\phi) \wedge \phi'_{\text{fin}}$ is satisfiable for $\text{QCTL}_{\mathbf{XF}}^t$.

As far as TOWER-hardness is concerned, for any formula ϕ in $\text{QCTL}_{\mathbf{XF}}^t$, one can show that ϕ is satisfiable for $\text{QCTL}_{\mathbf{XF}}^t$ iff $\phi \wedge \mathbf{EX} \mathbf{EF} \top \wedge \mathbf{AXAG} \mathbf{EXEF} \top$ is satisfiable in $\text{QCTL}_{\mathbf{XF}}^{gt}$. The two last conjuncts simply state that from any node, there is a child, which enforces that all the maximal branches are infinite. As the satisfiability problem for $\text{QCTL}_{\mathbf{XF}}^t$ is TOWER-hard (Theorem 5.4), this concludes the proof. \square