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Understanding mathematical work and mathematical thinking through individuals' actions analyses: a networking approach

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The theory of Mathematical Working Space (ThMWS) provides theoretical and methodological tools to describe and characterize mathematical work. A particular methodology has been developed to analyse actions produced by subjects when they perform mathematical tasks. In this paper, this methodology is introduced by an example of a geometrical task. Then, we show how two theories on knowledge construction based on actions were combined with ThMWS. Finally, we present the contributions drawn from these networking cases which provide insight into mathematical work as well as the thinking of subjects by articulating their knowledge and actions.

Keywords: Mathematical Working Space, mathematical work and mathematical thinking, action and knowledge, methodology.

Introduction

Within the theory of Mathematical Working Space (ThMWS), the understanding of mathematical work is mainly supported by the analysis of the actions, which students and teachers perform in the school context. In this paper, we wonder how to better understand the cognitive dimension of mathematical work and we explore this issue through common analysis of actions with some cognitive approaches.

We first justify why actions can be used as a common key-point for discussion between ThMWS and cognitive theories. After giving some of the key-constructs of the ThMWS used in the paper, we provide an example to introduce the methodology we have implemented to account for the mathematical work of individuals performing mathematical tasks. We then consider how research using both ThMWS and Abstraction in Context (AiC), or APOS theory, has opened up other avenues to describe and characterize the articulations between actions of individuals and their knowledge. It is thus possible to better understand subjects' mathematical work and mathematical thinking.

Articulating mathematical work and mathematical thinking through subjects' actions analysis

The approach of teaching and learning issues using the notion of mathematical work is relatively new in the field of research on mathematics education. Indeed, the great majority of research on teaching and learning refers to the idea of mathematical thinking as CERME conferences indicate through the topics of the working groups on mathematical contents. These topics have long been expressed in the form of specific modes of thinking such as algebraic thinking, geometric thinking, and probabilistic thinking. The focus on "mathematical thinking" urges to attend "thinking" with the real risk of dissolving the mathematical specificity in all the fields, which are concerned with

thinking. Indeed, the study of thinking supposes a multiplicity of approaches, which are often far apart from mathematical concerns.

Recently, cognitive sciences have also approached thinking through cognition with the objective of being "the science of thinking". The cognitivist current is itself a composite combination of disciplines such as neuroscience, artificial intelligence, the phenomenology of perception, etc. This current is probably at the origin of the collapse of the primacy of mental representations for the benefit of action in the field of mathematics education. In this view, action is prior to mental representation as the neurobiologist Berthoz (2008) endorses it, and thinking and consciousness relate directly to action and perception. This viewpoint is not new in the field and was already pointed out in his *genetic epistemology* by Piaget who considered that mathematical thinking is not "pre-formed but a creation constantly pursued by actual and new actions which open a new set of possibilities hitherto inconceivable" (Piaget, 1957, p. 34). The support of cognitive sciences gives this approach a new visibility.

This change of perspective on mathematical thinking is valued from the point of view of ThMWS and more generally of studies that favour an entry through mathematical work. In this case, the focus on work naturally places action as a key theoretical and methodological tool in the study of mathematics teaching and learning in school contexts. However, considering mathematical thinking and mathematical work as complementary seems to be necessary to account for the complexity of school mathematics. In particular, some cognitive theories, such as AiC and APOS, focus on the construction of abstract mathematical knowledge and can both complement and benefit from approaches to mathematical work that focus more on the outcomes and processes of work.

Describing and characterizing mathematical work: the ThMWS approach

The theory of MWS: motivations and issues

In the theory of MWS (Kuzniak et al., 2016), mathematical work is considered as intellectual work that involves significant cognitive activity of which the orientation and purpose are defined and supported by mathematics. In this framework, mathematics is envisaged as a specific human work and therefore, mathematical work is seen in both its cognitive and epistemological dimensions.

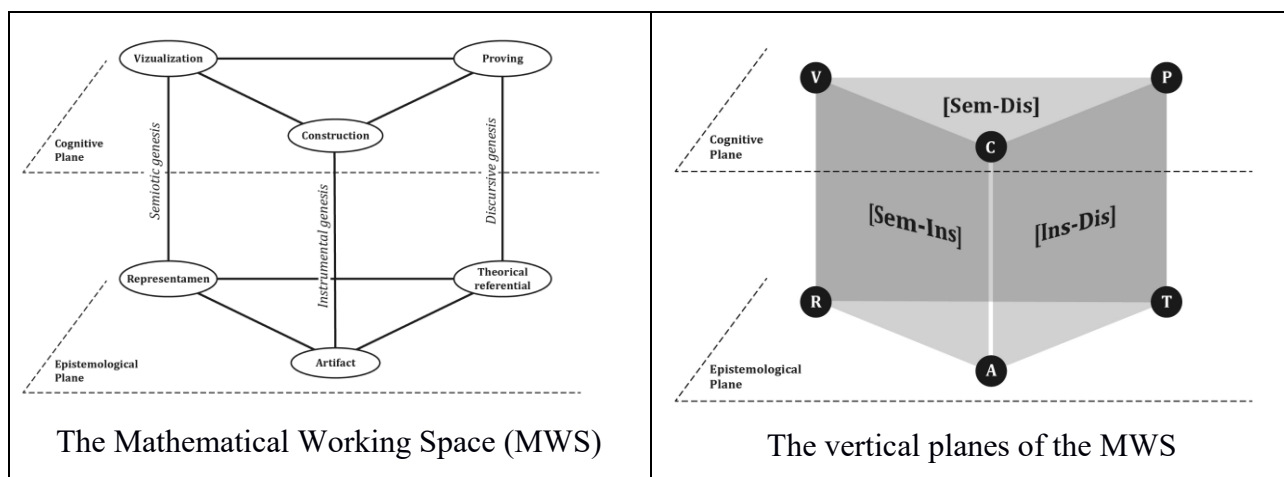


Figure 1: The MWS diagrams

To grasp the specific activity of students solving problems in mathematics, the two epistemological and cognitive facets of mathematics are present and articulated in two planes: one of an epistemological nature in close relation to the mathematical content; the other of a cognitive nature, related to the visible action of the individual solving problems (Figure 1, left). Three components in interaction are introduced for the purpose of describing the work in its epistemological dimension: a set of concrete and tangible signs (R), a set of artefacts (A), and a theoretical system of reference (T). The second level is centred in the subject; three cognitive components are introduced as follows: visualisation (V) related to deciphering and interpreting signs; construction (C) depending on the used artefacts and the associated techniques; proving (P) conveyed through discourses producing validations, and based on the theoretical system of reference.

Analysing mathematical work through ThMWS aims at tracking down the process of bridging the epistemological plane and the cognitive plane in accordance with three different geneses: An instrumental genesis (Ins) related to the use of artefacts, a discursive genesis (Dis) related to theoretical part of work, and finally a semiotic genesis (Sem) that expresses the importance of signs for mathematical work. The elaboration of mathematical work is perceived as the harmonious networking and intertwining of these geneses within the vertical planes of the diagram (Figure 1, right).

To achieve the description and characterization of mathematical work, studies performed within ThMWS theory are based on the analysis of the subject's various actions. Thus, the methodology of studying subjects' actions is central in the approach and is developed in the next section.

A cognitive analysis of subject's actions: The example of Alphonse's Parcel

Our starting point is that cognitive activities of a subject at work are not directly observable but they can be deduced from the visible actions of the subjects when they are performing mathematical tasks. The methodology we use must both consider mathematical content and give access to subjects' cognitive processes. We have chosen to introduce our methodology by an example, which is used along the text to show the combination of the different theories involved in the networking study: Alphonse's parcel (Kuzniak & Nechache, 2021). The task statement is given in the form of a text to be read.

Alphonse has just returned from a trip in Périgord where he saw a parcel of land in the shape of a quadrilateral that had interested his family. He would like to estimate its area. To do this, during his trip, he successively measured the four sides of the plot and found, approximately, 300 m, 900 m, 610 m, 440 m. He's having a hard time finding the area. Can you help him by showing him the method to be followed?

Data given in the task does not enable to find a unique solution to the problem because quadrilaterals with the same side lengths can have different forms and areas. However, the majority of students used the following very common Theorem in Act (TiA): figures with the same perimeter or the same side lengths have the same area. The main objective of the whole didactic situation explicitly destabilized the students and helped them to banish this false Theorem in Act.

To give a general overview of our methodology, we provide an analysis of Francis's solution of the problem. It is based on a corpus of data collected during the resolution of the mathematical task.

This corpus consists of his written and oral productions. We start dividing his activity into episodes. Each of the episodes is related to a sub-task self-prescribed by the student to achieve the global task: it includes a sequence of mathematical actions that lead to the realization of the sub-task. All the actions are analysed in detail by using ThMWS. The MWS diagrams are also used to visualize the circulation of work during the resolution of the task (Kuzniak & Nechache, 2021).

Table 1. Francis' work division in three episodes each with two actions.

<p>Episode 1: Construction of the figure</p>	<p>Episode 2: Justification of the nature of the figure</p>
	<p>I took the large base 900 m then from the two ends of the 900m with the compass, I made 440 on one side and 610 on each side and then with the ruler I tried to find the 300 with the two arcs of circle.</p> <p>I drew a perpendicular to the large base and this line perpendicular to the large base was also perpendicular to the small base of 300m and as two lines are perpendicular to the same line, they are parallel so it makes a trapezoid.</p>
<p>Action 1: Choice of scale Action 2. Construction of the figure</p>	<p>Action 3. Francis' explanation of his construction Action 4. Francis' description of the fact that it is a trapezium.</p>
<p>Episode 3: Determining the area of the figure. Action 5: Measurement of the height h of the trapezoid Action 6: Calculation of the area from the formula $(B+b) \cdot h/2$</p>	

Then a bottom-up analysis gives a summary overview of the different episodes, and helps to deduce the logical organization of Francis' mathematical actions. The analysis seeks to identify how the work-generating processes interact within the components of MWS. From the data, we surmise that mathematical work produced by Francis is triggered by the referential and initiated in the [Sem-Ins] plane to construct a convex quadrilateral with the shape of a trapezoid using technological tools. These tools are used by the student as semiotic tools to collect data and make measurements. Construction of the quadrilateral is then justified with the use of theoretical tools characterizing the trapezoid. The formula for calculating the area of a trapezoid (technological tool) is applied to produce the result (in this case, the area of the land). To conclude the analysis, observations made during the two previous steps are used to characterize the mathematical work. In Francis' case, we may consider that his work circulates through the different geneses and planes of the MWS diagram. He uses drawing and measure tools to support his reasoning and argues about a particular figure, so his work is compliant with a geometrical paradigm based on measurement and drawing

(Geometry I, see Kuzniak & Nechache, 2021). Finally, his mathematical work is not mathematically correct, as the result is false and based on a construction of an inappropriate figure and TiA theorem.

The description above provides the most accurate account possible of the student's choices and actions. However, it is sometimes difficult to be sure, of what really triggered and motivated these actions. Some of Francis' actions are material and easy to interpret, such as constructing the trapezium, but others are more challenging. Why did he choose to use a scale for the construction and then measure in the drawing? When did he focus on the trapezoid? Why didn't he finally see his mistake? To progress with the interpretation of subjects' actions, we explore other methods of action analysis based on theories, which are rather focused on mathematical thinking.

Combining ThMWS with cognitive theories

Several studies have combined ThMWS and cognitive theories. In this paper, we only refer on the studies with AiC and APOS. For each research, we first present its general purpose and introduce the method, which the authors used. Then, we apply this method to Alphonse's task and show the new contribution that can be drawn from the approaches to advance the analysis of Francis' actions and progress in its understanding.

A joint study between the AiC and MWS theories

In their study of the development of functional thinking of students working in four different MWS, Psycharis et al. (2021) combined Abstraction in Context (AiC theory, Dreyfus et al., 2015) and ThMWS. Using a task on the optimal shape of a gutter, they analysed students' progress in their understanding of covariation by having them work successively on four gutter models in different environments (sheet of paper, DGS, etc.). Each of these models is associated with a different MWS. The connections between these MWS are described and explained thanks to the MWS geneses involved. This first analysis through the MWS theory is completed with the AiC theory to account for the processes of constructing abstract mathematical knowledge. In this theory, abstraction is defined as an activity of vertical reorganization of previous mathematical constructs into a new abstract construct. According to the authors, the process of abstraction passes through three stages of knowledge construction shaped by three epistemic actions: recognizing (R), building-with (B) and constructing (C). These actions are nested. They allow tracing the process of knowledge construction and are used as a methodological tool for analysis.

Using AiC for Alphonse's solution of the task, we first focus on *recognizing*. If we look again at episode 1 (figure construction), we surmise that the student *recognized* a construction problem as a previous meta-construct (R0) for which he knew mathematical knowledge involved in the figure construction. While constructing, using the given side lengths but ignoring the word “successive”, he recognized (R1) a partially correct construct, a trapezoid, and with it its area formula, another previous construct (R2). Thus, he realized its shape by an approximation and calculated its area via measuring, the latter being a building-with (B) action built on the recognized constructs. The mathematical object "trapezoid" determines the follow-up of Francis' work and as it is only partially correct, he cannot construct the expected knowledge about the various areas of quadrilaterals having the same side lengths –construction of a new construct (C). In the example of the work carried out

by Francis, the mathematical object named “trapezoid” is considered under different aspects during the completion of the task and referring to the three components of the epistemological plane: its drawing aspect (representamen), the material artefact aspect (ruler, compass, square) or symbolic artefact (area calculation formula) associated with the trapezoid, and its property and definition. All these elements constitute what we call the Epistemological Entity associated to the trapezoid overlapping with what is called Context in AiC.

The subject’s visualization of the "trapezoid" object as being a quadrilateral with two parallel lines, his construction of a figure and the area with techniques associated with the artefacts to develop the trapezoid, and his justification of these constructions by citing the properties that characterize the trapezoid are all part of what we name the Cognitive Unit associated with the mathematical object “trapezoid”. Identifying the existence and role of the Cognitive Unit and Epistemological Entity associated with the mathematical object “trapezoid” enables to characterize the mathematical work and understand mathematical thinking in terms of epistemic actions developed by AiC.

A joint study between the APOS (Action, Process, Object, Schemas) and MWS theories

In her research, Camacho (2021) uses the APOS (Arnon et al., 2014) and ThMWS together to study the linear algebraic concepts of eigenvalues and eigenvectors in relation to their geometrical representation in the plane. This study helps to further consider *action* by underlining the diversity of meanings of this term. Indeed, in the APOS theory, which assumes the heritage of Piaget, the term Action is one of the basic terms of the theory. The acronym APOS stands for Action, Process, Object and Schema. The action to which it refers is a mental structure that must be distinguished from "material action", which Dubinsky interpreted as actions that are performed by a subject but are external to the subject (Arnon et al., 2014, p. 7). From a set of actions on objects, the subject can derive a property arriving at a more general Action that allows passing from Action to Process through the mental mechanism of *interiorization*. The mechanism of *encapsulation* then transforms the Process into an Object as a cognitive entity. A Schema is then build by acting on the mathematical object. In her research methodology, Camacho applies an approach, specific to APOS theory, which consists of discovering the cognitive functioning of subjects according to their failure or success about an Object in some mathematics tasks. It is then a matter of recognizing, which of the components of the quadruplet (APOS) is operative or not. In Alphonse task, the starting point of the analysis is an initial reflection on the mathematical Object, with its Schemas, which is at stake in the situation. For us, it is the Area of a quadrilateral (Object) and this area is associated to a decomposition by triangulation (Schema) or more generally to the triangulation of any polygon, (Kuzniak & Nechache, 2021). Francis however, activates another scheme acting on another mathematical object: he constructs a trapezoid by approximation and calculates its area with a formula via measuring. The actions as conducted do not lead to processes of building the area of a quadrilateral with the given side lengths because he elaborates the schema of an already existing mathematical object (trapezoid area), which is insufficient for the task.

In APOS, authors seek to elaborate the genetic decomposition of the Object of concern to be more precise in their analysis. Camacho elaborates the genetic decomposition of the Object "collinearity" within a digital environment and in terms of the different MWS geneses. When we analyse Francis'

work with the MWS, we may notice that he relies on his old and well-established knowledge of the trapezium of which he knows both the different elements (signs, artefacts, properties) in relation to what we referred to as the Epistemological Entity associated with the trapezium. He also knows the cognitive processes related to the use of the corresponding cognitive unit. He knows well the Object "trapezoid", unfortunately, it is not the right Object, which should be used to perform this task about the area of a generic quadrilateral. The case of Francis shows that students do not really regulate their work once they have embarked on a solution that seems to be in line with their knowledge. An illusion of knowing-how that fosters certainty in action emerges. In the example, it was however possible to see that the order of the sides used in the construction was not the one indicated by the statement (semiotic control). It was also possible to see that different quadrilaterals fulfilled the same condition since Francis searched, by approximation, among these figures the one with parallel sides (semiotic and instrumental control). An internal control of the instruments (instrumental control) should have been based on another construction without approximation. Finally, it is interesting to see that his proof reasoning is biased to reach the desired result (discursive control).

Conclusion and perspectives

All of the studies we reviewed emphasise that ThMWS and its organization around the three geneses was an important structuring element in the analysis of actions and helped to describe and better understand the relationship between cognitive and epistemological aspects. Conversely, the triadic and prismatic geometric structure of the MWS generates theoretical notions that are not always easy to interpret as a tool for studying issues of mathematics teaching and learning. However, these joint studies on students' mathematical actions focussing on cognitive aspects enrich and flesh out a number of emerging notions in ThMWS. In particular, they have provided us with a better understanding of the importance and value of the following ideas:

1. The Epistemological Entities and Cognitive Units account for the importance of knowledge and cognitive functioning of individuals who perform a task using particular mathematical objects. To achieve description and characterization of mathematical work, which captures adequately interactions between cognitive and epistemological aspects, it is important to make visible the mathematical knowledge involved in the solution by identifying precisely the components of the Epistemological Entity and the Cognitive Unit associated with a mathematical object
2. These Cognitive units and Epistemological Entities depend on forms of work implemented, which can be described by different paradigms associated with the teaching contents. Based on AiC epistemic actions and on APOS contributions, it is thus possible to describe how the observed work conforms to particular modes of mathematical reasoning.
3. Finally, we have also made progress in understanding the regulation and control of outcomes through the importance of knowledge and cognitive repository as tools for regulating actions. However, we also met the attracting force of an "illusion of knowing" on acting.

The question of actions regulation and control is a research direction that we plan to develop further. Controls and regulation questions are not new in the field of mathematics education, it has existed since at least the first research on problem solving, seeking to make students more autonomous. Particularly Arzarello and Sabena (2011), who also emphasise the variety of actions

implemented by students and the teacher, have recently addressed this. They have analysed the way in which gestures, considered as a semiotic resource, intervene in the learning and teaching process of mathematics. This extension of research to gestures and the body raises the question of how to analyse and identify their real impact on students' actions and beyond on their mathematical work and thinking. We are now engaged in a new project (E-ESMEA) dealing with enacting in astronomy education. We are mainly interested in the mathematical part of the project: relations between proportionality and velocity. The project focuses on the concept of embodied action, which assigns a key role to perceptual and bodily processes in the formation of abstract concepts and knowledge. The sensory-motor engagement of the body carries these processes in situations out that constitute the functional system, on which actions are based. From this study, we expect that it will help us to identify the limits and the right levels of analysis of mathematical action that are really necessary to advance in the mastery of mathematical work and in the learning of mathematical knowledge. This is, in our opinion, a fundamental question to develop a didactic approach in phase with the school reality.

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