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Place value plays a pivotal role in understanding numbers and arithmetic operations that universally cut across the primary and secondary curriculum. This study investigates the nature of 40 qualified teachers' knowledge in the context of articulating the role of place value in understanding arithmetic operations. The participants have 3-45 years of teaching experience who currently teach grades 1-8 (ages 6-14) in public or private schools in Turkey. The data consists of the videotapes of the semi-structured interviews in which we asked the teachers to analyse five different scenarios consisting of alternative student solutions regarding arithmetic operations. Our initial findings suggest that teachers who operate with (partial) specialised content knowledge or without it analyse the given alternative solutions qualitatively differently. The analysis also reveals that the teachers rely mostly on common content knowledge that has little or no connections to a solid place value understanding.

Keywords: Place value, specialised content knowledge, mathematics teacher knowledge, arithmetic operations

Introduction

Place value plays a pivotal role in understanding number structure and arithmetic operations (Nuerk et al., 2015) that universally cut across the primary and secondary curriculum. Arithmetic and symbolic calculations require a solid understanding of place value. However, numbers and arithmetic operations are mostly treated procedurally in schools and the teaching that fosters rote learning is a common practice among teachers (Pesek & Kirshner, 2000). As a result, we see students treating numbers as a combination of concatenated single digits (Fuson et al., 1997) or having difficulty understanding the meaning of each digit in a number (Kamii, 1986). Furthermore, understanding number requires one to form groups and think about groups (or groups of groups, etc.) as single entities or composite units (Hiebert & Wearne, 1996), which is also problematic among students (Thanheiser, 2015). Such difficulties affect students' understanding of the procedures in making sense of arithmetic operations (Verschaffel et al., 2007) and algorithms (Kamii & Dominic, 1998).

Even teachers struggle to clarify their rationale for algorithms as they primarily operate from procedural aspects (Ma, 1999). They have difficulty articulating number concepts (Thanheiser et al., 2013) and place value (Southwell & Penglase, 2005). Primary school teachers are not even aware of "the impact of place value understanding on the learning of mathematics" (Houdement & Tempier, 2019, p.36). Such a crucial area requires teachers to have the necessary knowledge and understanding to teach conceptually. The research literature piles up in articulating student understandings or misunderstandings (e.g., McClain, Cobb, & Bowers, 1998), whereas it falls short in delineating teacher knowledge and the nature of that knowledge. Several studies focused on the understandings

of trainee teachers (e.g., Lo, Grant & Flowers, 2008; McClain, 2003), but our experience suggests that qualified teachers do not have a solid knowledge of concepts, as mentioned earlier. Knowing more about the source of teacher difficulties and the nature of their knowledge specific to teaching would inform educators in developing well-designed professional development programmes and give them opportunities to improve the quality of instruction in schools. As Thames and Ball (2010, p.228) pointed out, identifying the mathematical demands of mathematics teaching "allows us to identify the mathematical knowledge needed for teaching", which will help us improve the quality of instruction in schools. Therefore, the current study investigates the nature of qualified teachers' knowledge specific to teaching place value as it "should be at the core of teachers' education" (Houdement & Tempier, 2019, p.36). We pursue the following specific research question: 'What does it mean to operate with/without specialised content knowledge in the context of place value and arithmetic operations?'

Theoretical Framework

It is hard to argue against the criticality of teacher knowledge for effective teaching. Shulman (1987) characterised teacher knowledge as a combination of two major domains, subject matter knowledge (SMK) and pedagogical content knowledge (PCK). Researchers who followed the footsteps of Shulman (e.g., Grossman, 1990; Fennema & Franke, 1992; Ball, Thames & Phelps, 2008) have refined these domains over the last few decades. Recently, Deborah Ball and her colleagues (2008) defined SMK as a combination of three subcategories: common content knowledge (CCK), specialised content knowledge (SCK) and horizon knowledge. In this framework, they described CCK as "mathematical knowledge and skill used in settings other than teaching" (e.g., simple calculations, solving mathematical problems correctly) (Ball et al., 2008, 399). SCK is "the mathematical knowledge and skill unique to teaching" to respond to everyday tasks of teaching (e.g., mathematical knowledge required to analyse alternative student solutions, sizing up the nature of a nonfamiliar error), and horizon knowledge is "an awareness of how mathematical topics are related over the span of mathematics included in the curriculum" (Ball et al., 2008, p.403). The framework also considers pedagogical content knowledge as a combination of knowledge of content and students (KCS), knowledge of content and curriculum (KCC) and knowledge of content and teaching (KCT). KCS is the "knowledge that combines knowing about students and knowing about mathematics" (e.g., student thinking, common student errors), and KCT is an amalgam of knowing about teaching and knowing about mathematics (e.g., different instructional models of place value and their effective deployment) (Ball et al., 2008, p.401). Finally, KCC is about combining knowledge of content and the curriculum.

To distinguish between these knowledge types, Ball et al. (2008, p.404) gave the following examples for the task of "selecting a numerical example to investigate students' understanding of decimal numbers". In this case, they consider "ordering a list of decimals" as CCK, "generating a list to be ordered that would reveal key mathematical issues" as SCK, "recognizing which decimals would cause students the most difficulty" as KCS, and "deciding what to do about their difficulties" as KCT (Ball et al., 2008, p.404). In this example, knowing what grade level is appropriate for teaching decimal ordering would be labelled as KCC. The development of decimals throughout the mathematics curriculum would be an example of horizon knowledge.

We adopted the Ball et al. (2008) framework to analyse teacher knowledge. We specifically zoomed in on how teachers operate with SCK or in the absence of SCK to pursue the aforementioned research question.

Method

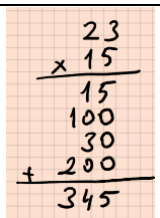
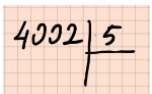
Participants

Participants were 40 teachers, 35 of whom worked as primary school teachers in public or private schools (teaching ages 6-11). Four worked as mathematics teachers in middle schools (teaching age 11-15) by the time this study was conducted. These teachers got their degrees from 20 different institutions from different regions of Turkey. Their teaching experiences range in years: 6 of them 1-5 years, 10 of them 6-10 years, 7 of them 11-15 years, 5 of them 16-20 years, 2 of them 21-25 years, 2 of them 26-30 years, 8 of them more than 30 years.

Data Collection Instrument

The data comes from one-on-one semi-structured interviews. We recruited these teachers using convenience sampling on volunteers by considering their varying teaching experiences. We videotaped the interviews without revealing their identities. We gave the participants previously piloted scenarios consisting of alternative student solutions during the interviews and asked them to analyse these scenarios. Each scenario, applied in separate pages, focused on a single arithmetic operation requiring the teachers to draw on SCK regarding place value. Note that we did not train these teachers about SCK or any other knowledge component. SCK is the mathematical knowledge required to respond to everyday tasks of teaching mathematics. For example, the mathematical knowledge required to analyse students' alternative solutions falls under the umbrella of SCK. Each scenario provided teachers with an alternative student solution requiring them to draw on their mathematical knowledge of place value, and arithmetic operations to some extent. Last two of these scenarios are illustrated in Table 1.

Table 1: Last two scenarios given to teachers during the interviews

	**4.** What would you say about the method used by Nazlı, who did the multiplication as illustrated on the left? If you were the teacher in this situation, how would you respond to this student's work once she made this explanation?
	**5.** Emre, who starts to do the following division (4002÷5), objects using the following reasoning: "When solving such problems, we first ask 'how many 5's are in 4?' Here, there is no 5 within 4. Well, isn't 4 actually 4000? And isn't it the case that there are 800 fives within 4000? Why don't we write 800 in the quotient section then?" How would you respond to this student's work as his teacher?

The scenarios allowed us to evaluate the degree of teachers' operating with SCK, the nature of SCK and the teacher knowledge in the context of place value and multidigit arithmetic. This paper will only provide a brief analysis of teacher responses for Scenario 4 and Scenario 5. A mathematical analysis of these scenarios is provided below.

Scenario 4 requires a solid mathematical knowledge of the multiplication algorithm as it is applied in four steps rather than two, making it an alternative solution. In the given solution, 23×15 is calculated using distributive law as follows (also see Figure 1):

$$23\times15 = (20+3)\times(5+10) = (5\times3)+(5\times20)+(10\times3)+(10\times20) = 15+100+30+200 = 345$$

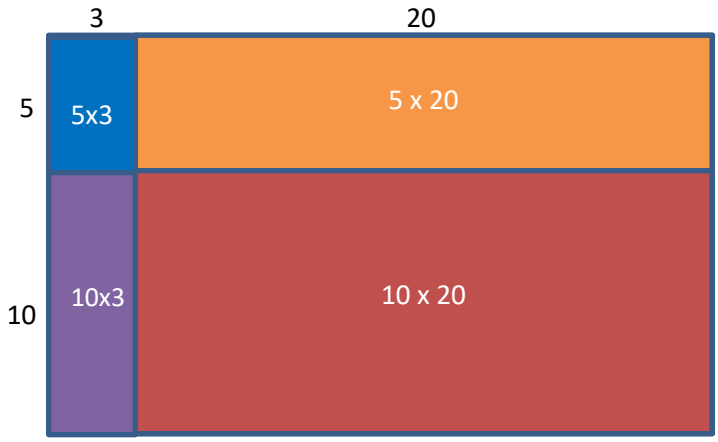


Figure 1: Analysis of 23×15 using distributive law geometrically

The multiplication algorithm includes the calculation of '(5×3)+(5×20)' as a first step yielding 115 and '(10×3)+(10×20)' as a second step resulting in 230 – in fact, procedurally, 230 is written as 23 by aligning 2 with 'Hundreds' column, 3 with 'Tens' column and ignoring 0 in the 'Ones' column. In contrast, the given alternative solution treats these two steps as a combination of four seemingly isolated steps, requiring teachers to analyse the reduction of four steps to two. Therefore, in responding to this alternative solution, teachers need to draw on the following mathematical knowledge: the role of place value in making sense of numbers (e.g., thinking about 23 as a combination of partitions 20 and 3), the connection between distributive law and multiplication algorithm (e.g., thinking about why and how to distribute 5 and 10 onto 20 and 3), and the relationship between four steps described above and two-step algorithm (e.g., how 15+100+30+200 is reduced to 115 and 230 (or reduced to 23, one place shifted to the left)).

Scenario 5 also provides an alternative solution for 4002÷5 by investigating whether the first question to ask in approaching such a problem is 'how many 5s are in 4' (quotative division) as usually taught in schools. Here, "4" within "4002" can mean its face value (4) and place value (4000). To respond to such an alternative solution, teachers need to know that "4 thousand" is to be shared among five parties (partitive division) as thousands, and since this is not possible, it needs to be converted to "40 Hundreds" to be shared as hundreds among five parties. This distribution results in "8 Hundreds", giving the result of the division as 800 with a remainder of 2, or '800R2'. Because of these reasons, the question of 'how many 5's are in 4?' is not mathematically appropriate, and the long division algorithm is to be interpreted by teachers with partitive division meaning (an amount being equally shared among several groups and determining group size).

Data Analysis Procedure

Data analysis is carried out in the form of content analysis and is still ongoing. In analysing the data, we mainly focus on the kind of knowledge the teachers draw on (e.g., SCK, CCK) and its nature. First, we go through all the videotapes and identify instances in which teachers do (not) refer to SCK

and the rationale behind their analyses. As we go through the data and as we gain more insight into how teachers operate, we begin to provide a fine-grained description of the nature of SCK for arithmetic operations and place value. After the initial run of the data, we check whether there are similarities among teacher interpretations and how they refer to different knowledge types. We will finally develop categories that characterise the nature of teacher knowledge regarding the place value concept and arithmetic operations. In this paper, we only share some preliminary findings.

Results

The data has given us certain clues about how weak and detached from the conceptual ties the teachers perceive the place value concept and how they drew on their knowledge. In addition, the data led us to believe that inquiring about the validity of the given alternative solution or even having a suspicion about the validity of the given solution help teachers operate more closely with SCK than other knowledge categories. Because of space limitations, we give two examples of this finding below.

I. Evaluation of multiplication scenario: When working on Scenario #4, the teachers considered the alternative solution as a brand-new way of doing multiplication that they had not seen before and pursued one or more of the following methods or arguments:

1. The teachers initially ignored the core of the given alternative solution and labelled it as "wrong" without a thorough analysis. They compared the alternative solution to the routine/procedural application of the multiplication algorithm they already knew (CCK) and used it in their practice. A typical teacher (out of those 12 teachers) reaction to this scenario was, "*If the student comes to me with something like this, I will tell that it can't be done this way, and then I will delete this method*". In this sense, teachers did not focus on the nature of the given solution as they operated with CCK. They were not even suspicious until they solved the given multiplication problem by referring to CCK and realised that their result matched the given result in the alternative solution.

2. The teachers solved the problems in the given scenarios independently of the given alternative student analysis by operating from CCK in a very procedural way – a typical teacher reaction (out of 14 teachers) was, "*Now, I need to find the result of this calculation first*". For example, in multiplying two two-digit numbers (23×15), their initial reaction was to follow the standard multiplication algorithm: applying the 'Ones' column of the multiplier to both 'Ones' and then 'Tens' column of the multiplicand (5×3 and 5×2 without considering 2 as 20), and then doing the same for 'Tens' column of the multiplier and multiplicand (1×3 and 1×2 without considering 1 as 10 and 2 as 20). This way of operating is about the use of CCK others also use in settings other than teaching (mathematics). This way of operating might be convenient to teachers, and therefore they did not feel the need to start with an analysis of the given alternative solution. Finding the matching results fed a suspicion for teachers about the validity of the given alternative solution. Once they became suspicious, they began to analyse the given solution superficially by analysing each given step, which also led them to pursue the question, "*would it be right*?" rather than sticking to their initial rigid observation, "*it is wrong*". What seemed to help teachers move away from operating with CCK and getting close to the use of SCK was the change of perspective (Ball & Bass, 2009) from the "*it is wrong*" argument to "*would it be right*?" argument.

3. Twenty-one teachers referred to one of the other knowledge types (e.g., KCT, KCC, KCS) as if they wanted to fill the void of SCK with those components. For example, teachers referring to

KCT did it by referring to what they would typically do in their teaching and the procedural aspects of the algorithm they would highlight. (e.g., always starting from 'Ones', moving to 'Tens').

II. Evaluation of division scenario. When it comes to Scenario #5 (about division), all the teachers except very few could not sufficiently articulate the dilemma between "*There is no 5 within 4*" and "*there are 800 fives within 4000*" on a conceptual basis by associating the issue with place value. They all, except three, referred to the long division algorithm and ignored the dilemma mentioned earlier. More specifically, a typical teacher's (from among the 37 teachers) initial analysis was:

> Teacher: I need to explain them that we need to work with the face value, we need to have them accept this. … I can take out 4 Turkish Liras [*referring to 1TL coins*] from my pocket and ask, can you give me 5 Turkish Liras from here, can you give or not?

When further probed about the meaning of leftmost two digits, 40, within the number "4002", these teachers were not able to talk about the place value and its role either. Such an approach focused on the face value instead of place value resulting in dependency on CCK or KCT rather than on SCK.

Only three teachers out of 40, when probed to mentally or physically use base-ten blocks, modelled the given division process with the help of base-ten blocks and explained this issue by mentioning that 4 thousand blocks cannot be shared fairly among 5 parties since there would not be enough thousands for each party. This is illustrated for one of those teachers in the below dialogue.

> Interviewer: When we think about the materials here [*referring to base 10 blocks*], what does it mean to say there are no 5 within 4?
> Teacher: Since these are 'Thousands' as wholes, I cannot divide them into 5 people. I need to convert them to 'Hundreds' to be distributed to everyone as 8 'Hundreds'. The remaining 2 would be mine.
> Interviewer: All right, here we say there are 8 fives within 40. Is there a connection between the division algorithm and the application you made with base-ten blocks?
> Teacher: […] Since I cannot distribute 'Thousands' as 'Thousands', I distribute them as 'Hundreds' [*circling 40 within 4002 on paper*]. So, in a way, I convert this [*referring to 4 thousand*] into 'Hundreds'.
> Interviewer: What does that 40 represent?
> Teacher: Let that 40 represent 40 'Hundreds'. Then 5 times 8 makes 40 – 40 'Hundreds'. I mean, when we say algorithm, if we sift through it, it is 40 [*groups of]* 'Hundreds'.
> […]
> Interviewer: What does that 8 represent [*referring to the 8 in quotient*]?
> Teacher: 8? It represents 8 'Hundreds' per person.
> Interviewer: What does it mean to put zero [*next to 8*] with base-ten blocks?
> Teacher: I do not have any 'Tens'. [*adds 0 next to 80*] I am distributing 'Tens', but I give you zero 'Tens'. I will also share 2, but since 2 is not enough for 5 people, I give you zero 'Ones'. In other words, we confirm that it is '8 Hundreds'.

In this dialogue, the teacher used several components to analyse the given alternative student solution. We consider these components as part of his SCK. These are:

- Investigation of the conditions for which the given alternative solution is correct,
- Referring to place value and its role in division using base-ten blocks,
- Operating from the sharing meaning of division.

This teacher did make a sufficient analysis of the given alternative solution by using all these components together. However, such reasoning was not apparent in this teacher's explanations until the base-ten blocks were shown. In this sense, manipulatives played a particular role in triggering this participant's thinking to carefully evaluate the act of sharing '4 Thousand' units among 5 parties. Base-

ten blocks allowed this teacher to question the validity of the given argument. The other 37 teachers, when probed, could not resolve this conundrum, and they continued to operate with CCK and/or KCT even with base-ten blocks. Therefore, the three SCK components need to be abstracted as a totality to sufficiently analyse a given alternative student solution.

Conclusion

Our initial findings suggest that teachers who operate with (partial) SCK or without SCK analyse the given alternative solutions qualitatively differently. The ones operating without SCK are the ones who are under the complete influence of CCK (e.g., procedural application of multiplication or division algorithms) and who analyse the given alternative solutions by purely comparing them to the algorithms. Teachers who operate with partial SCK try to fill the void by bringing in examples from other teacher knowledge types as excuses such as KCS (e.g., this will be quite difficult for students), KCT (e.g., I normally teach it this way) or KCC (e.g., this is not appropriate for lower grades). The teachers operating with SCK approach the given alternative solutions by thinking about the question, 'what makes this solution valuable/reasonable?' and then refer to the role of place value as the core of the given arithmetic operations of multiplication and division.

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