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TOPOLOGY OF IRREGULAR ISOMONODROMY TIMES ON A FIXED POINTED CURVE

JEAN DOUÇOT AND GABRIELE REMBADO

ABSTRACT. We will define and study moduli spaces of deformations of irregular classes on Riemann surfaces, which provide an intrinsic viewpoint on the “times” of irregular isomonodromy systems in general. Our aim is to study the deeper generalisation of the G-braid groups that occur as fundamental groups of such deformation spaces, with particular focus on the generalisation of the full G-braid groups.

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INTRODUCTION

Classically, the theory of isomonodromy constitutes a collection of nonlinear integrable differential equations, whose unknown is a (linear) meromorphic connection on a vector bundle over the Riemann sphere. Geometrically, these are flat Ehresmann connections on a bundle whose fibres are moduli spaces of such meromorphic connections.

The underlying deformation parameters, the “times”, have recently been given an intrinsic formulation, leading to a generalisation of the moduli of pointed curves (in any genus). This framework is especially useful when considering the generalised deformations, beyond the generic case: recall [34] set up a theory of “generic” isomonodromic deformations of meromorphic connections on vector bundles over

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a Riemann surface Σ , where the leading coefficient at each pole has distinct eigenvalues (building on [8]; cf. [36, 5]). This has been extended in two directions: i) replacing vector bundles by principal G -bundles [10], leading to the appearance of G -braid groups for complex reductive groups G , and ii) considering *nongeneric* admissible deformations [13], where the irregular type of the connection is *arbitrary*, leading to cabled braid groups [23].

In particular the spaces of generalised monodromy data, the wild character varieties (a.k.a. wild Betti spaces), have been proved to form a local system of Poisson varieties [13]¹

$$\mathcal{M}_{\overline{\mathbf{B}}} \longrightarrow \overline{\mathbf{B}}, \quad (1)$$

over any space $\overline{\mathbf{B}}$ of admissible deformations. These are important because they give a purely topological description of the nonlinear isomonodromy differential equations via the Riemann–Hilbert–Birkhoff correspondence.

Our purpose in this paper is to study the fundamental groups of the base spaces $\overline{\mathbf{B}}$ of such admissible deformations, the groups that will act by algebraic Poisson automorphisms on the wild character varieties (the fibres of (1)) from the parallel transport of the isomonodromy connection—i.e. the monodromy of the nonlinear differential equations. This builds on our previous paper [23], which used a fixed marking: here we will quotient by the Weyl group action and get to the full version of “wild” mapping class groups, in analogy to forgetting the ordering of marked points on the underlying pointed curve.

Importantly this encompasses the much-studied case of regular singular connections, involving the (tame) complex character varieties, which is the entry point for the standard mapping-class- and braid-group-actions in classical/quantum 2d gauge theories—via deformations of pointed curves, e.g. [35, 24, 37, 4] in the quantum case.

In this series of papers we rather fix the underlying pointed curve, and vary the rest of the wild Riemann surface structure [13], i.e. the irregular types/classes, controlling principal parts of irregular singular connections beyond their (formal) residues. More precisely [23] constructs a moduli scheme of irregular types for the split Lie algebra $(\mathfrak{g}, \mathfrak{t}) := (\mathrm{Lie}(G), \mathrm{Lie}(T))$, where $T \subseteq G$ is a maximal torus, while here we consider irregular classes.

Recall in brief an irregular type Q at a point $a \in \Sigma$ is the germ of a \mathfrak{t} -valued meromorphic function based there, defined up to holomorphic terms:

$$Q = \sum_j A_j z^{-j} \in \mathfrak{t}((z))/\mathfrak{t}[[z]], \quad A_j \in \mathfrak{t}, \quad z(a) = 0. \quad (2)$$

Then the Weyl group $W_{\mathfrak{g}} = \mathrm{N}(T)/T$ acts on the left tensor factor of

$$\mathfrak{t} \otimes_{\mathbb{C}} (\mathbb{C}((z))/\mathbb{C}[[z]]) \simeq \mathfrak{t}((z))/\mathfrak{t}[[z]],$$

and the irregular class underlying (2) is its projection \overline{Q} in the quotient—viz. the Weyl-orbit through Q [13, Rk. 10.6].²

The important fact is the fibres of (1) only depend on the collection of irregular classes underlying the irregular types at each marked point, and thus the base $\overline{\mathbf{B}}$ provides an intrinsic topological description of the local (irregular) isomonodromy times. In the generic case, where the leading coefficient of (2) is out of all root hyperplanes, the homotopy type of $\overline{\mathbf{B}}$ brings about the G -braid group: in this

¹Basically speaking, a bundle of Poisson manifolds equipped with a complete flat connection: the (Betti) isomonodromy connection.

²E.g. if $G = \mathrm{GL}_n(\mathbb{C})$ we thus consider the coefficients A_j in (3) up to simultaneous permutations of their eigenvalues.

paper we shall encounter a generalisation in the nongeneric case, which we will relate to braid cabling in type-A.

Main results. In § 1 we give the main definition: to a one-pointed (bare) wild Riemann surface $\Sigma = (\Sigma, \alpha, \bar{Q})$ we associate a full/nonpure local “wild” mapping class group (WMCG), viz. the fundamental group of a space $\bar{\mathbf{B}}$ of admissible deformations of the irregular class \bar{Q} (cf. Def. 1.1). This latter is the topological quotient of the (universal) admissible deformation space \mathbf{B} of Q , modulo the natural action of the Weyl group $W_{\mathfrak{g}}$, where Q is any irregular type lifting \bar{Q} .

In § 2 we describe the subgroup of the Weyl group preserving the space of admissible deformations of the irregular type, and further its quotient $W_{\mathfrak{g}|\mathfrak{h}}$, acting freely on \mathbf{B} . The relevant statements are proven inductively along the sequence of fission (root) subsystems of $\Phi_{\mathfrak{g}}$ associated to Q (cf. [23]): first in the case of a single semisimple element $A \in \mathfrak{t}$ (in § 2.1), and then in the general case (in § 2.2).

Theorem (Cf. Thm. 2.1). *The space \mathbf{B} is a Galois covering of $\bar{\mathbf{B}}$ with $\text{Gal}(\mathbf{B}, \bar{\mathbf{B}}) = W_{\mathfrak{g}|\mathfrak{h}}$, so the full/nonpure local WMCG is an extension of this latter by the pure local WMCG.*

In § 3 we describe all full/nonpure local WMCGs for the irreducible rank-2 root systems, after explaining it is enough to consider a simple Lie algebra.

Finally in § 4 we explicitly describe the full/nonpure local WMCGs when $\mathfrak{g} \in \{ \mathfrak{gl}_n(\mathbb{C}), \mathfrak{sl}_n(\mathbb{C}) \}$, in the nonabelian case $n \geq 2$. This means identifying the “effective” subquotient of the Weyl group that controls the Galois covering $\mathbf{B} \rightarrow \bar{\mathbf{B}}$ (a Coxeter-type group), and then compute the fundamental group of the base (an Artin-type group): the inductive step is in § 4.1, where we prove the following.

Theorem (Cf. Prop. 4.1, Cor. 4.1 and Prop. 4.2). *The Weyl-stabiliser of \mathbf{B} is a direct product of (restricted) wreath products of symmetric groups, and $W_{\mathfrak{g}|\mathfrak{h}}$ is a direct product of symmetric groups; then $\pi_1(\bar{\mathbf{B}})$ is the subgroup of braids whose underlying permutation lies in $W_{\mathfrak{g}|\mathfrak{h}}$ —an extension of this latter by the pure braid group.*

In the general case instead we introduce a family of trees (T, \mathfrak{r}) with some decoration, called “ranked” fission trees, which depend on the choice of the irregular class \bar{Q} (cf. Def. 4.1, and compare with the unranked fission trees of [23, § 5]). Their automorphisms control the Coxeter-type groups in the general type-A case:

Theorem (Cf. Thm. 4.1 and Prop. 4.3). *The automorphism group $\text{Aut}(T, \mathfrak{r})$ of the ranked fission tree is isomorphic to $W_{\mathfrak{g}|\mathfrak{h}}$.*

Finally we attach a (full/nonpure) “cabled” braid group $\text{CBr}(T, \mathfrak{r})$ to any ranked fission tree, in Def. 4.4, with a recursive algorithm (along maximal subtrees): this relies on the operadic composition of the symmetric and braid group operads, extending the pure cabled braid group of [23]—which rests in turn on the pure braid group operad.

The main result of § 4 is that the elements of type-A full/nonpure local WMCGs are precisely such “cabled” braids.

Theorem (Cf. Thm. 4.2). *The full/nonpure type-A local WMCG of Σ is isomorphic to $\text{CBr}(T, \mathfrak{r})$, where (T, \mathfrak{r}) is the ranked fission tree associated with the irregular class \bar{Q} .*

All Lie algebras and tensor products are defined over \mathbb{C} .

Some basic notions and conventions, used throughout the body of the paper, are collected in § A, while § B contains the proof of few lemmata. Finally in § C we spell out the relation of wild Riemann surfaces with the much-studied Hamiltonian viewpoint on isomonodromic deformations.

The end of remarks/examples is signaled by a \triangle .

1. FULL/NONPURE LOCAL WMCG

Let Σ be a Riemann surface, G a finite-dimensional connected reductive Lie group over \mathbb{C} , $\mathfrak{g} = \text{Lie}(G)$ its Lie algebra, $T \subseteq G$ a maximal (algebraic) torus, and $\mathfrak{t} = \text{Lie}(T) \subseteq \mathfrak{g}$ the associated Cartan subalgebra. Denote then $\Phi_{\mathfrak{g}} = \Phi(\mathfrak{g}, \mathfrak{t}) \subseteq \mathfrak{t}^{\vee}$ the root system of the split Lie algebra $(\mathfrak{g}, \mathfrak{t})$, and $W_{\mathfrak{g}} = W(\Phi_{\mathfrak{g}})$ the Weyl group.

Choose a point $a \in \Sigma$, and let

$$Q \in \mathfrak{t} \otimes \mathcal{T}_{\Sigma, a}, \quad \mathcal{T}_{\Sigma, a} := \widehat{\mathcal{K}}_{\Sigma, a} / \widehat{\mathcal{O}}_{\Sigma, a}, \quad (3)$$

be an untwisted irregular type based there, introducing the completed local ring $\widehat{\mathcal{O}}_{\Sigma, a}$ of the surface and its fraction field $\widehat{\mathcal{K}}_{\Sigma, a}$. Recall if z is a local coordinate on Σ with $z(a) = 0$ then (3) becomes

$$Q = \sum_{i=1}^p A_i z^{-i} \in z^{-1} \mathfrak{t}[z^{-1}] \simeq \mathfrak{t}((z)) / \mathfrak{t}[[z]],$$

for suitable (semisimple) coefficients $A_i \in \mathfrak{t}$ and for an integer $p \geq 1$.

As explained in the introduction, the moduli spaces attached to (3) (the de Rham/Betti spaces [11]) only depend on the Weyl-orbit of (3), denoted

$$\overline{Q} \in (\mathfrak{t} \otimes \mathcal{T}_{\Sigma, a}) / W_{\mathfrak{g}}. \quad (4)$$

Here the Weyl group acts on the Cartan subalgebra—and trivially on the other tensor factor; the element (4) defines an irregular class, a.k.a. “bare” irregular type [13, Rem. 10.6] (cf. [14] for a definition in the twisted case).

If Q is a “starting” irregular type, then we have associated to it the deformation space $\mathbf{B} = \mathbf{B}(Q)$ in [23]; recall this is $\mathbf{B} = \prod_{i=1}^p \mathbf{B}_i$, with

$$\mathbf{B}_i = \mathbf{B}(A_i) := \bigcap_{d_{\alpha} \leq i} \text{Ker}(\alpha) \cap \bigcap_{d_{\alpha} = i} (\mathfrak{t} \setminus \text{Ker}(\alpha)) \subseteq \mathfrak{t},$$

where

$$d_{\alpha} = \text{ord}(q_{\alpha}), \quad q_{\alpha} = \alpha \circ Q, \quad \alpha \in \Phi_{\mathfrak{g}},$$

taking the pole order at $a \in \Sigma$.

But two deformations are then equivalent (that is, they define the same irregular class) if they lie in the same $W_{\mathfrak{g}}$ -orbit, which leads to admissible deformations of the “starting” irregular class \overline{Q} . The main definition is thus the following.

Definition 1.1. The *local wild mapping class group* of the (bare) wild Riemann surface $\Sigma = (\Sigma, a, \overline{Q})$ is

$$\overline{\Gamma}(\Phi_{\mathfrak{g}}, \mathbf{d}) := \pi_1(\overline{\mathbf{B}}, \overline{Q}), \quad \mathbf{d} = (d_{\alpha})_{\alpha \in \Phi_{\mathfrak{g}}}, \quad (5)$$

where $\overline{\mathbf{B}} = \mathbf{B}/\sim$ is the topological quotient with respect to the equivalence relation

$$Q_1 \sim Q_2 \quad \text{if} \quad W_{\mathfrak{g}} Q_1 = W_{\mathfrak{g}} Q_2 \subseteq \mathfrak{t} \otimes \mathcal{T}_{\Sigma, a}. \quad (6)$$

This yields in general a larger fundamental group than the pure case—some paths in \mathbf{B} become loops in $\overline{\mathbf{B}}$.

Remark 1.1. The space \mathbf{B} itself depends on the irregular type Q , not just on the underlying irregular class. However if $w \in W_{\mathfrak{g}}$ then \mathbf{B} is homeomorphic to $w(\mathbf{B})$, and there is a further canonical homeomorphism $\overline{\mathbf{B}} \simeq \overline{w(\mathbf{B})}$ obtained by matching orbits: thus (5) only depends on \overline{Q} . \triangle

Now the Weyl action does *not* preserve \mathbf{B} in the nongeneric case, i.e. the case where A_p is not regular, so we first need to describe the subset

$$W_{\mathfrak{g}} Q \cap \mathbf{B} \subseteq W_{\mathfrak{g}} Q,$$

and further understand the Weyl-stabiliser of the irregular type.

2. WEYL GROUP FISSION

2.1. Inductive step. We first consider the case of a single coefficient, i.e. $Q = Az^{-1}$. In the general case the irregular type is transformed along the diagonal Weyl action on each coefficient.

Choose then $A \in \mathfrak{t}$, and let $\mathfrak{h} = \mathfrak{Z}_{\mathfrak{g}}(A) \subseteq \mathfrak{g}$ be the centraliser: it is the (reductive) Levi factor of a parabolic subgroup of G . The associated deformation space becomes

$$\mathbf{B} = \text{Ker}(\Phi_{\mathfrak{h}}) \cap \bigcap_{\Phi_{\mathfrak{g}} \setminus \Phi_{\mathfrak{h}}} (\mathfrak{t} \setminus \text{Ker}(\alpha)) \subseteq \text{Ker}(\Phi_{\mathfrak{h}}), \quad (7)$$

and we set $\mathbf{U} := \text{Ker}(\Phi_{\mathfrak{h}}) \subseteq \mathfrak{t}$.

Now if $w \in \text{Stab}_{W_{\mathfrak{g}}}(\mathbf{B}) \subseteq W_{\mathfrak{g}}$ then certainly $wA \in \mathbf{B}$, but the converse is true. To state this let $\mathcal{O}_A \subseteq \mathbf{B}$ be the orbit of A under the action of $\text{Stab}_{W_{\mathfrak{g}}}(\mathbf{B})$; then:

Lemma 2.1. *One has $\text{Stab}_{W_{\mathfrak{g}}}(\mathbf{U}) = \text{Stab}_{W_{\mathfrak{g}}}(\mathbf{B})$, and $(W_{\mathfrak{g}}A) \cap \mathbf{B} = \mathcal{O}_A$.*

Proof. The Weyl group permutes the root hyperplanes via

$$w(\text{Ker}(\alpha)) = \text{Ker}(w\alpha), \quad w \in W_{\mathfrak{g}}, \alpha \in \Phi_{\mathfrak{g}},$$

i.e. along the permutation of the roots. (Recall we identify $W(\Phi_{\mathfrak{g}}) \subseteq \text{GL}(\mathfrak{t}^{\vee})$ and $W(\Phi_{\mathfrak{g}}^{\vee}) \subseteq \text{GL}(\mathfrak{t})$, cf. § A.) Hence $w(\mathbf{B}) \subseteq \mathbf{B}$ if and only if $w \in W_{\mathfrak{g}}$ preserves the partition $\Phi_{\mathfrak{g}} = \Phi_{\mathfrak{h}} \cup (\Phi_{\mathfrak{g}} \setminus \Phi_{\mathfrak{h}})$, by (7). In turn this is equivalent to $w(\Phi_{\mathfrak{h}}) \subseteq \Phi_{\mathfrak{h}}$, proving the first statement.

Analogously if $wA \in \mathbf{B}$ then w preserves the above partition, whence the inclusion $(W_{\mathfrak{g}}A) \cap \mathbf{B} \subseteq \mathcal{O}_A$ —and the opposite one is tautological. \square

Thus the restriction of orbits to the deformation space is controlled by the setwise stabiliser of $\mathbf{U} \subseteq \mathfrak{t}$.

Remark 2.1. The extremal cases are $A = 0$, in which case $\mathbf{U} = \mathfrak{t}$ and $\text{Stab}_{W_{\mathfrak{g}}}(\mathbf{U}) = 1$; and $A \in \mathfrak{t}_{\text{reg}}$, in which case $\mathbf{U} = (0)$ and $\text{Stab}_{W_{\mathfrak{g}}}(\mathbf{U}) = W_{\mathfrak{g}}$. \triangle

Now the Weyl group $W_{\mathfrak{h}} = W(\Phi_{\mathfrak{h}}) \subseteq W_{\mathfrak{g}}$ of the Levi factor lies in the setwise stabiliser of \mathbf{U} , but in general the inclusion is proper. Namely by definition

$$\text{Ker}(\alpha) = \text{Ker}(\sigma_{\alpha} - 1) \subseteq \mathfrak{t}, \quad \alpha \in \Phi_{\mathfrak{g}},$$

and the subgroup $W_{\mathfrak{h}}$ is generated by the reflections along the hyperplanes of the subsystem $\Phi_{\mathfrak{h}} \subseteq \Phi_{\mathfrak{g}}$: hence automatically any element of $W_{\mathfrak{h}}$ acts trivially on $\mathbf{U} = \text{Ker}(\Phi_{\mathfrak{h}})$, i.e.

$$W_{\mathfrak{h}} \subseteq (W_{\mathfrak{g}})_{\mathbf{U}} \subseteq \text{Stab}_{W_{\mathfrak{g}}}(\mathbf{U}).$$

However it is possible to show the first inclusion is an equality, and more precisely that $W_{\mathfrak{h}}$ is the (maximal) parabolic subgroup fixing the given semisimple element $A \in \mathfrak{t}$.

Lemma 2.2. *One has $W_{\mathfrak{h}} = (W_{\mathfrak{g}})_{\mathbf{U}} = (W_{\mathfrak{g}})_{\mathbf{A}}$.*

Proof. In principle

$$(W_{\mathfrak{g}})_{\mathbf{U}} \subseteq (W_{\mathfrak{g}})_{\mathbf{B}} \subseteq (W_{\mathfrak{g}})_{\mathbf{A}},$$

since $\mathbf{A} \in \mathbf{B} \subseteq \mathbf{U}$, so it is enough to show the inclusion $(W_{\mathfrak{g}})_{\mathbf{A}} \subseteq W_{\mathfrak{h}}$, i.e. that any element of $W_{\mathfrak{g}}$ fixing A lies in the Weyl group of $\Phi_{\mathfrak{h}}$.

To this end recall the Lie-group-theoretic definition of the Weyl group is

$$W_{\mathfrak{h}} = N_H(T)/T \subseteq N_G(T)/T = W_{\mathfrak{g}},$$

using the normalisers $N_H(T) \subseteq N_G(T)$ of the given maximal torus, where $H \subseteq G$ is the reductive subgroup integrating $\mathfrak{h} \subseteq \mathfrak{g}$. Hence an element $w \in W_{\mathfrak{g}}$ such that

$w(A) = A$ corresponds to an element $g \in N_G(T)$ —defined up to the T -action—such that $\text{Ad}_g(A) = A$: this means

$$g \in N_G(T) \cap H = N_H(T),$$

whence $w \in W_{\mathfrak{h}}$. □

Finally we have an identification $\mathcal{O}_A \simeq W_{\mathfrak{g}|\mathfrak{h}}A$, introducing the quotient group

$$W_{\mathfrak{g}|\mathfrak{h}} := \text{Stab}_{W_{\mathfrak{g}}}(U) / W_{\mathfrak{h}}. \quad (8)$$

In turn there is a homeomorphism

$$\overline{\mathbf{B}} \simeq \mathbf{B} / W_{\mathfrak{g}|\mathfrak{h}}, \quad (9)$$

by Def. 1.1, which leads to the following.

Proposition 2.1. *The fundamental group $\pi_1(\overline{\mathbf{B}}, \overline{A})$ is an extension of (8) by $\pi_1(\mathbf{B}, A)$, where $\overline{A} \in \overline{\mathbf{B}}$ is the Weyl-orbit of the base point.*

Proof. By construction the $W_{\mathfrak{g}|\mathfrak{h}}$ -action on \mathbf{B} is free: indeed the stabilisers of all points are conjugated, and by Lem. 2.2 the stabiliser of the base point is trivial. Moreover the action is automatically properly discontinuous ($W_{\mathfrak{g}|\mathfrak{h}}$ is finite), and the spaces involved are Hausdorff: hence every point of \mathbf{B} has a neighbourhood $O \subseteq \mathbf{B}$ such that $\overline{w_1}(O) \cap \overline{w_2}(O) \neq \emptyset$ implies $\overline{w_1} = \overline{w_2} \in W_{\mathfrak{g}|\mathfrak{h}}$.

It follows that the canonical projection $p: \mathbf{B} \rightarrow \overline{\mathbf{B}}$ is a Galois covering, with automorphisms provided by the monodromy action of $W_{\mathfrak{g}|\mathfrak{h}}$. The choice of the base point $A \in p^{-1}(\overline{A}) \subseteq \mathbf{B}$ in the fibre yields an identification $p^{-1}(\overline{A}) \simeq W_{\mathfrak{g}|\mathfrak{h}}$ between the torsor and the group, so there is a (principle) fibre bundle

$$W_{\mathfrak{g}|\mathfrak{h}} \hookrightarrow \mathbf{B} \xrightarrow{p} \overline{\mathbf{B}}.$$

Then the resulting exact sequence of homotopy groups contains the short sequence

$$1 \longrightarrow \pi_1(\mathbf{B}, A) \xrightarrow{\pi_1(p)} \pi_1(\overline{\mathbf{B}}, \overline{A}) \longrightarrow W_{\mathfrak{g}|\mathfrak{h}} \longrightarrow 1, \quad (10)$$

identifying $\pi_0(W_{\mathfrak{g}|\mathfrak{h}}) \simeq W_{\mathfrak{g}|\mathfrak{h}}$ for the discrete space. The connecting map in this case is a group morphism: it corresponds to the monodromy action at the base point—up to turning it into a left action, i.e. exchanging $W_{\mathfrak{g}|\mathfrak{h}}$ with its opposite. □

Example 2.1. For example if $A \in \mathfrak{t}_{\text{reg}}$ then $W_{\mathfrak{h}}$ is trivial, so $W_{\mathfrak{g}|\mathfrak{h}} = W_{\mathfrak{g}}$. Hence (9) generalises the generic “configuration” space $\mathfrak{t}_{\text{reg}}/W_{\mathfrak{g}}$, and in turn $\pi_1(\overline{\mathbf{B}}, \overline{A})$ generalises the (full/nonpure) \mathfrak{g} -braid group. △

Remark 2.2 (Reduced reflection groups). At first one may think (8) is the reflection group of the “restricted” hyperplane arrangement

$$\mathcal{H} = \left\{ \text{Ker}(\alpha) \cap U = \text{Ker}(\alpha|_U) \mid \alpha \in \Phi_{\mathfrak{g}} \setminus \Phi_{\mathfrak{h}} \right\} \subseteq \mathbb{P}(U^{\vee}) : \quad (11)$$

however this is *not* the case in general.

For example if $\Phi_{\mathfrak{h}} = A_1 \subseteq A_2 = \Phi_{\mathfrak{g}}$ then (8) is trivial (see § 3), while the reduced arrangement is of type A_1 —so has a Weyl group of order two. The point is there are reflections of (11) which do not come as restrictions of elements in $\text{Stab}_{W_{\mathfrak{g}}}(U)$ (see § 4). △

2.2. General case. The space (9) corresponds to one factor of (5), i.e. to an irregular type with a simple pole only: here we extend this to an arbitrary irregular type.

Suppose thus to have an increasing sequence of fission subsystems

$$\Phi_{\mathfrak{h}_1} \subseteq \cdots \subseteq \Phi_{\mathfrak{h}_{p+1}} := \Phi_{\mathfrak{g}}, \quad (12)$$

for some integer $p \geq 1$, associated to an irregular type $Q = \sum_{i=1}^p A_i z^{-i}$. Then $\mathfrak{h}_1 = \mathfrak{Z}_{\mathfrak{g}}(\{A_1, \dots, A_p\})$ is the centraliser of Q , which generalises the Levi factor \mathfrak{h} of the previous section.

Namely there is a filtration of Weyl (sub)groups

$$W_{\mathfrak{h}_1} \subseteq \cdots \subseteq W_{\mathfrak{h}_{p+1}} = W_{\mathfrak{g}},$$

and the deformation space is a product, with each factor $\mathbf{B}_i \subseteq \mathfrak{t}$ determined as in (7) (it is the space of admissible deformations of $A_i \in \mathfrak{t}$). Then $W_{\mathfrak{g}}$ acts diagonally on $\mathbf{B} \subseteq \mathfrak{t}^p$.

Now $w(Q) \in \mathbf{B}$ means that $w(A_i) \in \mathbf{B}_i$ for $i \in \{1, \dots, p\}$, and this condition can be described recursively using Lem. 2.1. To this end define a sequence of subgroups

$$W_1 \subseteq \cdots \subseteq W_p \subseteq W_{\mathfrak{g}},$$

as follows. Set as above $\mathfrak{u}_i := \text{Ker}(\Phi_{\mathfrak{h}_i})$, and then

$$W_p := \text{Stab}_{W_{\mathfrak{g}}}(\mathfrak{u}_p), \quad W_{i-1} := \text{Stab}_{W_i}(\mathfrak{u}_{i-1}) \subseteq W_i, \quad i \in \{2, \dots, p\}. \quad (13)$$

Denote then \mathcal{O}_Q the orbit of the irregular type under the action of the smallest group $W_1 \subseteq W_{\mathfrak{g}}$.

Lemma 2.3. *One has $W_{i-1} = \text{Stab}_{W_i}(\mathbf{B}_i)$ for $i \in \{1, \dots, p\}$, and $(W_{\mathfrak{g}}Q) \cap \mathbf{B} = \mathcal{O}_Q$.*

Proof. First $w(A_p) \in \mathbf{B}_p$ if and only if $w \in W_p$, and the first statement has been proven in Lem. 2.1—for $i = p$.

Then we can replace $(\mathfrak{h}_p, \mathfrak{g})$ with $(\mathfrak{h}_{p-1}, \mathfrak{h}_p)$, and repeat the same construction: we need $w \in W_p$ such that $w(A_{p-1}) \in \mathbf{B}_{p-1}$, where

$$\mathbf{B}_{p-1} = \mathfrak{u}_{p-1} \cap \bigcap_{\Phi_{\mathfrak{h}_p} \setminus \Phi_{\mathfrak{h}_{p-1}}} (\mathfrak{t} \setminus \text{Ker}(\alpha)) \subseteq \mathfrak{t},$$

by (7). Reasoning as in the proof of Lem. 2.1 this requires $w(\Phi_{\mathfrak{h}_{p-1}}) \subseteq \Phi_{\mathfrak{h}_{p-1}}$, which is equivalent to preserving the partition $\Phi_{\mathfrak{h}_p} = \Phi_{\mathfrak{h}_{p-1}} \cup (\Phi_{\mathfrak{h}_p} \setminus \Phi_{\mathfrak{h}_{p-1}})$, since by (recurrence) hypothesis $w(\Phi_{\mathfrak{h}_p}) \subseteq \Phi_{\mathfrak{h}_p}$. Hence $w \in W_{p-1}$ and

$$W_{p-1} = \text{Stab}_{W_p}(\mathbf{B}_{p-1}) \subseteq W_p.$$

Descending until $i = 1$ shows that $w(Q) \in \mathbf{B}$ if and only if $w \in \bigcap_i W_i = W_1$, and proves the first statement—inductively. \square

Note $W_1 \subseteq W_{\mathfrak{g}}$ is determined by the flag of kernels

$$\mathbf{U} = (\mathfrak{t} \supseteq \mathfrak{u}_1 \supseteq \cdots \supseteq \mathfrak{u}_p), \quad (14)$$

by (13). Indeed consider the (parabolic) stabiliser of (14), within $W_{\mathfrak{g}}$, i.e.

$$\text{Stab}_{W_{\mathfrak{g}}}(\mathbf{U}) := \bigcap_i \text{Stab}_{W_{\mathfrak{g}}}(\mathfrak{u}_i) \subseteq W_{\mathfrak{g}}, \quad (15)$$

which coincides with $W_{\mathfrak{g}} \cap \text{Stab}_{\text{GL}(\mathfrak{t})}(\mathbf{U})$; then:

Lemma 2.4. *One has $W_1 = \text{Stab}_{W_{\mathfrak{g}}}(\mathbf{U})$.*

Proof. Postponed to B. \square

Thus the restriction of orbits to \mathbf{B} is controlled by the action of the setwise Weyl-stabiliser of the kernel flag (14), generalising the inductive step.

Remark 2.3. Beware however it is *not* true in general that $\text{Stab}_{W_{\mathfrak{g}}}(\mathcal{U}_i) = \text{Stab}_{W_{\mathfrak{g}}}(\mathbf{B}_i)$: e.g. for the fission $\emptyset = \Phi_{\mathfrak{h}_1} \subseteq \Phi_{\mathfrak{h}_2} \subseteq \Phi_{\mathfrak{g}}$ one has $\mathcal{U}_1 = \mathfrak{t}$, so $\text{Stab}_{W_{\mathfrak{g}}}(\mathcal{U}_1) = W_{\mathfrak{g}}$; but

$$\mathbf{B}_1 = \mathfrak{t} \setminus \bigcup_{\Phi_{\mathfrak{h}_2}} \text{Ker}(\alpha) \subseteq \mathfrak{t},$$

which is not stabilised by $W_{\mathfrak{g}}$ if $\mathfrak{h}_2 \subseteq \mathfrak{g}$ is a proper Lie subalgebra. \triangle

Analogously we can identify the subgroup fixing the irregular type.

Lemma 2.5. *One has $W_{\mathfrak{h}_1} = (W_{\mathfrak{g}})_{\mathcal{U}_1} = (W_{\mathfrak{g}})_{\mathcal{Q}}$.*³

Proof. By definition $w(\mathcal{Q}) = \mathcal{Q}$ if and only if $w(A_i) = A_i$ for $i \in \{1, \dots, p\}$, i.e. $w \in \bigcap_i (W_{\mathfrak{g}})_{A_i}$.

Now the argument of Lem. 2.2 yields the inductive step for the proof of the identity

$$\bigcap_{j \leq i \leq p} (W_{\mathfrak{g}})_{A_i} = (W_{\mathfrak{g}})_{\{A_j, \dots, A_p\}} = W_{\mathfrak{h}_j} \subseteq W_{\mathfrak{g}}, \quad j \in \{1, \dots, p\},$$

whence

$$(W_{\mathfrak{g}})_{\mathcal{Q}} = W_{\mathfrak{h}_1} \subseteq W_{\mathfrak{g}}.$$

On the other hand

$$W_{\mathfrak{h}_1} = \bigcap_i W_{\mathfrak{h}_i} \subseteq \bigcap_i (W_{\mathfrak{g}})_{\mathcal{U}_i} = (W_{\mathfrak{g}})_{\mathcal{U}_1},$$

since $\mathcal{U}_1 = \sum_i \mathcal{U}_i \subseteq \mathfrak{t}$, and $W_{\mathfrak{h}_i}$ acts as the identity on $\mathcal{U}_i = \text{Ker}(\Phi_{\mathfrak{h}_i})$.

Finally if w acts as the identity on \mathcal{U}_1 then it also fixes (pointwise) $\mathbf{B}_i \subseteq \mathcal{U}_1$ for $i \in \{1, \dots, p\}$, thus

$$(W_{\mathfrak{g}})_{\mathcal{U}_1} \subseteq (W_{\mathfrak{g}})_{\mathbf{B}} \subseteq (W_{\mathfrak{g}})_{\mathcal{Q}},$$

proving the remaining inclusion. \square

In particular by Lemmata 2.4 and 2.5 we also have an inclusion $W_{\mathfrak{h}_i} \subseteq W_i$, since

$$W_{\mathfrak{h}_i} = (W_{\mathfrak{g}})_{\mathcal{U}_i} \subseteq \text{Stab}_{W_{\mathfrak{g}}}(\mathcal{U}_j), \quad j \in \{i, \dots, p\},$$

as $\mathcal{U}_i = \bigcup_{j \geq i} \mathcal{U}_j \subseteq \mathfrak{t}$.

It follows that $W_{\mathfrak{h}_1}$ is a normal subgroup of $\text{Stab}_{W_{\mathfrak{g}}}(\mathcal{U}_1)$, hence a fortiori of (15), and we consider again the quotient group

$$W_{\mathfrak{g}|\mathfrak{h}} := \text{Stab}_{W_{\mathfrak{g}}}(\mathbf{U}) / W_{\mathfrak{h}_1}. \quad (16)$$

Note the numerator of (16) depends on the whole sequence $\mathfrak{h} = (\mathfrak{h}_1, \dots, \mathfrak{h}_p)$, while the denominator only depends on the last term—the pointwise stabiliser of a flag/filtration only depend on its union/sum, contrary to the setwise stabiliser.

Example 2.2 (Complete fission and generic case). In particular if the fission is “complete”, which means that $H_1 = \text{Stab}_{\mathbf{G}}(\mathcal{Q}) = \mathbf{T} \subseteq \mathbf{G}$ is the maximal torus, then $\Phi_{\mathfrak{h}_1} = \emptyset$; in this case $\mathcal{U}_1 = \mathfrak{t}$ and $W_{\mathfrak{h}_1}$ is trivial, so $W_{\mathfrak{g}|\mathfrak{h}} \simeq \text{Stab}_{W_{\mathfrak{g}}}(\mathbf{U})$.

If further we are in the generic case where $A_p \in \mathfrak{t}_{\text{reg}}$ then \mathbf{U} is stationary at \mathfrak{t} , so $\text{Stab}_{W_{\mathfrak{g}}}(\mathbf{U}) = W_{\mathfrak{g}}$. \triangle

Finally by construction there is a topological identification $\overline{\mathbf{B}} \simeq \mathbf{B} / W_{\mathfrak{g}|\mathfrak{h}}$, and the same argument of the proof of Prop. 2.1 yields the following.

Theorem 2.1. *The projection $\mathbf{B} \rightarrow \overline{\mathbf{B}}$ is a Galois covering, and the local WMCG (5) is an extension of $W_{\mathfrak{g}|\mathfrak{h}}$ by the pure local WMCG.*

Of course if $W_1 = W_{\mathfrak{h}_1}$ then $\mathbf{B} = \overline{\mathbf{B}}$, in which case the local WMCG is pure (and has been studied in [23]).

³Note in the rightmost identity we consider two different actions of $W_{\mathfrak{g}}$: the former is an action on \mathfrak{t} , the latter on $\mathfrak{t} \otimes \mathcal{F}_{\Sigma, \alpha}$.

3. LOW-RANK EXAMPLES

Analogously to the pure case we provide examples of local WMCGs for low-rank Lie algebras, after proving we can reduce to the *simple* case.

3.1. Reduction to the simple case. Suppose $\mathfrak{g} = \bigoplus_i^\perp \mathfrak{J}_i$ is a decomposition of \mathfrak{g} into orthogonal ideals, with respect to an Ad_G -invariant nondegenerate symmetric bilinear form $\mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathbb{C}$ (such as $(X | Y) = \text{Tr}(XY)$ for $X, Y \in \mathfrak{gl}_n(\mathbb{C})$). Choose then a root subsystem $\Phi \subseteq \Phi_{\mathfrak{g}}$ —not necessarily obtained from fission.

Introduce $\mathfrak{t}_i := \mathfrak{t} \cap \mathfrak{J}_i$ (a Cartan subalgebra of \mathfrak{J}_i), and let $\Phi_{\mathfrak{J}_i} = \Phi(\mathfrak{J}_i, \mathfrak{t}_i) \subseteq \Phi_{\mathfrak{g}}$ be the associated root system; this way there are two other decompositions:

$$\mathfrak{t} = \bigoplus_i^\perp \mathfrak{t}_i, \quad \Phi_{\mathfrak{g}} = \bigoplus_i \Phi_{\mathfrak{J}_i}.$$

Further let $\Phi^{(i)} := \Phi \cap \Phi_{\mathfrak{J}_i}$, which is a root subsystem of $\Phi_{\mathfrak{J}_i}$.

Then one can show [23] the deformation space (7) splits as a product $\mathbf{B} = \prod_i \mathbf{B}_i$, where

$$\mathbf{B}_i = \text{Ker}(\Phi^{(i)}) \cap \bigcap_{\Phi_{\mathfrak{J}_i} \setminus \Phi^{(i)}} (\mathfrak{t}_i \setminus \text{Ker}(\alpha)) \subseteq \mathfrak{t}_i. \quad (17)$$

Finally the Weyl group also splits as a product $W_{\mathfrak{g}} = \prod_i W_{\mathfrak{J}_i} \subseteq \prod_i \text{GL}(\mathfrak{t}_i)$, where the i -th factor (the Weyl group of $\Phi_{\mathfrak{J}_i}$) acts trivially on the complementary direct summands. It follows that every $W_{\mathfrak{g}}$ -orbit (inside \mathfrak{t}) splits as a product of $W_{\mathfrak{J}_i}$ -orbits (inside \mathfrak{t}_i), so the previous discussion of setwise/pointwise stabilisers can be carried over factorwise, and:

Corollary 3.1. *The deformation space $\overline{\mathbf{B}}$ decomposes as a (topological) product $\prod_i \overline{\mathbf{B}}_i$, where $\overline{\mathbf{B}}_i$ is the topological quotient of (17) with respect to the equivalence relation (6)—with $W_{\mathfrak{J}_i}$ replacing $W_{\mathfrak{g}}$.*

In particular the factor corresponding to the centre $\mathfrak{Z}_{\mathfrak{g}} \subseteq \mathfrak{g}$ is contractible, and can be removed; and further if \mathfrak{g} is semisimple then the full/nonpure local WMCG is a direct product of the groups associated with its simple ideals.

Hence hereafter we will assume \mathfrak{g} is simple.

Remark 3.1. In this case all Cartan subalgebras $\mathfrak{t} \subseteq \mathfrak{g}$ are conjugated by (inner) Lie-algebra automorphisms of \mathfrak{g} , which in turn induces homeomorphisms of the resulting deformation spaces \mathbf{B} . Hence the pure local WMCG does *not* depend on the choice of the Cartan subalgebra, and so in turn neither does the full/nonpure one—since the latter is an extension of the former by a subquotient of the Weyl group. \triangle

3.2. Rank one. If $\text{rk}(\mathfrak{g}) = 1$ then the only nontrivial fission is the “generic” one, so $\overline{\Gamma}(\Phi_{\mathfrak{g}}, \mathbf{d})$ is either trivial or isomorphic to the \mathfrak{g} -braid group. This is of type A_1 , i.e. the braid group on 2-strands, and (10) becomes

$$1 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1,$$

with $\Gamma(\Phi_{\mathfrak{g}}, \mathbf{d}) \simeq \mathbb{Z} \simeq \overline{\Gamma}(\Phi_{\mathfrak{g}}, \mathbf{d})$, matching up with a particular case of (32):

$$1 \longrightarrow \text{PBr}_2 \longrightarrow \text{Br}_2 \longrightarrow \text{S}_2 \longrightarrow 1.$$

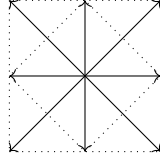
Equivalently up to homotopy we have $\mathbf{B} \simeq \mathbb{S}^1$, and the arrow $\mathbf{B} \rightarrow \overline{\mathbf{B}}$ is the two-sheeted covering of the circle over itself.

3.3. Rank 2. Suppose now $\text{rk}(\mathfrak{g}) = 2$: since \mathfrak{g} is simple then $\Phi_{\mathfrak{g}}$ is isomorphic to A_2 , B_2/C_2 or G_2 , and $W_{\mathfrak{g}}$ is isomorphic to $\text{Dih}_3 \simeq S_3$, Dih_4 or Dih_6 , respectively (i.e. the symmetries of a triangle, a square, or a hexagon). Here Dih_n denotes the dihedral group of order $2n$, for an integer $n \geq 1$ —i.e. we use the “geometric” convention rather than the “algebraic” one.

The generic fission is $\emptyset \subseteq \Phi_{\mathfrak{g}}$, in which case we obtain the \mathfrak{g} -braid group, while the nongeneric (incomplete) fission is $\Phi_{\mathfrak{h}} \subseteq \Phi_{\mathfrak{g}}$, with $\Phi_{\mathfrak{h}} = \{\pm\theta\}$ for some $\theta \in \Delta_{\mathfrak{g}}$ —here $\Delta_{\mathfrak{g}} \subseteq \Phi_{\mathfrak{g}}$ is a choice of simple roots; this corresponds to the deformation space $\mathbf{B} = \mathbb{C} \setminus \{0\}$. With the usual notation we find $\mathbf{U} = \text{Ker}(\theta)$ and $W_{\mathfrak{h}} \simeq \mathbb{Z}/2\mathbb{Z}$, and we must describe $\text{Stab}_{W_{\mathfrak{g}}}(\mathbf{U}) \subseteq W_{\mathfrak{g}}$ —acting on \mathbf{B} . This is the same as the setwise stabiliser of the line $\mathbb{C}\theta \subseteq \mathfrak{t}^{\vee}$ for the dual action, and the difference among the three types is due to the parity of the corresponding dihedral group.

Namely for type A the Weyl group yields the standard permutation action of S_3 on $\mathbb{C}^3 \supseteq \mathfrak{t}^{\vee}$ —identified with the standard dual Cartan subalgebra for $\mathfrak{gl}_3(\mathbb{C}) \supseteq \mathfrak{sl}_3(\mathbb{C})$. Then the only nontrivial permutation fixing the line generated by either simple root is the associated (simple) reflection. It follows that $\text{Stab}_{W_{\mathfrak{g}}}(\mathbf{U}) = W_{\mathfrak{h}}$, so (8) is trivial and the local WMCG is pure: it is thus infinite cyclic.

For type B the long roots are vertices of a square centered at the origin of $\mathfrak{t}_{\mathbb{R}}^{\vee} \simeq \mathbb{R}^2$, while the short roots are vertices of a smaller square obtained by taking midpoints of each side:

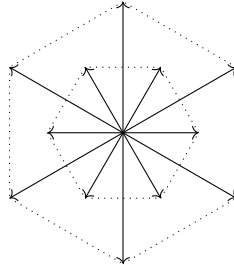


The Weyl group acts by preserving both squares, and operates as the group of their symmetries. In both cases a diagonal is fixed by the subgroup generated by the (simple) reflection along the corresponding axis, but also by a rotation of π . This means the stabiliser is always the Klein four-group $K_4 \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, hence (8) becomes

$$W_{\mathfrak{g}|\mathfrak{h}} \simeq \mathbb{Z}/2\mathbb{Z},$$

acting as the antipode on the punctured plane, and $\mathbf{B} \rightarrow \overline{\mathbf{B}}$ is again a two-sheeted covering of the circle onto itself (up to homotopy equivalence). In particular $\overline{\Gamma}(\Phi_{\mathfrak{g}}, \mathbf{d})$ is infinite cyclic.

Finally type G yields to an analogous situation. Long/short roots assemble into two Weyl-invariant hexagons in the real plane, and the action of the Klein group (within Dih_6) fixes any given diagonal within each hexagon:



Then we can extend to the complete (nongeneric) fission $\emptyset = \Phi_{\mathfrak{t}} \subseteq \Phi_{\mathfrak{h}} \subseteq \Phi_{\mathfrak{g}}$, with the middle term as above. The associated kernel flag is $\mathbf{U} = (\mathfrak{t} \supseteq \mathfrak{t} \supseteq \text{Ker}(\theta))$, so the setwise stabiliser stays the same; but this time the irregular type is centralised by the maximal torus only, so $W_{\mathfrak{g}|\mathfrak{h}}$ will be isomorphic to the group of order two

(for type A), or to the Klein group (for type B/C and G). The result is a covering

$$\mathbb{C}^* \times \mathbb{C}^* = \mathbf{B} \longrightarrow \overline{\mathbf{B}}, \quad \mathbb{C}^* = \mathbb{C} \setminus \{0\},$$

with either two or four sheets, and $\overline{\Gamma}(\Phi_{\mathfrak{g}}, \mathbf{d})$ is an extension of the monodromy group by $\Gamma(\Phi_{\mathfrak{g}}, \mathbf{d}) \simeq \mathbb{Z}^2$.

4. TYPE A

Importantly we can explicitly describe local WMCGs for the special/general linear Lie algebra in full generality, building on [23].

Let $n \geq 2$ be an integer and $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$. The Weyl group $W_{\mathfrak{g}} \simeq S_n$ acts naturally on $V := \mathbb{C}^n$, so we will use the vector representation $\mathfrak{g} \hookrightarrow \mathfrak{gl}(V) \simeq \mathfrak{gl}_n(\mathbb{C})$.

Remark 4.1 (General linear description). Using the basis we identify V with the standard Cartan subalgebra of $\mathfrak{gl}(V)$, so $\mathfrak{t} = V \cap \mathfrak{g}$ (the standard Cartan subalgebra of \mathfrak{g}) becomes the subspace of n -tuples of (possibly coalescing) points of the complex plane with vanishing barycentre.

The resulting inclusion $\mathbb{C}^{n-1} \simeq \mathfrak{t} \hookrightarrow V$ induces a homotopy equivalence

$$\mathfrak{t}_{\text{reg}} \simeq \text{Conf}_n = \mathbb{C}^n \setminus \bigcup_{1 \leq i \neq j \leq n} H_{ij},$$

using the notation of (31), which moreover is compatible with the Weyl group action—since $\mathfrak{gl}_n(\mathbb{C}) = \mathfrak{g} \oplus^{\perp} \mathbb{C} \text{Id}_V$, and the orthogonal line $\mathbb{C} \text{Id}_V = \mathfrak{z}_{\mathfrak{gl}(V)}$ is fixed (pointwise) by $W_{\mathfrak{g}}$. Hence there is a second homotopy equivalence

$$\mathfrak{t}_{\text{reg}}/W_{\mathfrak{g}} \simeq \text{UConf}_n,$$

and whenever useful we will work within the general linear Lie algebra. \triangle

4.1. Inductive step. If $\Phi_{\mathfrak{h}} \subseteq \Phi_{\mathfrak{g}} = A_{n-1}$ is a fission subsystem, we have an associated J -partition $\underline{n} = \coprod_{j \in J} I_j$, and

$$\Phi_{\mathfrak{h}} \simeq \bigoplus_J A_{|I_j|-1} \subseteq A_{n-1},$$

with the usual convention that $A_0 = \emptyset$ (cf. [23] and § A). Namely for $i \in \underline{n}$ we set

$$I_i := \{i\} \cup \left\{ j \in \underline{n} \mid \pm \alpha_{ij}^- \in \Phi_{\mathfrak{h}} \right\} \subseteq \underline{n}, \quad \alpha_{ij}^- = e_i^{\vee} - e_j^{\vee} \in \mathfrak{t}^{\vee},$$

and then

$$J := \{ \min(I_i) \mid i \in \underline{n} \} \subseteq \underline{n}.$$

The Weyl group of \mathfrak{h} thus comes with a natural factorisation

$$W_{\mathfrak{h}} \simeq \prod_J S_{I_j} \subseteq S_n = W_{\mathfrak{g}},$$

with trivial factors corresponding to the trivial components of $\Phi_{\mathfrak{h}}$. The setwise stabiliser of $U = \text{Ker}(\Delta_{\mathfrak{h}}) \subseteq \mathfrak{t}$ is bigger in general, since we can also permute components of $\Phi_{\mathfrak{h}}$ of the same rank.

To state this precisely consider two nonempty finite sets I and K , and suppose $I = \coprod_{k \in K} I_k$ is a K -partition of I with parts $I_k \subseteq I$ of equal cardinality $m \geq 1$ —so $|I| = m|K|$. Then the symmetric group S_I contains the subgroup

$$N := \{ \tau \in S_I \mid \tau(I_k) \subseteq I_k \text{ for } k \in K \} \simeq (S_m)^{|K|},$$

which stabilises all parts (and permutes their elements). If I has a total order then there is a “complementary” subgroup $P \subseteq S_I$, which permutes all parts (fixing their elements): more precisely, if $I_k = (i_1^{(k)}, \dots, i_m^{(k)}) \subseteq I$ for $k \in K$, then any element of $\sigma \in S_K \simeq P$ acts as

$$\sigma: i_j^{(k)} \longmapsto i_j^{(\sigma_k)} \in I, \quad j \in \{1, \dots, m\}.$$

By construction $N \cap P = 1$ inside S_I , and P acts on N by conjugation.

Lemma 4.1 (Cf. [54], Lem. 3.2.8). *If $\tau = \prod_K \tau^{(k)} \in N$, with $\tau^{(k)} \in S_{I_k} \simeq S_m$, then*

$$\sigma\tau\sigma^{-1} = \prod_K \tau^{(\sigma_k^{-1})} \in N, \quad \sigma \in P. \quad (18)$$

Hence we have an *inner* semidirect product $P \ltimes N \subseteq S_I$, and it follows that $(P \ltimes N)/N \simeq P$ canonically. Equivalently the *outer* semidirect product of P and N , with respect to the action (18), comes with a natural group embedding $P \ltimes N \hookrightarrow S_I$. This latter is also the wreath product $S_K \wr S_m \hookrightarrow S_I$, cf. § A. ⁴

Remark 4.2. One has

$$|P \ltimes N| = (m!)^{|K|} |K|! \leq (m|K|)! = |S_I|,$$

with strict inequality if $1 < m < |I|$ (proven e.g. by induction on m). Thus the embedding $P \ltimes N \hookrightarrow S_I$ is *proper* for nontrivial partitions of I . \triangle

Let us apply this to the present situation: for an integer $i \geq 1$ denote

$$K_i := \left\{ j \in J \mid |I_j| = i \right\} \subseteq J.$$

If $i \geq 2$, the integer $|K_i| \geq 0$ is thus the multiplicity of A_{i-1} as an irreducible component of $\Phi_{\mathfrak{h}}$. Instead for $i = 1$ one has a natural bijection

$$K_1 \xrightarrow{\simeq} \left\{ i \in \underline{n} \mid \pm \alpha_{ij}^- \notin \Phi_{\mathfrak{h}} \text{ for any } j \right\}, \quad I_j = \{i\} \mapsto i.$$

The subgroups $P_i \simeq S_{K_i}$ and $N_i \simeq (S_i)^{|K_i|}$ of S_J are defined as the above—noting J inherits a natural total order from \underline{n} .

Proposition 4.1. *There is a canonical group isomorphism*

$$\text{Stab}_{W_{\mathfrak{g}}}(\mathcal{U}) \simeq \prod_{i \geq 0} (S_{K_i} \wr S_i) \subseteq W_{\mathfrak{g}}. \quad (19)$$

Proof. The statement is the algebraic rewriting of the following claim: the setwise stabiliser of $\mathcal{U} = \text{Ker}(\Delta_{\mathfrak{h}})$ is the subgroup of $W_{\mathfrak{g}} = S_n$ that permutes parts $I_j \subseteq \underline{n}$ of the same cardinality, and that further permutes the elements within each part. By the above discussion this yields the direct product (19)—as permutations of disjoint parts commute.

To prove the claim recall the “extended” kernel $\widetilde{\text{Ker}}(\Delta_{\mathfrak{h}}) \subseteq V$ is defined by the condition that the coordinate of any vector are equal within each part $\mathbb{C}^{I_j} \subseteq V$, so its setwise stabiliser is given by the above condition. Thus to conclude it is enough to show that the setwise Weyl-stabiliser of the “essential” kernel $\mathcal{U} = \widetilde{\text{Ker}}(\Delta_{\mathfrak{h}}) \cap \mathfrak{t}$ is the same; but by construction

$$\widetilde{\text{Ker}}(\Delta_{\mathfrak{h}}) = \mathcal{U} \oplus^{\perp} \mathbb{C} \text{Id}_V \subseteq \mathfrak{gl}(V),$$

and the orthogonal line is fixed (pointwise) by the whole of $W_{\mathfrak{g}}$ (cf. Rem. 4.1). \square

Corollary 4.1. *One has*

$$W_{\mathfrak{gl}(\mathfrak{h})} \simeq \prod_{i \geq 0} S_{K_i}.$$

⁴This is a particular example of application of the operadic composition of the symmetric group operad, cf. [54, § 3.1] and below.

Proof. By definition $W_{\mathfrak{g}|\mathfrak{h}}$ is the quotient (8), which is readily computed in this case using Prop. 4.1 and the factorisation

$$W_{\mathfrak{h}} \simeq \prod_{i \geq 1} (S_i)^{|K_i|}, \quad (20)$$

as quotients and direct products commute. ⁵ \square

This corresponds to the fact that $W_{\mathfrak{g}|\mathfrak{h}}$ is naturally identified with the subgroup permuting parts of equal cardinality (and fixing the elements within each part). In particular in this case the exact group sequence

$$1 \longrightarrow W_{\mathfrak{h}} \longrightarrow \text{Stab}_{W_{\mathfrak{g}}}(\mathcal{U}) \longrightarrow W_{\mathfrak{g}|\mathfrak{h}} \longrightarrow 1$$

splits.

Further (20) is naturally a subgroup of the Weyl group of the “reduced” root system

$$\Phi_{\mathfrak{g}|\mathcal{U}} = \left\{ \alpha|_{\mathcal{U}} \mid \alpha \in \Phi_{\mathfrak{g}} \right\} \simeq A_{|J|-1},$$

viz. a subgroup of S_J —corresponding to certain “admissible” permutations.

On the whole there is a Galois covering $\mathbf{B} \rightarrow \overline{\mathbf{B}}$ with $\prod_{i \geq 0} (|K_i|!)$ sheets, and to go further let us work within $\mathfrak{gl}(V) \supseteq \mathfrak{g}$. Recall from [23] that there is a canonical vector space isomorphism $\widetilde{\mathcal{U}} := \widetilde{\text{Ker}}(\Delta_{\mathfrak{h}}) \simeq \mathbb{C}^J$, and by Rem. 4.1 the $W_{\mathfrak{g}}$ -equivariant inclusion $\mathcal{U} \hookrightarrow \widetilde{\mathcal{U}}$ yields homotopy equivalences

$$\mathbf{B} \simeq \text{Conf}_{|J|}, \quad \mathbf{B}/W_{\mathfrak{g}} \simeq \text{UConf}_{|J|},$$

with fundamental groups $\text{PBr}_{|J|}$ and $\text{Br}_{|J|}$, respectively. What we have here is an “intermediate” covering, since it is only $W_{\mathfrak{g}|\mathfrak{h}}$ that acts (freely) on \mathbf{B} .

To simplify the notation let us contemplate the following abstract situation. For an integer $d \geq 1$ consider the ordered configuration space $Y_d := \text{Conf}_d \subseteq \mathbb{C}^d$, as well as an I -partition $\varphi: \underline{d} \rightarrow I$ with parts $I_i = \varphi^{-1}(i) \subseteq \underline{d}$, for $i \in I$. Then there is a natural group embedding

$$S_{\varphi} := \prod_I S_{I_i} \hookrightarrow S_d,$$

obtained by juxtaposing permutations, and we let $X_{\varphi} := Y_d/S_{\varphi}$ (the “semioordered” configuration space): this is the space of configurations of $d = \sum_i |I_i|$ points in the complex plane, such that two of them are indistinguishable if they lie within the same part of the I -partition.

To identify the fundamental group recall there is an “augmentation” group morphism $p_d: \text{Br}_d \rightarrow S_d$, with kernel $\pi_1(Y_d) = \text{PBr}_d \subseteq \text{Br}_d$.

Proposition 4.2. *There is a group isomorphism*

$$\pi_1(X_{\varphi}) \simeq \text{Br}_{\varphi} \subseteq \text{Br}_d, \quad \text{Br}_{\varphi} := p_d^{-1}(S_{\varphi}),$$

and Br_{φ} is an extension of S_{φ} by PBr_d .

Here Br_{φ} is thus the “semipure” braid group of the partition, i.e. the group of braids whose underlying permutation lies within $S_{\varphi} \subseteq S_d$.

Proof. There are Galois coverings $Y_d \rightarrow X_d := \text{UConf}_d$ and $Y_d \rightarrow X_{\varphi}$, and it follows the induced map $X_{\varphi} \rightarrow X_d$ is a covering (with $[S_d: S_{\varphi}]$ sheets). Up to identifying groups and torsors (the discrete fibres), after a suitable choice of base points, this

⁵Note (20) has a trivial factor corresponding to K_1 .

yields a commutative diagram of pointed topological spaces, with (principle) fibre bundles in each row:

$$\begin{array}{ccccc} S_\varphi & \hookrightarrow & Y_d & \twoheadrightarrow & X_\varphi \\ \downarrow & & \parallel & & \downarrow \\ S_d & \hookrightarrow & Y_d & \twoheadrightarrow & X_d \end{array} .$$

In turn this leads to a morphism of (short) exact group sequences, proving the statement:

$$\begin{array}{ccccccc} & & \pi_1(X_\varphi) & \longrightarrow & S_\varphi & & \\ & \swarrow & \downarrow & & \downarrow & \searrow & \\ 1 & \longrightarrow & \text{PBr}_d & & & & 1 \\ & \searrow & \downarrow & & \downarrow & \swarrow & \\ & & \text{Br}_d & \xrightarrow{p_d} & S_d & & \end{array} .$$

Note indeed $\text{Ker}(p_d|_{\text{Br}_\varphi}) = \text{Ker}(p_d) \cap \text{Br}_\varphi = \text{PBr}_d$. \square

In our situation we thus find a group isomorphism

$$\pi_1(\overline{\mathbf{B}}) \simeq \text{Br}_\varphi \subseteq \text{Br}_{|J|},$$

where $\varphi: J \rightarrow I \subseteq \mathbb{Z}_{\geq 0}$ is the I -partition obtained from $J = \coprod_{i \geq 0} K_i$ by removing the empty parts.

Remark 4.3. The extreme cases are $S_\varphi = 1$, where $X_\varphi = Y_d$, and $S_\varphi = S_d$, where $X_\varphi = X_d$ is the (fully) unordered configuration space. In our setting this means either no two irreducible components of Φ_h have the same rank, or conversely they all have the same rank—respectively. \triangle

Remark 4.4. Note there is also a different subgroup of Br_d associated with the partition and projecting onto S_φ , namely $\prod_i \text{Br}_{I_i} \hookrightarrow \text{Br}_d$: this is the subgroup obtained by juxtaposing $|I|$ braids, each on $|I_i|$ strands. However in general the inclusion $\prod_i \text{Br}_{I_i} \subseteq \text{Br}_\varphi$ is *proper*. E.g. $\text{Br}_1 \times \text{Br}_1 \subseteq \text{Br}_2$ is trivial, while

$$p_2^{-1}(S_1 \times S_1) = p_2^{-1}(1) = \text{PBr}_2 \simeq \mathbb{Z}.$$

This simple example shows the fundamental group of the semiordered configuration space is *not* just the direct product of the corresponding braid groups: it is possible two points in different parts braid across each other (along a loop in X_φ), provided they are not swapped by the underlying permutation of the overall braid. \triangle

Remark 4.5. By the Galois correspondence the isomorphism class of the covering $X_\varphi \rightarrow X_d$ matches up with the conjugacy class of a subgroup of $\text{Br}_d = \pi_1(X_d)$.⁶ This is precisely the conjugacy class of $\text{Br}_\varphi \subseteq \text{Br}_d$, which is generically nontrivial—as $S_\varphi \subseteq S_d$ is generically *not* a normal subgroup. \triangle

4.2. General case: ranked fission trees. Suppose now to have an increasing filtration

$$\Phi_{h_1} \subseteq \dots \subseteq \Phi_{h_p} \subseteq \Phi_{h_{p+1}} := \mathcal{A}_{n-1}.$$

of fission subsystems. As in [23] this corresponds to a “fission” tree $T = (T_0, \Phi)$ of height $p \geq 1$ (cf. § A). The set $J_l = J_{h_l}$ is as above, for $l \in \underline{p}$, and then we add a tree root at level $p+1$. By definition $\Phi(i) = j \in J_{l+1}$ means that the irreducible component of $h_l \subseteq h_{l+1}$ corresponding to $i \in J_l$ lies within the irreducible component of h_{l+1} corresponding to j .

This was enough to encode the pure local WMCG in type A, while in the nonpure case we must retain more data, according to the results of the previous section.

⁶All spaces involved are (locally) path-connected and semi-locally simply-connected [32, Thm. 1.38].

Definition 4.1 (Ranked fission tree). A *ranked* fission tree is a fission tree $T = (T_0, \Phi)$ equipped with a *rank function* $\mathbf{r}: T_0 \rightarrow \mathbb{Z}_{\geq 1}$; in turn a rank function satisfies:

$$\mathbf{r}(i) = \sum_{\Phi^{-1}(i)} \mathbf{r}(j), \quad i \in T_0.$$

We then say $\mathbf{r}(T) := \mathbf{r}(*) \geq 1$ is the *rank* of the tree.

This means to each node we attach a positive rank, which equals the sum of its child-nodes'. In particular $\sum_{J_1} \mathbf{r}(i) = \mathbf{r}(T)$, independently of the level, and the rank function is determined by assigning ranks to the leaves—i.e. by $\mathbf{r}|_{J_1} \in \mathbb{Z}_{\geq 1}^{J_1}$.

The algorithm to associate a ranked fission tree (T, \mathbf{r}) to (12) is the following: the underlying fission tree is constructed as in [23], and we further set $\mathbf{r}(i) = k + 1$ if the node $i \in J_1$ corresponds to a type-A irreducible rank- k component of $\Phi_{\mathfrak{h}_{l+1}}$. Working within the general linear Lie algebra, this is the same as setting $\mathbf{r}(i) = k$ if i corresponds to an irreducible component isomorphic to $\Phi_{\mathfrak{gl}_k(\mathbb{C})}$ —including $\Phi_{\mathfrak{gl}_1(\mathbb{C})} = \emptyset$. It follows that $\mathbf{r}(T) = n$ if we work within $\mathfrak{gl}_n(\mathbb{C})$.⁷

By construction the Weyl group of $\mathfrak{h}_l \subseteq \mathfrak{g}$ comes with a canonical group isomorphism

$$W_{\mathfrak{h}_l} \simeq \prod_{J_1} S_{\mathbf{r}(i)}, \quad l \in \underline{p}, \quad (21)$$

and to construct the stabiliser of the kernel flag in terms of the tree we introduce the following.

Definition 4.2. An *isomorphism* $(T_0, \Phi, \mathbf{r}) \rightarrow (T'_0, \Phi', \mathbf{r}')$ of ranked fission trees is a bijection $f: T_0 \rightarrow T'_0$ matching roots, and such that there are commutative diagrams:

$$\begin{array}{ccc} T_0 \setminus \{*\} & \xrightarrow{f} & T'_0 \setminus \{*\} \\ \Phi \downarrow & & \downarrow \Phi' \\ T_0 & \xrightarrow{f} & T'_0 \end{array} \quad \text{and} \quad \begin{array}{ccc} T_0 & \xrightarrow{f} & T'_0 \\ \searrow \mathbf{r} & & \swarrow \mathbf{r}' \\ & \mathbb{Z}_{\geq 1} & \end{array}.$$

An *automorphism* of (T_0, Φ, \mathbf{r}) is an isomorphism $(T_0, \Phi, \mathbf{r}) \rightarrow (T_0, \Phi, \mathbf{r})$; their group is denoted $\text{Aut}(T, \mathbf{r})$.

This restricts the usual notion of isomorphism of (rooted) trees, by further asking that ranks be preserved. Note by definition an automorphism preserves the nodes at each level, and is uniquely determined by the image of the leaves.

4.3. General case: reflection groups. It is possible to compute the automorphism group of the tree recursively, and in turn this will control the monodromy action of the Galois covering $\mathbf{B} \rightarrow \overline{\mathbf{B}} = \mathbf{B}/W_{\mathfrak{gl}_l}$.

Choose then a ranked fission tree (T, \mathbf{r}) , and note its subtrees are equipped with restricted rank functions. In particular let $\mathcal{T} = \mathcal{T}(T, \mathbf{r})$ be the set of (ranked) *maximal* proper subtrees, i.e. the subtrees of T rooted at each child-node of the root, and choose a complete set of representatives $\tilde{\mathcal{T}} \subseteq \mathcal{T}$ of isomorphism classes. Finally denote $n(t) \geq 1$ the cardinality of the isomorphism class of any maximal proper subtree $t \in \tilde{\mathcal{T}}$.

Definition 4.3. The *extended* automorphism group $\widetilde{\text{Aut}}(T, \mathbf{r})$ of the ranked fission tree (T, \mathbf{r}) is defined recursively by

$$\widetilde{\text{Aut}}(T, \mathbf{r}) = \prod_{t \in \tilde{\mathcal{T}}} S_{n(t)} \wr \widetilde{\text{Aut}}(t, \mathbf{r}_0), \quad \mathbf{r}_0 := \mathbf{r}|_{t_0}, \quad (22)$$

⁷This choice is more natural since the (nonsemisimple) rank of the general linear Lie algebra controls the Weyl/braid groups *without* shifts.

with basis $\widetilde{\text{Aut}}(i, r(i)) := S_{r(i)}$ for $i \in J_1$.

Note at each step it is the global information of the subtree t that is required in the algorithm—rather than simply the data at each level of T .

A priori (22) depends on the choice of $\widetilde{\mathcal{T}} \subseteq \mathcal{T}$, but the following identification in particular shows it does not.

Theorem 4.1. *One has $\widetilde{\text{Aut}}(T, \mathbf{r}) = \text{Stab}_{W_{\mathfrak{g}}}(\mathbf{U})$. Further the automorphism group $\text{Aut}(T, \mathbf{r})$ is obtained recursively as in (22) but with recursion basis $\text{Aut}(i, r(i)) := S_1$ for $i \in J_1$ —the trivial group.*

Proof. The first item can be proven by induction on $p \geq 1$.

If $p = 1$ then a maximal proper subtree is a leaf, so $\mathcal{T} = J_1$: then two leaves are isomorphic (as ranked trees) if and only if they have the same rank. Hence for $i \in J_1$ the integer $n(i) \geq 1$ is the number of rank- $r(i)$ leaves, and in this case

$$\widetilde{\text{Aut}}(T, \mathbf{r}) = \prod_{i \in \widetilde{J}_1} S_{n(i)} \wr S_{r(i)} \subseteq S_{r(T)},$$

where $\widetilde{J}_1 \subseteq J_1$ is a set of representatives of leaves—of all possible ranks. The result follows from Prop. 4.1.

Now let $p \geq 2$. By the induction hypothesis the bases of the wreath products in (22) are the setwise stabilisers of the deformation space of the “sub-irregular types” obtained by focusing on each eigenspaces of the leading coefficient. In addition to that we are then permuting isomorphic maximal proper subtrees, i.e. eigenspaces of the leading coefficient whose nested decomposition (into eigenspaces for the subleading coefficients) plays a symmetric role: this yields the whole of $\text{Stab}_{W_{\mathfrak{g}}}(\mathbf{U})$, as any other permutation of the eigenvalues of the leading coefficient moves the irregular type out of the space of admissible deformations.

The second item is a straightforward extension from the unranked case, and can also be proven recursively on $p \geq 1$.

If $p = 1$ an automorphism is the data a permutation of the leaves which matches up ranks. Hence in this case

$$\text{Aut}(T, \mathbf{r}) = \prod_{i \in \widetilde{J}_1} S_{n(i)} = \prod_{i \in \widetilde{J}_1} S_{n(i)} \wr S_1,$$

using the above notation. (Note $S_{n(i)} \subseteq S_{J_1}$ is naturally identified with the symmetric group of rank- $r(i)$ leaves.)

Now let $p \geq 2$. By the induction hypothesis the bases of the wreath products in (22) are the automorphism groups of the maximal proper ranked subtrees. In addition to that we are then permuting isomorphic maximal proper subtrees: this yields the whole of $\text{Aut}(T, \mathbf{r})$, as any other permutation of child-nodes of the root, bringing along the corresponding subtrees, cannot restrict to an isomorphism of these latter. \square

Example 4.1. The most symmetric example is that in which \mathbf{r} is constant at each level: in this case any automorphism of the underlying tree $T = (T_0, \Phi)$ preserves the rank function.

If moreover T is a complete m -ary tree, viz. if all interior nodes have $m \geq 1$ child-nodes, then simply

$$\text{Aut}(T, \mathbf{r}) \simeq \underbrace{S_m \wr \cdots \wr S_m}_{p \text{ times}},$$

the p -fold wreath power—recall this example of wreath product is associative. The extended group instead is

$$\widetilde{\text{Aut}}(\mathbb{T}, \mathbf{r}) \simeq (S_n)^{\wr p} \wr S_r,$$

where $r \geq 1$ is the rank of any leaf.

On the opposite end, if \mathbf{r} is injective at each level, the group $\text{Aut}(\mathbb{T}, \mathbf{r})$ is trivial—and $\widetilde{\text{Aut}}(\mathbb{T}, \mathbf{r}) \simeq \prod_{J_1} S_{r(i)} \subseteq S_{r(\mathbb{T})}$. In this case the type-A local WMCG is pure. \triangle

Now by (recursive) construction $\text{Aut}(\mathbb{T}, \mathbf{r}) \subseteq \widetilde{\text{Aut}}(\mathbb{T}, \mathbf{r})$ is a subgroup, so by Thm. 4.1 it can be identified with a subgroup of the kernel-flag stabiliser.

Indeed choose $\mathbf{f} \in \text{Aut}(\mathbb{T}, \mathbf{r})$, so by definition $\mathbf{f}: T_0 \rightarrow T_0$ yields rank-preserving permutations $f_l := \mathbf{f}|_{J_l} \in S_{J_l}$ of the nodes at each level $l \in \underline{p+1}$. In particular f_1 permutes subsets of leaves (of constant rank), and we can map it to an element of $W_{\mathfrak{g}} \simeq S_{r(\mathbb{T})}$ along the group embedding

$$\prod_{\tilde{J}_1} S_{n(i)} \hookrightarrow \prod_{\tilde{J}_1} S_{n(i)} \wr S_{r(i)} \subseteq S_{r(\mathbb{T})},$$

keeping the notation of the proof of Thm. 4.1. This yields an injective group morphism $\iota: \text{Aut}(\mathbb{T}, \mathbf{r}) \hookrightarrow \text{Stab}_{W_{\mathfrak{g}}}(\mathbf{U})$ —since \mathbf{f} is determined by f_1 .

By construction the image of ι is disjoint from $W_{\mathfrak{h}_1} = \prod_{J_1} S_{r(i)}$ (using (21)), and acts on it by conjugation, so there is a second group embedding

$$\tilde{\iota}: \text{Aut}(\mathbb{T}, \mathbf{p}) \times W_{\mathfrak{h}_1} \hookrightarrow \widetilde{\text{Aut}}(\mathbb{T}, \mathbf{r}). \quad (23)$$

Proposition 4.3. *The group morphism (23) is surjective.*

Hence the exact group sequence

$$1 \longrightarrow W_{\mathfrak{h}_1} \longrightarrow \text{Stab}_{W_{\mathfrak{g}}}(\mathbf{U}) \longrightarrow W_{\mathfrak{g}/\mathfrak{h}} \longrightarrow 1$$

splits in the general type-A fission, generalising the recursive step.

Proof. It is equivalent to show that $W_{\mathfrak{h}_1} \subseteq \widetilde{\text{Aut}}(\mathbb{T}, \mathbf{r})$ is a normal subgroup, and that

$$\widetilde{\text{Aut}}(\mathbb{T}, \mathbf{r})/W_{\mathfrak{h}_1} \simeq \text{Aut}(\mathbb{T}, \mathbf{r}).$$

This can be proven recursively on $p \geq 1$, the base being the content of Cor. 4.1.

If $p \geq 2$ consider a maximal proper subtree $\mathfrak{t} \in \mathcal{T}$: its leaves yield a subset $J_1(\mathfrak{t}) \subseteq J_1$, and there is a partition

$$J_1 = \coprod_{\mathcal{T}} J_1(\mathfrak{t}),$$

of the leaves of \mathbb{T} . Accordingly the centraliser of the irregular type splits as

$$W_{\mathfrak{h}_1} \simeq \prod_{\mathcal{T}} W_{\mathfrak{h}_1}(\mathfrak{t}) = \prod_{\tilde{\mathcal{T}}} (W_{\mathfrak{h}_1}(\mathfrak{t}))^{n(\mathfrak{t})},$$

where $W_{\mathfrak{h}_1}(\mathfrak{t}) := \prod_{J_1(\mathfrak{t})} S_{r(i)}$ is the Weyl group of the Lie algebra $\mathfrak{h}_1 \cap \mathfrak{gl}_{r(\mathfrak{t})}(\mathbb{C}) \subseteq \mathfrak{h}_1$, and in turn $\mathfrak{gl}_{r(\mathfrak{t})}(\mathbb{C}) \subseteq \mathfrak{gl}_{r(\mathbb{T})}(\mathbb{C})$ matches up with the eigenspace of the leading coefficient corresponding to the root of the subtree \mathfrak{t} —viz. a child-node of the root of \mathbb{T} .

Hence, using the decomposition (22), and the fact that direct products and quotients commute, the result follows from Lem. 4.2; indeed in particular

$$\begin{aligned} \widetilde{\text{Aut}}(\mathbb{T}, \mathbf{r})/W_{\mathfrak{h}_1} &\simeq \left(\prod_{\tilde{\mathcal{T}}} S_{n(\mathfrak{t})} \wr \widetilde{\text{Aut}}(\mathfrak{t}, \mathbf{r}) \right) / W_{\mathfrak{h}_1} = \prod_{\tilde{\mathcal{T}}} \left(S_{n(\mathfrak{t})} \wr \widetilde{\text{Aut}}(\mathfrak{t}, \mathbf{r}) / W_{\mathfrak{h}_1(\mathfrak{t})} \right) \\ &= \prod_{\tilde{\mathcal{T}}} S_{n(\mathfrak{t})} \wr \text{Aut}(\mathfrak{t}, \mathbf{r}) = \text{Aut}(\mathbb{T}, \mathbf{r}), \end{aligned}$$

by the recursive hypothesis—and definition of the automorphism group. \square

Lemma 4.2. *Let $m \geq 0$ be an integer and P a group, and choose a normal subgroup $N \subseteq P$. Then $1 \wr N \subseteq S_m \wr P$ is a normal subgroup, and in this identification there is a canonical group isomorphism*

$$(S_m \wr P)/N \simeq S_m \wr (P/N). \quad (24)$$

Proof. Postponed to § B. \square

Hence in brief there is an explicit (finite) algorithm to compute the “effective” subquotient of the Weyl group acting freely on the deformation space of any type-A irregular type, to yield the deformation space of the associated irregular class.

Example 4.2. Let us look at the examples of type-A irregular type considered in [23], which all had pure local WMCG isomorphic to $PBr_2 \times PBr_3^2 \times PBr_4$. We will see their associated stabilisers are *not* all isomorphic.

First consider

$$Q = T_1x + T_2x^2 + T_3x^3, \quad T_i \in \mathfrak{gl}_9(\mathbb{C}),$$

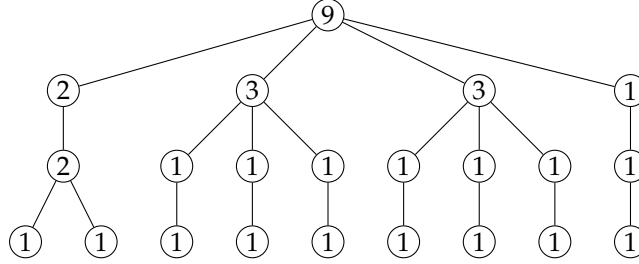
with

$$T_1 = \text{diag}(4, 3, 2, 1, 0, -1, -2, -3, -4),$$

$$T_2 = \text{diag}(4, 4, 3, 2, 1, 0, -3, -4, -7),$$

$$T_3 = \text{diag}(2, 2, 1, 1, 1, 0, 0, 0, -7).$$

The corresponding ranked fission tree (T, r) is drawn below:



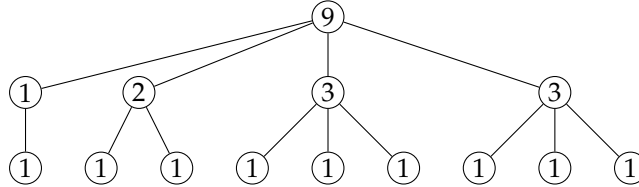
From the recursive algorithm we get

$$\text{Aut}(T, r) = S_2 \times (S_2 \wr S_3).$$

This is the same as the automorphism group of the ranked tree for the irregular type $Q = T_1x + T_2x^2$, with $T_i \in \mathfrak{gl}_9(\mathbb{C})$ given by

$$T_1 = \text{diag}(4, 3, 2, 1, 0, -1, -2, -3, -4), \quad T_2 = \text{diag}(4, 1, 1, 0, 0, 0, -2, -2, -2).$$

Indeed in that case the tree is as follows:



Finally let us consider $Q = T_1x + T_2x^2 + T_3x^3 + T_4x^4$, with $T_i \in \mathfrak{gl}_9(\mathbb{C})$ given by

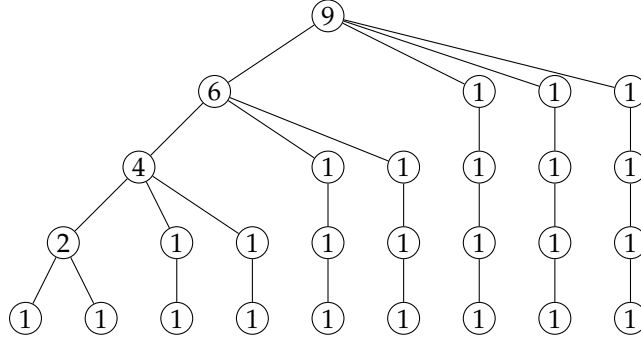
$$T_1 = \text{diag}(4, 3, 2, 1, 0, -1, -2, -3, -4),$$

$$T_2 = \text{diag}(4, 4, 3, 2, 1, 0, -3, -4, -7),$$

$$T_3 = \text{diag}(2, 2, 2, 2, 1, 0, -3, -3, -3),$$

$$T_4 = \text{diag}(1, 1, 1, 1, 1, 1, 0, -2, -4).$$

The ranked fission tree is then:



Its automorphism group is now $\text{Aut}(T, \mathbf{r}) = S_3 \times S_2^3$, which is *not* the same as in the previous two cases.

This is not contradictory: the pure local WMCG only depends on the whole set of unordered configuration spaces attached to the (unranked) fission tree, while the full/nonpure local WMCG also depends on their positions in the tree. \triangle

4.4. General case: (cabled) braid group. Write now $\mathbf{B} = \prod_{i=1}^p \mathbf{B}_i$ the deformation space, in the decomposition of [23]. This means $\mathbf{B}_i \subseteq \mathbb{C}^{J^i}$ is a product of ordered configuration spaces (type-A root-hyperplane complements), attached to the nodes at the above level J_{i+1} : namely

$$\mathbf{B}_i = \prod_{J_{i+1}} \text{Conf}_{k_i} \subseteq \mathbb{C}^{J^i}, \quad k_i = |\Phi^{-1}(i)| \geq 1,$$

counting the number of child-nodes. Thus globally

$$\mathbf{B} = \prod_{T_0} \text{Conf}_{k_i} \subseteq \mathbb{C}^{T_0 \setminus \{*\}}, \quad \text{and} \quad \pi_1(\mathbf{B}) \simeq \prod_{T_0} \text{PBr}_{k_i}. \quad (25)$$

Now recall from [23, § 6] that the *cabling* of pure braid group operad, viz. the operadic composition

$$\gamma^{\mathcal{PB}}: \text{PBr}_n \times \prod_{i=1}^n \text{PBr}_{k_i} \longrightarrow \text{PBr}_k, \quad k = \sum_i k_i, \quad (26)$$

for $n, k_1, \dots, k_n \geq 0$, yields a group embedding $\pi_1(\mathbf{B}) \hookrightarrow \text{PBr}_{J_1}$. More precisely recursive cabling along the (unranked) fission tree $T = (T_0, \Phi)$ leads to a pure “cabled braid group” $\text{PCBr}(T) \subseteq \text{PBr}_{|J_1|}$, and there is a canonical group isomorphism $\pi_1(\mathbf{B}) \simeq \text{PCBr}(T)$.⁸ The point is that in the pure case one finds a “noncrossed” group/action operad, which in particular implies (26) is a group morphism equipping the domain with the direct product structure.

Here instead we naturally encountered the operadic composition of the *symmetric group operad* $\mathcal{S} = (\mathbf{S}_\bullet, 1 \in \mathbf{S}_1, \gamma^{\mathcal{S}})$, viz. the function

$$\gamma^{\mathcal{S}}: \mathbf{S}_n \times \prod_{i=1}^n \mathbf{S}_{k_i} \longrightarrow \mathbf{S}_k, \quad (\sigma, \boldsymbol{\tau}) \longmapsto \gamma^{\mathcal{S}}(\sigma; \boldsymbol{\tau}), \quad (27)$$

for $\sigma \in \mathbf{S}_n$ and $\boldsymbol{\tau} = \prod_{i=1}^n \tau^{(i)}$. Its definition is a generalisation of the above construction to arbitrary partitions $\underline{k} = \coprod_{i=1}^n I_i$, where $k_i = |I_i|$. Namely there is a “block permutation” operation

$$\mathbf{S}_n \longrightarrow \mathbf{S}_k, \quad \sigma \longmapsto \sigma \langle k_1, \dots, k_n \rangle,$$

⁸See op. cit. for an explanation of terminology, due to the nested braiding of eigenspaces for the coefficients of the irregular type.

which consists in the permutation of all parts by fixing their elements, and then

$$\gamma^{\mathcal{S}}(\sigma; \tau) := \sigma \langle k_1, \dots, k_n \rangle \cdot \tau \in S_k,$$

with tacit use of the natural group embedding $\prod_i S_{I_i} \hookrightarrow S_k$ on the right factor.

Lemma 4.3. *If $k_1 = \dots = k_n$ then (27) is an injective morphism, equipping the domain with the semidirect product structure.*

Proof. Postponed to B. □

This means the operadic compositions yields in particular group embeddings $S_n \wr S_m \hookrightarrow S_{mn}$, which were used above.

Thus the recursive definition (22) can be rewritten in this language, and amounts to:

$$\text{Aut}(T, \mathbf{r}) = \prod_{t \in \tilde{\mathcal{T}}} \gamma^{\mathcal{S}} \left(S_{n(t)} \times \text{Aut}(t, \mathbf{r}_0)^{n(t)} \right) \subseteq S_{J_1}, \quad \mathbf{r}_0 = \mathbf{r}|_{t_0}, \quad (28)$$

starting again from the trivial group at each leaf. But finally this can reformulated to exhibit the relation with braid groups.

Lemma 4.4. *Let $\{P_i\}_{i \in I}$ and $\{N_i\}_{i \in I}$ be finite collections of groups, and $\rho_i: P_i \rightarrow \text{Aut}(N_i)$ group morphisms. Then there is a canonical group isomorphism*

$$P \ltimes N \simeq \prod_I P_i \ltimes N_i, \quad P = \prod_I P_i, \quad N = \prod_I N_i,$$

using the production action on the left-hand side:

$$\rho: P \longrightarrow \prod_I \text{Aut}(N_i) \subseteq \text{Aut}(N), \quad \rho = \prod_I \rho_i.$$

Proof. Postponed to § B. □

It follows that (28) is equivalent to the recursive definition

$$\text{Aut}(T, \mathbf{r}) = S_\varphi \times \prod_{\tilde{\mathcal{T}}} \text{Aut}(T, \mathbf{r}_0)^{n(t)}, \quad (29)$$

introducing the $\tilde{\mathcal{T}}$ -partition $\varphi: J_p \rightarrow \tilde{\mathcal{T}}$ induced from the isomorphism classes of maximal proper subtrees; this means $S_\varphi = \prod_{\tilde{\mathcal{T}}} S_{n(t)} \subseteq S_{J_p}$.

The expression (29) clarifies the natural definition of an analogous (full/nonpure) “cabled” braid groups: one should “lift” this through the (augmentation) operad morphism $\mathbf{p}: \mathcal{B} \rightarrow \mathcal{S}$, where $\mathcal{B} = (\text{Br}_\bullet, 1 \in \text{Br}_1, \gamma^{\mathcal{B}})$ is the (full/nonpure) braid group operad.

Definition 4.4. The *cabled braid group* of the ranked fission tree (T, \mathbf{r}) is the group recursively defined by

$$\text{CBr}(T, \mathbf{r}) = \text{Br}_\varphi \times \prod_{\tilde{\mathcal{T}}} \text{CBr}(t, \mathbf{r}_0)^{n(t)} \subseteq \text{Br}_{|J_1|}, \quad (30)$$

with basis $\text{CBr}(i, \mathbf{r}(i)) := \text{Br}_1$, for $i \in J_1$, using the semipure braid group of Prop. 4.2.

Here $\text{Br}_\varphi \subseteq \text{Br}_{|J_p|}$ is the subgroup corresponding to the braiding of the maximal proper subtrees (i.e. the equal-dimensional eigenspaces of the leading coefficient), acting by conjugation of the cabled braid group of any such subtree.

Remark 4.6. The recursive definition of the pure cabled braid group of [23] can also be given in terms of subtrees, but in that context it simplifies to a recursion on the level—since it only involves direct products. △

Finally we can prove that (30) is the correct definition, i.e. that indeed this is the group controlling the topology of admissible deformations of type-A irregular classes.

Theorem 4.2. *There is a group isomorphism $\pi_1(\overline{\mathbf{B}}) \simeq \text{CBr}(\mathbf{T}, \mathbf{r})$, and $\text{CBr}(\mathbf{T}, \mathbf{r})$ is an extension of $\text{Aut}(\mathbf{T}, \mathbf{r})$ by $\text{PCBr}(\mathbf{T})$.*

Proof. By Prop. 4.3 the projection $\mathbf{B} \rightarrow \overline{\mathbf{B}}$ amounts to the Galois covering over the quotient $\mathbf{B}/\text{Aut}(\mathbf{T}, \mathbf{r})$, so the claimed group extension will follow from the first statement—as $\text{PCBr}(\mathbf{T}) \simeq \pi_1(\mathbf{B})$, proven in [23].

The first statement instead can be proven by induction on $p \geq 1$. If $p = 1$ then $\text{PCBr}(\mathbf{T}) = \text{PBr}_{|J_p|}$, and $\text{CBr}(\mathbf{T}, \mathbf{r}) \subseteq \text{Br}_{|J_p|}$ is the “semipure” braid group of partition of the leaves into equal-rank nodes: the result then follows from Prop. 4.2.

Now suppose $p \geq 2$, and write $\mathbf{B}(\mathbf{T})$ the space determined by the (unranked) tree as in (25). By definition

$$\mathbf{B}(\mathbf{T}) = \text{Conf}_{|J_p|} \times \prod_{t \in \tilde{\mathcal{T}}} \mathbf{B}(t)^{n(t)},$$

with the usual notation for representatives of maximal proper subtrees, and for the cardinality of their isomorphism classes. Then for $t \in \tilde{\mathcal{T}}$ the base of the wreath product $S_{n(t)} \wr \text{Aut}(t, \mathbf{r}_0)$ acts on the rightmost factor, while $S_{n(t)}$ is naturally a subgroup of permutations of the child-nodes of the roots—permuting the (isomorphic) subtrees rooted there.

Assume first $\tilde{\mathcal{T}} = \{t\}$ is a singleton, i.e. all maximal proper subtrees are isomorphic, and let $n := |J_p|$. Then simply $\mathbf{B}(\mathbf{T}) = \text{Conf}_n \times \mathbf{B}(t)^n$, and accordingly $\text{Aut}(\mathbf{T}, \mathbf{r}) = S_n \wr \text{Aut}(t, \mathbf{r}_0)$ by (22).

Now we have a natural surjective map $\mathbf{B}(\mathbf{T}) \rightarrow \text{UConf}_n$, composing the canonical projection $\mathbf{B}(\mathbf{T}) \rightarrow \text{Conf}_n$ with the Galois covering $\text{Conf}_n \rightarrow \text{UConf}_n$; and there is also a Galois covering $\mathbf{B}(\mathbf{T}) \rightarrow \overline{\mathbf{B}}(\mathbf{T}) = \mathbf{B}(\mathbf{T})/\text{Aut}(\mathbf{T}, \mathbf{r})$. By construction the former factorises through the latter, so there is a commutative triangle of topological spaces:

$$\begin{array}{ccc} & \mathbf{B}(\mathbf{T}) & \\ p \swarrow & & \searrow \pi \\ \overline{\mathbf{B}}(\mathbf{T}) & \xrightarrow{\bar{\pi}} & \text{UConf}_n \end{array} .$$

The difference from the case $p = 1$ is that the arrows onto the unordered configuration space are *not* coverings, but rather (locally trivial) fibre bundles with positive-dimensional fibres. For π this is clear (it is the composition of a trivial bundle and a locally trivial one), while for $\bar{\pi}$ it can be proven as follows. If $O \subseteq \text{UConf}_n$ is an open trivialising set for π then $\pi^{-1}(O) = \tilde{O} \times \mathbf{B}(t)^n$, where $\tilde{O} \simeq S_n \times U$ is the preimage of O under the standard Galois covering, and

$$\bar{\pi}^{-1}(O) = p(\pi^{-1}(O)) = (\tilde{O} \times \mathbf{B}(t)^n) / \text{Aut}(\mathbf{T}, \mathbf{r}).$$

Now the latter quotient can be taken in two steps: first the action of the base yields

$$(\tilde{O} \times \mathbf{B}(t)^n) / (1 \wr \text{Aut}(t, \mathbf{r}_0)) \simeq \tilde{O} \times \overline{\mathbf{B}}(t)^n,$$

and then the space $O \times \overline{\mathbf{B}}(t)^n \subseteq O \times S_n \times \overline{\mathbf{B}}(t)^n \simeq \tilde{O} \times \overline{\mathbf{B}}(t)^n$ is a slice for the action of the “complement” subgroup $S_n \wr 1$. In conclusion $\bar{\pi}^{-1}(O) \simeq O \times \overline{\mathbf{B}}(t)^n$, proving we have a locally trivial fibre bundle

$$\overline{\mathbf{B}}(t)^n \hookrightarrow \overline{\mathbf{B}}(\mathbf{T}) \xrightarrow{\bar{\pi}} \text{UConf}_n .$$

This yields the exact group sequence

$$1 \longrightarrow \text{CBr}(t, \mathbf{r}_0)^n \longrightarrow \pi_1(\overline{\mathbf{B}}(T)) \xrightarrow{\pi_1(\overline{\pi})} \text{Br}_n \longrightarrow 1,$$

by the recursive hypothesis, using that UConf_n is a $\pi(K, 1)$ -space and that fibres are connected. Moreover any continuous function $\text{UConf}_n \rightarrow \overline{\mathbf{B}}(t)^n$ yields a global section, so in conclusion there is a semidirect product decomposition

$$\pi_1(\overline{\mathbf{B}}(T)) \simeq \text{Br}_n \wr \text{CBr}(t, \mathbf{r}_0) =: \text{Br}_n \ltimes \text{CBr}(t, \mathbf{r}_0)^n,$$

in accordance with the recursive definition (30).

Finally consider the general case where $\tilde{\mathcal{T}}$ is not a singleton. Then we can generalise the above argument using the composition of projections

$$\mathbf{B}(T) \longrightarrow \overline{\mathbf{B}}(T) \longrightarrow X_\varphi = \text{Conf}_n / S_\varphi,$$

onto the semioordered configuration space, where $\varphi: J_p \rightarrow \tilde{\mathcal{T}}$ is as above. Again a two-step quotient (over any open trivialising subspace $O \subseteq X_\varphi$) can be taken with respect to the actions of the subgroups

$$\prod_{\tilde{\mathcal{T}}} (1 \wr \text{Aut}(t, \mathbf{r}_0)^{n(t)}), \quad \prod_{\tilde{\mathcal{T}}} (S_{n(t)} \wr 1) \subseteq \text{Aut}(T, \mathbf{r}),$$

whose (inner) product gives the whole of $\text{Aut}(T, \mathbf{r})$ in view of Lem. 4.4 The analogous commutative triangle of topological spaces then yields the fibre bundle

$$\prod_{\tilde{\mathcal{T}}} \mathbf{B}(t)^{n(t)} \hookrightarrow \overline{\mathbf{B}}(T) \xrightarrow{\overline{\pi}} X_\varphi,$$

whence the exact group sequence

$$1 \longrightarrow \prod_{\tilde{\mathcal{T}}} \text{CBr}(t, \mathbf{r}_0)^{n(t)} \longrightarrow \pi_1(\overline{\mathbf{B}}(T)) \xrightarrow{\pi_1(\overline{\pi})} \text{Br}_\varphi \longrightarrow 1,$$

using the recursive hypothesis, Prop. 4.2, the fact that fibres are connected, and that X_φ is a covering of a $K(\pi, 1)$ -space—so it is also a $K(\pi, 1)$. Again this has global sections, since there are continuous maps $X_\varphi \rightarrow \prod_{\tilde{\mathcal{T}}} \overline{\mathbf{B}}(t)^{n(t)}$, proving the statement. \square

Remark 4.7. Note the fibre bundles $\overline{\mathbf{B}}(T) \rightarrow \text{UConf}_n$, in the first part of the above proof, is *not* trivial in general. Indeed the total space is homeomorphic to $(\text{Conf}_n \times \overline{\mathbf{B}}(t)^n) / S_n \wr 1$, but this is not simply $\text{UConf}_n \times \overline{\mathbf{B}}(t)$ because there is no global slice for the quotient $\text{Conf}_n \rightarrow \text{UConf}_n$ —but rather only “local” ones, over a trivialising cover. \triangle

Remark 4.8. The auxiliary fibre bundle

$$S_n \times \mathbf{B}(t)^n \hookrightarrow \mathbf{B}(T) \xrightarrow{\pi} \text{UConf}_n,$$

which appears in the above proof when $\tilde{\mathcal{T}} = \{t\}$, yields the following exact group sequence:

$$1 \longrightarrow \text{PCBr}(t)^n \longrightarrow \text{PCBr}(T) \xrightarrow{\pi_1(\pi)} \text{Br}_n \longrightarrow S_n \longrightarrow 1.$$

This is recovering the fact that $\text{PCBr}(T) / \text{PCBr}(t)^n \simeq \text{PBr}_n$, considering the direct product over the nodes of the (unranked) fission tree. \triangle

OUTLOOK

There is a “twisted” version of irregular types/classes [14, 15, 22]; and there exists “global” deformations of (twisted, bare) wild Riemann surfaces, defined as in [23]: we plan to consider these elsewhere.

Using the standard generators/relations of braid groups should in principle lead to a presentation of their cabled versions, which in turn might be used to compute their monodromy action on wild character varieties.

Finally recall [23] constructed a smooth *affine* moduli scheme $\mathcal{I}\mathcal{T}_d^{\leq p}$ of irregular types of bounded pole order $p \in \mathbb{Z}_{\geq 1}$, and with given pole order $d_\alpha \in \{0, \dots, p\}$ after evaluation at each root $\alpha \in \Phi_g$ —for any complex reductive group G . In particular the \mathbb{C} -points $\mathcal{I}\mathcal{T}_d^{\leq p}(\text{Spec } \mathbb{C})$ recover the above “universal” deformation space $\mathbf{B} = \mathbf{B}(Q)$, for any irregular type $Q \in \mathfrak{t} \otimes \mathcal{T}_{\Sigma, a}^{\leq p}$ with $\text{ord}(q_\alpha) = d_\alpha$. Then one may take a quotient in the (complex) algebraic category, rather than the analytic/holomorphic one (e.g. the affine GIT quotient), to try to construct a moduli space of irregular classes at the point $a \in \Sigma$: we also plan to study this further, both in the twisted and nonlocal setting.

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APPENDIX A. BASIC NOTIONS/NOTATIONS

Permutations and partitions. For an integer $n \geq 0$ we denote $\underline{n} = \{1, \dots, n\}$ (so $0 = \emptyset$), and S_n the symmetric group of permutations of \underline{n} (so S_0 and S_1 are trivial). The action of a permutation $\sigma \in S_n$ is denoted $j \mapsto \sigma_j \in \underline{n}$, and we compose them right-to-left. More generally if I is a finite set we denote S_I its symmetric group of permutations, so $S_I \simeq S_{|I|}$ choosing a total order on I —coherently with $S_{\underline{n}} = S_n$.

A *J-partition* of a finite set I is a surjection $\phi: I \rightarrow J$ onto a (finite) set J , which is equivalent to giving a decomposition

$$I = \coprod_{j \in J} I_j, \quad I_j := \phi^{-1}(j) \subseteq I,$$

of I , with nonempty parts indexed by J .

If all parts have the same cardinality $m = |I_j| \geq 1$ then we can consider the subgroup permuting all parts, and further the elements within each part: this is the (restricted) *wreath* product

$$S_J \wr S_m = S_J \wr_J S_m,$$

using the natural S_J -action on J . In turn the wreath product is the same as

$$S_J \wr S_m = S_J \times (S_m)^{|J|},$$

with respect to the action of S_J given by

$$\sigma \cdot \tau = \prod_J \tau^{(\sigma_j^{-1})}, \quad \sigma \in S_J, \quad \tau = \prod_J \tau^{(j)} \in (S_m)^{|J|},$$

where $\tau^{(j)} \in S_{I_j} \simeq S_m$ for $j \in J$. Elements of $S_J \wr S_m$ are then written $(\sigma; \tau)$.

Weyl actions. Let (V, Φ) be a root system in the finite-dimensional complex vector space V , and $W = W(\Phi) \subseteq GL(V)$ the Weyl group. If $S \subseteq V$ is a subset, its *setwise* Weyl-stabiliser is the subgroup

$$\text{Stab}_W(S) = \{ w \in W \mid w(S) \subseteq S \} \subseteq W,$$

and its *pointwise* Weyl-stabiliser is the subgroup

$$W_S = \{ w \in W \mid S \subseteq \text{Ker}(w - \text{Id}_V) \} \subseteq \text{Stab}_W(S).$$

The latter is a *parabolic* subgroup of W —thinking of the Weyl group as a reflection group—and is a normal subgroup of the former. Clearly $W_S = W_{\mathbb{C}S}$, and if $\mathbb{C}S_1 \subseteq \mathbb{C}S_2$ then $W_{S_1} \subseteq W_{S_2}$; further if $U_1, U_2 \subseteq V$ are subspaces then

$$W_{U_1+U_2} = W_{U_1} \cap W_{U_2} \subseteq W.$$

On the other hand, tautologically, if $W_i = \text{Stab}_W(U_i)$ then

$$\text{Stab}_{W_1}(U_2) = W_1 \cap \text{Stab}_W(U_2) = W_2 \cap \text{Stab}_W(U_1) = \text{Stab}_{W_2}(U_1).$$

We identify W with the Weyl group $W(\Phi^\vee) \subseteq GL(V^\vee)$ for the dual/inverse root system, via $w \mapsto {}^t w^{-1}$ [16, Ch. VI, § 1.1].

Braid groups. For an integer $n \geq 0$ we denote PBr_n the *pure braid group* on n strands—so PBr_0 and PBr_1 are trivial. It is the fundamental group of the space

$$\text{Conf}_n = \text{Conf}_n(\mathbb{C}) := \mathbb{C}^n \setminus \bigcup_{1 \leq i \neq j \leq n} H_{ij}, \quad (31)$$

where

$$H_{ij} = \{ (z_1, \dots, z_n) \in \mathbb{C}^n \mid z_i = z_j \} \subseteq \mathbb{C}^n.$$

In particular $\text{Conf}_1 = \mathbb{C}$, and in general this yields the space of *ordered* configurations of n points in the complex plane. The symmetric group S_n acts naturally on (31), and the projection

$$\text{Conf}_n \longrightarrow \text{UConf}_n := \text{Conf}_n / S_n,$$

to the space of *unordered* configurations, is a Galois cover. The (full/nonpure) *braid group* is $\text{Br}_n = \pi_1(\text{UConf}_n)$, and the associated exact group sequence

$$1 \longrightarrow \text{PBr}_n \longrightarrow \text{Br}_n \xrightarrow{p_n} S_n \longrightarrow 1 \quad (32)$$

corresponds to the braid group “augmentation”, i.e. the morphism taking the permutation underlying the braiding of the n strands.

More generally for a split Lie algebra $(\mathfrak{g}, \mathfrak{t})$ we consider the root-hyperplane complement

$$\mathfrak{t}_{\text{reg}} = \mathfrak{t} \setminus \bigcup_{\alpha \in \Phi_{\mathfrak{g}}} \text{Ker}(\alpha) \subseteq \mathfrak{t}, \quad \Phi_{\mathfrak{g}} = \Phi(\mathfrak{g}, \mathfrak{t}),$$

generalising (31) in type A , and $\text{PBr}_{\mathfrak{g}} = \pi_1(\mathfrak{t}_{\text{reg}})$ is the *pure \mathfrak{g} -braid group*, a.k.a. the generalised (Artin–Tits) braid group of type \mathfrak{g} [17, 19, 21, 18]. The Weyl group $W_{\mathfrak{g}} = W(\Phi_{\mathfrak{g}})$ acts freely on $\mathfrak{t}_{\text{reg}}$, and $\mathfrak{t}_{\text{reg}} \rightarrow \bar{\mathfrak{t}}_{\text{reg}} := \mathfrak{t}_{\text{reg}} / W_{\mathfrak{g}}$ is a Galois covering. Then $\text{Br}_{\mathfrak{g}} := \pi_1(\bar{\mathfrak{t}}_{\text{reg}})$ is the *full/nonpure \mathfrak{g} -braid group*, and there is an exact group sequence generalising (32):

$$1 \longrightarrow \text{PBr}_{\mathfrak{g}} \longrightarrow \text{Br}_{\mathfrak{g}} \longrightarrow W_{\mathfrak{g}} \longrightarrow 1.$$

Trees. A (finite) *tree* $T = (T_0, \Phi)$ of height $p \geq 1$ is the data of a finite set T_0 with a partition $T_0 = \coprod_{l=1}^{p+1} J_l$, such $J_{p+1} = \{*\}$ is a singleton, and a function $\Phi: T_0 \setminus \{*\} \rightarrow T_0$ such that $\Phi(J_l) \subseteq J_{l+1}$ for $l \in \{1, \dots, p\}$. The elements of T_0 and the *nodes* of the tree, and $\Phi(i)$ is the *parent-node* of $i \in T_0 \setminus \{*\}$ —so $*$ $\in J_{p+1}$ is the *root*, while $J_1 \subseteq T_0$ contains the *leaves*. Conversely $\Phi^{-1}(i) \subseteq T_0$ is the set of *child-nodes* of $i \in T_0$.

APPENDIX B. MISSING PROOFS

Proof of Lem. 2.4. We can recursively prove that

$$W_i = \bigcap_{i \leq j \leq p} \text{Stab}_{W_g}(\mathcal{U}_j), \quad i \in \{1, \dots, p\}.$$

The base $i = p$ is tautological, and then

$W_{i-1} = \{ w \in W_i \mid w(\mathcal{U}_{i-1}) \subseteq \mathcal{U}_{i-1} \} = \{ w \in W_g \mid w(\mathcal{U}_j) \subseteq \mathcal{U}_j \text{ for } j \geq i-1 \}$,
using (13) and the recursive hypothesis. \square

Proof of Lem. 4.2. By definition $S_m \wr P = S_m \ltimes P^m$, with respect to the natural permutation action $S_m \rightarrow \text{Aut}(P^m)$. Then $1 \wr N = 1 \ltimes N^m \subseteq S_m \ltimes P^m$, and it is normal since it is normalised by $1 \wr P = 1 \ltimes P^m$ and stabilised by the permutation action. Hence the quotient on the left-hand of (24) is well defined.

Now there is an induced action $S_m \rightarrow \text{Aut}(Q^m)$, where $Q := P/N$, and finally the natural surjective group morphism $S_m \wr P \rightarrow S_m \wr Q$ vanishes on $1 \wr N$. \square

Proof of Lem. 4.3. The compatibility with the product follows from (18) (which in turn is equivalent to the action-operad axiom for \mathcal{S} [54, Eq. 4.1.2]), and from the fact that the block permutation operation is a group morphism in this case.

Injectivity follows from the identity

$$S_n \langle \mathbf{k} \rangle \cap (S_n)^n = 1 \subseteq S_{n\mathbf{k}},$$

where $S_n \langle \mathbf{k} \rangle \subseteq S_{n\mathbf{k}}$ is the image of the block permutation operation $S_n \rightarrow S_{n\mathbf{k}}$. \square

Proof of Lem. 4.4. There is a natural bijection

$$\prod_I (p_i, n_i) \longmapsto \left(\prod_I p_i, \prod_I n_i \right),$$

between the underlying sets, and one can show it is compatible with the semidirect multiplication.

Indeed choose elements $p'_i, p_i \in P_i$ and $n'_i, n_i \in N_i$ for $i \in I$, so that

$$\begin{aligned} \prod_I (p'_i, n'_i) \prod_I (p_i, n_i) &= \prod_I (p'_i, n'_i) \bullet_i (p_i, n_i) \\ &= \prod_I (p'_i p_i, \rho_i(p_i) n'_i n_i) \in \prod_I P_i \times N_i, \end{aligned}$$

which is mapped to $\left(\prod_I p'_i p_i, \prod_I \rho_i(p_i) n'_i n_i \right) \in P \times N$. Conversely

$$\left(\prod_I p'_i, \prod_I n'_i \right) \bullet \left(\prod_I p_i, n_i \right) = \left(\prod_I p'_i \prod_I p_i, \rho\left(\prod_I p_i\right) \prod_I n'_i \prod_I n_i \right) \in P \times N,$$

which coincides with the above—using the product action and the direct product multiplication. \square

APPENDIX C. RELATIONS TO ISOMONODROMY SYSTEMS

On the other side of the Riemann–Hilbert–Birkhoff correspondence there is a Poisson fibre bundle analogous to (1), viz.

$$\mathcal{M}_{\text{dR}} \xrightarrow{\pi} \overline{\mathbf{B}}, \quad (33)$$

whose fibres (the de Rham spaces) are moduli spaces of irregular singular algebraic connections on principal G -bundles. This is equipped with the pullback (flat nonlinear algebraic) isomonodromy connection; see in particular [9, Fig. 1], which spells out this picture.

Now one can choose a local trivialisation of (33), i.e. an isomorphism of fibre bundles

$$\begin{array}{ccc} \underline{\mathcal{M}}_{\text{dR}}|_{\mathcal{O}} := \pi^{-1}(\mathcal{O}) & \xrightarrow{\simeq} & \mathcal{O} \times M \\ & \searrow \pi & \swarrow p_1 \\ & \mathcal{O} & \end{array}, \quad (34)$$

over an open subspace $\mathcal{O} \subseteq \overline{\mathcal{B}}$, for a *fixed* Poisson manifold $(M, \{\cdot, \cdot\})$. Then the isomonodromy connection (on the upper-left corner of (34)) can be given by explicit nonlinear first-order partial differential equations in local coordinates $\mathbf{t} = (t_1, \dots, t_d)$ on \mathcal{O} , where $d = \dim(\overline{\mathcal{B}})$, for local sections over the trivialising locus.

Moreover the difference between the isomonodromy connection and the trivial connection (on the upper-right corner of (34)) can be “integrated”⁹ to a non-autonomous Hamiltonian system

$$\mathbf{H} = (H_1, \dots, H_d): M \times \mathcal{O} \longrightarrow \mathbb{C}^d. \quad (35)$$

In this Hamiltonian viewpoint the symplectic nature of isomonodromic deformations, as considered in [9] (and extended in successive degree of generality in subsequent work), is equivalent to the integrability of (35); this amounts to the identities

$$\{H_i, H_j\} + \frac{\partial H_i}{\partial t_j} - \frac{\partial H_j}{\partial t_i} = 0, \quad i, j \in \{1, \dots, d\}.$$

Hence the local coordinates become times of isomonodromic deformations over $\mathcal{O} \subseteq \overline{\mathcal{B}}$: but in principle they are *not* intrinsically associated to isomonodromic deformations, contrary to the flat Ehresmann connections on (1) and (33) (one needs a choice of “initial” trivialisation [9, Rk. 7.1]).

Examples of such isomonodromy systems abound, with far-reaching applications already in the genus-zero case, famously encompassing (generalisations of) the Painlevé equations [42, 43, 41, 7, 20] and the Schlesinger system [47]. The (Harnad-)dual version of the Schlesinger system, on the other side of the Fourier–Laplace transform [30, 51], was considered in [10], and the combination of Schlesinger and its dual yield the system of Jimbo–Miwa–Môri–Sato (JMMS) [33]. Note [30] also links previous papers about *isospectral* deformations [2, 1] to the *isomonodromic* deformations of JMMS.

Finally a generalisation of all the above was derived in [12]. This latter setup brings about nongeneric isomonodromic deformations, considering connections with several levels¹⁰ which extend examples of the seminal paper [34]. Importantly this is more symmetric than op. cit., which in turn is one of our main motivations for studying the “deeper” nongeneric case: in particular in [12] the group $\text{SL}_2(\mathbb{C})$ acts on the bundle of de Rham spaces via automorphisms of the 1-dimensional Weyl algebra, and contains the Fourier–Laplace transform as the element $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

See [52, 28] for more recent developments, with the latter also on the quantum side.

Importantly all these Hamiltonian systems have far-reaching applications in mathematical physics, notably in integrable hierarchies of differential equations

⁹The difference of the corresponding horizontal distributions—inside $\text{TO} \times \text{TM} \xrightarrow{\text{Tp}_1} \text{TO}$ —is given by \mathcal{O} -dependent (vertical) vector fields $X_i: \mathcal{O} \times M \rightarrow \text{TM}$ on the fibre, and one has $dH_i = \langle \omega, X_i \rangle$ (cf. [12, § 5]).

¹⁰The nonzero pole orders of the irregular types evaluated at each root.

such as KdV [29], and in 2d conformal field theory after quantisation (e.g. [39, 50, 3, 27, 26], opening to “irregular” conformal blocks and the AGT correspondence). More precisely the quantisation of the Schlesinger system leads to the Knizhnik–Zamolodchikov connection (KZ) [46, 31], while the quantisation of the dual Schlesinger system (as in [10]) leads to the Casimir connection of de Concini/Mills–Toledano Laredo [38]. The quantisation of the JMMS systems, generalising the above and recovering the connection of Felder–Markov–Tarasov–Varchenko [25], was considered in [44]; further op. cit. constructed a quantisation of the more general “simply-laced” systems of [12] (this is resumed in the table in the introduction of [44]); cf. [53] for a different construction of “quantum” simply-laced isomonodromy systems, and [39, 40, 28] for a “confluence” viewpoint on the quantisation of irregular singularities.

Finally encoding the irregular moduli in the base curve, and constructing bundles over their (admissible) deformations, is also helpful for the quantisation of the extended “classical” symmetries of isomonodromy systems. In particular the quantised $SL_2(\mathbb{C})$ -symmetries [45] generalise the Howe duality [6], which in turn were used in [49] to compute the monodromy of the Casimir connection in terms of that of KZ (cf. also [48]): this latter example of generic “quantum” monodromy action brings about the G-braid groups which we generalise in this series of papers.

APPENDIX D. LIST OF SOME NONSTANDARD NOTATION (IN ROUGH ORDER OF APPEARANCE)

Σ	Riemann surface
G	connected complex reductive Lie group
$\mathcal{M}_{\mathbb{B}}$	Poisson/symplectic fibration of Betti spaces
$\overline{\mathbb{B}}$	space of admissible deformations of bare wild Riemann surfaces
\mathfrak{g}	Lie algebra of G
\mathfrak{t}	Cartan subalgebra of \mathfrak{g}
T	maximal (algebraic) torus in G
Q	irregular type
a	point of Σ
A_j	coefficients of Q
$W_{\mathfrak{g}}$	Weyl group of $(\mathfrak{g}, \mathfrak{t})$
\overline{Q}	irregular class underlying Q
Σ	wild Riemann surface
$W_{\mathfrak{g} \mathfrak{h}}$	subquotient of $W_{\mathfrak{g}}$ acting freely on \mathbb{B}
(T, \mathfrak{r})	ranked fission tree
$\text{Aut}(T, \mathfrak{r})$	automorphisms of (T, \mathfrak{r})
$\text{CBr}(T, \mathfrak{r})$	full/nonpure cabled braid group of (T, \mathfrak{r})
$\Phi_{\mathfrak{g}}$	root system of $(\mathfrak{g}, \mathfrak{t})$
$\widehat{\mathcal{O}}_{\Sigma, a}$	completed local ring of Σ at a
$\widehat{\mathcal{H}}_{\Sigma, a}$	completed fraction field of $\widehat{\mathcal{O}}_{\Sigma, a}$
$\mathcal{T}_{\Sigma, a}$	quotient of $\widehat{\mathcal{H}}_{\Sigma, a}$ modulo $\widehat{\mathcal{O}}_{\Sigma, a}$
$\mathbb{B} = (Q)$	universal space of admissible deformations of Q
$\mathbb{B}_i = \mathbb{B}(A_i)$	factors of $\mathbb{B} =$ universal spaces of admissible deformations of the A_i
d_{α}	pole order of $q_{\alpha} = (\alpha \otimes 1)Q$
\mathbf{d}	tuple of the d_{α}
$\overline{\Gamma}(\Phi_{\mathfrak{g}}, \mathbf{d})$	full/nonpure local WMCG
$\mathfrak{t}_{\text{reg}}$	regular part of \mathfrak{t}
\mathfrak{h}_i	nested centralisers of the coefficients of Q
H_i	subgroup of G with Lie algebra \mathfrak{h}_i

$\Phi_{\mathfrak{h}_i}$	sequence of fission root subsystems of $\Phi_{\mathfrak{g}}$
$W_{\mathfrak{h}_i}$	Weyl group of $(\mathfrak{h}_i, \mathfrak{t})$
\mathcal{U}_i	intersection of the root hyperplanes of $\Phi_{\mathfrak{h}_i}$
W_i	nested setwise stabilisers of the \mathcal{U}_i
$\text{Stab}_{W_i}(\mathbf{B}_i)$	setwise stabiliser of \mathbf{B}_i in W_i
\mathbf{U}	flag of the subspaces \mathcal{U}_i in \mathfrak{t}
$(W_{\mathfrak{g}})_{\mathcal{U}_i}$	pointwise stabiliser of \mathcal{U}_i in $W_{\mathfrak{g}}$
$(W_{\mathfrak{g}})_Q$	stabiliser of Q in $W_{\mathfrak{g}}$
\mathcal{J}_i	orthogonal ideals of \mathfrak{g}
\mathfrak{t}_i	intersection of \mathfrak{t} with \mathcal{J}_i
$\Phi_{\mathcal{J}_i}$	root system of $(\mathcal{J}_i, \mathfrak{t}_i)$
$W_{\mathcal{J}_i}$	Weyl group of $\Phi_{\mathcal{J}_i}$
$\mathfrak{Z}_{\mathfrak{g}}$	centre of \mathfrak{g}
PBr_n	pure braid group on n strands
Br_n	full/nonpure braid group on n strands
Dih_n	dihedral group of order $2n$
$\Delta_{\mathfrak{g}}$	choice of simple roots for $\Phi_{\mathfrak{g}}$
Conf_n	configuration space of n ordered points in \mathbb{C}
UConf_n	configuration space of n unordered points in \mathbb{C}
\underline{n}	the set $\{1, \dots, n\}$
S_I	group of permutations of a set I
I_i	parts of \underline{n} defined by a root subsystem of A_{n-1}
J	index set for the parts I_i
K_i	collection of parts I_j with $i \geq 0$ elements
$\widetilde{\text{Ker}}(\Delta_{\mathfrak{h}})$	extended kernel of $\Delta_{\mathfrak{h}}$
S_{φ}	group of permutations preserving a partition φ
p_n	augmentation group morphism of Br_n
Br_{φ}	group of braids with underlying permutations in S_{φ}
T	fission tree
J_i	levels of T
T_0	nodes of T
Φ	parent-node function of T
\mathbf{r}	rank function of T
\mathcal{T}	set of maximal proper subtrees of T
$\widetilde{\mathcal{T}}$	choice of representatives for the isomorphism classes inside \mathcal{T}
$n(\mathfrak{t})$	cardinality of the isomorphism class containing $\mathfrak{t} \in \widetilde{\mathcal{T}}$
$\widetilde{\text{Aut}}(T, \mathbf{r})$	extended automorphism group of (T, \mathbf{r})
\mathfrak{t}_0	nodes of \mathfrak{t}
\mathbf{r}_0	restriction of \mathbf{r} to \mathfrak{t}_0
\mathcal{S}	symmetric braid group operad
$\gamma^{\mathcal{S}}$	composition of \mathcal{S}
\mathcal{B}	full/nonpure braid group operad
$\gamma^{\mathcal{B}}$	composition of \mathcal{B}
$\text{PCBr}(T)$	pure cabled braid group of T
$\mathcal{I}\mathcal{T}_d^{\leq p}$	moduli scheme of irregular types of bounded pole order, and given pole order along any root
\mathcal{M}_{dR}	Poisson/symplectic fibration of de Rham spaces
\mathfrak{t}_i	local coordinates on $\overline{\mathbf{B}}$: isomonodromy times
\mathfrak{t}	tuple of the \mathfrak{t}_i
H_i	isomonodromy Hamiltonian
\mathbf{H}	tuple of the H_i : isomonodromy system

CONFLICT OF INTEREST

The authors declare that they have no conflict of interest.

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