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# Sp(1)-SYMMETRIC HYPER-KÄHLER QUANTISATION

JØRGEN ELLEGAARD ANDERSEN, ALESSANDRO MALUSÀ, GABRIELE REMBADO

ABSTRACT. We provide a new general scheme for the geometric quantisation of Sp(1)-symmetric hyper-Kähler manifolds, considering Hilbert spaces of holomorphic sections with respect to the complex structures in the hyper-Kähler 2-sphere. Under properness of an associated moment map, or other finiteness assumptions, we construct unitary quantum (super) representations of central extensions of certain subgroups of Riemannian isometries preserving the 2-sphere, and we study their decomposition in irreducible components. We apply this quantisation scheme to hyper-Kähler vector spaces, the Taub–NUT metric on  $\mathbb{R}^4$ , moduli spaces of framed SU( $r$ )-instantons on  $\mathbb{R}^4$ , and partly to the Atiyah–Hitchin manifold of magnetic monopoles in  $\mathbb{R}^3$ .

## CONTENTS

1. Introduction	1
Acknowledgements	5
2. Abstract Sp(1)-symmetric hyper-Kähler quantisation	6
3. Examples of applications	15
4. Outlook and further perspectives	25
Appendix A. Comparison with the standard approach	26
References	26

## 1. INTRODUCTION

The constructions of geometric quantisation offer a recipe for addressing problems related to the quantum mechanics of an object moving in an arbitrary, possibly curved, phase space [48, 93]. The process, essentially mimicking canonical quantisation, is fundamentally based on the structure of a symplectic manifold. Two of the main goals are to obtain operators subject to commutation relations prescribed by the Poisson bracket, and unitary representations of groups associated to Hamiltonian flows, although there are strong limitations to the extent to which these can be achieved in general. One of the most typical problematic points of the construction is the need for a polarisation, whose existence is generally not guaranteed, nor is its uniqueness ever satisfied. Further, the choice of a particular polarisation poses serious constraints on which functions and Hamiltonian flows can be quantised.

One way to obtain a polarisation is to assume that there exist a complex structure compatible with the symplectic form, thus it is rather common in geometric quantisation to work with Kähler manifolds [30, 93]. For instance, this

approach to quantisation has been successfully applied to Chern–Simons theory [1, 3, 17, 51], also in combination with its relation with deformation quantisation [2, 26, 59, 82, 83, 6] and other approaches [9, 10, 11, 12, 64, 5].

In this paper instead we address the problem of applying geometric quantisation to a *hyper*-Kähler manifold, so in particular we have a family of Kähler forms parametrised by the space of unit-norm imaginary quaternions: This is beyond the usual setup of Kähler quantisation, where a preferred symplectic structure is fixed and the polarisation varies. The problem is somewhat dual to that of choosing a polarisation and study the dependence of geometric quantisation on that choice [2, 8]. Indeed in our situation the ambiguity resides in the symplectic form itself: Having picked one, a Kähler polarisation is automatically assigned.

The first motivation for considering this situation originates from certain moduli spaces arising in differential and algebraic geometry, often related to interesting quantisation problems: to mention a few, flat connections [1, 39, 51, 90, 91] and vector bundles [4, 50] over smooth Riemann surfaces, (hyper-)polygons [58, 29], and quiver varieties [42, 76]. Some of these objects, such as polygons and flat  $SU(n)$ -connections, give rise to real symplectic spaces whose geometric quantisation has been broadly studied [1, 51, 17].

Other moduli spaces however naturally arise as *holomorphic* symplectic spaces (e.g. flat  $SL_n(\mathbb{C})$ -connections on a smooth surface [46]), which sometimes are actually hyper-Kähler manifolds read in a particular complex structure (e.g. Higgs bundles on a complex projective curve [50]), and the aforementioned difficulties appear.

Furthermore, many of these hyper-Kähler spaces carry  $U(1)$ -actions which are holomorphic and Hamiltonian for some Kähler structures, but not for all. One important example is the  $U(1)$ -action on the Hitchin moduli space, which is Hamiltonian for the structure traditionally denoted by  $\omega_I$ , something which has been recently explored in the context of quantisation in [4]; on the other hand, Witten’s description of quantum  $SL_n(\mathbb{C})$ -Chern–Simons theory at level  $t = k + is$  [91, 3] is based on the symplectic structure corresponding via nonabelian Hodge theory to  $\omega_t = k\omega_I - s\omega_K$  [50, 35, 84, 31], for which the action is not even symplectic. Interest for this particular setting was the main motivation of other works by the authors [7, 69, 79], but importantly further extensions are possible since new hyper-Kähler metrics have been attached to (wild generalisations of) Riemann surfaces, leading to a "wild" nonabelian Hodge correspondence [81, 20] (see also Witten’s viewpoint [92] and § 4 below).

We will not address the case of hyper-Kähler manifolds with  $U(1)$ -actions in the present paper: Instead, we will consider a different kind of symmetry. Namely, we shall focus on the problem of constructing unitary representations of a suitable group of symmetries of a hyper-Kähler manifold  $(M, g, I, J, K)$  endowed with an isometric  $Sp(1)$ -action which transitively permutes the Kähler forms.

We will thus consider more general transformations than the Riemannian isometries which are symplectic/holomorphic for all the Kähler structures, since this is too restrictive to include the aforementioned  $U(1)$ -actions: We will instead relax this condition and consider isometries preserving the vector space generated by the Kähler forms—rather than each form individually. Let us denote

$\text{Hk}(M) = \text{Hk}(M, g, I, J, K)$  the group of such transformations, which by definition acts on the hyper-Kähler 2-sphere of complex structures (a copy of  $\mathbb{C}P^1$ ) by positive isometries.

**1.1. Description of the main construction.** Since no preferred symplectic form is given on  $M$  it makes little sense to talk about a pre-quantum line bundle over the hyper-Kähler manifold. Instead one may fix a Hermitian line bundle  $(L, h)$  on  $M$  and a family of Hermitian connections  $\nabla_q$  on  $L \rightarrow M$ , making it a pre-quantum line bundle for the symplectic form  $\omega_q \in \Omega^2(M)$  corresponding to  $q \in \mathbb{C}P^1$ . Further, let  $\text{Hk}'(M)$  be the group of automorphisms of  $L \rightarrow M$  covering the  $\text{Hk}(M)$ -action: We assume it is a central  $U(1)$ -extension of  $\text{Hk}(M)$  (see § 2.2 for a cohomological reformulation of this assumption).

If  $M_q$  denotes the Kähler manifold corresponding to  $q \in \mathbb{C}P^1$ , its geometric quantisation consists of the Hilbert space  $\mathcal{H}_q$  of  $L^2$  holomorphic sections of the pre-quantum line bundle. One may then attempt, as customary, to identify the spaces  $\mathcal{H}_q$  by constructing a 1-cocycle of unitary isomorphisms parametrised by pairs of points in  $\mathbb{C}P^1$ . We have however chosen to proceed differently: We will exploit the compactness of the base space  $\mathbb{C}P^1$  and make sense of *holomorphic* dependence of our construction, obtaining non-trivial representations of a subgroup of  $\text{Hk}'(M)$ .<sup>1</sup>

The first step is, for each  $q \in \mathbb{C}P^1$ , to consider the action of the  $\text{Sp}(1)$ -stabiliser  $T_q \subseteq \text{Sp}(1)$ —a maximal torus. This gives an orthogonal  $L^2$ -completed decomposition of  $\mathcal{H}_q$  into subspaces  $\mathcal{H}_q^{(d)} \subseteq \mathcal{H}_q$  on which  $T_q$  acts by the  $d$ 'th power of the standard representation (see Eq. (11)). We then consider a *compact* subgroup  $\text{Sp}_0'(M) \subseteq \text{Hk}'(M)$  which commutes with the  $\text{Sp}(1)$ -action, getting analogous splittings of  $\mathcal{H}_q^{(d)}$  into subspaces  $\mathcal{H}_{a,q}^{(d)}, \mathcal{H}_{\lambda,q}^{(d)}$  indexed by characters  $a \in T^\vee$  of a maximal torus  $T \subseteq \text{Sp}_0'(M)$  and highest weights  $\lambda$  of simple  $\text{Sp}_0'(M)$ -modules, respectively (see Eqq. (13) and (12)).

Our main assumption is that the subspaces  $\mathcal{H}_{\lambda,q}^{(d)} \subseteq \mathcal{H}_q$  are finite-dimensional, which is satisfied in a number of situations, and (at first) that they assemble into a finite-rank smooth vector bundle over  $\mathbb{C}P^1$ . In this case we establish that the bundles  $\mathcal{H}_\lambda^{(d)} \rightarrow \mathbb{C}P^1$  carry natural  $\text{Sp}(1)$ -invariant Hermitian connections, isomorphic to a finite direct sum of bundles of the form  $\mathcal{L}^d \otimes V_\lambda$ : Here  $\mathcal{L} \rightarrow \mathbb{C}P^1$  is the  $\text{Sp}(1)$ -equivariant line bundle with connection associated to the standard representation of  $T_q$  for any  $q \in \mathbb{C}P^1$ , and  $V_\lambda$  is the simple  $\text{Sp}_0'(M)$ -module of highest weight  $\lambda$  (see Thm. 2.1). In the general case, still assuming finite-dimensionality, we will thus directly work with the above tensor product (see Rem. 2.4).

Then we define  $H_{\lambda,j}^{(d)}$  to be the finite-dimensional super Hilbert space obtained by taking the holomorphic cohomology of  $\mathcal{L}^d \otimes V_\lambda \rightarrow \mathbb{C}P^1$ , where  $j$  indexes the multiplicity of  $V_\lambda$  as sub- $\text{Sp}_0'(M)$ -module of  $\mathcal{H}_{\lambda,q}^{(d)}$ . This is isomorphic

<sup>1</sup>We discuss in § A the traditional identifications of the Hilbert spaces via holonomy of a flat connection; this approach trivialises the representations of the  $\text{Sp}(1)$ -factor produced by the main construction.

to the irreducible super representation  $W^{(d)} \otimes V_\lambda$  of  $\text{Hk}_0'(M)$ , where  $W^{(d)} := H^0(\mathbb{C}P^1, \mathcal{L}^d) \oplus H^1(\mathbb{C}P^1, \mathcal{L}^d)$ .

Finally we consider the  $L^2$ -completed orthogonal direct sum of the superspaces  $H_{\lambda,j}^{(d)}$  over  $j, \lambda$  and  $d \in \mathbb{Z}$ , getting to a Hilbert space  $H$  equipped with a unitary representation of the subgroup  $\text{Hk}_0'(M) \subseteq \text{Hk}'(M)$  generated by  $\text{Sp}_0'(M)$  and  $\text{Sp}(1)$ : This is the end of the construction of § 2, which can be summarised in the following main result (when the suitable hypotheses apply, e.g. §§ 3 and 4).

**Theorem.** *Suppose  $(M, g, I, J, K)$  is a hyper-Kähler manifold carrying a faithful isometric  $\text{Sp}(1)$ -action which transitively permutes the hyper-Kähler 2-sphere, identified to  $\mathbb{C}P^1$ . Then:*

- *let  $\text{Sp}_0(M) \subseteq \text{Iso}(M, g)$  be a (connected) compact subgroup commuting with the  $\text{Sp}(1)$ -action and preserving  $(I, J, K)$ ;*
- *choose an  $\text{Sp}(1)$ -equivariant family of connections on a fixed Hermitian line bundle  $L \rightarrow M$  prequantising all Kähler forms, and consider the central extension  $\text{Sp}_0'(M) \rightarrow \text{Sp}_0(M)$  given by automorphisms of the line bundle covering the  $\text{Sp}_0(M)$ -action;*
- *decompose the Kähler-quantisation Hilbert spaces into isotypical components for the action of  $\text{Sp}(1)$ -stabilisers of points  $q \in \mathbb{C}P^1$ , and for the commuting  $\text{Sp}_0'(M)$ -action;*
- *finally assemble these components into a finite-rank vector bundle  $\mathcal{H}_\lambda^{(d)} \rightarrow \mathbb{C}P^1$ .*

*Then  $\mathcal{H}_\lambda^{(d)} \rightarrow \mathbb{C}P^1$  carries a canonical Hermitian connection (cf. Thm. 2.1), and its holomorphic cohomology yields all tensor products of unitary simple  $\text{Sp}_0'(M)$ - and super  $\text{Sp}(1)$ -modules (cf, Thm. 2.2).*

Hence this in particular involves decompositions of quantum Hilbert spaces by circle actions, as in [4], but contrary to op. cit. we do not work in a preferred Kähler structure (and quantise all of them).

Moreover we consider the generating series for the ranks of the families  $\mathcal{H}^{(d)}$ ,  $\mathcal{H}_a^{(d)}$  of Hilbert spaces, and for the multiplicities of the representations  $W^{(d)} \otimes V_\lambda$  in the Hilbert space  $H$ . We establish algebraic relations between these generating series, as well as localisation formulæ under certain assumptions: See Thm. 2.6).

We further discuss the case where  $M$  carries a hyper-Kähler potential which is also a moment map for the  $T_q$ -actions (see § 2.9). If the hyper-Kähler potential is proper then we get finite-dimensional isotypical components as requested, but we also discuss other ways to ensure this: one version of the general principle that “quantisation and reduction commute”, involving proper moment maps, and Hartog’s theorem (see § 2.7).

**1.2. Applications and further directions.** In § 3 we showcase applications of the main construction: The first one is a hyper-Kähler vector space  $V$  of real dimension  $4n$ , with  $n \in \mathbb{Z}_{\geq 1}$ . In this case

$$\text{Hk}(V) \simeq \text{Sp}(n) \cdot \text{Sp}(1),$$

identifying  $V \simeq \mathbb{H}^n$  (cf. Rem. 3.1). Indeed under this isomorphism  $\text{Sp}(1)$  acts on  $V$  via right multiplication of unit-norm imaginary quaternions, and commutes with the natural  $\text{Sp}(n)$ -action. Further the norm associated to the hyper-Kähler

metric provides a hyper-Kähler potential and we can apply the abstract construction (see Thm. 3.1 and 3.2).

But importantly there are many more examples of (nonflat) Sp(1)-symmetric hyper-Kähler manifolds. Examples include moduli spaces of magnetic monopoles on  $\mathbb{R}^3$  by the work of Atiyah–Hitchin and Taubes [15] or equivalently, by the work of Donaldson [33], the moduli spaces of based rational maps from  $\mathbb{C}P^1$  to itself; moduli spaces of framed SU(r)-instantons on  $\mathbb{R}^4$ , by the work of Maciocia [68]; the hyper-Kähler structure on nilpotent orbits, by the work of Kronheimer [62], and more generally the hyper-Kähler Swann bundle over any quaternionic Kähler manifold [86]. In four dimensions a complete classification of Sp(1)-symmetric hyper-Kähler manifolds is given (up to finite covers) by the work of Gibbons–Pope [41] and by Atiyah–Hitchin [15]: The three examples are the flat metric on  $\mathbb{H}$ , the Taub–NUT metric, and the hyper-Kähler metric on the moduli space of charge-2 monopoles, i.e. the Atiyah–Hitchin manifold.

We establish in § 3.2 and 3.3 that the theorems 2.1, 2.2 and 2.6 (or slight modifications thereof) apply to some of these examples, producing a quantisation and corresponding irreducible unitary (super) representations of distinguished groups of hyper-Kähler isometries.

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2. ABSTRACT  $\mathrm{Sp}(1)$ -SYMMETRIC HYPER-KÄHLER QUANTISATION

**2.1. Hyper-Kähler manifolds and their symmetry groups.** Let  $n$  be a positive integer and  $M$  a smooth manifold of dimension  $4n$ .

**Definition 2.1.** A hyper-Kähler structure on  $M$  is a Riemannian metric  $g$  and an ordered triple  $(I, J, K)$  of covariant constant orthogonal automorphism of  $TM$  satisfying the quaternionic identities  $I^2 = J^2 = K^2 = IJK = -\mathrm{Id}_{TM}$ .

It follows that  $I, J, K$  are  $g$ -skew-symmetric global sections of  $\mathrm{End}(TM) \rightarrow M$ , and we denote  $\mathfrak{su}(2)_M$  the three-dimensional real Lie algebra they span.

The hyper-Kähler 2-sphere of complex structures of  $(M, g, I, J, K)$  is

$$\mathfrak{S}_{IJK} := \left\{ I_q = aI + bJ + cK \mid q = (a, b, c) \in \mathbb{R}^3, a^2 + b^2 + c^2 = 1 \right\} \subseteq \mathfrak{su}(2)_M. \quad (1)$$

As usual we identify (1) with the 2-sphere of unit-norm imaginary quaternions, i.e.  $\mathbb{C}P^1$  as a Kähler manifold. In particular for  $q \in \mathbb{C}P^1$  there is a (real) symplectic form on  $M$  defined by

$$\omega_q(v, w) := g(I_q v, w), \quad \text{for } v, w \in TM.$$

The triple  $M_q := (M, I_q, \omega_q)$  is a Kähler manifold, and for further use we let  $\mu_q = \mathrm{dvol} \in \Omega^{\mathrm{top}}(M)$  the Liouville volume form—independent of  $q \in \mathbb{C}P^1$  as it agrees with the Riemannian volume form of  $(M, g)$ .

*Remark 2.1.* The above data can be encoded in a fibration  $\pi_{\mathbb{C}P^1}: Z \rightarrow \mathbb{C}P^1$  of Kähler manifolds over the Riemann sphere, the *twistor space* of  $(M, g, I, J, K)$  [52]. This family comes with a global trivialisation  $Z \simeq M \times \mathbb{C}P^1$  as a smooth fibre bundle, but not as a symplectic/complex fibre bundle; nonetheless the natural complex structure on  $Z$  makes  $Z \rightarrow \mathbb{C}P^1$  into a *holomorphic* fibre bundle (cf. op. cit.), and further  $TZ \rightarrow \mathbb{C}P^1$  has a natural Hermitian structure defined by the fibrewise Hermitian metrics.  $\triangle$

Now consider the group  $\mathrm{Sp}(M) = \mathrm{Sp}(M, g, I, J, K) \subseteq \mathrm{Iso}(M, g)$  of Riemannian isometries of  $(M, g)$  preserving the Kähler forms  $\omega_q$  for  $q \in \mathbb{C}P^1$  (or equivalently the complex structures  $I_q$ ), often referred to as the *hyper-unitary* group. Denoting  $\mathrm{Aut}_0(Z)$  the group of holomorphic automorphisms of  $Z \rightarrow \mathbb{C}P^1$  over the identity that preserve the Hermitian structure, there is a natural morphism

$$\rho: \mathrm{Sp}(M) \longrightarrow \mathrm{Aut}_0(Z), \quad (2)$$

given by the fibrewise action of  $\mathrm{Sp}(M)$ .

We shall consider a group of isometries that preserve the hyper-Kähler structure in a looser sense, relaxing the condition that differentials should commute with  $I, J$ , and  $K$  individually.

**Definition 2.2.** Let  $\mathrm{Hk}(M) \subseteq \mathrm{Iso}(M, g)$  be the subgroup stabilising the Lie algebra  $\mathfrak{su}(2)_M$ :

$$\mathrm{Hk}(M) = \mathrm{Hk}(M, g, I, J, K) := \left\{ \varphi \in \mathrm{Iso}(M, g) \mid \mathrm{Ad}_{d\varphi}(\mathfrak{su}(2)_M) = \mathfrak{su}(2)_M \right\}.$$

Hence  $\mathrm{Hk}(M)$  acts on  $\mathfrak{su}(2)_M$ , and  $\mathrm{Sp}(M) \subseteq \mathrm{Hk}(M)$  is the subgroup doing so trivially. Moreover the Adjoint action on  $\mathfrak{su}(2)_M \simeq \mathbb{R}^3$  is by positive isometries for the standard Euclidean structure, resulting in a group morphism

$$\mathrm{Ad}: \mathrm{Hk}(M) \longrightarrow \mathrm{SO}(3), \quad \mathrm{Ad}: \varphi \longmapsto \mathrm{Ad}_{d\varphi},$$

and an action on the hyper-Kähler 2-sphere (1)—simply denoted  $q \mapsto \varphi.q$ .

As mentioned in the introduction, in this paper we suppose to have a faithful isometric Sp(1)-action on  $(M, g)$  permuting transitively (1), i.e. an injective group morphism

$$\sigma: \mathrm{Sp}(1) \longrightarrow \mathrm{Hk}(M), \quad (3)$$

such that  $\mathrm{Ad} \circ \sigma: \mathrm{Sp}(1) \rightarrow \mathrm{SO}(3)$  is the natural surjection. We may identify  $\mathrm{Sp}(1) \simeq \sigma(\mathrm{Sp}(1)) \subseteq \mathrm{Hk}(M)$ , and in particular there is an exact sequence

$$1 \longrightarrow \mathrm{Sp}(M) \longrightarrow \mathrm{Hk}(M) \xrightarrow{\mathrm{Ad}} \mathrm{SO}(3) \longrightarrow 1.$$

Consider now a maximal *compact* connected subgroup  $\mathrm{Sp}_0(M) \subseteq \mathrm{Sp}(M)$  commuting with (3):

$$\mathrm{Sp}_0(M) \subseteq \{ \varphi \in \mathrm{Sp}(M) \mid \sigma(g)\varphi = \varphi\sigma(g) \text{ for } g \in \mathrm{Sp}(1) \},$$

and let  $\mathrm{Hk}_0(M) \subseteq \mathrm{Hk}(M)$  be the subgroup generated by  $\mathrm{Sp}_0(M)$  and  $\mathrm{Sp}(1)$ . Finally assume that  $\sigma(-\mathrm{Id}) \in \mathrm{Sp}_0(M)$ , which yields another exact sequence

$$1 \longrightarrow \mathrm{Sp}_0(M) \longrightarrow \mathrm{Hk}_0(M) \xrightarrow{\mathrm{Ad}} \mathrm{SO}(3) \longrightarrow 1.$$

By construction there is a surjective morphism  $\mathrm{Sp}_0(M) \times \mathrm{Sp}(1) \rightarrow \mathrm{Hk}_0(M)$  with kernel  $\{ (\mathrm{Id}, \mathrm{Id}), (-\mathrm{Id}, -\mathrm{Id}) \}$ , whence an abstract identification

$$\mathrm{Hk}_0(M) = \mathrm{Sp}_0(M) \cdot \mathrm{Sp}(1) \simeq (\mathrm{Sp}_0(M) \times \mathrm{Sp}(1)) / \mathbb{Z}/2\mathbb{Z}. \quad (4)$$

Hence we are globalising the Riemannian holonomy group of quaternion-Kähler manifolds (on tangent spaces; cf. § 3.1 for the linear case).

Finally, if  $\mathrm{Aut}(Z)$  denotes the group of automorphisms of  $Z \rightarrow \mathbb{C}P^1$  preserving the fibrewise Hermitian structures, covering arbitrary biholomorphisms of  $\mathbb{C}P^1$ , we have a group morphism

$$\rho^Z: \mathrm{Hk}(M) \rightarrow \mathrm{Aut}(Z), \quad (5)$$

extending (2).

**2.2. Families of pre-quantum line bundles.** Now assume that

$$[\omega_q] \in H^2(M, \mathbb{Z}), \quad \text{for } q \in \mathbb{C}P^1.$$

It follows by continuity that the cohomology class  $[\omega_q]$  is independent of  $q$ . Hence we can fix a Hermitian line bundle  $(L, h)$  over  $M$ , and a family of Hermitian connections  $\nabla_q$  on  $L$  with curvatures  $F_q = -i\omega_q$  for  $q \in \mathbb{C}P^1$ .

Then for any  $\varphi \in \mathrm{Hk}(M)$  the line bundle  $\varphi^*L \otimes L^{-1} \rightarrow M$  comes with a family of *flat* Hermitian connections, and since isomorphism classes of such objects are parametrised by  $G := H^1(M, U(1))$  this yields a 1-cocycle

$$u: \mathrm{Hk}(M) \longrightarrow C^\infty(\mathbb{C}P^1, G), \quad u: \varphi \longmapsto \varphi^*L \otimes L^{-1}.$$

By construction the cohomology class  $[u] \in H^1(\mathrm{Hk}(M), C^\infty(\mathbb{C}P^1, G))$  vanishes if and only if there exists  $\Lambda \in C^\infty(\mathbb{C}P^1, G)$  such that  $\varphi^*(L \otimes \Lambda^{-1}) \simeq L \otimes \Lambda^{-1}$ . This is equivalent to saying that, after twisting  $L$  by a family of flat connections, every element of  $\mathrm{Hk}(M)$  can be lifted to an automorphism of the family of pre-quantum line bundles: If  $\mathrm{Hk}'(M)$  denotes the group of automorphisms of  $L \rightarrow M$  covering the  $\mathrm{Hk}(M)$ -action we thus have a central extension

$$1 \longrightarrow U(1) \longrightarrow \mathrm{Hk}'(M) \longrightarrow \mathrm{Hk}(M) \longrightarrow 1, \quad (6)$$



as in the introduction.

The simplest vanishing  $[u] = 0$  is obtained if  $G$  is trivial (which we will see in some examples) or by the existence of a hyper-Kähler potential (which we discuss in § 2.9).

In what follows we thus assume to have fixed a family of pre-quantum connections satisfying the property above. Restricting the central extension (6) to  $\text{Hk}_0(M) \subseteq \text{Hk}(M)$  yields a second central extension

$$1 \longrightarrow \text{U}(1) \longrightarrow \text{Hk}_0'(M) \longrightarrow \text{Hk}_0(M) \longrightarrow 1, \quad (7)$$

where  $\text{Hk}_0'(M) \subseteq \text{Hk}'(M)$  acts on  $L \rightarrow M$  preserving its structure.

Since  $\text{Sp}(1)$  has no nontrivial central  $\text{U}(1)$ -extensions, it follows from the faithfulness of (3) that (7) is trivial when restricted to  $\text{Sp}(1)$ . Hence we can lift (3) to a group morphism  $\sigma': \text{Sp}(1) \rightarrow \text{Hk}_0'(M)$ . We denote the induced central extension of  $\text{Sp}_0(M)$  by  $\text{Sp}_0'(M) \subseteq \text{Hk}_0'(M)$ , fitting into an exact sequence

$$1 \longrightarrow \text{U}(1) \longrightarrow \text{Sp}_0'(M) \longrightarrow \text{Sp}_0(M) \longrightarrow 1.$$

We see  $\text{Sp}_0'(M)$  is a *compact* Lie group commuting with the  $\text{Sp}(1)$ -action on the pre-quantum line bundle, and  $\text{Hk}_0'(M) \subseteq \text{Hk}'(M)$  is the subgroup generated by  $\sigma'(\text{Sp}(1))$  and  $\text{Sp}_0'(M)$ .

The resulting action on the family of pre-quantum line bundles is a group morphism

$$\rho^L: \text{Hk}'(M) \longrightarrow \text{Aut}(L),$$

lifting (5) and restricting to an action of  $\text{Hk}_0'(M)$ .

Generic elements of the central extension will be denoted  $\varphi' \in \text{Hk}'(M)$ , projecting to elements  $\varphi \in \text{Hk}(M)$  by forgetting the action on the fibres of  $L \rightarrow M$ .

**2.3. Geometric quantisation.** Following the prescription of geometric quantisation, for  $q \in \mathbb{C}P^1$  consider the separable Hilbert space

$$\mathcal{H}_q := \left\{ \psi \in H^0(M_q, L) \mid \int_M h(\psi, \psi) \, \text{dvol} < \infty \right\} \subseteq L^2(M, L), \quad (8)$$

using the holomorphic structure  $\bar{\partial}_q = \nabla_q^{0,1}$  and the standard  $L^2$  Hermitian product:

$$\langle \psi \mid \psi' \rangle := \int_M h(\psi, \psi') \, \text{dvol}, \quad \text{for } \psi, \psi' \in \mathcal{H}_q. \quad (9)$$

Let us denote  $\mathcal{H}$  the family of Hilbert spaces thus defined over  $\mathbb{C}P^1$ .

By construction there are unitary isomorphisms

$$\rho_{\varphi'}^{\mathcal{H}}: \mathcal{H}_q \longrightarrow \mathcal{H}_{\varphi, q}, \quad \text{for } q \in \mathbb{C}P^1 \text{ and } \varphi' \in \text{Hk}'(M),$$

explicitly given by:

$$\rho_{\varphi'}^{\mathcal{H}} \psi(m) := \rho_{\varphi'}^L(\psi(\rho_{\varphi'}^Z m)), \quad \text{for } m \in M. \quad (10)$$

**2.4. Decomposition of  $\mathcal{H}_q$ .** We will now consider the decompositions of the spaces (8) induced by the action (10) of various subgroups of  $\text{Hk}_0'(M) \subseteq \text{Hk}'(M)$ .

For  $q \in \mathbb{C}P^1$  consider the stabiliser maximal torus  $T_q \subseteq \text{Sp}(1)$ , so the subgroup  $\sigma'(T_q) \subseteq \text{Hk}_0'(M)$  acts on  $\mathcal{H}_q$  by unitary operators. Its action defines an  $L^2$ -completed orthogonal (inner) direct sum

$$\mathcal{H}_q = \overline{\bigoplus_{d \in \mathbb{Z}} \mathcal{H}_q^{(d)}}}, \quad (11)$$

where  $\mathcal{H}_q^{(d)} \subseteq \mathcal{H}_q$  is the isotypical component corresponding to the character

$$U(1) \simeq \sigma'(T_q) \longrightarrow \mathbf{C}^\times, \quad z \longmapsto z^d,$$

after fixing an identification with the standard torus  $U(1) \subseteq \mathbf{C}^\times$ .

Then we refine (11) using the  $\mathrm{Sp}_0'(M)$ -action on  $\mathcal{H}_q^{(d)}$ . Namely, choosing a set of representatives  $\{V_\lambda\}_{\lambda \in \Lambda}$  for isomorphism classes of unitary simple  $\mathrm{Sp}_0'(M)$ -modules, the Peter–Weyl theorem [60, Thm. 1.12] provides a splitting

$$\mathcal{H}_q^{(d)} = \overline{\bigoplus_{\lambda \in \Lambda} \mathcal{H}_{\lambda,q}^{(d)}}}. \quad (12)$$

Further we let  $\Lambda^{(d)} \subseteq \Lambda$  be the subset of active representations.<sup>2</sup>

Finally we may also consider a maximal torus  $T \subseteq \mathrm{Sp}_0'(M)$  and find

$$\mathcal{H}_q^{(d)} = \overline{\bigoplus_{\alpha \in T^\vee} \mathcal{H}_{\alpha,q}^{(d)}}}, \quad (13)$$

where  $\mathcal{H}_{\alpha,q}^{(d)} \subseteq \mathcal{H}_q^{(d)}$  is the isotypical component of the character  $\alpha: T \rightarrow \mathbf{C}^\times$ .

We denote  $\mathcal{H}^{(d)}$ ,  $\mathcal{H}_\alpha^{(d)}$  and  $\mathcal{H}_\lambda^{(d)}$  the families of Hilbert spaces thus defined over  $\mathbf{CP}^1$ , so that we have  $L^2$ -completed orthogonal direct sums

$$\mathcal{H} = \overline{\bigoplus_{d \in \mathbf{Z}} \mathcal{H}^{(d)}}, \quad \overline{\bigoplus_{\lambda \in \Lambda} \mathcal{H}_\lambda^{(d)}} = \mathcal{H}^{(d)} = \overline{\bigoplus_{\alpha \in T^\vee} \mathcal{H}_\alpha^{(d)}}}.$$

**2.5. Structure of  $\mathcal{H}_\lambda^{(d)}$ .** Our main assumption is the following, unless otherwise stated:

The spaces  $\mathcal{H}_{\lambda,q}^{(d)}$  are finite-dimensional.

Now assume further (and temporarily, cf. Rem. 2.4) that the family of Hilbert spaces  $\mathcal{H}_\lambda^{(d)}$  forms a *smooth* vector bundle over the Riemann sphere, hence a smooth vector sub-bundle of the trivial Hilbert bundle  $L^2(M, L) \rightarrow \mathbf{CP}^1$ . We can then differentiate local sections  $\psi = \psi(q)$  of  $\mathcal{H}_\lambda^{(d)} \rightarrow \mathbf{CP}^1$  along vector fields on  $\mathbf{CP}^1$ ; then, since  $\mathcal{H}_{\lambda,q}^{(d)} \subseteq L^2(M, L)$  is a closed/Hilbert subspace, there are orthogonal projections

$$\pi_{\lambda,q}^{(d)}: L^2(M, L) \longrightarrow \mathcal{H}_{\lambda,q}^{(d)}.$$

**Definition 2.3.** For any tangent vector  $X \in T_q \mathbf{CP}^1$  set

$$\nabla_X^{\mathcal{H}_\lambda^{(d)}} \psi \Big|_q := \pi_{\lambda,q}^{(d)}(X[\psi](q)) \in \mathcal{H}_{\lambda,q}^{(d)}. \quad (14)$$

*Remark 2.2.* The same definition (of the standard  $L^2$ -connection) can be given verbatim in the case where the families  $\mathcal{H}_\alpha^{(d)} \subseteq \mathcal{H}^{(d)}$  also constitute smooth vector bundles.  $\triangle$

**Proposition 2.1.** *The covariant-derivative operators (14) define a  $\rho^{\mathcal{H}}$ -invariant Hermitian connection  $\nabla^{\mathcal{H}_\lambda^{(d)}}$  on the Hermitian vector bundle  $\mathcal{H}_\lambda^{(d)} \rightarrow \mathbf{CP}^1$ .*

<sup>2</sup>The subset  $\Lambda^{(d)}$  is independent of  $q \in \mathbf{CP}^1$  since the  $\mathrm{Sp}_0'(M)$ -modules  $\mathcal{H}_q^{(d)}$  are isomorphic under the  $\mathrm{Sp}(1)$ -action.

*Proof.* Given  $\varphi' \in \text{Hk}_0'(\mathbb{M})$ , a local section  $\psi$  of  $\mathcal{H}_\lambda^{(d)} \rightarrow \mathbb{C}\mathbb{P}^1$ , and a tangent vector  $X \in T\mathbb{C}\mathbb{P}^1$ , we have

$$X[\rho_{\varphi'}\psi] = \rho_{\varphi'}((\varphi_*^{-1}X)[\psi]),$$

denoting  $\rho = \rho^{\mathcal{H}_\lambda^{(d)}}$ . The covariant derivative  $\nabla_X = \nabla_X^{\mathcal{H}_\lambda^{(d)}}$  is characterised by the following identity, for every vector  $\psi' \in \mathcal{H}_{\lambda, \varphi, q}^{(d)}$ :

$$\langle \nabla_X \psi \mid \psi' \rangle = \langle X[\psi] \mid \psi' \rangle.$$

On the other hand a change of variables in (9) shows that

$$\langle \rho_{\varphi'}\psi \mid \psi' \rangle = \langle \psi \mid \rho_{\varphi'}^{-1}\psi' \rangle$$

which combined with the above yields  $\nabla_X(\rho_{\varphi'}\psi) = \rho_{\varphi'}(\nabla_{\varphi_*^{-1}X}[\psi])$ .  $\square$

Let now  $\mathcal{L} \rightarrow \mathbb{C}\mathbb{P}^1$  be the  $\text{Sp}(1)$ -equivariant Hermitian line bundle over  $\mathbb{C}\mathbb{P}^1$  equipped with the (unique)  $\text{Sp}(1)$ -invariant connection  $\nabla$  associated to the standard representation of the standard maximal torus  $\text{U}(1) \subseteq \text{Sp}(1)$ . Finally denote  $\nabla^{\text{Tr}}$  the trivial connection in the trivial vector bundle  $\underline{V}_\lambda \rightarrow \mathbb{C}\mathbb{P}^1$  with fibre  $V_\lambda$  for  $\lambda \in \Lambda^{(d)}$ , and let  $m_\lambda^{(d)} \in \mathbb{Z}_{>0}$  be the multiplicity of the simple  $\text{Sp}_0'(\mathbb{M})$ -module  $V_\lambda$  inside  $\mathcal{H}_q^{(d)}$ .<sup>3</sup>

**Theorem 2.1.** *There is an  $\text{Hk}_0'(\mathbb{M})$ -invariant direct sum decomposition of Hermitian vector bundles with connections*

$$\mathcal{H}_\lambda^{(d)} = \bigoplus_{j=1}^{m_\lambda^{(d)}} \mathcal{H}_{\lambda, j}^{(d)}, \quad \nabla^{\mathcal{H}_\lambda^{(d)}} = \bigoplus_{j=1}^{m_\lambda^{(d)}} \nabla^{\mathcal{H}_{\lambda, j}^{(d)}}, \quad (15)$$

and an  $\text{Hk}_0'(\mathbb{M})$ -equivariant isomorphism

$$\mathcal{H}_{\lambda, j}^{(d)} \simeq \mathcal{L}^d \otimes \underline{V}_\lambda, \quad \nabla^{\mathcal{H}_{\lambda, j}^{(d)}} = \nabla \otimes \text{Id} + \text{Id} \otimes \nabla^{\text{Tr}}.$$

*Proof.* For fixed  $\lambda, d$  consider the vector bundle  $\text{Hom}(\underline{V}_\lambda, \mathcal{H}_\lambda^{(d)})$  with the induced  $\text{Hk}_0'(\mathbb{M})$ -action, Hermitian structure, and connection. Since all of this data is  $\text{Sp}_0'(\mathbb{M})$ -equivariant, it restricts naturally to

$$E_\lambda^{(d)} := \text{Hom}_{\text{Sp}_0'(\mathbb{M})}(\underline{V}_\lambda, \mathcal{H}_\lambda^{(d)}),$$

the vector bundle of  $\text{Sp}_0'(\mathbb{M})$ -linear morphisms, and  $\text{rk}(E_\lambda^{(d)}) = m_\lambda^{(d)}$ . Furthermore, it is clear that each stabiliser  $T_q$  acts on  $E_{\lambda, q}^{(d)}$  via multiplication by the degree- $d$  character, so there is an identification

$$E_\lambda^{(d)} \simeq \bigoplus_{j=1}^{m_\lambda^{(d)}} \mathcal{L}^d. \quad (16)$$

Now consider the evaluation map

$$E_\lambda^{(d)} \otimes \underline{V}_\lambda \longrightarrow \mathcal{H}_\lambda^{(d)}, \quad f \otimes v \longmapsto f(v),$$

<sup>3</sup>The integer  $m^{(d)}$  is independent of  $q \in \mathbb{C}\mathbb{P}^1$  (cf. the previous footnote).

which by construction of  $E_\lambda^{(d)}$  respects the group action, Hermitian structure, and connection. By the definition of  $\mathcal{H}_{\lambda,q}^{(d)}$ , it restricts to an isomorphism on every fibre: By (16), this is enough to conclude.  $\square$

*Remark 2.3.* Note the decomposition (15) is not canonical, while the sub-bundles  $\mathcal{H}_\lambda^{(d)} \rightarrow \mathbb{C}P^1$  are uniquely determined by the  $\text{Hk}_0'(M)$ -action.  $\triangle$

*Remark 2.4.* Thm. 2.1 yields an alternative definition of the bundles of isotypical components, without smoothness assumptions.

Indeed for  $d \in \mathbb{Z}$ ,  $\lambda \in \Lambda^{(d)}$  and  $j \in \{1, \dots, m_\lambda^{(d)}\}$  one can define

$$\mathcal{H}_{\lambda,j}^{(d)} := \mathcal{L}^d \otimes \underline{V}_\lambda, \quad \mathcal{H}_\lambda^{(d)} := (\mathcal{L}^d \otimes \underline{V}_\lambda)^{\oplus m_\lambda^{(d)}}.$$

This yields finite-rank smooth  $\text{Hk}_0'(M)$ -equivariant Hermitian vector bundles over the Riemann sphere, equipped with Hermitian connections, defined from the combinatorial data of the multiplicities of  $\mathcal{H}_q$  as a representation, as soon as the main assumption is verified.  $\triangle$

**2.6. Quantum super Hilbert spaces and unitary representations.** We now let  $H_{\lambda,j}^{(d)}$  be the super vector space obtained by taking the holomorphic cohomology of the bundles of isotypical components:

$$H_{\lambda,j}^{(d)} := H^*(\mathbb{C}P^1, \mathcal{H}_{\lambda,j}^{(d)}), \quad \text{for } j \in \{1, \dots, m_\lambda^{(d)}\}.$$

If  $W^{(d)} = W_+^{(d)} \oplus W_-^{(d)}$  is the unitary super  $\text{Sp}(1)$ -representation defined by

$$W_+^{(d)} := H^0(\mathbb{C}P^1, \mathcal{L}^d), \quad W_-^{(d)} := H^1(\mathbb{C}P^1, \mathcal{L}^d),$$

then  $H_{\lambda,j}^{(d)} \simeq W^{(d)} \otimes V_\lambda$  as super  $\text{Hk}_0'(M)$ -representations.

*Remark 2.5.* One has  $W_-^{(d)} = W_+^{(-d)} = (0)$  for  $d \in \mathbb{Z}_{<0}$ , and  $W_+^{(0)} \simeq \mathbb{C} \simeq W_-^{(0)}$ .  $\triangle$

Finally consider the nested  $L^2$ -completed orthogonal direct sums

$$H := \overline{\bigoplus_{d \in \mathbb{Z}} H^{(d)}}, \quad H^{(d)} := \overline{\bigoplus_{\lambda \in \Lambda^{(d)}} H_\lambda^{(d)}}, \quad (17)$$

setting as above

$$H_\lambda^{(d)} := \bigoplus_{j=1}^{m_\lambda^{(d)}} H_{\lambda,j}^{(d)}. \quad (18)$$

The endpoint of our construction is the following, which is our model for the hyper-Kähler quantisation of the  $\text{Sp}(1)$ -symmetric manifold  $(M, g, I, J, K)$ .

**Theorem 2.2.** *There is a unitary action  $\text{Hk}_0'(M) \rightarrow \text{U}(H)$  preserving the splittings (17) and (18), and the restriction  $\text{Hk}_0'(M) \rightarrow \text{U}(H_{\lambda,j}^{(d)})$  is isomorphic to  $W^{(d)} \otimes V_\lambda$  as simple super  $\text{Hk}_0'(M)$ -module.*

**2.7. Finite-rank conditions.** We shall now consider conditions which entail finite-dimensionality for the isotypical components of § 2.4, starting from the case where there is a *proper* moment map for the action of  $T_q, T'_q := T_q \times T$  or  $Sp_0'(M)$ .

In general, if  $K$  is a compact Lie group with Lie algebra  $\mathfrak{k} = \text{Lie}(K)$ , acting on a Kähler manifold  $X$  with a lifted  $K$ -action on a pre-quantum line bundle  $(L, \nabla)$ , there is a natural moment map  $\mu: X \rightarrow \mathfrak{k}^\vee$  defined by Kostant's formula

$$2\pi i \langle \mu, \xi \rangle \frac{\partial}{\partial \theta} = \xi_X^H - \xi_L \quad (19)$$

for every  $\xi \in \mathfrak{k}$ , where  $\xi_L$  is the vector field corresponding to  $\xi$  on  $L$ ,  $\xi_X^H$  the one on  $X$  lifted horizontally, and  $\frac{\partial}{\partial \theta}$  is the fibre-wise "angular" vector field. In this setup, we will make use of the following version of the general principle that "quantisation commutes with reduction".

**Theorem 2.3** ([49, 85]). *If the  $K$ -action extends holomorphically to the complexified group  $K^C$ , and if the moment map (19) is proper, then for every dominant weight  $\gamma$  of  $K$  there is an identification*

$$\text{Hom}_K(V_\gamma, H^0(X, L)) \simeq H^0(X_\gamma, L_\gamma),$$

where  $V_\gamma$  denotes a simple  $K$ -module of highest weight  $\gamma$ ,  $X_\gamma = X //_\gamma K$  is the symplectic reduction of  $X$  at level  $\gamma$ , and  $L_\gamma$  is the induced ( $V$ -)bundle on  $X_\gamma$ .

This result was first established by Guillemin and Sternberg [49] in the case  $X$  is compact, with additional regularity conditions, and then extended by Sjamaar [85]. The statement has been subsequently generalised in various works including those of Meinrenken [73, 74], Meinrenken–Sjamaar [75], Vergne [88, 89], Ma [66], Ma–Zhang [67], and Hochs–Song [54].

Returning to our setting we have moment maps

$$\mu_q: M \rightarrow i\mathbb{R}, \quad \mu'_q: M \rightarrow i\mathbb{R} \times \mathfrak{t}^\vee, \quad \tilde{\mu}_q: M \rightarrow \mathfrak{sp}_0'(M)^\vee, \quad (20)$$

identifying  $\text{Lie}(T_q)^\vee \simeq i\mathbb{R}$ , and where  $\mathfrak{t} = \text{Lie}(T)$  and  $\mathfrak{sp}_0'(M) = \text{Lie}(Sp_0'(M))$ . Then Thm. 2.3 yields the following.

**Theorem 2.4.** *Suppose either of the moment maps (20) is proper for some  $q \in \mathbb{C}P^1$ , and the action of the corresponding group has a holomorphic complexification. Then the corresponding family  $\mathcal{H}^{(d)}$ ,  $\mathcal{H}_\alpha^{(d)}$ , or  $\mathcal{H}_\lambda^{(d)}$  has finite rank.*

*Proof.* Thm. 2.3 applies when  $X_\gamma$  is a compact complex analytic space and  $L_\gamma$  a coherent sheaf on it, which includes our setup.

Indeed properness of the moment map implies the complex analytic reduction  $(M_q)_\gamma$  is compact for any dominant weight  $\gamma$ , hence the space of section of  $L_\gamma$  is finite-dimensional [47]. It follows that each irreducible representation has finite multiplicity inside  $H^0(M_q, L_q)$ , so *a fortiori* inside  $\mathcal{H}_q$ .  $\square$

*Remark 2.6.* Another way to ensure finite-dimensionality is to assume there are (at least) two-codimensional compactifications of the symplectic reductions, having rational singularities; then Hartogs's theorem applies on the reduction (see e.g. [87] for such generalizations).  $\triangle$

Finally finite-dimensionality can be controlled using the results of [94].

**Theorem 2.5** (op. cit.). *Suppose the action of  $T_q$  (resp.  $T_q'$ ) is meromorphic in the sense of Def. 2.12 of op. cit, and that it satisfies the assumption 2.14 therein. Then  $\mathcal{H}^{(d)}$  (resp.  $\mathcal{H}_a^{(d)}$ ) has finite rank.*

**2.8. Rank-generating series and localisation formulæ.** If either  $\mathcal{H}^{(d)}$ ,  $\mathcal{H}_a^{(d)}$  or  $\mathcal{H}_\lambda^{(d)}$  have finite rank, we consider the (formal) generating series

$$H(t) = \sum_d \text{rk}(\mathcal{H}^{(d)}) \cdot t^d, \quad (21)$$

and

$$H'(t, \tilde{t}) = \sum_{d,a} \text{rk}(\mathcal{H}_a^{(d)}) \cdot t^d \tilde{t}^a, \quad (22)$$

as well as

$$G(t, \tilde{t}) = \sum_{d \in \mathbb{Z}} \sum_{\lambda \in \Lambda^{(d)}} m_\lambda^{(d)} \cdot t^d \tilde{t}^\lambda. \quad (23)$$

Note that if  $\mathcal{H}_a^{(d)}$  and  $\mathcal{H}_\lambda^{(d)}$  are both finite-rank then (22) can be obtained from (23) via the substitution  $\tilde{t}^\lambda \mapsto \chi_\lambda(\tilde{t})$ , where

$$\chi_\lambda(\tilde{t}) = \sum_{\alpha \in E_\lambda} n_\alpha^{(\lambda)} \cdot \tilde{t}^\alpha,$$

and where  $E_\lambda$  is the set of weights of  $V_\lambda$ —with multiplicities  $n_\alpha^{(\lambda)} \in \mathbb{Z}_{\geq 0}$ .

If in particular  $\text{Sp}_0'(M)$  is semisimple then the Weyl Character formula yields

$$\chi_\lambda(\tilde{t}) = \frac{\sum_{w \in W} \epsilon(w) \tilde{t}^{w(\lambda + \rho)}}{\sum_{w \in W} \epsilon(w) \tilde{t}^{w(\rho)}},$$

where  $W = N(T)/T$  is the Weyl group and  $\rho \in t^\vee$  the half-sum of positive roots.

Conversely (23) can be recovered from (22) (when both are defined) as follows. Fix  $d \in \mathbb{Z}$  and let  $H_d(\tilde{t})$  be the coefficient of  $t^d$  in (22). Let  $\lambda$  be maximal among the weights such that  $\tilde{t}^\lambda$  appears in  $H_d(\tilde{t})$ . In particular, the weight  $\lambda_d^{(0)}$  can only appear in an irreducible component of  $\mathcal{H}_q^{(d)}$  (as a  $\text{Sp}_0'(M)$ -module) if it is the highest. Therefore, the coefficient of  $\tilde{t}^\lambda$  in  $H_d(\tilde{t})$  is equal to  $m_\lambda^{(d)}$ . One may now consider  $H_d(\tilde{t}) - m_\lambda^{(d)} \chi_\lambda(\tilde{t})$  and repeat the procedure inductively. Since each step strictly decreases one of the maximal weights the process terminates—exactly when the polynomial vanishes. This results in a decomposition

$$H_d(\tilde{t}) = \sum_{\lambda \in \Lambda^{(d)}} m_\lambda^{(d)} \chi_\lambda(\tilde{t}),$$

recovering all multiplicities and ultimately (23).

Further the generating series (21), (22), and (23) can sometimes be computed by localisation formulæ. We refer to [55] for general results, and we review here the simpler versions used in what follows.

Suppose the action of  $T_q$  on  $M_q$  has a finite number of fixed points  $|M_q| \subseteq M_q$ , and let  $\bar{R}(T_q)$  be the formal completion of the character ring  $R(T_q)$  of  $T_q$ —i.e.  $\bar{R}(T_q) \simeq \mathbb{Z}[[t^{\pm 1}]]$  canonically.

Since the fixed points  $p \in |M_q|$  are isolated we see that  $\Lambda_{-1}(T_p M_q) \in \overline{\mathcal{R}}(T_q)$  is invertible. Suppose now we have a decomposition

$$H^i(M_q, L_q) = \overline{\bigoplus_{d \in \mathbb{Z}} H^i(M_q, L_q)^{(d)}},$$

such that  $T_q$  acts on  $H^i(M_q, L_q)^{(d)}$  via the  $d$ 'th power of the standard representation, and such that the spaces  $H^i(M_q, L_q)^{(d)}$  are finite-dimensional.

**Proposition 2.2** ([27, 55]). *The following formula holds:*

$$\sum_{i=0}^{2n} (-1)^i \dim H^i(M_q, L_q)^{(d)} t^d = \sum_{p \in |M_q|} \frac{L_{q,p}}{\Lambda_{-1}(T_p M_q)}.$$

Hence if  $H^i(M_q, L_q) = (0)$  for  $i > 0$  then simply

$$H(t) = \sum_{p \in |M_q|} \frac{L_{q,p}}{\Lambda_{-1}(T_p M_q)}. \quad (24)$$

Considering the action of  $T'_q$  on  $M_q$  we get an analogous result, provided  $T'_q$  has finitely many fixed points and all spaces  $H^i(M_q, L_q)_{\alpha, q}^{(d)}$  are finite-dimensional—and interpreting the right-hand side as an element of  $\overline{\mathcal{R}}(T) \simeq \mathbb{Z}[[t^{\pm 1}, \tilde{t}_i^{\pm 1}]]$ . In particular

$$H'(t, \tilde{t}) = \sum_{p \in |M_q|} \frac{L_{q,p}}{\Lambda_{-1}(T_p M_q)}. \quad (25)$$

Now recall that if  $M_q$  is a Stein space, or has the structure of an affine scheme, then Cartan's theorem yields the vanishing of higher cohomology groups [28]. Thus putting together the previous results we have established the following.

**Theorem 2.6.** *Suppose there exists  $q \in \mathbb{C}P^1$  such that  $M_q$  is a Stein space, or has the structure of an affine scheme, and that the  $T_q$ -action (resp.  $T'_q$ -action) has finitely many fixed points. Assume further that one of the following holds:*

- *there is a proper moment map for the  $T_q$ -action (resp.  $T'_q$ -action);*
- *there exists a compactification of the symplectic reductions with rational singularities, with boundary of codimension at least two (cf. Rem. 2.6).*

*Then the family  $\mathcal{H}^{(d)}$  (resp.  $\mathcal{H}_\alpha^{(d)}$ ) has finite rank, and the associated localisation formula (24) (resp. (25)) holds for the rank-generating series (21) (resp. (22)).*

**Remark 2.7.** If the higher cohomology groups do not vanish one could replace (8) by the super space

$$\tilde{\mathcal{H}}_q = H^{\text{even}}(M_q, L) \oplus H^{\text{odd}}(M_q, L),$$

in which case case formulæ (24) and (25) hold for the super representations  $\tilde{\mathcal{H}}_q$  of  $T_q$  and  $T'_q$ . (In this setup one need not assume that  $M_q$  be a Stein space or an affine scheme.)  $\triangle$

**Remark 2.8.** Alternatively if the assumptions of Thm. 2.5 are satisfied then Wu's localisation theorem [94, Thm. 3.14] yields the generating series (21) and (22) by an index computation of the fixed-point locus for the  $T_q$ - and  $T'_q$ -action, respectively.  $\triangle$

### 2.9. Sp(1)-symmetric hyper-Kähler potentials.

**Definition 2.4.** A hyper-Kähler potential on the hyper-Kähler manifold  $(M, g, I, J, K)$  is a smooth map  $\mu: M \rightarrow \mathbb{R}$  such that  $\omega_q = i\partial_q\bar{\partial}_q\mu$  for every  $q \in \mathbb{C}P^1$ .

One can also use such potentials to obtain equivariant pre-quantum data, as discussed below. Assume further that  $\mu$  is Sp(1)-invariant and that it generates the  $T_q$ -actions, i.e.  $i\mu: M_q \rightarrow i\mathbb{R} \simeq \mathfrak{t}_q^\vee$  is a moment map.

In this case we consider the trivial Hermitian line bundle, and lift the  $\text{Hk}_0(M)$ -action by the identity on each fibre; i.e.  $\text{Hk}_0'(M) = U(1) \times \text{Hk}_0(M)$ . Natural symplectic potentials are given by

$$\theta_q = \frac{1}{2}(\bar{\partial}_q\mu - \partial_q\mu) \in \Omega^1(M),$$

hence  $\nabla_q = d + \frac{\theta_q}{\hbar}$  is a pre-quantum connection for all  $q \in \mathbb{C}P^1$ , and the resulting prequantum data are  $\text{Hk}_0'(M)$ -equivariant since  $\mu$  is Sp(1)-invariant.

Now if  $\text{grad}(\mu)$  is complete then each  $T_q$ -action extends holomorphically to  $\mathbb{C}^*$ , and if in addition  $\mu$  is proper then the subspaces  $\mathcal{H}_q^{(d)}$  are finite-dimensional by Thm 2.4.

**Proposition 2.3.** *Suppose that  $\mu$  is bounded below and that it has finitely many critical values. Then the function  $\psi_0 := e^{-\mu/2\hbar}$  is square-summable and a holomorphic frame for every  $q \in \mathbb{C}P^1$ .*

*Proof.* Nonvanishing and holomorphicity are straightforward consequence of the definition.

On the other hand, the  $L^2$ -square-norm of  $\psi_0$  can be expressed as

$$\|\psi_0\|_{L^2}^2 = \int_M e^{-\frac{\mu}{\hbar}} d\text{vol} = \int_B e^{-\frac{\xi}{\hbar}} \mu_*(d\text{vol}), \quad (26)$$

where  $B \in \mathbb{R}$  is a lower bound for  $\mu$  and  $\mu_*(d\text{vol})$  the push-forward of the Liouville measure. By the Duistermaat-Heckman theorem [37] the push-forward admits a density which restricts to a polynomial on every interval  $I \subset \mathbb{R}$  not containing critical values for  $\mu$ . Since there are finitely many such values, (26) splits as a finite sum of converging integrals, proving the claim.  $\square$

Under the assumptions of Prop. 2.3 we thus get an isomorphism

$$\Psi: L^2 H^0(M_q, \mathcal{O}, e^{-\mu/\hbar} d\text{vol})^{(d)} \longrightarrow \mathcal{H}_q^{(d)},$$

given by  $\Psi(f) = f\psi_0$ , where the left-hand side denotes the space of  $d$ -homogeneous holomorphic functions with finite  $L^2$ -norm with respect to  $e^{-\mu/\hbar} d\text{vol}$ .

## 3. EXAMPLES OF APPLICATIONS

**3.1. Hyper-Kähler vector spaces.** Let  $n > 0$  be integer and  $V$  a real vector space of dimension  $fn$ .

**Definition 3.1.** A linear hyper-Kähler structure on  $V$  is a scalar product  $g$  and an ordered triple  $(I, J, K)$  of orthogonal automorphisms of  $V$  satisfying the quaternionic identities  $I^2 = J^2 = K^2 = IJK = -\text{Id}_V$ .



Equivalently, a linear hyper-Kähler structure on  $V$  may be regarded as a Hermitian representation of the quaternion algebra

$$\mathbb{H} = \{ q = d + ai + bj + ck \mid a, b, c, d \in \mathbb{R} \},$$

on  $V$ , where the quaternionic Hermitian form is

$$\mathbf{h} := g - i\omega_I - j\omega_J - k\omega_K,$$

with

$$\omega_{\bullet}(v, w) := g(\bullet v, w) \quad \text{for } \bullet \in \{I, J, K\}.$$

It follows that  $I, J, K$  are  $g$ -skew-symmetric, hence they span a real Lie subalgebra  $\mathfrak{su}(2)_V \subseteq \mathfrak{o}(V, g)$ .

Attached to the hyper-Kähler vector space is the group  $\mathrm{Sp}(V, \mathbf{h}) \subseteq \mathrm{O}(V, g)$  of  $\mathbb{R}$ -linear endomorphisms of  $V$  preserving  $\mathbf{h}$ —hence  $g$  and each of the forms  $\omega_I, \omega_J, \omega_K$ . As above we are interested in transformations that preserve the hyper-Kähler structure in a looser sense, but here we restrict to *linear* ones:

$$\mathrm{Hk}(V) = \mathrm{Hk}(V, g, I, J, K) := \left\{ A \in \mathrm{O}(V, g) \mid \mathrm{Ad}_A(\mathfrak{su}(2)_V) = \mathfrak{su}(2)_V \right\}. \quad (27)$$

As a subgroup of  $\mathrm{O}(V, g)$ , the above is compact.

*Remark 3.1.* We are thus slightly abusing the notation from § 2.

Indeed if  $V$  is regarded as a smooth hyper-Kähler manifold then the group of *all* transformations preserving  $g$  and  $\mathfrak{su}(2)_V$  also contains the translations, and it is in fact generated by these two kinds of transformations. In the framework of § 2 the group (27) thus rather coincides with  $\mathrm{Hk}_0(V)$ , but we shall denote it  $\mathrm{Hk}(V)$  in the linear case to simplify the notation.  $\triangle$

*Remark 3.2.* In this case the twistor space is a rank- $2n$  holomorphic vector bundle  $\pi_{\mathbb{C}P^1}: Z \rightarrow \mathbb{C}P^1$  isomorphic to  $\underline{\mathbb{C}}^{2n} \otimes \mathcal{O}(1)$  [52]. The family comes with a preferred global trivialisation  $Z \simeq V \times \mathbb{C}P^1$  as a smooth (rank- $4n$ ) vector bundle, but not as a symplectic vector bundle or as a Hermitian vector bundle.  $\triangle$

**Lemma 3.1.** *There is an exact sequence of Lie groups*

$$1 \longrightarrow \mathrm{Sp}(V, \mathbf{h}) \longrightarrow \mathrm{Hk}(V) \xrightarrow{\mathrm{Ad}} \mathrm{SO}(3) \longrightarrow 1,$$

and an embedding  $\sigma: \mathrm{Sp}(1) \rightarrow \mathrm{Hk}(V)$  such that  $\mathrm{Ad} \circ \sigma: \mathrm{Sp}(1) \rightarrow \mathrm{SO}(3)$  is the natural surjection.

*Proof.* The natural  $\mathrm{Sp}(1)$ -action on  $\mathbb{H}$  by multiplication on the right induces the standard  $\mathrm{Sp}(1)$ -action on the unit sphere of complex structures  $S_{IJK}$ . The conclusion follows from a choice of identification  $V \simeq \mathbb{H} \otimes_{\mathbb{R}} \mathbb{R}^n$  as  $\mathbb{H}$ -module.  $\square$

Hence a choice of orthonormal basis for  $(V, \mathbf{h})$  (as a left  $\mathbb{H}$ -module) yields an identification

$$\mathrm{Hk}(V) = \mathrm{Sp}(n) \cdot \mathrm{Sp}(1) \simeq (\mathrm{Sp}(n) \times \mathrm{Sp}(1)) / \mathbb{Z}/2\mathbb{Z},$$

a particular case of (4). In the notation of the introduction we can thus choose

$$\mathrm{Sp}'_0(V) := \mathrm{Sp}(V, \mathbf{h}) \subseteq \mathrm{Hk}(V).$$

3.1.1. *Geometric quantisation.* Geometric quantisation on a Kähler vector space is straightforward and essentially unique up to the choice of a symplectic potential, which corresponds to a gauge choice on the pre-quantum line bundle. For  $\hbar \in \mathbb{R}_{>0}$  one considers the triple  $(L, \mathfrak{h}, \nabla_q)$ , consisting of the trivial complex line bundle  $L := V \times \mathbb{C} \rightarrow V$  with the tautological Hermitian metric  $\mathfrak{h}$ , and the connection  $\nabla_q := d - \frac{i}{\hbar} \theta_q$  defined by the invariant symplectic potential

$$\theta_q(v)(X) = \frac{1}{2} \omega_q(v, X),$$

for  $v \in V$  a point and  $X$  a tangent vector there. The above yields pre-quantum data for  $(V, \omega_q)$  at level  $\hbar^{-1}$ . We may denote  $L_q \rightarrow V$  the line bundle to emphasize the structure we are prequantising on  $V$ .

The bundle  $L_q$  comes endowed with a natural holomorphic frame

$$\psi_0(q, v) := \exp\left(-\frac{1}{4\hbar} g(v, v)\right),$$

which is manifestly independent of  $q \in \mathbb{C}P^1$ . For each  $q$ , the resulting quantum Hilbert space consists of sections  $\psi = f\psi_0$ , with  $f: V \rightarrow \mathbb{C}$  an  $I_q$ -holomorphic function with finite  $L^2$ -norm with respect to the Gaussian measure. This space is well known to be densely generated by the polynomial functions, which induces a grading on each  $\mathcal{H}_q$ —the Fock grading.

This setting is a particular case of the one discussed in § 2.9. Indeed, on a Kähler vector space, the function  $\mu(v) = \frac{1}{2} \|v\|^2$  is a moment map for the  $U(1)$ -action by scalar multiplication and a Kähler potential, and moreover

$$-\frac{i}{2}(\partial - \bar{\partial})\mu = \theta$$

is the invariant symplectic potential. Additionally, for each  $q \in \mathbb{C}P^1$  the action of  $T_q$  is the standard one.

Furthermore  $d$ -homogeneous holomorphic functions on a complex vector space are  $d$ -homogeneous polynomials, whence the decomposition of  $\mathcal{H}_q$  into isotypical components as a  $T_q$ -module reduces to the well-known Fock grading. By the identification of the space of such homogeneous polynomials with  $\text{Sym}^d V_q^\vee$ , the finite-dimensional spaces  $\mathcal{H}_q^{(d)}$  assemble into finite-rank Hermitian sub-bundles  $\mathcal{H}^{(d)} \rightarrow \mathbb{C}P^1$  of the trivial  $L^2(V, L)$ -bundle, with a natural isomorphism

$$\text{Sym}^d Z^\vee \longrightarrow \mathcal{H}^{(d)}$$

of vector bundles over the Riemann sphere.

3.1.2. *Group action on quantum spaces.* The action  $\rho^Z: \text{Hk}(V) \rightarrow \text{Aut}(Z)$  has a natural lift to  $L = Z \times \mathbb{C}$  as  $\rho^Z \times \text{Id}$ . Since  $A^*\theta_q = \theta_{A,q}$  for  $A \in \text{Hk}(V)$  and  $q \in \mathbb{C}P^1$ , it follows that this action preserves the structure of  $L$  as a family of pre-quantum line bundles. This defines an action  $\rho^{\mathcal{H}}$  on sections of  $\mathcal{H}^{(d)}$  by pull-back, as in (10), and it is easy to check this is a graded fibrewise unitary  $\text{Hk}$ -action—covering that on hyper-Kähler 2-sphere.

**Theorem 3.1.** *For  $q \in \mathbb{C}P^1$  there is a canonical isomorphism  $\mathcal{H}_q^{(d)} \simeq \text{Sym}^d(V)$  of simple  $\text{Sp}(V, \mathfrak{h})$ -modules, and the bundle with connection  $(\mathcal{H}^{(d)}, \nabla^{\mathcal{H}^{(d)}})$  is  $\text{Hk}(V)$ -equivariantly isomorphic to  $\mathcal{L}^d \otimes \text{Sym}^d(V) \rightarrow \mathbb{C}P^1$ .*

*Proof.* This follows directly from the above discussion and from Thm. 2.1: The metric  $g$ , and hence the section  $\psi_0$ , are fixed by  $\mathrm{Sp}(V, \mathfrak{h})$ . It is known the natural action on  $\mathrm{Sym}^d V_q^\vee$  is irreducible [80].  $\square$

Altogether the statements of this sections establish the assumptions needed to apply Thm. 2.2, which in this particular case yields the following.

**Theorem 3.2** (cf. Thm. 2.2). *The  $\mathrm{Sp}(1)$ -symmetric geometric quantisation of the hyper-Kähler vector space  $V$  yields the super Hilbert space*

$$H = \overline{\bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^{(d)}},$$

analogously to § 2.6. This carries a unitary  $\mathrm{Hk}(V)$ -representation preserving the splitting, and there is an isomorphism  $H^{(d)} \simeq W^{(d)} \otimes \mathrm{Sym}^d(V)$  of simple  $\mathrm{Hk}(V)$ -modules. For every  $d \geq 0$  we thus have

$$\dim(W_+^{(d)}) = (d+1), \quad \dim(W_-^{(d)}) = 0, \quad \dim(H^{(d)}) = (d+1) \binom{2n+d}{d}.$$

The generating series (21) and (22) are obtained explicitly from the above

$$H(t) = \frac{1}{(1-t)^{2n}}, \quad H'(t, \tilde{t}) = \frac{1}{\prod_{i=1}^n (1 - t\tilde{t}_i)(1 - t\tilde{t}_i^{-1})}.$$

On the other hand, since  $V_q \simeq \mathbb{C}^{2n}$  is a Stein space, and since the actions of  $T_q$  and  $T'_q$  only fix the origin, Thm. 2.6 also applies, and the result from (24) and (25) yields the same formulæ. Now by Thm. 3.2 we see that  $m_{\mathrm{Sym}^d(V)}^{(d)} = 1$  for  $d \in \mathbb{Z}_{\geq 0}$ , whence

$$G(t, \tilde{t}) = \sum_{d=0}^{\infty} t^d \tilde{t}^{\lambda_{\mathrm{Sym}^d(V)}}.$$

**3.2. Four-dimensional examples.** As mentioned in the introduction, in dimension 4 there is a complete classification of  $\mathrm{Sp}(1)$ -symmetric hyper-Kähler manifolds up to finite quotients: Besides  $\mathbb{H}$  with its flat metric there are the Taub–NUT metrics on either  $\mathbb{R} \times \mathbb{S}^3$  or  $\mathbb{R}^4$ , and then the hyper-Kähler metric on the moduli space of charge-2 monopoles, the Atiyah–Hitchin manifold  $M_{\mathrm{AH}}$ .

**3.2.1. Taub–NUT metrics.** Consider the case of  $M = \mathbb{R}^4$  with the Taub–NUT metric  $g^a$  corresponding to a positive real parameter  $a$ —the case  $a = 0$  corresponds to the standard flat metric on  $\mathbb{H}$ , which we already discussed. We will denote  $\omega_q^a$  the corresponding symplectic structures. It is well-known (e.g. [40, Rem. 1]) that

$$\mathrm{Hk}(M) \simeq (\mathrm{Sp}(1) \times \mathrm{U}(1))/\mathbb{Z}_2 \simeq \mathrm{U}(2).$$

In particular there is a faithful  $\mathrm{Sp}(1)$ -action rotating the sphere of hyper-Kähler structures, while  $\mathrm{Sp}(M) = \mathrm{U}(1)$  is compact and commutes with  $\mathrm{Sp}(1)$ , so we may choose  $\mathrm{Sp}_0(M) = \mathrm{U}(1)$ , satisfying all the assumptions from § 2. Furthermore there exists—up to isomorphism—a unique family of pre-quantum line bundles for  $M$ , since  $H^2(M, \mathbb{Z}) = 0 = H^1(M, \mathrm{U}(1))$ .

The action of  $T'_q = (\mathrm{U}(1) \times \mathrm{U}(1))/\mathbb{Z}_2 \subseteq \mathrm{Hk}(M)$  is studied explicitly in [40, § 3.2] for the complex structure  $J_+$  corresponding to  $q = i$ . The subgroup is identified in that context with  $\mathrm{U}(1) \times \mathrm{U}(1)$  via the isomorphism  $(t, s) \mapsto (ts, ts^{-1})$ .

From Eqq. (3.10) and (3.19) of op. cit. one concludes the action of  $T_q = U(1) \times \{1\}$  on  $M_q$  is Hamiltonian with moment map  $\mu_q = \mu_1^+ + \mu_2^+$  (borrowing the notation from Gauduchon), which is easily seen to be proper from the definitions. Finally Prop. 1 (ibidem) provides a biholomorphism  $\Phi_+^q = \Phi: (M, J_+) \rightarrow \mathbb{C}^2$ , and by a straightforward check this map intertwines the  $T_q$ -action on  $M_q$  with the standard  $U(1)$ -action on  $\mathbb{C}^2$ . In particular the  $T_q$ -action extends holomorphically to  $\mathbb{C}^*$ , and the hypotheses of Thm. 2.4 are verified.

Thus decomposing  $\mathcal{H}_q$  with respect to the  $T_q$ -action yields

$$\mathcal{H}_q = \overline{\bigoplus_{d \in \mathbb{Z}_{\geq 0}} \mathcal{H}_q^{(d)}},$$

where the subspaces  $\mathcal{H}_q^{(d)} \subseteq \mathcal{H}_q$  are finite-dimensional. Then we consider the action of the commuting compact group  $\text{Sp}_0(M) = \{1\} \times U(1)$  on  $\mathcal{H}_q^{(d)}$  to refine:

$$\mathcal{H}_q^{(d)} = \bigoplus_{d' \in \Lambda^{(d)}} \mathcal{H}_{d',q}^{(d)}$$

where  $\Lambda^{(d)} \subseteq \mathbb{Z}_{\geq 0}$  is finite. But more can be proven, as in the following statement.

**Proposition 3.1.** *For  $q = i$ , the pre-quantum line bundle  $L_q$  admits a  $T_q$ -invariant holomorphic frame  $\psi_q$  such that  $\Phi^*(f) \cdot \psi_q$  is  $L^2$  for every polynomial function  $f$  on  $\mathbb{C}^2$ .*

*Proof.* Recall that, again in the notations of [40],  $x_1, x_2$ , and  $x_3$  are three real-valued functions on  $M$  whose span is preserved by  $\text{Sp}(1)$ , which acts on them by rotations in the standard way. Furthermore, all three functions are fixed by the action of  $U(1) = \text{Sp}(M)$ . Writing  $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$ , it follows from Eqq. (3.19), (2.11), and (2.12) (ibidem) that the aforementioned moment map  $\mu$  can be expressed as

$$\mu = r + a^2(x_2^2 + x_3^2).$$

Since  $\mu_i := \mu$  is a moment map for  $T_i$  with respect to  $\omega_i^a$ , it follows that for every  $g \in \text{Sp}(1)$  the function  $(g^{-1})^*\mu$  generates the  $T_{g,i}$ -action with respect to  $\omega_{g,i}^a$ . In particular, if  $g.i = j$ , then the flow associated to

$$\mu_j := (g^{-1})^*\mu = r + a^2(x_1^2 + x_3^2)$$

with respect to  $\omega_j^a$  rotates the circle spanned by  $\omega_3^a$  and  $\omega_1^a$ . It is therefore a Kähler potential for  $\omega_i^a$  [53], and repeating the argument when  $g.i = k$  so is  $\mu_k := r + a^2(x_1^2 + x_2^2)$ . Therefore the  $T_i$ -invariant function

$$\varphi = \varphi_i := \frac{\mu_j + \mu_k}{2} = r + \frac{a^2}{2}(r^2 + x_1^2)$$

is also a Kähler potential. It follows that for every  $g \in \text{Sp}(1)$  the function  $(g^{-1})^*\varphi_i$  is completely determined by  $q = g.i$ , so that

$$\varphi_q := (g^{-1})^*\varphi_i$$

is well-defined, and a potential for  $\omega_q^a$ . From this we obtain an explicit realisation of the family of pre-quantum line bundles, for which the functions  $\psi_{0,q} = e^{-\frac{1}{2\hbar}\varphi_q}$  define holomorphic frames.

Now note the function  $\mu$  is bounded below, and its only critical point is the origin—the only fixed point of the induced action. We may then apply the Duistermaat–Heckman theorem [37] as in Prop. 2.3 to conclude that  $e^{-\alpha\mu}$  is integrable with respect to the Taub-NUT volume  $d\text{vol}^\alpha$  for every parameter  $\alpha \in \mathbb{R}_{>0}$ . The same clearly applies to  $\mu_j$  and  $\mu_k$ , and from this it is easily deduced that

$$e^{-\alpha\varphi_q} \in L^2(M, d\text{vol}^\alpha)$$

for every  $q \in \mathbb{C}\mathbb{P}^1$  and  $\alpha > 0$ . In particular, the holomorphic frames constructed above are  $L^2$ .

To conclude we recall that the two components of  $\Phi$  are defined as

$$w_1 = e^{\alpha^2 x_1 z_1}, \quad w_2 = e^{-\alpha^2 x_2 z_2},$$

where  $z_1$  and  $z_2$  are the standard  $i$ -holomorphic coordinates on  $M = \mathbb{H}$  with respect to the usual flat metric (cf. [40, Eq. (3.4)]). We need to show that, for every  $n, m \in \mathbb{Z}$ , the function  $w_1^n w_2^m \psi_0$  is also  $L^2$ . Expanding the definition of  $\varphi$  yields

$$\begin{aligned} \frac{1}{\hbar} \varphi - 2\alpha^2(n-m)x_1 &= \frac{r}{\hbar} + \frac{\alpha^2}{2\hbar}(r^2 + x_1^2) - 2\alpha^2(n-m)x_1 \\ &\geq \frac{r}{2\hbar} + \frac{\alpha^2}{4\hbar}(r^2 + x_1^2) - C = \frac{1}{2\hbar} \varphi - C \geq \frac{1}{4\hbar} \mu_j - C \end{aligned}$$

provided  $C \in \mathbb{R}_{>0}$  is large enough. Furthermore, it is a simple consequence of the definitions in [40] that  $|z_1|^2 + |z_2|^2 = 2r$ , whence

$$|z_1^n z_2^m|^2 \leq (2r)^{2(n+m)} \leq \mu_j^{2(n+m)}.$$

Collecting the estimates and using again the Duistermaat–Heckman theorem we conclude:

$$\int_M |w_1^n w_2^m|^2 \psi_0^2 d\text{vol}^\alpha \leq e^C \int_M \mu_j^{2(m+n)} e^{-\frac{1}{4\hbar} \mu_j} d\text{vol}^\alpha < \infty.$$

□

As a consequence of this result we have  $\dim \mathcal{H}_{d'}^{(d)} = 1$  for all  $d \in \mathbb{Z}_{\geq 0}$  and  $d' = d - 2j$  with  $j \in \{0, \dots, d\}$ , and we conclude that  $\mathcal{H}_{d'}^{(d)} \simeq \mathcal{L}^d$  for such values of  $d$  and  $d'$ .

**Theorem 3.3.** *The generating series (22) is the same as for the flat metric, namely*

$$H'(\mathfrak{t}, \tilde{\mathfrak{t}}) = \frac{1}{(1 - \mathfrak{t}\tilde{\mathfrak{t}})(1 - \mathfrak{t}\tilde{\mathfrak{t}}^{-1})}.$$

We thus have

$$H = \overline{\bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^{(d)}}, \quad H^{(d)} = \bigoplus_{d' \in \Lambda^{(d)}} H_{d'}^{(d)} = H^0(\mathbb{C}\mathbb{P}^1, \mathcal{L}^d)^{\oplus (d+1)}.$$

**3.2.2. The Atiyah–Hitchin manifold.** Let us consider the Atiyah–Hitchin manifold  $M_{\text{AH}}$ , the last four-dimensional case: We shall discuss the extent to which our methods apply here.

The Atiyah–Hitchin manifold can be realised as the moduli space of charge-2 centred magnetic monopoles in  $\mathbb{R}^3$ , and it comes with a natural Riemannian metric preserved by the  $\text{SO}(3)$ -action induced by rotating monopoles. The quaternionic nature of the Bogomolny equation, of which the monopoles represented

by  $M_{\text{AH}}$  are a particular class of solutions, induces a family of almost complex structures, which can be better understood via Donaldson's description in terms of rational maps [33]. More precisely, the choice of an oriented direction (affine line) in  $\mathbb{R}^3$  induces an identification

$$\widetilde{M}_{\text{AH}} = \left\{ S(z) = \frac{uz+v}{z^2-w} \in \mathbb{C}(z) \mid v^2 - wu^2 = 1 \right\} =: \mathbb{R}_2^0,$$

where the left-hand side denotes the (two-fold) universal cover of  $M_{\text{AH}}$ . The Atiyah–Hitchin manifold is recovered from the monodromy action, generated by  $(u, v, w) \mapsto (-u, -v, w)$ . The resulting map is a biholomorphism with respect to one of the aforementioned almost complex structures, establishing that the latter is integrable and the former is Kähler. Rotations around the preferred direction induce a  $U(1)$ -action of  $\mathbb{R}_2^0$  by

$$t.(u, v, w) = (tu, v, t^{-2}w). \quad (28)$$

As the preferred direction changes across all possible choices, this results in a family of Kähler structures parametrised by  $\mathbb{C}P^1$ , which is clearly rotated by the  $SO(3)$ -action (see [16, Ch. 2]).

The above identification is not isometric with respect to the Riemannian embedding  $\mathbb{R}_2^0 \subseteq \mathbb{C}^3$ ; nonetheless, the Riemannian structure on  $M_{\text{AH}}$  can be described by studying the  $SO(3)$ -orbits [16, Chs. 8-11]. The generic stabiliser of a monopole is the Klein four-group  $K_4$ , while orbits are parametrised by  $k = \sin(\alpha)$  for an angle  $\alpha \in [0, \frac{\pi}{2}]$ , resulting in a description of an open dense of  $M_{\text{AH}}$  as the product  $(0, 1) \times SO(3)/K_4$ ; further as  $k \rightarrow 0$  the orbit degenerates to a diffeomorphic copy of  $\mathbb{R}P^2$ , onto which  $M_{\text{AH}}$  deformation-retracts.

Swann's work [86] provides an obstruction to the existence of a hyper-Kähler potential. Furthermore, one sees from (28) that the stabiliser of each Kähler structure has exactly one fixed point, and since the manifold has the homotopy type of  $\mathbb{R}P^2$  there can be no proper moment map. Nonetheless the above homotopy equivalence yields

$$H^1(M_{\text{AH}}, U(1)) \simeq H^2(M_{\text{AH}}, \mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z},$$

hence by § 2.2 there are exactly two inequivalent  $SO(3)$ -equivariant families of pre-quantum line bundles: They differ by a twist by a family of flat connections on the non-trivial complex line bundle on  $M_{\text{AH}}$ .

The family supported on the trivial bundle can be constructed by means of the Kähler potentials of Olivier [78]. Namely the metric on the Atiyah–Hitchin manifold is the completion of

$$ds^2 = \frac{\beta^2 \gamma^2 \delta^2}{(4k^2(1-k^2)K^2)^2} dm^2 + \beta^2 \sigma_x^2 + \gamma^2 \sigma_y^2 + \delta^2 \sigma_z^2, \quad (29)$$

defined on  $(0, \frac{\pi}{2}) \times SO(3)/K_4$ . We follow the conventions of op. cit. Namely,  $m = k^2$  is used as a coordinate in place of  $k$ , while  $(\sigma_x, \sigma_y, \sigma_z)$  is an orthonormal frame of  $T^*SO(3) \rightarrow SO(3)$  and the coefficients  $\beta, \gamma, \delta \in \mathbb{R}$  are determined by

$$\beta\gamma = -EK, \quad \gamma\delta = -EK + K^2, \quad \beta\delta = -EK + (1-k^2)K^2,$$

where

$$K := K(k) = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}, \quad E := E(k) = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 \phi} d\phi$$

are the complete elliptic integrals of the first and second kind, respectively.

Op. cit. then uses the Euler angles  $(\varphi, \theta, \psi)$  as coordinates on  $SO(3)$  to give an explicit Kähler potential  $\Omega$  for one of the complex structures, say  $I_3$ , preserved by rotations in the angle  $\varphi$ . This is given in of [78, Eq. 55] and can be written explicitly using Eqq. (6), (24), (25) and (36) therein, getting the formula

$$\Omega = \frac{\beta\gamma + \gamma\delta + \delta\beta}{8} + \frac{1}{8} \left( \gamma\delta \sin^2 \theta \cos^2 \psi + \delta\beta \sin^2 \theta \sin^2 \psi + \gamma\beta \cos^2 \theta \right).$$

Note for  $k \in (0, 1)$  this function extends continuously to the whole of  $SO(3)$ , and the trigonometric functions of  $(\theta, \psi)$  descend to the projective space at  $k = 0$ ; hence the potential extends to the completion  $M_{\text{AH}}$ . Finally, we emphasise that this potential is independent of the variable  $\varphi$ , which is to say that it is invariant under the action of the  $I_3$ -stabiliser. It follows that  $\Omega$  defines an equivariant family of potentials under the  $SO(3)$ -action, whence an equivariant family of pre-quantum line bundles by the usual construction, together with a holomorphic frame  $\psi_0 = e^{-\frac{1}{2\hbar}\Omega}$  for  $I_3$ .

**Proposition 3.2.** *The function  $e^{-\alpha\Omega}$  is integrable on  $M_{\text{AH}}$  for  $\alpha \in \mathbb{R}_{>0}$ .*

*Proof.* From (29) we obtain the following expression for the volume form on (the complement of a negligible set in)  $M_{\text{AH}}$ :

$$d\text{vol} = \frac{\beta^2 \gamma^2 \delta^2}{4k^2(1-k^2)K^2} d\sigma_x d\sigma_y d\sigma_z.$$

We need to show that

$$\int_{(0,1) \times SO(3)} e^{-\alpha\Omega} \frac{\beta^2 \gamma^2 \delta^2}{4k^2(1-k^2)K^2} d\sigma_x d\sigma_y d\sigma_z < \infty.$$

Note that  $\beta\gamma \leq 0$ ,  $\gamma\delta \geq 0$ , and  $\beta\delta \leq 0$  yield

$$\Omega \geq \frac{\gamma\delta}{8}.$$

We may then use these bounds and the Fubini-Tonelli theorem to reduce the statement to

$$\int_0^1 e^{-\frac{\alpha}{8}\gamma\delta} \frac{\beta^2 \gamma^2 \delta^2}{k^2(1-k^2)K^2} dk < \infty.$$

We will proceed by studying the asymptotic behaviour of the integrand in the limit  $k \rightarrow 1$ —the integral is necessarily regular for  $k \rightarrow 0$ . It is well-known that

$$K \sim \frac{1}{2} \log(1 - k^2),$$

and since  $E(1) = 1$  we find that

$$\beta\gamma \sim -\frac{1}{2} \log(1 - k^2), \quad \gamma\delta \sim \frac{1}{4} \log^2(1 - k^2), \quad \beta\delta \sim -\frac{1}{2} \log(1 - k^2),$$

hence the integral converges by comparison with

$$\int_0^1 \exp\left(-\frac{\alpha}{32} \log^2(1-k^2)\right) \frac{\log^2(1-k^2)}{(1-k^2)} dr = \int_0^\infty e^{-\frac{\alpha}{32} x^2} x^2 dx < \infty.$$

□

For  $\alpha = 1/\hbar$  this implies the holomorphic frame  $\psi_0$  is  $L^2$ , hence an element of  $\mathcal{H}_{I_3}^{(0)}$ ; in principle more  $L^2$  holomorphic sections may be found considering functions of the holomorphic coordinates  $(u, v, w)$  on  $\mathbb{R}_0^2$ . If all the monomials that descend to  $M_{\text{AH}}$  are  $L^2$ , then one concludes that  $\mathcal{H}_q^{(d)}$  has infinite rank for every integer  $d$ , since  $u^a v^b w^c$  is  $(a-2c)$ -homogeneous and well-defined on  $M_{\text{AH}}$  if  $a+b$  is even: We obtain a partial result in this direction, showing that all powers of  $w$  are  $L^2$ .

The problem of describing  $u$ ,  $v$ , and  $w$  in terms of the setup above is addressed in [16, Chapters 6-7], by making use of the twistor description and spectral curves [56]. Introducing parameters

$$k_1 = \frac{\sqrt{k\sqrt{1-k^2}}K}{2}, \quad k_2 = \frac{1-2k^2}{3k\sqrt{1-k^2}},$$

consider the elliptic curve

$$y^2 = 4k_1^2(x^3 - 3k_2x^2 - x)$$

and let  $\wp, \zeta$  be its corresponding Weierstrass functions,  $\eta$  the real period of  $\zeta$ . Suppose that  $a, b \in \mathbb{C}$  are the entries of a matrix in  $SU(2)$ , thought of as a parametrisation of  $SO(3)/K_4$ , and let  $\xi \in \mathbb{C}$  be such that

$$\wp(\xi) = \frac{b}{a} - k_2. \quad (30)$$

Then the corresponding point in  $M_{\text{AH}}$  has holomorphic coordinates

$$\begin{aligned} u &= \frac{\sinh(2k_1\zeta(\xi) - \frac{\eta\xi}{2} + k_1\bar{a}\bar{b}\wp'(\xi))}{k_1\bar{a}^2\wp'(\xi)}, \\ v &= \cosh(2k_1\zeta(\xi) - \frac{\eta\xi}{2} + k_1\bar{a}\bar{b}\wp'(\xi)), \\ w &= k_1^2\bar{a}^4\wp'(\xi)^2, \end{aligned}$$

up to the sign ambiguity resulting from the monodromy. Substituting (30) in the differential equation for  $\wp$ , and using  $g_2$  and  $g_3$  as given in [56], we obtain

$$w = k_1^2\bar{a}(-12\bar{a}b^2k_2 + 4b^3 - 4\bar{a}^2b).$$

Now since  $|a|^2 + |b|^2 = 1$  a straightforward check shows that

$$|w|^2 \leq 16k_1^4(9k_2^2 + 2) \sim 4K^2 \sim \log^2(1-k^2),$$

for  $k \rightarrow 1$ . Adapting the proof of Prop. 3.2 and using (28) we obtain the following.

**Proposition 3.3.** *For every integer  $n \geq 0$  the holomorphic section  $w^n\psi_0$  is  $L^2$  and therefore an element of  $\mathcal{H}_{I_3}^{(-2n)}$ .*



The analysis is more delicate for the functions  $u$  and  $v$ . Using (30) one can express  $\overline{ab}$  in terms of  $\xi$ , and write the argument of the hyperbolic functions as

$$\Phi(\xi) = 2k_1\zeta(\xi) - \frac{\eta\xi}{2} + k_1\wp'(\xi) \frac{k_2 + \overline{\wp(\xi)}}{1 + |k_2 + \wp(\xi)|^2}.$$

It follows from the definitions and the Legendre relation that this function is periodic for the real period of  $\wp$  and quasi-periodic for the imaginary period, with step  $\pi i$ , whence the sign ambiguity of  $u$  and  $v$ . Moreover one can show the poles of the summands cancel out, leaving a non-holomorphic analytic function—hallmark of the fact that the  $SO(3)$ -action does not preserve the complex structure. In particular its real part is bounded for fixed “ $k$ ”.

**3.3. Moduli spaces of framed  $SU(r)$ -instantons.** Let  $r \geq 2$  and  $k \geq 0$  be integers, and consider the moduli space  $M_{k,r}$  of charge- $k$  framed  $SU(r)$ -instantons on  $\mathbb{R}^4$ , which is a hyper-Kähler manifold [14, 34]. Each of its complex structures can be described in terms of the ADHM construction as follows, after fixing an identification  $\mathbb{R}^4 \simeq \mathbb{C}^2$ . Consider the product

$$\mathbb{M} := \text{End}(\mathbb{C}^k)^2 \times \text{Hom}(\mathbb{C}^k, \mathbb{C}^r) \times \text{Hom}(\mathbb{C}^r, \mathbb{C}^k),$$

with  $GL(\mathbb{C}^k)$ -action given by

$$g \cdot (\alpha_0, \alpha_1, a, b) = (g\alpha_0g^{-1}, g\alpha_1g^{-1}, ga, bg^{-1}).^4$$

Let  $\mathbb{M}_0$  denote the set of elements of  $\mathbb{M}$  satisfying the additional conditions

- (i)  $[\alpha_0, \alpha_1] + ab = 0$ ;
- (ii) For all  $\lambda, \mu \in \mathbb{C}$ ,  $\begin{pmatrix} \alpha_0 + \lambda \\ \alpha_1 + \mu \\ a \end{pmatrix}$  is injective and  $\begin{pmatrix} \lambda - \alpha_0 & \alpha_1 - \mu & b \end{pmatrix}$  is surjective.

Then the restricted  $U(r)$ -action is Hamiltonian with moment map

$$\mu(\alpha_0, \alpha_1, a, b) := [\alpha_1, \alpha_1^*] + [\alpha_2, \alpha_2^*] + bb^* - a^*a,$$

and there is an identification

$$M_{k,r} \simeq \mathbb{M}_0 //_{\mu} U(k). \quad (31)$$

The rotation group  $SO(4)$  acts on  $M_{k,r}$ , and in particular the subgroup  $Sp(1)$ , in identification  $\mathbb{R}^4 \simeq \mathbb{H}$ , transitively permutes the complex structures. Furthermore, the work of Maciocia [68] shows that for each  $q \in \mathbb{C}P^1$  the  $T_q$ -action has moment map

$$m_2(A) = \frac{1}{16\pi^2} \int_{\mathbb{R}^4} \|x\|^2 \text{tr} F_{\lambda}^2.$$

This function is clearly  $Sp(1)$ -invariant, and therefore a hyper-Kähler potential, so one can construct an  $Sp(1)$ -invariant family of pre-quantum line bundle endowed with holomorphic frames as in § 2.9.

The function  $m_2$  is not, however, a proper map: By op. cit. it corresponds under the identification (31) to the norm-squared function  $f: \mathbb{M}_0 \rightarrow \mathbb{R}$ , which is

<sup>4</sup>Note  $\mathbb{M}$  is a space of representations of a quiver on two nodes—having a double loop-edge at one node, and a pair of opposite arrows between the two nodes—and that the action naturally extends to  $GL(\mathbb{C}^k) \times GL(\mathbb{C}^r)$  (which controls isomorphisms of representations).

$U(k)$ -invariant but not proper, on account of the open condition (ii). However Donaldson's work [34] identifies the symplectic reduction (31) with the GIT quotient of  $\mathbb{M}_0$  by  $GL(k, \mathbb{C})$ , whereupon (ii) translates into a stability condition. One may then include the semi-stable points to obtain a partial compactification

$$\overline{\mathbb{M}}_{k,r} := \mathbb{M} //_{\text{GIT}} GL(k, \mathbb{C}),$$

which is smooth by the work of Nakajima and Yoshioka [77, Cor. 2.2]. The map  $f$  descends then to a proper one on  $\overline{\mathbb{M}}_{k,r}$ ; it is also clear that its gradient is complete on the quotient, showing that geometric quantisation on this space yields finite-rank isotypical components by Thm. 2.4. On the other hand, the codimension of the boundary  $\overline{\mathbb{M}}_{k,r} \setminus M_{k,r}$  is greater than 2, so that Hartogs's theorem allows for the extension of holomorphic functions on  $M_{k,r}$ , which yields the finite-dimensionality of the isotypic components over this latter space.

#### 4. OUTLOOK AND FURTHER PERSPECTIVES

There are more spaces that fit some of the requirements for our quantisation scheme.

By the work of Kronheimer [62] the nilpotent (co)adjoint orbits of complex semisimple (1-connected) Lie groups are hyper-Kähler, and Swann's discussion [86] on such spaces implies in particular that they admit hyper-Kähler potentials. This is a particular instance of hyper-Kähler moduli spaces of solutions of Nahm's equations, specifically on a half-line with nilpotent boundary conditions.<sup>5</sup> Since Nahm's equations come naturally with a quaternionic structure and  $Sp(1)$ -action, the resulting manifolds have symmetries of the kind considered in this paper, and different choices of domain and boundary conditions give rise to different hyper-Kähler structures. For instance, semisimple boundary conditions on a half-line result in orbits of semisimple elements [63], while the study of Nahm's equations on a compact interval leads to the cotangent bundle  $T^*G$  [61, 32]. By the works of Mayrand [71, 70, 72], the latter comes with natural  $Sp(1)$ -equivariant families of Kähler potentials and moment maps for the stabilizers  $T_q$ , rather than a hyper-Kähler one, and they enjoy interesting properties that might lead to a variation of our main construction.

Also, as mentioned in the introduction, many new interesting hyper-Kähler metrics can be defined on moduli spaces of irregular connections/Higgs bundles over (wild generalisations of) Riemann surfaces [20, 92], with simple examples reviewed in [25]: the "multiplicative" versions of the Eguchi–Hanson space and Calabi's examples (whose standard "additive" versions are quiver varieties on two nodes). This fits into a more general (new) multiplicative theory of quiver varieties [24], involving a "fission" gluing operation generalising the TQFT construction of moduli spaces of flat connections [21, 23]; note that conjecturally this produces a lot more new hyper-Kähler manifolds [22], beyond (wild) nonabelian Hodge spaces.

Finally the example of § 3.3, i.e. the moduli spaces of framed  $SU(r)$ -instantons, opens the way for further discussion on the relation between the generating series produced by this new quantisation scheme and the well-known Nekrasov partition functions.

<sup>5</sup>The hyper-Kähler metric on general orbits was constructed in [19, 57].

## APPENDIX A. COMPARISON WITH THE STANDARD APPROACH

In this section we shall correct the family of quantum Hilbert spaces  $\mathcal{H}_q$  to obtain finite-rank flat vector bundles of isotypical components (under the main assumption), as well as unitary equivalences between the quantisation of  $M$  with respect to the given Kähler polarisations.

Based on Thm. 2.1, we do this by a correcting twist of the finite-rank bundles  $\mathcal{H}_\lambda^{(d)} \rightarrow \mathbb{C}P^1$ ; namely consider the tensor product

$$\tilde{\mathcal{H}}_\lambda^{(d)} := \mathcal{H}_\lambda^{(d)} \otimes \mathcal{L}^{-d}, \quad \text{for } d \in \mathbb{Z}, \lambda \in \Lambda^{(d)}.$$

This new vector bundle comes with a  $\text{Hk}_0'(M)$ -action, and we denote  $\nabla^{\tilde{\mathcal{H}}_\lambda^{(d)}}$  the resulting  $\text{Hk}_0'(M)$ -invariant flat connection.

Since  $\mathbb{C}P^1$  is simply-connected the parallel transport defines canonical unitary isomorphisms

$$\tilde{\mathcal{H}}_{\lambda,q}^{(d)} \longrightarrow \tilde{\mathcal{H}}_{\lambda,q'}^{(d)}, \quad \text{for } q, q' \in \mathbb{C}P^1, \quad (32)$$

satisfying 1-cocycle identities. Analogously to the above we then define

$$\overline{\bigoplus_{\lambda \in \Lambda^{(d)}} \tilde{\mathcal{H}}_{\lambda,q}^{(d)}} := \tilde{\mathcal{H}}_q^{(d)} \subseteq \tilde{\mathcal{H}}_q := \overline{\bigoplus_{d \in \mathbb{Z}} \tilde{\mathcal{H}}_q^{(d)}},$$

and these families of Hilbert spaces carry a 1-cocycle of unitary isomorphisms induced from (32): This is the usual geometric quantisation construction.

Now we can introduce super Hilbert spaces  $\tilde{\mathbb{H}}_{\lambda,j}^{(d)}$  analogously to § 2.6, taking the holomorphic cohomology of the twisted vector bundles  $\tilde{\mathcal{H}}_\lambda^{(d)} \rightarrow \mathbb{C}P^1$ .

**Theorem A.1** (cf. Thm. 2.2). *There is a unitary action  $\text{Hk}_0'(M) \rightarrow \text{U}(\tilde{\mathbb{H}})$  preserving the nested splittings:*

$$\tilde{\mathbb{H}} := \overline{\bigoplus_{d \in \mathbb{Z}} \tilde{\mathbb{H}}^{(d)}}, \quad \tilde{\mathbb{H}}^{(d)} := \overline{\bigoplus_{\lambda \in \Lambda^{(d)}} \tilde{\mathbb{H}}_\lambda^{(d)}}, \quad \tilde{\mathbb{H}}_\lambda^{(d)} := \bigoplus_{j=1}^{m_\lambda^{(d)}} \tilde{\mathbb{H}}_{\lambda,j}^{(d)}.$$

Finally we can compare this representation with the one constructed in § 2.6, finding that twisting trivializes part of the action: Namely, the present super Hilbert space  $\tilde{\mathbb{H}}_{\lambda,j}^{(d)} \simeq V_\lambda \otimes W^{(0)}$  replaces the original  $\mathbb{H}_{\lambda,j}^{(d)} \simeq V_\lambda \otimes W^{(d)}$  as a  $\text{Hk}_0'(M)$ -module—recalling that  $W^{(0)}$  is the trivial one-dimensional  $\text{Sp}(1)$ -module. This should be compared with the (more) interesting irreducible representations of  $\text{Hk}_0'(M)$  obtained from the main Thm. 2.2.

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