



**HAL**  
open science

## Local wild mapping class groups and cabled braids

Jean Douçot, Gabriele Rembado, Matteo Tamiozzo

► **To cite this version:**

Jean Douçot, Gabriele Rembado, Matteo Tamiozzo. Local wild mapping class groups and cabled braids. 2024. hal-03748600v2

**HAL Id: hal-03748600**

**<https://hal.science/hal-03748600v2>**

Preprint submitted on 12 Jun 2024

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# LOCAL WILD MAPPING CLASS GROUPS AND CABLED BRAIDS

JEAN DOUÇOT, GABRIELE REMBADO, AND MATTEO TAMIOZZO

**ABSTRACT.** We define and study some generalisations of pure  $\mathfrak{g}$ -braid groups, for any complex reductive Lie algebra  $\mathfrak{g}$ . They naturally occur in the theory of isomonodromic deformations for meromorphic connections with irregular singularities on principal bundles over Riemann surfaces, covering the general untwisted case, going beyond the case of generic irregular types. These generalised braid groups make up local pieces of the wild mapping class groups, which in turn extend the usual mapping class groups and govern the braiding of Stokes data.

We establish a general product decomposition for these local wild mapping class groups, and in all classical cases define a fission tree governing the decomposition; in particular in type D we will find a factor which is not isomorphic to any pure braid group coming from a root system. In type A, the fission tree and the pure braid group operad yields a proof of the corresponding “multi-scale” braiding conjecture.

## CONTENTS

1. Introduction	2
2. Admissible deformations of wild Riemann surfaces	10
3. Pure local wild mapping class groups	14
4. Filtrations and fission	15
5. General results	18
6. Type A	20
7. Type B/C	24
8. Type D	27
9. Pure cabled braid groups	30
10. Outlook	33
Appendix A. Some notions/notations we use	33
Appendix B. Some remarks about quantisation	34
References	35

---

2020 *Mathematics Subject Classification.* 14D22,14F35,17B22,32S22.

*Key words and phrases.* Braid groups, braid group operads, isomonodromic deformations, root systems, reflection groups, hyperplane arrangements, fundamental groups, moduli spaces.

J. D. is funded by FCIências.ID..

G. R. was supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany’s Excellence Strategy - GZ 2047/1, Projekt-ID 390685813; he is now funded by the European Commission under the grant agreement n. 101108575 (HORIZON-MSCA project [QuantMod](#)).

M. T. was funded by ERC grant 804176.

## 1. INTRODUCTION

**1.1. General aim.** In this article, we initiate a series of works with the aim of systematically studying the topology of the spaces of times for isomonodromic deformations of meromorphic connections with irregular singularities of any irregular type (encompassing in particular all the Painlevé equations).

Our main goal is to extend the classical mapping class group actions on character varieties, which may be seen as arising from isomonodromic deformations of meromorphic connections with at most *regular* singularities (i.e. basically simple poles) on principal  $G$ -bundles. We aim to develop an analogous theory in the setting of isomonodromic deformations of meromorphic connections with *irregular* singularities (i.e. higher-order poles), so as to obtain wild mapping class group actions on wild character varieties.

As a first step in this direction, we will consider in this paper *pure local* wild mapping class groups—defined in § 3—for *untwisted* irregular types. They turn out to yield generalisations of pure  $\mathfrak{g}$ -braid groups, and in many cases are controlled by the data of a *fission tree* which encodes the irregular type of the connection.

In this introduction, we will first explain the context and motivation of our work; we will then describe the structure of this paper and state our main results. Finally, we will illustrate some of these results in a concrete example.

**1.2. Monodromy actions in 2d gauge theory.** Let  $\Sigma$  be a compact Riemann surface,  $\mathfrak{a} \subseteq \Sigma$  a finite subset with complement  $\Sigma^\circ := \Sigma \setminus \mathfrak{a}$  and  $G$  a complex reductive group (for instance,  $GL_n(\mathbb{C})$ ). Much is known about the (tame) character variety

$$(1) \quad \mathcal{M}_{\mathbf{B}}(\Sigma, \mathfrak{a}) := \mathrm{Hom}(\pi_1(\Sigma^\circ), G) // G,$$

parametrising  $G$ -local systems on  $\Sigma^\circ$ . The Riemann-Hilbert correspondence [58], sending a meromorphic connection on a principal  $G$ -bundle over  $\Sigma$  with regular singularities at  $\mathfrak{a} \subseteq \Sigma$  to its monodromy representation, is a transcendental map relating the *de Rham* moduli space  $\mathcal{M}_{\mathrm{dR}}(\Sigma, \mathfrak{a})$  (parametrising such connections) and the *Betti* moduli space  $\mathcal{M}_{\mathbf{B}}(\Sigma, \mathfrak{a})$ . The character variety (1) is an algebraic Poisson variety, and the mapping class group of  $(\Sigma, \mathfrak{a})$  acts on it by algebraic Poisson automorphisms. In fact, for any smooth family  $\underline{\Sigma} \rightarrow \mathbf{B}$  of Riemann surfaces with marked points the character varieties of the fibres assemble into a fibration endowed with a flat Ehresmann connection (the nonabelian Gauß–Manin connection [89], [90]). The fundamental group of the base  $\mathbf{B}$  hence acts on the character variety (1) of any fibre, by algebraic Poisson automorphisms.

In recent years, several works have extended many features of 2d gauge theory to the case of connections with *irregular/wild* singularities, as advocated for instance in [72]. See (among others) [62, 70, 71, 74, 86]. This led to the discovery of new algebraic Poisson varieties generalising (1), called “wild” character varieties [17, 21, 26], parameterising refined monodromy data known as Stokes data [93, 12, 88]—cf. [24] for a modern (topological) view. Wild character varieties are endowed with algebraic Poisson automorphisms (coming for instance from  $\mathfrak{g}$ -braid groups [18, 21], the fundamental groups of root-hyperplane complements for the Lie algebra  $\mathfrak{g} = \mathrm{Lie}(G)$ <sup>1</sup>). As above, such automorphisms arise from the action of the fundamental group of suitable manifolds  $\mathbf{B}$  carrying families of wild character varieties. A key point, which we will explain in more detail below, is that a “new braiding” appears

<sup>1</sup>The “generalised” braid groups [31, 32, 41], i.e. the Artin(–Tits) groups of type  $\mathfrak{g}$  [94, 33] (cf. § A).

in the wild setting: namely, one has interesting group actions even for manifolds  $\mathbf{B}$  carrying constant families of pointed Riemann surfaces.

Part of this story was given a quantum field theory interpretation by Witten [102], and further the symplectic leaves of these Poisson varieties were shown to be new (complete) hyperkähler manifolds in the Biquard–Boalch extension of the nonabelian Hodge correspondence on Riemann surfaces [11]; see § B for further discussion.

The viewpoint discussed above subsumes many previous examples of monodromy actions in 2d gauge theory, including:

- (1)  $\mathbf{B} = \text{Conf}_m(\mathbb{C})$ , the genus-zero tame case, leading to the action of the pure braid group  $\text{PB}_m$  on the (tame) character variety. This is the first example to have been investigated, dating back to Hurwitz’s work [60];
- (2)  $\mathbf{B} = \mathcal{M}_g$ , the nonsingular genus- $g$  case, leading to the action of the mapping class group  $\Gamma_g$  of  $\Sigma$  (cf. [13] for the relation with braid groups);
- (3) and  $\mathbf{B} = \mathfrak{t}_{\text{reg}}$ , related to irregular connections with a “generic” fixed pole of order two in genus zero,<sup>2</sup> where  $\mathfrak{t}_{\text{reg}}$  is the regular part of a Cartan subalgebra  $\mathfrak{t} \subseteq \mathfrak{g}$ . This leads to the action of the pure  $\mathfrak{g}$ -braid group  $\text{PB}_{\mathfrak{g}}$  on  $G^*$  [18] (viz. the semiclassical action of the quantum Weyl group of De Concini–Kac–Procesi [40]).

The latter case is a first instance of the new braiding mentioned above, having to do with the structure group of the connections (e.g. if  $G = \text{GL}_n(\mathbb{C})$  we can deform the noncoalescing eigenvalues of an  $n$ -by- $n$  diagonal matrix) and not with the motion of poles on the surface, nor with the deformation of the complex structure of the surface. In this simplest generic example, one can in fact switch to the classical tame case, with varying points on  $\Sigma$ , via the Fourier–Laplace transform (a.k.a. Harnad’s duality [55], cf. [20]); however, this is not possible in general.

Most of the current paper is concerned with studying fundamental groups of spaces carrying families of wild character varieties related to non-generic connections. We will show that going beyond the generic case leads to new “multi-scale” braiding phenomena, envisaged by previous authors over the last decade (see for example the conjectures in Ramis’ talk [80], slides n. 148/196). This phenomenon is first visible when studying deformations of non-generic connections with poles of order three, as in [20], to which we will come back below.

*Wild Riemann surfaces.* The starting point are Boalch’s “wild” Riemann surfaces [21, Def. 8.1], generalising Riemann surfaces with distinct marked points (cf. § 2). Wild Riemann surfaces are endowed with “irregular types” at the marked points, describing the meromorphic part of a connection. More precisely, recall that locally around a point  $a \in \Sigma$  a meromorphic connection is encoded by a  $\mathfrak{g}$ -valued meromorphic 1-form  $\mathcal{A}$  on  $\Sigma$ , where  $\mathfrak{g}$  is the Lie algebra of  $G$ . We assume that up to a local gauge transformation and holomorphic terms one has

$$(2) \quad \mathcal{A} = dQ + \frac{\wedge}{z} dz,$$

---

<sup>2</sup>Generic isomonodromic deformations involve regular semisimple leading terms; in type A, these are diagonal matrices with distinct entries. See [6, 62, 70], developing the subject started in [12], and cf. Ex. 3.2.

where  $z$  is a local coordinate with  $z(a) = 0$ , and where

$$\Lambda \in \mathfrak{g}, \quad Q = \sum_{i=1}^p A_i z^{-i} \in z^{-1}\mathfrak{t}[z^{-1}],$$

for a Cartan subalgebra  $\mathfrak{t} \subseteq \mathfrak{g}$  and an integer  $p \geq 1$ . Then  $Q$  is the irregular type at the point  $a \in \Sigma$  (and it corresponds to “very good” orbits [23]). It is worth emphasising here that, throughout this paper, we only consider untwisted/unramified irregular types (their underlying irregular classes, possibly twisted, have now been considered in the subsequent works [44], [25]).

Furthermore, there is a natural notion of an “admissible family” of wild Riemann surfaces (cf. Def. 2.2), roughly ensuring that the irregular types on each fibre are such that the Stokes data have the “same shape”. Deforming irregular types subject to this admissibility condition, one obtains spaces whose fundamental groups are responsible for the genuinely new actions on wild character varieties. Indeed, one main result of [21] (extending those obtained in the generic case in [17, 18, 19]) shows that any admissible family of wild Riemann surfaces over a base space  $\mathbf{B}$  determines a (nonlinear) fibre bundle  $\underline{\mathcal{M}}_{\mathbf{B}} \rightarrow \mathbf{B}$  of Poisson wild character varieties, equipped with a complete flat Ehresmann connection (the *wild* nonabelian Gauß–Manin connection [17], in the wild analogue of the “symplectic nature” of  $\pi_1(\Sigma^\circ)$  [53], cf. [42]). In turn the fundamental group  $\pi_1(\mathbf{B})$  acts by algebraic Poisson automorphisms on any fibre.

*Goal of the paper.* The discussion in the previous paragraph involves the choice of an admissible family of wild Riemann surfaces over a base  $\mathbf{B}$ . If one wants to focus on the new phenomena arising from the variation of irregular types, a natural choice consists in taking a constant family of pointed Riemann surfaces, and deforming only the irregular types at the marked points. It turns out that such admissible families are parameterised by a universal deformation space (cf. § 2), whose fundamental group we call a *local wild* mapping class group. The main aim of this paper is to give a description of local wild mapping class groups which on the one hand works for arbitrary reductive groups (and for deformations of non-generic irregular types) and on the other hand is concrete and explicit enough to be useful for studying the action on wild character varieties (cf. Ex. 9.1). In particular, we will study the aforementioned “braiding of braids”, cf. Thm. 4 and the discussion preceding it.

**1.3. Main results and layout of the paper.** Let us now outline the structure and main results of this paper (we refer the reader to the body of the document for more precise statements). Let  $(\mathfrak{g}, \mathfrak{t})$  be a finite-dimensional split reductive Lie algebra defined over  $\mathbb{C}$ , and let  $\Sigma = (\Sigma, a, Q)$  be a one-pointed<sup>3</sup> wild Riemann surface.

In § 2 we define a universal space of admissible deformations  $\mathbf{B}_Q$  of the irregular type  $Q$  (keeping the underlying pointed Riemann surface  $(\Sigma, a)$  fixed), cf. Def. 2.3. We then define the (pure) *local wild mapping class group* (WMCG, cf. [22, § 8])  $\Gamma_Q$  to be the fundamental group of the space of admissible deformations, cf. § 3. The notion of admissible deformation depends on the root system  $\Phi_{\mathfrak{g}} = \Phi(\mathfrak{g}, \mathfrak{t})$ , and on the pole orders  $d_\alpha \in \{0, \dots, p\}$  of the meromorphic function germ  $q_\alpha = \alpha \circ Q$ ,

<sup>3</sup>The many-point case amounts to repeating the present discussion independently at each marked point, cf. Rem. 2.6.

for all  $\alpha \in \Phi_{\mathfrak{g}}$ : for a deformation to be admissible, one asks that the pole orders after evaluation at each  $\alpha$  remain constantly equal to  $d_{\alpha}$ . One of the main aims of this paper is to describe  $\Gamma_Q$ —which in principle depends on  $Q$ ,  $p$  and the integers  $d_{\alpha}$ —in terms of the couple  $(\Phi_{\mathfrak{g}}, \mathfrak{t})$ .

We will see (cf. (14)) that  $\Gamma_Q$  breaks into a product

$$(3) \quad \Gamma_Q \simeq \prod_{i=1}^p \pi_1(\mathbf{B}_i, \mathbf{A}_i),$$

of fundamental groups of certain hyperplane complements arising as deformation spaces of each of the coefficients of  $Q$ . This simple observation will be the starting point to construct, in § 4, an increasing filtration of root subsystems of  $\Phi_{\mathfrak{g}}$ , obtained by *fission*, which will be our key tool to describe  $\Gamma_Q$ . In the generic case our filtration is trivial and one obtains the pure  $\mathfrak{g}$ -braid group; in general, we find a generalisation thereof controlled by a nested sequence of (Levi) root subsystems. See § 1.4 for a concrete illustration of this phenomenon.

In § 5 we prove some general properties of local wild mapping class groups. First we explain how their description can be reduced to the case of simple Lie algebras, cf. § 5.2. Secondly, we obtain a uniform bound on the number of nontrivial factors of the local WMCG in (3).

**Theorem 1** (§ 5.4). *The number of nontrivial factors of (3) is at most the semisimple rank of  $\mathfrak{g}$  (i.e., the rank of  $\Phi_{\mathfrak{g}}$ ).*

Note that the bound in the previous theorem is independent of the order  $p$  of the pole of the irregular type, and only depends on  $\mathfrak{g}$ . This bound—which is not evident a priori—rests on the description of the local WMCG via fission, and is a first piece of evidence for the significance of fission in the description of the structure of  $\Gamma_Q$ .

Finally, as another application of fission we classify local WMCGs for low-rank Lie algebras.

**Theorem 2** (§ 5.5).

- *If the semisimple rank of  $\mathfrak{g}$  is one, then the local WMCG is either trivial or infinite cyclic (i.e., isomorphic to the pure  $\mathfrak{g}$ -braid group).*
- *If the semisimple rank of  $\mathfrak{g}$  is two, then the local WMCG is either trivial or isomorphic to one of the groups  $\mathbb{Z}$ ,  $\mathbb{Z}^2$ , or the pure  $\mathfrak{g}$ -braid group.*

In particular this classifies the local WMCGs for the exceptional simple Lie algebra of type  $G_2$ , while from § 6 we focus on *classical* simple Lie algebras, and give a complete explicit description of the local WMCG.

Beginning with type A, we attach a tree to any sequence of root subsystems obtained from fission, which we thus call a *fission tree* (see Def. 6.2 and cf. [16, App. C]). An example of this construction is given in § 1.4, to which we refer the reader for an illustration of the next theorem. Similarly, in § 7, we attach a *bichromatic tree* to any irregular type of type B/C (cf. Def. 7.1); and finally a generalisation thereof in type D, in § 8 (cf. Def. 8.1). This leads to the following description of local wild mapping class groups, for *all* classical simple Lie algebras.

**Theorem 3** (Thmm. 6.2, 7.2 and 8.2). *The generalised fission tree uniquely determines the local WMCG, as follows: at each node of the tree one attaches the pure braid group of an explicit hyperplane arrangement, and the local WMCG is the product of those factors.*

Let us point out that for types A and B/C all factors correspond to root-hyperplane arrangements, while for type D there is an “exotic” factor (which is *not* crystallographic) further studied in Prop. 8.1. (The simplest example involves seven hyperplanes in  $\mathbb{C}^3$ .)

Finally, in § 9 we relate the (monochromatic) fission trees of type A to *cabled* braids—as in the title. This formalises the driving idea that local wild mapping class groups ought to be described in terms of multi-scale braiding, cf. the example of § 1.4. More precisely, to any tree  $\mathcal{T}$  we attach a pure *cabled* braid group  $\mathcal{PB}(\mathcal{T})$  by using the compositions of the pure braid group operad (cf. Def. 9.1), and we prove the following.

**Theorem 4** (Thm. 9.1). *If  $\mathcal{T}$  is the fission tree of a type-A irregular type  $Q$ , then there is a group isomorphism  $\Gamma_Q \simeq \mathcal{PB}(\mathcal{T})$ .*

To sum up, our results show that  $\Gamma_Q$  is obtained via the following steps (the last one being performed via operads only in type A; a conjectural generalisation is given in Conj. 9.1):

$$\mathfrak{g} \xrightarrow{\text{fission}} \Phi_1 \subseteq \cdots \subseteq \Phi_{p+1} = \Phi_{\mathfrak{g}} \xrightarrow{\text{fission tree}} \mathcal{T} \xrightarrow{\text{braid cabling}} \mathcal{PB}(\mathcal{T}).$$

Let us finally note that the use of fission trees goes much beyond a description of pure local wild mapping class groups: they will also be used to study the local full/nonpure case [44], also twisted [25], and they have applications beyond the local case (see particularly §§ 3.7 and 5 of [25]).

*Conventions.* All Lie algebras, commutative (associative, unitary) algebras, and tensor products are defined over  $\mathbb{C}$ : a few more basic notions/notations, used throughout the body of the paper, are summarised in App. A.

**1.4. An example in type A.** To conclude our introduction, we would like to work out explicitly a simple example in type A illustrating the main general phenomena which will be studied later. We hope that this toy model can serve both as a motivation and as a test case for the general results proved in the paper.

Fix a pointed Riemann surface  $(\Sigma, a)$ , and choose a local coordinate  $z$  vanishing at the marked point. In the case of meromorphic connections on a holomorphic rank- $n$  vector bundle  $E \rightarrow \Sigma$ , an (untwisted) irregular type  $Q$  at this point is simply an element  $Q \in z^{-1}\mathfrak{t}[z^{-1}]$ , where  $\mathfrak{t} \subseteq \mathfrak{gl}_n(\mathbb{C})$  is the standard Cartan subalgebra of diagonal matrices. More precisely a connection  $\nabla$  on  $E$  with irregular type  $Q$  at the point  $a \in \Sigma$  can be locally written

$$\nabla = d - \mathcal{A}, \quad \mathcal{A} = dQ + \frac{\Lambda}{z} dz,$$

in some (local) trivialisation of  $E$ . Here the residue  $\Lambda$  is a constant block-diagonal matrix, centralising  $Q$ .

Consider in particular the irregular type

$$Q = \frac{A_2}{z^2} + \frac{A_1}{z}, \quad A_1, A_2 \in \mathfrak{t},$$

corresponding to a pole of order 3 for  $\nabla$  at the marked point. Such an irregular type is generic when the leading coefficient  $A_2$  has  $n$  distinct eigenvalues (see [62, 70]). We are rather interested in the non-generic situation, hence let us suppose that  $Q$  is *not* generic.

*Admissible deformations.* Deforming the irregular type  $Q$  means varying the coefficients of  $A_2$  and  $A_1$ . What does it mean for such a deformation to be admissible?

It goes as follows. First decompose the fibre  $E_\alpha \simeq \mathbb{C}^n$  of the vector bundle over  $\alpha \in \Sigma$ , into eigenspaces for  $A_2 \in \text{End}_{\mathbb{C}}(E_\alpha)$ :

$$(4) \quad \mathbb{C}^n = \bigoplus_{i=1}^k V_i, \quad V_i = \text{Ker}(A_2 - \lambda_i \text{Id}_{\mathbb{C}^n}) \subseteq \mathbb{C}^n,$$

where  $\{\lambda_1, \dots, \lambda_k\} \subseteq \mathbb{C}$  is the spectrum of  $A_2$ . Then use  $[A_2, A_1] = 0$ , whence  $A_1(V_i) \subseteq V_i$ , and split further into eigenspaces for the restriction  $A_{1,i} := A_1|_{V_i} \in \text{End}_{\mathbb{C}}(V_i)$ :

$$(5) \quad V_i = \bigoplus_{j=1}^{k_i} W_{ij}, \quad W_{ij} = \text{Ker}(A_{1,i} - \lambda_{j,i} \text{Id}_{V_i}) \subseteq V_i,$$

where  $\{\lambda_{1,i}, \dots, \lambda_{k_i,i}\} \subseteq \mathbb{C}$  is the spectrum of  $A_{1,i}$ . An *admissible* deformation of  $(A_1, A_2) \in \mathfrak{t}^2$  is another pair  $(A'_1, A'_2)$  inducing the same decompositions (4)–(5). It follows that the space of admissible deformations  $\mathbf{B}_Q$  will depend on the multiplicities of the eigenvalues of  $A_1$  and  $A_2$ . For arbitrary reductive Lie algebras, we will generalise this looking at the positions of the coefficients of irregular types relative to the root-hyperplanes of  $(\mathfrak{g}, \mathfrak{t})$ .

*Local WMCG and cabling of braids.* Keeping the notation from § 1.4, we will now give an example of braid cabling to describe elements of the pure local wild mapping class group  $\pi_1(\mathbf{B}_Q)$ .

Concretely, consider first a loop in the configuration space of the (ordered) eigenvalues of  $A_2 \in \text{End}(\mathbb{C}^n)$ , keeping them distinct, which yields a (pure) braid  $\sigma \in \text{PB}_k$ ; then the  $i$ -th strand of  $\sigma$  can be replaced by another braid  $\tau_i \in \text{PB}_{k_i}$ , corresponding to braiding the eigenvalues of  $A_{1,i} \in \text{End}(V_i)$  for  $i \in \{1, \dots, k\}$ . The result of this operation is a cabled braid

$$(6) \quad (\sigma, \tau_1, \dots, \tau_k) \longmapsto \gamma(\sigma; \tau_1, \dots, \tau_k) \in \text{PB}_k, \quad k = \sum_i k_i,$$

as in the theory of (action) operads, cf. § 9 and [105, Chp. 5].

Let us now give the example: consider the (traceless) rank-3 irregular type with

$$(7) \quad A_2 = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 2 \end{pmatrix}, \quad A_1 = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 0 \end{pmatrix} \in \mathfrak{sl}_3(\mathbb{C}),$$

whose admissible deformations look like

$$A'_2 = \begin{pmatrix} a & & \\ & a & \\ & & a' \end{pmatrix}, \quad A'_1 = \begin{pmatrix} b & & \\ & b' & \\ & & c \end{pmatrix},$$

with  $a, a', b, b', c \in \mathbb{C}$  such that  $a \neq a'$  and  $b \neq b'$ .

With the above notation, we have  $(k; k_1, k_2) = (2; 2, 1)$ , and we can take e.g.



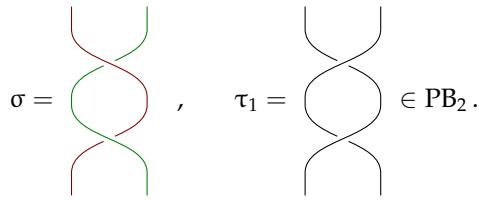


FIGURE 1. Before cabling

(Note  $\sigma \neq \tau_1$ , and colours are just for comparison with the next figure.)  
Then  $\tau_2 \in \text{PB}_1$  is necessarily trivial, and the resulting cabled braid is

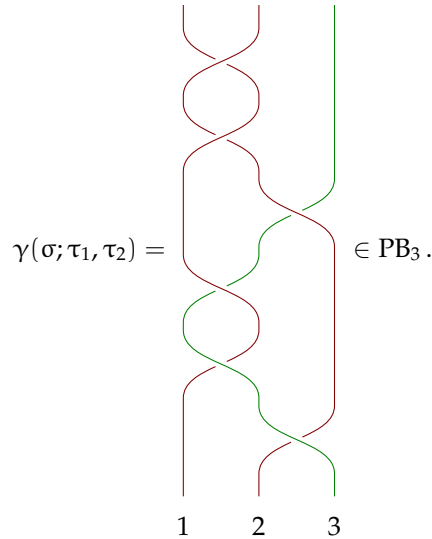
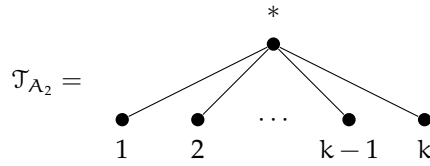


FIGURE 2. Example of (pure) braid cabling

Note particularly how the strands 1 and 2 in Fig. 2 move in parallel, and are braided with the strand 3: this corresponds to the (distinct) eigenvalues  $(a, a')$  looping around each other, following the braid  $\sigma$ ; then the (distinct) eigenvalues  $(b, b')$  do the same, following  $\tau_1$ , so the strands 1 and 2 are also eventually braided—on the top left corner of the diagram. The latter phenomenon is only possible because one eigenspace of the leading coefficient  $A_2$  is 2-dimensional, so it can break in the subleading coefficient  $A_1$ .

*Local WMCG and trees.* Back to the general setting of this section, encoding the operation (6) in terms of its inputs/outputs naturally yields trees. Namely, the splitting (4) can be pictorially represented as



This diagram is a tree of height 1, whose root is the node  $*$ , and with a choice of ordering for the leaves. Analogously can be done for the splittings (5), getting (labelled) trees  $\mathcal{T}_{A_{1,1}}, \dots, \mathcal{T}_{A_{1,k}}$ . Gluing each of those at the corresponding leaf of  $\mathcal{T}_{A_2}$  then yields a tree  $\mathcal{T}_{A_2, A_1}$  of height 2; for instance (7) corresponds to

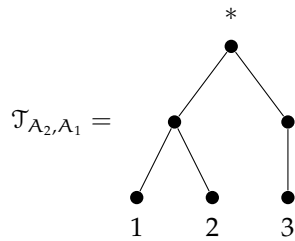


FIGURE 3. Example of fission tree (in type A)

Iterating this procedure associates a tree  $\mathcal{T}_Q$  with any irregular type  $Q$ , cf. [16, App. C], and see 6.1 below for more examples. Finally, the pure braid group operad can be evaluated on  $\mathcal{T}_Q$ , and the result is a group isomorphic to the pure local wild mapping class group  $\Gamma_Q$ . For instance, the tree in Figure 3 yields a group isomorphic to  $PB_2 \times PB_2$ : its generators are the elements  $(\sigma, \tau_1)$  used in the cabling of Fig. 2. One can use them to compute the braiding of Stokes data, as we do in Ex. 9.1. In general, the procedure we just discussed formalises conveniently and precisely the intuition of a “multi-scale braiding” description of local WMCGs.

*The general case.* Let us highlight some of the main differences between the general setting considered in the body of the paper and the example discussed above.

- (1) We consider throughout the text principal bundles with arbitrary (reductive) structure groups  $G$ . We do not use faithful  $G$ -modules to define admissible deformations; analogously, we will not assume that  $\mathfrak{g}$  is a matrix Lie algebra. Avoiding this assumption is necessary in order to work with general structure groups, cf. e.g. [18, Lem A.2].
- (2) We shall use the root system of  $(\mathfrak{g}, \mathfrak{t})$  to construct deformation spaces of irregular types, generalising (differences of) eigenvalues of semisimple endomorphisms of  $\mathbb{C}^n$ ; these are akin to root valuation strata considered, for different purposes, in [54].
- (3) We will introduce more general *fission* trees for Lie algebras of classical types B/C and D: to do so we will have to deal with several possible “fissions” at each stage, related to the irreducible components of all possible Levi subsystems of the root systems at hand, and we will accordingly introduce new decoration of the trees to encode them. In particular this will allow us

to formulate a more precise version of the multilevel braiding conjecture, for any classical Lie algebra: see Conj. 9.1.

**Acknowledgements.** We thank P. Boalch, F. Naef and G. Paolini for listening/answering to some of our questions.

## 2. ADMISSIBLE DEFORMATIONS OF WILD RIEMANN SURFACES

We will start from recalling the general definition of admissible families of wild Riemann surfaces in the untwisted setting, following [21].

Let  $\Sigma$  be a Riemann surface,  $G$  a connected complex reductive Lie group,  $\mathfrak{g} = \text{Lie}(G)$  its Lie algebra,  $T \subseteq G$  a maximal torus, and  $\mathfrak{t} = \text{Lie}(T) \subseteq \mathfrak{g}$  the corresponding Cartan subalgebra. A (dressed, untwisted) wild Riemann surface structure on  $\Sigma$  is the choice of a finite *ordered* set  $\mathbf{a} = (a_1, \dots, a_m) \in \Sigma^m$  of  $m \geq 0$  distinct marked points, and *untwisted* irregular types  $\mathbf{Q} = (Q_1, \dots, Q_m)$  based there; in turn an untwisted irregular type is the germ of a  $\mathfrak{t}$ -valued meromorphic function, defined up to holomorphic terms.

More precisely let  $\mathcal{O}_{\Sigma, a}$  be the local ring of germs of functions at a point  $a \in \Sigma$ ,  $\widehat{\mathcal{O}}_{\Sigma, a}$  its completion, and  $\widehat{\mathcal{K}}_{\Sigma, a}$  the field of fractions of this latter. Consider the quotient  $\mathcal{T}_{\Sigma, a} := \widehat{\mathcal{K}}_{\Sigma, a} / \widehat{\mathcal{O}}_{\Sigma, a}$ , consisting of “tails” of formal Laurent series. By definition, an untwisted irregular type at  $a \in \Sigma$  is an element

$$Q \in \mathfrak{t} \otimes \mathcal{T}_{\Sigma, a}.$$

If  $z$  is a local coordinate with  $z(a) = 0$  then

$$\widehat{\mathcal{O}}_{\Sigma, a} \simeq \mathbb{C}[[z]], \quad \widehat{\mathcal{K}}_{\Sigma, a} \simeq \mathbb{C}((z)),$$

and  $\mathcal{T}_{\Sigma, a} \simeq \mathbb{C}((z))/\mathbb{C}[[z]]$ , so we can write

$$(8) \quad Q = \sum_{i=1}^p A_i z^{-i} \in z^{-1} \mathfrak{t}[[z^{-1}]] \simeq \mathfrak{t}((z))/\mathfrak{t}[[z]], \quad A_1, \dots, A_p \in \mathfrak{t},$$

for some integer  $p \geq 1$ . Hereafter we simply refer to such elements as “irregular types”—all implicitly untwisted/unramified.

*Remark 2.1.* Recall that the motivation behind the definition of irregular types is to single out (and give intrinsic meaning to) the time-variables for isomonodromic deformations of irregular singular meromorphic connections on principal  $G$ -bundles over  $\Sigma$ . Notably, they control norms forms for (the principal of) local connection 1-forms, as in (2).

This way (8) controls the exponential factors of the fundamental solutions of the associated system of *linear* differential equations (i.e. the local horizontal sections of the connection), which ultimately lead to the Stokes data encoding their exponential growth/decay along prescribed infinitesimal directions at the pole. As mentioned in the introduction, this is the starting point to define the (universal) parameter spaces such that the overall deformations of the meromorphic connections are isomonodromic, i.e. Stokes data are locally constant.  $\triangle$

Let  $\Sigma = (\Sigma, \mathbf{a}, \mathbf{Q})$  be a wild Riemann surface, which we want to deform in *admissible* fashion. To this end let  $\mathbf{B}$  be a complex manifold and

$$\Sigma_{\mathbf{b}} = \pi^{-1}(\mathbf{b}) \hookrightarrow \Sigma \xrightarrow{\pi} \mathbf{B}, \quad \mathbf{b} \in \mathbf{B}$$

a holomorphic family of Riemann surfaces fibering over  $\mathbf{B}$ . Choose an  $m$ -tuple

$$\underline{\mathbf{a}} = (\underline{a}_1, \dots, \underline{a}_m) : \mathbf{B} \longrightarrow \underline{\Sigma}^m$$

of global sections. Finally consider a holomorphic  $\mathbf{B}$ -family of irregular types<sup>4</sup>  $\mathbf{b} \mapsto \underline{Q}_i(\mathbf{b})$ , based at the marked points  $\underline{a}_i(\mathbf{b}) \in \Sigma_{\mathbf{b}}$ , and let  $\underline{\mathbf{Q}} = (\underline{Q}_1, \dots, \underline{Q}_m)$  be their collection.

**Definition 2.1.** The triple  $(\underline{\Sigma}, \underline{\mathbf{a}}, \underline{\mathbf{Q}})$  is a holomorphic  $\mathbf{B}$ -family of wild Riemann surfaces. Denoting it  $\underline{\Sigma} \rightarrow \mathbf{B}$ , the “fibre” at  $\mathbf{b} \in \mathbf{B}$  is the wild Riemann surface

$$\Sigma_{\mathbf{b}} = (\Sigma_{\mathbf{b}}, \mathbf{a}_{\mathbf{b}}, \mathbf{Q}_{\mathbf{b}}), \quad \mathbf{a}_{\mathbf{b}} := (\underline{a}_1(\mathbf{b}), \dots, \underline{a}_m(\mathbf{b})), \quad \mathbf{Q}_{\mathbf{b}} := (\underline{Q}_1(\mathbf{b}), \dots, \underline{Q}_m(\mathbf{b})).$$

If  $0 \in \mathbf{B}$  is a base point, and  $\mathbf{B}$  is connected, the family  $\underline{\Sigma} \rightarrow (\mathbf{B}, 0)$  is a *deformation* of the “starting” wild Riemann surface  $\Sigma_0$ .

To introduce the admissibility condition let  $\Phi_{\mathfrak{g}} = \Phi(\mathfrak{g}, \mathfrak{t}) \subseteq \mathfrak{t}^\vee$  be the root system of the split Lie algebra  $(\mathfrak{g}, \mathfrak{t})$ , and consider an irregular type as (8). Then consider the meromorphic function germ (defined up to holomorphic terms)

$$q_\alpha := \alpha(Q) \in \mathcal{F}_{\Sigma, \mathbf{a}}, \quad \alpha \in \Phi_{\mathfrak{g}},$$

obtained by evaluating the irregular type on a root.<sup>5</sup> Hence a collection of  $\mathbf{B}$ -families of irregular types yields  $\mathbf{B}$ -families

$$\mathbf{b} \longmapsto \underline{q}_{j, \alpha}(\mathbf{b}) := \alpha \circ (\underline{Q}_j(\mathbf{b})) \in \mathcal{F}_{\Sigma_{\mathbf{b}}, \mathbf{a}_{\mathbf{b}}},$$

for  $j \in \{1, \dots, m\}$  and  $\alpha \in \Phi_{\mathfrak{g}}$ .

Finally, if  $q$  is the germ of a meromorphic function at  $\mathbf{a} \in \Sigma$ , let  $\text{ord}(q) \in \mathbb{Z}_{\geq 0}$  be its pole order at the base point, with the convention that  $\text{ord}(q) = 0$  if  $q$  is holomorphic—this integer is well defined up to adding holomorphic terms.

**Definition 2.2** ([21], Def. 10.1). An *admissible* deformation of  $\Sigma_0$  is a deformation  $\underline{\Sigma} \rightarrow (\mathbf{B}, 0)$  of  $\Sigma_0$  such that for all  $\mathbf{b} \in \mathbf{B}$ :

- $\Sigma_{\mathbf{b}}$  is smooth;
- the marked points  $\mathbf{a}_{\mathbf{b}} \in (\Sigma_{\mathbf{b}})^m$  are distinct;
- one has

$$(9) \quad \text{ord}(\underline{q}_{\alpha, j}(\mathbf{b})) = \text{ord}(\underline{q}_{\alpha, j}(0)) \in \mathbb{Z}_{\geq 0}, \quad \text{for all } \alpha \in \Phi_{\mathfrak{g}}.$$

In words: the genus of each (smooth) Riemann surface, the cardinality of each set of marked points, and the pole orders of the irregular types evaluated at each root, are *constant* along the deformation.

Recall the set of nonzero pole orders which occur at  $\underline{a}_i(\mathbf{b}) \in \Sigma_{\mathbf{b}}$  is the set of *levels* of the irregular type  $\underline{Q}_i(\mathbf{b})$ : this paper is about the multilevel case.

**2.1. Wild deformations.** By Def. 2.2 we can deform the complex structure of the Riemann surface underlying a wild one, and move the marked points inside their configuration space. These are the tame isomonodromy times (controlled by the stack  $\mathcal{M}_{g, m}$ ); as explained in the introduction, we are rather interested in the additional local wild moduli of the irregular types—freezing the underlying pointed surface.

<sup>4</sup>In the setting of this paper the marked points will be fixed, hence we will have a fixed target space of irregular types, and  $\underline{\mathbf{Q}}$  will be a holomorphic map with domain  $\mathbf{B}$ .

<sup>5</sup>This means evaluating  $\alpha \otimes \text{Id} : \mathfrak{t} \otimes \mathcal{F}_{\Sigma, \mathbf{a}} \rightarrow \mathbb{C} \otimes \mathcal{F}_{\Sigma, \mathbf{a}} = \mathcal{F}_{\Sigma, \mathbf{a}}$  at  $Q$ .

Consider thus simply an irregular type as (8). Introducing the local coordinate  $x = z^{-1}$  (on a punctured neighbourhood of the marked point) yields a polynomial

$$(10) \quad Q = \sum_{i=1}^p A_i x^i \in x \mathfrak{t}[x], \quad A_i \in \mathfrak{t}.$$

To avoid discussing trivial cases, assume  $A_p \neq 0$  in what follows.

Now  $q_\alpha \in x\mathbb{C}[x] \cup \{0\}$  for all roots  $\alpha \in \Phi_{\mathfrak{g}}$ , and the pole orders are controlled by the function

$$\mathbf{d}_Q : \alpha \mapsto d_\alpha = \deg_x(q_\alpha), \quad \alpha \in \Phi_{\mathfrak{g}},$$

with the analogous convention that  $\deg_x(0) = 0$ . (On the whole  $\mathbf{d}_Q(\Phi_{\mathfrak{g}}) \subseteq \{0, \dots, p\}$ , and  $d_\alpha = 0$  if and only if  $q_\alpha = 0$ .)

Thus (9) becomes

$$(11) \quad \deg_x(\alpha \circ Q') = d_\alpha, \quad \alpha \in \Phi_{\mathfrak{g}},$$

where  $Q' = \sum_{i>0} A'_i x^i$  is another polynomial with coefficients  $A'_i \in \mathfrak{t}$ .

*Remark 2.2.* In principle we should consider polynomials of arbitrary degree, but only those with  $\deg_x(Q') \leq p$  will contribute to the topology of the (universal) space of admissible deformations.

Indeed imposing (11) yields  $A'_i \in \text{Ker}(\Phi_{\mathfrak{g}}) \subseteq \mathfrak{t}$  for  $i > p$ , so all coefficients of higher degree live in a contractible space. (Note  $\text{Ker}(\Phi_{\mathfrak{g}}) = \bigcap_{\alpha \in \Phi_{\mathfrak{g}}} \text{Ker}(\alpha) = \mathfrak{Z}_{\mathfrak{g}}$ , which is *not* zero in the nonsemisimple case.)  $\triangle$

Hence we pose:

**Definition 2.3** (Cf. [21], Ex. 10.1). The *universal deformation space* of (10) is

$$(12) \quad \mathbf{B}_Q := \{ Q' \in x \mathfrak{t}[x] \mid \deg_x(Q') \leq p, \deg_x(\alpha \circ Q') = d_\alpha \text{ for } \alpha \in \Phi_{\mathfrak{g}} \}.$$

Note this depends on  $Q$  via the integer  $p \geq 1$  and the tuple  $\mathbf{d}_Q \in \mathbb{Z}_{\geq 0}^{\Phi_{\mathfrak{g}}}$ ; the use of the term “universal” is justified in Remark 2.3.

To describe  $\mathbf{B}_Q$  let us use the natural identification

$$\{ Q' \in x \mathfrak{t}[x] \mid \deg_x(Q') \leq p \} \xrightarrow{\cong} \mathfrak{t}^p,$$

mapping

$$Q' = \sum_{i=1}^p A'_i x^i \mapsto (A'_1, \dots, A'_p) \in \mathfrak{t}^p.$$

We obtain an inclusion  $\mathbf{B}_Q \subseteq \mathfrak{t}^p$ , and the tuple  $\mathbf{A} = (A_1, \dots, A_p) \in \mathbf{B}_Q$  corresponds to the base point (10).

**Proposition 2.1.** *There is a product decomposition  $\mathbf{B}_Q = \prod_{i=1}^p \mathbf{B}_i \subseteq \mathfrak{t}^p$ , where*

$$(13) \quad \mathbf{B}_i := \bigcap_{d_\alpha < i} \text{Ker}(\alpha) \cap \bigcap_{d_\alpha = i} (\mathfrak{t} \setminus \text{Ker}(\alpha)) \subseteq \mathfrak{t}.$$

*Proof.* By definition  $(A'_1, \dots, A'_p) \in \mathbf{B}_Q$  if and only if for any root  $\alpha \in \Phi_{\mathfrak{g}}$  one has

$$\alpha(A'_{d_\alpha}) \neq 0, \quad \alpha(A'_i) = 0, \quad i > d_\alpha,$$

and there are no conditions on  $A'_1, \dots, A'_{d_\alpha-1} \in \mathfrak{t}$ . Hence

$$\mathbf{B}_Q = \bigcap_{\alpha \in \Phi_{\mathfrak{g}}} \left( \left( \prod_{1 \leq i < d_\alpha} \mathfrak{t} \right) \times (\mathfrak{t} \setminus \text{Ker}(\alpha)) \times \prod_{d_\alpha < i \leq p} \text{Ker}(\alpha) \right) \subseteq \mathfrak{t}^p,$$

and the conclusion follows by swapping products/intersections.  $\square$

We conclude this section with few observations.

*Remark 2.3* (Universal deformation). The pointed space  $(\mathbf{B}_Q, \mathbf{A})$  is a fine moduli space of admissible deformations of the irregular type  $Q$  on the “starting” wild Riemann surface  $\Sigma = (\Sigma, \mathfrak{a}, Q)$ .

Indeed, let  $\mathbf{B}$  be a connected pointed complex manifold, and consider the constant family of Riemann surfaces  $\underline{\Sigma} := \Sigma \times \mathbf{B} \rightarrow \mathbf{B}$ , with the constant section  $\underline{a} = (a, \text{Id}) : \mathbf{B} \rightarrow \Sigma \times \mathbf{B}$ . The projection on the first factor of  $\Sigma \times \mathbf{B}$  induces isomorphisms  $\Sigma_b \simeq \Sigma$  and  $\mathcal{F}_{\Sigma_b, \underline{a}(b)} \simeq \mathcal{F}_{\Sigma, a}$  for all  $b \in \mathbf{B}$ . Let  $\mathcal{F}_{\Sigma, a}^{\leq p} \subseteq \mathcal{F}_{\Sigma, a}$  be the subspace of tails of Laurent series of pole order at most  $p$ . Via the previous identification, a holomorphic  $\mathbf{B}$ -family of irregular types  $\underline{Q}$  of pole order at most  $p$  at  $\underline{a}$  is a holomorphic function  $\mathbf{B} \rightarrow \mathfrak{t} \otimes \mathcal{F}_{\Sigma, a}^{\leq p}$ . Let us identify  $\mathfrak{t} \otimes \mathcal{F}_{\Sigma, a}^{\leq p}$  with  $\mathfrak{t}^p$  sending  $\sum_{i=1}^p A'_i z^{-i}$  to  $(A'_1, \dots, A'_p)$ . Then, by construction, a map of pointed complex manifolds  $f : (\mathbf{B}, 0) \rightarrow (\mathfrak{t}^p, \mathbf{A})$  yields an admissible deformation  $(\Sigma \times \mathbf{B}, \underline{a}, \underline{Q})$  of  $(\Sigma, a, Q)$  if and only if  $f(\mathbf{B}) \subseteq \mathbf{B}_Q$ . By definition, this means that  $(\mathbf{B}_Q, \mathbf{A})$  is the fine moduli space of admissible deformations of  $Q$  (on the fixed pointed Riemann surface  $(\Sigma, a)$ ) with pole order bounded by  $p$ . In particular, the family of irregular types

$$\underline{Q} : \mathbf{B}_Q \longrightarrow \mathfrak{t} \otimes \mathcal{F}_{\Sigma, a}^{\leq p}, \quad (A'_1, \dots, A'_p) \mapsto \sum_{i=1}^p A'_i z^{-i}$$

is the universal admissible deformation of  $Q$  with pole order bounded by  $p$ .  $\triangle$

*Remark 2.4* (Trivial deformations). One can add an element of the centre to any coefficient, i.e.  $(\mathfrak{Z}_{\mathfrak{g}})^p$  acts on  $\mathbf{B}_Q$  by factorwise translations.

Hence in principle one could consider the quotient space  $\mathbf{B}_Q / \mathfrak{Z}_{\mathfrak{g}}^p$ , which yields the same fundamental group, and amounts to considering the semisimple part of  $\mathfrak{g}$ : this will be done later on, but at this stage the main definitions are cleaner without such restrictions. Moreover we need reductive Lie algebras to discuss fission recursively, see § 4.  $\triangle$

*Remark 2.5* (Intrinsic definition). The pole order of (the germ of) a meromorphic function on  $\Sigma$  is well defined up to local biholomorphisms. Hence the integers  $d_\alpha \in \{0, \dots, p\}$  depend on  $Q$  only, and not on the identifications  $\widehat{\mathcal{O}}_{\Sigma, a} \simeq \mathbb{C}[[z]] \subseteq \mathbb{C}((z)) \simeq \widehat{\mathcal{H}}_{\Sigma, a}$ : the space (12) is well defined.  $\triangle$

*Remark 2.6* (Many-point case). It is straightforward to extend (12) to the case of several *fixed* marked points on  $\Sigma$ .

Namely if  $\mathfrak{a} = (a_1, \dots, a_m) \in \Sigma^m$  we still consider a trivial family  $\underline{\Sigma} = \Sigma \times \mathbf{B}_Q \rightarrow \mathbf{B}_Q$ , equipped with the corresponding global constant sections, and this time  $\mathbf{B}_Q$  is a space of (simultaneous) admissible deformations of irregular types at each marked point. More precisely  $\mathbf{Q} = (Q_1, \dots, Q_m)$ , with

$$Q_j \in \mathfrak{t} \otimes \mathcal{F}_{\Sigma, a_j}, \quad j \in \{1, \dots, m\},$$

each with a pole of order  $p_j \geq 0$ , and then

$$\mathbf{B}_Q := \prod_{j=1}^m \mathbf{B}_{Q_j} \subseteq \prod_{j=1}^m t^{p_j}. \quad \triangle$$

### 3. PURE LOCAL WILD MAPPING CLASS GROUPS

The main definition is the following:

**Definition 3.1.** The *pure local wild mapping class group* (WMCG) of the wild Riemann surface  $\Sigma = (\Sigma, a, Q)$  is

$$\Gamma_Q := \pi_1(\mathbf{B}_Q, \mathbf{A}),$$

where as before  $\mathbf{A} = (A_1, \dots, A_p) \in t^p$  and  $Q = \sum_{i=1}^p A_i x^i$ . (We will also say that  $\Gamma_Q$  is a pure local WMCG of type  $\mathfrak{g}$ .)

Once more, this does not depend on the underlying pointed surface  $(\Sigma, a)$ , but only on (the integers associated with)  $Q$ . The terminology is chosen with a view towards the *global* WMCG, to be defined elsewhere (cf. § 10).

Importantly by Prop. 2.1 there is a product decomposition

$$(14) \quad \Gamma_Q \simeq \prod_{i=1}^p \pi_1(\mathbf{B}_i, A_i),$$

and further the many-point case yields a product of such groups, with a factor at each marked point, as in Rem. 2.6.

In the rest of the paper we will study the pure local WMCGs, particularly aiming at a classification of (the isomorphism class of) the factors (14).

*Example 3.1* (Abelian case). Suppose  $\mathfrak{g}$  is abelian. Then  $\Phi_{\mathfrak{g}} = \emptyset$ , and  $\mathbf{B}_Q = t^p$ : all pure local WMCGs of type  $\mathfrak{g}$  are trivial.  $\triangle$

The next example is less trivial; nonetheless it shows there is not much to say about the generic case.

*Example 3.2* (Generic case). Suppose  $\mathbf{d}$  is *constant*, say  $d_{\alpha} = d \in \{0, \dots, p\}$  for all  $\alpha \in \Phi_{\mathfrak{g}}$ . If  $d = 0$  then  $Q \in z^{-1} \mathfrak{Z}_{\mathfrak{g}}[z^{-1}]$ , and (12) is the contractible space  $\mathfrak{Z}_{\mathfrak{g}}^p \subseteq t^p$ :  $\Gamma_Q$  is trivial.

Else  $d > 0$ , which is precisely the case of meromorphic connections with a single level. One finds

$$\mathbf{B}_i = t, \quad \mathbf{B}_d = t_{\text{reg}}, \quad \mathbf{B}_{i'} = \mathfrak{Z}_{\mathfrak{g}},$$

for  $i < d < i'$ . Hence  $\mathbf{B}_d$  is a strong deformation retract of  $\mathbf{B}_Q$ , and

$$\Gamma_Q \simeq \pi_1(t_{\text{reg}}, A_d) = \text{PB}_{\mathfrak{g}},$$

the pure  $\mathfrak{g}$ -braid group.  $\triangle$

As explained in the introduction, in the next sections we will describe the topology of the deformation space in the case where several levels occur, going *beyond* pure  $\mathfrak{g}$ -braid groups.

## 4. FILTRATIONS AND FISSION

In this section we will rewrite (13) as the complement of a hyperplane arrangement, involving the root system  $\Phi_{\mathfrak{g}}$  in an essential way. This makes it possible to prove the general results of § 5. Moreover we will introduce Dynkin diagrams, which will be crucial for the classification statements of §§ 6–8.

While the material of this section is standard, we chose to spell it out for the sake of self-contained exposition.

The starting point is noticing that there is an increasing sequence of subsets

$$(15) \quad \Phi_1 \subseteq \cdots \subseteq \Phi_{p+1} = \Phi_{\mathfrak{g}}, \quad \text{with} \quad \Phi_i := \{ \alpha \in \Phi_{\mathfrak{g}} \mid d_{\alpha} < i \}.$$

(In particular  $\Phi_1 = \{ \alpha \in \Phi_{\mathfrak{g}} \mid q_{\alpha} = 0 \}$ .)

But there is more structure, which follows from the triangular inequality of the standard valuation of the field of formal Laurent series in one variable:

**Lemma 4.1.** *Every term of (15) is a root subsystem of  $\Phi_{\mathfrak{g}}$ .*

A different proof follows from the discussion below, were we identify each subset  $\Phi_i \subseteq \Phi_{\mathfrak{g}}$  with the root system of a reductive subalgebra  $\mathfrak{h}_i \subseteq \mathfrak{g}$  (containing  $\mathfrak{t}$ ).

*Remark 4.1.* Different filtrations (15) may yield pure local WMCGs of type  $\mathfrak{g}$  which are isomorphic, e.g. acting on each term by an automorphism of the root system—or in particular by the Weyl group. This is part of the classification problem.  $\triangle$

In particular we can now rewrite (13) as

$$(16) \quad \mathbf{B}_i = \text{Ker}(\Phi_i) \cap \bigcap_{\Phi_{i+1} \setminus \Phi_i} (\mathfrak{t} \setminus \text{Ker}(\alpha)) \subseteq \mathfrak{t},$$

so in principle the factors of  $\Gamma_Q$  are controlled by nested root subsystems of  $\Phi_{\mathfrak{g}}$ . However not all subsystems will appear, but rather only *Levi subsystems*: these arise by taking nested centralisers, as we will momentarily explain.

**4.1. Fission: Lie groups/algebras.** The sequence (15) is associated with a filtration of complex reductive subgroups of  $G$  (cf. [21, Eq. 33]). In turn their Lie algebras are (reductive) Levi factors of parabolic subalgebras of  $\mathfrak{g}$ , which we use to give a more explicit description of (16).

Let us then define the “fission” subgroups

$$H_i := \{ g \in G \mid \text{Ad}_g(A_k) = A_k, i \leq k \leq p \} \subseteq G, \quad i \in \{1, \dots, p\},$$

fitting into an increasing sequence  $H_1 \subseteq \cdots \subseteq H_p \subseteq G$  of connected complex reductive groups.

*Remark 4.2.* As mentioned in the introduction, the terminology is due to the “breaking” of the structure group of the principal bundle at the boundary of the (real, oriented) blowup of  $(\Sigma, a)$ , from  $G$  down to  $H_1$ . This phenomenon is only visible in the wild case, and it is different from the usual “fusion” operation (= sewing surfaces with boundaries, along their boundaries).  $\triangle$

In particular

$$H_1 = \{ g \in G \mid \text{Ad}_g(Q) = Q \}$$

is the centraliser of the irregular type in  $G$ —the stabiliser for the diagonal Adjoint action  $G \rightarrow \text{GL}(\mathfrak{g})$  on each coefficient. Note  $T \subseteq H_1$ , and we allow a strict inclusion.



*Example 4.1* (Generic fission). In the generic case of Ex. 3.2 we find

$$H_i = T, \quad H'_i = G,$$

for  $i \leq d < i'$ . Namely the structure group breaks down to the maximal torus as soon as the generic coefficient is encountered.

It is only in the nongeneric/multilevel case that we encounter nontrivial fissions.  $\triangle$

Denote now  $\mathfrak{h}_i := \text{Lie}(H_i)$  the  $i$ -th “fission” subalgebra, which by construction is the centraliser of the coefficients  $A_i, \dots, A_p \in \mathfrak{t}$  in  $\mathfrak{g}$ . In particular

$$\mathfrak{h}_1 = \{ X \in \mathfrak{g} \mid [X, Q] = 0 \}$$

is the centraliser of  $Q$  in  $\mathfrak{g}$ . As expected  $\mathfrak{h}_1$  contains  $\mathfrak{t} = \text{Lie}(T)$ , and in turn  $\mathfrak{t} \subseteq \mathfrak{h}_i$  is a Cartan subalgebra for  $i \in \{1, \dots, p\}$ .

**Lemma 4.2.** *One has  $\Phi_i = \Phi_{\mathfrak{h}_i} := \Phi(\mathfrak{h}_i, \mathfrak{t})$  for  $i \in \{1, \dots, p\}$ , in the notation of (15).*

*Proof.* By induction on  $i \in \{p, \dots, 1\}$ . The base is the identity

$$(17) \quad \Phi_{\mathfrak{h}_p} = \Phi_{\mathfrak{g}} \cap \{A_p\}^\perp \subseteq \Phi_{\mathfrak{g}},$$

which follows by observing that  $\mathfrak{g}_\alpha \cap \mathfrak{h}_p \neq (0)$  if and only if  $\alpha(A_p) = 0$ .

Then replacing  $(H_p, G, A_p)$  with  $(H_i, H_{i+1}, A_i)$  at each step proves the claim.  $\square$

Analogous “descending” inductions will be a common theme in the rest of the paper. In particular here let us consider the centraliser  $\mathfrak{h} = \text{Ker}(\text{ad}_A) \subseteq \mathfrak{g}$  of an element  $A \in \mathfrak{t}$  (in  $\mathfrak{g}$ ).

*Remark 4.3* (Reductive centralisers). It is important here that  $\mathfrak{h}$  is reductive.

Beware however it need *not* be (semi)simple, even if  $\mathfrak{g}$  is: e.g. if  $A \in \mathfrak{t}_{\text{reg}}$  then  $\mathfrak{h} = \mathfrak{t}$  is even abelian.  $\triangle$

Denote  $\Phi_{\mathfrak{h}} = \Phi(\mathfrak{h}, \mathfrak{t}) \subseteq \Phi_{\mathfrak{g}}$ , which is the subset of roots vanishing on  $A$ —as in (17). Then, up to repeating all constructions by replacing  $\mathfrak{g}$  with  $\mathfrak{h}$  (and keeping  $\mathfrak{t}$ ), to understand (16) it is enough to study the space

$$(18) \quad \mathbf{B}(\Phi_{\mathfrak{h}}, \Phi_{\mathfrak{g}}) := \text{Ker}(\Phi_{\mathfrak{h}}) \setminus \bigcup_{\Phi_{\mathfrak{g}} \setminus \Phi_{\mathfrak{h}}} \text{Ker}(\alpha) \subseteq \mathfrak{t}.$$

Later we will show (18) is never empty, so indeed it is a hyperplane complement: it is obtained by “restricting” the hyperplane arrangement of  $\Phi_{\mathfrak{g}} \setminus \Phi_{\mathfrak{h}}$  to  $\text{Ker}(\Phi_{\mathfrak{h}}) \subseteq \mathfrak{t}$ . Note this generalises  $\mathfrak{t}_{\text{reg}}$ , which in turn corresponds to the generic case  $\Phi_{\mathfrak{h}} = \emptyset$ . In particular we do *not* expect that the factors of  $\Gamma_Q$  will be pure braid groups of Lie algebras, and indeed there is a counterexample in type D (cf. § 8).

*Remark 4.4* (Dimensions). One has

$$(19) \quad \dim(\text{Ker}(\Phi_{\mathfrak{h}})) = \text{rk}(\mathfrak{g}) - \text{rk}(\Phi_{\mathfrak{h}}) = \text{rk}(\mathfrak{g}) - \text{rk}(\mathfrak{h}'),$$

where  $\mathfrak{h}' = [\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$  is the semisimple part.  $\triangle$

**4.2. Fission: Dynkin diagrams.** Let again  $\mathfrak{h} \subseteq \mathfrak{g}$  be the centraliser of an element  $A \in \mathfrak{t}$ . Then  $\Phi_{\mathfrak{h}} \subseteq \Phi_{\mathfrak{g}}$  is a Levi subsystem, and it is known that for all such there exists a base of simple roots  $\Delta_{\mathfrak{g}} \subseteq \Phi_{\mathfrak{g}}$  such that  $\Phi_{\mathfrak{g}}$  corresponds to a subdiagrams of the Dynkin diagram of  $\Phi_{\mathfrak{g}}$ .

Namely, consider the following subspace of the complex plane:

$$\mathfrak{C} = \{ \lambda \in \mathbb{C} \mid \operatorname{Re}(\lambda) \geq 0; \text{ if } \operatorname{Re}(\lambda) = 0 \text{ then } \operatorname{Im}(\lambda) \geq 0 \}.$$

Note the identities  $\mathfrak{C} \cup (-\mathfrak{C}) = \mathbb{C}$  and  $\mathfrak{C} \cap (-\mathfrak{C}) = \{0\}$  are a natural “complexification” of the analogous one for the subspace  $\mathbb{R}_{\geq 0} \subseteq \mathbb{R}$ . Building on this, one can prove the following subspace is a fundamental domain for the action of the Weyl group, for any choice of a base  $\Delta_{\mathfrak{g}} \subseteq \Phi_{\mathfrak{g}}$ :

$$\mathfrak{C}_{\Delta_{\mathfrak{g}}} = \left\{ A' \in \mathfrak{t} \mid \langle \theta, A' \rangle \in \mathfrak{C} \text{ for all } \theta \in \Delta_{\mathfrak{g}} \right\} \subseteq \mathfrak{t}.$$

It is thus a natural “complexification” of the  $\Delta_{\mathfrak{g}}$ -dominant Weyl chamber—in the real part of the Cartan subalgebra [36]. (Note by definition  $\mathfrak{Z}_{\mathfrak{g}} \subseteq \mathfrak{C}_{\Delta_{\mathfrak{g}}}$ , and the Weyl group acts trivially there.)

Hence, up to acting via the Weyl group on the choice of base, we can assume that  $A \in \mathfrak{C}_{\Delta_{\mathfrak{g}}}$ . In turn one can now prove that the subset  $\Delta_{\mathfrak{h}} := \Delta_{\mathfrak{g}} \cap \{A\}^{\perp} \subseteq \Phi_{\mathfrak{h}}$  is a base for  $\Phi_{\mathfrak{h}}$ . This is essentially equivalent to proving that  $\mathfrak{h}$  is the Levi factor of the (standard) parabolic subalgebra of  $\mathfrak{g}$  corresponding to  $\Delta_{\mathfrak{h}} \subseteq \Delta_{\mathfrak{g}}$ , cf. [39, Prop. 5.6].

Hence in brief  $\Phi_{\mathfrak{h}}$  admits a base given by the simple roots of  $\Phi_{\mathfrak{g}}$  which vanish on  $A$ .

*Remark 4.5.* We can rewrite (18) using  $\operatorname{Ker}(\Phi_{\mathfrak{h}}) = \operatorname{Ker}(\Delta_{\mathfrak{h}})$ . (This is the centre of  $\mathfrak{h}$ .)

On the contrary, it is *not* enough to remove the hyperplanes of  $\Delta_{\mathfrak{g}} \setminus \Delta_{\mathfrak{h}}$ ; rather those of  $\Phi_{\mathfrak{g}}^+ \setminus \Phi_{\mathfrak{h}}^+ \subseteq \Phi_{\mathfrak{g}} \setminus \Phi_{\mathfrak{h}}$ , where  $\Phi_{\mathfrak{h}}^+ = \Phi_{\mathfrak{g}}^+ \cap \{A\}^{\perp} \subseteq \Phi_{\mathfrak{h}}$ —which is a system of positive roots for  $(\mathfrak{h}, \mathfrak{t})$ .  $\triangle$

It follows that the Dynkin diagram  $\mathcal{D}_{\mathfrak{h}}$  of  $(\Phi_{\mathfrak{h}}, \Delta_{\mathfrak{h}})$  is obtained by choosing a subset of nodes of the Dynkin diagram  $\mathcal{D}_{\mathfrak{g}}$  of  $(\Phi_{\mathfrak{g}}, \Delta_{\mathfrak{g}})$ , keeping all edges among them (and their decoration, i.e. possible doubling/tripling and orientation). Repeating this procedure at each step, as in [35, Lem. 3.2.5], finally yields a nested sequence of Dynkin diagrams:

$$(20) \quad \mathcal{D}_{\mathfrak{h}_1} \subseteq \cdots \subseteq \mathcal{D}_{\mathfrak{h}_p} \subseteq \mathcal{D}_{\mathfrak{g}}.$$

Namely, at each step one finds the complete subdiagram on a subset of nodes (up to relabeling them for a new choice of basis). In this viewpoint, “fission” refers to how a connected component of  $\mathcal{D}_{\mathfrak{h}_{i+1}}$  breaks into connected components of  $\mathcal{D}_{\mathfrak{h}_i}$ .

Importantly, this will enable the classification of §§ 6–8. Namely denote as customary  $A_n, B_n, C_n$  and  $D_n$  the irreducible rank- $n$  root systems of the simple Lie algebras of classical type. Let then  $\mathcal{D}_{A_n}, \mathcal{D}_{B_n}, \mathcal{D}_{C_n}$  and  $\mathcal{D}_{D_n}$  be their Dynkin diagrams with respect to the standard bases. Then:

- all components of a Dynkin subdiagram  $\mathcal{D} \subseteq \mathcal{D}_{A_n}$  are of type  $A$ ;
- at most one component of a Dynkin subdiagram  $\mathcal{D} \subseteq \mathcal{D}_{B_n}$  (resp.  $\mathcal{D} \subseteq \mathcal{D}_{C_n}$ ,  $\mathcal{D} \subseteq \mathcal{D}_{D_n}$ ) is of type  $B$  (resp.  $C, D$ ), and the others are of type  $A$ .

We will thus be able to encode a sequence such as (20) into a decorated tree. Roughly, each node at the  $i$ -th level of the tree will correspond to a component of the Dynkin diagram  $\mathcal{D}_{\mathfrak{h}_i}$ —with some subtlety, already treated in type  $A$ .

## 5. GENERAL RESULTS

Before jumping into the classification we will prove a few abstract results, building on the material of the previous section.

**5.1. Hyperplane arrangements.** Let us start from some general observation.

Given any root subsystem  $\Phi \subseteq \Phi_{\mathfrak{g}}$  let  $\mathcal{U} := \text{Ker}(\Phi) \subseteq \mathfrak{t}$ . It is natural to ask whether the “restricted” hyperplane arrangement

$$(21) \quad \mathcal{H} = \left\{ \text{Ker}(\alpha) \cap \mathcal{U} = \text{Ker}(\alpha|_{\mathcal{U}}) \mid \alpha \in \Phi_{\mathfrak{g}} \setminus \Phi \right\} \subseteq \mathbb{P}(\mathcal{U}^{\vee})$$

is crystallographic, i.e. if it comes from a root system: our results below imply this is false even when  $\mathfrak{g}$  is simple, and  $\Phi$  is a Levi subsystem. The basic obstruction of course is that  $\Phi_{\mathfrak{g}} \setminus \Phi \subseteq \Phi_{\mathfrak{g}}$  is *not* a root (sub)system in general.

*Remark 5.1.* Still writing  $\mathcal{U} = \text{Ker}(\Phi)$ , note the set

$$(22) \quad \Phi_{\mathfrak{g}}|_{\mathcal{U}} = \left\{ \alpha|_{\mathcal{U}} \mid \alpha \in \Phi_{\mathfrak{g}} \right\} \subseteq \mathcal{U}^{\vee}$$

is naturally identified with the quotient  $\Phi_{\mathfrak{g}}/\mathcal{U}^{\perp} \subseteq \mathfrak{t}^{\vee}/\mathcal{U}^{\perp}$ .

In turn

$$\mathcal{U}^{\perp} = (\text{Ker}(\Phi))^{\perp} = \text{span}_{\mathbb{C}}(\Phi) \subseteq \mathfrak{t}^{\vee},$$

so this is the same as considering the quotient set  $\Phi_{\mathfrak{g}}/\text{span}_{\mathbb{C}}(\Phi) \subseteq \mathfrak{t}^{\vee}/\text{span}_{\mathbb{C}}(\Phi)$ .  $\triangle$

*Remark 5.2.* Studying the reflection group of (21) goes towards the *full/nonpure* local WMCGs, which will be defined elsewhere (cf. § 10). (This is subtler than simply restricting the reflections associated with  $\alpha \in \Phi_{\mathfrak{g}} \setminus \Phi$  to  $\mathcal{U}$ , even in type A.)  $\triangle$

One of the insights of this work is that such restrictions/quotients of root systems, and their hyperplane arrangements, naturally arise in the theory of isomonodromic deformations for wild connections on principal bundles.

**5.2. Reduction to the simple case.** Suppose there is a decomposition  $\mathfrak{g} = \bigoplus_i \mathfrak{J}_i$  into mutually commuting ideals  $\mathfrak{J}_i \subseteq \mathfrak{g}$ , and let  $\mathfrak{t}_i = \mathfrak{t} \cap \mathfrak{J}_i$ —a Cartan subalgebra of  $\mathfrak{J}_i$ . There is then a second decomposition  $\mathfrak{t} = \bigoplus_i \mathfrak{t}_i$ , which induces an analogous one on the dual  $\mathfrak{t}^{\vee} \simeq \bigoplus_i^{\perp} \mathfrak{t}_i^{\vee}$ , by identifying  $\mathfrak{t}_i^{\vee}$  with the subspace

$$\bigcap_{j \neq i} \mathfrak{t}_j^{\perp} = (\mathfrak{t} \ominus \mathfrak{t}_i)^{\perp} \subseteq \mathfrak{t}^{\vee}, \quad \mathfrak{t} \ominus \mathfrak{t}_i := \bigoplus_{j \neq i} \mathfrak{t}_j.$$

Denote now  $\Phi_{\mathfrak{J}_i} = \Phi(\mathfrak{J}_i, \mathfrak{t}_i)$ , so there is a splitting of root systems

$$(\mathfrak{t}, \Phi_{\mathfrak{g}}) \simeq \bigoplus_i (\mathfrak{J}_i, \Phi_{\mathfrak{J}_i}).$$

Finally, for any root subsystem  $\Phi \subseteq \Phi_{\mathfrak{g}}$  set

$$\Phi^{(i)} := \Phi \cap \mathfrak{t}_i^{\vee} \subseteq \Phi,$$

finding a disjoint union  $\Phi = \bigsqcup_i \Phi^{(i)}$ . Then the subset  $\Phi^{(i)} \subseteq \Phi_{\mathfrak{J}_i}$  is also a root subsystem, cf. [30, Ch. VI, § 1.2].

**Proposition 5.1.** *In the notation of (18), there is a product decomposition*

$$\mathbf{B}(\Phi, \Phi_{\mathfrak{g}}) = \prod_i \mathbf{B}(\Phi^{(i)}, \Phi_{\mathfrak{J}_i}) \subseteq \mathfrak{t}.$$

*Proof.* If  $\alpha \in \Phi_{\mathcal{J}_i} \subseteq \Phi_{\mathfrak{g}}$  then

$$\mathrm{Ker}(\alpha) = (\mathfrak{t} \ominus \mathfrak{t}_i) \oplus \mathrm{Ker}(\alpha_i) \subseteq \mathfrak{t}, \quad \alpha_i := \alpha|_{\mathcal{J}_i} \in \mathfrak{t}_i^\vee.$$

Intersecting along the partition  $\Phi = \coprod_i \Phi^{(i)}$  then yields

$$\mathrm{Ker}(\Phi) = \prod_i \left( \bigcap_{\Phi^{(i)}} \mathrm{Ker}(\alpha_i) \right) \subseteq \prod_i \mathfrak{t}_i = \mathfrak{t}.$$

Analogously

$$\bigcap_{\Phi_{\mathfrak{g}} \setminus \Phi} (\mathfrak{t} \setminus \mathrm{Ker}(\alpha)) = \prod_i \left( \bigcap_{\Phi_{\mathcal{J}_i} \setminus \Phi^{(i)}} (\mathfrak{t}_i \setminus \mathrm{Ker}(\alpha_i)) \right) \subseteq \prod_i \mathfrak{t}_i,$$

and the statement follows by intersecting the two.  $\square$

Applying this to a filtration of root subsystems yields a product decomposition for pure local WMCGs of type  $\mathfrak{g}$ , into pure local WMCGs of type  $\mathcal{J}_i$ . In particular the splitting  $\mathfrak{g} = \mathfrak{z}_{\mathfrak{g}} \oplus \bigoplus_i \mathcal{J}_i$ , of a reductive Lie algebra into its simple ideals and its centre, means it is enough to work with *simple* Lie algebras—as  $\Gamma_Q$  is trivial in the abelian case.

**Hereafter we will thus assume  $\mathfrak{g}$  to be simple.**

**5.3. Nonempty complements.** Choose again  $A \in \mathfrak{t}$ , and let  $\mathfrak{h} \subseteq \mathfrak{g}$  be the centraliser.

**Lemma 5.1.** *The complement (18) is nonempty.*

*Proof.* Suppose  $\beta \in \Phi_{\mathfrak{g}} \setminus \Phi_{\mathfrak{h}}$ , and by contradiction  $\mathrm{Ker}(\Phi_{\mathfrak{h}}) \subseteq \mathrm{Ker}(\beta)$ . This happens if and only if  $\mathbb{C}\beta \subseteq \mathbb{C}\Phi_{\mathfrak{h}}$ , which implies  $\beta(A) = 0$ : this is absurd, as  $\Phi_{\mathfrak{h}} \subseteq \Phi_{\mathfrak{g}}$  is the subset of roots vanishing on  $A$ .  $\square$

*Remark 5.3.* The statement of Lem. 5.1 is false for general root subsystems  $\Phi \subseteq \Phi_{\mathfrak{g}}$ . E.g. the subsystem of short/long roots inside the root system of type  $G_2$  yields an empty complement. This corresponds to the proper inclusion  $A_2 \subseteq G_2$ , which in turn does *not* correspond to an inclusion of the (finite) Dynkin diagrams.

Again, the point is that here we have Levi subsystems, i.e. the inclusion  $\Phi_{\mathfrak{h}} \subseteq \mathrm{span}_{\mathbb{C}}(\Phi_{\mathfrak{h}}) \cap \Phi_{\mathfrak{g}}$  is an equality.  $\triangle$

**5.4. Descending ranks.** If  $\mathrm{rk} \mathfrak{h}' = \mathrm{rk} \mathfrak{g}$ , it follows by (19) that the complement (18) is homotopically trivial—and nonempty by Lem. 5.1. Hence to have a nontrivial fundamental group we need the rank to diminish at each step, and we find that:

**Corollary 5.1.** *The number of nontrivial factors of (14) is at most  $\mathrm{rk}(\mathfrak{g})$*

Note this bound is independent of the pole order  $p \geq 1$  of  $Q$ , and of the pole orders  $d_{\alpha}$  of the functions  $q_{\alpha} = \alpha \circ Q$ .

**5.5. Low-rank cases.** Suppose further  $\mathrm{rk}(\mathfrak{g}) - \mathrm{rk}(\mathfrak{h}') = 1$ . Then  $\mathrm{Ker}(\Phi_{\mathfrak{h}}) \subseteq \mathfrak{t}$  is a line, and since the relative complement cannot be empty it must be homeomorphic to  $\mathbb{C} \setminus \{0\}$ ; thus:

**Corollary 5.2.** *If  $\mathrm{rk}(\mathfrak{g}) - \mathrm{rk}(\mathfrak{h}') = 1$ , then the fundamental group of (18) is infinite cyclic.*

This corresponds to the pure braid group of type  $A_1$ , i.e. the pure braid group on 2 strands.

An easy extension then yields:

**Corollary 5.3.** *Suppose  $\text{rk}(\mathfrak{g}) = 2$ . Then  $\Gamma_Q$  is isomorphic to  $\mathbb{Z}$ , to  $\mathbb{Z}^2$ , or to the pure  $\mathfrak{g}$ -braid group.*

*Proof.* The possible filtrations of Levi subsystems are listed as

$$\emptyset \subseteq \Phi_{\mathfrak{g}}, \quad \Phi_{\mathfrak{h}} \subseteq \Phi_{\mathfrak{g}}, \quad \emptyset \subseteq \Phi_{\mathfrak{h}} \subseteq \Phi_{\mathfrak{g}},$$

with  $\text{rk}(\Phi_{\mathfrak{h}}) = 1$ . The first (generic) one leads to the pure  $\mathfrak{g}$ -braid group, and the other two are controlled by Cor. 5.2.  $\square$

For instance this completely classifies  $\Gamma_Q$  in the exceptional type  $G_2$ .

## 6. TYPE A

Starting from the section, we will explicitly describe the pure local WMCGs of classical type.

Let  $n \geq 1$  be an integer, and  $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$ . The standard Cartan subalgebra of traceless diagonal matrices is naturally a subspace of  $V := \mathbb{C}^{n+1}$ . We will use the shorthand notation  $\underline{k} := \{1, \dots, k\}$ , for an integer  $k \geq 1$ , in all that follows.

Denote then  $e_1, \dots, e_{n+1} \in V$  the vectors of the canonical basis, and  $\alpha_i = e_i^\vee \in V^\vee$  the associated dual coordinates. Then we write

$$\alpha_{ij}^- := \alpha_i - \alpha_j, \quad i \neq j \in \underline{n+1},$$

so that the root system is

$$A_n = \left\{ \pm \alpha_{ij}^- \mid i < j \in \underline{n+1} \right\} \subseteq \mathfrak{t}^\vee,$$

with standard basis  $\Delta_{\mathfrak{g}} = \{\theta_1, \dots, \theta_n\}$ ,  $\theta_i = \alpha_{i, i+1}^-$  [30, Ch. VI, § 4.7].

**6.1. Dynkin diagrams.** Choose an element  $\Lambda \in \mathfrak{t}$  and let  $\Phi_{\mathfrak{h}} \subseteq A_n$  be the Levi subsystem of its centraliser  $\mathfrak{h} \subseteq \mathfrak{g}$ . Reasoning as in § 4, we can assume it has base  $\Delta_{\mathfrak{h}} = \{\theta \in \Delta_{\mathfrak{g}} \mid \langle \theta, \Lambda \rangle = 0\}$ , which yields a subdiagram

$$\mathcal{D}_{\mathfrak{h}} \subseteq \mathcal{D}_{A_n} = \bullet_{\theta_1} \text{---} \bullet_{\theta_2} \text{---} \dots \text{---} \bullet_{\theta_{n-1}} \text{---} \bullet_{\theta_n}.$$

Keeping all edges (among adjacent nodes) results in a disjoint union of connected components

$$\mathcal{D}_{\mathfrak{h}} = \coprod_i \mathcal{D}_i \subseteq \mathcal{D}_{A_n}.$$

All components  $\mathcal{D}_i$  are of type A: a component on  $k \geq 1$  nodes corresponds to an irreducible root subsystem  $A_k \subseteq \Phi_{\mathfrak{h}}$ , whose simple roots form an unbroken string of length  $k$  inside  $\Delta_{\mathfrak{g}}$ . Consider then the subset

$$J' := \left\{ i \in \underline{n} \mid \text{there is an unbroken string of maximal length starting at } \theta_i \right\}.$$
<sup>6</sup>

For  $i \in J'$  we find an irreducible component  $A_{I'_i} \subseteq \Phi_{\mathfrak{h}}$  of rank  $|I'_i| > 0$ .

This results in a partition  $\Delta_{\mathfrak{h}} = \coprod_{i \in J'} I'_i$ , and it will be helpful to introduce the following versatile terminology:

**Definition 6.1.** If  $S$  and  $J$  are finite sets, a  $J$ -partition of  $S$  is a surjection  $\phi : S \twoheadrightarrow J$ . (This is the same as giving a partition  $S = \coprod_J I_j$  indexed by  $J$ , with nonempty parts  $I_j := \phi^{-1}(j) \subseteq S$ .)

<sup>6</sup>Note that “starting at” relies on the natural ordering of nodes from left to right (as the Dynkin diagram is not oriented per se). Moreover, if such an unbroken string exists, then it is unique.

We thus have a  $J'$ -partition  $\Delta_{\mathfrak{h}} \twoheadrightarrow J'$ . This does not quite control the complement (18), since one must take into account the roots of  $A_n$  that have been left out, as we now set out to do. (While the discussion can be slightly simplified in type  $A$ , we keep it going for the sake of streamlining the exposition through all classical types.)

**6.2. Kernels.** The space (18) is controlled by a partition “extending”  $\Delta_{\mathfrak{h}} \twoheadrightarrow J'$ . To clarify this, for  $i \in \underline{n+1}$  define the subset

$$(23) \quad I_i = I_i^{\mathfrak{h}} := \{i\} \cup \left\{ j \in \underline{n+1} \mid \pm \alpha_{ij}^- \in \Phi_{\mathfrak{h}} \right\} \subseteq \underline{n+1}.$$

**Lemma 6.1.** *The subsets (23) provide a  $J$ -partition  $\underline{n+1} \twoheadrightarrow J$ , and the set of parts  $J$  has cardinality*

$$|J| = n + 1 - \text{rk}(\mathfrak{h}').$$

*Proof.* First we must show that  $I_i \cap I_j \neq \emptyset$  implies  $I_i = I_j$ , for  $i, j \in \underline{n+1}$ . This is because the nontrivial root reflections act by

$$\sigma_{ij}^-(\alpha_{jk}^-) = \alpha_{ik}^-, \quad \sigma_{ij}^- = \sigma_{\alpha_{ij}^-},$$

for distinct indices  $i, j, k \in \underline{n+1}$ , and by hypothesis  $\Phi_{\mathfrak{h}}$  is a root subsystem.

As for the cardinality of  $J$ , if  $\Phi_{\mathfrak{h}} = \emptyset$  the identity is clear; and adding an irreducible component  $A_k \subseteq \Phi_{\mathfrak{h}}$  reduces it exactly by  $k$ .  $\square$

Now by construction the irreducible component  $A_{I'_i} \subseteq \Phi_{\mathfrak{h}}$  consists of the set of roots

$$\left\{ \pm \alpha_{jl}^- \mid i \leq j < l \leq i + |I'_i| \right\} \subseteq A_n,$$

for  $i \in J'$ , and in turn  $I_i = \{i, \dots, i + |I'_i|\}$ . Then there is a natural injection  $J' \hookrightarrow J$  which induces a one-to-one correspondence between  $I'_i$  and  $I_i$ .

We can thus also denote  $A_{I_i} \subseteq \Phi_{\mathfrak{h}}$  the irreducible components, but now the notation makes sense in general: if  $I_i = \{i\}$  then  $A_{I_i} \simeq A_0$  stands for the trivial (nonspanning) “rank-zero” root systems  $A_0 = \emptyset \subseteq \mathbb{C}e_i$ . This simply means that  $\mathbb{C}e_i \subseteq \text{Ker}(\Phi_{\mathfrak{h}})$  (see below).<sup>7</sup>

We now can describe  $\text{Ker}(\Phi_{\mathfrak{h}})$  in 2 steps: first we consider  $\Phi_{\mathfrak{h}}$  as a Levi subsystem of  $\Phi_{\mathfrak{gl}_{n+1}(\mathbb{C})} \subseteq V^\vee$ , and then we restrict to the trace-free case. Concretely, denote  $\widetilde{\text{Ker}}(\Phi_{\mathfrak{h}}) \subseteq V$  the kernel in the general linear case, so that  $\text{Ker}(\Phi_{\mathfrak{h}}) = \widetilde{\text{Ker}}(\Phi_{\mathfrak{h}}) \cap \mathfrak{t}$  is what we are after—in the special linear case.

**Proposition 6.1.** *There is a linear isomorphism  $\mathbb{C}^J \xrightarrow{\cong} \widetilde{\text{Ker}}(\Phi_{\mathfrak{h}})$ , given by mapping*

$$\bar{e}_I \mapsto e_I := \sum_{i \in I} e_i \in V,$$

for all  $I \in J$ , where  $\bar{e}_I$  is a vector of the canonical basis of  $\mathbb{C}^J$ .

*Proof.* By Lem. 6.1 there is a splitting

$$V \simeq \bigoplus_{I \in J} \mathbb{C}^I, \quad \mathbb{C}^I = \bigoplus_{i \in I} \mathbb{C}e_i,$$

and the kernel decomposes accordingly.

<sup>7</sup>Note  $J$  is naturally a subset of  $\underline{n+1}$  by mapping  $I_j \mapsto \min(I_j)$ . In particular it inherits a total order, intrinsically coming from the ordering of the eigenvalues of a (traceless) diagonal matrix.

In turn each component contains a full copy of the type-A root system, hence the kernel there is spanned by the line through the vector  $e_I \in V$  of the statement.  $\square$

In brief the vectors of the canonical basis corresponding to each part  $I \in J$  are fused in the kernel. In the end:

$$(24) \quad \text{Ker}(\Phi_h) = \left\{ \sum_{I \in J} \lambda_I e_I \in V \mid \sum_J \lambda_I = 0 \right\} \subseteq V.$$

We compute

$$\dim(\text{Ker}(\Phi_h)) = |J| - 1 = n - \text{rk}(h'),$$

using Lem. 6.1, in accordance with (19).

**6.3. Restricted subsystem and fundamental group.** Denote  $U := \text{Ker}(\Phi_h)$ . To conclude we must remove from it the root-hyperplanes corresponding to (positive) roots  $\alpha \in A_n \setminus \Phi_h$ , and it turns out this still yields a root system of type A.

**Theorem 6.1.** *Under the isomorphism of Prop. 6.1 there is an identification of root systems*

$$A_d \simeq A_n|_U \subseteq U^\vee,$$

where  $d = \dim(U)$ , using the notation of (22).

*Proof.* Introduce the dual basis  $\bar{\alpha}_I = \bar{e}_I^\vee$  of  $(\mathbb{C}^J)^\vee$ . By construction  $\alpha_{\tilde{I}}^- \in A_n \setminus \Phi_h$  if and only  $I_i \neq I_j$ , and restricting such covectors to (24) yields all linear functional  $\bar{\alpha}_I - \bar{\alpha}_{\hat{I}} \in U^\vee$ , with  $I \neq \hat{I} \in J$ .  $\square$

It follows that  $\pi_1(\mathbf{B}(\Phi_h, \Phi_g), A) \simeq \text{PB}_{d+1}$  in this case.

**6.4. Fission trees.** Finally we can reason recursively: consider a nested Levi subsystem  $\Phi_{\tilde{J}} \subseteq \Phi_h$ . Splitting into irreducible components, it follows that the partition  $\underline{n+1} \rightarrow \tilde{J}$  (associated with  $\Phi_{\tilde{J}}$  as in (23)) is a refinement of the  $J$ -partition associated with  $\Phi_h$ . More precisely, for any  $i \in \tilde{J}$  there exists  $\phi(i) \in J$  such that  $I_i^{\tilde{J}} \subseteq I_{\phi(i)}^h$ , i.e. there is a (new)  $J$ -partition  $\phi : \tilde{J} \rightarrow J$ .

Hence a filtration

$$\Phi_{h_1} \subseteq \cdots \subseteq \Phi_{h_p} \subseteq \Phi_{h_{p+1}} := A_n$$

of Levi subsystems corresponds to a decreasing sequence of sets

$$(25) \quad J_1 \xrightarrow{\phi_1} J_2 \longrightarrow \cdots \longrightarrow J_p \xrightarrow{\phi_p} J_{p+1} := \{*\},$$

where  $J_l$  is the set of parts corresponding to  $\Phi_{h_l}$ , as in Lem. 6.1—for  $l \in \underline{p}$ . This is the same as considering the disjoint union  $\mathcal{T}_0 := \coprod_{l=1}^{p+1} J_l$ , and giving a single function

$$\phi : \mathcal{T}_0 \setminus \{*\} \longrightarrow \mathcal{T}_0, \quad J_l \ni i \longmapsto \phi_l(i) \in J_{l+1}.$$

**Definition 6.2** (Fission tree). The *fission tree*  $\mathcal{T}_Q$  of (25) is the tree with nodes  $\mathcal{T}_0$ , such that  $\phi(i) \in \mathcal{T}_0$  is the parent-node of  $i \in \mathcal{T}_0 \setminus \{*\}$ .

Hence  $J_l \subseteq \mathcal{T}_0$  are the leaves and  $* \in J_{p+1}$  is the root, while  $|J_l|$  is the number of nodes at level  $l \in \underline{p+1}$ ; note by construction  $|J_1| \leq n+1$ . (The equality corresponds to  $H_1 = T$ , in the notation of § 4.) Set finally

$$k_i := |\phi^{-1}(i)| \in \mathbb{Z}_{\geq 0}, \quad i \in \mathcal{T}_0,$$

which is the number of child-nodes of  $i \in \mathcal{T}_0$ .<sup>8</sup>

**Theorem 6.2.** *There is a group isomorphism  $\Gamma_Q \simeq \prod_{\mathcal{T}_0} \text{PB}_{k_i}$ , and conversely pure local WMCGs of type A exhaust finite products of pure braid groups.*

*Proof.* By construction a node  $i \in J_1$  corresponds to an irreducible component of  $\Phi_{h_1}$ , which splits into  $k_i \geq 0$  irreducible components inside  $\Phi_{h_{l-1}} \subseteq \Phi_{h_l}$ , corresponding to the child-nodes  $j \in \Phi^{-1}(i) \subseteq J_{l-1}$ . By Thm. 6.1 this yields a pure braid group on  $k_i$  strands, sitting inside  $\Gamma_Q$ . This gives independent factors at each level of the tree, and the conclusion follows from the splitting (14)—taking the product over all levels.

For the second statement, given any sequence of integers  $n_i \geq 1$  with finite support we can construct a fission tree having precisely  $n_i$  nodes with  $i \geq 1$  child-nodes (in many ways, cf. Ex. 6.1). In that case

$$(26) \quad \Gamma_Q \simeq \prod_{i \geq 1} \text{PB}_i^{n_i}. \quad \square$$

*Remark 6.1* (Low-order and irreducible presentations). The theorem implies that different trees can lead to isomorphic groups, e.g. a “presentation” by a tree of minimal height is obtained by splitting nodes as quick as possible.

Conversely we can consider a tree with a single node splitting at each level: all fission subsystems are then *irreducible*, so there are integers  $n_1 \leq \dots \leq n_p \leq n$  such that (15) becomes

$$A_{n_1} \subseteq \dots \subseteq A_{n_p} \subseteq A_n,$$

with embedding on the first slots. △

*Example 6.1* (Examples of presentations). For  $n = 8$  let us consider the irregular type  $Q^1 = T_1x + T_2x^2 + T_3x^3$ , taking the following coordinate vectors  $\alpha_i = (\alpha_1(T_i), \dots, \alpha_9(T_i)) \in \mathbb{R}^9$ :

$$\alpha_1 = (4, 3, 2, 1, 0, -1, -2, -3, -4), \quad \alpha_2 = (4, 4, 3, 2, 1, 0, -3, -4, -7),$$

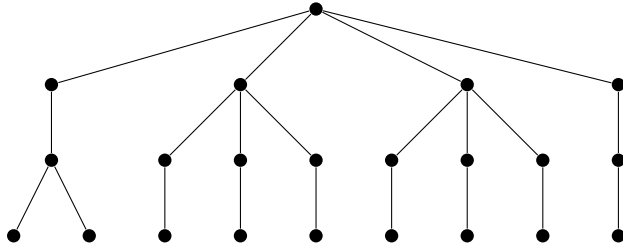
and

$$\alpha_3 = (2, 2, 1, 1, 1, 0, 0, 0, -7).$$

Then the sequence of Levi subsystems is

$$\emptyset \subseteq A_1 \subseteq A_1 \oplus A_2 \oplus A_2 \subseteq A_8,$$

and the associated pure local WMCG is  $\Gamma_{Q^1} \simeq \text{PB}_2 \times \text{PB}_3^2 \times \text{PB}_4$ . The corresponding fission tree is:



<sup>8</sup>All sets  $\Phi^{-1}(i) \subseteq \mathcal{T}_0$  come with a total order: cf. the previous footnote.



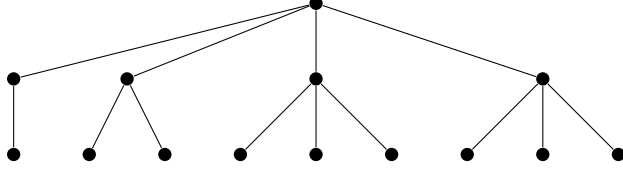
But this is also the fundamental group of the space of admissible deformations of  $Q^{\text{II}} = T_1x + T_2x^2$ , taking coordinate vectors

$$\alpha_1 = (4, 3, 2, 1, 0, -1, -2, -3, -4), \quad \alpha_2 = (4, 1, 1, 0, 0, 0, -2, -2, -2),$$

which yields the low-order presentation, and has associated filtration

$$\emptyset \subseteq A_1 \oplus A_2 \oplus A_2 \subseteq A_8.$$

The (minimal-height) fission tree is then:



Finally, this pure local WMCG is also the fundamental group of the space of admissible deformations of  $Q^{\text{III}} = T_1x + T_2x^2 + T_3x^3 + T_4x^4$ , taking coordinate vectors

$$\alpha_1 = (4, 3, 2, 1, 0, -1, -2, -3, -4), \quad \alpha_2 = (4, 4, 3, 2, 1, 0, -3, -4, -7),$$

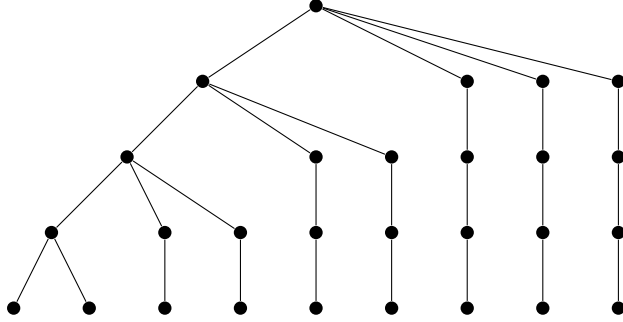
and

$$\alpha_3 = (2, 2, 2, 2, 1, 0, -3, -3, -3), \quad \alpha_4 = (1, 1, 1, 1, 1, 1, 0, -2, -4).$$

This yields the irreducible presentation, with filtration

$$\emptyset \subseteq A_1 \subseteq A_3 \subseteq A_5 \subseteq A_8,$$

and the fission tree is as follows:



△

## 7. TYPE B/C

Let still  $n \geq 1$  be an integer, and  $\mathfrak{g} = \mathfrak{so}_{2n+1}(\mathbb{C})$ . The standard Cartan subalgebra  $\mathfrak{t} \subseteq \mathfrak{g}$  is identified with  $\mathbb{C}^n = \bigoplus_i \mathbb{C}e_i$ , and we retain the notations of § 6.

The usual choice of basis for the root system is  $\Delta_{\mathfrak{g}} = \{\theta_1, \dots, \theta_n\}$  with

$$\theta_i = \alpha_{i,i+1}^-, \quad i \in \underline{n-1},$$

as for  $A_{n-1}$ , plus the *short* root  $\theta_n = \alpha_n \in \mathfrak{t}^\vee$ . We then have

$$B_n = \left\{ \pm \alpha_{ij}^-, \pm \alpha_{ij}^+ \mid i < j \in \underline{n} \right\} \cup \{ \alpha_i \mid i \in \underline{n} \} \subseteq \mathfrak{t}^\vee,$$

writing  $\alpha_{ij}^+ := \alpha_i + \alpha_j$  [30, Ch. VI, § 4.5].

**7.1. Dynkin diagrams.** Let as above  $\Phi_{\mathfrak{h}} \subseteq B_n$  be the Levi subsystem associated to an element  $A \in \mathfrak{t}$ , and suppose  $\Delta_{\mathfrak{h}} = \Delta_{\mathfrak{g}} \cap \{A\}^{\perp} \subseteq \Delta_{\mathfrak{g}}$ . We must now consider whether the Dynkin (sub)diagram  $\mathcal{D}_{B_n}$  contains the rightmost node of

$$\mathcal{D}_{B_n} = \bullet_{\theta_1} - \bullet_{\theta_2} - \cdots - \bullet_{\theta_{n-1}} - \bullet_{\theta_n}.$$

If it does not, then  $\Delta_{\mathfrak{h}}$  only contains long roots, and  $\Phi_{\mathfrak{h}} \subseteq A_{n-1}$ . Else there exists an integer  $m \leq n$  such that

$$\Delta_{\mathfrak{h}} = \Delta_{\mathfrak{h}}^A \cup \Delta_{\mathfrak{h}}^B, \quad \Delta_{\mathfrak{h}}^B = \{\theta_{n-m+1}, \dots, \theta_n\} \subseteq \Delta_{\mathfrak{g}}.$$

Accordingly one finds  $\Phi_{\mathfrak{h}} \simeq \Phi_{\mathfrak{h}}^A \oplus B_m$ , with  $\Phi_{\mathfrak{h}}^A \subseteq A_{n-m}$  (inside  $\bigoplus_{i=1}^{n-m} \mathbb{C}\alpha_i$ , while  $B_m \subseteq \bigoplus_{i=n-m+1}^n \mathbb{C}\alpha_i$ ).

**7.2. Kernels.** Let again  $\mathfrak{U} = \text{Ker}(\Phi_{\mathfrak{h}}) \subseteq \mathfrak{t}$ , so by construction

$$\mathfrak{U} = \text{Ker}(\Phi_{\mathfrak{h}}^A) \cap \text{Ker}(B_m).$$

With the above notation one has  $\text{Ker}(B_m) = \mathbb{C}^{n-m} \times (0) \subseteq \mathfrak{t}$ , and we conclude by Prop. 6.1: there is a linear isomorphism

$$(27) \quad \mathbb{C}^J \simeq \mathfrak{U} \subseteq \mathbb{C}^{n-m} \times (0),$$

where  $J$  is the index set of the partition associated with  $\Phi_{\mathfrak{h}}^A$ , as in (23).

**7.3. Restricted arrangement and fundamental group.** Finally we will describe the hyperplane arrangement inside the kernel, provided by the kernel of the (positive) roots  $\alpha \in B_n \setminus \Phi_{\mathfrak{h}}$  after restriction to  $\mathfrak{U}$ .

**Theorem 7.1.** *The hyperplane arrangement in the kernel is of type  $B_{d+1}/C_{d+1}$ , where  $d = \dim(\mathfrak{U})$ .*

*Moreover, if no component of the  $J$ -partition is trivial, under the isomorphism (24) there is an identification of root systems*

$$BC_{d+1} \simeq B_n|_{\mathfrak{U}} \subseteq \mathfrak{U}^{\vee},$$

using the notation of (22).

*Proof.* Computing the restrictions of all roots shows one always has the inclusion

$$\left\{ \pm(\bar{\alpha}_I - \bar{\alpha}_{\hat{I}}), \pm(\bar{\alpha}_I + \bar{\alpha}_{\hat{I}}) \mid I \neq \hat{I} \in J \right\} \cup \{ \bar{\alpha}_I \mid I \in J \} \subseteq B_n|_{\mathfrak{U}},$$

using the notation in the proof of Thm. 6.1. Further the covector  $2\bar{\alpha}_I \in \mathfrak{U}^{\vee}$  appears from the restriction of  $\alpha_{ij}^{\pm} \in \mathfrak{t}^{\vee}$  if and only if there exists a pair  $(i, j)$  with  $i \in J$  and  $j \in I_i \setminus \{i\}$ .

These are all covectors obtained upon restriction to  $\mathfrak{U}$ , so the hyperplane arrangement is always of type  $B/C$ .  $\square$

*Remark 7.1.* The hyperplanes arrangements are always those of a root system, so their reflection groups are crystallographic; but the set of restricted functional themselves are *not* root systems in general.  $\triangle$

It follows that  $\pi_1(\mathbf{B}(\Phi_{\mathfrak{h}}, \Phi_{\mathfrak{g}}), A) \simeq PB_{d+1}^{BC}$  in this case.

**7.4. Bichromatic fission trees.** Again we can now reason recursively. In brief, a component of type B will produce a pure braid group of type B/C upon breaking to a Levi subsystem, and this will continue until such subsystems still contain a type-B component. At some point this might stop, in which case we will get back to only having (sub)components of type A—and pure braid groups.

This leads to the following natural generalisation of Def. 6.2. Denote  $\{g, b\}$  the set of colours “green” and “blue”, ordered by  $g \leq b$ ; then:

**Definition 7.1** (Bichromatic fission tree). A *bichromatic fission tree* is a fission tree  $\mathcal{T} = \mathcal{T}_Q$  equipped with a *colour function*  $c : \mathcal{T}_0 \rightarrow \{g, b\}$ ; in turn a colour function satisfies:

- $c(\Phi(i)) \geq c(i)$  for  $i \in \mathcal{T}_0 \setminus \{*\}$ ;
- $|\Phi^{-1}(i) \cap c^{-1}(b)| \leq 1$  for  $i \in \mathcal{T}_0$ .

The conditions mean that green nodes have green child-nodes, and that any node has at most one blue child-node, respectively.

Now consider a fission sequence in type B, i.e.

$$\Phi_{\eta_1} \subseteq \cdots \subseteq \Phi_{\eta_p} \subseteq \Phi_{\eta_{p+1}} = B_n.$$

As above this leads to a type-A filtration

$$\Phi_{\eta_1}^A \subseteq \cdots \subseteq \Phi_{\eta_p}^A \subseteq A_{n-1},$$

and to a filtration by irreducible subsystems

$$B_{m_1} \subseteq \cdots \subseteq B_{m_p} \subseteq B_n,$$

with embeddings on the last slots (at each step).

The algorithm to assign a bichromatic fission tree to such double filtrations is the following. A node  $i \in J_l$  corresponds to an irreducible component of the subsystem  $\Phi_{\eta_l} \subseteq B_n$ : put a green node for each type-A component, and a blue node if  $m_l > 0$ ; then define  $j = \Phi(i) \in J_{l+1}$  if the irreducible component of  $\Phi_{\eta_{l+1}}$  associated with  $j$  contains the irreducible component of  $\Phi_{\eta_l} \subseteq \Phi_{\eta_{l+1}}$  associated with  $i \in J_l$ .

Finally we extend the notation of § 6 by redefining

$$k_i := |\Phi^{-1}(i) \cap c^{-1}(g)| \geq 0, \quad i \in \mathcal{T}_0,$$

which is the number of *green* child-nodes of  $i$ .

**Theorem 7.2.** *There is a group isomorphism*

$$\Gamma_Q \simeq \prod_{c^{-1}(g)} \text{PB}_{k_i} \times \prod_{c^{-1}(b)} \text{PB}_{k_i}^{\text{BC}}.$$

*Conversely, pure local WMCGs of type B exhaust finite products of pure braid groups of types A and B/C.*

*Proof.* For the first statement, the new situation (with respect to Thm. 6.2) is that a blue node  $i \in \mathcal{T}_0$  yields a pure braid group of type  $B_k/C_k$ , where  $k$  is the number of its green child-nodes—corresponding to the decomposition of  $B_{m_l} \subseteq \Phi_{\eta_{l+1}}$  into type-A irreducible components for  $\Phi_{\eta_l} \subseteq \Phi_{\eta_{l+1}}$ .

For the second statement consider trees where no green node splits. If there are  $n_i \geq 1$  blue nodes with  $i \geq 1$  green child-nodes this yields

$$\Gamma_Q \simeq \prod_{i \geq 0} (\text{PB}_i^{\text{BC}})^{n_i},$$

which is an arbitrary finite product (analogously to (26)), and type-A factors are then obtained by splaying some green node.  $\square$

7.5. **Type C.** Taking  $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{C})$ , with root system

$$C_n = \left\{ \pm\alpha_{ij}^-, \pm\alpha_{ij}^+ \mid i < j \in \underline{n} \right\} \cup \{2\alpha_i \mid i \in \underline{n}\} \subseteq \mathfrak{t}^\vee,$$

yields the same situation (see [30, Ch. VI, § 4.6] for the construction of the root system).

In brief this is because  $C_n$  is the dual/inverse of  $B_n$ , and the Dynkin diagram has an analogous shape:

$$\mathcal{D}_{C_n} = \begin{array}{c} \bullet \quad \bullet \quad \cdots \quad \bullet \quad \bullet \quad \bullet \\ \theta_1 \quad \theta_2 \quad \quad \quad \theta_{n-1} \quad \theta_n \end{array},$$

where now  $\theta_n = 2\alpha_n$  is the long simple root.

This leads to the same hyperplane arrangements, and slight variations of the above arguments yield proofs of theorems analogous to 7.1–7.2. Hence bichromatic fission trees control pure local WMCGs of types A, B, and C.

*Remark 7.2.* Let us nonetheless stress a difference: the partition introduced in Lem. 6.1, in type A, classifies *all* root subsystem, thereby proving they are all obtained from fission.

This is false in types B/C: suffices to take a root subsystem which has more than an irreducible component of type B/C (respectively). Nonetheless the idea of Lem. 6.1 can be extended to treat all the classical types, generalising the partition  $\underline{n} + \underline{1} \rightarrow J$  to other combinatoric objects (which retain more information than the Dynkin diagram, cf. [81]).  $\triangle$

In the next section we will see that yet another generalisation is necessary for the last classical type, which finally leads to *noncrystallographic* arrangements.

## 8. TYPE D

For an integer  $n \geq 1$ , let  $\mathfrak{g} = \mathfrak{so}_{2n}(\mathbb{C})$ . The standard Cartan subalgebra  $\mathfrak{t} \subseteq \mathfrak{g}$  is identified with  $\mathbb{C}^n = \bigoplus_i \mathbb{C}e_i$ , and we retain the notations of §§ 6–7.

The usual choice of basis is  $\Delta_{\mathfrak{g}} = \{\theta_1, \dots, \theta_{n-1}\}$  with

$$\theta_i = \alpha_{i,i+1}^-, \quad i \in \underline{n-1},$$

as for  $A_{n-1}$ , and  $\theta_n = \alpha_{n-1,n}^+ \in \mathfrak{t}^\vee$  [30, Ch. VI, § 4.8]. We then have

$$D_n = \left\{ \pm\alpha_{ij}^-, \pm\alpha_{ij}^+ \mid i < j \in \underline{n} \right\}.$$

8.1. **Dynkin diagrams.** Let  $\Phi_{\mathfrak{h}} \subseteq D_n$  be a Levi subsystem.

Analogously to the types B/C, the question is whether the Dynkin diagram  $\mathcal{D}_{\mathfrak{h}} \subseteq \mathcal{D}_{D_n}$  has a component of type D or not, looking at

$$\mathcal{D}_{D_n} = \begin{array}{c} \bullet \quad \bullet \quad \cdots \quad \bullet \quad \bullet \quad \bullet \\ \theta_1 \quad \theta_2 \quad \quad \quad \theta_{n-1} \\ \theta_n \end{array}.$$

If not, then  $\Phi_{\mathfrak{h}} \subseteq A_{n-1}$ . Else  $\Phi_{\mathfrak{h}} \simeq \Phi_{\mathfrak{h}}^A \oplus D_m$ , with  $\Phi_{\mathfrak{h}}^A \subseteq A_{n-m}$  and  $D_m \subseteq (\mathbb{C}^m)^\vee$  for some integer  $m \leq n$ , as in § 7.

## 8.2. Kernels. One finds

$$\text{Ker}(\Phi_h) = \text{Ker}(\Phi_h^\Lambda) \cap \text{Ker}(D_m),$$

and again  $\text{Ker}(D_m) = \mathbb{C}^{n-m} \times (0) \subseteq \mathfrak{t}$ . Thus  $U = \text{Ker}(\Phi_h)$  only depends on the type-A irreducible components of  $\Phi_h$ , as in (27).

**8.3. Restricted arrangement and fundamental group.** To introduce our new example, consider two integers  $r, s \geq 0$ . Consider the following hyperplane arrangement inside  $\mathbb{C}^{r+s}$ : it contains the hyperplanes  $\text{Ker}(\alpha_{ij}^\pm)$  for  $i \neq j \in \underline{r+s}$  (i.e. the root hyperplanes of  $D_{r+s}$ ) but also the hyperplanes  $\text{Ker}(\alpha_i)$  for  $i \in \underline{r}$ . Hence  $\mathbb{C}^r \times \{0\} \subseteq \mathbb{C}^{r+s}$  contains the root hyperplanes of type  $B_r/C_r$ , but there is no splitting.

We will say this is an hyperplane arrangement of “exotic” type  $(B_r/C_r)D_s$ .

*Remark 8.1.* Note the reflection group generated by this hyperplane arrangement is the Weyl group of type  $B_{r+s}/C_{r+s}$  if  $r > 0$ , else it is the Weyl group of type  $D_s$ ; this is thus always crystallographic, but the hyperplane arrangement itself is *not* that of a root system in general: e.g. the easiest nontrivial example yields 7 hyperplanes in  $\mathbb{C}^3$ . (Note that there are no irreducible, reduced, rank-3 root systems with 14 roots.)  $\triangle$

**Theorem 8.1.** *There are two cases:*

- If  $m > 0$  then the hyperplane arrangement in the kernel is of type  $B_{d+1}/C_{d+1}$ , where  $d = \dim(U)$ ;
- if  $m = 0$  then the hyperplane arrangement in the kernel is of type  $(B_r/C_r)D_s$ , where  $r \leq |J|$  is the number of nontrivial irreducible components of  $\Phi_h^\Lambda \subseteq \Phi_h$ , and  $s = |J| - r$ .

*Proof.* In the notation of the proof of Thm. 6.1, one always has

$$\left\{ \pm(\bar{\alpha}_I - \bar{\alpha}_{\hat{I}}), \pm(\bar{\alpha}_I + \bar{\alpha}_{\hat{I}}) \mid I \neq \hat{I} \in J \right\} \subseteq D_n|_U,$$

but further some of the covectors  $\bar{\alpha}_I, 2\bar{\alpha}_I \in U^\vee$  may appear.

Namely if  $I \in J$  is not a singleton then  $2\bar{\alpha}_I \in D_n|_U$ , and further if  $m > 0$  then  $\bar{\alpha}_I \in D_n|_U$  for all  $I \in J$ , leading to the classification in the statement.  $\square$

Hence we have 2 cases: if  $m > 0$  one has  $\pi_1(\mathbf{B}(\Phi_h, \Phi_g), \Lambda) \simeq \text{PB}_{d+1}^{\text{BC}}$ , while if  $m = 0$  then

$$\pi_1(\mathbf{B}(\Phi_h, \Phi_g), \Lambda) = \text{PB}_{r,s}^{\text{BC,D}},$$

denoting  $\text{PB}_{r,s}^{\text{BC,D}}$  the pure (Artin) braid group of the hyperplane arrangement of type  $(B_r/C_r)D_s$ .

To study the latter further, write  $\mathbf{z} = (z_1, \dots, z_r) \in \mathbb{C}^r$ ,  $\mathbf{w} = (w_1, \dots, w_s) \in \mathbb{C}^s$ . Then explicitly the “exotic” hyperplane complement is

$$(28) \quad X_{r,s} = \left\{ (\mathbf{z}, \mathbf{w}) \in \mathbb{C}^{r+s} \mid z_i \neq 0, z_i \neq \pm z_j, z_i \neq \pm w_k, w_k \neq \pm w_l \right\} \subseteq \mathbb{C}^{r+s}.$$

Denote then  $F_i$  the free group on  $i \geq 0$  generators.

**Proposition 8.1.** *There is a group isomorphism*

$$\text{PB}_{r,1}^{\text{BC,D}} \simeq \text{PB}_r^{\text{BC}} \times F_{2r}.$$

*Proof.* Consider the subspace  $X_r := X_{r,1} \cap (\mathbb{C}^r \times \{0\}) \subseteq \mathbb{C}^{r+1}$ , so that

$$X_r \simeq \{z \in \mathbb{C}^r \mid z_i \neq 0, z_i \neq \pm z_j\} \subseteq \mathbb{C}^r,$$

which is the root-hyperplane complement of type  $B_r/C_r$ . Then there is a canonical projection  $p : X_{r,1} \rightarrow X_r$  with fibres

$$p^{-1}(z) \simeq \{w \in \mathbb{C} \mid w \neq \pm z_i\} \subseteq \mathbb{C},$$

i.e. a locally trivial fibration

$$(29) \quad Y_r \hookrightarrow X_{r,1} \xrightarrow{p} X_r, \quad Y_r := \mathbb{C} \setminus \{\pm 1, \dots, \pm r\}.$$

Now  $Y_r$  and  $X_r$  are path-connected, and further  $X_r$  is a  $K(\pi, 1)$ -space [41, 33]. Hence (29) induces an exact sequence of fundamental groups (omitting base points):

$$(30) \quad 1 \longrightarrow F_{2r} \longrightarrow \text{PB}_{r,1}^{\text{B/C,D}} \xrightarrow{\pi_1(p)} \text{PB}_r^{\text{B/C}} \longrightarrow 1.$$

Finally there is a canonical global (zero) section  $X_r \rightarrow X_{r,1}$  splitting (30).  $\square$

*Remark 8.2* (Exceptional isomorphism). If further  $r = 1$  then (30) simplifies to

$$1 \longrightarrow F_2 \longrightarrow \text{PB}_{1,1}^{\text{B/C,D}} \longrightarrow \mathbb{Z} \longrightarrow 1,$$

and in this case we can identify the extension.

Namely the space  $X_{1,1} \subseteq \mathbb{C}^2$  is isomorphic to the root-hyperplane complement of type  $A_2$ , essentially in view of the exceptional isomorphism  $D_3 \simeq A_3$  and the results of § 6.

Hence  $\text{PB}_{1,1}^{\text{B/C,D}} \simeq \text{PB}_3$ , and using  $\text{PB}_2 \simeq \mathbb{Z}$  we see (30) becomes the usual split extension

$$1 \longrightarrow F_2 \longrightarrow \text{PB}_3 \longrightarrow \text{PB}_2 \longrightarrow 1. \quad \triangle$$

**8.4. Generalised fission trees.** Once more a filtration of fission subsystems splits into  $\Phi_{h_1}^A \subseteq \dots \subseteq \Phi_{h_p}^A \subseteq A_{n-1}$  and  $D_{m_1} \subseteq \dots \subseteq D_{m_p} \subseteq D_n$ , for an increasing sequence of integers  $m_i \leq n$ .

To encode  $\Gamma_Q$  we now need to retain more information, according to the statement of Thm. 8.1: namely at each level we must recall the number of trivial/nontrivial type-A irreducible components of  $\Phi_{h_l} \cap D_{m_{l+1}} \subseteq \Phi_{h_{l+1}}$ .

This leads to the following generalisation of Def. 7.1. Introduce the set  $\{s, l\}$  of diameters “small” and “large”, ordered by  $s \leq l$ ; then:

**Definition 8.1** (Generalised fission tree). A *generalised fission tree* is a bichromatic fission tree  $(\mathcal{T}_Q, c)$  equipped with a *diameter function*  $d : \mathcal{T}_0 \rightarrow \{s, l\}$ ; in turn a diameter function satisfies:

- $d(i) = l$  if  $c(i) = b$ ;
- $d(\Phi(i)) \geq d(i)$  for  $i \in \mathcal{T}_0 \setminus \{*\}$ ;
- $k_i \leq 1$  if  $d(i) = s$ .

Hence green nodes can be small or large; large green nodes can have (green) child-nodes of any diameter, while small green nodes cannot split.

The algorithm to attach a generalised fission tree to a double filtration as above is the following. A node  $i \in J_l$  corresponds to an irreducible component of the subsystem  $\Phi_{h_l} \subseteq D_n$ : put a large green node for each *nontrivial* type-A component,

a small green node for each *trivial* type-A component, and finally a large blue node if  $m_1 > 0$ . The parent-node function is determined as in the bichromatic case.

To compute  $\Gamma_Q$  in terms of the tree, note there exists a unique blue node  $i_0 \in \mathcal{T}_0$  with no blue child-nodes: let  $r_0, s_0 \geq 0$  be the number of large and small child-nodes of  $i_0$ , respectively, and let  $\mathcal{T}'_0 := \mathcal{T}_0 \setminus \{i_0\}$ . Then retain the notation of § 7.

**Theorem 8.2.** *There is a group isomorphism*

$$\Gamma_Q \simeq \prod_{c^{-1}(g)} \text{PB}_{k_i} \times \text{PB}_{r_0, s_0}^{\text{B/C, D}} \times \prod_{c^{-1}(b) \cap \mathcal{T}'_0} \text{PB}_{k_i}^{\text{B/C}}.$$

*Conversely, pure local WMCGs of type D are obtained by adding any one exotic factor to a pure local WMCG of type B/C.*

*Proof.* The new factor, with respect to Thm. 7.2, comes from irreducible type-D components which decompose into irreducible components of type A (cf. Thm. 8.1).

As for the second statement, if the special node  $i_0 \in \mathcal{T}_0$  is a leaf then  $\Gamma_Q$  only depends on the underlying bichromatic fission tree, and yields any pure local WMCG of type B/C. Then adding a new level where only  $i_0$  splits, and has no blue child-nodes, adds an exotic factor (28) of any kind.  $\square$

This is the most general pure local WMCG for a classical simple Lie algebra, containing a factor which is *not* in general the pure braid group of a simple Lie algebra.

## 9. PURE CABLED BRAID GROUPS

Here we prove the “multi-scale” (pure) braiding conjecture in type A, cf. [80]. We will do it using the fission trees and the pure braid group operad.

In brief one can express elements of  $\Gamma_Q$  as braids on as many strands as the number of leaves of the tree, formalising the driving intuition of the introduction; more precisely, *cabling* will provide an injective group morphism  $\Gamma_Q \hookrightarrow \text{PB}_{|J_1|}$ , where  $J_1 \subseteq \mathcal{T}_0$  are the leaves of  $\mathcal{T}_Q$ . The final statement is that pure local WMCGs generalise pure *cabled* braid groups (see Def. 9.1), and we still conjecture they are given by “braiding of braids” (see Conj. 9.1).

**9.1. Pure cabling.** There are two natural operations on (pure) braids:

- (1) the “direct sum”, i.e. the canonical group embedding

$$\prod_i \text{PB}_{m_i} \longrightarrow \text{PB}_m, \quad (\sigma_i)_i \longmapsto \bigoplus_i \sigma_i,$$

with  $m_i \in \mathbb{Z}_{\geq 0}$  and  $m = \sum_i m_i$ ;

- (2) the “block braid”

$$\text{PB}_m \longrightarrow \text{PB}_k, \quad \sigma \longmapsto \sigma \langle k_1, \dots, k_m \rangle,$$

with  $m, k_1, \dots, k_m \in \mathbb{Z}_{\geq 0}$  and  $k = \sum_i k_i$ , which is the function obtained by replacing the  $i$ -th strand of a braid by  $k_i$  parallel copies of it.

Then the *cabling* of a braid  $\tau \in \text{PB}_m$  onto the  $i$ -th strand of a braid  $\sigma \in \text{PB}_n$  is

$$(31) \quad \sigma \circ_i \tau := \sigma \left( \underbrace{1, \dots, 1}_{i-1 \text{ times}}, \underbrace{m, 1, \dots, 1}_{n-i \text{ times}} \right) \cdot \left( \underbrace{\text{Id}_1 \oplus \dots \oplus \text{Id}_1}_{i-1 \text{ times}} \oplus \tau \oplus \underbrace{\text{Id}_1 \oplus \dots \oplus \text{Id}_1}_{n-i \text{ times}} \right) \in \text{PB}_{m+n-1},$$

where on the rightmost factor  $\text{Id}_1 \in \text{PB}_1$ . In words this means replacing the  $i$ -th strand of  $\sigma$  with the braid  $\tau$ .

One can show the data of the sets  $P(n) = \text{PB}_n$ , the unit  $\text{Id}_1 \in \text{PB}_1$ , and the maps (31), satisfies the associativity/unity axioms of an operad (as introduced in [28, 76, 29]), leading to the *pure braid group operad*  $\mathcal{PB}$  [105, § 5]. In particular “simultaneous” cabling yields the operadic composition

$$\gamma : \text{PB}_n \times \prod_{i=1}^n \text{PB}_{k_i} \longrightarrow \text{PB}_m, \quad (\sigma, \tau_1, \dots, \tau_n) \longmapsto \gamma(\sigma; \tau_1, \dots, \tau_n),$$

where  $m = \sum_i k_i$ , defined by

$$(32) \quad \gamma(\sigma; \tau_1, \dots, \tau_n) := \sigma \langle k_1, \dots, k_n \rangle \cdot (\tau_1 \oplus \dots \oplus \tau_n).$$

In principle this is only a function of sets, but if we equip the domain with the direct-product group structure then:

**Lemma 9.1.** *The operadic composition (32) is an injective group morphism.*

*Proof.* The compatibility with products follows from [105, Lem. 5.2.4].<sup>9</sup>

To show injectivity we can prove that if

$$\sigma' = \sigma \langle 1, \dots, 1, m, 1, \dots, 1 \rangle = \text{Id}_1 \oplus \text{Id}_1 \oplus \tau \oplus \text{Id}_1 \cdots \oplus \text{Id}_1 \in \text{PB}_{m+n-1},$$

for some  $(\sigma, \tau) \in \text{PB}_n \times \text{PB}_m$ , then both  $\sigma$  and  $\tau$  are trivial. This identity implies the first  $i-1$  and the last  $n-i$  strands of  $\sigma'$  have trivial braiding, so the same is true of all the strands of  $\sigma$  except at most the  $i$ -th one; but if this had nontrivial braiding then the “central”  $m$  strands of  $\sigma'$  would cross the “peripheral” ones, and  $\sigma' = \sigma \langle 1, \dots, 1, m, 1, \dots, 1 \rangle$  is impossible.  $\square$

Let now  $\mathcal{T}$  be a tree with nodes  $\mathcal{T}_0$ , and parent-node function  $\phi : \mathcal{T}_0 \setminus \{*\} \rightarrow \mathcal{T}_0$ , as in § 6. Retain the notation for the levels  $J_l \subseteq \mathcal{T}_0$  and the number  $k_i \geq 0$  of child-nodes of  $i \in \mathcal{T}_0$ .

**Definition 9.1.** The *pure cabled braid group*  $\mathcal{PB}(\mathcal{T})$  of  $\mathcal{T}$  is the group obtained at the end of the following sequence of applications of (32):

- start at the root and set  $\mathcal{PB}(\mathcal{T})_{p+1} := \text{PB}_1$  (the trivial group);
- for each level  $l \in \{p, \dots, 1\}$  define recursively

$$\mathcal{PB}(\mathcal{T})_l := \gamma \left( \mathcal{PB}(\mathcal{T})_{l+1} \times \prod_{J_{l+1}} \text{PB}_{k_i} \right) \subseteq \text{PB}_{|J_l|}.$$

By construction  $\mathcal{PB}(\mathcal{T}) = \mathcal{PB}(\mathcal{T})_1 \subseteq \text{PB}_{|J_1|}$  is a subgroup of the pure braid group on as many strands as the leaves of  $\mathcal{T}$ , and finally matching up fission/cabling trees yields the following.

**Theorem 9.1.** *In type A there is a group isomorphism  $\Gamma_Q \simeq \mathcal{PB}(\mathcal{T}_Q)$ .*

*Proof.* By induction on  $p \geq 1$ , the height of  $\mathcal{T}_Q$ .

The base uses part of the operad unity axiom, namely the identity

$$\gamma(\text{PB}_1 \times \text{PB}_k) = \text{PB}_k, \quad k \in \mathbb{Z}_{\geq 1}.$$

<sup>9</sup>This is thus a (noncrossed) “group” operad [107, Ex. 2.11] (cf. [106, 96]), a.k.a. an “action” operad [37] (cf. [105, Def. 4.1.1]).



For the inductive step, the recursive hypothesis yields

$$\mathcal{PB}(\mathcal{T})_2 = \prod_{\mathcal{T}'_0} \text{PB}_{k_i} \subseteq \text{PB}_{|J_2|},$$

where  $\mathcal{T}'_0 := \mathcal{T}_0 \setminus J_1$  are the nodes of the (sub)tree obtained by pruning the leaves; thus

$$\mathcal{PB}(\mathcal{T})_1 = \gamma \left( \prod_{\mathcal{T}'_0} \text{PB}_{k_i} \times \prod_{J_2} \text{PB}_{k_i} \right) \simeq \prod_{\mathcal{T}_0} \text{PB}_{k_i} \simeq \Gamma_Q,$$

using both Lem. 9.1 and Thm. 6.2.  $\square$

*Example 9.1* (Braiding of Stokes data). To showcase future applications, we will write here an explicit formula for an example of braiding of Stokes data, i.e. an action of a pure cabled braid group on a wild character variety.

Consider again the irregular type  $Q$  of (7), for  $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$ . Recall

$$\mathbf{B}_Q \simeq \{ a, a', b, b', c \in \mathbb{C} \mid a \neq a', b \neq b' \} \simeq \text{Conf}_2(\mathbb{C})^2 \times \mathbb{C},$$

and the fission tree  $\mathcal{T}_Q$  appears in Fig. 3. Then the pure cabled braid group  $\mathcal{PB}(\mathcal{T}_Q)$ , is generated by the braids of Fig. 1.

Now, using the description of Stokes data by level [21, § 7.2], the space of Stokes representations is identified with tuples

$$(33) \quad \rho = (h, B_1^1, B_3^1, B_1^2, B_2^2, B_3^2, B_4^2) \in \text{SL}_3(\mathbb{C})^7, \quad \text{such that} \quad h \cdot (B_3^1 B_1^1) \cdot (B_4^2 B_3^2 B_2^2 B_1^2) = 1.$$

Here  $h \in \mathbb{T}$  is in the maximal torus, and the rest are unipotent elements.

Then the explicit action of the “level-2” generator is

$$(34) \quad \sigma : \rho \mapsto (h, B_1^1, B_3^1, B_3^2, B_4^2, h_1^{-1} B_1^2 h_1, h_1^{-1} B_2^2 h_1), \quad h_1 := h B_3^1 B_1^1 \in \text{SL}_3(\mathbb{C}),$$

while the “level-1” generator acts by

$$(35) \quad \tau_1 : \rho \mapsto (h, B_3^1, h^{-1} b_1 h, b_1 B_1^2 b_1^{-1}, b_1 B_2^2 b_1^{-1}, b_1 B_3^2 b_1^{-1}, b_1 B_4^2 b_1^{-1}), \quad b_1 := B_1^1.$$

It is straightforward to check that these actions commute—and that the (quasi) moment-map condition (33) is preserved.

Further (34)–(35) commute with the diagonal conjugation action of  $\mathbb{T} \subseteq G$  on the space of Stokes representations, hence they descend to an action on the quasi-Hamiltonian  $\mathbb{T}$ -quotient, which is precisely the wild character variety  $\mathcal{M}_B$ . This is analogous to the fact that the  $\text{PB}_m$ -action commutes with the diagonal  $G$ -action on the space of monodromy representations of a punctured sphere, thus descending to an action on the tame character variety of the introduction.  $\triangle$

We conclude with a precise formulation of the multilevel/nongeneric braiding conjecture, beyond type A:

**Conjecture 9.1** (Classical pure local cabled braid groups). *There exists a 3-coloured (action) operad  $\mathcal{P}$ , whose evaluations on (a variation of) the generalised fission trees of Def. 8.1 recovers all pure local WMCGs of classical type:  $\mathcal{P}(\mathcal{T}_Q) \simeq \Gamma_Q$ , where  $Q \in \mathfrak{t} \otimes \mathcal{T}_{\Sigma, \alpha}$  is an irregular type of type A, B, C or D.<sup>10</sup>*

<sup>10</sup>We also expect that two colours should be enough to treat type B/C, according to Def. 7.1, e.g. in Willwacher’s “moperads” [98]—seeing the corresponding operads as modules for the type-A pure braid group operad  $\mathcal{PB}$ . Note also Coron’s relations with Orlik–Solomon algebras of hyperplane arrangements [38] should be relevant here.

## 10. OUTLOOK

There is a *full/nonpure* version of local WMCGs, which involves taking out the Weyl action on irregular types, leading to the notion of a “bare” irregular type [21, Rk. 10.6] (a.k.a. an “irregular class”); and moreover one can define *twisted* irregular types/classes [26], leading to “twisted” (dressed/bare) wild Riemann surfaces. A diagram-theoretic description of twisted irregular classes for  $G = \mathrm{GL}_n(\mathbb{C})$  was given in [27], and more generally in [43]: their admissible deformations will be considered elsewhere.<sup>11</sup>

Further the admissible deformations of wild Riemann surfaces allow for varying the underlying pointed Riemann surface, as in Def. 2.2, and we plan to study the topology of the relevant (universal) “global” deformation spaces.

## APPENDIX A. SOME NOTIONS/NOTATIONS WE USE

We collect here some standard material used throughout the paper, also fixing notation. Besides Bourbaki, see e.g. [36, § 2] and [59].

**About Lie algebras.** Let  $\mathfrak{g}$  be a finite-dimensional reductive Lie algebra.

The *centraliser* of a subset  $S \subseteq \mathfrak{g}$  is

$$\mathfrak{Z}_{\mathfrak{g}}(S) = \{ X \in \mathfrak{g} \mid [X, S] = 0 \} \subseteq \mathfrak{g},$$

and in particular  $\mathfrak{Z}_{\mathfrak{g}} = \mathfrak{Z}_{\mathfrak{g}}(\mathfrak{g})$  is the centre. There is a Lie algebra decomposition  $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{Z}_{\mathfrak{g}}$ , where  $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$  is the *semisimple part* of  $\mathfrak{g}$ .

A *Cartan subalgebra*  $\mathfrak{t} \subseteq \mathfrak{g}$  is a maximal abelian subalgebra consisting of semisimple elements, so  $\mathfrak{Z}_{\mathfrak{g}} \subseteq \mathfrak{t}$ . The Cartan subalgebra decomposes as  $\mathfrak{t} = \mathfrak{t}' \oplus \mathfrak{Z}_{\mathfrak{g}}$ , where  $\mathfrak{t}' = \mathfrak{t} \cap \mathfrak{g}'$ , which is a Cartan subalgebra of  $\mathfrak{g}'$ . Conversely the Cartan subalgebras *split*  $\mathfrak{g}$  and  $\mathfrak{g}'$ , and the pairs  $(\mathfrak{g}, \mathfrak{t})$  and  $(\mathfrak{g}', \mathfrak{t}')$  are *split* Lie algebras.

The *rank* of  $\mathfrak{g}$  is the dimension of a Cartan subalgebra, so in particular

$$\mathrm{rk}(\mathfrak{g}) = \dim(\mathfrak{Z}_{\mathfrak{g}}) + \mathrm{rk}(\mathfrak{g}'),$$

while  $\mathrm{rk}(\mathfrak{g}')$  is the *semisimple rank* of  $\mathfrak{g}$ .

Given a (reductive) split Lie algebra  $(\mathfrak{g}, \mathfrak{t})$ , a *root* is a linear functional  $\alpha \in \mathfrak{t}^{\vee} \setminus \{0\}$  such that the following subspace is nonzero:

$$\mathfrak{g}_{\alpha} := \{ X \in \mathfrak{g} \mid [A, X] = \langle \alpha, A \rangle X \text{ for } A \in \mathfrak{t} \} \subseteq \mathfrak{g}.$$

The (finite) set of roots is denoted  $\Phi_{\mathfrak{g}} = \Phi(\mathfrak{g}, \mathfrak{t}) \subseteq \mathfrak{t}^{\vee}$ , and is called the *root system* of  $(\mathfrak{g}, \mathfrak{t})$ . Note all roots vanish on the centre, so the root system does *not* span  $\mathfrak{t}^{\vee}$  in the nonsemisimple case. Conversely, if  $\Phi_{\mathfrak{g}'} \subseteq (\mathfrak{t}')^{\vee}$  is the (spanning) root system of the semisimple part  $\mathfrak{g}' \subseteq \mathfrak{g}$ , then its elements are precisely the restriction (to  $\mathfrak{t}'$ ) of the elements of  $\Phi_{\mathfrak{g}}$ .

**About root systems.** Hence we will consider root systems  $\Phi \subseteq V$ , where  $V$  is a finite-dimensional vector space, which are *crystallographic* (= they have integer Cartan numbers), but not necessarily irreducible or spanning. The subspace  $\mathrm{span}_{\mathbb{C}}(\Phi) \subseteq V$  is the *essential* part of  $(V, \Phi)$ ,<sup>12</sup> and the *rank* of  $\Phi$  is the dimension of the essential part. (Beware this coincides with the *semisimple* rank of the associated Lie algebra.)

<sup>11</sup>The general (nongeneric) full/nonpure untwisted case is now studied in [44], while the type-A twisted case (both pure and full) is the subject of [25].

<sup>12</sup>On this space the root-hyperplane arrangement is “essential” [34, § 2.B].

A *root subsystem*  $\Phi' \subseteq \Phi$  is a subset which is preserved by the reflections associated with the roots it contains. Such subsystems are permuted by automorphisms of  $\Phi$ , so in particular by the Weyl group, cf. [79].

A root subsystem  $\Phi' \subseteq \Phi$  is *Levi* if it is closed under  $\mathbb{Q}$ -linear combinations of its elements, provided these are still roots: in the case of a split reductive Lie algebra  $(\mathfrak{g}, \mathfrak{t})$ , it is equivalent to ask that  $\Phi'$  is the annihilator of an element  $X \in \mathfrak{t}$ —intersected with  $\Phi$ .

The *direct sum* of two root systems  $(\Phi_1, V_1), (\Phi_2, V_2)$  is

$$\Phi_1 \oplus \Phi_2 = (V_1, \Phi_1) \oplus (V_2, \Phi_2) := (V_1 \oplus V_2, \Phi_1 \amalg \Phi_2).$$

We will also encounter *nonreduced* root systems. There exists a unique (spanning) irreducible nonreduced rank- $n$  root system, up to isomorphism, denoted  $BC_n$ : it consists of the vectors

$$\{ \pm(e_i - e_j), \pm(e_i + e_j) \mid 1 \leq i \neq j \leq n \} \cup \{ e_i, 2e_i \mid 1 \leq i \leq n \} \subseteq V = \mathbb{C}^n,$$

using the canonical basis of  $\mathbb{C}^n = \bigoplus_i \mathbb{C}e_i$  [30, Ch. VI, § 4.14].

**About braid groups and hyperplane arrangements/complements.** We denote  $PB_n$  the *pure braid group* on  $n \geq 0$  strands [4]. (So  $PB_0$  and  $PB_1$  are trivial.) It is the fundamental group of the space

$$(36) \quad \text{Conf}_n(\mathbb{C}) = \mathbb{C}^n \setminus \bigcup_{1 \leq i \neq j \leq n} H_{ij}, \quad H_{ij} := \{ (z_1, \dots, z_n) \in \mathbb{C}^n \mid z_i = z_j \} \subseteq \mathbb{C}^n,$$

i.e. the space of configurations of (ordered)  $n$ -tuples points in the complex plane [50]. These are thus the fundamental group of complements of hyperplane arrangements, i.e. “hyperplane complements” for short.

More generally for a (reductive) split Lie algebra  $(\mathfrak{g}, \mathfrak{t})$  we consider the *pure  $\mathfrak{g}$ -braid group*  $PB_{\mathfrak{g}}$ , which is the fundamental group of the space

$$(37) \quad \mathfrak{t}_{\text{reg}} = \mathfrak{t} \setminus \bigcup_{\Phi_{\mathfrak{g}}} \text{Ker}(\alpha) \subseteq \mathfrak{t},$$

viz. the complement of the root-hyperplane arrangement—the “root-hyperplane complement” [31, 32, 41]. Such arrangements are said to be *crystallographic*, and in particular (36) corresponds to a simple Lie algebra of type  $A_{n-1}$ .

In the case of simple Lie algebras of type  $B_n/C_n$  (resp.  $D_n$ ) we will denote  $PB_n^{BC}$  the pure  $\mathfrak{g}$ -braid group (resp.  $PB_n^D$ ). (Note types  $B_n$  and  $C_n$  yield the same complement (37).)

The Weyl groups are the reflection groups generated by the root-hyperplane arrangements. Conversely a reflection group is *crystallographic* if it is the Weyl group of a root system [30, Ch. VI, § 2.5].

## APPENDIX B. SOME REMARKS ABOUT QUANTISATION

For completeness we review here some of the literature about the quantum analogue of the background material. While we do not need/use in this paper, it inspires part of this work.

The main idea is that the nonlinear monodromy actions of mapping class/braid groups on wild character varieties have linear analogues obtained after *quantisation*. In turn, this means considering the Poisson varieties as the phase-spaces of classical mechanical systems (parameterising pure states), or as spaces of classical gauge

fields, and replace them by their analogue in quantum mechanics/field theory (see e.g. [82, App. A] and [49] for the basic mathematical dictionary).

Several constructions are possible, crossing the boundary between mathematics and theoretical physics: rigorous mathematical approaches include *geometric* quantisation, born out of the work of Kirillov, Kostant and Souriau on coadjoint orbits [63, 67, 92], and *deformation* quantisation [7, 8]—which concentrates on the quantisation of observables. In any sensible formalism, quantisation replaces a Poisson/symplectic fibre bundle by a family of vector spaces: the mathematician’s aim is to prove these assemble into a vector bundle, and equip it with a flat (projective) connection, whose monodromy finally provides (projective) “quantum” representations of the fundamental group of the base.

In this framework we notably find the Knizhnik–Zamolodchikov connection (KZ) [65, 48], in the genus-zero Wess–Zumino–Novikov–Witten model (WZNW) [97, 78, 99, 100] for 2d conformal field theory [10, 87], quantising the Schlesinger system [85, 56]. But also the connection of Felder–Markov–Tarasov–Varchenko (FMTV) [52], quantising the system of JMMS [83].

Note the dual version of the Schlesinger system, which is a particular case of JMMS, was quantised earlier [18], recovering the “Casimir” connection of De Concini/Millson–Toledano Laredo (DMT) [77]. (Cf. the introduction of [83].)

The monodromy of KZ then features in the Kohn–Drinfel’d theorem [66, 46]. Analogously, the monodromy of DMT is tantamount to a Kohn–Drinfel’d theorem for  $q$ -Weyl groups [95], and recovers the action of Lusztig/Soibelman/Kirillov–Reshetikhin [69, 91, 64] on the Jimbo–Drinfel’d quantum group  $U_{qg}$  [61, 45]. This Hopf algebras quantises the algebra of functions on  $G^*$ : as mentioned above, this is precisely the wild character variety in this example, and the semiclassical limit of the action coincides with that of De Concini–Kac–Procesi. (The monodromy of FMTV was considered in [103].)

These examples are in the regular singular case, and in the generic irregular singular case:<sup>13</sup> the main result of [83, 84, 104] is that one can also do quantisation in the nongeneric case, namely quantising the system of [20], generalising all the above. (Cf. [35] for an extension to arbitrary polar divisors/structure groups.)

In all these important cases one sees the local wild moduli, viz. the irregular types, behave as the moduli of the underlying pointed surface even after quantisation. But the “semiclassical” theory of isomonodromic deformations goes beyond these examples, and it is still expressed in a language that lends itself to quantisation, so it provides a guide to prove analogous statements in a (much) more general context.

## REFERENCES

1. J. E. Andersen, *Asymptotic faithfulness of the quantum  $SU(n)$  representations of the mapping class groups*, Ann. of Math. (2) **163** (2006), no. 1, 347–368. 35

<sup>13</sup>We said the high-genus nonsingular case also fits this story: indeed one finds the connection of Witten [101, 2] in *complex* quantum Chern–Simons gauge theory, which is equivalent to a “complexified” Hitchin connection in genus one [3]. The original Hitchin connection [57, 5] is for *compact* Chern–Simons, so it requires starting from a (maximal) compact subgroup  $K \subseteq G$ ; nonetheless it yields “quantum” representations of mapping class groups, cf. e.g. [75, 1, 73].

Note an identification of the projectively flat vector bundles of nonabelian  $\theta$ -functions and WZWN conformal blocks was first given in [9, 51, 68], in absence of marked points—thus missing KZ. Recently this has been extended to the case of marked points, see [47, 15, 14] and references therein.

2. J. E. Andersen and N. L. Gammelgaard, *The Hitchin–Witten connection and complex quantum Chern–Simons theory*, [arXiv:1409.1035](#). 35
3. J. E. Andersen, A. Malusà, and G. Rembado, *Genus-one complex quantum Chern–Simons theory*, to appear in *J. Symplectic Geom.* 35
4. E. Artin, *Theorie der Zöpfe*, *Abh. Math. Sem. Univ. Hamburg* **4** (1925), no. 1, 47–72. 34
5. S. Axelrod, S. Della Pietra, and E. Witten, *Geometric quantisation of Chern–Simons gauge theory*, *J. Differential Geom.* **33** (1991), no. 3, 787–902. 35
6. W. Balsler, W. B. Jurkat, and D. A. Lutz, *Birkhoff invariants and Stokes’ multipliers for meromorphic linear differential equations*, *J. Math. Anal. Appl.* **71** (1979), no. 1, 48–94. 3
7. F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer, *Deformation theory and quantisation. I. Deformations of symplectic structures*, *Ann. Physics* **111** (1978), no. 1, 61–110. 35
8. ———, *Deformation theory and quantisation. II. Physical applications*, *Ann. Physics* **111** (1978), no. 1, 111–151. 35
9. A. Beauville and Y. Laszlo, *Conformal blocks and generalized theta functions*, *Comm. Math. Phys.* **164** (1994), no. 2, 385–419. 35
10. A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov, *Infinite conformal symmetry in two-dimensional quantum field theory*, *Nuclear Phys. B* **241** (1984), no. 2, 333–380. 35
11. O. Biquard and P. P. Boalch, *Wild nonabelian Hodge theory on curves*, *Compos. Math.* **140** (2004), no. 1, 179–204. 3
12. G. D. Birkhoff, *The generalized Riemann problem for linear differential equations and allied problems for linear difference and q-difference equations*, *Proc. Amer. Acad. Arts and Sci.* **49** (1913), 531–568. 2, 3
13. J. S. Birman, *Braids, links, and mapping class groups*, *Annals of Mathematics Studies*, No. 82, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1974. 3
14. I. Biswas, S. Mukhopadhyay, and R. A. Wentworth, *Geometrization of the TUY/WZW/KZ connection*, [arXiv:2110.00430](#). 35
15. ———, *Ginzburg algebras and the Hitchin connection for parabolic g-bundles*, [arXiv:2103.03792](#). 35
16. P. P. Boalch, *Irregular connections and Kac–Moody root systems*, [arXiv:0806.1050](#). 5, 9
17. ———, *Symplectic manifolds and isomonodromic deformations*, *Adv. Math.* **163** (2001), no. 2, 137–205. 2, 4
18. ———, *G-bundles, isomonodromy, and quantum Weyl groups*, *Int. Math. Res. Not.* (2002), no. 22, 1129–1166. 2, 3, 4, 9, 35
19. ———, *Quasi-Hamiltonian geometry of meromorphic connections*, *Duke Math. J.* **139** (2007), no. 2, 369–405. 4
20. ———, *Simply-laced isomonodromy systems*, *Publ. Math. Inst. Hautes Études Sci.* **116** (2012), 1–68. 3, 35
21. ———, *Geometry and braiding of Stokes data; fission and wild character varieties*, *Ann. of Math. (2)* **179** (2014), no. 1, 301–365. 2, 3, 4, 10, 11, 12, 15, 32, 33
22. ———, *Poisson varieties from Riemann surfaces*, *Indag. Math. (N.S.)* **25** (2014), no. 5, 872–900. 4
23. ———, *Wild character varieties, meromorphic Hitchin systems and Dynkin diagrams*, *Proceedings, Nigel Hitchin’s 70th Birthday Conference*, Oxford University Press, 2017. 4
24. ———, *Topology of the Stokes phenomenon*, *Integrability, quantization, and geometry. I. Integrable systems*, *Proc. Sympos. Pure Math.*, vol. 103.1, Amer. Math. Soc., Providence, RI, [2021] ©2021, pp. 55–100. 2
25. P. P. Boalch, J. Douçot, and G. Rembado, *Twisted local wild mapping class groups: configuration spaces, fission trees and complex braids*, [arXiv:2209.12695](#). 4, 6, 33
26. P. P. Boalch and D. Yamakawa, *Twisted wild character varieties*, [arXiv:1512.08091](#). 2, 33
27. ———, *Diagrams for nonabelian Hodge spaces on the affine line*, *C. R. Math. Acad. Sci. Paris* **358** (2020), no. 1, 59–65. 33
28. J. M. Boardman and R. M. Vogt, *Homotopy-everything H-spaces*, *Bull. Amer. Math. Soc.* **74** (1968), 1117–1122. 31
29. ———, *Homotopy invariant algebraic structures on topological spaces*, *Lecture Notes in Mathematics*, Vol. 347, Springer-Verlag, Berlin–New York, 1973. 31
30. N. Bourbaki, *Éléments de mathématique. Fasc. XXXVII. Groupes et algèbres de Lie. Chapitres IV–VI*, Hermann, Paris, 1968. 18, 20, 24, 27, 34
31. E. Brieskorn, *Die Fundamentalgruppe des Raumes der regulären Orbits einer endlichen komplexen Spiegelungsgruppe*, *Invent. Math.* **12** (1971), 57–61. 2, 34
32. ———, *Sur les groupes de tresses [d’après V. I. Arnol’d]*, *Séminaire Bourbaki*, Exp. No. 401, 1973, pp. 21–44. *Lecture Notes in Math.*, Vol. 317. 2, 34

33. E. Brieskorn and K. Saito, *Artin-Gruppen und Coxeter-Gruppen*, *Invent. Math.* **17** (1972), 245–271. [2](#), [29](#)
34. M. Broué, G. Malle, and R. Rouquier, *Complex reflection groups, braid groups, Hecke algebras*, *J. Reine Angew. Math.* **500** (1998), 127–190. [33](#)
35. D. Calaque, G. Felder, G. Rembado, and R. Wentworth, *Wild orbits and generalised singularity modules: stratifications and quantisation*, [arXiv:2402.03278](#). [17](#), [35](#)
36. D. H. Collingwood and W. M. McGovern, *Nilpotent orbits in semisimple Lie algebras*, Van Nostrand Reinhold Mathematics Series, Van Nostrand Reinhold Co., New York, 1993. [17](#), [33](#)
37. A. S. Corner and N. Gurski, *Operads with general groups of equivariance, and some 2-categorical aspects of operads in Cat*, [arXiv:1312.5910](#). [31](#)
38. B. Coron, *Matroids, Feynman categories, and Koszul duality*, [arXiv:2211.12370](#). [32](#)
39. P. Crooks, *Complex adjoint orbits in Lie theory and geometry*, *Expo. Math.* **37** (2019), no. 2, 104–144. [17](#)
40. C. De Concini, V. G. Kac, and C. Procesi, *Quantum coadjoint action*, *J. Amer. Math. Soc.* **5** (1992), no. 1, 151–189. [3](#)
41. P. Deligne, *Les immeubles des groupes de tresses généralisés*, *Invent. Math.* **17** (1972), 273–302. [2](#), [29](#), [34](#)
42. P. Deligne, B. Malgrange, and J.-P. Ramis, *Singularités irrégulières*, Documents Mathématiques (Paris) [Mathematical Documents (Paris)], vol. 5, Société Mathématique de France, Paris, 2007, Correspondance et documents. [Correspondence and documents]. [4](#)
43. J. Douçot, *Diagrams and irregular connections on the Riemann sphere*, [arXiv:2107.02516](#). [33](#)
44. J. Douçot and G. Rembado, *Topology of irregular isomonodromy times on a fixed pointed curve*, *Transformation Groups* (2023). [4](#), [6](#), [33](#)
45. V. G. Drinfel'd, *Quantum groups*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986), Amer. Math. Soc., Providence, RI, 1987, pp. 798–820. [35](#)
46. ———, *Quasi-Hopf algebras*, *Algebra i Analiz* **1** (1989), no. 6, 114–148. [35](#)
47. J. K. Egsgaard, *Hitchin connections for genus 0 quantum representations*, Ph.D. thesis, Centre for Quantum Geometry of Moduli Spaces, University of Aarhus, 2015. [35](#)
48. P. I. Etingof, I. B. Frenkel, and A. A. Kirillov, Jr., *Lectures on representation theory and Knizhnik–Zamolodchikov equations*, Mathematical Surveys and Monographs, vol. 58, American Mathematical Society, Providence, RI, 1998. [35](#)
49. L. D. Faddeev and O. A. Yakubovskii, *Lectures on quantum mechanics for mathematics students*, Student Mathematical Library, vol. 47, American Mathematical Society, Providence, RI, 2009, Translated from the 1980 Russian original by Harold McFaden, With an appendix by Leon Takhtajan. [35](#)
50. E. Fadell and L. Neuwirth, *Configuration spaces*, *Math. Scand.* **10** (1962), 111–118. [34](#)
51. G. Faltings, *A proof for the Verlinde formula*, *J. Algebraic Geom.* **3** (1994), no. 2, 347–374. [35](#)
52. G. Felder, Y. Markov, V. Tarasov, and A. Varchenko, *Differential equations compatible with KZ equations*, *Math. Phys. Anal. Geom.* **3** (2000), no. 2, 139–177. [35](#)
53. W. M. Goldman, *The symplectic nature of fundamental groups of surfaces*, *Adv. in Math.* **54** (1984), no. 2, 200–225. [4](#)
54. M. Goresky, R. Kottwitz, and R. MacPherson, *Codimensions of root valuation strata*, *Pure Appl. Math. Q.* **5** (2009), no. 4, Special Issue: In honor of John Tate. Part 1, 1253–1310. [9](#)
55. J. Harnad, *Dual isomonodromic deformations and moment maps to loop algebras*, *Comm. Math. Phys.* **166** (1994), no. 2, 337–365. [3](#)
56. ———, *Quantum isomonodromic deformations and the Knizhnik–Zamolodchikov equations*, *Symmetries and integrability of difference equations* (Estérel, PQ, 1994), CRM Proc. Lecture Notes, vol. 9, Amer. Math. Soc., Providence, RI, 1996, pp. 155–161. [35](#)
57. N. J. Hitchin, *Flat connections and geometric quantization*, *Comm. Math. Phys.* **131** (1990), no. 2, 347–380. [35](#)
58. ———, *Frobenius manifolds*, Gauge theory and symplectic geometry (Montreal, PQ, 1995), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 488, Kluwer Acad. Publ., Dordrecht, 1997, pp. 69–112. [2](#)
59. J. E. Humphreys, *Introduction to Lie algebras and representation theory*, Springer-Verlag, New York-Berlin, 1972, Graduate Texts in Mathematics, Vol. 9. [33](#)
60. A. Hurwitz, *Ueber Riemann'sche Flächen mit gegebenen Verzweigungspunkten*, *Math. Ann.* **39** (1891), no. 1, 1–60. [3](#)
61. M. Jimbo, *A q-difference analogue of  $U(\mathfrak{g})$  and the Yang–Baxter equation*, *Lett. Math. Phys.* **10** (1985), no. 1, 63–69. [35](#)
62. M. Jimbo, T. Miwa, and K. Ueno, *Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. I. General theory and  $\tau$ -function*, *Phys. D* **2** (1981), no. 2, 306–352. [2](#), [3](#), [6](#)

63. A. A. Kirillov, *Unitary representations of nilpotent Lie groups*, Uspehi Mat. Nauk **17** (1962), no. 4 (106), 57–110. [35](#)
64. A. N. Kirillov and N. Y. Reshetikhin, *q-Weyl group and a multiplicative formula for universal R-matrices*, Comm. Math. Phys. **134** (1990), no. 2, 421–431. [35](#)
65. V. G. Knizhnik and A. B. Zamolodchikov, *Current algebra and Wess–Zumino model in two dimensions*, Nuclear Phys. B **247** (1984), no. 1, 83–103. [35](#)
66. T. Kohno, *Monodromy representations of braid groups and Yang–Baxter equations*, Ann. Inst. Fourier (Grenoble) **37** (1987), no. 4, 139–160. [35](#)
67. B. Kostant, *Quantization and unitary representations. I. Prequantization*, (1970), 87–208. Lecture Notes in Math., Vol. 170. [35](#)
68. Y. Laszlo, *Hitchin’s and WZW connections are the same*, J. Differential Geom. **49** (1998), no. 3, 547–576. [35](#)
69. G. Lusztig, *Quantum groups at roots of 1*, Geom. Dedicata **35** (1990), no. 1-3, 89–113. [35](#)
70. B. Malgrange, *Sur les déformations isomonodromiques. II. Singularités irrégulières*, Mathematics and physics (Paris, 1979/1982), Progr. Math., vol. 37, Birkhäuser Boston, Boston, MA, 1983, pp. 427–438. [2, 3, 6](#)
71. ———, *Équations différentielles à coefficients polynomiaux*, Progress in Mathematics, vol. 96, Birkhäuser Boston, Inc., Boston, MA, 1991. [2](#)
72. ———, *Déformations isomonodromiques, forme de liouville, fonction  $\tau$* , Ann. Inst. Fourier **54** (2004), no. 5, 1371–1392. [2](#)
73. J. Marché, *Introduction to quantum representations of mapping class groups*, [arXiv:1812.03888](#). [35](#)
74. J. Martinet and J.-P. Ramis, *Elementary acceleration and multisummability. I*, Ann. Inst. H. Poincaré Phys. Théor. **54** (1991), no. 4, 331–401. [2](#)
75. G. Masbaum, *Quantum representations of mapping class groups*, Groupes et géométrie, SMF Journ. Annu., vol. 2003, Soc. Math. France, Paris, 2003, pp. 19–36. [35](#)
76. J. P. May, *The geometry of iterated loop spaces*, Lecture Notes in Mathematics, Vol. 271, Springer-Verlag, Berlin-New York, 1972. [31](#)
77. J. J. Millson and V. Toledano Laredo, *Casimir operators and monodromy representations of generalised braid groups*, Transform. Groups **10** (2005), no. 2, 217–254. [35](#)
78. S. P. Novikov, *The Hamiltonian formalism and a multivalued analogue of Morse theory*, Uspekhi Mat. Nauk **37** (1982), no. 5(227), 3–49, 248. [35](#)
79. T. Oshima, *A classification of subsystems of a root system*, [arXiv:math/0611904](#). [34](#)
80. J.-P. Ramis, *Iso-irregular deformations of linear O.D.E and dynamics of Painlevé equations*, 2012, Talk at the International workshop on Integrability in Dynamical Systems and Control: [slides](#). [3, 30](#)
81. G. Rembado, *A colourful classification of (quasi) root systems and hyperplane arrangements*, [arXiv:2206.03779](#), submitted. [27](#)
82. ———, *Quantisation of moduli spaces and connections*, Ph.D. thesis, Université de Paris-Sud/Saclay, 2018, [tel-02004685](#). [35](#)
83. ———, *Simply-laced quantum connections generalising KZ*, Comm. Math. Phys. **368** (2019), no. 1, 1–54. [35](#)
84. ———, *Symmetries of the simply-laced quantum connections and quantisation of quiver varieties*, SIGMA Symmetry Integrability Geom. Methods Appl. **16** (2020), Paper No. 103, 44. [35](#)
85. N. Y. Reshetikhin, *The Knizhnik–Zamolodchikov system as a deformation of the isomonodromy problem*, Lett. Math. Phys. **26** (1992), no. 3, 167–177. [35](#)
86. C. Sabbah, *Harmonic metrics and connections with irregular singularities*, Ann. Inst. Fourier (Grenoble) **49** (1999), no. 4, 1265–1291. [2](#)
87. G. B. Segal, *The definition of conformal field theory*, Differential geometrical methods in theoretical physics (Como, 1987), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 250, Kluwer Acad. Publ., Dordrecht, 1988, pp. 165–171. [35](#)
88. Y. Sibuya, *Linear differential equations in the complex domain: problems of analytic continuation*, Translations of Mathematical Monographs, vol. 82, American Mathematical Society, Providence, RI, 1990, Translated from the Japanese by the author. [2](#)
89. C. T. Simpson, *Moduli of representations of the fundamental group of a smooth projective variety. I*, Inst. Hautes Études Sci. Publ. Math. (1994), no. 79, 47–129. [2](#)
90. ———, *Moduli of representations of the fundamental group of a smooth projective variety. II*, Inst. Hautes Études Sci. Publ. Math. (1994), no. 80, 5–79 (1995). [2](#)
91. Y. S. Soibelman, *Algebra of functions on a compact quantum group and its representations*, Algebra i Analiz **2** (1990), no. 1, 190–212. [35](#)

92. J.-M. Souriau, *Structure des systèmes dynamiques*, Maîtrises de mathématiques, Dunod, Paris, 1970. [35](#)
93. G. G. Stokes, *On the discontinuity of arbitrary constants which appear in divergent developments*, *Cam. Phil. Trans.* **X.1** (1857), 105–128. [2](#)
94. J. Tits, *Le problème des mots dans les groupes de Coxeter*, *Symposia Mathematica (INDAM, Rome, 1967/68)*, Vol. 1, Academic Press, London, 1969, pp. 175–185. [2](#)
95. V. Toledano Laredo, *A Kohno–Drinfeld theorem for quantum Weyl groups*, *Duke Math. J.* **112** (2002), no. 3, 421–451. [35](#)
96. N. Wahl, *Ribbon braids and related operads*, Ph.D. thesis, University of Oxford, 2001. [31](#)
97. J. Wess and B. Zumino, *Consequences of anomalous Ward identities*, *Phys. Lett.* **37B** (1971), 95–97. [35](#)
98. T. Willwacher, *The homotopy braces formality morphism*, *Duke Math. J.* **165** (2016), no. 10, 1815–1964. [32](#)
99. E. Witten, *Global aspects of current algebra*, *Nuclear Phys. B* **223** (1983), no. 2, 422–432. [35](#)
100. ———, *Nonabelian bosonization in two dimensions*, *Comm. Math. Phys.* **92** (1984), no. 4, 455–472. [35](#)
101. ———, *Quantization of Chern–Simons gauge theory with complex gauge group*, *Comm. Math. Phys.* **137** (1991), no. 1, 29–66. [35](#)
102. ———, *Gauge theory and wild ramification*, *Anal. Appl. (Singap.)* **6** (2008), no. 4, 429–501. [3](#)
103. X. Xu, *Stokes phenomenon and Yang–Baxter equations*, *Comm. Math. Phys.* **377** (2020), no. 1, 149–159. [35](#)
104. D. Yamakawa, *Quantization of simply-laced isomonodromy systems by the quantum spectral curve method*, *SUT J. Math.* **58** (2022), no. 1, 23–50. [35](#)
105. D. Yau, *Infinity operads and monoidal categories with group equivariance*, [arXiv:1903.03839](#). [7](#), [31](#)
106. J. Yoshida, *Group operads as crossed interval groups*, [arXiv:1806.03012](#). [31](#)
107. W. Zhang, *Group operads and homotopy theory*, [arXiv:1111.7090](#). [31](#)

(J. Douçot) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF LISBON, CAMPO GRANDE, EDIFÍCIO C6, PT-1749-016 LISBOA (PORTUGAL)

*Email address:* [jeandoucot@gmail.com](mailto:jeandoucot@gmail.com)

(G. Rembado) HAUSDORFF CENTRE FOR MATHEMATICS, UNIVERSITY OF BONN, 60 ENDENICHER ALLEE, D-53115 BONN (GERMANY)

*Current address:* Institut Montpellierain Alexander Grothendieck, University of Montpellier, Place Eugène Bataillon, 34090, Montpellier (France)

*Email address:* [gabriele.rembado@hcm.uni-bonn.de](mailto:gabriele.rembado@hcm.uni-bonn.de)

(M. Tamiozzo) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WARWICK, ZEEMAN BUILDING, CV47AL COVENTRY (UK)

*Current address:* Université Sorbonne Paris Nord, Institut Galilée, 99 Av. Jean Baptiste Clément, 93430 Villetaneuse (France)

*Email address:* [tamiozzo@math.univ-paris13.fr](mailto:tamiozzo@math.univ-paris13.fr)