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► **To cite this version:**

Jean Douçot, Gabriele Rembado, Matteo Tamiozzo. LOCAL WILD MAPPING CLASS GROUPS AND CABLED BRAIDS. 2022. hal-03748600v1

**HAL Id: hal-03748600**

**<https://hal.science/hal-03748600v1>**

Preprint submitted on 9 Aug 2022 (v1), last revised 12 Jun 2024 (v2)

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# LOCAL WILD MAPPING CLASS GROUPS AND CABLED BRAIDS

JEAN DOUÇOT, GABRIELE REMBADO, AND MATTEO TAMIOZZO

ABSTRACT. We will define and study some generalisations of pure  $\mathfrak{g}$ -braid groups that occur in the theory of connections on curves, for any complex reductive Lie algebra  $\mathfrak{g}$ . They make up local pieces of the wild mapping class groups, which are fundamental groups of (universal) deformations of wild Riemann surfaces, underlying the braiding of Stokes data and generalising the usual mapping class groups. We will establish a general product decomposition for the local wild mapping class groups, and in many cases define a fission tree controlling this decomposition. Further in type A we will show one obtains cabled versions of braid groups, related to braid operads.

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## INTRODUCTION AND MAIN RESULTS

Let  $\Sigma$  be a compact Riemann surface,  $\mathfrak{a} \subseteq \Sigma$  a finite subset, and  $G$  a complex reductive group—such as  $GL_n(\mathbb{C})$ . Much is known about the character variety

$$\mathcal{M}_{\mathfrak{B}} = \text{Hom}(\pi_1(\Sigma^\circ), G) // G, \quad (1)$$

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2020 *Mathematics Subject Classification.* 14D22,14F35,17B22,32S22.

*Key words and phrases.* Braid groups, braid group operads, isomonodromic deformations, root systems, reflection groups, hyperplane arrangements, fundamental groups, moduli spaces.

parametrising holomorphic connections on principal  $G$ -bundles over the punctured surface  $\Sigma^\circ := \Sigma \setminus \mathfrak{a}$  via their monodromy data, which is equivalent to considering algebraic connections with regular/tame singularities on  $\Sigma^\circ$ . In particular (1) is an algebraic Poisson variety, and the mapping class group of  $(\Sigma, \mathfrak{a})$  acts on it by algebraic Poisson automorphisms; in fact for any smooth family

$$\underline{\Sigma} \rightarrow \mathbf{B}$$

of Riemann surfaces with marked points the fundamental group of the base  $\mathbf{B}$  acts on the character variety (1) of any fibre, by algebraic Poisson automorphisms.

But in recent years many of the features of 2d gauge theory have been extended to the case of connections with *irregular/wild* singularities [45, 50, 51, 52, 59], which led to new algebraic Poisson varieties generalising the character varieties, the so-called “wild” character varieties [9, 17, 20], and in turn new algebraic Poisson automorphisms. These latter involve the  $\mathfrak{g}$ -braid groups [10, 17]: the fundamental groups of root-hyperplane complements for the Lie algebra  $\mathfrak{g} = \text{Lie}(G)$ .<sup>1</sup>

Part of this story was given a quantum field theory interpretation by Witten [66], and further the symplectic leaves of these Poisson varieties were shown to be new (complete) hyperkähler manifolds in the Biquard–Boalch extension of the nonabelian Hodge correspondence on Riemann surfaces [6].

An elegant way of approaching this subject is to define a generalisation of the notion of Riemann surface, and view their deformations as responsible for the new braiding. Some of the local parameters at an irregular singularity, the “irregular types” (see below and § 1), can be isolated and behave just like the moduli of the underlying pointed Riemann surface: adding the choice of irregular types then leads to the notion of a “wild” Riemann surface [17, Def. 8.1], generalising a compact Riemann surface with distinct marked points. Our main aim here is to define and study the fundamental group of the (fine) moduli space of irregular types, complementing the deformations of pointed Riemann surfaces.

The main result of op. cit. (extending [9, 10, 12] in the generic case) shows that any admissible family of wild Riemann surfaces over a base space  $\mathbf{B}$  determines a (nonlinear) fibre bundle  $\mathcal{M}_{\mathbf{B}} \rightarrow \mathbf{B}$  of Poisson wild character varieties generalising (1), equipped with a complete flat Ehresmann connection: this is the wild analogue of the “symplectic nature” of  $\pi_1(\Sigma^\circ)$  [39], and of the nonabelian Gauß–Manin connection [61]. In turn the fundamental group  $\pi_1(\mathbf{B})$  acts by algebraic Poisson automorphisms on any fibre, putting this subject within the context of dynamical systems (discrete group actions on Poisson spaces); in this paper we will compute examples of such groups.

Importantly this viewpoint subsumes many previous examples of monodromy actions in 2d gauge theory, including:

- (1)  $\mathbf{B} = \text{Conf}_m(\mathbb{C})$ , the genus-zero tame case, leading to the Hurwitz action of the pure braid group  $\text{PB}_m$  on the character variety (originated from [44]);
- (2)  $\mathbf{B} = \mathcal{M}_g$ , the nonsingular genus- $g$  case, leading to the action of the mapping class group  $\Gamma_g$  of  $\Sigma$  (cf. [8] for the relation with braid groups);

<sup>1</sup>The “generalised” braid groups [26, 27, 32], i.e. the Artin(–Tits) groups of type  $\mathfrak{g}$  [62, 28] (cf. § A).

- (3)  $\mathbf{B} = \mathfrak{t}_{\text{reg}}$ , for a “generic” pole of order two in genus zero,<sup>2</sup> leading to the action of the pure  $\mathfrak{g}$ -braid group  $\text{PB}_{\mathfrak{g}}$  on  $G^*$  [10] (viz. the classical action of the quantum Weyl group [31]).

In the latter case  $\mathfrak{t}_{\text{reg}}$  is the regular part of a Cartan subalgebra  $\mathfrak{t} \subseteq \mathfrak{g}$ , showcasing one of the key features: there is a “new” braiding which has to do with the structure group of the connections, and not with the motion of poles on the surface; e.g. if  $G = \text{GL}_n(\mathbb{C})$  we can deform the (noncoalescing) eigenvalues of a diagonal matrix. In this simplest generic example this is equivalent to the braiding of the marked points under the Fourier–Laplace transform [42],<sup>3</sup> but not so in the general case.

Moreover the above monodromy actions have “quantum” analogues, obtained by replacing the fibre bundle  $\underline{\mathcal{M}}_{\mathbf{B}} \rightarrow \mathbf{B}$  with a (projectively) flat vector bundle (over  $\mathbf{B}$ ), via fibrewise deformation/geometric quantisation: we thus find the monodromy of the Knizhnik–Zamolodchikov connection (KZ) [47, 48, 34], of the connection of Witten [65, 1] (equivalent to a “complexified” Hitchin connection in genus one [2]), and of the “Casimir” connection [63, 54]. Note a version of KZ involving irregular singularities has been considered in [58, 37, 36], but this still only allows for deformations of the positions of the marked points—in the quantisation of the isomonodromy system of [46].

On the contrary there exist new flat “quantum” connections corresponding to deformations of the irregular types [56, 57], showing the new parameters behave as the moduli of the underlying pointed surface even after quantisation (cf. [38] for a different, related viewpoint). Our main point is the “classical” theory goes beyond these examples, and is expressed in a language that lends itself to quantisation, providing a guide to prove analogous statements in a (much) more general context.

Indeed there have been more recent developments in understanding the boundary conditions for meromorphic connections on Riemann surfaces (in part related to the moduli of the surface), going beyond the generic case, which are precisely our focus here.

To introduce them, recall that locally around a point  $\mathfrak{a} \in \Sigma$  a meromorphic connection is encoded by a  $\mathfrak{g}$ -valued meromorphic 1-form  $\mathcal{A}$  on  $\Sigma$ : we assume that up to a formal gauge transformation and holomorphic terms one has

$$\mathcal{A} = dQ + \frac{\Lambda}{z} dz,$$

where  $z$  is a local coordinate with  $z(\mathfrak{a}) = 0$ , and where

$$\Lambda \in \mathfrak{g}, \quad Q = \sum_{i=1}^p \Lambda_i z^{-i} \in z^{-1}\mathfrak{t}[z^{-1}].$$

for an integer  $p \geq 1$ . Then  $Q$  is the irregular type at the point  $\mathfrak{a} \in \Sigma$ , and corresponds to “very good” orbits [19].

Now there is a notion of admissible deformations for the wild Riemann surface  $\Sigma = (\Sigma, \mathfrak{a}, Q)$  (see § 1), and if we fix the underlying pointed Riemann surface

<sup>2</sup>Generic isomonodromic deformations involve regular semisimple leading terms [5, 45, 50], developing the subject started in [7] (cf. Ex. 2.2).

<sup>3</sup>See [11, § 3] for the description of the braid group action on the dual/wild side.

$(\Sigma, a)$  this becomes a condition on the coefficients  $A_i \in \mathfrak{t}$  of the irregular type: in the generic case it just amounts to restricting  $A_p \in \mathfrak{t}_{\text{reg}}$ , but in general yields a (universal) deformation space  $\mathbf{B} \subseteq \mathfrak{t}^p$ , which is the main object of study here.

For example [16] considers nongeneric connections with poles of order three for  $G = \text{GL}_n(\mathbb{C})$ , corresponding to an irregular type  $Q = Tz^{-1} + Az^{-2}$ . Then the admissible deformations of the leading coefficient  $A$  (a diagonal matrix) are obtained by requiring the associated eigenspace decomposition  $\mathbb{C}^n = \bigoplus_j W_j$  be fixed, leading to the braiding of  $\text{Spec}(A) \subseteq \mathbb{C}$ . But further we can decompose  $W_j$  into eigenspaces for the restriction  $T|_{W_j}$ , and braid those as well: this yields a product decomposition of the space  $\mathbf{B} \subseteq \mathfrak{t}^2$ , and intuitively a *cablings* of each strand of the braid associated with the spectrum of  $A$ .

In this paper we will consider the topology of the moduli space of an arbitrary irregular type, for any complex reductive group, generalising the above example and proving the relation to braid cablings in the general linear case. The monodromy of the resulting local systems of wild character varieties thus yields algebraic Poisson actions of the (fundamental) groups we are constructing here, and “quantum” linear representations after deformation/geometric quantisation.

**Main results and layout of the paper.** Let then  $\Sigma = (\Sigma, a, Q)$  be a one-pointed wild Riemann surface, and  $(\mathfrak{g}, \mathfrak{t})$  a finite-dimensional split reductive Lie algebra over  $\mathbb{C}$ . (The many-point case amounts to repeating the present discussion independently at each marked point, cf. Rem. 1.3.)

In § 1 we define a (universal) space of admissible deformations  $\mathbf{B}(Q)$  of the irregular type  $Q$ , by keeping the underlying pointed Riemann surface  $(\Sigma, a)$  fixed (Def. 1.3). This depends on the root system  $\Phi_{\mathfrak{g}} = \Phi(\mathfrak{g}, \mathfrak{t})$ , and on the pole order  $d_{\alpha} \in \{0, \dots, p\}$  of the meromorphic function germ  $q_{\alpha} = \alpha \circ Q$ , for  $\alpha \in \Phi_{\mathfrak{g}}$ . This generalises the root system of type  $A_{n-1}$ , where we find the functions  $q_i - q_j$  for  $1 \leq i \neq j \leq n$ , if  $Q = \text{diag}(q_1, \dots, q_n)$ .

In § 2 we then define the *local wild mapping class group* (WMCG, cf. [18, § 8]) of  $\Sigma$ , denoted  $\Gamma(\Phi_{\mathfrak{g}}, \mathbf{d})$ , as the fundamental group of the space of admissible deformations—depending on  $\mathbf{d} = (d_{\alpha})_{\alpha \in \Phi}$ . We explain how it breaks into fundamental groups of spaces of admissible deformations of the coefficients of  $Q$  (see (10)), and in § 3 we use this decomposition to explain that the local WMCG is associated with an increasing filtration of root subsystems of  $\Phi_{\mathfrak{g}}$ , obtained by *fission*. This generalises the pure  $\mathfrak{g}$ -braid group, arising from the generic case (viz. the trivial filtration  $\emptyset \subseteq \Phi_{\mathfrak{g}}$ ), and is in the end controlled by a nested sequence of Dynkin (sub)diagrams.

In § 4 we prove some general statement, e.g. that we can reduce the study to *simple* Lie algebras.

**Theorem 1** (Cf. § 4.1). *Let  $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g}$  be the semisimple part of  $\mathfrak{g}$ : then  $\Gamma(\Phi_{\mathfrak{g}}, \mathbf{d})$  is canonically isomorphic to a local WMCG for  $\mathfrak{g}'$ —with the same pole orders.*

*Further, if  $\mathfrak{g}' = \bigoplus_i \mathfrak{J}_i$  is the decomposition of  $\mathfrak{g}'$  into simple ideals, then there is a canonical group isomorphism  $\Gamma(\Phi_{\mathfrak{g}}, \mathbf{d}) \simeq \prod_i \Gamma(\Phi_{\mathfrak{J}_i}, \mathbf{d}_i)$ , where  $\Phi_{\mathfrak{J}_i} \subseteq \Phi_{\mathfrak{g}}$  are the root systems of the simple ideals and  $\mathbf{d}_i$  the pole orders of  $q_{\alpha}$  for  $\alpha \in \Phi_{\mathfrak{J}_i}$ .*

Further we will obtain a uniform bound on the number of nontrivial factors of the local WMCG, independent of the order of the pole of the irregular type.

**Theorem** (Cf. § 4.3). *The number of nontrivial factors of (10) is at most the semisimple rank of  $\mathfrak{g}$ —i.e. the rank of  $\Phi_{\mathfrak{g}}$ .*

Finally, as a last general remark, we will classify local WMCGs for low-rank Lie algebras.

**Theorem 2** (Cf. § 4.4).

- *If the semisimple rank of  $\mathfrak{g}$  is one, then the local WMCG is either trivial or infinite cyclic (i.e. isomorphic to the pure  $\mathfrak{g}$ -braid group).*
- *If the semisimple rank of  $\mathfrak{g}$  is two, then the local WMCG is either trivial, isomorphic to  $\mathbb{Z}$ , to  $\mathbb{Z}^2$ , or to the pure  $\mathfrak{g}$ -braid group.*

In particular this classifies the local WMCGs for the exceptional Lie algebra of type  $G_2$ . Starting from § 5 instead we consider *classical* simple Lie algebras, and give a complete explicit description of the local WMCG.

Beginning with type A, we associate a tree to any sequence of root subsystems obtained from fission, thus called a *fission tree* (see Def. 5.2 and cf. [13, App. C]). Similarly for types B/C in §§ 7 and 8 we associate a *bichromatic tree* to any irregular type (Def. 7.1), and finally a generalisation thereof for type D in § 9 (Def. 9.1).

This leads to the following statement, for *all* classical types.

**Theorem 3** (Cf. Thmm. 5.2, 7.2 and 9.2). *The generalised fission tree uniquely determines the local WMCG, as follows: at each node of the tree one attaches the pure braid group of a hyperplane arrangement, and the local WMCG is the product of those factors.*

*For types A and B/C all factors correspond to root-hyperplane arrangements, while for type D there is an “exotic” factor (which is not crystallographic) further studied in Prop. 9.1.*

Then in § 6 we relate the (monochromatic) fission tree of type A to cabled braids, formalising the above driving intuition—whence the second part of the title. More precisely to any tree  $T$  we associate a *pure cabled braid group*  $\text{PCB}(T)$ , using the compositions of the pure braid operad (i.e. multi-cabling, see Def. 6.1), and we prove the following.

**Theorem 4** (Cf. Thm. 6.1). *If  $T$  is the fission tree of a type-A irregular type  $Q$  then there is a canonical group isomorphism  $\Gamma(\Phi_{\mathfrak{g}}, \mathbf{d}) \simeq \text{PCB}(T)$ .*

Finally in § 10 we work in the category of schemes (rather than complex manifolds) to construct a moduli space of irregular types with prescribed pole orders at each root (Def. 10.2). In particular we define irregular types over any affine scheme  $\text{Spec } R$ , where  $R$  is a commutative  $\mathbb{C}$ -algebra, generalising the standard notion for  $R = \mathbb{C}$ ; then we let  $X := \text{Spec } \text{Sym}(t^\vee)$ , and prove the following.

**Theorem 5** (Cf. Prop. 10.1). *The functor taking  $\text{Spec } R$  to the set of irregular types over  $\text{Spec } R$ , of bounded pole orders at any root, is representable by a closed subscheme of  $X^{\mathbb{P}}$ . Further prescribing the pole order at each root yields a stratification thereof, and the complex points of each (affine) stratum are identified with the deformation space of Def. 1.3.*

Hence the local WMCG is the fundamental group of an analytified moduli space of irregular types. In Rk. 10.4 we will explain how this viewpoint relates to the above admissible deformations/families of wild Riemann surfaces.

All Lie algebras, commutative (associative, unitary) algebras, and tensor products are defined over  $\mathbb{C}$ : some basic notions, notations and conventions, used throughout the body of the paper, are summarised in App. A.

#### ACKNOWLEDGEMENTS

- J. D. is funded by the NCCR SwissMAP: The Mathematics of Physics (SNF project 141869);
- G. R. is supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy - GZ 2047/1, Projekt-ID 390685813;
- M. T. is supported by the ERC Grant 804176.

We thank P. Boalch, F. Naef and G. Paolini for listening/answering to some of our questions.

#### 1. ADMISSIBLE DEFORMATIONS OF WILD RIEMANN SURFACES

Let  $\Sigma$  be a smooth compact Riemann surface,  $G$  a finite-dimensional connected complex reductive Lie group,  $\mathfrak{g} = \text{Lie}(G)$  its Lie algebra,  $T \subseteq G$  a maximal (algebraic) torus and  $\mathfrak{t} = \text{Lie}(T) \subseteq \mathfrak{g}$  the corresponding Cartan subalgebra.

Recall a (dressed) wild Riemann surface structure on  $\Sigma$  is the choice of a finite ordered set  $\mathbf{a} = (a_1, \dots, a_m) \subseteq \Sigma^m$  of  $m \geq 0$  distinct marked points, and “untwisted” irregular types  $\mathbf{Q} = (Q_1, \dots, Q_m)$  based there [17, Def. 8.1]; in turn an untwisted irregular type at the point  $a \in \Sigma$  is the germ of a  $\mathfrak{t}$ -valued meromorphic function based there, defined up to holomorphic terms.

More precisely let  $\mathcal{O}_{\Sigma, a}$  be the local ring at  $a \in \Sigma$ ,  $\widehat{\mathcal{O}}_{\Sigma, a}$  its completion, and  $\widehat{\mathcal{K}}_{\Sigma, a}$  the field of fractions of this latter: then by definition an untwisted irregular type based at  $a \in \Sigma$  is an element

$$Q \in \mathfrak{t} \otimes \mathcal{T}_{\Sigma, a}, \quad \mathcal{T}_{\Sigma, a} := \widehat{\mathcal{K}}_{\Sigma, a} / \widehat{\mathcal{O}}_{\Sigma, a}.$$

The space  $\mathcal{T}_{\Sigma, a}$  contains “tails” of  $\mathfrak{t}$ -valued formal Laurent series: indeed if  $z$  is a local coordinate with  $z(a) = 0$  then

$$\widehat{\mathcal{O}}_{\Sigma, a} \simeq \mathbb{C}[[z]], \quad \widehat{\mathcal{K}}_{\Sigma, a} \simeq \mathbb{C}((z)),$$

and  $\mathcal{T}_{\Sigma, a} \simeq \mathbb{C}((z))/\mathbb{C}[[z]]$ , so can write

$$Q = \sum_{i=1}^p A_i z^{-i} \in z^{-1} \mathfrak{t}[[z^{-1}]] \simeq \mathfrak{t}((z))/\mathfrak{t}[[z]], \quad A_i \in \mathfrak{t}, \quad (2)$$

for some integer  $p \geq 1$ . Hereafter we simply refer to such elements (2) as “irregular types” (cf. § 11).

Denote  $\Sigma = (\Sigma, \mathbf{a}, \mathbf{Q})$  the resulting wild Riemann surface, which we want to deform in an “admissible” way. To this end let  $\mathbf{B}$  be a complex manifold and

$$\Sigma_b = \pi^{-1}(b) \hookrightarrow \underline{\Sigma} \xrightarrow{\pi} \mathbf{B}, \quad b \in \mathbf{B},$$

a holomorphic family of Riemann surfaces fibering over  $\mathbf{B}$ . Choose an  $m$ -tuple

$$\mathbf{a} = (a_1, \dots, a_m): \mathbf{B} \longrightarrow \underline{\Sigma}^m$$



of global sections, as well as a holomorphic  $\mathbf{B}$ -family of irregular types  $b \mapsto \underline{Q}_i(b)$  at the marked points  $\underline{a}_i(b) \in \Sigma_b$ , and let  $\underline{Q} = (\underline{Q}_1, \dots, \underline{Q}_m)$  be their collection.<sup>4</sup>

**Definition 1.1.** The tuple  $(\underline{\Sigma}, \mathbf{B}, \underline{a}, \underline{Q})$  is a  $\mathbf{B}$ -family of wild Riemann surfaces, denoted  $\underline{\Sigma} \rightarrow \mathbf{B}$ ; its fibre at  $b \in \mathbf{B}$  is the wild Riemann surface

$$\Sigma_b = (\Sigma_b, \underline{a}(b), \underline{Q}(b)).$$

If  $0 \in \mathbf{B}$  is a base point, and  $\mathbf{B}$  is connected, the family  $\underline{\Sigma} \rightarrow (\mathbf{B}, 0)$  is a *deformation* of the “starting” wild Riemann surface  $\Sigma_0$ .

To restrict such deformations let  $\Phi_{\mathfrak{g}} = \Phi(\mathfrak{g}, \mathfrak{t}) \subseteq \mathfrak{t}^\vee$  be the root system of the split Lie algebra  $(\mathfrak{g}, \mathfrak{t})$ , and consider an irregular type  $Q \in \mathfrak{t} \otimes \mathcal{T}_{\Sigma, \alpha}$  on a one-pointed Riemann surface  $(\Sigma, \alpha)$ . Then introduce

$$q_\alpha := (\alpha \otimes \text{Id})(Q) \in \mathcal{T}_{\Sigma, \alpha}, \quad \alpha \in \Phi_{\mathfrak{g}},$$

which is the meromorphic function germ obtained by “evaluating” the irregular type at a root (defined up to holomorphic terms). A collection of  $\mathbf{B}$ -families of irregular types thus yields  $\mathbf{B}$ -families of meromorphic function germs

$$b \mapsto \underline{q}_{i, \alpha}(b) := (\alpha \otimes \text{Id})(\underline{Q}_i(b)) \in \mathcal{T}_{\Sigma_b, \underline{a}_i(b)},$$

for  $i \in \{1, \dots, m\}$  and  $\alpha \in \Phi_{\mathfrak{g}}$ .

Further, if  $q$  is the germ of a meromorphic function on a Riemann surface, let  $\text{ord}(q) \in \mathbb{Z}_{\geq 0}$  be its pole order at the base point, with the convention that  $\text{ord}(q) = 0$  if  $q$  is holomorphic; this only depends on  $q$  up to adding holomorphic terms.

**Definition 1.2** ([17], Def. 10.1). An *admissible* deformation of  $\Sigma_0$  is a deformation  $\underline{\Sigma} \rightarrow (\mathbf{B}, 0)$  of  $\Sigma_0$  such that for all  $b \in \mathbf{B}$ :

- (1)  $\Sigma_b$  is smooth;
- (2) the marked points  $\underline{a}_i(b) \in \Sigma_b$  are distinct;
- (3) one has

$$\text{ord}(\underline{q}_{\alpha, i}(b)) = \text{ord}(\underline{q}_{\alpha, i}(0)) \in \mathbb{Z}_{\geq 0}.$$

This means the genus of each Riemann surface, the cardinality of each set of marked points, and the pole orders of the irregular types evaluated at each root are *constant* along the deformation. The set of nonzero orders of poles which occur at each marked point is the set of “levels” of the irregular type: this paper is essentially about the multi-level case, i.e. the case where the leading coefficients are *not* regular.

**1.1. Wild deformations.** Denote  $\Sigma$  a “starting” Riemann surface.

By Def. 1.2 we can deform the Riemann surface structure of  $\Sigma$ , and move the marked points  $\underline{a} \subseteq \Sigma$  inside the configuration space of ordered  $m$ -tuples of points in  $\Sigma$ . These are the “tame” deformations, but as explained in the introduction there are additional local “wild” moduli, which are our main focus: these are the parameters of the irregular types, which can be taken independently at each marked point (cf. Rem. 1.3).

<sup>4</sup>In the examples we look at in this section we will have an explicit target space of irregular types, and  $\underline{Q}$  will be a holomorphic map with domain  $\mathbf{B}$  (see (8)).



Suppose thus  $m = 1$ , whence  $\mathbf{a} = \mathbf{a} \in \Sigma$  and  $\mathbf{Q} = Q \in \mathfrak{t}((z))/\mathfrak{t}[[z]]$ , with  $z(\mathbf{a}) = 0$ . We have a starting wild Riemann surface  $\Sigma = (\Sigma, \mathbf{a}, Q)$ , and we will *not* deform the underlying pointed Riemann surface  $(\Sigma, \mathbf{a})$ —but only the irregular type  $Q$ .

To this end introduce the local coordinate  $x = z^{-1}$  (on a punctured neighbourhood of the marked point), so we find a  $\mathfrak{t}$ -valued polynomial

$$Q = \sum_{i=1}^p A_i x^i \in x \mathfrak{t}[[x]], \quad A_i \in \mathfrak{t}, \quad (3)$$

for some  $p \in \mathbb{Z}_{\geq 1}$ , and with  $A_p \neq 0$ . Hence  $q_\alpha \in \mathbb{C}[[x]]$  for all roots  $\alpha \in \Phi_{\mathfrak{g}}$ , and the levels are controlled by the function

$$\mathbf{d}: \alpha \mapsto d_\alpha = \deg_x(q_\alpha), \quad \alpha \in \Phi_{\mathfrak{g}}.$$

We take the convention that  $\deg_x(0) = 0$ , so  $\mathbf{d}(\Phi_{\mathfrak{g}}) \subseteq \{0, \dots, p\}$ , and  $d_\alpha = 0$  corresponds to  $q_\alpha = 0$ .

We will now consider (universal, degree-bounded) admissible deformations of (3), as follows.<sup>5</sup> Define  $\mathfrak{g}_{p+1} := \mathfrak{g}[[x]]/x^{p+1}\mathfrak{g}[[x]]$ , which is a Lie algebra (of truncated positive currents). It comes with a vector space identification

$$\mathfrak{g}_{p+1} \simeq \prod_{i=0}^p \mathfrak{g} \cdot x^i \simeq \mathfrak{g}^{p+1}, \quad (4)$$

where the  $i$ -th factor

$$\mathfrak{g} \cdot x^i = \left\{ Y \cdot x^i \mid Y \in \mathfrak{g} \right\} \simeq \mathfrak{g}$$

corresponds to the coefficient of  $x^i$  in a  $\mathfrak{g}$ -valued polynomial.

Analogously we have a “deeper” Cartan subalgebra  $\mathfrak{t}_{p+1} \subseteq \mathfrak{g}_{p+1}$ , and we further consider the Birkhoff Lie algebra

$$\mathfrak{b}_{p+1} = x\mathfrak{g}[[x]]/x^{p+1}\mathfrak{g}[[x]] \subseteq \mathfrak{g}_{p+1}.$$

Then by definition  $Q \in \mathfrak{t}_{p+1} \cap \mathfrak{b}_{p+1}$ , and we will take admissible deformations inside the intersection. To this end note (4) restricts to the identification

$$\mathfrak{t}_{p+1} \cap \mathfrak{b}_{p+1} \simeq \prod_{i=1}^p \mathfrak{t} \cdot x^i \simeq \mathfrak{t}^p. \quad (5)$$

**Definition 1.3.** The *universal deformation space* of (3) is

$$\mathbf{B}(Q) = \prod_{i=1}^p \mathbf{B}(A_i) \subseteq \mathfrak{t}^p, \quad (6)$$

where

$$\mathbf{B}(A_i) := \bigcap_{d_\alpha < i} \text{Ker}(\alpha) \cap \bigcap_{d_\alpha = i} (\mathfrak{t} \setminus \text{Ker}(\alpha)) \subseteq \mathfrak{t}. \quad (7)$$

For the rest of this section, and in the next section, we will simply denote  $\mathbf{B} = \mathbf{B}(Q)$  and  $\mathbf{B}_i = \mathbf{B}(A_i)$ . Note  $\mathbf{B}_i$  is an open subspace of a vector subspace of  $\mathfrak{t}$ , so it is naturally a complex manifold, and so is  $\mathbf{B}$ : in § 10 we will show this latter is the analytification of the moduli scheme of irregular types with prescribed pole orders at each root.

<sup>5</sup>Higher-degree terms lead to adding a contractible factor, so are homotopically negligible. See § 10 about universality.

We now let  $\mathbf{A} := (A_1, \dots, A_p) \in \mathbf{B}$ , and we turn the (connected) pointed space  $(\mathbf{B}, \mathbf{A})$  into a base space for an admissible deformation of the “starting” wild Riemann surface  $\Sigma = (\Sigma, \alpha, Q)$ . Namely consider the trivial holomorphic family  $\underline{\Sigma} := \Sigma \times \mathbf{B} \rightarrow \mathbf{B}$  of Riemann surfaces, and let  $\underline{\alpha}: \mathbf{B} \rightarrow \Sigma$  be the global constant  $\alpha$ -valued section. Then the varying irregular type is defined by

$$\underline{Q}: \mathbf{B} \longrightarrow \mathfrak{t}_{p+1} \cap \mathfrak{b}_{p+1}, \quad \underline{Q}(\mathbf{A}') := \sum_{j=1}^p A'_j \chi^j, \quad \mathbf{A}' \in \mathbf{B}. \quad (8)$$

This yields a  $\mathbf{B}$ -family  $\underline{\Sigma} \rightarrow (\mathbf{B}, \mathbf{A})$  of wild Riemann surfaces starting at  $\mathbf{A} \in \mathbf{B}$ .

**Proposition 1.1.** *The family  $\underline{\Sigma} \rightarrow (\mathbf{B}, \mathbf{A})$  is an admissible deformation of  $\Sigma$ .*

*Proof.* By construction (8) is a restriction of the identification (5), so it is holomorphic.

Thus we must only check that

$$\deg_x((\alpha \otimes 1)(\underline{Q})) = d_\alpha, \quad \alpha \in \Phi_{\mathfrak{g}},$$

everywhere on  $\mathbf{B}$ . But indeed by (7)

$$\alpha(A'_{d_\alpha}) \neq 0, \quad \alpha(A'_i) = 0, \quad i > d_\alpha,$$

for  $\alpha \in \Phi_{\mathfrak{g}}$ , so all degrees are preserved along the deformation.  $\square$

*Remark 1.1.* Equivalently by definition

$$\mathbf{B} = \bigcap_{\Phi_{\mathfrak{g}}} \left( \left( \prod_{1 \leq i < d_\alpha} \mathfrak{t} \right) \times (\mathfrak{t} \setminus \text{Ker}(\alpha)) \times \prod_{d_\alpha < i \leq p} \text{Ker}(\alpha) \right) \subseteq \mathfrak{t}^p,$$

after swapping products and intersections.  $\triangle$

*Remark 1.2* (Trivial deformations). To all coefficients one can add an element of

$$\text{Ker}(\Phi_{\mathfrak{g}}) = \mathfrak{Z}_{\mathfrak{g}} \subseteq \mathfrak{t},$$

i.e.  $(\mathfrak{Z}_{\mathfrak{g}})^p$  acts on  $\mathbf{B}$  by factorwise translations. In principle one could thus consider the quotient space  $\mathbf{B}/\mathfrak{Z}_{\mathfrak{g}}^p$ , which yields the same fundamental group, and amounts to considering the semisimple part of  $\mathfrak{g}$  (see § 4.1).  $\triangle$

Note the definition of irregular types is *intrinsic*, i.e. independent of the choice of local coordinate, while the deformation space (6) in principle uses a local coordinate to get to meromorphic functions in one variable—and take their pole order. However the order of the pole of (the germ of) a meromorphic function on  $\Sigma$  is well defined up to local biholomorphisms, so the integers  $d_\alpha \in \{0, \dots, p\}$  depend on  $Q$  only, and not on the identifications  $\widehat{\mathcal{O}}_{\Sigma, \alpha} \simeq \mathbb{C}[[z]] \subseteq \mathbb{C}((z)) \simeq \widehat{\mathcal{H}}_{\Sigma, \alpha}$  (cf. Rk. 10.1).

*Remark 1.3* (Many-point case). It is easy to extend (6) to the case of several *fixed* marked points on  $\Sigma$ .

Namely if  $\mathbf{a} = (a_1, \dots, a_m) \in \Sigma^m$  we still consider the family  $\underline{\Sigma} = \Sigma \times \mathbf{B} \rightarrow \mathbf{B}$ , equipped with the corresponding global constant sections, and this time  $\mathbf{B}$  is a space of (simultaneous) admissible deformations of irregular types at each marked point. More precisely we will have  $\mathbf{Q} = (Q_1, \dots, Q_m)$ , with

$$Q_j \in \mathfrak{t} \otimes \mathcal{T}_{\Sigma, a_j}, \quad j \in \{1, \dots, m\},$$

each with a pole of order  $p_j \geq 0$ , and then

$$\mathbf{B}(\mathbf{Q}) := \prod_{j=1}^m \mathbf{B}(Q_j) \subseteq \prod_{j=1}^m \mathfrak{t}^{p_j}.$$

△

## 2. LOCAL WILD MAPPING CLASS GROUPS

The main definition is the following.

**Definition 2.1.** The (pure) *local wild mapping class group* (WMCG) of the wild Riemann surface  $\Sigma = (\Sigma, \alpha, Q)$  is

$$\Gamma(\Phi_{\mathfrak{g}}, \mathbf{d}) := \pi_1(\mathbf{B}, \mathbf{A}). \quad (9)$$

Note this only depends on the function  $\mathbf{d}: \Phi_{\mathfrak{g}} \rightarrow \mathbb{Z}_{\geq 0}$ —and not on  $p \geq 1$ . By construction there is a product decomposition

$$\Gamma(\Phi_{\mathfrak{g}}, \mathbf{d}) \simeq \prod_{i=1}^p \pi_1(\mathbf{B}_i, \mathbf{A}_i), \quad (10)$$

and further the many-point case yields a product of such groups, with a factor at each marked point (cf. Rem. 1.3).

In what follows we will study and classify the factors (10).

*Example 2.1* (Abelian case). Suppose  $\mathfrak{g}$  is Abelian. Then the root system is empty, and  $\mathbf{B} = \mathfrak{t}^p$  is contractible: the local WMCG is trivial. △

*Example 2.2* (Generic case). Suppose  $\mathbf{d}$  is constant, say  $d_{\alpha} = d \in \{0, \dots, p\}$  for all  $\alpha \in \Phi_{\mathfrak{g}}$ .

If  $d = 0$  then  $Q \in z^{-1}\mathfrak{Z}_{\mathfrak{g}}[z^{-1}]$ , and (6) is the contractible space  $\mathfrak{Z}_{\mathfrak{g}}^p \subseteq \mathfrak{t}^p$ —with trivial fundamental group.

Else  $d > 0$ : in this case there is only one level, whence

$$\mathbf{B}_i = \mathfrak{t}, \quad \mathbf{B}_d = \mathfrak{t}_{\text{reg}}, \quad \mathbf{B}_j = \text{Ker}(\Phi_{\mathfrak{g}}),$$

for  $i < d < j$ . Hence  $\mathbf{B}_d$  is a strong deformation retract of  $\mathbf{B}$ , and

$$\Gamma(\Phi_{\mathfrak{g}}, \mathbf{d}) \simeq \pi_1(\mathfrak{t}_{\text{reg}}, \mathbf{A}_d) = \text{PB}_{\mathfrak{g}},$$

the pure  $\mathfrak{g}$ -braid group. △

The point here is to describe the topology of the deformation space in the multi-level case, generalising pure  $\mathfrak{g}$ -braid groups from the generic case.

## 3. ROOT FILTRATIONS AND FISSION

There is an increasing filtration

$$\Phi_1 \subseteq \dots \subseteq \Phi_{p+1} = \Phi_{\mathfrak{g}}, \quad \text{with} \quad \Phi_i := \{ \alpha \in \Phi_{\mathfrak{g}} \mid d_{\alpha} < i \}, \quad (11)$$

so in particular  $\Phi_1 = \{ \alpha \in \Phi_{\mathfrak{g}} \mid \alpha \circ Q = 0 \}$ .

The sequence (11) is by root (sub)systems, since the degree of a linear combination of polynomials is at most the maximum of their degrees. Hence the factors (7) can be written

$$\mathbf{B}(\Phi', \Phi) = \text{Ker}(\Phi') \cap \bigcap_{\Phi \setminus \Phi'} (\mathfrak{t} \setminus \text{Ker}(\alpha)) \subseteq \mathfrak{t}, \quad (12)$$

where  $\Phi' \subseteq \Phi$  are suitable nested root systems—within  $\Phi_{\mathfrak{g}}$ .

*Remark 3.1.* The local WMCG is associated with this filtration of root subsystems, but different filtrations may give isomorphic groups. E.g., choosing an element  $\varphi \in \text{Aut}(\Phi_{\mathfrak{g}}) \subseteq \text{GL}(\mathfrak{t}^{\vee})$ , then  ${}^t\varphi: \mathfrak{t} \rightarrow \mathfrak{t}$  maps (12) homeomorphically to another subspace of the Cartan subalgebra. In particular the local WMCG only depends on the  $W_{\mathfrak{g}}$ -orbit of the filtration of root subsystems, where  $W_{\mathfrak{g}} = W(\Phi_{\mathfrak{g}}) \simeq W(\Phi_{\mathfrak{g}}^{\vee})$  is the Weyl group.

Conversely not all root subsystems will appear in our construction, but rather only those obtained by taking consecutive centralisers of semisimple elements (the coefficients of the irregular type), as we will momentarily explain.  $\triangle$

**3.1. Fission.** The sequence (11) is associated with a filtration of connected complex reductive subgroups of  $G$  (cf. [17, Eq. 33]).

Namely let  $H_i := \bigcap_{j=i}^p \text{Stab}_{\text{Ad}_G}(A_j)$ , i.e.

$$H_i = \{ g \in G \mid \text{Ad}_g(A_j) = A_j, i \leq j \leq p \} \subseteq G,$$

whence  $H_1 \subseteq \cdots \subseteq H_p$ . Denote then  $\mathfrak{h}_i := \text{Lie}(H_i)$ , so that  $\mathfrak{h}_1 \subseteq \cdots \subseteq \mathfrak{h}_p$  and

$$\mathfrak{h}_i = \mathfrak{Z}_{\mathfrak{g}}(\{A_i, \dots, A_p\}) \subseteq \mathfrak{g}.$$

Note  $\mathfrak{t} \subseteq \mathfrak{h}_i$ , and it is still a Cartan subalgebra there; let now  $\Phi_{\mathfrak{h}_i} = \Phi(\mathfrak{h}_i, \mathfrak{t})$ .

**Lemma 3.1.** *One has  $\Phi_i = \Phi_{\mathfrak{h}_i} \subseteq \Phi_{\mathfrak{g}}$  (in the notation of (11)).*

*Proof.* This can be seen inductively.

Namely start from  $H_p = \text{Stab}_{\text{Ad}_G}(A_p)$ , and  $\mathfrak{h}_p = \mathfrak{Z}_{\mathfrak{g}}(A_p)$ : then by definition  $\Phi_{\mathfrak{h}_p} = \Phi_{\mathfrak{g}} \cap \{A_p\}^{\perp} \subseteq \Phi_{\mathfrak{g}}$ . Indeed if  $[X, T] = \alpha(T)X$  for some nonzero  $X \in \mathfrak{h}_p$  and  $\alpha \in \mathfrak{t}^{\vee}$ , and for all  $T \in \mathfrak{t}$ , then  $\alpha \in \Phi_{\mathfrak{g}}$  must vanish on  $A_p$  since  $[X, A_p] = 0$ ; and conversely if there is  $X \in \mathfrak{g}$  nonzero such that the same holds, with  $\alpha(A_p) = 0$ , then  $[X, A_p] = 0$ —so  $X \in \mathfrak{h}_p$ .

Then replacing  $(H_p, G, A_p)$  with  $(H_i, H_{i+1}, A_i)$  at each step proves the claim.  $\square$

Hereafter we say  $\Phi_{\mathfrak{h}_i} \subseteq \Phi_{\mathfrak{g}}$  is a *fission* (root) subsystem, and in the rest of this section we describe them. Again, up to replacing  $H_p \subseteq G$  with  $H_i \subseteq H_{i+1}$  at each step, it will be enough to consider the centraliser of a semisimple element in  $\mathfrak{g}$ .

*Remark 3.2.* Such filtrations naturally occur in 2d gauge theory: the terminology is due to the “breaking” of the structure group at the boundary of the real oriented blowup of  $(\Sigma, a)$ , from  $G$  to  $H_1$  [14] (note earlier in [12] fission is “complete”, i.e.  $G$  breaks down to the maximal torus  $T$ ).

This is different to the usual “fusion” operation (of sewing “tame” boundaries) in the regular singular case.  $\triangle$

Choose thus  $A \in \mathfrak{t}$ , and set  $H = \text{Stab}_{\text{Ad}}(A)$ ,  $\mathfrak{h} = \mathfrak{Z}_{\mathfrak{g}}(A)$ . By definition  $\mathfrak{h} = \text{Ker}(\text{ad}_A)$ , and one can show that  $\mathfrak{g} = \mathfrak{h} \oplus \text{ad}_A(\mathfrak{g})$ ; more precisely decomposing  $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\Phi_{\mathfrak{g}}} \mathfrak{g}_{\alpha}$  as a vector space yields

$$\mathfrak{h} = \mathfrak{t} \oplus \bigoplus_{\alpha(A)=0} \mathfrak{g}_{\alpha}, \quad \text{ad}_A(\mathfrak{g}) = \bigoplus_{\alpha(A) \neq 0} \mathfrak{g}_{\alpha}.$$

To go further it is helpful to choose a basis  $\Delta_{\mathfrak{g}} \subseteq \Phi_{\mathfrak{g}}$  (of simple roots) which is “adapted” to  $\mathfrak{h}$ . To this end, since we are only interested in root-vanishing

conditions, we can first suppose that  $A \in \mathfrak{t}' = \mathfrak{t} \cap \mathfrak{g}'$  lies in the Cartan subalgebra of the semisimple part. (It is only the semisimple part of  $\mathfrak{g}$  that contributes to the local WMCG, cf. § 4.1.) Further we can assume  $A$  lies in the real form  $\mathfrak{t}'_{\mathbb{R}} = \mathbb{R}\Phi_{\mathfrak{g}}^{\vee} \subseteq \mathfrak{t}'$ , i.e. the subspace such that  $\Phi_{\mathfrak{g}}(\mathfrak{t}'_{\mathbb{R}}) \subseteq \mathbb{R}$ , and finally (up to changing basis) that  $A \in \overline{\mathfrak{C}(\Delta_{\mathfrak{g}})} \subseteq \mathfrak{t}'_{\mathbb{R}}$  lies in the closure of the fundamental Weyl chamber—it is *dominant*. Recall

$$\overline{\mathfrak{C}(\Delta_{\mathfrak{g}})} \subseteq \{ T \in \mathfrak{t}'_{\mathbb{R}} \mid \alpha(T) \geq 0 \text{ for } \alpha \in \Delta_{\mathfrak{g}} \} \subseteq \mathfrak{t}'_{\mathbb{R}},$$

so the basis is determined by the condition that  $\alpha(A) \geq 0$  for all  $\alpha \in \Delta_{\mathfrak{g}}$ .

Let then  $\Phi_{\mathfrak{g}}^+ \subseteq \Phi_{\mathfrak{g}}$  be the system of positive roots associated with  $\Delta_{\mathfrak{g}}$ , i.e.

$$\Phi_{\mathfrak{g}}^+ = \mathbb{Z}_{\geq 0}\Delta_{\mathfrak{g}} \cap \Phi_{\mathfrak{g}}.$$

It follows that  $\alpha(A) \geq 0$  for all  $\alpha \in \Phi_{\mathfrak{g}}^+$ , but there are also negative roots vanishing on  $A$ : namely the  $\mathbb{Z}_{<0}$ -linear combinations of simple roots vanishing on  $A$ . Conversely if  $\alpha(A) > 0$  then  $\alpha$  is necessarily positive.

**Lemma 3.2.** *The subset  $\Phi_{\mathfrak{h}}^+ := \Phi_{\mathfrak{g}}^+ \cap \{A\}^{\perp} \subseteq \Phi_{\mathfrak{h}}$  is a positive root system for  $(\mathfrak{h}, \mathfrak{t})$ , and the subset  $\Delta_{\mathfrak{h}} := \Delta_{\mathfrak{g}} \cap \{A\}^{\perp} \subseteq \Phi_{\mathfrak{h}}^+$  is the associated basis of  $\Phi_{\mathfrak{h}}$ .*

Hence  $\Phi_{\mathfrak{h}}$  admits a basis given by the simple roots of  $\Phi_{\mathfrak{g}}$  vanishing on  $A$ . The extreme case where  $A \in \mathfrak{t}_{\text{reg}}$  corresponds to  $\mathfrak{h} = \mathfrak{t}$ , in which case  $\Phi_{\mathfrak{h}} = \emptyset$ ; on the opposite end  $A = 0$  yields  $\mathfrak{h} = \mathfrak{g}$ .

*Proof.* The first statement is a general fact about root (sub)systems.

As for the second statement, any element of  $\Delta_{\mathfrak{g}} \cap \{A\}^{\perp}$  is an indecomposable positive root for  $(\mathfrak{g}, \mathfrak{t})$ , so a fortiori for  $(\mathfrak{h}, \mathfrak{t})$ : hence the intersection is contained within the basis of  $\Phi_{\mathfrak{h}}$  associated with the positive root system  $\Phi_{\mathfrak{h}}^+$ .

Conversely choose an indecomposable positive root  $\theta \in \Phi_{\mathfrak{h}}^+$ : to conclude we must show it is also indecomposable in  $\Phi_{\mathfrak{g}}^+$ . Now if one had

$$\theta = \sum_i n_i \alpha_i, \quad \alpha_i \in \Phi_{\mathfrak{g}}^+, n_i \in \mathbb{Z}_{\geq 0},$$

then in particular

$$0 = \theta(A) = \sum_i n_i \alpha_i(A) \geq 0.$$

This yields  $n_i = 0$  whenever  $\alpha_i(A) \neq 0$ , in which case the decomposition takes place inside  $\Phi_{\mathfrak{h}}^+$ .  $\square$

*Remark 3.3 (Nested diagrams).* This way the Dynkin diagram  $\mathcal{DD}(\Phi_{\mathfrak{h}})$  of  $\Phi_{\mathfrak{h}}$  is obtained by choosing a subset of nodes of the Dynkin diagram  $\mathcal{DD}(\Phi_{\mathfrak{g}})$  of  $\Phi_{\mathfrak{g}}$ , and keeping all edges among them. Repeating this procedure at each step finally yields a nested sequence of Dynkin (sub)diagrams, describing fission in general:

$$\mathcal{DD}(\Phi_{\mathfrak{h}_1}) \subseteq \cdots \subseteq \mathcal{DD}(\Phi_{\mathfrak{h}_p}) \subseteq \mathcal{DD}(\Phi_{\mathfrak{g}}).$$

$\triangle$

*Remark 3.4 (Parabolic subalgebras).* Note there is a decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{u}_+ \oplus \mathfrak{u}_-, \quad \mathfrak{u}_{\pm} := \bigoplus_{\pm \alpha(A) > 0} \mathfrak{g}_{\alpha} \subseteq \mathfrak{g}.$$

Since  $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$  the subspaces  $u_\pm$  are (nilpotent) subalgebras.

Then one can show that  $[\mathfrak{h}, u_\pm] \subseteq u_\pm$ , so there is a (inner) semidirect product  $\mathfrak{p}_\pm := u_\pm \rtimes \mathfrak{h} \subseteq \mathfrak{g}$ ; by construction this comes with a vector space decomposition

$$\mathfrak{p}_\pm = \mathfrak{h} \oplus u_\pm = \mathfrak{t} \oplus \bigoplus_{\pm\alpha(\lambda) \geq 0} \mathfrak{g}_\alpha. \quad (13)$$

This is a standard parabolic subalgebra of  $\mathfrak{g}$ , viz. a subalgebra containing the standard Borel subalgebra  $\mathfrak{b}_\pm \subseteq \mathfrak{g}$  associated with the system of positive/negative roots.

By construction  $\mathfrak{p}_\pm$  has nilradical  $\mathfrak{Nil}(\mathfrak{p}_\pm) = u_\pm$  (the maximal nilpotent ideal), and  $\mathfrak{h} \simeq \mathfrak{p}_\pm / u_\pm$  is (a lift of) its *reductive* Levi factor. We obtain a standard parabolic subalgebra, rather than simply a parabolic subalgebra, because we have already imposed the semisimple element lies in the fundamental Weyl chamber.

Of course given  $A \in \mathfrak{t}_\mathbb{R}$  we could equivalently start from the parabolic subalgebra (13). Then  $\mathfrak{h} := \mathfrak{p}_\pm / u_\pm$  is recovered as the reductive Levi factor, where  $u_\pm = \mathfrak{Nil}(\mathfrak{p}_\pm)$ ; and in this case the sequence

$$0 \longrightarrow u_\pm \longrightarrow \mathfrak{p}_\pm \longrightarrow \mathfrak{h} \longrightarrow 0$$

splits, so we can identify  $\mathfrak{h}$  with a subalgebra of  $\mathfrak{p}_\pm$ .  $\triangle$

We see we are led to study the following hyperplane complements:

$$\mathbf{B}(\Phi_\mathfrak{h}, \Phi_\mathfrak{g}) = \text{Ker}(\Delta_\mathfrak{h}) \cap \bigcap_{\Phi_\mathfrak{g}^+ \setminus \Phi_\mathfrak{h}^+} (\mathfrak{t} \setminus \text{Ker}(\alpha)) \subseteq \mathfrak{t}, \quad (14)$$

comparing with (12), since  $\mathbb{C}\Delta_\mathfrak{h} = \mathbb{C}\Phi_\mathfrak{h} \subseteq \mathfrak{t}$ . Conversely it is *not* enough to remove the hyperplanes corresponding to the simple roots inside  $\Delta_\mathfrak{g} \setminus \Delta_\mathfrak{h}$ .

Later we will show (14) is never empty, so it is a hyperplane complement: it is obtained by “restricting” the root-hyperplane complement of  $\Phi_\mathfrak{g} \setminus \Phi_\mathfrak{h}$  to  $\text{Ker}(\Delta_\mathfrak{h}) \subseteq \mathfrak{t}$  (which is a *flat* of the root-hyperplane arrangement of  $\Phi_\mathfrak{g}$ ).

Note also (14) is controlled by the semisimple part  $\mathfrak{h}' = [\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$  alone, i.e. by the Levi factor of  $\mathfrak{p}_\pm$ .<sup>6</sup>

**3.2. Dimensions and semisimple parts.** By construction  $\mathfrak{t} \subseteq \mathfrak{h}$  is a Cartan subalgebra, so  $\text{rk}(\mathfrak{h}) = \text{rk}(\mathfrak{g})$ . However  $|\Delta_\mathfrak{h}| = \dim(\mathbb{C}\Phi_\mathfrak{h}) = \text{rk}(\mathfrak{h}') \leq \text{rk}(\mathfrak{h})$ , which yields the *semisimple* rank of  $\mathfrak{h}$ .

It follows that

$$\dim(\text{Ker}(\Delta_\mathfrak{h})) = \text{rk}(\mathfrak{g}) - \text{rk}(\mathfrak{h}'), \quad (15)$$

while  $\text{rk}(\mathfrak{g}) = \text{rk}(\mathfrak{g}') + \dim(\mathfrak{z}_\mathfrak{g})$ . In particular if  $\mathfrak{g}$  is semisimple then the dimension of (14) equals the number of nodes that have been removed from  $\mathcal{DD}(\Phi_\mathfrak{g})$  to obtain  $\mathcal{DD}(\Phi_\mathfrak{h})$ —the semisimple *corank*.

#### 4. GENERAL RESULTS

Given a subsystem  $\Phi \subseteq \Phi_\mathfrak{g}$  it is natural to ask whether the hyperplane arrangement

$$\mathcal{H} = \left\{ \text{Ker}(\alpha) \cap \text{Ker}(\Phi) = \text{Ker}(\alpha|_{\text{Ker}(\Phi)}) \mid \alpha \in \Phi_\mathfrak{g} \setminus \Phi \right\} \subseteq \mathbb{P}(\text{Ker}(\Phi)^\vee) \quad (16)$$

<sup>6</sup>The “reductive” Levi factor is *not* the same as the (semisimple) Levi factor  $\mathfrak{p}_\pm / \mathfrak{Rad}(\mathfrak{p}_\pm) \simeq \mathfrak{h} / \mathfrak{z}_\mathfrak{h}$ .

is crystallographic, or at least if its reflection group is: we will show the latter holds when  $\mathfrak{g}$  is a simple Lie algebra of classical type, and  $\Phi$  is obtained by fission; but the former is false already in that case (looking at type D, see § 9). The point is that  $\Phi_{\mathfrak{g}} \setminus \Phi \subseteq \Phi_{\mathfrak{g}}$  is *not* a root system in general.

First we discuss briefly the hyperplane arrangement (16) in abstract terms. If  $\mathcal{U} := \text{Ker}(\Phi) \subseteq \mathfrak{t}$ , the “restricted” system

$$(\Phi_{\mathfrak{g}})|_{\mathcal{U}} = \left\{ \alpha|_{\mathcal{U}} \mid \alpha \in \Phi_{\mathfrak{g}} \right\} \subseteq \mathcal{U}^{\vee} \quad (17)$$

is naturally identified with the quotient set  $\Phi_{\mathfrak{g}}/\mathcal{U}^{\perp} \subseteq \mathfrak{t}^{\vee}/\mathcal{U}^{\perp}$ : two roots  $\alpha, \beta \in \Phi_{\mathfrak{g}}$  define the same linear functional on  $\mathcal{U}$  if and only if  $\alpha - \beta \in \mathcal{U}^{\perp}$ . In turn

$$\mathcal{U}^{\perp} = (\text{Ker}(\Phi))^{\perp} = \mathbb{C}\Phi \subseteq \mathfrak{t}^{\vee},$$

so this is the same as considering the quotient set  $\Phi_{\mathfrak{g}}/\mathbb{C}\Phi \subseteq \mathfrak{t}^{\vee}/\mathbb{C}\Phi$ —in particular, if  $\Phi = \{\pm\alpha\} \simeq A_1$  for some  $\alpha \in \Phi_{\mathfrak{g}}$ , this is the same as classifying the  $\alpha$ -strings in  $\Phi_{\mathfrak{g}}$ .

This is intrinsic and does not use a scalar product. Using a scalar product then (17) dually corresponds to a “projected” system  $\pi_{\mathcal{U}}(\Phi_{\mathfrak{g}}^{\vee}) \subseteq \mathcal{U}$ , where  $\pi_{\mathcal{U}}: \mathfrak{t} \rightarrow \mathcal{U}$  is the  $(\cdot | \cdot)$ -orthogonal projection.

Hence admissible deformations of wild Riemann surfaces naturally lead to such restrictions/projections of root systems.

*Remark 4.1* (Restricted reflection groups). The reflection group of (16) is in general *not* given by elements of  $W_{\mathfrak{g}}$  preserving the hyperplane complement—identified when they coincide there.

This discrepancy can already be seen in type A, and will be discussed elsewhere (it is related to *nonpure* local WMCGs, cf. 11); nonetheless all the reflection groups arising for simple Lie algebras of classical types are explicitly classified in §§ 5, 7, and 9.  $\triangle$

**4.1. Reduction to the simple case.** In this section let us assume the Weyl-invariant scalar product on  $\mathfrak{t}$  comes from an  $\text{Ad}_G$ -invariant scalar product  $(\cdot | \cdot): \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathbb{C}$  (cf. § A), such as the “trace” product  $(X | Y) = \text{Tr}(XY)$  for  $X, Y \in \mathfrak{gl}_n(\mathbb{C})$ . Suppose then there is a Lie algebra decomposition  $\mathfrak{g} = \bigoplus_i^{\perp} \mathfrak{J}_i$  into mutually orthogonal ideals  $\mathfrak{J}_i \subseteq \mathfrak{g}$ , and let  $\mathfrak{t}_i = \mathfrak{t} \cap \mathfrak{J}_i$ : this is a Cartan subalgebra of  $\mathfrak{J}_i$ .

There is a second orthogonal decomposition  $\mathfrak{t} = \bigoplus_i^{\perp} \mathfrak{t}_i$ , which induces an analogous one on the dual  $\mathfrak{t}^{\vee} \simeq \bigoplus_i^{\perp} \mathfrak{t}_i^{\vee}$ , by identifying  $\mathfrak{t}_i^{\vee}$  with the subspace

$$\bigcap_{j \neq i} \mathfrak{t}_j^{\perp} = (\mathfrak{t} \ominus \mathfrak{t}_i)^{\perp} \subseteq \mathfrak{t}^{\vee}, \quad \mathfrak{t} \ominus \mathfrak{t}_i := \bigoplus_{j \neq i} \mathfrak{t}_j.$$

Denote now  $\Phi_{\mathfrak{J}_i} = \Phi(\mathfrak{J}_i, \mathfrak{t}_i)$ . Then there is a decomposition

$$(\mathfrak{t}, \Phi_{\mathfrak{g}}) \simeq \bigoplus_i (\mathfrak{J}_i, \Phi_{\mathfrak{J}_i}),$$

underlying a disjoint union  $\Phi_{\mathfrak{g}} = \coprod_i \Phi_{\mathfrak{J}_i}$ —where  $\Phi_{\mathfrak{J}_i} \simeq \Phi_{\mathfrak{g}} \cap \mathfrak{t}_i^{\vee} \subseteq \Phi_{\mathfrak{g}}$  as above (cf. (31)). In particular the Weyl group  $W_{\mathfrak{g}}$  decomposes into the product of Weyl groups  $W_{\mathfrak{J}_i} \subseteq \text{GL}(\mathfrak{t}_i^{\vee})$ , and  $W_{\mathfrak{J}_i}$  corresponds to the (parabolic) subgroup acting as the identity on the subspace  $\mathfrak{t} \ominus \mathfrak{t}_i$ .



Now for a root subsystem  $\Phi \subseteq \Phi_{\mathfrak{g}}$  set

$$\Phi^{(i)} := \Phi \cap \mathfrak{t}_i^{\vee} \subseteq \Phi,$$

so that there is a disjoint union  $\Phi = \coprod_i \Phi^{(i)}$ . Then the subset  $\Phi^{(i)} \subseteq \Phi_{\mathfrak{J}_i}$  is also a root subsystem: indeed if  $\alpha \in \Phi^{(i)}$  then  $\sigma_{\alpha} \in W_{\mathfrak{J}_i}$  preserves  $\Phi$  while acting trivially on  $\coprod_{j \neq i} \Phi^{(j)}$ , hence it preserves  $\Phi^{(i)}$ .

Finally we can prove the following:

**Proposition 4.1.** *There is a canonical homeomorphism*

$$\mathbf{B}(\Phi, \Phi_{\mathfrak{g}}) \simeq \prod_i \mathbf{B}(\Phi^{(i)}, \Phi_{\mathfrak{J}_i}) \subseteq \mathfrak{t}.$$

*Proof.* By (12) we have

$$\mathbf{B}(\Phi, \Phi_{\mathfrak{g}}) = \text{Ker}(\Phi) \cap \bigcap_{\Phi_{\mathfrak{g}} \setminus \Phi} (\mathfrak{t} \setminus \text{Ker}(\alpha)) \subseteq \mathfrak{t}.$$

Now if  $\alpha \in \Phi_{\mathfrak{J}_i} \subseteq \Phi_{\mathfrak{g}}$  then

$$\text{Ker}(\alpha) = (\mathfrak{t} \ominus \mathfrak{t}_i) \times \text{Ker}(\alpha_i) \subseteq \mathfrak{t}, \quad \alpha_i := \alpha|_{\mathfrak{J}_i} \in \mathfrak{t}_i^{\vee},$$

and analogously

$$\mathfrak{t} \setminus \text{Ker}(\alpha) = (\mathfrak{t} \ominus \mathfrak{t}_i) \times (\mathfrak{t}_i \setminus \text{Ker}(\alpha_i)) \subseteq \mathfrak{t}.$$

Then intersecting along the partition  $\Phi = \coprod_i \Phi^{(i)}$  yields

$$\begin{aligned} \text{Ker}(\Phi) &= \bigcap_{\Phi} \text{Ker}(\alpha) = \bigcap_i \left( \bigcap_{\Phi^{(i)}} \text{Ker}(\alpha) \right) = \bigcap_i \left( (\mathfrak{t} \ominus \mathfrak{t}_i) \times \bigcap_{\Phi^{(i)}} \text{Ker}(\alpha_i) \right) \\ &= \prod_i \left( \bigcap_{\Phi^{(i)}} \text{Ker}(\alpha_i) \right) \subseteq \prod_i \mathfrak{t}_i = \mathfrak{t}, \end{aligned}$$

since products and intersections commute.

Analogously, looking at the partition

$$\Phi_{\mathfrak{g}} \setminus \Phi = \left( \prod_i \Phi_{\mathfrak{J}_i} \right) \setminus \Phi = \prod_i (\Phi_{\mathfrak{J}_i} \setminus \Phi^{(i)}),$$

one finds

$$\bigcap_{\Phi_{\mathfrak{g}} \setminus \Phi} (\mathfrak{t} \setminus \text{Ker}(\alpha)) = \prod_i \left( \bigcap_{\Phi_{\mathfrak{J}_i} \setminus \Phi^{(i)}} (\mathfrak{t}_i \setminus \text{Ker}(\alpha_i)) \right) \subseteq \prod_i \mathfrak{t}_i.$$

The result follows by taking intersections.  $\square$

We can apply this in the particular case where  $\Phi = \Phi_{\mathfrak{h}}$  is a fission subsystem; and then repeating at each step if we have a filtration of such. This means (9) is a direct product

$$\Gamma(\Phi_{\mathfrak{g}}, \mathbf{d}) = \prod_i \Gamma(\Phi_{\mathfrak{J}_i}, \mathbf{d}_i),$$

where  $\mathbf{d}_i = (d_{\alpha})_{\alpha \in \Phi_{\mathfrak{J}_i}}$ .

In particular the splitting  $\mathfrak{g} = \mathfrak{g}' \oplus^{\perp} \mathfrak{J}_{\mathfrak{g}}$  of a reductive Lie algebra shows  $\Gamma(\Phi_{\mathfrak{g}}, \mathbf{d})$  is isomorphic to a local WMCG for  $\mathfrak{g}' \subseteq \mathfrak{g}$ , since the deformation

space associated with the centre is contractible. Hence we can always assume  $\mathfrak{g}$  is semisimple.

By the same token we can decompose  $\mathfrak{g}$  into simple ideals, and compute the local WMCGs for each of them: so in the end it is enough to study the case of a simple Lie algebra.

Hereafter we thus always assume  $\mathfrak{g}$  is *simple*—so  $\mathfrak{g} = \mathfrak{g}'$ ,  $\mathfrak{t} = \mathfrak{t}'$ , etc.

**4.2. Nonempty complements.** We can exclude the case that (14) be empty in our situation. Namely choose again a dominant semisimple element  $A \in \mathfrak{t}_{\mathbb{R}}$ .

**Lemma 4.1.** *The complement (14) is nonempty.*

*Proof.* Suppose  $\beta \in \Phi_{\mathfrak{g}} \setminus \Phi_{\mathfrak{h}'}$ , and by contradiction  $\text{Ker}(\Delta_{\mathfrak{h}'}) \subseteq \text{Ker}(\beta)$ . This happens if and only if  $\mathbb{C}\beta \subseteq \mathbb{C}\Delta_{\mathfrak{h}'} = \mathbb{C}\Phi_{\mathfrak{h}'}$ , which implies  $\beta(A) = 0$ : absurd.  $\square$

*Remark 4.2.* Note this is false for general root subsystems  $\Phi \subseteq \Phi_{\mathfrak{g}}$ : e.g. the subsystem of short/long roots inside the root system of type  $G_2$  yields an empty complement. This corresponds to the proper inclusion  $A_2 \subseteq G_2$ , which in turn does *not* correspond to a proper inclusion of (finite) Dynkin diagrams since  $\text{rk}(A_2) = 2 = \text{rk}(G_2)$ . (It rather corresponds to an inclusion of *affine* Dynkin diagrams, and is related to *parahoric* structures—instead of *parabolic*.)  $\triangle$

**4.3. Descending ranks.** We can further obtain a natural bound on the number of factors of local WMCGs.

Namely if  $\text{rk } \mathfrak{h}' = \text{rk } \mathfrak{g}$  it follows that (14) is trivial, since  $\dim(\text{Ker}(\Delta_{\mathfrak{h}'})) = 0$  by (15). Hence to have a nontrivial fundamental group we need the rank to diminish at each step, and we find that:

**Proposition 4.2.** *The number of nontrivial factors of (10) is at most  $\text{rk}(\mathfrak{g})$ —independently of  $p \geq 1$  and  $\mathbf{d} \in \mathbb{Z}_{\geq 0}^{\Phi}$ .*

This means any local WMCG can be realised with a uniform bound on the pole order of the irregular type.

**4.4. Low-rank cases.** Suppose further we have minimal nontrivial semisimple co-rank, i.e. that  $\text{rk}(\mathfrak{g}) - \text{rk}(\mathfrak{h}') = 1$ . Then  $\text{Ker}(\Delta_{\mathfrak{h}'}) \subseteq \mathfrak{t}$  is a line, and since the relative complement cannot be empty it must be homeomorphic to  $\mathbb{C} \setminus \{0\}$ ; thus:

**Corollary 4.1.** *If  $\text{rk}(\mathfrak{g}) - \text{rk}(\mathfrak{h}') = 1$  then the fundamental group of (14) is infinite cyclic.*

This corresponds to the pure braid group of type  $A_1$ , i.e. the pure braid group on 2 strands—see below.

An easy extension of this argument then leads the following.

**Corollary 4.2.** *Suppose  $\text{rk}(\mathfrak{g}) = 2$ . Then the local WMCG is either trivial, isomorphic to  $\mathbb{Z}$ , isomorphic to  $\mathbb{Z}^2$ , or isomorphic to the pure  $\mathfrak{g}$ -braid group.*

*Proof.* The nontrivial factors of (10) occur for a Levi factor of rank zero or one.  $\square$

For instance this completely classifies all local WMCGs of exceptional type  $G_2$ .

Starting from the next section we will instead go through the classification of *classical* simple Lie algebras.

## 5. TYPE A

Let  $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$ , for an integer  $n \geq 1$ —to avoid the nonsimple Abelian case. We choose the standard Cartan subalgebra  $\mathfrak{t} \subseteq \mathfrak{g}$  of (traceless) diagonal matrices, and the root system will be simply denoted  $A_n$ .

Identify  $\mathfrak{t} \simeq \mathbb{C}^n$  with the subspace

$$\mathfrak{t} = \left\{ A = \sum_{i=1}^{n+1} \alpha_i(A) e_i \in \mathbb{C}^{n+1} \mid \sum_i \alpha_i(A) = 0 \right\} \subseteq V := \mathbb{C}^{n+1},$$

where  $e_i \in V$  are the vectors of the canonical (orthonormal) basis. Hence  $(A_n, V)$  is the nonspanning root system of the reductive Lie algebra  $\mathfrak{gl}_{n+1}(\mathbb{C}) \supseteq \mathfrak{g}$ , and  $\mathfrak{t} \subseteq V$  is the essential space.<sup>7</sup>

The linear coordinates  $\alpha_i = e_i^\vee \in \mathfrak{t}^\vee$  provide an orthonormal basis of  $V^\vee$  for the dual scalar product, and we set

$$\alpha_{ij}^- := \alpha_i - \alpha_j, \quad 1 \leq i \neq j \leq n+1.$$

The root system is then

$$A_n = \left\{ \pm \alpha_{ij}^- \mid 1 \leq i < j \leq n+1 \right\} \subseteq \mathfrak{t}^\vee,$$

spanning the dual Cartan subalgebra. The standard choice of basis is  $\Delta_{\mathfrak{g}} = \{\theta_i\}_i$ , with  $\theta_i = \alpha_{i,i+1}^-$  for  $i \in \{1, \dots, n\}$  [25, Ch. VI, § 4.7].

**5.1. Fission subsystems.** If  $A \in \mathfrak{t}_{\mathbb{R}}$  is dominant then

$$\alpha_1(A) \geq \dots \geq \alpha_{n+1}(A) \in \mathbb{R}.$$

By the above discussion, the centraliser  $\mathfrak{h} = \mathfrak{z}_{\mathfrak{g}}(A)$  is then a reductive Lie (sub)algebra with root subsystem  $\Phi_{\mathfrak{h}} \subseteq A_n$ , whose simple roots are

$$\Delta_{\mathfrak{h}} = \left\{ \theta_i \in \Delta_{\mathfrak{g}} \mid \alpha_i(A) = \alpha_{i+1}(A) \right\}.$$

This corresponds to a subset of nodes of Dynkin diagram  $\mathcal{DD}(A_n)$ : keeping all edges (among adjacent nodes) yields a disjoint union

$$\mathcal{DD}(\Phi_{\mathfrak{h}}) = \coprod_i \mathcal{DD}(A_{k_i})$$

of Dynkin diagrams of type A. Any connected component on  $k \geq 1$  nodes corresponds to an (irreducible) root subsystem  $A_k \subseteq \Phi_{\mathfrak{h}}$ , whose simple roots form an unbroken string  $I' = \{\theta_i, \dots, \theta_{i+k-1}\} \subseteq \Delta_{\mathfrak{g}}$ . This yields a partition of  $\Delta_{\mathfrak{h}}$ , and accordingly a decomposition of  $\Phi_{\mathfrak{h}}$ .

**Proposition 5.1.** *For any nonempty fission subsystem  $\Phi_{\mathfrak{h}} \subseteq A_n$  there exists “multiplicities”  $n_i \in \mathbb{Z}_{\geq 1}$  such that*

$$\Phi_{\mathfrak{h}} \simeq \bigoplus_{i \geq 1} n_i A_i, \quad \text{with} \quad \sum_i i \cdot n_i \leq n.$$

Below we will show that *all* root subsystems arise from fission in this case (cf. Rem. 5.2).

<sup>7</sup>The centre of the general linear Lie algebra is spanned by the identity matrix, generating the orthogonal line to  $\mathfrak{g}$  for the “trace” scalar product.

*Proof.* Up to ordering the irreducible components by increasing rank, and combining them when equal, the statement follows from

$$n = \text{rk}(A_n) \geq \text{rk}(\mathfrak{h}') = \sum_i n_i \text{rk}(A_i).$$

□

It is then helpful to introduce the following piece of terminology.

**Definition 5.1.** If  $S$  and  $J$  are finite sets, a  $J$ -partition of  $S$  is a surjection  $\phi: S \twoheadrightarrow J$ .

The term is justified, since it is the same to give a partition  $S = \coprod_J I_j$  indexed by  $J$ , with nonempty parts  $I_j := \phi^{-1}(j) \subseteq S$ .

In particular, if  $J' = J'_\mathfrak{h}$  denotes the set of (nonempty) strings of simple roots contained in  $\Delta_\mathfrak{h} \subseteq \Delta_\mathfrak{g}$ , we find a  $J'$ -partition  $\Delta_\mathfrak{h} \twoheadrightarrow J'$ , with parts  $I'_j \subseteq \Delta_\mathfrak{h}$ . Denote then  $A_{I'_j} \subseteq \Phi_\mathfrak{h}$  the irreducible component associated with the string  $I'_j$ , for  $j \in J'$ ; by construction it has rank  $|I'_j| \geq 1$ .

*Remark 5.1.* Note  $J'$  is naturally identified with a subset of  $\{1, \dots, n+1\}$ , mapping

$$I'_j \longmapsto \min(I'_j).$$

△

**5.2. Kernels.** Now we must study the space (14), and it turns out this is controlled by a partition “extending”  $\Delta_\mathfrak{h} \twoheadrightarrow J'$ .

Introduce for brevity the notation  $\underline{n+1} := \{1, \dots, n+1\}$ , and for  $i \in \underline{n+1}$  define

$$I_i = I_i^\mathfrak{h} := \{i\} \cup \left\{ j \in \underline{n+1} \mid \pm \alpha_{ij}^- \in \Phi_\mathfrak{h} \right\} \subseteq \underline{n+1}. \quad (18)$$

**Lemma 5.1.** *The subsets (18) provide a  $J$ -partition  $\underline{n+1} \twoheadrightarrow J$ , where  $J = J_\mathfrak{h}$  is a finite set extending  $J'$ , of cardinality*

$$|J| = n+1 - \text{rk}(\mathfrak{h}').$$

At the extreme cases we have  $J = \underline{n+1}$  for  $\Phi_\mathfrak{h} = \emptyset$ , and  $J = \{*\}$  for  $\Phi_\mathfrak{h} = A_n$ .

*Proof.* First we must show that  $I_i \cap I_j \neq \emptyset$  implies  $I_i = I_j$ —for  $i, j \in \underline{n+1}$ : but this is because the nontrivial root reflections act by

$$\sigma_{ij}^-(\alpha_{jk}^-) = \alpha_{ik}^-, \quad \sigma_{ij}^- = \sigma_{\alpha_{ij}^-},$$

for distinct indices  $i, j, k \in \underline{n+1}$ , and  $\Phi_\mathfrak{h}$  is a root subsystem.

Now by construction the irreducible component  $A_{I'_i} \simeq A_{k_i}$  consists of the set of roots

$$\left\{ \pm \alpha_{jl}^- \mid i \leq j < l \leq i + k_i \right\} \subseteq A_n,$$

if  $k_i = |I'_i| \geq 1$ , and in turn  $I_i = \{i, \dots, i + k_i\}$ . Then there is a natural injective map  $J' \hookrightarrow J$  sending  $I'_i$  to  $I_i$  when  $k_i \geq 2$ .

As for the cardinality, adding a nontrivial irreducible component  $A_{k_i} \subseteq \Phi_\mathfrak{h}$  as above reduces  $|J|$  exactly by  $k_i$ : the singletons  $\{i\}, \dots, \{i + k_i\} \subseteq \underline{n+1}$  are then fused into the subset  $\{i, \dots, i + k_i\}$ . □

Hence we can also denote  $A_{I_i} \subseteq \Phi_{\mathfrak{h}}$  the irreducible components, for  $i \in J' \subseteq J$ . If instead  $I_i = \{i\}$  then  $A_{I_i} \simeq A_0$  is the trivial (nonspanning) “rank-zero” root systems  $A_0 = \emptyset \subseteq \mathbb{C}e_i$ . Concretely this means that  $\mathbb{C}e_i \subseteq \text{Ker}(\Delta_{\mathfrak{h}})$  (see below).

Note in all cases  $\text{rk}(A_{I_i}) = |I_i| - 1$ , and again  $J$  is naturally a subset of  $\underline{n+1}$  by mapping  $I_j \mapsto \min(I_j)$ .

*Remark 5.2* (Fission gives everything). In the above definition of the  $J$ -partition  $\underline{n+1} \twoheadrightarrow J$ , and in the proof of Lem. 5.1, we have not used that  $\Phi_{\mathfrak{h}} \subseteq A_n$  was obtained from fission.

The same argument for any root subsystem  $\Phi \subseteq A_n$  then yields the decomposition of Prop. 5.1: this means all root subsystems can be read from the Dynkin diagram in this case—but not so for other classical types (cf. Rem. 7.1).  $\triangle$

It is now easy to describe the “extended” kernel of  $\Delta_{\mathfrak{h}}$  inside the ambient space  $V \supseteq \mathfrak{t}$ , i.e. considering  $\Phi_{\mathfrak{h}}$  as a fission subsystem for  $\mathfrak{gl}_{n+1}(\mathbb{C})$ . Denote  $\widetilde{\text{Ker}}(\Delta_{\mathfrak{h}}) \subseteq V$  this subspace, so that  $\text{Ker}(\Delta_{\mathfrak{h}}) = \widetilde{\text{Ker}}(\Delta_{\mathfrak{h}}) \cap \mathfrak{t}$ .

**Proposition 5.2.** *There is a canonical vector space isomorphism  $\mathbb{C}^J \simeq \widetilde{\text{Ker}}(\Delta_{\mathfrak{h}})$ . For  $i \in J$  it is defined by*

$$e_i \mapsto \frac{e_{I_i}}{|I_i|}, \quad e_{I_i} := \sum_{j \in I_i} e_j \in V,$$

where  $e_i \in \mathbb{C}^J$  is a vector of the canonical (orthonormal) basis.

*Proof.* There is an orthogonal decomposition

$$V \simeq \bigoplus_{i \in J}^{\perp} \mathbb{C}^{I_i}, \quad \mathbb{C}^{I_i} = \bigoplus_{j \in I_i} \mathbb{C}e_j,$$

and since each summand contains the irreducible component  $A_{I_i} \subseteq \Phi_{\mathfrak{h}}$  the kernel splits accordingly.

Now consider the linear map

$$\pi_i: \mathbb{C}^{I_i} \longrightarrow \mathbb{C}^{I_i}, \quad \pi_i: e_j \mapsto \frac{e_{I_i}}{|I_i|}. \quad (19)$$

By construction (19) is an idempotent with  $\pi_i(\mathbb{C}^{I_i}) = \mathbb{C}e_{I_i}$ , and

$$(\pi_i(e_j) | e_{I_i}) = (e_j | e_{I_i}), \quad j \in I_i,$$

so it is the orthogonal projection onto the line generated by  $e_{I_i}$ . But this line is precisely  $\text{Ker}(A_{I_i}) \subseteq \mathbb{C}^{I_i}$ , since this is the subspace where all coordinates are equal.  $\square$

Then the “essential” kernel we are after is the subspace

$$\mathfrak{u} := \text{Ker}(\Delta_{\mathfrak{h}}) = \left\{ A = \sum_{i \in J} \alpha_{I_i}(A) e_{I_i} \in V \mid \sum_i \alpha_{I_i}(A) = 0 \right\} \subseteq V,$$

introducing the dual basis  $\alpha_{I_i} = e_{I_i}^{\vee}$ . It has dimension  $|J| - 1 = n - \text{rk}(\mathfrak{h}')$ , in accordance with (15).

**5.3. Restricted subsystem and fundamental group.** Finally we must remove the root-hyperplanes corresponding to (positive) roots  $\alpha \in \mathcal{A}_n \setminus \Phi_{\mathfrak{h}}$ , and it turns out this still yields a root system of type A.

**Theorem 5.1.** *There is a canonical isomorphism of root systems*

$$\mathcal{A}_{|J|-1} \simeq \mathcal{A}_n|_{\mathcal{U}} \subseteq \mathcal{U}^\vee.$$

*Equivalently, in dual terms, there is a canonical isomorphism of root systems*

$$\mathcal{A}_{|J|-1} \simeq \pi_{\mathcal{U}}(\mathcal{A}_n^\vee) \subseteq \mathcal{U},$$

where  $\pi_{\mathcal{U}}: \mathfrak{t} \rightarrow \mathcal{U}$  is the orthogonal projection.

*Proof.* The restriction of  $\alpha_{ij}^- \in \mathfrak{t}^\vee$  to this kernel is precisely the linear functional  $\alpha_{I_i I_j}^- = \alpha_{I_i} - \alpha_{I_j} \in \mathcal{U}^\vee$ , provided that  $I_i \neq I_j$ —else it vanishes; and conversely  $\alpha_{ij}^- \in \mathcal{A}_n \setminus \Phi_{\mathfrak{h}}$  if and only if the indices  $i$  and  $j$  lie in different parts of the  $J$ -partition.  $\square$

It follows that (14) is the complement of the type-A hyperplane arrangement of rank  $|J| - 1$ , whence

$$\pi_1(\mathbf{B}(\Phi_{\mathfrak{h}}, \Phi_{\mathfrak{g}}), \mathcal{A}) \simeq \text{PB}_{|J|},$$

the pure braid group on  $|J|$  strands.

**5.4. Fission trees.** Finally we can reason recursively. By the above any fission subsystem  $\Phi_{\tilde{\mathfrak{h}}} \subseteq \Phi_{\mathfrak{h}}$  split as  $\Phi_{\tilde{\mathfrak{h}}} = \bigoplus_{\tilde{J}} \Phi^{(i)}$ , where  $\Phi^{(i)} \subseteq \mathcal{A}_{I_i}$  is an irreducible component; and in turn  $\Phi^{(i)}$  will decompose according to Prop. 5.1, replacing  $n = \text{rk}(\mathfrak{g})$  with  $|I_i| - 1 = \text{rk}(\mathcal{A}_{I_i})$ . Hence the partition  $\underline{n+1} \rightarrow \tilde{J}$ , associated with  $\Phi_{\tilde{\mathfrak{h}}}$  as in (18), is a refinement of the  $J$ -partition associated with  $\Phi_{\mathfrak{h}}$ : for any  $i \in \tilde{J}$  there exists  $\phi(i) \in J$  such that  $I_i^{\tilde{\mathfrak{h}}} \subseteq I_{\phi(i)}^{\mathfrak{h}}$ , i.e. there is a (new)  $J$ -partition  $\phi: \tilde{J} \rightarrow J$ .

In conclusion an increasing sequence

$$\Phi_{\mathfrak{h}_1} \subseteq \cdots \subseteq \Phi_{\mathfrak{h}_p} \subseteq \Phi_{\mathfrak{h}_{p+1}} := \mathcal{A}_n$$

of fission subsystems corresponds to a decreasing sequence of sets

$$J_1 \xrightarrow{\phi_1} J_2 \longrightarrow \cdots \longrightarrow J_p \xrightarrow{\phi_p} J_{p+1} := \{*\}, \quad (20)$$

where  $J_l = J_{\mathfrak{h}_l}$  is the set of parts  $I_i^{\mathfrak{h}_l}$  as in (18), for  $l \in \{1, \dots, p\}$ . This is the same as considering the disjoint union

$$T_0 := \coprod_{l=1}^{p+1} J_l,$$

and giving a single function

$$\phi: T_0 \setminus \{*\} \longrightarrow T_0, \quad J_l \ni i \longmapsto \phi_l(i) \in J_{l+1}.$$

It is natural to encode this data into a tree.

**Definition 5.2** (Fission tree). The *fission tree*  $T = T(\phi)$  of (20) is the tree with nodes  $T_0$ , such that  $\phi(i) \in T_0$  is the parent-node of  $i \in T_0 \setminus \{*\}$ .

Hence  $J_1 \subseteq T_0$  are the leaves and  $*$  in  $J_{p+1}$  is the root, while  $|J_l|$  is the number of nodes at level  $l \in \{1, \dots, p+1\}$ ; note by construction  $|J_1| \leq n+1$ .<sup>8</sup> Set finally

$$k_i := \left| \Phi^{-1}(i) \right| \in \mathbb{Z}_{\geq 0}, \quad i \in T_0,$$

where by convention  $k_i := 0$  if  $i \in J_1$ . This is the number of child-nodes of  $i \in T_0$ . This construction yields the main result of this section.

**Theorem 5.2.** *The fission tree determines the type-A local WMCG via the canonical group isomorphism*

$$\Gamma(A_n, \mathbf{d}) \simeq \prod_{T_0} \text{PB}_{k_i}.$$

*Conversely type-A local WMCGs exhaust finite products of pure braid groups.*

*Proof.* By construction a node  $i \in J_l$  corresponds to an irreducible component of  $\Phi_{h_l}$ , which splits into  $k_i \geq 0$  irreducible components inside  $\Phi_{h_{l-1}} \subseteq \Phi_{h_l}$ —corresponding to the child-nodes  $j \in \Phi^{-1}(i) \subseteq J_{l-1}$ , with  $l \in \{1, \dots, p+1\}$ . By the above discussion this leads to a root-hyperplane complement of type  $A_{k_i-1}$ , i.e. a pure braid group factor on  $k_i$  strands within the local WMCG. Hence

$$\pi_1(\mathbf{B}(\Phi_{h_{l-1}}, \Phi_{h_l}), A_l) \simeq \prod_{J_l} \text{PB}_{k_i},$$

for an irregular type  $Q = \sum_{l=1}^p A_l z^{-1}$ . The conclusion follows from the splitting (10), i.e. the fact that these products arise independently at each level of the tree.

For the second statement, given any sequence of integers  $n_i \geq 1$  with finite support we can construct a fission tree having precisely  $n_i$  nodes with  $i \geq 1$  child-nodes (in many ways, cf. Ex. 5.1). In that case

$$\Gamma(A_n, \mathbf{d}) \simeq \prod_{i \geq 1} \text{PB}_i^{n_i}. \quad (21)$$

□

*Remark 5.3* (Low-order and irreducible presentations). Thus trees of arbitrary height (i.e. irregular types of arbitrary pole order) can give a presentation of one and the same local WMCG.

The minimal height/pole order is obtained by putting as many splittings as possible, as soon as possible, starting from the root.

Conversely, neglecting trees with no splitting at some level, we can consider a tree with a single node splitting at each level: the associated sequence of fission subsystems is then by *irreducible* ones, i.e. there are integers  $n_1 \leq \dots \leq n_p \leq n$  such that (11) becomes

$$A_{n_1} \subseteq \dots \subseteq A_{n_p} \subseteq A_n,$$

where  $A_{n_i} = A_n \cap \bigoplus_{i=1}^{n_i+1} \mathbb{C} \alpha_i \subseteq V^\vee$ —embedded on the first slots.

Moreover in this case  $n_i \geq 1$  is equal to the number of leaves of the subtree rooted at the (unique) splitting node at level  $J_i \subseteq T_0$ .  $\triangle$

<sup>8</sup>The equality  $|J_1| = n+1$  corresponds to the case where no root annihilates the irregular type  $Q$ . In this case the fission is “complete”, i.e.  $H_1 = \text{Stab}_{\text{Ad}_G}(Q) = T \subseteq G$  (cf. § 3).



*Example 5.1* (Examples of presentations). For  $n = 8$  let us consider the irregular type

$$Q = A_1x + A_2x^2 + A_3x^3, \quad \text{where} \quad A_i = \sum_{j=1}^9 \alpha_j(A_i)e_j \in \mathfrak{t}_{\mathbb{R}},$$

taking the following coordinate vectors  $\alpha_i = (\alpha_1(A_i), \dots, \alpha_9(A_i)) \in \mathbb{R}^9$ :

$$\alpha_1 = (4, 3, 2, 1, 0, -1, -2, -3, -4), \quad \alpha_2 = (4, 4, 3, 2, 1, 0, -3, -4, -7),$$

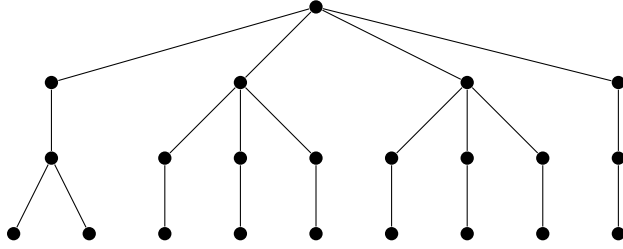
and

$$\alpha_3 = (2, 2, 1, 1, 1, 0, 0, 0, -7).$$

Then the sequence of fission subsystems is

$$\emptyset \subseteq A_1 \subseteq A_1 \oplus A_2 \oplus A_2 \subseteq A_8,$$

and the associated local WMCG is  $\Gamma(A_n, \mathbf{d}) \simeq \text{PB}_2 \times \text{PB}_3^2 \times \text{PB}_4$ . The corresponding fission tree is drawn just below:



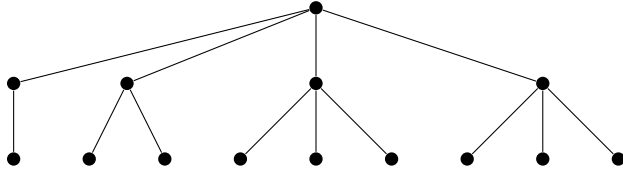
But  $\Gamma(A_n, \mathbf{d})$  is also the fundamental group of the space of admissible deformations of  $Q = A_1x + A_2x^2$ , taking coordinate vectors

$$\alpha_1 = (4, 3, 2, 1, 0, -1, -2, -3, -4), \quad \alpha_2 = (4, 1, 1, 0, 0, 0, -2, -2, -2),$$

which yields the low-order presentation, and has associated filtration

$$\emptyset \subseteq A_1 \oplus A_2 \oplus A_2 \subseteq A_8.$$

The (minimal-height) fission tree is then:



Finally  $\Gamma(A_n, \mathbf{d})$  is also the fundamental group of the space of admissible deformations of  $Q = A_1x + A_2x^2 + A_3x^3 + A_4x^4$ , taking coordinate vectors

$$\alpha_1 = (4, 3, 2, 1, 0, -1, -2, -3, -4), \quad \alpha_2 = (4, 4, 3, 2, 1, 0, -3, -4, -7),$$

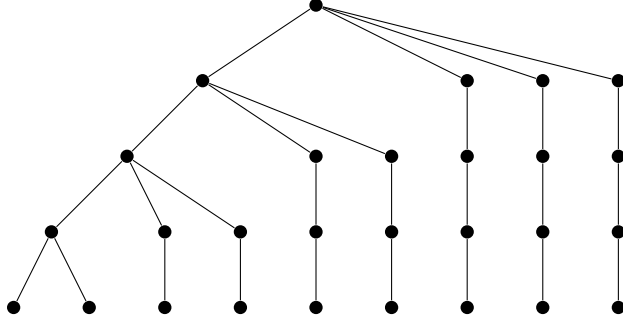
and

$$\alpha_3 = (2, 2, 2, 2, 1, 0, -3, -3, -3), \quad \alpha_4 = (1, 1, 1, 1, 1, 1, 0, -2, -4).$$

This yields the irreducible presentation, with filtration

$$\emptyset \subseteq A_1 \subseteq A_3 \subseteq A_5 \subseteq A_8,$$

and the fission tree is as follows:



Note at each level only the leftmost node splits.  $\triangle$

*Remark 5.4.* In the examples above we see the sequence of restricted root systems (17) is uniquely determined by the corresponding local WMCG, and conversely determines its factors.

In particular the sum of such root systems in Ex. 5.1 is  $A_1 \oplus 2A_2 \oplus A_3$ , and in general a (finite) sum  $\bigoplus_{i \geq 1} n_i A_i$  will correspond to the identification  $\Gamma(A_n, \mathbf{d}) = \prod_{i \geq 1} \text{PB}_{i+1}^{n_i}$ —i.e. (21), up to shifting the ranks of the irreducible components to match it up with the number of strands.  $\triangle$

## 6. CABLED BRAID GROUPS

The fission tree will also give a way to express an element of the local WMCG as a braid on as many strands as the number of leaves of the tree, formalising the driving intuition of the introduction; intuitively what happens is the following.

If  $Q = \sum_{i=1}^p A_i z^{-i}$  is an irregular type, we can first braid the spectrum  $\text{Spec}(A_p) \subseteq \mathbb{C}$  of the leading coefficient  $A_p \in \mathfrak{sl}(V)$ , in the given vector representation: a deformation of  $A_p$  will be admissible if distinct eigenvalues do not coalesce, i.e. if the eigenspace decomposition

$$V = \bigoplus_{\text{Spec}(A_p)} V_\lambda^{(p)}, \quad V_\lambda^{(p)} = \text{Ker}(A_p - \lambda \text{Id}_V)$$

is preserved. We thus find a pure braid group  $\text{PB}_{n_p}$ , on as many strands as  $n_p = |\text{Spec}(A_p)| \geq 1$ .

But the  $A_p$ -eigenspaces are stable for the subleading coefficient  $A_{p-1} \in \mathfrak{t}$ , so each of those can be further split into eigenspaces for the restriction

$$A_{p-1}[\lambda] := A_{p-1}|_{V_\lambda^{(p)}} \in \text{End}(V_\lambda^{(p)}).$$

Then we can braid the spectrum  $\text{Spec}(A_{p-1}[\lambda]) \subseteq \mathbb{C}$  of each restriction in an admissible way, on top of the braiding of the eigenvalues of  $A_p$ : this means replacing the strands of  $\text{PB}_{n_p}$  with other braids, i.e. *cabling* them. Repeating at each order finally leads to a (multi)cabled braid inside a many-strand pure braid group  $\text{PB}_k$ , and  $k \geq 1$  is precisely the number of leaves of the associated fission tree.

This way the fission tree of Def. 5.2 provides a map  $\iota: \Gamma(A_n, \mathbf{d}) \rightarrow \text{PB}_k$ , where  $\text{PB}_k$  corresponds to the (uncabled) braid group associated with the “generic” fission: in this section we show  $\iota$  is an injective group morphism, which follows from the *operadic* nature of the cabling operation, and identify its image as a

(pure) “cabled braid group”. In turn this latter is constructed recursively using the operadic composition of the pure braid group operad (cf. Def. 6.1).

6.1. **Pure cabling.** There are two natural operations on (pure) braids:

(1) the “direct sum”

$$\prod_i \text{PB}_{m_i} \longrightarrow \text{PB}_m, \quad (\sigma_i)_i \longmapsto \bigoplus_i \sigma_i,$$

with  $m_i \in \mathbb{Z}_{\geq 0}$  and  $m = \sum_i m_i$ , which is the group morphism obtained by sending a tuple of braids into a many-strand braid where these are juxtaposed;

(2) the “block braid”

$$\text{PB}_m \longrightarrow \text{PB}_k, \quad \sigma \longmapsto \sigma \langle k_1, \dots, k_m \rangle,$$

with  $m, k_1, \dots, k_m \in \mathbb{Z}_{\geq 0}$  and  $k = \sum_i k_i$ , which is the function (set morphism) obtained by replacing the  $i$ -th strand of a braid by  $k_i$  parallel copies of it—keeping the same crossings. In particular we simply delete the  $i$ -th strand if  $k_i = 0$ , and below we will recall that this a group morphism in the pure case.

Then the *cabling* (or  *$i$ -composition*) of a braid  $\tau \in \text{PB}_m$  onto the  $i$ -th strand of a braid  $\sigma \in \text{PB}_n$  is the operation

$$\sigma \circ_i \tau := \sigma \left( \underbrace{1, \dots, 1}_{i-1 \text{ times}}, \underbrace{m, 1, \dots, 1}_{n-i \text{ times}} \right) \cdot \left( \underbrace{\text{Id}_1 \oplus \dots \oplus \text{Id}_1}_{i-1 \text{ times}} \oplus \tau \oplus \underbrace{\text{Id}_1 \oplus \dots \oplus \text{Id}_1}_{n-i \text{ times}} \right) \in \text{PB}_{m+n-1}, \quad (22)$$

where on the rightmost factor  $\text{Id}_1 \in \text{PB}_1$ —the unique element. In words this means replacing the  $i$ -th strand of  $\sigma$  with the braid  $\tau$ .

Now one can show the data of the sets  $P(n) = \text{PB}_n$ , the unit  $\text{Id}_1 \in \text{PB}_1$ , and the maps (22), satisfies the associativity/unity axioms of an operad (as introduced in [22, 53, 23]), leading to the *pure braid group operad*  $\mathcal{PB}$  [67, § 5]. In particular one can cable each strand in any order, and the result will not change, so “simultaneous” cabling yields the operadic composition

$$\gamma^{\mathcal{PB}} : \text{PB}_n \times \prod_{i=1}^n \text{PB}_{k_i} \longrightarrow \text{PB}_m, \quad (\sigma, \tau_1, \dots, \tau_n) \longmapsto \gamma^{\mathcal{PB}}(\sigma; \tau_1, \dots, \tau_n),$$

where  $m = \sum_i k_i$ , defined by

$$\gamma^{\mathcal{PB}}(\sigma; \tau_1, \dots, \tau_n) := \sigma \langle k_1, \dots, k_n \rangle \cdot (\tau_1 \oplus \dots \oplus \tau_n). \quad (23)$$

In principle this is only a function of sets, but if we equip the domain with the direct-product group structure then:

**Lemma 6.1.** *The operadic composition (23) is an injective group morphism.*<sup>9</sup>

*Proof.* It will be enough to prove this statement for the  $i$ -th strand cabling (22). By definition this is the composition of the function

$$\text{PB}_n \times \text{PB}_m \longrightarrow \text{PB}_{m+n-1} \times \text{PB}_{m+n-1},$$

<sup>9</sup>We thus find a (noncrossed) “group” operad [69, Ex. 2.11] (cf. [68, 64]), a.k.a. “action” operad [30] (cf. [67, Def. 4.1.1]).

defined by

$$(\sigma, \tau) \mapsto (\sigma \langle 1, \dots, 1, m, 1, \dots, 1 \rangle, \text{Id}_1 \oplus \text{Id}_1 \oplus \tau \oplus \text{Id}_1 \cdots \oplus \text{Id}_1),$$

with the group multiplication  $\text{PB}_{m+n-1} \times \text{PB}_{m+n-1} \rightarrow \text{PB}_{m+n-1}$ . Hence to see (22) is a group morphism (from the direct product) it is enough to show that the block braid operation is a group morphism, and further that the images of the two operations commute: this follows from [67, Lem. 5.2.4].

To show injectivity instead we can prove that if

$$\sigma' = \sigma \langle 1, \dots, 1, m, 1, \dots, 1 \rangle = \text{Id}_1 \oplus \text{Id}_1 \oplus \tau \oplus \text{Id}_1 \cdots \oplus \text{Id}_1 \in \text{PB}_{m+n-1},$$

for some  $(\sigma, \tau) \in \text{PB}_n \times \text{PB}_m$ , then both  $\sigma$  and  $\tau$  are trivial. Now the identity

$$\sigma' = \text{Id}_1 \oplus \text{Id}_1 \oplus \tau \oplus \text{Id}_1 \cdots \oplus \text{Id}_1$$

implies the first  $i-1$  and the last  $n-i$  strands of  $\sigma'$  have trivial braiding, so the same is true of all the strands of  $\sigma$  except at most the  $i$ -th one; but if this had non-trivial braiding then the “central”  $m$  strands of  $\sigma'$  would cross the “peripheral” ones, and  $\sigma' = \sigma \langle 1, \dots, 1, m, 1, \dots, 1 \rangle$  is impossible. Hence  $\sigma$  and  $\sigma'$  are trivial, so  $\tau$  as well—it is a copy of the  $m$  “central” strands of  $\sigma'$ .  $\square$

*Remark 6.1.* In general (pure) cabling is *not* an isomorphism: e.g.  $\text{PB}_2 \times \text{PB}_2 \rightarrow \text{PB}_3$  cannot be, since the source is abelian while the target is not. Nonetheless the argument about injectivity also applies verbatim to the nonpure case—in which case the cabling is *not* a group morphism in general.  $\triangle$

We then introduce *cabled* braid groups, as follows. Let  $T = T(\phi)$  be a tree with  $\phi: T_0 \setminus \{*\} \rightarrow T_0$  as in § 5, and retain the notation for the levels  $J_l \subseteq T_0$  and the number of child-nodes  $k_i \geq 0$  of  $i \in T_0$ . This is a “cabling” tree, rather than a fission tree, in this context (but they both describe the same class of groups, cf. Thm. 6.1 below).

**Definition 6.1.** The *pure cabled braid group*  $\text{PCB} = \text{PCB}(T)$  of the cabling tree is the group obtained at the end of the following sequence of applications of (23):

- start at the root and set  $\text{PCB}_{p+1} := \text{PB}_1$  (the trivial group);
- for each level  $l \in \{p, \dots, 1\}$  define recursively

$$\text{PCB}_l := \gamma^{\mathcal{PB}} \left( \text{PCB}_{l+1} \times \prod_{J_{l+1}} \text{PB}_{k_i} \right) \subseteq \text{PB}_{|J_l|}.$$

By construction  $\text{PCB}(T) = \text{PCB}_1 \subseteq \text{PB}_{|J_1|}$  is a subgroup of the pure braid group on as many strands as the leaves of  $T$ , and finally matching up fission/cabling trees yields the following.

**Theorem 6.1.** *If  $T$  is the fission tree associated with an irregular type for  $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$ , then there is a canonical group isomorphism  $\Gamma(A_n, \mathbf{d}) \simeq \text{PCB}(T)$ .*

*Proof.* This follows from the factorisation of  $\Gamma(A_n, \mathbf{d})$  of Thm. 5.2, by induction on  $p \geq 1$ —i.e. on the height of  $T$ .

The base  $p = 1$  corresponds to the identity

$$\gamma^{\mathcal{PB}}(\text{PB}_1 \times \text{PB}_k) = \text{PB}_k, \quad k \in \mathbb{Z}_{\geq 1},$$

which is part of the operad unity axiom.

For the inductive step, the recursive hypothesis yields

$$\text{PCB}_2 = \prod_{T'_0} \text{PB}_{k_i} \subseteq \text{PB}_{|J_2|},$$

where  $T'_0 := T_0 \setminus J_1 \subseteq T_0$  are the nodes of the (sub)tree obtained by pruning the leaves of  $T$ ; thus

$$\text{PCB}_1 = \gamma^{\mathcal{PB}} \left( \prod_{T'_0} \text{PB}_{k_i} \times \prod_{J_2} \text{PB}_{k_i} \right) \simeq \prod_{T_0} \text{PB}_{k_i} \subseteq \text{PB}_{|J_1|},$$

by Lem. 6.1. □

In the next sections we explore generalisation of this situation to other simple Lie algebras of classical type.

## 7. TYPE B

Here we consider  $\mathfrak{g} = \mathfrak{so}_{2n+1}(\mathbb{C})$ , for  $n \geq 1$ . The standard Cartan subalgebra  $\mathfrak{t} \subseteq \mathfrak{g}$  is identified with  $\mathbb{C}^n = \bigoplus_i \mathbb{C}e_i$  with canonical scalar product, and we retain the notations of § 5. The root system will be simply denoted  $B_n$ .

The usual choice of basis is  $\Delta_{\mathfrak{g}} = \{\theta_i\}_i$  with

$$\theta_i = \alpha_{i,i+1}^-, \quad i \in \{1, \dots, n-1\},$$

as for  $A_{n-1}$ , plus the *short* root  $\theta_n = \alpha_n \in \mathfrak{t}^\vee$ . It follows that

$$B_n = \left\{ \pm \alpha_{ij}^-, \pm \alpha_{ij}^+ \mid 1 \leq i < j \leq n \right\} \cup \{ \alpha_i \mid 1 \leq i \leq n \} \subseteq \mathfrak{t}^\vee,$$

writing  $\alpha_{ij}^+ := \alpha_i + \alpha_j$  [25, Ch. VI, § 4.5].

**7.1. Fission subsystems.** If  $\Lambda = \sum_i \alpha_i(A)e_i \in \mathfrak{t}_{\mathbb{R}}$  is dominant then

$$\alpha_1(A) \geq \dots \geq \alpha_n(A) \in \mathbb{R}_{\geq 0},$$

and we consider again its centraliser  $\mathfrak{h} = \text{Ker}(\text{ad}_{\Lambda}) \subseteq \mathfrak{g}$ .

Now we have two cases: either  $\theta_n(A) > 0$  or not. If the last coordinate is positive then  $\Delta_{\mathfrak{h}}$  only contains long simple roots, and  $\Phi_{\mathfrak{h}} \subseteq A_{n-1}$ . In this case we can appeal to Prop. 5.1 to classify the fission subsystem.

If instead  $\alpha_n(A) = 0$  then there exists an integer  $m \leq n$  such that

$$\theta_n(A) = \dots = \theta_{n-m+1}(A) = 0, \quad \text{but} \quad \theta_{n-m}(A) \neq 0 :$$

then we can split  $\Delta_{\mathfrak{h}} = \Delta_{\mathfrak{h}}^A \cup \Delta_{\mathfrak{h}}^B$ , with  $\Delta_{\mathfrak{h}}^B = \{\theta_{n-m+1}, \dots, \theta_n\}$ , and accordingly

$$\Phi_{\mathfrak{h}} \simeq \Phi_{\mathfrak{h}}^A \oplus B_m,$$

with  $\Phi_{\mathfrak{h}}^A \subseteq A_{n-m}$ . Finally we can apply Prop. 5.1 to the summand  $\Phi_{\mathfrak{h}}^A$ , so on the whole:

**Corollary 7.1.** *For any nontrivial fission subsystem  $\Phi_{\mathfrak{h}} \subseteq B_n$  there exists unique multiplicities  $m \geq 0$  and  $n_i \geq 1$  such that*

$$\Phi_{\mathfrak{h}} \simeq B_m \oplus \bigoplus_{i \geq 1} n_i A_i, \quad \text{with} \quad m + \sum_i n_i \leq n.$$

*Remark 7.1* (Fission does not give everything). The above decomposition corresponds to the fact that the subdiagram  $\mathcal{DD}(\Phi_{\mathfrak{h}}) \subseteq \mathcal{DD}(B_n)$  can have at most one component isomorphic to a Dynkin diagram of type B, namely the “rightmost” one.

However this is not true of any subsystem: for example we can have systems of type  $B_m \oplus B_{m'} \subseteq B_n$  such that neither  $B_m$  nor  $B_{m'}$  is isomorphic to a root system of type A, and we can also have subsystems of type D. It is nonetheless possible to show that *any* root subsystem  $\Phi \subseteq B_n$  decomposes as a direct sum of systems of type A, B or D (coherently with the tables of [55, § 10.1]).<sup>10</sup>  $\triangle$

**7.2. Kernels.** The next step is identifying  $\mathfrak{U} = \text{Ker}(\Delta_{\mathfrak{h}}) \subseteq \mathfrak{t}$ , and here we can appeal to Prop. 5.2. Namely one has

$$\mathfrak{U} = \text{Ker}(\Delta_{\mathfrak{h}}^A) \cap \text{Ker}(\Delta_{\mathfrak{h}}^B),$$

so one need only further understand the kernel of a component  $B_m \subseteq \Phi_{\mathfrak{h}}$ . However this component spans the subspace  $(\mathbb{C}^m)^\vee = \bigoplus_{i=n-m+1}^n \mathbb{C}\alpha_i \subseteq \mathfrak{t}^\vee$  (it is “essential”), whence

$$\text{Ker}(\Delta_{\mathfrak{h}}^B) = \mathbb{C}^{n-m} \times (0) = \bigoplus_{i=1}^{n-m} \mathbb{C}e_i \subseteq \mathfrak{t}.$$

It follows that  $\mathfrak{U} \simeq \mathbb{C}^{|J|} \subseteq \mathbb{C}^{n-m} \times (0)$ , where J is the index set of the partition associated with  $\Phi_{\mathfrak{h}}^A \subseteq (\mathbb{C}^{n-m})^\vee$  as in (18). We will write this as

$$\mathfrak{U} = \left\{ \sum_{i \in J} \alpha_{I_i}(A) e_{I_i} \right\} \subseteq \mathfrak{t}, \quad (24)$$

where again  $e_{I_i} = \sum_{j \in I_i} e_j$ , and  $I_i \subseteq \underline{n-m}$  is one part of the J-partition.

**7.3. Restricted arrangement and fundamental group.** Finally we must describe the hyperplane arrangement inside the kernel, provided by the kernel of the (positive) roots  $\alpha \in B_n \setminus \Phi_{\mathfrak{h}}$  after restriction to  $\mathfrak{U}$ .

To this end introduce again the notation  $\alpha_{I_i I_j}^\pm = \alpha_{I_i} \pm \alpha_{I_j} \in \mathfrak{U}^\vee$  for the linear functionals associated with the given coordinates, with  $i, j \in J$ —but for  $i \in \underline{n-m}$  we also denote  $I_i \subseteq \underline{n-m}$  the subset containing, as in (18).

**Theorem 7.1.** *The hyperplane arrangement in the kernel is of type  $B_{|J|}/C_{|J|}$ .*

*Moreover if no component of the J-partition is trivial then there is a canonical isomorphism of root systems*

$$BC_{|J|} \simeq B_n|_{\mathfrak{U}} \subseteq \mathfrak{U}^\vee.$$

*Equivalently, in dual terms, there is a canonical isomorphism of root systems*

$$BC_{|J|} \simeq \pi_{\mathfrak{U}}(B_n^\vee) \subseteq \mathfrak{U},$$

*where  $\pi_{\mathfrak{U}}: V \rightarrow \mathfrak{U}$  is the orthogonal projection.*

<sup>10</sup>This can e.g. be proven by introducing a class of graphs with edges of two different colours, and admitting loop edges, generalising the partition (18); but it will not be discussed here.

*Proof.* Computing all restrictions yields

$$\alpha_{ij}^-|_{\mathcal{U}} = \begin{cases} \alpha_{I_i I_j}^-, & I_i \neq I_j \subseteq \underline{n-m}, \\ 0, & I_i = I_j \subseteq \underline{n-m} \\ \alpha_{I_i}, & i \leq n-m, j \geq n-m+1, \\ 0, & i, j \geq n-m+1, \end{cases}$$

and

$$\alpha_{ij}^+|_{\mathcal{U}} = \begin{cases} \alpha_{I_i I_j}^+, & I_i \neq I_j \subseteq \underline{n-m}, \\ 2\alpha_{I_i}, & I_i = I_j \subseteq \underline{n-m} \\ \alpha_{I_i}, & i \leq n-m, j \geq n-m+1, \\ 0, & i, j \geq n-m+1, \end{cases}$$

while simply

$$\alpha_i|_{\mathcal{U}} = \begin{cases} \alpha_{I_i}, & i \in \underline{n-m}, \\ 0, & i \in \{n-m+1, \dots, n\}. \end{cases}$$

Hence one always has the inclusion

$$\left\{ \pm \alpha_{I_i I_j}^-, \pm \alpha_{I_i I_j}^+ \mid I_i \neq I_j \right\} \cup \{ \alpha_{I_i} \mid i \in J \} \subseteq B_n|_{\mathcal{U}}.$$

Further the root  $2\alpha_{I_i} \in \mathcal{U}^\vee$  appears from the restriction of  $\alpha_{ij}^+ \in \mathcal{U}^\vee$  for some  $j \neq i \in I_i$ , i.e. if and only if  $I_i \neq \{i\}$ .

These are all the linear functionals obtained via restriction to the kernel, so the hyperplane arrangement is always of type B/C.  $\square$

*Remark 7.2.* The hyperplanes arrangements are always those of a root system, so their reflection groups are crystallographic; but the set of restricted functional themselves are *not* root systems in general.

Namely if  $I_i = \{i\}$  is a trivial component of the J-partition then  $2\alpha_{I_i} \notin B_n|_{\mathcal{U}}$ ; but a nontrivial component  $I_j \subseteq \underline{n-m}$  further yields

$$\sigma_{I_i I_j}^-(2\alpha_{I_j}) = 2\alpha_{I_i}, \quad \sigma_{I_i I_j}^- = \sigma_{\alpha_{I_i I_j}^-},$$

hence the subset is not closed under mutual reflections in this case.  $\triangle$

It follows that the fundamental group of the hyperplane complement (14) is

$$\pi_1(\mathbf{B}(\Phi_{\mathfrak{h}}, \Phi_{\mathfrak{g}}), A) \simeq \text{PB}_{|J|}^{\text{B/C}},$$

the pure braid group of type B/C. Note this only depends on the number of type-A irreducible components of  $\Phi_{\mathfrak{h}}$ .

**7.4. Bichromatic fission trees.** Finally we can reason recursively, as in the previous section. A filtration

$$\Phi_{\mathfrak{h}_1} \subseteq \dots \subseteq \Phi_{\mathfrak{h}_{p+1}} = B_n,$$

of fission subsystems, gives rise to two filtrations, by splitting  $\Phi_{\mathfrak{h}_i} = \Phi_{\mathfrak{h}_i}^A \oplus B_{m_i}$  as above—with  $m_1 \leq \dots \leq m_p \leq n$ . Namely there is a type-A filtration

$$\Phi_{\mathfrak{h}_1}^A \subseteq \dots \subseteq \Phi_{\mathfrak{h}_p}^A \subseteq A_{n-1},$$

and a filtration by irreducible subsystems

$$B_{m_1} \subseteq \dots \subseteq B_{m_p} \subseteq B_n,$$



with embeddings on the last slots (at each step).

Hence one can introduce a fission tree with *green* nodes to encode the type-A fission/filtration, and *blue* nodes to encode the type-B components, getting to a natural generalisation of Def. 5.2. Denote  $\{g, b\}$  the set of colours “green” and “blue”, with total order  $g \leq b$ ; then:

**Definition 7.1** (Bichromatic fission tree). A *bichromatic fission tree* is a fission tree  $T = T(\Phi)$  equipped with a *colour function*  $c: T_0 \rightarrow \{g, b\}$ ; in turn a colour function satisfies:

- $c(\Phi(i)) \geq c(i)$  for  $i \in T_0 \setminus \{*\}$ ;
- $|\Phi^{-1}(i) \cap c^{-1}(b)| \leq 1$  for  $i \in T_0$ .

The conditions mean that green nodes have green child-nodes, and that any node has at most one blue child-node, respectively. In particular if  $c(*) = g$  we find a (green/monochromatic) fission tree as in Def. 5.2.

The algorithm to assign a bichromatic fission tree to a double filtration as above is the following. A node  $i \in J_l$  corresponds to an irreducible component of the subsystem  $\Phi_{h_l} \subseteq B_n$ : put a green node for each type-A component, and a blue node if  $m_l > 0$ —i.e. if there is a type-B component at all; then  $j := \Phi(i) \in J_{l+1}$  if the irreducible component of  $\Phi_{h_{l+1}}$  associated with  $j$  contains the irreducible component of  $\Phi_{h_l} \subseteq \Phi_{h_{l+1}}$  associated with  $i \in J_l$ . (So the tree starts with a blue root, corresponding to  $B_n$  itself.)

Finally we can compute the local WMCG in terms of the (generalised) fission tree. We extend the notation of § 5 by redefining

$$k_i := |\Phi^{-1}(i) \cap c^{-1}(g)| \geq 0, \quad i \in T_0 :$$

this is the number of *green* child-nodes of each node—which coincides with the number of child-nodes if  $c(i) = g$ .

**Theorem 7.2.** *The bichromatic fission tree determines the type-B local WMCG via the canonical group isomorphism*

$$\Gamma(B_n, \mathbf{d}) \simeq \prod_{c^{-1}(g)} \text{PB}_{k_i} \times \prod_{c^{-1}(b)} \text{PB}_{k_i}^{\text{B/C}}. \quad (25)$$

*Conversely type-B local WMCGs exhaust finite products of pure braid groups of types A and B/C.*

*Proof.* Again there is a pure braid group factor at each green node, with as many strands as its (green) child-nodes.

Then further by the above discussion a blue node  $i \in T_0$  yields a pure braid group of type  $B_k/C_k$ , where  $k$  is the number of its green child-nodes—corresponding to the decomposition of  $B_{m_l} \subseteq \Phi_{h_{l+1}}$  into type-A irreducible components for  $\Phi_{h_l} \subseteq \Phi_{h_{l+1}}$ .

For the second statement consider bichromatic trees where no green node ever splits; if there are  $n_i \geq 1$  blue nodes with  $i \geq 1$  green child-nodes this yields

$$\Gamma(B_n, \mathbf{d}) \simeq \prod_{i \geq 0} (\text{PB}_i^{\text{B/C}})^{n_i},$$

which is an arbitrary finite product (analogously to (21)): any bichromatic fission tree is then obtained by splaying some green node, and this can add on any finite product of pure braid groups of type A by Thm. 5.2.  $\square$

## 8. TYPE C

Let  $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{C})$ , for an integer  $n \geq 1$ . The standard Cartan subalgebra  $\mathfrak{t} \subseteq \mathfrak{g}$  is identified with  $V = \bigoplus_{i=1}^n \mathbb{C}e_i$  with canonical basis and scalar product, and the root system is simply denoted  $C_n$ . We retain the notation of §§ 5 and 7.

The usual choice of basis is  $\Delta_{\mathfrak{g}} = \{\theta_i\}_i$  with

$$\theta_i = \alpha_{i,i+1}^-, \quad i \in \{1, \dots, n-1\},$$

as for  $A_{n-1}$ , plus the *long* root  $\theta_n = 2\alpha_n \in \mathfrak{t}^\vee$  [25, Ch. VI, § 4.6]. It follows that

$$C_n = \left\{ \pm\alpha_{ij}^-, \pm\alpha_{ij}^+ \mid 1 \leq i < j \leq n \right\} \cup \{2\alpha_i \mid 1 \leq i \leq n\} \subseteq \mathfrak{t}^\vee.$$

**8.1. Fission subsystems.** The fundamental chamber is the same as that of type B, and we consider again the centraliser  $\mathfrak{h} = \text{Ker}(\text{ad}_A) \subseteq \mathfrak{g}$  of an element  $A \in \mathfrak{t}_{\mathbb{R}}$  lying there.

Reasoning as in § 7 we either have  $\Phi_{\mathfrak{h}} \subseteq A_{n-1}$  (if  $\theta_n(A) > 0$ ), or else there exists an integer  $m \leq n$  such that  $\Phi_{\mathfrak{h}} \simeq \Phi_{\mathfrak{h}}^A \oplus C_m$ , with  $\Phi_{\mathfrak{h}}^A \subseteq A_{n-m}$ .

Then again by Prop. 5.1:

**Corollary 8.1.** *For any nontrivial fission subsystem  $\Phi_{\mathfrak{h}} \subseteq C_n$  there exists unique multiplicities  $m \geq 0$  and  $n_i \geq 1$  such that*

$$\Phi_{\mathfrak{h}} \simeq C_m \oplus \bigoplus_{i \geq 1} n_i A_i, \quad \text{with} \quad m + \sum_i n_i \leq n.$$

*Remark 8.1.* Again these are not all root subsystems: a generic subsystem decomposes as a direct sum of classical systems of type A, C or D (cf. Rem. 7.1).  $\triangle$

**8.2. Kernels.** As in § 7 we find

$$\mathfrak{U} = \text{Ker}(\Delta_{\mathfrak{h}}^A) \cap \text{Ker}(\Delta_{\mathfrak{h}}^C),$$

with  $\Delta_{\mathfrak{h}}^A = \Delta_{\mathfrak{h}} \cap \Phi_{\mathfrak{h}}^A$  and  $\Delta_{\mathfrak{h}}^C = \{\theta_{n-m+1}, \dots, \theta_n\} \subseteq \Delta_{\mathfrak{h}}$ .

The simple roots in  $\Delta_{\mathfrak{h}}^C$  span the rightmost addend in the decomposition  $V = \mathbb{C}^{n-m} \oplus^{\perp} \mathbb{C}^m$ , so we have the same description (24) of  $\mathfrak{U} \subseteq V$ .

**8.3. Restricted arrangement and fundamental group.** The proof of Thm. 7.1 yields the following.

**Theorem 8.1.** *The hyperplane arrangement in the kernel is of type  $B_{|J|}/C_{|J|}$ .*

*Moreover if  $m > 0$  then there is a canonical isomorphism of root systems*

$$BC_{|J|} \simeq C_n|_{\mathfrak{U}} \subseteq \mathfrak{U}^\vee.$$

*Equivalently, in dual terms, there is a canonical isomorphism of root systems*

$$BC_{|J|} \simeq \pi_{\mathfrak{U}}(C_n^\vee) \subseteq \mathfrak{U},$$

*where  $\pi_{\mathfrak{U}}: V \rightarrow \mathfrak{U}$  is the orthogonal projection.*

*Proof.* The only difference from type B is that

$$2\alpha_i|_{\mathcal{U}} = \begin{cases} 2\alpha_{I_i}, & i \in \underline{n-m}, \\ 0, & i \in \{n-m+1, \dots, n\}. \end{cases}$$

Hence

$$\left\{ \pm\alpha_{I_i I_j}^-, \pm\alpha_{I_i I_j}^+ \mid I_i \neq I_j \right\} \cup \{2\alpha_{I_i} \mid i \in J\} \subseteq C_n|_{\mathcal{U}}.$$

Further the root  $\alpha_{I_i} \in \mathcal{U}^\vee$  appears from the restriction of  $\alpha_{ij}^\pm \in \mathfrak{t}^\vee$  for some  $j \in \{n-m+1, \dots, n\}$ , i.e. if and only if  $m > 0$ .  $\square$

It follows again that  $\pi_1(\mathbf{B}(\Phi_{\mathfrak{h}}, \Phi_{\mathfrak{g}}), \mathcal{A}) \simeq \text{PB}_{|J|}^{\text{B/C}}$ .

**8.4. Bichromatic fission trees (again).** As in § 7 we can now split any filtration of fission subsystems  $\Phi_{\mathfrak{h}_i} \subseteq C_n$  as

$$\Phi_{\mathfrak{h}_1}^\Lambda \subseteq \dots \subseteq \Phi_{\mathfrak{h}_p}^\Lambda \subseteq A_{n-1},$$

plus a filtration by irreducible subsystems

$$C_{m_1} \subseteq \dots \subseteq C_{m_p} \subseteq C_n, \quad m_1 \leq \dots \leq m_p \leq n.$$

These can still be coded by the bichromatic fission trees of Def. 7.1: we start from a blue root (corresponding to the irreducible system  $C_n$  at the top), and every other node of  $J_1 \subseteq T_0$  will correspond to an irreducible component of a subsystem  $\Phi_{\mathfrak{h}_i} \subseteq C_n$ , as in § 7, where now a blue node stands for a type-C component.

The computation of the local WMCG works the same, so there is a canonical group isomorphism analogous to (25).

## 9. TYPE D

Finally let  $\mathfrak{g} = \mathfrak{so}_{2n}(\mathbb{C})$ , for  $n \geq 1$ . The standard Cartan subalgebra  $\mathfrak{t} \subseteq \mathfrak{g}$  is identified with  $\mathbb{C}^n = \bigoplus_i \mathbb{C}e_i$  with canonical scalar product, and we retain the notations of §§ 5 and 7. The root system will be simply denoted  $D_n$ .

The usual choice of basis is  $\Delta_{\mathfrak{g}} = \{\theta_i\}_i$  with

$$\theta_i = \alpha_{i, i+1}^-, \quad i \in \{1, \dots, n-1\},$$

as for  $A_{n-1}$ , plus  $\theta_n = \alpha_{n-1, n}^+ \in \mathfrak{t}^\vee$  [25, Ch. VI, § 4.8]. It follows that

$$D_n = \left\{ \pm\alpha_{ij}^-, \pm\alpha_{ij}^+ \mid 1 \leq i < j \leq n \right\}.$$

**9.1. Fission subsystems.** If  $\lambda = \sum_i \alpha_i(\lambda)e_i \in \mathfrak{t}_{\mathbb{R}}$  is dominant then

$$\alpha_1(\lambda) \geq \dots \geq \alpha_{n-1}(\lambda) \geq |\alpha_n(\lambda)| \in \mathbb{R}_{\geq 0},$$

and we consider again its centraliser  $\mathfrak{h} = \text{Ker}(\text{ad}_\lambda) \subseteq \mathfrak{g}$ .

We find once more two cases (which can be read from the Dynkin diagram): if  $\theta_n(\lambda) > 0$  then  $\Phi_{\mathfrak{h}} \subseteq A_{n-1}$ , else

$$\Phi_{\mathfrak{h}} \simeq \Phi_{\mathfrak{h}}^\Lambda \oplus D_m,$$

with  $\Phi_{\mathfrak{h}}^\Lambda \subseteq A_{n-m}$ , for some integer  $m \leq n$ .

Thus Prop. 5.1 yields:

**Corollary 9.1.** *For any nontrivial fission subsystem  $\Phi_{\mathfrak{h}} \subseteq D_n$  there exists unique multiplicities  $m \geq 0$  and  $n_i \geq 1$  such that*

$$\Phi_{\mathfrak{h}} \simeq D_m \oplus \bigoplus_{i \geq 1} n_i A_i, \quad \text{with} \quad m + \sum_i n_i \leq n.$$

*Remark 9.1.* Once more there are more subsystems: in general they decompose as a direct sum of classical systems of type A or D (cf. Rem. 7.1).  $\triangle$

**9.2. Kernels.** With the usual notation one has

$$U = \text{Ker}(\Delta_{\mathfrak{h}}^A) \cap \text{Ker}(\Delta_{\mathfrak{h}}^D),$$

and  $D_m \subseteq \mathbb{C}^m$  is essential. So  $U \subseteq V$  still only depends on the type-A irreducible components, as in (24).

**9.3. Restricted arrangement and fundamental group.** The computations in the proof of Thm. 7.1 yield the following situation.

For two integers  $r, s \geq 0$  we define the following (essential) hyperplane arrangement inside  $\mathbb{C}^{r+s}$ : it contains the hyperplanes  $\text{Ker}(\alpha_{ij}^{\pm})$  for  $i \leq i \neq j \leq r+s$  (i.e. the root hyperplanes of  $D_{r+s}$ ) plus the hyperplanes  $\text{Ker}(\alpha_i)$  for  $i \leq i \leq r$ . Hence  $\mathbb{C}^r \times \{0\} \subseteq \mathbb{C}^{r+s}$  contains the root hyperplanes of type  $B_r/C_r$ , but there is no splitting since there are hyperplanes involving the coordinates on either factors of  $\mathbb{C}^{r+s} = \mathbb{C}^r \times \mathbb{C}^s$ . We will say this is an hyperplane arrangement of “exotic” type  $(B/C)_r D_s$ .<sup>11</sup>

*Remark 9.2.* Note the reflection group generated by this hyperplane arrangement is the Weyl group of type  $B_{r+s}/C_{r+s}$  if  $r > 0$ , else it is the Weyl group of type  $D_s$ ; this is thus always crystallographic, but the hyperplane arrangement itself is not that of a root system if  $r, s > 0$ .  $\triangle$

**Theorem 9.1.** *There are two cases:*

- *If  $m > 0$  then the hyperplane arrangement in the kernel is of type  $B_{|J|}/C_{|J|}$ ;*
- *if  $m = 0$  then the hyperplane arrangement in the kernel is of type  $(B/C)_r D_s$ , where  $r \leq |J|$  is the number of nontrivial irreducible components of  $\Phi_{\mathfrak{h}}^A \subseteq \Phi_{\mathfrak{h}}$ , and  $s = |J| - r$  (the number of trivial components).*

*Proof.* One always has

$$D_{|J|} \simeq \left\{ \pm \alpha_{I_i I_j}^-, \pm \alpha_{I_i I_j}^+ \mid I_i \neq I_j \right\} \subseteq D_n|_U,$$

but further some functional  $\alpha_{I_i}, 2\alpha_{I_i} \in U^{\vee}$  may appear.

Namely if  $I_i \neq \{i\}$  then  $2\alpha_{I_i} \in D_n|_U$ , and if  $m > 0$  then  $\alpha_{I_i} \in D_n|_U$  for all  $i \in J$ , leading to the classification in the statement.  $\square$

We thus see local WMCGs go beyond the class of finite products of pure  $\mathfrak{g}$ -braid groups. Namely for  $m > 0$  one has  $\pi_1(\mathbf{B}(\Phi_{\mathfrak{h}}, \Phi_{\mathfrak{g}}), A) \simeq \text{PB}_{|J|}^{B/C}$ , while if  $m = 0$  then

$$\pi_1(\mathbf{B}(\Phi_{\mathfrak{h}}, \Phi_{\mathfrak{g}}), A) = \text{PB}_{r,s}^{B/C,D},$$

denoting  $\text{PB}_{r,s}^{B/C,D}$  the fundamental group of the hyperplane complement of type  $(B/C)_r D_s$  above.

<sup>11</sup>With tautological identities  $(B/C)_r D_0 = B_r/C_r$  and  $(B/C)_0 D_s = D_s$ .

To study this further we revert to more common notation for the standard coordinates, and simply write  $\mathbf{z} = (z_1, \dots, z_r) \in \mathbb{C}^r$ ,  $\mathbf{w} = (w_1, \dots, w_s) \in \mathbb{C}^s$ . Then the “exotic” hyperplane complement is

$$X_{r,s} = \{ (\mathbf{z}, \mathbf{w}) \in \mathbb{C}^{r+s} \mid z_i \neq 0, z_i \neq \pm z_j, z_i \neq \pm w_k, w_k \neq \pm w_l \} \subseteq \mathbb{C}^{r+s}. \quad (26)$$

In some case it is easy to compute the fundamental group of this space. To this end denote  $F_i$  the free group on  $i \geq 0$  generators.

**Proposition 9.1.** *There is a canonical group isomorphism*

$$\text{PB}_{r,1}^{\text{B/C,D}} \simeq \text{PB}_r^{\text{B/C}} \rtimes F_{2r}.$$

*Proof.* Consider the subspace  $X_r := X_{r,1} \cap (\mathbb{C}^r \times \{0\}) \subseteq \mathbb{C}^{r+1}$ , so that

$$X_r \simeq \{ \mathbf{z} \in \mathbb{C}^r \mid z_i \neq 0, z_i \neq \pm z_j \} \subseteq \mathbb{C}^r,$$

which is the root-hyperplane complement of type  $B_r/C_r$ . Then there is a canonical projection  $p: X_{r,1} \rightarrow X_r$  with fibres

$$p^{-1}(\mathbf{z}) \simeq \{ \mathbf{w} \in \mathbb{C} \mid w \neq \pm z_i \} \subseteq \mathbb{C},$$

i.e. a locally trivial fibration

$$Y_r \hookrightarrow X_{r,1} \xrightarrow{p} X_r, \quad Y_r := \mathbb{C} \setminus \{ \pm 1, \dots, \pm r \}. \quad (27)$$

Now  $Y_r$  and  $X_r$  are path-connected, and further  $X_r$  is a  $K(\pi, 1)$ -space [32, 28]—it has trivial higher homotopy groups. Hence (27) induces an exact sequence of fundamental groups (omitting base points):

$$1 \longrightarrow F_{2r} \longrightarrow \text{PB}_{r,1}^{\text{B/C,D}} \xrightarrow{\pi_1(p)} \text{PB}_r^{\text{B/C}} \longrightarrow 1. \quad (28)$$

Finally there is a canonical global (zero) section  $X_r \rightarrow X_{r,1}$  splitting (28).  $\square$

*Remark 9.3* (Exceptional isomorphism). If further  $r = 1$  then (28) simplifies to

$$1 \longrightarrow F_2 \longrightarrow \text{PB}_{1,1}^{\text{B/C,D}} \longrightarrow \mathbb{Z} \longrightarrow 1,$$

and in this case we can identify the extension.

Namely the space  $X_{1,1} \subseteq \mathbb{C}^2$  is isomorphic to the root-hyperplane complement of type  $A_2$ . Indeed in our notation this complement is obtained from a restriction  $D_3|_{\mathcal{U}}$ , where  $\mathcal{U} = \text{Ker}(\Delta_{\mathfrak{h}})$  is the kernel of a rank-1 fission subsystem  $\Phi_{\mathfrak{h}} \subseteq D_3$ ; but in view of the exceptional isomorphism  $D_3 \simeq A_3$  this is isomorphic to a restriction of  $A_3$ , i.e. to  $A_2$  by applying Thm. 5.1.

Hence  $\text{PB}_{1,1}^{\text{B/C,D}} \simeq \text{PB}_3$  and (28) is the usual split extension

$$1 \longrightarrow F_2 \longrightarrow \text{PB}_3 \longrightarrow \text{PB}_2 \longrightarrow 1,$$

since  $\text{PB}_2 \simeq \mathbb{Z}$ —generated by the square of the simple braiding of two distinct points in  $\mathbb{C}$ .  $\triangle$

**9.4. Generalised fission trees.** Once more a filtration of fission subsystems splits into  $\Phi_{h_1}^\Lambda \subseteq \cdots \Phi_{h_p}^\Lambda \subseteq A_{n-1}$  and  $D_{m_1} \subseteq \cdots \subseteq D_{m_p} \subseteq D_n$ , for an increasing sequence of integers  $m_i \leq n$ .

To encode the associated local WMCG we now need to retain more information, according to the statement of Thm. 9.1: namely at each level we must recall the number of trivial/nontrivial type-A irreducible components of  $\Phi_{h_l} \cap D_{m_{l+1}} \subseteq \Phi_{h_{l+1}}$ , to which we will associate nodes of different diameters in a tree.

This leads to the following natural generalisation of Def. 7.1. Introduce the set  $\{s, l\}$  of diameters “small” and “large”, with total order  $s \leq l$ ; then:

**Definition 9.1** (Generalised fission tree). A *generalised fission tree* is a bichromatic fission tree  $(T(\Phi), c)$  equipped with a *diameter function*  $d: T_0 \rightarrow \{s, l\}$ ; in turn a diameter function satisfies:

- $d(i) = l$  if  $c(i) = b$ ;
- $d(\Phi(i)) \geq d(i)$  for  $i \in T_0 \setminus \{*\}$ ;
- $k_i \leq 1$  if  $d(i) = s$ .

Hence green nodes can be small or large; large green nodes can have (green) child-nodes of any diameter, while small green nodes cannot split.

The algorithm to attach a generalised fission tree to a double filtration as above is the following. A node  $i \in J_l$  corresponds to an irreducible component of the subsystem  $\Phi_{h_l} \subseteq D_n$ : put a large green node for each *nontrivial* type-A component, a small green node for each *trivial* type-A component, and finally a large blue node if  $m_l > 0$ —i.e. if there is a type-D component at all. The parent-node function is determined as in the bichromatic case

To compute the local WMCG in terms of the tree, note there exists a unique blue node  $i_0 \in T_0$  with no blue child-nodes (possibly a leaf): let  $r_0, s_0 \geq 0$  be the number of large and small child-nodes of  $i_0$ , respectively, and let  $T'_0 := T_0 \setminus \{i_0\}$ . Then retain the notation of § 7.

**Theorem 9.2.** *The generalised fission tree determines the type-D local WMCG via the canonical group isomorphism*

$$\Gamma(D_n, \mathbf{d}) \simeq \prod_{c^{-1}(g)} \text{PB}_{k_i} \times \text{PB}_{r_0, s_0}^{\text{B/C, D}} \times \prod_{c^{-1}(b) \cap T'_0} \text{PB}_{k_i}^{\text{B/C}}.$$

*Conversely type-D local WMCGs are obtained by adding any one exotic factor to a type-B/C local WMCG.*

*Proof.* The first statement is a rewriting of Thm. 9.1 (building upon Thm. 7.2): the new fact is that any irreducible type-D component inside  $\Phi_{h_{l+1}}$  leads to an exotic fundamental group precisely if it decomposes into irreducible components of type A only—for  $\Phi_{h_l} \subseteq \Phi_{h_{l+1}}$ .

As for the second statement, if the special node  $i_0 \in T_0$  is a leaf then  $\Gamma(D_n \mathbf{d})$  only depends on the underlying bichromatic fission tree  $(T_0, \Phi, c)$ —rather the generalised fission tree  $(T, \Phi, c, d)$ —and yields any local WMCG of type B/C. Then adding a new level where only  $i_0$  splits, and has no blue child-nodes, adds an exotic factor (26) of any kind.  $\square$

This yields the most general local WMCG for a classical simple Lie algebra.

## 10. LOCAL WILD RIEMANN MODULI SPACES

The aim of this section is to explain in which sense the deformation space (6) is the holomorphic/analytic counterpart of a (fine) *moduli space/scheme* of irregular types at the point  $a \in \Sigma$ , with prescribed pole orders at each root, formalising the main intuition that it is a universal space of deformations thereof. An elegant way to do this is to provide an “algebraic” definition of irregular types, define an associated moduli problem, and construct a (moduli) scheme representing the resulting functor; nonetheless in Rk. 10.4 we will explain how this viewpoint naturally relates to the (holomorphic) admissible families/deformations of § 1.

Fix then a pointed Riemann surface  $(\Sigma, a)$ , and retain the local data introduced in § 1. In particular an irregular type based at  $a \in \Sigma$  is an element  $Q \in \mathfrak{t} \otimes \mathcal{T}_{\Sigma, a}$ .

Since we can (and will) bound the pole order of  $Q$  by an integer  $p \geq 1$ , let us introduce the maximal ideal  $\mathfrak{m}_{\Sigma, a} \subseteq \widehat{\mathcal{O}}_{\Sigma, a}$  of the completed local ring of  $\Sigma$  at the point  $a$ , and set

$$\mathcal{T}_{\Sigma, a}^{\leq p} := (\mathfrak{m}_{\Sigma, a}^{-p} \widehat{\mathcal{O}}_{\Sigma, a}) / \widehat{\mathcal{O}}_{\Sigma, a} \subseteq \widehat{\mathcal{H}}_{\Sigma, a} / \widehat{\mathcal{O}}_{\Sigma, a} = \mathcal{T}_{\Sigma, a}.$$

Then we consider elements  $Q \in \mathfrak{t} \otimes \mathcal{T}_{\Sigma, a}^{\leq p}$ .

*Remark 10.1.* As we vary the irregular type only, keeping  $(\Sigma, a)$  fixed, the global geometry of the Riemann surface plays no role in what follows. Indeed the definition of an irregular type based at  $a \in \Sigma$  only involves “infinitesimal” information about the Riemann surface around  $a$ , which is captured by the completed local ring of  $\Sigma$  at the marked point  $a$ . In the constructions below—as well as in § 1—one could thus work at the closed point of the infinitesimal disc  $D = \text{Spec } \mathbb{C}[[z]]$  (rather than on  $(\Sigma, a)$ ).

In particular for any choice of a uniformiser  $\varpi \in \mathfrak{m}_{\Sigma, a}$ , i.e. a generator of the maximal ideal, there is a vector space isomorphism

$$\mathcal{T}_{\Sigma, a}^{\leq p} \simeq (z^{-p} \mathbb{C}[[z]]) / \mathbb{C}[[z]],$$

mapping the class of  $\varpi^{-1}$  inside  $\mathcal{T}_{\Sigma, a}^{\leq p}$  to the class of  $z^{-1}$  inside  $(z^{-p} \mathbb{C}[[z]]) / \mathbb{C}[[z]]$ . Thus we will simply denote  $\mathcal{T}^{\leq p} = \mathcal{T}_{\Sigma, a}^{\leq p}$ —but importantly we need *not* choose a uniformiser, and our definitions will be intrinsic.  $\triangle$

Note  $\mathcal{T}^{\leq p}$  is isomorphic to the  $(p-1)$ -th infinitesimal neighbourhood of the marked point, and regarding it as a scheme will be convenient (in particular since our construction involves nilpotents). To define moduli spaces we thus change category and work with schemes rather than complex manifolds: in particular  $\Sigma$  is now a smooth complex algebraic curve—and  $\Sigma = (\Sigma, a, Q)$  is an “irregular” curve, viz. the algebraic analogue of a wild Riemann surface [15].

Denote then  $\mathbf{Alg}_{\mathbb{C}}$  the category of commutative (unitary)  $\mathbb{C}$ -algebras. For an object  $R$  of  $\mathbf{Alg}_{\mathbb{C}}$  consider the completed tensor product

$$\widehat{\mathcal{O}}_{\Sigma, a, R} := R \widehat{\otimes} \widehat{\mathcal{O}}_{\Sigma, a} = \varprojlim_{n \geq 0} R \otimes (\widehat{\mathcal{O}}_{\Sigma, a} / \mathfrak{m}_{\Sigma, a}^n),$$

and denote by  $\widehat{\mathcal{H}}_{\Sigma, a, R}$  the localisation of  $\widehat{\mathcal{O}}_{\Sigma, a, R}$  at a uniformiser. Finally set

$$\mathcal{T}_R := \widehat{\mathcal{H}}_{\Sigma, a, R} / \widehat{\mathcal{O}}_{\Sigma, a, R} \quad \text{and} \quad \mathcal{T}_R^{\leq p} := (\mathfrak{m}_{\Sigma, a}^{-p} \widehat{\mathcal{O}}_{\Sigma, a, R}) / \widehat{\mathcal{O}}_{\Sigma, a, R}.$$

Once more the choice of a uniformiser yields identifications

$$\widehat{\mathcal{O}}_{\Sigma, \alpha, \mathbb{R}} \simeq \mathbb{R}[[z]], \quad \mathcal{T}_{\mathbb{R}} \simeq \mathbb{R}((z))/\mathbb{R}[[z]], \quad \mathcal{T}_{\mathbb{R}}^{\leq p} \simeq (z^{-p}\mathbb{R}[[z]])/\mathbb{R}[[z]],$$

generalising the case  $\mathbb{R} = \mathbb{C}$  above.

Choose now a collection of nonnegative integers

$$\mathbf{d} = (d_{\alpha})_{\alpha \in \Phi_{\mathfrak{g}}} \in \mathbb{Z}_{\geq 0}^{\Phi_{\mathfrak{g}}},$$

attached to the roots of  $(\mathfrak{g}, \mathfrak{t})$ , such that  $\max(\mathbf{d}) \leq p$ .

**Definition 10.1.** An (untwisted) *irregular type* over  $\text{Spec } \mathbb{R}$  is an element

$$Q \in \mathfrak{t} \otimes \mathcal{T}_{\mathbb{R}}.$$

We say  $Q$  has *pole order bounded by  $p$*  if  $Q \in \mathfrak{t} \otimes \mathcal{T}_{\mathbb{R}}^{\leq p}$ , and further it has *pole order bounded by  $d_{\alpha}$* , at a root  $\alpha \in \Phi_{\mathfrak{g}}$ , if  $q_{\alpha} \in \mathcal{T}_{\mathbb{R}}^{\leq d_{\alpha}}$ .

*Remark 10.2.* Let

$$\Gamma_{\mathbf{a}}: \text{Spec } \mathbb{R} \longrightarrow \Sigma_{\mathbb{R}}, \quad \Sigma_{\mathbb{R}} := \text{Spec } \mathbb{R} \times_{\text{Spec } \mathbb{C}} \Sigma, \quad (29)$$

be the graph of the constant  $\mathbf{a}$ -valued map. The map (29) is a closed immersion, and  $\text{Spec } \mathbb{R}[[z]]/z^p\mathbb{R}[[z]]$  is (isomorphic to) the  $(p-1)$ -th infinitesimal neighbourhood of the image of  $\text{Spec } \mathbb{R}$  in  $\Sigma_{\mathbb{R}}$ . Def. 10.1 therefore generalises the notion of an irregular type (of bounded pole order) from  $\text{Spec } \mathbb{C}$  to an arbitrary affine scheme (cf. Rem. 10.1).

Moreover an irregular type  $Q \in \mathfrak{t} \otimes \mathcal{T}_{\mathbb{R}}$  gives rise, for every complex point of  $\text{Spec } \mathbb{R}$ , to an irregular type in the sense of § 1.  $\triangle$

Let now  $\mathbf{Aff}_{\mathbb{C}}$  be the category of affine schemes over  $\text{Spec } \mathbb{C}$  (i.e. the opposite of  $\mathbf{Alg}_{\mathbb{C}}$ ). The irregular types of Def. 10.1 can be pulled back along morphisms  $\text{Spec } S \rightarrow \text{Spec } \mathbb{R}$  of complex affine schemes: our goal is to show that the resulting (contravariant) functor sending  $\text{Spec } \mathbb{R}$  to the set of irregular types over  $\text{Spec } \mathbb{R}$ , of pole order bounded by  $p \geq 1$ , and further by  $d_{\alpha} \in \{0, \dots, p\}$  at each root  $\alpha \in \Phi_{\mathfrak{g}}$ , is representable by an algebraic variety. We will then define a suitable subvariety thereof parametrising irregular types of pole order exactly  $d_{\alpha}$  at  $\alpha \in \Phi_{\mathfrak{g}}$ , whose complex points are naturally identified with the deformation space (6).

**Proposition 10.1.** *Let  $X := \text{Spec } \text{Sym}(\mathfrak{t}^{\vee})$ ; then:*

- (1) *the (pole-order-bounded) “irregular type” functor*

$$\mathcal{T}^{\leq p}: \mathbf{Aff}_{\mathbb{C}} \longrightarrow \mathbf{Set}, \quad \text{Spec } \mathbb{R} \longmapsto \mathfrak{t} \otimes \mathcal{T}_{\mathbb{R}}^{\leq p},$$

*is representable by the affine scheme  $X^p$ ;*

- (2) *the subfunctor  $\mathcal{T}_{\mathbf{d}}^{\leq p}: \mathbf{Aff}_{\mathbb{C}} \rightarrow \mathbf{Set}$ , defined by*

$$\text{Spec } \mathbb{R} \longmapsto \left\{ Q \in \mathfrak{t} \otimes \mathcal{T}_{\mathbb{R}}^{\leq p} \mid q_{\alpha} \in \mathcal{T}_{\mathbb{R}}^{\leq d_{\alpha}} \text{ for } \alpha \in \Phi_{\mathfrak{g}} \right\},$$

*is representable by a closed subscheme of  $X^p$ .*

*Proof.* We start by pointing to the functorial bijections

$$\mathfrak{t} \otimes \mathbb{R} = \text{Hom}_{\mathbf{Vect}_{\mathbb{C}}}(\mathfrak{t}^{\vee}, \mathbb{R}) = \text{Hom}_{\mathbf{Alg}_{\mathbb{C}}}(\text{Sym}(\mathfrak{t}^{\vee}), \mathbb{R}),$$

for any commutative  $\mathbb{C}$ -algebra  $\mathbb{R}$ , passing through the category  $\mathbf{Vect}_{\mathbb{C}}$  of  $\mathbb{C}$ -vector spaces; thus the functor  $\text{Spec } \mathbb{R} \mapsto \mathfrak{t} \otimes \mathbb{R}$  is represented by  $X$ .



It follows that  $\mathcal{I}\mathcal{T}^{\leq p}$  is the restriction of scalars, from  $\text{Spec } \mathcal{T}^{\leq p}$  to  $\text{Spec } \mathbb{C}$ , of the functor represented by  $X$ . The first statement then follows as in the proof of [24, Thm 4, p. 194].<sup>12</sup> More precisely, identifying  $\widehat{\mathcal{O}}_{\Sigma, \alpha} \simeq \mathbb{C}[[z]]$  yields a direct sum decomposition

$$\mathcal{T}^{\leq p} \simeq \bigoplus_{i=1}^p \mathbb{C}z^{-i},$$

whence

$$\mathfrak{t} \otimes (\mathbb{R} \otimes \mathcal{T}^{\leq p}) \simeq \bigoplus_{i=1}^p (\mathfrak{t} \otimes \mathbb{R})z^{-i}.$$

This yields a bijection, functorial in  $\mathbb{R}$ , between  $\mathcal{I}\mathcal{T}^{\leq p}(\text{Spec } \mathbb{R})$  and

$$(\mathfrak{t} \otimes \mathbb{R})^p = X^p(\text{Spec } \mathbb{R}).$$

As for the second statement, fix a  $\mathbb{C}$ -algebra  $\mathbb{R}$  and an irregular type

$$Q \in \bigoplus_{i=1}^p (\mathfrak{t} \otimes \mathbb{R})z^{-i},$$

over  $\text{Spec } \mathbb{R}$ . Given  $\alpha \in \Phi_{\mathfrak{g}}$ , we have  $\text{ord}(q_{\alpha}) \leq d_{\alpha}$  if and only if the coefficient of  $z^{-i}$  lies in  $\ker(\alpha) \otimes \mathbb{R}$  for  $d_{\alpha} < i \leq p$ . Moreover every  $\alpha \in \Phi_{\mathfrak{g}}$  induces a morphism of group schemes

$$\text{ev}_{\alpha}: X \longrightarrow \mathbb{A}_{\mathbb{C}}^1,$$

whose kernel is canonically isomorphic to  $\text{Spec } \text{Sym}(\ker(\alpha)^{\vee}) \subseteq X$ .

Therefore the subfunctor of  $\mathcal{I}\mathcal{T}^{\leq p}$  parametrising irregular types with pole order bounded by  $d_{\alpha}$ —at  $\alpha$ —is represented by the closed subscheme

$$\left( \prod_{1 \leq i \leq d_{\alpha}} X \right) \times \prod_{d_{\alpha} < i \leq p} \text{Spec } \text{Sym}(\ker(\alpha)^{\vee}) \subseteq \prod_{1 \leq i \leq p} X = X^p.$$

Finally taking the intersection of these subschemes along  $\Phi_{\mathfrak{g}}$  yields the scheme representing  $\mathcal{I}\mathcal{T}_{\leq \mathbf{d}}^{\leq p}$ .  $\square$

Hereafter we identify the functors of Prop. 10.1 with the schemes representing them. Then we can write the scheme  $\mathcal{I}\mathcal{T}^{\leq p}$  as the union of closed subschemes  $\mathcal{I}\mathcal{T}_{\geq \mathbf{d}}^{\leq p}$ —for  $\mathbf{d} = (d_{\alpha}) \in \mathbb{Z}_{\geq 0}^{\Phi_{\mathfrak{g}}}$ .

To replace upper bounds by equalities let us introduce the product (partial) order on  $\Phi_{\mathfrak{g}}$ -tuples: we write  $\mathbf{d}' \leq \mathbf{d}$  if  $d'_{\alpha} \leq d_{\alpha}$  for every  $\alpha \in \Phi_{\mathfrak{g}}$ , and further  $\mathbf{d}' < \mathbf{d}$  if  $\mathbf{d}' \neq \mathbf{d}$ —where  $\mathbf{d}' = (d'_{\alpha})_{\alpha \in \Phi_{\mathfrak{g}}}$ . Thus for  $\mathbf{d}' \leq \mathbf{d}$  we have a closed immersion  $\mathcal{I}\mathcal{T}_{\leq \mathbf{d}'}^{\leq p} \hookrightarrow \mathcal{I}\mathcal{T}_{\leq \mathbf{d}}^{\leq p}$ .

We arrive at the main definition of this section.

**Definition 10.2.** The *moduli space* of irregular types of pole order *bounded* by  $p \geq 1$ , and of pole order *equal* to  $d_{\alpha} \leq p$  at each root  $\alpha \in \Phi_{\mathfrak{g}}$ , is the locally closed subscheme

$$\mathcal{I}\mathcal{T}_{\mathbf{d}}^{\leq p} := \mathcal{I}\mathcal{T}_{\leq \mathbf{d}}^{\leq p} \setminus \bigcup_{\mathbf{d}' < \mathbf{d}} \mathcal{I}\mathcal{T}_{\leq \mathbf{d}'}^{\leq p} \subseteq \mathcal{I}\mathcal{T}^{\leq p}. \quad (30)$$

<sup>12</sup>We consider  $\mathcal{T}^{\leq p}$  as a commutative  $\mathbb{C}$ -algebra in the identification  $\mathcal{T}^{\leq p} \simeq \mathbb{C}[[z]]/z^p\mathbb{C}[[z]]$ .

Note the locally closed subschemes (30) define a stratification of  $\mathcal{I}\mathcal{T}_d^{\leq p}$  as  $\mathbf{d}$  varies. Each stratum is affine and smooth, being an intersection of hyperplane complements in an affine space.

By construction the  $(\text{Spec } R)$ -points of  $\mathcal{I}\mathcal{T}_d^{\leq p}$ , for a commutative  $\mathbb{C}$ -algebra  $R$ , are in functorial bijection with irregular types  $Q \in \mathfrak{t} \otimes \mathcal{T}_R^{\leq p}$  such that the pole order of  $q_\alpha \in \mathcal{T}_R^{\leq p}$  equals  $d_\alpha$ —for every  $\alpha \in \Phi_{\mathfrak{g}}$ .<sup>13</sup> This was our goal.

We conclude this section with two remarks, making the connection with the viewpoint of (universal) deformations of the introduction and § 1.

*Remark 10.3 (Universal family).* For the reader’s convenience we spell out some consequences of the above description of the functor of points of  $\mathcal{I}\mathcal{T}_d^{\leq p}$  (refer to [40, § 8] for a general discussion of representable functors).

The proof of Prop. 10.1 yields bijections, functorial in  $\text{Spec } R \in \mathbf{Aff}_{\mathbb{C}}$ ,

$$\psi_{\text{Spec } R}: \left\{ Q \in \mathfrak{t} \otimes \mathcal{T}_R^{\leq p} \mid \text{ord}(q_\alpha) = d_\alpha \text{ for } \alpha \in \Phi_{\mathfrak{g}} \right\} \xrightarrow{\cong} \mathcal{I}\mathcal{T}_d^{\leq p}(\text{Spec } R).$$

In particular there is a canonical irregular type  $Q^{\text{un}} := \psi_{\mathcal{I}\mathcal{T}_d^{\leq p}}^{-1}(\text{Id})$ , over  $\mathcal{I}\mathcal{T}_d^{\leq p}$  (of pole order  $d_\alpha$  at every root).

This irregular type is *universal* in the sense that any other can be obtained from it via pullback. More precisely let  $R$  be a commutative  $\mathbb{C}$ -algebra and  $Q \in \mathfrak{t} \otimes \mathcal{T}_R^{\leq p}$  an irregular type of pole order  $d_\alpha$  at  $\alpha \in \Phi_{\mathfrak{g}}$ . There is then a map

$$f_Q: \text{Spec } R \longrightarrow \mathcal{I}\mathcal{T}_d^{\leq p}, \quad f_Q := \psi_{\text{Spec } R}(Q),$$

and by construction  $Q$  is the pullback of  $Q^{\text{un}}$  under  $f_Q$ .

Finally this universal property determines  $\mathcal{I}\mathcal{T}_d^{\leq p}$  uniquely up to isomorphism—by Yoneda’s lemma.  $\triangle$

As a particular case of the above discussion we find the following description of the complex points  $\mathcal{I}\mathcal{T}_d^{\leq p}(\text{Spec } \mathbb{C}) \subseteq \mathcal{I}\mathcal{T}^{\leq p}(\text{Spec } \mathbb{C}) \simeq \mathfrak{t}^p$ :

$$\mathcal{I}\mathcal{T}_d^{\leq p}(\text{Spec } \mathbb{C}) \simeq \bigcap_{\Phi_{\mathfrak{g}}} \left( \left( \prod_{1 \leq i < d_\alpha} \mathfrak{t} \right) \times (\mathfrak{t} \setminus \ker(\alpha)) \times \prod_{d_\alpha < i \leq p} \ker(\alpha) \right),$$

recovering (6) (in view of Rk. 1.1).

Hence the local WMCG of Def. 2.1 is the fundamental group of the analytification of the moduli space of irregular types (with given pole orders at each root, and with bounded global pole order).<sup>14</sup>

*Remark 10.4 (Admissible deformations).* Finally let us point out the relation between the “algebraic” notion of irregular type of Def. 10.1 and the “holomorphic/analytic” notion of admissible deformations of wild Riemann surfaces in Def. 1.2.

<sup>13</sup>Beware this means the following in general (which is relevant when  $R$  is not a field): writing  $\mathcal{T}_R^{\leq p} \simeq \bigoplus_i R z^{-i}$  as in the proof of Prop. 10.1, the coefficient of  $z^{-i}$  is zero if  $i > d_\alpha$ , and is a *unit* in  $R$  if  $i = d_\alpha$ .

<sup>14</sup>The *algebraic/étale* fundamental group of the scheme (30) is instead the profinite completion of the fundamental group of its analytification [41, Cor. 5.2], so for the purposes of this paper it is enough to consider this latter.

Suppose we still have one (fixed) marked point  $a \in \Sigma$ , and let now  $\mathbf{B}$  be the analytification of a smooth, complex affine variety  $\text{Spec } R$  (i.e. if  $R = \mathbb{C}[X_1, \dots, X_n]/I$  for some ideal  $I = (P_1, \dots, P_r)$  then  $\mathbf{B} \subseteq \mathbb{C}^n$  is the complex manifold cut out by the polynomial equations  $P_1 = \dots = P_r = 0$ ); this applies in particular to the analytification of the moduli space  $\mathcal{S}_d^{\leq p}$ .

Identify as above  $\mathcal{S}_R^{\leq p} \simeq \bigoplus_{i=1}^p R z^{-i}$ . An irregular type over  $\text{Spec } R$  of pole order bounded by  $p$  is therefore an element  $Q \in \bigoplus_{i=1}^p (R \otimes \mathfrak{t}) z^{-i}$ . Fix now a tuple  $\mathbf{d} = (d_\alpha)_{\alpha \in \Phi_g}$ , and assume that  $Q$  has pole order  $d_\alpha$  at  $\alpha$  (so that  $Q$  corresponds to a  $(\text{Spec } R)$ -point of  $\mathcal{S}_d^{\leq p}$ ). Evaluating the irregular type  $Q$  at the complex points of  $\text{Spec } R$ , we obtain a function

$$\underline{Q}: \mathbf{B} \longrightarrow \mathfrak{t} \otimes \left( \bigoplus_{i=1}^p \mathbb{C} z^{-i} \right),$$

which is holomorphic, since elements of  $R$  give rise to polynomial functions on  $\mathbf{B} \subseteq \mathbb{C}^n$ . Hence, letting  $\underline{\Sigma} = \Sigma \times \mathbf{B}$  and  $\underline{a}: \mathbf{B} \rightarrow \underline{\Sigma}$  be the global constant  $a$ -valued section, the tuple  $\underline{\Sigma} = (\underline{\Sigma}, \mathbf{B}, \underline{a}, \underline{Q})$  is a holomorphic  $\mathbf{B}$ -family of wild Riemann surfaces. Let us assume in addition that  $\mathbf{B}$  is connected (equivalently,  $\text{Spec } R$  is connected, cf [41, Prop. 2.4]); then, since  $Q$  has fixed pole order at each root,  $\underline{\Sigma} \rightarrow \mathbf{B}$  is an admissible deformation of any of its fibres in the sense of Def. 1.2.

In particular, we can apply the present discussion to the deformation space  $\mathbf{B}(Q)$  of (6) viz. the analytification of the smooth moduli space  $\mathcal{S}_d^{\leq p}$ . We thus obtain a (holomorphic) admissible deformation  $(\underline{\Sigma}, \mathbf{B}(Q), \underline{a}, \underline{Q}^{\text{un}})$ , which coincides with the one constructed in Prop. 1.1: finally *every* admissible deformation  $\underline{\Sigma} \rightarrow \mathbf{B}$  obtained as above will be a pullback of it, as explained in Rk. 10.3.  $\triangle$

## 11. OUTLOOK

There is a *nonpure* versions of local WMCGs, which involves taking out the Weyl action on irregular types, leading to the notion of a “bare” irregular type [17, Rk. 10.6] (a.k.a. “irregular class”); and moreover one can define *twisted* irregular types/classes [20], leading to “twisted” (dressed/bare) wild Riemann surfaces. A diagram-theoretic description of twisted irregular classes for  $G = \text{GL}_n(\mathbb{C})$  was given in [21], and more generally in [33]: their admissible deformations will be considered elsewhere.

Further the admissible deformations of wild Riemann surfaces allow for varying the underlying pointed Riemann surface, as in Def. 1.2, and we plan to study the topology of such “global” deformations generalising the material of § 10. Contrary to the present “local” situation, this will also involve the automorphisms of families of wild Riemann surfaces—i.e. *stacks* in general.

## APPENDIX A. BASIC NOTIONS/NOTATIONS

In this appendix we collect some standard material which is used throughout the body of the paper.

**Lie algebras.** All (complex) Lie algebras  $\mathfrak{g}$  in this paper are reductive and finite-dimensional.

The *centraliser* of a subset  $S \subseteq \mathfrak{g}$  is

$$\mathfrak{Z}_{\mathfrak{g}}(S) = \{ X \in \mathfrak{g} \mid [X, S] = 0 \} \subseteq \mathfrak{g},$$

and in particular the *centre* of  $\mathfrak{g}$  is  $\mathfrak{Z}_{\mathfrak{g}} = \mathfrak{Z}_{\mathfrak{g}}(\mathfrak{g})$ . There is a Lie algebra decomposition  $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{Z}_{\mathfrak{g}}$ , where  $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$  is the *semisimple part* of  $\mathfrak{g}$ .

Given a Cartan subalgebra  $\mathfrak{t} \subseteq \mathfrak{g}$ , a *root* is an element  $\alpha \in \mathfrak{t}^{\vee} \setminus (0)$  such that

$$\mathfrak{g}_{\alpha} = \{X \in \mathfrak{g} \mid [A, X] = \alpha(A)X \text{ for } A \in \mathfrak{t}\} \subseteq \mathfrak{g}$$

is nonzero. Hence all roots vanish on  $\mathfrak{Z}_{\mathfrak{g}} \subseteq \mathfrak{t}$ , and the root system  $\Phi_{\mathfrak{g}} = \Phi(\mathfrak{g}, \mathfrak{t})$  does not span  $\mathfrak{t}^{\vee}$  if  $\mathfrak{g}$  is not semisimple. Further the Cartan subalgebra splits as  $\mathfrak{t} = \mathfrak{t}' \oplus \mathfrak{Z}_{\mathfrak{g}}$ , where  $\mathfrak{t}' = \mathfrak{t} \cap \mathfrak{g}'$ , which is a Cartan subalgebra of the semisimple part.

If  $\Phi(\mathfrak{g}', \mathfrak{t}') \subseteq (\mathfrak{t}')^{\vee}$  is the (spanning) root system of the semisimple part, then its elements are precisely the restriction (to  $\mathfrak{t}'$ ) of the elements of  $\Phi_{\mathfrak{g}}$ .

**Root systems.** In this paper we thus consider root systems  $\Phi \subseteq V$ , where  $V$  is a finite-dimensional complex vector space, which are *crystallographic*—i.e. with integer Cartan numbers—but not necessarily reduced, irreducible or spanning. We will have  $V = V' \oplus V''$ , with  $V' = \mathbb{C}\Phi$ , for some possibly nontrivial complement space  $V'' \subseteq V$ : we say  $V'$  is the *essential part* of  $(V, \Phi)$ ,<sup>15</sup> and admit the case  $\Phi = \emptyset$  (the root system of “type  $A_0$ ”), in which case  $V' = (0)$ .

Whenever useful we endow  $V$  with a nondegenerate symmetric bilinear form  $(\cdot | \cdot): V \otimes V \rightarrow \mathbb{C}$  such that the above is an orthogonal decomposition, and which moreover is invariant under the action of the Weyl group  $W(\Phi) \subseteq GL(V)$ . Such a form is a (complex bilinear) “scalar product”, and the dual vector space  $V^{\vee}$  is then equipped with the push-forward scalar product along the isomorphism  $(\cdot | \cdot)^{\flat}: V \rightarrow V^{\vee}$ .

In general the subset  $\Phi^{\vee} = (\cdot | \cdot)^{\flat}(\Phi) \subseteq V^{\vee}$  is a root system (the dual/inverse of  $\Phi$ ), and the Weyl group  $W(\Phi^{\vee}) \subseteq GL(V^{\vee})$  is canonically identified to  $W(\Phi)$  via  $w \mapsto {}^t w^{-1}$ ; both will be denoted  $W$ , and we simply talk of “the” Weyl group. Note  $W$  acts trivially on  $V''$ , hence we retain the same notation for the action of the essential space; if  $\alpha \in \Phi$  we denote  $\sigma_{\alpha} \in W$  the induced reflection.

More generally we can consider the group of *automorphisms*  $\text{Aut}(\Phi) \subseteq GL(V)$  of the root system, i.e. linear automorphisms of  $V$  preserving  $\Phi \subseteq V$ —hence automatically the Cartan integers [43, § 9.2].

A *root subsystem*  $\Phi' \subseteq \Phi$  is a subset such that  $\sigma_{\alpha}(\Phi') \subseteq \Phi'$  for  $\alpha \in \Phi'$ . They are permuted by  $\text{Aut}(\Phi)$  (so in particular by  $W \subseteq \text{Aut}(\Phi)$ ), since

$$\sigma_{\varphi(\alpha)}(\varphi(\beta)) = \varphi(\sigma_{\alpha}(\beta)), \quad \alpha, \beta \in \Phi, \varphi \in \text{Aut}(\Phi).$$

The (orthogonal) *direct sum* of two root systems  $(\Phi_1, V_1), (\Phi_2, V_2)$  is

$$\Phi_1 \oplus \Phi_2 = (V_1, \Phi_1) \oplus (V_2, \Phi_2) := (V_1 \oplus V_2, \Phi_1 \amalg \Phi_2), \quad (31)$$

in the canonical embeddings  $V_i \hookrightarrow V_1 \oplus V_2$ . We also set  $n \cdot \Phi := \Phi^{\oplus n}$  for an integer  $n \geq 0$ —implying  $0 \cdot \Phi = \emptyset$ .

As mentioned above we will also encounter *nonreduced* root systems, i.e. such that  $\{\pm\alpha\} \subsetneq \mathbb{C}\alpha \cap \Phi$  for some  $\alpha \in \Phi$ . There exists a unique (spanning) irreducible nonreduced rank- $n$  root system, up to isomorphism, denoted  $\text{BC}_n$ : it consists of the vectors

$$\{\pm(e_i - e_j), \pm(e_i + e_j) \mid 1 \leq i \neq j \leq n\} \cup \{e_i, 2e_i \mid 1 \leq i \leq n\} \subseteq V = \mathbb{C}^n,$$

using the canonical basis of  $\mathbb{C}^n = \bigoplus_i \mathbb{C}e_i$  [25, Ch. VI, § 4.14].

<sup>15</sup>On this space the root-hyperplane arrangement is “essential”, i.e.  $\bigcap_{\phi} \text{Ker}(\alpha) \cap V' = (0)$ .

**Annihilators and kernels.** Let again  $V$  be a finite-dimensional complex vector space.

If  $S \subseteq V$  is a subset, its *annihilator* is

$$S^\perp = \{ \varphi \in V^\vee \mid S \subseteq \text{Ker}(\varphi) \} \subseteq V^\vee.$$

It is a vector space, isomorphic to the  $(\cdot \mid \cdot)$ -orthogonal subspace  $(CS)^\perp = (\cdot \mid \cdot)^\sharp(CS^\perp) \subseteq V$ , where  $(\cdot \mid \cdot)^\sharp: V^\vee \rightarrow V$  is the inverse of  $(\cdot \mid \cdot)^\flat$ . One has  $S^\perp \subseteq (S')^\perp \subseteq V^\vee$  if and only if  $CS' \subseteq CS \subseteq V$ , and

$$(U + U')^\perp = U^\perp \cap (U')^\perp, \quad (U \cap U')^\perp = U^\perp + (U')^\perp,$$

if  $U, U' \subseteq V$  are vector subspaces. Also by definition  $\emptyset^\perp = (0)^\perp = V^\vee$ .

Conversely, if  $T \subseteq V^\vee$  is a subset, its *kernel/vanishing locus* is

$$\text{Ker}(T) = \left\{ v \in V \mid T \subseteq \{v\}^\perp \right\} \subseteq V.$$

In addition to the analogous identities for the annihilator, one has  $\text{Ker}(S^\perp) = CS$  and  $(\text{Ker}(T))^\perp = CT$ —all finite-dimensional vector subspaces are closed.

**Braid groups and hyperplane arrangements/complements.** For an integer  $n \geq 0$  we denote  $\text{PB}_n$  the *pure braid group* on  $n$  strands [3, 4, 49]. It is the fundamental group of the space

$$\mathbf{B} = \mathbb{C}^n \setminus \bigcup_{1 \leq i \neq j \leq n} H_{ij}, \quad H_{ij} = \{ (z_1, \dots, z_n) \in \mathbb{C}^n \mid z_i = z_j \} \subseteq \mathbb{C}^n, \quad (32)$$

i.e. the space of ordered configurations of  $n$  points in the complex plane [35]—so  $\text{PB}_0$  and  $\text{PB}_1$  are trivial. These are thus the fundamentals group of complements of hyperplane arrangements, i.e. “hyperplane complements”.

More generally for a split Lie algebra  $(\mathfrak{g}, \mathfrak{t})$  we consider the *pure  $\mathfrak{g}$ -braid group*  $\text{PB}_{\mathfrak{g}}$ , which is the fundamental group of the space

$$\mathfrak{t}_{\text{reg}} = \mathfrak{t} \setminus \bigcup_{\Phi_{\mathfrak{g}}} \text{Ker}(\alpha) \subseteq \mathfrak{t}, \quad (33)$$

viz. the complement of the root-hyperplane arrangement—the “root-hyperplane complement” [26, 27, 32]. Such hyperplane arrangements are said to be *crystallographic*, and in particular (32) corresponds to a simple Lie algebra of type  $A_{n-1}$ .

In the case of simple Lie algebras of type  $B_n/C_n$  (resp.  $D_n$ ) we will denote  $\text{PB}_n^{\text{B/C}}$  the pure  $\mathfrak{g}$ -braid group (resp.  $\text{PB}_n^{\text{D}}$ )—as types  $B_n$  and  $C_n$  yield the same complement (33).

The Weyl groups are the reflection groups generated by the root-hyperplane arrangements: for the classical types  $A_{n-1}$ ,  $B_n/C_n$  and  $D_n$  one thus recovers the groups  $G(1, 1, n)$ ,  $G(1, 2, n)$  and  $G(2, 2, n)$  of the Shephard–Todd series, respectively [60] (these examples are complexifications of real reflection groups, and were classified earlier than op. cit., cf. [29] and references therein). Conversely a reflection group is *crystallographic* if it is the Weyl group of a root system [25, Ch. VI, § 2.5].

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