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# Mathematical algorithms in civic contexts: mathematics education and algorithmic literacy

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*Mathematics is used to address civic problems, and mathematical skills also are also a basis for a critical attitude of responsible citizens. As algorithms are more and more presented as solutions to social problems, it is necessary to highlight how mathematics education can contribute to a reflected handling as well as an evaluation of the topic. For this purpose, exemplary algorithms from civic applications such as proportional representation, fair distribution, and algorithmic decision-making systems are the starting point for an analysis. The question is how discussing relevant algorithms promotes a deeper mathematical and contextual understanding. Linking both enfolds a mathematics-specific empowerment in an issue of general interest. As a conclusion, this results in a sketch of what algorithmic literacy could look like as an educational ideal of teaching algorithm in mathematics.*

*Keywords: Algorithmics, citizenship education, critical literacy, mathematical applications.*

## Motivation from the perspective of education theory

In a digital world where a lot of data emerges algorithms are presented as solutions to social problems, e.g. sentences based on algorithms as *fair* decisions. So, algorithms are too important to leave the concept obscure to most citizens who have a right to transparency and a say in the basic structures of society. According to Fischer (2012), citizen participation in a dialogue with experts requires basic knowledge and reflection skills on the part of lay citizen. How can mathematics education contribute to such basic knowledge and reflection skills regarding algorithms in civic contexts?

Considering the mathematical-contextual dichotomy of civic issues, a promotion of mathematical enlightenment must analyse both mathematical and contextual characteristics as well as their social relevance (Winter 1990). This paper will therefore explore how addressing civic extra-mathematical applications can foster a deeper understanding of algorithms and stimulate reflection on their use. Mathematics cannot only be used for a description of reality, but also for defining public quantifications, e.g. poverty line or tax rate. Since a major aspect of citizen empowerment through mathematics involves addressing the “formatting power of mathematics” (Skovsmose 1998, p. 197), the examples in this paper will focus on a prescriptive, hence non-descriptive, use of mathematics and algorithms. In a first step, classical algorithms are examined in their mathematical-contextual dichotomy regarding algorithmic literacy and citizen empowerment. An outlook will suggest why algorithmic decision-making systems pose an even greater societal challenge and raise new questions for this perspective. Beyond the reflection on algorithms currently envisaged in the context of algorithmics (Lagrange 2019, p. 33), there is a need for a fundamental awareness for the relevance of algorithms in everyday life, that is so necessary that “algorithmic literacy” (Oldridge 2017) refers terminologically to this ideal.

## Comparing algorithms within mathematics and in applications

A very accessible definition for an algorithm would be that of a problem-solving tool that given a certain input, produces an output using a finite set of successive instructions, but it lacks the complex and constructive traits of algorithms (Modeste et al. 2010, pp. 53–54). From an educational perspective the broader definition provides important insights, but the specifications have enlightening potential, too. The constructedness of algorithms directly leads to the responsibilities of their application as part of a socially constructed reality. While intra-mathematical algorithms, e.g. the Euclidean algorithm as a paradigm, provide purely mathematical results (ibid., p. 58), extra-mathematical algorithms are based on a real issue that has been mathematised, and their output has a meaning outside of mathematics. An example of experiencing, learning, and testing such extra-mathematical algorithms is the fair distribution of an amount less than the sum of two individual claims. The Talmud suggests the following algorithm (Young 1994, p. 67): Distribute the resource equally until one receives half of one's claim. Distribute the resource to the other one until the missing amount to the full claim equals the correspondent opponent's missing amount. Now continue to distribute the resource equally. The algorithm terminates as soon as the resource is spent. If e.g. a resource is worth 300 and Mara is entitled to 300 and Nele to 200, we obtain the distribution (200, 100). For comparison, the proportional approach would deliver (180, 120).

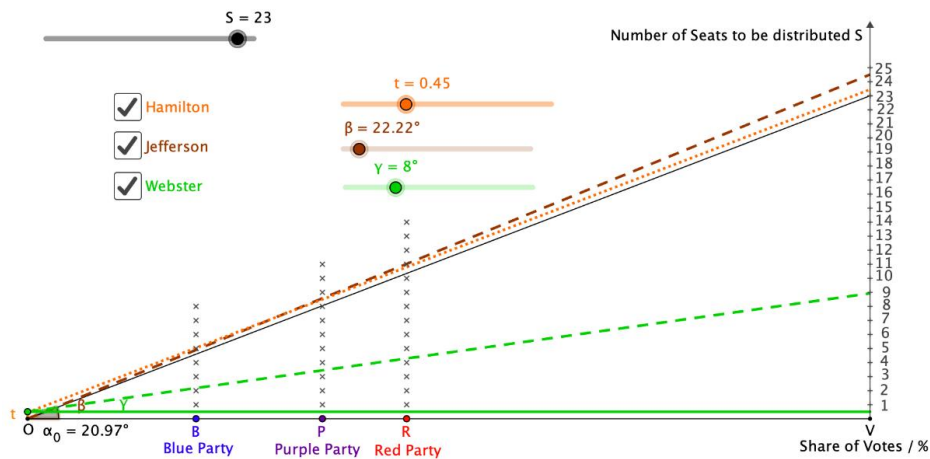
Comparing the Euclidean algorithm with the Talmud algorithm reveals an important difference for the reflection of algorithms: In the first case, the algorithm is there to extract a well-defined result and to do so as elegantly as possible. In the second case, the result is not so clearly defined: What does *fair* mean? In addition, the calculation of the proportionality shows that the extra-mathematical meaning of an algorithm's output cannot only be vague in advance, but also ambiguous as different mathematical approaches (the alternative here is just a calculation) produce divergent solutions.

## Proportional representation as a starting point for mathematical investigations

The importance of a mathematical investigation of different algorithms for a social problem can be illustrated by the example of proportional representation. If one allows only integer allocations, proportionality is not a solution. Under this constraint, there are several methods to get the output of a parliament composition, considering the share of votes as the input (e.g. Balinski & Young 2001). Three of them (Hamilton, Jefferson and Webster) are currently used in different German parliaments: Why is there no unitary solution and where are mathematical differences?

In order to distribute  $M$  seats, the Hamilton method starts with the rule of three and gives each party  $i$  the rounded down quota  $q_i = M \cdot \frac{v_i}{V}$  ( $v_i$ : votes for party  $i$ ,  $V$ : total votes); if there are still  $n$  seats that haven't been distributed yet, each of the  $n$  parties with the largest remainder gets one more seat. The divisor methods, on the other hand, are based on the following algorithm: Each  $v_i$  is divided by the same random starting divisor and the quotient is rounded with a certain rule (the Jefferson method rounds down, while the Webster method rounds to the nearest). If the sum of all these results is greater/less than  $M$ , the past step is repeated with a greater/smaller divisor. If the sum equals  $M$ , each party gets the rounded quotient as the number of seats (if  $M$  is missed due to

the same rounding limit for several parties, the decision between these parties is made, for example, by drawing lots).



**Figure 1: Visualisation of the algorithms of proportional apportionment**

The obviously significant contextual consequence of a different parliament motivates the mathematical comparison of the algorithms. While their outputs can be easily contrasted, it is more difficult to compare the procedures of the algorithms. The intercept theorem offers a geometrical, dynamical visualisation (Figure 1). The length from  $O$  to the point of a party in relation to the horizontal leg corresponds to the party's vote share. Therefore, the length of the vertical leg through the point of a party indicates proportional seat share of this party. As there are normally less crosses for integer seats under the hypotenuse than seats to be distributed, the algorithm of the Hamilton method is consistent with the *translation* of the hypotenuse. The *rotation* of the hypotenuse can easily be derived from the visualisation as the common principle of the methods of Jefferson and Webster. This initiates a discussion on mathematical characteristics of the algorithms as well as contextual effects: The divisor methods preserve the original proportions, while in Hamilton method the triangle resolves in  $O$ . What does that mean mathematically? When does a method (not) favor larger parties? In which way can such a bias be desirable?

Since all algorithms try to approximate the mathematical concept of proportionality with integers, a comparison with mathematical means like the intercept theorem is possible and accessible. Nevertheless, this doesn't lead to an obvious best solution. It is a human decision which method should be prescribed in the electoral law. The importance lies in the fact that the methods can lead to slightly different results, which can even be decisive for the government in the case of narrow majorities. The existent pluralism in Germany document that the political choice between the methods is not so unequivocal as well. The fact that there are also other possible calculations of the divisor methods, each of which leads in each case to the same result, is not to be deepened here, even if it does enable a discussion on the result equivalence of algorithms.

### **Difficulty of the distinction and transparency: algorithms for fair distribution**

With recourse to the distribution of a scarce resource, there is another algorithm with a variant of the „first come, first serve“-principle: Firstly one calculates all permutations of the different claims. For every permutation the resource is allocated along the order, until it is spent. The actual

allocation is the arithmetic mean of all these allocations: If a resource is worth 300 and Mara is entitled to 300 and Nele to 200, we obtain the distribution (300, 0) if Mara is first; otherwise, it is (100, 200). The solution

**Table 1: Talmud (Shapley) solution to the claims of three wives (Young 1994, pp. 71–72)**

	$c_1 = 100$	$c_2 = 200$	$c_3 = 300$
$R = 100$	$33\frac{1}{3}$	$33\frac{1}{3}$	$33\frac{1}{3}$
$R = 200$	50 $(33\frac{1}{3})$	75 $(83\frac{1}{3})$	75 $(83\frac{1}{3})$
$R = 300$	50	100	150

of this so-called Shapley value would be equivalent to the Talmud algorithm. Applying both algorithms to other examples with two claims leads to the correct hypothesis that this algorithm coincides with the Shapley solution for two claimants (Young 1994, pp. 67–70). However, the Talmud presents a second example of a fair distribution, but this time with three claims and varying resource values (Tab. 1). Obviously, this distribution doesn't follow the proportional logic ( $R \neq 300$ ), nor is the Shapley algorithm behind it ( $R = 200$ ). The algorithm behind this centuries old Talmudic allocation rule has only been discovered in 1985 (Young 1994, p. 72). Its implementation for two claims has been described above, although there is a simpler and older solution in this special case, that doesn't work for more than two claims. The long time of not knowing the algorithm exemplifies how difficult and lengthy it can be to reconstruct an unknown algorithm, even if the known input and output are rather accessible. At the same time, it becomes obvious how dissatisfying it is to obtain only the result without insight into the process or even an explanation.

Further conclusions can be drawn: Firstly, for certain inputs it is not possible to differentiate between some algorithms (Shapley vs. Talmud algorithm for two claims), so the constraints for this limited result equivalence should not be neglected. Secondly, even for a supposedly simple distribution there is no single solution, and one must choose between the alternatives. As there is no algorithm for objective fairness, the extra- and intra-mathematical reasons for a particular algorithm and against the other are significant. All ends up to the idea that both the mathematical and contextual aspects of socially relevant algorithms should be potentially transparent for interested citizens. But we have seen a degradation of transparency so far. In the case of proportional representation, the idea of proportionality determines the various algorithms and can easily be communicated. However, fairness has not such a clear mathematical landmark. Thus, the results appear even more subjective and debatable. Algorithms are used to create a fair process, in the sense that there are no arbitrary changing rules. The results are legitimated only by the *incorruptible* algorithm. But the choice of an algorithm itself is subjective and ambiguous. This contradicts the view that algorithms are neutral, which is ultimately a deduction from the myths about mathematics (Hersh 1991). For the aim of a critical citizenship, the daily experience of being confronted with established algorithms and its results can be facilitating by exemplarily implementing algorithms in a concrete context. An online game about fake news has shown that simulating the producer's

perspective promotes awareness for a further reflected consumption of fake news (Roozenbeek & van der Linden 2019). In analogy, students should formulate and compare different algorithms to solve social issues. In this way, they become sensitised for the important mathematical and contextual aspects in the face of existing or newly deployed algorithms. Until now, algorithms have been a tool that helped humans respond to civic problems. As seen above, the use of algorithms does not automatically lead to one suitable solution though. To find appropriate solutions to a problem, one must consider the mathematical properties and contextual consequences of the algorithm. Therefore, this aspect of algorithmic literacy refers to ‘literacy through mathematics’ since mathematics helps to understand the algorithm. Moreover, citizens should critically reflect the use of algorithms in the world and their stance on the issue. This requires ‘literacy towards mathematics’. A simple example would be that if two friends must share a scarce resource, they can dismiss all algorithmic approaches and decide to not use algorithms or even mathematics to solve their problem. A more elaborate position would comprise the awareness, that the use of an algorithm doesn’t form itself an indisputable legitimacy and that certainty in computation is not equivalent with unambiguity in an extra-mathematical context.

### **Automated decisions through algorithms: playful insights**

It is precisely because of their constructive character and the clear step-by-step instructions that algorithms are potentially transparent and assessable. The use of artificial intelligence and algorithmic decision-making systems (ADMS) changes this. Nowadays, ADMS take over the decision and share only the final results without disclosing the finding process. Examples are decisions from banks, insurances, employers as well as from the justice system; common hope is to improve the fairness of the respective processes (MacCarthy 2019). These non-transparent techniques can no longer be completely retraced, but in a democracy a majority should be able to control and accept at least their fundamental principles. It includes the critical evaluation of cultural practices involving algorithms in ADMS. The didactic problem is that there can be no enlightenment if nothing can be made transparent. One approach is to analyze the functionality in easy, transparent examples and then to reflect further on the consequences if not all processes were longer traceable. One good teaching unit to do so is the Good-Monkey-Bad-Monkey-Game under creative common license (Lindner & Seegerer n.d., pp. 4–7). The basis is a data set of images of monkeys with different characteristics (smiling, eye shape etc.) and the information whether this monkey bites or not (for examples of such images see Figure 2).

By means of a training set, students must construct a decision tree as an algorithm to determine whether a monkey bites. The algorithmic decision trees are then checked on a test set. Even for the simplest version of the game, more than one decision tree is a fitting solution. Therefore, students are confronted with the ambiguity of a possible algorithm. In the extended version of the game, it is more difficult, first, to find a splitting for the training set, that, second, fits in the test set. In addition, there is one monkey in the test set that doesn’t fit because of different, so far unknown properties. This undecidable case shows the limitations of this method, especially concerning the quality of the training and test set, from which reality may still differ. Instead of deterministic predictions as “every non-smiling monkey with x-shaped eyes always bites”, such decision trees rather result in good probabilities. Figure 2 shows a constructed variation where x-shaped eyes still

implicate biting, but there is only a stochastic correlation between smiling and biting (original data in the last column of Figure 2). A non-smiling monkey without x-shaped eyes bites with a probability of  $\frac{2}{3}$ : should it be

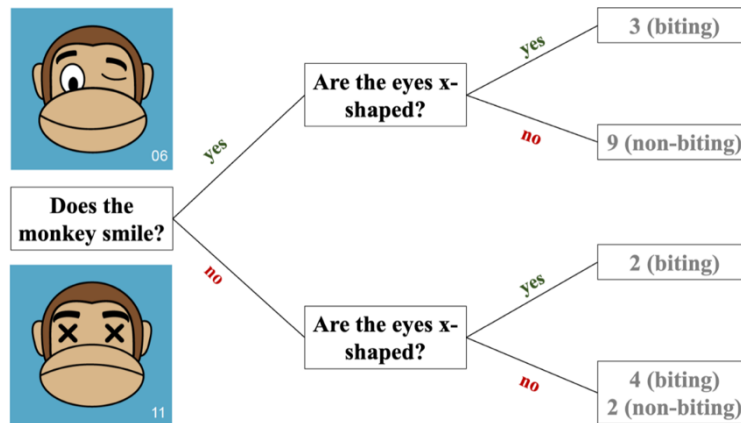


Figure 2: Decision tree for a non-deterministic variant of the monkey-game

isolated or incarcerated? Mathematically, the mistake is smaller, if such an animal is treated as a biting one. Socially, one can argue that punishing an innocent monkey is faultier than sparing the troubled one. If one omits the distinction of the eyes in this decision, a non-smiling monkey bites with a probability of  $\frac{3}{4}$ . The probability of punishing an innocent has decreased, although we have less information and fewer steps in the algorithms to the detriment of more biting monkeys that could be spared. Generally, less information or fewer steps go hand in hand with a greater uncertainty, but less work could justify a greater deviation regarding the cost-benefit calculation. E.g., not distinguishing the eyes of smiling monkeys leads to a  $\frac{3}{4}$  chance of non-biting instead of a deterministic statement: Is this good enough in the related context? However, the characteristic of the x-shaped eyes is more compelling than smiling. If a monkey has x-shaped eyes, it is without any further distinction and without doubt clear that it bites. Hence, a decision tree starting with the x-shaped eyes would need less steps than Figure 2. The efficacy of an algorithm could be theoretically verified – in practice, the permutation possibilities grow fast with additional information –, but how can an algorithm take the other arguments into account?

Via such decision trees students deal with the construction and validation aspect of algorithms. By thinking in terms of case distinctions and conditional probabilities, they experience how a computer generates decisions themselves. While the algorithms so far have been independent from the context (e.g. the parliament size doesn't change the principle of the algorithms), the decision tree relies on the data of the context. In a second zoo with other monkeys the decision tree could be totally different. This raises the question of how objective and unambiguous such algorithms are for automated decision-making. With reference to statistical literacy, the critical evaluation can be extended to the origin of data (Gal 2002, p. 11): Were the monkeys with the x-shaped eyes hungry or have they been mistreated because of existing prejudices? The algorithm can't be neutral if the reference data is biased. This may not seem so central to the example of the biting monkeys, but what if job applications are pre-sorted using ADMS or recidivism is predicted for criminals?

## Algorithmic literacy: towards a reflection of algorithms and ADMS

Algorithms and ADMS are used to solve real civic problems. This can be a starting point for a detailed mathematical analysis. It becomes clear that algorithms are neither unambiguous nor objective, e.g., a proportional representation by different algorithms can lead to different compositions of the parlia-

**Table 2: Categorisation for a critical evaluation of socially relevant algorithms**

		instruction independent from context	
		+	-
well-defined output	+	Euclidean algorithm	Good-Monkey-Bad-Monkey
	Algorithms for proportional representation		
	-	Algorithms for fair distributions	<i>unknown-unknown</i>

ment. Mathematical explorations (such as geometric representations) can help to answer the question of whether two algorithms produce the same output and how they differ. The extra-mathematical interpretation draws students' attention to the existence of such a pluralism.

Following Jablonka's (2017, p. 44) categories of explicitness for the relation between mathematics and the context as well as for the intra-mathematical model, we would like to classify the discussed examples in terms to the explicitness of selected aspects of algorithms. The vertical axis is used to categorise the relationship between mathematics and context: while the greatest common divisor is explicitly defined as a mathematical output, 'fairness' of a distribution is a fuzzy concept. This category is a continuum rather than a dichotomy, since proportionality is well-defined within mathematics, but its application to seat distributions with integer seats is ambiguous. For the monkey game, 'well-defined' means whether the monkey bites (or with what probability), the consequences of how to deal with the monkey are yet not so clear. The second axis makes a difference whether an instruction is independent of the context: the Euclidean algorithm works the same for every two natural numbers. Decision trees, as in the monkey game, are based on data; with new information about other monkeys, the rule for decision making will change. The societal challenge is the fourth category of the "unknown-unknown" (ibid.). Indeed, there are more dimensions to what is unknown. Furthermore, one can differentiate whether humans don't see the algorithm behind a decision or are not able to understand it. Maybe an understanding of some ADMS and its application is even not possible in a semantic way. To raise awareness for the inaccessible unknown-unknown, the only possibility is to discuss the other categories and to imagine losing their defining feature.

Teaching algorithms must thematize the consequences of implementations in social contexts. Zweig et al. (2018) propose general competencies for such a necessary algorithmic literacy. Respective mathematic-specific competencies with a citizen empowering potential are concretised, analogously



to statistical literacy, in “worry questions” (Gal 2002, p. 17), e.g.: (1) What is the specific output and purpose of the algorithm and who has developed it? (2) Where did the data come from and what happens if the input changes? (3) How is the result derived? What is the underlying mathematical model of the algorithm? Are there alternative or equivalent models? (4) What is known and what is *not*? Is this adequate? In the sense of the mathematical-contextual dichotomy, mathematics education can contribute to an algorithmic literacy by focusing on the intra- and extra-mathematical characteristics of algorithms in civic contexts and on the interdependence between mathematics and the context.

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