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Fair games – A case study on a negotiation of meaning on the concept of fairness in probability in mathematics classes

Felicitas Pielsticker

University of Siegen, Germany; pielsticker@mathematik.uni-siegen.de

This paper describes a student's negotiation of meaning about the concept of fairness in a 10th grade mathematics class. Using the case study approach and the approach of empirical theories, key scenes in a learning unit about gambling are examined and knowledge structures are analyzed. A 3D printed dice tower plays a special role and is used to reflect on fairness in probability calculation.

Keywords: Empirical theories, knowledge development process, case study, empirical-oriented mathematics classes

Introduction and motivation.

Fairness is a much-discussed concept in various contexts. It often seems as if everyone has clear beliefs of fairness and "whether something is fair". Fairness for us people have quite different facets (Möbius, 2018). So we can say we want everyone to be rewarded equally. Or we can say that the person who has done more gets more. But we can also say we want those who have higher needs to get more (Möbius, 2018). A frequently made statement is "That is unfair!" and not "That is fair!" But how can we decide whether something is fair? Questions of fairness and justice (used synonymously in most contexts, e.g. Duden online, 2021) are dealt with by different disciplines. The concept of fairness thus has many facets, even beyond a simple mathematical distribution model (Möbius, 2018). In mathematics, the concept of fairness plays a crucial role. Looking at the work "An Introduction to Probability Theory and Its Applications" by Feller (1968), we can see that a "theory of 'fair' games" (Feller, 1968, p. 248) is treated here. Textbooks also address fairness for mathematics education. Often we can read: A game is said to be fair if the expected value for winning is 0. The expected value seems to be important in connection with fairness. A fair game is often discussed in connection with probability calculation. The Common Core for High School: Statistics & Probability. Using Probability to Make Decisions also states: "Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator)" (Common Core State Standards Initiative, 2021). This paper describes what beliefs of fairness can be analyzed in a student (10th grade of a high school) of our case study and what discussion impulses on "fair" and "unfair" can be gained from it. For this goal, the teaching unit "A fair game of chance?!" was developed (section: Material and data collection).¹

Theoretical background – empirical-oriented mathematics classes.

If students acquire mathematical concepts by dealing with empirical objects (visual materials and physical objects, e.g., folded and drawn figures) in the constructivist sense, we assume according to the approach of empirical theories (Burscheid & Struve, 2020a; 2020b) that they develop a so-called empirical belief system (Schoenfeld, 1985). We describe the knowledge of students in empirical

¹ We note that parts of the results of this paper have also been published in the German-language volume "Neue Perspektiven auf mathematische Lehr-Lernprozesse mit digitalen Medien" by Springer.

theories which are constituted by dealing with these empirical. According to studies of Burscheid and Struve (2020a) and Witzke (2019), this "'non-abstract' point of view [...] is a reasonable one for the developing of mathematical knowledge" (Witzke, 2019, p. 199). Students construct their mathematical knowledge by themselves in interaction with their environment. With Hefendehl-Hebeker (2016) we can state that the concepts and contents of school mathematics have their phenomenological sources predominantly in our surrounding reality. For our analysis of the cognitive aspects in students' knowledge development processes, the concept of empirical-oriented mathematics class is crucial. An empirical-oriented mathematics class builds on, on the approach of empirical theories. Empirical-oriented mathematics classes are classes in which the teacher intendedly makes decisions to work with empirical objects as the mathematical objects of math classes. In mathematics classes, the empirical objects (e.g. dice or drawn figures) do not intend to illustrate mathematical concepts that are abstract in nature, but they are rather the subject of the lesson, where students acquire their knowledge on (Pielsticker, 2020). Empirical objects that play a role in this paper are 3D printed dice towers (Figure 1 and 6) and commercially available dice. Studies have shown that the use of 3D printing seems particularly useful for probability. For example, Pielsticker and Witzke (2022) argue that with a creation and use of 3D printed manipulated dice, profitable negotiation processes on the theoretical concept of probability can be initiated among learners. An empirical object such as the 3D printed dice tower described in this article (Figure 1 and 6) then serves in the classroom as an object of instruction (Pielsticker, 2020). 3D printing as a digital tool in a teaching conception of an empirical-oriented mathematics class creates student-oriented occasions to exchange ideas about mathematics. Students negotiate mathematics, which means that the digital tool 3D printing can be seen as a motor for the further development of student knowledge in the classroom. A self-construction of the dice tower in the CAD program in mathematics lessons is quite desirable and a crucial part also of the teaching unit "A fair game of chance?!" considered here. In this process, the learners negotiate the construction and thus also the functioning of the 3D printed dice tower. In the following, a teaching-learning process from a mathematics class that takes this conception into account is described.

Design and methodological decisions.

Research questions.

In this paper, we describe the beliefs of fairness that can be analyzed in a student in a 10th grade class at a high school. The research of Burscheid and Struve (2020a; 2020b) on questions of fairness of games of chance (Burscheid & Struve, 2020b) in historical reference is also guiding for this. There are many different games of chance. Often there was agreement on what the ratio of stake to winnings had to be in order for the game to be fair. In the "force majeure" problem, however, mathematicians pursued different solutions with respect to the question: How is the stake to be distributed fairly? Pascal and Fermat propose a solution to this in 1654, which corresponds to the modern probabilistic view (Burscheid & Struve, 2020b). Leibniz, knowing the solution of Fermat and Pascal, makes a different proposal for division in 1678. For normative problems there are just no right or wrong solutions, but only more or less appropriate proposed solutions (Burscheid & Struve, 2020a). In this context, probability theory nowadays plays an authoritative role for an assessment of the fairness of games of chance. This was not the case before, because probability theory has a root in the theory of

the fairness of games of chance (Burscheid & Struve, 2020b). Thus, it is interesting to note that the concept of probability, which is fundamental to probability theory, has historically been developed within an application-related context: precisely the question of the fairness of games of chance (Burscheid & Struve, 2020a). For us, it is interesting to see how the student in our case study deals with the concept of fairness in the instructional context of probability. We pursue two research questions in this paper: 1. Which beliefs does the student of our case study develop about the concept of fairness? 2. To what extent can (students') theories about the concept of fairness be (further) developed when dealing with the 3D printed dice tower?

Material and data collection.

Data collection for the analysis of knowledge structures was based on student documents created before, during and after the implementation of the lesson “A fair game of chance?!”. These include worksheets and a questionnaire. Both tasks of the worksheets and the questionnaire are part of the data collection and will be analyzed in this paper. The topic was introduced by the students' report about their experiences with gambling. Afterwards, there was a worksheet where cheating and fraud were discussed. This was intended to create a deeper awareness of the problem. In ancient times, for example, the use of a dice tower was considered a guarantee of fairness in dice games. The task of constructing a dice tower using 3D printing technology was intended to create an action-oriented approach in relation to a game situation. 3D printing is a quite new tool for mathematics education. It is an additive manufacturing method which allows digital models constructed with CAD-software to be transformed layer by layer into a real model (e.g. out of liquid plastic) (Gibson et al., 2014). This technology makes the development as well as the production of empirical objects in a variety of mathematical contexts quite easy and thus may facilitate embodied (Tall, 2013) approaches to mathematical content. In ancient times, a dice tower was seen as a way to prevent cheating. In the learning unit the students designed their dice towers on their own using the CAD software Tinkercad™ and 3D printed them out of plastic (Figure 4). To ensure that the tower had the conditions required for fair play, common criteria had to be defined within the student groups. In partner work, the students discussed the criteria. Subsequently, the students conducted test series with their dice towers and calculated the absolute and relative frequencies of their throws. In the worksheet with the sentence beginning “For me, fairness is ...”, the students of the class summarized their view of the concept of fairness once again.

Methods.

In our instrumental case study (Stake, 1995), we describe the student Ardelin (a 10th grade student, 16 years old, name changed) in relation to our two research questions (see section: Research questions). „Case studies are undertaken to make the case understandable” (Stake, 1995, p. 85). With our case study we need to understand the case in order to understand our two research questions (Stake, 1995). We use the case study approach to identify our key scenes in terms of the research questions and to process the data material. We use the approach of empirical theories according to Burscheid and Struve (2020a) for informal description of the student's knowledge structures. We use the following terms (Table 1) as the basis for our analysis in terms of our research questions.

Terms	Explanation
intended applications	The phenomena of reality described and explained by an empirical theory.
empirical objects	In this study, empirical objects are understood as items and objects of reality that are immediately accessible to students, especially in a tactile or visual way.
theoretical concepts	Terms whose meaning can only be clarified by setting up or within a theory and which do not have an empirical reference object.

Table 1: Terms of empirical theories for analysis (Pielsticker, 2020; Burscheid & Struve, 2020a)

We use the terms from Table 1 as an analysis tool. In terms of our research concern, this is an informal description using the terminology of the empirical theories approach according to Burscheid and Struve (2020a).

Analysis and Results.

In order to achieve a deeper understanding of the case, key scenes have been selected, which we describe below. In the learning unit, the game of chance – dice game – was considered and knowledge about fairness was acquired. The question of when a game is fair should be addressed. In the learning unit, a link between ancient and digital tools – the dice tower – was used for this purpose. With reference to the construction of an (ancient) 3D printed dice tower (section: Material and data collection), the aim was to reflect on fairness. For the case of fair play, we will first discuss the results of the student Ardelin. To do this, we will start by looking at the student's answers in the questionnaire and then describe her answers on the worksheets. In the questionnaire, Ardelin is asked: "What does 'fairness' mean to you?". The student states, "That everyone is treated or valued equally, so that everyone is equal and no one is favored" (Figure 1).

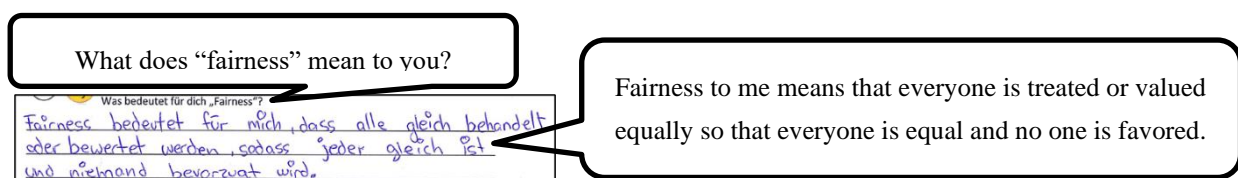


Figure 1: Answer to 2) from Ardelin in the questionnaire

The student bases her concept of fairness on dealing with people ("everyone" and "no one"). She thus refers to (her) fellow players. Ardelin is now asked to relate her concept of fairness to a game. She is asked, "When is a game fair for you?". The student states, "when everyone is treated equally and everyone gets the same chances and everyone gets the same rules" (Figure 2).

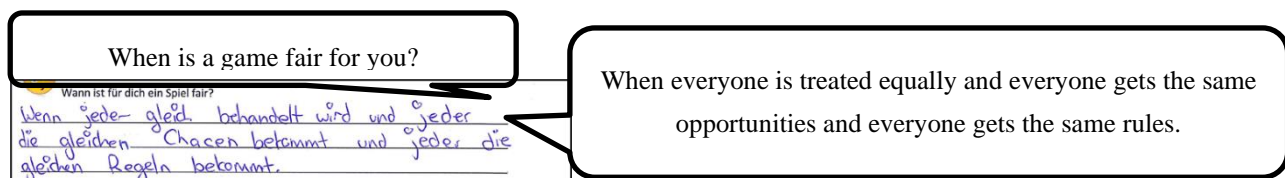


Figure 2: Answer to 3) from Ardelin in the questionnaire

Ardelin sticks to the reference ("everyone"), emphasizing that "everyone gets the same chances" and "everyone gets the same rules" (Figure 3). The student thus makes a fair game once to a game context in which "the same rules" apply, also she connects a fair game with "equal chances" and finally she names the players, "everyone", who should be "treated equally". When Ardelin is asked for her opinion in the 4th question of the questionnaire, "Is chance fair? Give reasons for your opinion.", the student first makes this point, about the chance of winning a game. "If the chance of winning and losing is the same, then it is fair" (Figure 3). At the same time, Ardelin also refers back to the players by noting, "It also depends on whether the person knows it's chance."

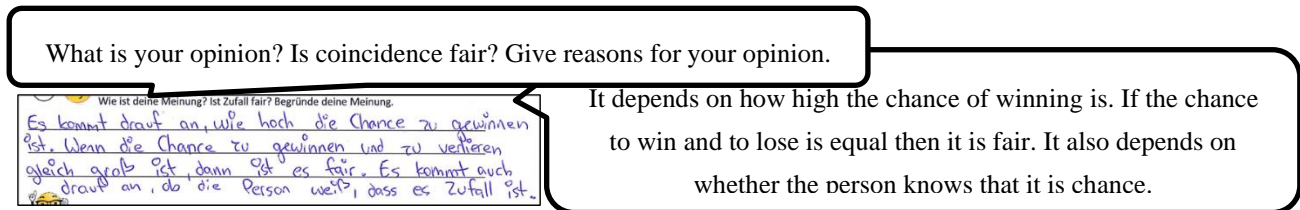


Figure 3: Answer to 4) from Ardelin in the questionnaire

In her answers, the student thus distinguishes between the game, the people involved (players), and the chance of winning the game. At this point, we would like to discuss the results of the student's worksheets. Ardelin created a 3D printed dice tower and tried it out in the further course of the learning unit (Figure 4). Figure 4 shows the 3D printed cube tower. Ardelin has constructed some inclined levels (the student herself writes "steps") inside the cube tower and further a cube tower wall to be attached to the rest of the cube tower with tape afterwards. The student explained in the classroom situation that she only wanted to tape the cube tower wall because she wanted to control how the cube rolled along the inclined planes.

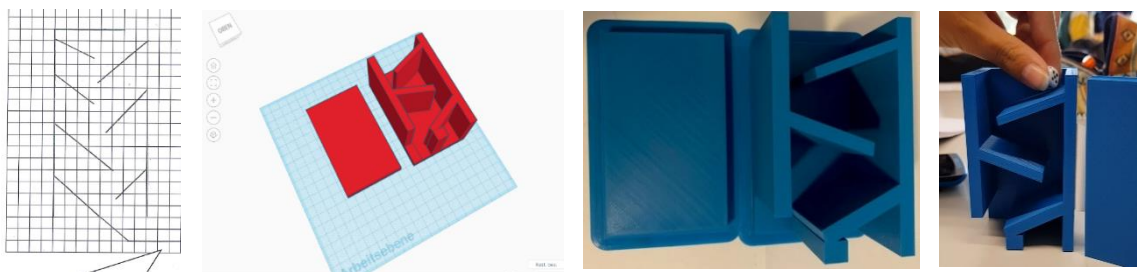


Figure 4: Ardelin's 3D printed dice tower

On her worksheet, the student describes regarding the criteria that should apply to her dice tower, "It must be fair" and "It must always be dropped with the "1" from the top and give more of these steps." The empirical object to which Ardelin refers at this point is the 3D printed dice tower. For the game with the dice tower, special rules should continue to be recorded, such as, "It [the dice] must always be dropped with the 1 from the top [into the dice tower]," there must be "more of these steps," which should be placed "at even intervals," and ultimately, "nothing should fall out of the sides." Using the example of the 3D printed dice tower, as an empirical object, Ardelin argues how it must be designed for fair play. To do this, she first states that the game must be fair, and then simultaneously formulates what rules must apply. Ardelin also describes her view once again on her worksheet (Figure 5). Here, Ardelin describes a "fair chance [...], that is, [...] equal chance" - she probably uses the two

expressions synonymously - with "equal rules and conditions." She argues this with the help of the intentional applications "dice game" and "game of chance: even - odd" - as intentional applications of the student theory about games of chance. The dice can be regarded as the empirical object for the intended application dice game. Ardelin emphasizes here that "every player should have the same die" (Figure 5).

Fairness ist für mich, wenn die gleiche Chance besteht zu gewinnen, also, wenn jeder Teilnehmer die gleiche Chance hat und jeder die gleichen Regeln und Bedingungen hat. Ein Beispiel ist ein Würfelspiel. Fair ist, wenn jeder Spieler den gleichen Würfel hat und die gleiche Chance besteht zu gewinnen. Fair ist, wenn alle Spieler die gleichen Regeln und Bedingungen haben. Ein Beispiel ist ein Würfelspiel. Fair ist, wenn jeder Spieler den gleichen Würfel hat und die gleiche Chance besteht zu gewinnen. Fair ist, wenn alle Spieler die gleichen Regeln und Bedingungen haben. Ein Beispiel ist ein Würfelspiel.

Fairness for me is when there is a fair chance to win. That is, when every player has the same chance, and everyone has the same rules and conditions. An example is a dice game. Fair is when every player has the same dice and the same chance to win. For example, if the players agree that one wins on even numbers and the other wins on odd numbers. Then everyone has the same chance to win. But if you say that one wins on the number 6 and the other wins on all other numbers, the chance is no longer equal. This is not fair, because the chance is higher not to get a 6.

Figure 5: Adrelin's 3D printed dice tower

In relation to the theory of the fairness of games of chance, the "game of chance: even - odd" is an intended application with even and odd numbers as failures of a random experiment. At this point, the student gives a counterexample, which she relates either to the intended application of the dice game or to another game of chance: "But if you say that one wins with the number 6 and the other with the other numbers, the chance is no longer equal and also not fair, because the chance is higher not to get a 6". If we assume the intended application of the dice game (where "one wins on the number 6 and the other on the other numbers"), the dice can be seen as an empirical object on which Ardelin explains her counterexample to "equal chance". It is also clear that she is oriented to the game context in her negotiation of meaning regarding the concept of fairness. The intended applications in relation to the student's theory of fairness of games of chance are games of chance such as "dice game" (with or without 3D printed dice tower) or "even - odd". In this context, Ardelin gives meaning to the concept of fairness in the context of gambling. Whether a game is fair is decided for the student by this game context, with the associated rules and "prerequisite" (Figure 5), such as, for example, that "every player has the same dice" (Figure 5). It should also be mentioned that Ardelin's examples probably tacitly assume that each player pays the same stake.

Discussion

1. Which beliefs does the student of our case study develop about the concept of fairness?

We can describe for Ardelin that she behaves as if she developed an empirical beliefs system, an empirical theory about the fairness of gambling. In doing so, the student repeatedly uses words like "chance" and "equal chance" (Figure 2 & Figure 5), with which a connection to probability theory can be assumed. This would not be surprising, because until today (not from the beginning!), however, probability theory has remained the authoritative theory for assessing the fairness of games of chance (Burscheid & Struve, 2020a). In relation to a fair game, the student – in relation to her theory on the fairness of games of chance – can describe the intended applications "dice game" and "game of chance: even - odd". Ardelin makes the "same rules and preconditions" (Figure 5), which apply to a fair game in the student's sense of the word, on empirical objects such as the 3D printed

dice tower or dice. For a fair game, Ardeline emphasizes that "every player should have the same dice" (Figure 5). In her negotiation of the meaning of the concept of fairness, the student orients herself to the game context, and she gives her concept of fairness a meaning in this application-related game context. Whether a game is fair is determined for the student by this game context, with the associated rules and "prerequisite" (Figure 5), such as that "every player has the same dice" (Figure 5). The notion of fairness is seen in connection with the expectation value, as for example in Christian Huygens, who defines the notion of expectation in terms of fairness as equal expectation prevails in a fair game (Gigerenzer & Krüger, 1999). Burscheid and Struve (2020a) also make it clear with historical reference that the concept of expected value, which is fundamental for probability theory, did not develop within a formal theory, but within an application-related context (Burscheid & Struve, 2020a). In this context, expected value can be identified and described as a theoretical concept. Ardeline decides whether a game is fair along the chance of winning (and the expectation of winning) of the players in the game situation. In doing so, the concept of fairness takes on meaning for her in this context.

2. To what extent can (students') theories about the concept of fairness be (further) developed when dealing with the 3D printed dice tower?

Ardeline seems to add the dice tower as a (further) possibility of throwing the dice in her student theory about the fairness of gambling. The 3D printed dice tower is not crucial for the student to decide whether fair game prevails in a gambling situation (such as the "dice game"). The 3D printed dice tower is integrated by the student into her empirical theory as a further dice-rolling possibility and thus expands her knowledge of dice-rolling possibilities. The notion of fairness can be identified as an epistemological obstacle (Sierpiska, 1992) in probability theory and in connection with expected value (in relation to our case study). Here, parallels to the historical development of probability theory can also be identified on an epistemological level, in line with the remarks of Burscheid and Struve (2020b). With a development of meaning of the concept of fairness, as a theoretical concept in relation to a (school) theory of probability, special challenges are therefore associated in mathematics education. In the practical application (of probability theory in schools), time should not only be given for the development of the meaning of the concept of fairness, but it should also be addressed in game situations, e.g. in the sense of fair play and fair game. This impulse should also be followed up in further research and it should be considered whether and how both theories of fair game and fair behavior of persons in gambling are related.

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