Chinese teachers' professional noticing of students' reasoning in the context of Lakatos-style proving activity

Mei Yang, Andreas J Stylianides, Mateja Jamnik

Chinese teachers' professional noticing of students' reasoning in the context of Lakatos-style proving activity

Mei Yang[1], Andreas J. Stylianides[2] and Mateja Jamnik[3]

[1] my370@cam.ac.uk; [2] as899@cam.ac.uk; [3] mateja.jamnik@cl.cam.ac.uk

Despite the pedagogical values of Lakatos-style proving activity in school mathematics, little is known about how teachers react to students' reasoning in this kind of activity. As an attempt to unpack the underlying decision-making processes, this study examines teachers' noticing of students' justification and refutation of conjectures in the context of Lakatos-style proving activity. Twelve Chinese pre-service and in-service secondary mathematics teachers participated in semi-structured, vignette-based interviews where they were presented with realistic classroom scenarios. These were based on actual classroom episodes reported in the literature and covered various aspects of Lakatos-style reasoning in the context of geometry. Findings show that teachers can better notice students' justifications than students' refutations, and better notice students' valid arguments than invalid arguments. Key themes are identified to characterise their noticing of various student arguments.

Keywords: Proof, Lakatos, Justification, Refutation, Professional noticing.

Introduction

In the form of fictional classroom discussion, Lakatos (1976) described how mathematicians constructed and utilised mathematical knowledge through a zig-zag reasoning process. Some aspects of Lakatos-style reasoning, such as conscious guessing and the zig-zag path of reasoning, were suggested by mathematical educators (e.g., Lampert, 1990) to be applied in some school mathematical activities for engaging students in authentic mathematics. Some empirical studies also showed that school-age students can perform in line with Lakatos-style reasoning, but most of them paid attention to students' proving processes (e.g., Komatsu, 2016; Reid, 2002), while few studies focused on the role of teachers. It remains unclear how teachers deal with various types of students' responses throughout different phases of Lakatos-style proving. To address this research gap, one promising attempt is to investigate teachers' professional noticing by focusing on how they pay attention to and make sense of particular instructional situations (Jacobs et al., 2010). We explore teachers' professional noticing of students' reasoning in the context of Lakatos-style proving activity. In this paper, we focus specifically on how teachers notice students' uses of examples and counterexamples during the justification and refutation of conjectures, which potentially can inform students' further refinement of the conjectures or proofs thereof.

Theoretical framework

Lakatos-style proving process

Using Lakatos' (1976) book and some mathematics education studies that discussed the implementation of his approach in school mathematics (e.g., Reid, 2002; Komatsu, 2016; Deslis et al., 2021), we identified five phases of the Lakatos-style proving process to capture some aspects of Lakatos-style reasoning: First, a conjecture is formulated through conscious guessing (Phase 1). Then the conjecture is tested through examination of supportive examples (Phase 2). A proof may be constructed to further validate the conjecture (Phase 3). Yet, counterexamples may emerge that refute the conjecture or the respective proof (Phase 4), thus necessitating the refinement of the conjecture or the proof (Phase 5). Note that Lakatos' philosophy is more complex than this 5-phase framework. It also emphasises other aspects (e.g., the crucial role of the interplay between defining and proving).

Justification and refutation schemes

We utilised a framework of students' justification and refutation schemes to conceptualise how students justify and refute conjectures in diverse ways, reflecting different degrees of mathematical sophistication. This framework was constructed drawing on previous research on students' proof schemes (Balacheff, 1988; Harel & Sowder, 1998; K. Lee, 2016; Stylianides & Stylianides, 2009; Deslis et al., 2021). There are four levels of justification schemes. Students at the lowest level (Naïve empirical justification) believe that examining a few examples which are easy to check (e.g., examining the example "x = 1" for validating the conjecture "for any natural number, 2x is an even number") can prove a mathematical generalisation. Students at a higher level (Crucial experiment justification) also accept example-based proofs, but they believe that the examples need to be strategically identified following some rationale (e.g., examining a set of odd numbers "x = 1, 3, 5, 7, 9…" for the above-mentioned conjecture). Students at the next level (Nonempirical justification) believe that it is not sufficient to validate a conjecture based on a subset of examples, but unlike students at the most advanced level (Deductive justification), they may not recognise the role of deductive inferences in proof. There are also four levels of refutation schemes. Students at the least advanced level (Naïve refutation) regard counterexample(s) as exception(s), and still consider a conjecture to be true regardless of the existence of counterexample(s). Students at the next level (Empirical refutation) think it is insufficient to refute a conjecture based on a single counterexample and need to see more counterexamples to be convinced that the conjecture is false. Students at the next level (Single counterexample refutation) believe that it is sufficient to refute a conjecture based on a single counterexample. Students at the most advanced level (General counterexample refutation) accept the sufficiency of a single counterexample in refuting a conjecture and recognise further that identifying the common properties of counterexamples can support the refinement of the conjecture.

Noticing

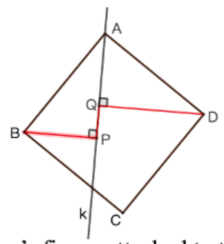
Despite various conceptualisations of teacher noticing, it is generally considered to involve at least two components, *Attending* and *Interpreting* (e.g., Es & Sherin, 2008). Jacobs et al. (2010) additionally introduced a third component, *Deciding*, which works with the other two components in integrated ways to lay the foundation for teachers' responses to students' mathematical thinking. This idea has been used in much later research (e.g., M. Y. Lee & Francis, 2018). Following Jacobs et al.'s (2010) conceptualisation, we define teacher professional noticing as an integrated set of three key processes: (1) selectively *attending* to noteworthy students' strategies in particular instructional events; (2) *interpreting* students' understanding reflected in these strategies; and (3) *deciding intended responses* to students (as opposed to executing actual responses).

Research methods

Data were drawn from semi-structured interviews with twelve Chinese teachers. For a diversity of teacher profiles, these participants included four pre-service teachers, four novice teachers with an average of 2.75 years of teaching experience, and four experienced teachers with an average of 17.75 years of teaching experience in junior high school for students aged 12-15. They were recruited through convenience sampling.

The one-hour interviews were conducted online. The participants were presented with a set of classroom vignette episodes showing how students solved a geometric proof task (see Figure 1). These episodes were adapted from actual classroom scenarios reported in Komatsu et al.'s (2014) research on Lakatos-style proving, guided by the above-mentioned theoretical frameworks. They reflect the five phases of Lakatos-style proving and diverse students' understandings in a format of classroom discussion, trying to make the episodes sufficiently realistic to elicit teachers' responses in the context of Lakatos-style activity (Skilling & Stylianides, 2020). We chose to present teachers with comic-style episodes because Lakatos-style proving activity is believed to be scarce in existing Chinese classrooms. Also, comic-style episodes allowed us to provide participants with sufficiently realistic but also abstract enough information, aiming to direct their attention to critical aspects of the classroom practices of interest, and allow them to form their interpretations of such context (Herbst et al., 2011). Note that we did not study what teachers actually noticed in real classrooms. Instead, we attempted to explore what they might notice in the context of Lakatos-style proving activity.

Draw a line k that passes through a point A of square ABCD and passes through the inside of square ABCD. Draw lines BP and DQ perpendicular to line k from points B and D, respectively. What is the length relation between BP, DQ, and QP? Prove your conjecture.



(The teacher's figure attached to the task)

Figure 1: The geometric proof task (Komatsu et al., 2014) presented in the vignette episodes

Based on the above rationales, we designed eleven episodes around this proof task, among which there were four episodes about justifications and four episodes about refutations of the conjecture (i.e., irrespective of the position of line k, $PQ = DQ - BP$). These eight episodes respectively reflected each level of Justification and Refutation schemes, covering Phases 2-4 in the Lakatos-style proving process. Figure 2 shows two sample episodes translated from Chinese to English.

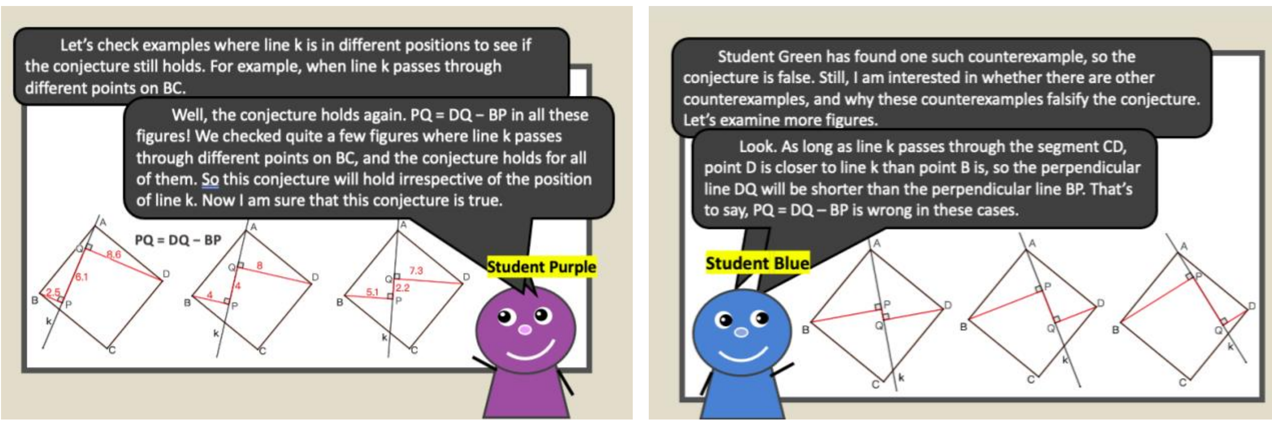


Figure 2: Crucial experiment justification (Left) and General counterexample refutation (Right)

After seeing each episode, participants were asked to describe (i) the students' thinking and/or actions that they attended to, (ii) how they interpreted the students' understandings, and (iii) how would they respond to the students, corresponding to the *Attending*, *Interpreting*, and *Deciding* aspects of teachers' professional noticing. Following Jacobs et al.'s (2010) coding scheme of teacher noticing, teachers' responses were coded based on the extent of evidence that teachers demonstrated in their consideration of students' reasoning. Specifically, responses about the *Attending* aspect were coded on a 2-point scale: Evidence (1) and Lack of evidence (0), and responses about the *Interpreting* and *Deciding* aspects were coded on a 3-point scale: Robust evidence (2), Limited evidence (1), and Lack of evidence (0) (Jacobs et al., 2010). After coding teachers' responses into the various categories, emerging themes were identified for each category to capture its characteristics (Corbin & Strauss, 1990). Double counting was applied for responses that related to more than one theme.

Findings

Overview of teachers' professional noticing

To capture participants' noticing of students' justifications and refutations, mean scores of the *Attending, Interpreting,* and *Deciding* aspects were calculated for each of 8 episodes. Given that the two lower levels and the two higher levels of Justification and Refutation schemes describe students' invalid and valid reasoning, respectively, we calculated the average scores of each pair of levels as a more stable measure of teachers' noticing of (in)valid justification/refutation arguments (Table 1).

Participants had on average higher scores in noticing students' justifications than refutations, across each aspect of teacher noticing – Attending (0.75 vs. 0.54), Interpreting (1.29 vs. 1.13), and Deciding (1.31 vs. 1.29). Although participants showed the same averages (1.29) in interpreting students' valid and invalid justifications, they showed higher averages in attending to (0.96 vs. 0.54) and deciding how to respond to (1.42 vs. 1.17) students' valid justifications than invalid justifications. A similar pattern emerged from participants noticing of students' refutations.

Table 1: Means (SD) for teaches' scores of noticing students' justification and refutation

Component skill (Scale)	Justification			Refutation		
	Invalid	Valid	Overall	Invalid	Valid	Overall
Attending (0-1)	0.54 (0.51)	0.96 (0.20)	0.75 (0.44)	0.46 (0.51)	0.63 (0.49)	0.54 (0.50)
Interpreting (0-2)	1.29 (0.55)	1.29 (0.46)	1.29 (0.50)	1.13 (0.68)	1.13 (0.54)	1.13 (0.61)
Deciding (0-2)	1.17 (0.56)	1.42 (0.65)	1.31 (0.59)	1.08 (0.65)	1.50 (0.66)	1.29 (0.68)

To supplement the average scores and give a more complete picture of teacher noticing, Table 2 shows the number of teacher responses which were coded as showing different extents of evidence.

Attending to students' strategies

Regarding the *Attending* aspect, responses with evidence mentioned exact mathematically important details of specific students' strategies, as described by our theoretical framework. Most participants gave evidence of attending to students' justifications, except to a student's Crucial experiment justification. Meanwhile, most participants demonstrated evidence of attending to Single counterexample refutation, but they did not demonstrate evidence of attending to other types of refutation. This may explain partly why participants had higher averages in attending to students' justifications, especially valid justifications, than other types of student reasoning.

Table 2: Number of teacher responses which were coded as showing the different extent of evidence

	Justification (Total: 48 teacher responses)		**Refutation** (Total: 48 teacher responses)	
	Invalid (Total: 24)	**Valid** (Total: 24)	**Invalid** (Total: 24)	**Valid** (Total: 24)
Attending	Evidence (N=13) Lack of evidence (N=11)	Evidence (N=23) Lack of evidence (N=1)	Evidence (N=11) Lack of evidence (N=13)	Evidence (N=15) Lack of evidence (N=9)
Interpreting	Robust evidence (N=8) Limited evidence (N=15) Lack of evidence (N=1)	Robust evidence (N=7) Limited evidence (N=17)	Robust evidence (N=7) Limited evidence (N=13) Lack of evidence (N=4)	Robust evidence (N=5) Limited evidence (N=17) Lack of evidence (N=2)
Deciding	Robust evidence (N=6) Limited evidence (N=16) Lack of evidence (N=2)	Robust evidence (N=12) Limited evidence (N=10) Lack of evidence (N=2)	Robust evidence (N=6) Limited evidence (N=14) Lack of evidence (N=4)	Robust evidence (N=13) Limited evidence (N=9) Lack of evidence (N=2)

Among the lack-of-evidence responses, some of them omitted mathematically important details of the students' strategies. This characteristic was evident in 8/12 teacher responses to a student's Crucial experiment justification. To illustrate, 7 teachers mentioned that this student tested *more examples* compared to another student who only tested one example, but they did not comment on the unique feature of this student's reasoning, that is, the *strategic selection* of examples. Some other lack-of-evidence responses included information that was inconsistent with the students' strategies. This was notable in teachers' responses to students' invalid refutations (9/24). For example, regarding the episode about Empirical refutation, 3 teachers wrongly noted that the student wanted to find *all* counterexamples, whereas the student just suggested finding *more* counterexamples.

Interpreting students' understandings

Concerning the *Interpreting* aspect, robust-evidence responses exactly described what the students did and did not understand (as suggested by our theoretical framework), citing details of the students' mathematically important strategies. Eight participants gave robust-evidence responses for interpreting a student's Naïve empirical justification. Yet, when interpreting other types of student arguments, no more than 4 participants provided robust evidence. This may partially explain teachers' slightly higher average scores in interpreting justifications (1.29) versus refutations (1.13).

The majority of limited-evidence responses omitted some details of students' (mis)understandings. This characteristic existed in more than half of teachers' responses, except those for interpreting Naïve empirical justification and Empirical refutation. For example, when interpreting a student's General counterexample refutation, a teacher expressed appreciation of this student's idea of comparing three counterexamples to examine the conjecture, but she did not elaborate on this idea.

Some limited-evidence responses contained descriptions inconsistent with specific student strategies, although they precisely mentioned some (mis)understandings of these students. To illustrate, commenting on a student's Empirical refutation, a teacher said this student's idea (i.e., finding more counterexamples so that we can be convinced) was acceptable in refutations, although she also mentioned that using one counterexample can already refute the conjecture. By contrast, another teacher misinterpreted that this student would like to test *all* counterexamples, and all descriptions this teacher provided were inconsistent with the student's strategy details. This was unclear whether this teacher understood the student's strategy, so this response was coded as "Lack of evidence".

Some teachers' limited-evidence responses contained descriptions as if based on teachers' assumptions rather than on the vignette provided. For example, a teacher assumed that the student with Naïve refutation ignored the counterexample because "this student was stubborn and was not willing to hear others' ideas", but this was hard to justify based on what the student said.

Deciding how to respond based on students' understanding

Around half of the teacher responses for valid justifications (12/24) and refutations (13/24) gave robust evidence when deciding how to respond to a student, while much fewer responses for invalid justifications (6/24) and refutations (6/24) showed robust evidence. In these robust-evidence responses, teachers demonstrated explicit consideration of the students' reasoning and how their proposed responses could further these students' thinking (as suggested by our theoretical framework). For example, in a robust-evidence response for a student's Crucial experiment justification, a teacher made use of one mathematically important aspect of this student's strategy (i.e., asking the student to analyse common properties of these strategically identified examples) to facilitate the student's progression from testing examples to giving a proof:

> Teacher: I will appreciate the student's spirit of exploration, and I will remind him these are only some examples…The student needs to learn how to analyse…whether there are common properties among different figures (i.e., examples). If there are, can we start proving this conclusion with geometric proof? He has drawn a lot of figures, and we need to find their common properties.

Limited-evidence responses demonstrated teachers' uses of students' reasoning in deciding how to respond but in a general or unproductive way. To illustrate, 4/12 teachers proposed very similar responses (e.g., simply asking students to give a proof) for students' Naïve empirical justification and Crucial experiment justification, even though both types of student reasoning indicated different levels of mathematical sophistication (as reflected in students' way of identifying examples to validate the conjecture). Some teachers suggested to the student who ignored the counterexample (Naive refutation) to find more counterexamples or accepted the idea of the student with Empirical refutation to find more counterexamples, in order to convince both students that this conjecture was refuted. But they did not remind the students that one counterexample can already refute the conjecture, and this was not conducive to students' development of refutation ability.

Unlike responses with robust or limited evidence, lack-of-evidence responses cited few or no details of the specific students' reasoning (e.g., "I will let the student think more and then examine based on his idea…I think it will be better if I allow students to inquire rather than directly telling them the answers"), leaving open the question of whether teachers consider these students' reasoning.

Discussion and Conclusion

To conclude, our results show that teachers may be less able to notice students' refutations than justifications. When the students in our vignettes gave invalid arguments (regardless of whether for justification or refutation), teachers were less likely to attend to and decide how to respond to their thinking. Noticing with non-robust-evidence may hinder teachers from making use of students' reasoning to engage students in Lakatos-style zig-zag processes of conjectures, proofs, and refutations.

For instance, many participants noticed that compared to the student with a Naïve empirical justification, the student with a Crucial experiment justification tested more examples. Yet, they did not point out that these examples were strategically identified. In other words, teachers may be sensitive to whether students' reasoning is example-based, but they may pay limited attention to and interpretation of students' ways of identifying supportive examples. Their neglect of such a mathematically important strategy may partly explain why some teachers suggested to both students a very similar next step (i.e., directly asking them to switch from example-based reasoning to giving a proof), without using each student's existing reasoning as a starting point for his/her further development. By contrast, like a teacher's robust-evidence response that we quoted above, one productive use of a student's Crucial experiment justification can be letting this student analyse the common properties of the strategically-identified examples for the further construction of a proof.

For students' refutations, despite the wide recognition of the role of a single counterexample in refutation, some teachers still allowed or even encouraged students who had a Naïve refutation or an Empirical refutation scheme to examine more counterexamples to confirm the conjecture was false. This may not be conducive to students' understandings of the minimally necessary and sufficient way of refutations. Yet if teachers can emphasise the role of a single counterexample and meanwhile support students' investigation of more counterexamples to find out their common properties that refute the conjecture (as described by the General counterexample refutation), students may have opportunities to experience the process of refining a conjecture based on analyses of counterexamples.

Overall, this study constructs a picture of teachers' professional noticing of students' thinking in justification and refutation in the context of Lakatos-style activity, which was seldom investigated in previous research. This can help us (as a field) to better understand in such context how teachers attend to and make sense of students' justifications and refutations in different ways, which in turn may shape what learning opportunities teachers offer to students in the follow-up refinement of the proof or the conjecture. To better prepare teachers to implement Lakatos-style proving activity, further training on their professional noticing, especially their noticing of students' refutations and invalid justifications, is needed. Finally, we acknowledge Sherin and Star's (2011) critique of research on teacher noticing that we indirectly learn what teachers notice from what they express in their comments, but the underlying mechanism of teacher noticing is still unclear. In a further study, we will try to unpack such a mechanism by considering how teachers' views (e.g., their views of proofs, teaching proof, and noticing) condition teachers to notice.

References

Balacheff, N. (1988). A study of students' proving processes at the junior high school level. *Second UCSMP International Conference on Mathematics Education*.

Corbin, J. M., & Strauss, A. (1990). Grounded theory research: Procedures, canons, and evaluative criteria. *Qualitative Sociology*, *13*(1), 3–21. https://doi.org/10.1007/BF00988593

Deslis, D., Stylianides, A. J., & Jamnik, M. (2021). Primary school teachers' mathematical knowledge for Lakatos-style proof instruction. In M. Inprasitha, N. Chansgri, & N. Boonsena (Eds.), *Proceedings of the 44th Conference of the International Group for the Psychology of Mathematics Education* (pp. 209–217). Khon Kaen, Thailand: PME.

Es, E. A. van, & Sherin, M. G. (2008). Mathematics teachers' "learning to notice" in the context of a video club. *Teaching and Teacher Education*, *24*(2), 244–276. https://doi.org/10.1016/j.tate.2006.11.005

Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. *American Mathematical Society*, *7*, 234–283. https://doi.org/10.1090/cbmath/007/07

Herbst, P., Chazan, D., Chia-Ling Chen, Chieu, V.-M., & Michael Weiss. (2011). Using comics-based representations of teaching, and technology, to bring practice to teacher education courses. *ZDM-Mathematics Education*, *43*(1), 91–103. https://doi.org/10.1007/s11858-010-0290-5

Jacobs, V. R., Lamb, L. L. C., & Philipp, R. A. (2010). Professional Noticing of Children's Mathematical Thinking. *Journal for Research in Mathematics Education*, *41*(2), 169–202. https://doi.org/10.5951/jresematheduc.41.2.0169

Komatsu, K. (2016). A framework for proofs and refutations in school mathematics: Increasing content by deductive guessing. *Educational Studies in Mathematics*, *92*(2), 147–162. https://doi.org/10.1007/s10649-015-9677-0

Komatsu, K., Tsujiyama, Y., & Sakamaki, A. (2014). Rethinking the discovery function of proof within the context of proofs and refutations. *International Journal of Mathematical Education in Science and Technology*, *45*(7), 1053–1067. https://doi.org/10.1080/0020739X.2014.902135

Lakatos, I. (1976). *Proofs and refutations: The logic of mathematical discovery*. Cambridge university press. https://doi.org/10.1017/CBO9781139171472

Lampert, M. (1990). When the Problem Is Not the Question and the Solution Is Not the Answer: Mathematical Knowing and Teaching. *American Educational Research Journal*, *27*(1), 29–63. https://doi.org/10.2307/1163068

Lee, K. (2016). Students' proof schemes for mathematical proving and disproving of propositions. *The Journal of Mathematical Behavior*, *41*, 26–44. https://doi.org/10.1016/j.jmathb.2015.11.005

Lee, M. Y., & Francis, D. C. (2018). Investigating the relationships among elementary teachers' perceptions of the use of students' thinking, their professional noticing skills, and their teaching practices. *The Journal of Mathematical Behavior*, *51*, 118–128. https://doi.org/10.1016/j.jmathb.2017.11.007

Reid, D. A. (2002). Conjectures and Refutations in Grade 5 Mathematics. *Journal for Research in Mathematics Education*, *33*(1), 5–29. https://doi.org/10.2307/749867

Sherin, B., & Star, J. R. (2011). Reflections on the study of teacher noticing. In *Mathematics teacher noticing* (pp. 96–108). Routledge.

Skilling, K., & Stylianides, G. J. (2020). Using vignettes in educational research: A framework for vignette construction. *International Journal of Research & Method in Education*, *43*(5), 541–556. https://doi.org/10.1080/1743727X.2019.1704243

Stylianides, G. J., & Stylianides, A. J. (2009). Facilitating the Transition from Empirical Arguments to Proof. *Journal for Research in Mathematics Education*, *40*(3), 314–352. https://doi.org/10.5951/jresematheduc.40.3.0314