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Swedish students' exploration of trigonometrical relationships: GeoGebra and protractors yield qualitatively different insights

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Trigonometry, an important pre-requisite for many advanced topics of school mathematics, links geometric, algebraic and graphical reasoning, but remains a difficult topic to teach and learn. The dynamic nature of many trigonometric functions is amenable to dynamic geometry software, which, in the form of GeoGebra, is the focus of this paper. However, both generally and in respect of trigonometry, research on GeoGebra's efficacy seems ambivalent. In this paper, we offer a case study of two groups of Swedish upper secondary students' solutions to the same tasks. One group was instructed to use GeoGebra and the other a protractor to investigate the sine and cosine functions in the interval $0^\circ \leq v \leq 180^\circ$. Analyses yielded qualitatively different outcomes; students using the protractor typically identified a geometrical relationship based on symmetry around the protractor's 90° line, while those using GeoGebra tended to identify only numerical relationships.

Keywords: Trigonometry, GeoGebra, Protractor, Sweden, Upper secondary school.

Introduction

For at least half a century, technophiles have asserted the transformative impact of various forms of computer software on the teaching and learning of mathematics. An early example can be seen in Papert's (1972) enthusiastic promotion of Turtle Geometry as a way of engaging young children with mathematics in ways that generate rather than replicate mathematics. Later, spreadsheets were presented as a means of introducing various aspects of both school mathematics (Healy & Sutherland, 1990) and university mathematics (Steward, 1994). More recently, attention has been focused on, inter alia, dynamic geometry software (DGS), particularly GeoGebra, the focus of this paper. Unfortunately, despite the undoubted quality and availability of the software, much GeoGebra-related research seems to have been underpinned by the technophiles' desire to 'prove' its educational efficacy and is often little more than instructions or tips for teachers to develop their own applets for teaching purposes (e.g., Little, 2011; Phan-Yamada & Yamada, 2012). Even when attempts have been made to establish a baseline understanding of its impact on learning, research has typically been equivocal in its outcomes, methodologically problematic or both. In the following, before introducing our study, we summarise these methodological problems.

GeoGebra and the teaching and learning mathematics

Our reading of the literature indicates an important distinction between the use of preprepared applets, designed to facilitate the exploration of the mathematics under scrutiny, and expectations that students construct their own applets. In respect of the former, which is more frequently reported than the latter, studies have reported on, for example, undergraduates' investigations of the definite integral (Tatar

& Zengin, 2016), circle theorems (Sigler et al., 2017) and Fermat and Steiner points (Flores & Park, 2018). Such studies, while supportive of the software's potential efficacy, seem not only to lack details of the ways in which such applets support learning but assume that such learning could not be achieved via other means. When control groups have been employed, as with studies of high school students' learning of functions (Zulnaidi et al., 2020) and exponential functions (Birgin & Yazici, 2020), researchers typically report higher achievement and deeper learning in the groups exposed to GeoGebra applets than the control groups. However, these studies seem not to be without problems, particularly with respect to details pertaining to the students' experiences. For example, Zulnaidi et al. (2020p. 54) wrote only that "students in the control group were taught using the traditional method whilst their counterparts in the treatment group utilised GeoGebra", while Birgin and Yazici (2020, p.5) wrote only that "the control group was taught by using textbook-based direct instruction". In other words, such studies seem problematic in their unarticulated assumptions about traditional teaching and the experiences of control group students.

GeoGebra and trigonometry

Understanding trigonometric functions is a pre-requisite for understanding many other topics in science and engineering, and as "one of the earliest mathematics topics that links algebraic, geometric, and graphical reasoning, it can serve as an important precursor towards understanding pre-calculus and calculus" (Weber, 2005, p.91). However, it is a difficult topic to teach, with approaches based on the right-angled triangle stressing procedural skills at the expense of any conceptual understanding of either sine or cosine as functions (Kendal & Stacey, 1997). In attempts to address such matters, a number of studies have exploited GeoGebra in the teaching of different aspects of trigonometry.

Unfortunately, much of this research is prone to the problems as discussed above. For example, Kepceoğlu and Yavuz (2016) investigated the teaching of periodicity of trigonometric functions with grade ten students. In one classroom, the teacher demonstrated by means of an applet, while in the other, the same teacher adopted a traditional exposition. The post-test found the experimental group performing better than the control, although no tests of statistical significance were used, and little detail was offered with respect to what students were invited to do in either classroom. In similar vein, Zengin et al. (2012) developed a five-week trigonometry course for high school students. A post-test found greater improvements in the experimental group than the control. However, beyond the implied use of applets, nothing was said with respect to the interventions, other than to assert that "GeoGebra prepared activities aimed to make the subject more dynamic, concrete and visual" (p. 185). By way of contrast, Mosese, and Ogbonnaya (2021) undertook a controlled experiment in which one group experienced GeoGebra-based applet led instruction, while the control group experienced, *inter alia*, a model of a rotating arm to underpin a series of lessons focused on basic trigonometric functions. In both instances, some detail was offered with respect to what students experienced, with the experimental group performing better on a post-test than the control. Finally, Nordlander's (2021) study drew on observations of upper secondary students working on the limit of $\sin\theta/\theta$. Her hope was that students would "explore, compare, and connect items leading to discovering relationships and learning through their own reflections and self-explaining" (p. 3). She found students' conceptual and procedural knowledge to be enhanced by the visualisations afforded by the software.

In conclusion, much GeoGebra-related research seems problematic, mainly because the tasks students receive are rarely adequately described. Even the authors of review papers seem unaware of such problems. For example, Chan and Leung's (2014) meta-analysis, despite identifying 428 articles, evaluated just nine that satisfied their selection criteria and, while acknowledging factors like length of intervention, student age and teacher role, concluded that DGS-based instruction had a significant impact on achievement in relation to traditional instruction. However, no attention was paid to the tasks students received or, importantly, what is meant by traditional instruction.

This study and its methods

Acknowledging what seems to be a significant gap in the literature, this paper presents an investigation of the impact of different 'technologies' on students' learning of trigonometry. It is conceptualised as an exploratory instrumental case study in which two student groups, one working with GeoGebra and the other with protractors, solve the same tasks. Exploratory case studies aim to identify hypotheses for further inquiry (Woodside & Wilson, 2010), while instrumental case studies aim to advance knowledge of the issue under scrutiny (Garner & Kaplan, 2019). The study is framed by the question

What differences emerge in upper secondary students' solutions to the same trigonometrical tasks when some students work with GeoGebra and others work with protractors?

The study involved 22 students in the second year of the Swedish upper secondary school's (17-18 years) natural science programme. Participants, who had previously encountered trigonometry only in relation to right-angled triangles, were randomly assigned to two groups, each split into pairs. Twelve students (6 pairs) worked with GeoGebra (G) and 10 (5 pairs) with the protractor (P). Each group completed the same tasks in, effectively, identical lessons of 60 minutes duration.

During the first part of each lesson, (ca 25-40 minutes), students undertook an investigative activity aimed at extending earlier trigonometric relationships beyond acute angles to arbitrary angles. During this time, both groups were allowed pocket calculators, paper and pencil, while group G uniquely used GeoGebra and group P uniquely used protractors and a unit circle drawn on paper. Next, participants in both groups worked through identical textbook-based tasks (ca 10-25 minutes), using whatever tools they preferred. Finally (ca 10 minutes), students solved a trigonometric equation, during which time they were allowed only a unit circle drawn on a paper, paper and pencil. At the beginning of the lesson, written instructions were distributed and afterwards the teacher clarified any misunderstandings for individual pairs. Throughout, students were asked to discuss their thinking, with each conversation being audio-recorded and every GeoGebra activity being digitally recorded.

In this paper, due to limitations of space, we focus on the results of students' work from the first and final phases of the lesson, which were structured by two tasks, presented in ways that would be amenable to either protractor or GeoGebra approaches.

Task 1, the introductory investigations, comprised several parts

- Draw a circle with radius 1 and its centre in the origin.
- Working from the centre of the circle O and the positive x -axis, draw an angle 125° and its corresponding radius OP.

- Note the coordinates for the point where the radius intersects the circle: the x -coordinate is the cosine value, the y -coordinate is the sine value.
- Find an angle for which $\sin v = 0.8$ in the interval
 - a) $0^\circ \leq v \leq 90^\circ$ b) $90^\circ \leq v \leq 180^\circ$
- Find all angles in the interval $0^\circ \leq v \leq 180^\circ$ for which $\sin v = 0.5$.
- Examine the results of the previous tasks and make a note of the relationship between them.

The solutions to the first part are a) $v \approx 53^\circ$ and b) $v \approx 127^\circ$, while those of the second are 30° and 150° . Finally, it was hoped that students would have noticed that the angles are symmetrically placed either side of 90° and that the sum of the angles is 180° .

Task 2 comprised one part, namely, determine, without using a calculator or digital tool, which angles in the interval $0^\circ \leq v \leq 180^\circ$ are solutions to $\sin v = \sin 56^\circ$. Its solutions are $v = 56^\circ$ and $v = 124^\circ$. During this time, it was hoped that students would exploit the relationship discovered during Task 1.

Data analysis

Analyses of qualitative data are typically either theory- or data-driven (Boyatzis, 1998). Our view is that the exploratory nature of this study is best served by the latter, as it privileges emergent insights that may be masked by the former (Andrews & Sayers, 2013). Consequently, participants' utterances and actions, including the GeoGebra digital recordings, were interpreted, and coded in ways that would expose similarities and differences in how students approached their tasks. For example, during their work on the investigative task, Epsilon pair (P), offered the following:

Epsilon (P): But it is 90 degrees. Or is it 180 minus that angle, is the same ... Look, it should be the same distance ... 90 degrees plus ... or ... Ok, check, ok, now, now I came up with it, check, if you have the angle here, you should add 90 degrees minus the angle you have here, because then you come here on the other side. So, if you have 90 degrees ... So, it's 0 plus the angle ... is the same sine value as 180 eh ... minus ... yes ... 0 minus the angle v .

In this excerpt, the various utterances indicate that the two students had identified a symmetrical relationship around the protractor's 90° line and were attempting to articulate a geometrical relationship. By way of contrast, during their work on the same task, one of the GeoGebra pairs, Gamma, suggested:

Gamma (G): So, 53 divided by 127 might be the same thing as... 30 divided by 180.

Such an utterance, focused on division, indicates little, if any, awareness of symmetry and, we infer, reflects a loosely formed interpretation of a proportional relationship. Later, continuing to work with numerical values, the same pair identified a relationship, although there remained no explicit evidence that they had noticed the symmetry embedded in the situation:

Gamma (G): Now I have found another connection, 53 degrees plus 127 degrees is 180 degrees and 150 plus 30 is 180. It is also a small connection.

Interestingly, their description of the relationship as 'small' indicates a view that they thought an additive relationship of such a form was, possibly, too trivial to be correct.

Results

In the following, we summarise the results for each of the two tasks respectively, highlighting important similarities and differences.

Task 1

Overall, ten of the 11 pairs identified a connection between the angles in Task 1. Five of these, four protractor pairs and one GeoGebra pair, discovered the expected symmetrical relationship. The protractor groups typically drew on expressions like ‘equally far’ or ‘mirror image’ to highlight the physical location of the angles in relation to the vertical associated with the 90° line of the protractor. With respect to the former, in addition to the comments made by Epsilon (P) above, another of the protractor pairs, Lambda, was heard to say:

Lambda (P): They are on each side of... both are equally as far away from 90 degrees... the line. If you take 90 minus those in the ... 90 ... 0 to 90 range. Yes. Then it will be as much as if you take... 90 plus ... Yes exactly. Between 180 and the 90 range. Yes... that is the connection. It is the same distance on both sides as well. Yes? The degrees are equally far from 90 degrees.

With respect to the latter, another protractor pair, Iota, seemed more explicit in their articulation of a symmetrical relationship. They said:

Iota (P): It looks like as if ... like a mirror image... Yes exactly. It's not plus 90 but ... but it is mirror-inverted, mirror-inverted ... Yes. Yes. So that it (the 90° line) is the line of symmetry.

In short, four of the five protractor pairs, drawing the physical characteristics of their given tool, were able to identify the expected symmetrical relationship.

By way of contrast, and in addition to the comments made by the Gamma pair discussed above, one of the Sigma (G), identified a numerical rather than geometrical relationship, was heard to say:

Sigma (G): Just that, if we add them, yes? Yes. Mm then it will be ... These, added to each other should always be 180. Yes. Is that the connection they want? I think that's the connection, it sounds like a nice connection.

Indeed, the utterance, “added to each other should always be 180”, seems to confirm that this pair was thinking numerically rather than geometrically. Moreover, the closing comment that “it sounds like a nice connection” indicates that they were not only content with their conclusion but that they were not expecting to think in anything but numerical terms.

Finally, in this section, one of the GeoGebra pairs, Omega, initially struggled to make sense of their results, as seen in their initially confused and confusing comment that:

Omega (G): Is it the same, em, number of degrees between here it is 120 degrees between this ... and this ... and what were the angles? Ok, no it is not, because then it is ... about 54 something, I think. And here it is ... no it is nothing, there is no connection, it seems...

The utterances “no it is not because then it is ... about 54 something” and “no it is nothing, there is no connection, it seems” indicate that the students of this pair were struggling to identify any form of

connection between the angles under scrutiny. However, after further thought, they shifted attention from the purely numerical to notice that:

Omega (G): I don't know if it's a connection but that they are just as far ... from here to there and from there, em, from the x -axis, yes. Yes exactly, it's true. And from the y -axis.

In such an utterance, despite their obvious hesitancy, this pair had come close to identifying and articulating the expected symmetrical result.

Task 2

Overall, nine of the 11 pairs, four protractor and five GeoGebra, were able to solve Task 2 and find the values $v = 56^\circ$ and $v = 124^\circ$ respectively. In most cases, the successful pairs exploited the result they had identified earlier. For example, drawing on the symmetrical relationship they had found previously, Iota were heard to say:

Iota (P): Do we have v at all?... Can't we just do our ninety tactics?... If we have... like one answer, we take the other. Yes? ...90 minus 56 is equal to 34. Yes? So, 34 plus 90, 124. Boom.

The somewhat rhetorical question, "Can't we just do our ninety tactics?" confirmed the pair's connection to their solution to Task 1. Afterwards, by performing "90 minus 56, is equal to 34... So, 34 plus 90, 124. Boom.", their task is solved efficiently and, it seems, with understanding.

However, not all protractor pairs made such an explicit connection between the two tasks. For example, uniquely among their colleagues, the protractor pair Epsilon solved Task 2 by drawing on a numerical interpretation of the symmetrical relationship they had discovered during Task 1. They began by repeating the task:

Epsilon (P): Determine without calculator which angles in the range are solutions to ... Ok. This one we know already; it is 180 minus 56. Yeah!

The utterance, "This one we know already, it is 180 minus 56", came shortly after they had read the task and shows how they connected the two tasks swiftly. Interestingly, their conversation, due to its brevity, indicates that they transformed the relationship they found in Task 1 to the process used here.

GeoGebra pairs using numerical approaches that had not found the symmetrical relationship at Task 1, solved Task 2 by working once again with purely numerical values, and, as expected, did not use symmetry when solving it. For example, the Gamma pair noted that:

Gamma (G): ... when it was sine, and it was 180 degrees, it was positive all the time.... So, sine 56 degrees ... must be the same as ... 180 minus 56 degrees what is ... Yes, 130, 124 ... it will be, because then the value is the same? Because it is always positive.

Finally, the Alfa pair connected the expectations of the task to their understanding that sine values were read-off the y -axis, before performing their calculation and commenting, with a degree of irony, that any connection had been obscured by the number of decimal places returned by the software.

Alfa (G): It's equal to 56 degrees? Yeah? Sine was the y -axis, right? Sine what x -axis ... y -axis. So, it should be 56 degrees.... Now, I do not understand anything. Wait, it's... So, first of all 180 minus 56. I was so confused, I just ... how are we going to do this? ... The connection is that there are unsatisfyingly many decimals.

Discussion

In this paper, acknowledging that trigonometry is a difficult topic to teach, we have examined the impact of two qualitatively different approaches, involving the same tasks, to the teaching of trigonometry to Swedish upper secondary students. The outcomes were unexpectedly different with, on the one hand, students using the protractor tending towards an awareness of a symmetrical relationship that allowed them to understand and exploit the sine of angles in the range 90-180 degrees, while, on the other hand, students using GeoGebra tending towards an awareness of a numerical relationship between such angles. Significantly, in contrast to Nordlander's (2021) study, there was little evidence of the G group's conceptual and procedural knowledge being enhanced by the visualisations afforded by the software. While further research would be necessary to establish the reasons for these differences, it is not improbable that the protractor scaffolds students' awareness of symmetry in ways that GeoGebra does not. Also, as noted by the Alfa pair of students, the decimal places offered by GeoGebra may interfere with students' interpretation and subsequent generalisation of their results, highlighting a need for teachers to ensure that the appropriate number of places is set.

In sum, the results indicate a need not only for further comparative studies but also for teachers to understand how different technologies may enable or hinder learning. If there are any generalisation to be inferred from a limited study such as this, they are, firstly, that teachers have a role in facilitating students' awareness of any teacher-expected outcomes and, secondly, if students are to make sense of their experiences with DGS, then the role of paper and pencil seems crucial (Komatsu & Jones, 2020); working without such tools may create unnecessary barriers.

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