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# Why introductory experiments on functional relationships should be qualitative to foster covariation

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*The covariation aspect of functional thinking (FT) is difficult for students but at the same time central for the development of a concept of function. Looking at high school mathematics, covariation is not in focus. Prevalent approaches put the well accessible correspondence aspect in foreground and support this imbalance with numeric processing. Although student experiments have proven to be a beneficial introduction to functional relationships, the measurements collection could contribute to this problem. In our pre-post-test intervention study, we contrast a numerical and a qualitative introductory learning environment to functional relationships with a combination of hands-on and digital experiments. Results (N = 332) show significant increases in functional thinking in both settings, but significantly higher gains in the qualitative setting. Effects of the constraints cooperation level and school form indicate possibly relevant influential factors on this lead in line with theory.*

*Keywords: covariation, digital technology, student experiments, concept of function.*

## Developing a concept of function

The concept of functions is a major concept and at the same time a major hurdle in mathematics at school. Hence a considerable amount of research has been dedicated to the teaching and learning of functions. This study tries to bring together several branches of evidence to a coherent approach to the concept of functions. Breidenbach et al. (1992) used the Action-Process-Object-Scheme (APOS) theory for a developmental perspective on students' conceptualization of functions. The action concept on the lowest level is limited to the assignment of single output values to an input. With the more generalized process concept students consider a functional relationship over a continuum, enabling the reflection on output variation corresponding to input variation. Finally, functions conceptualized as objects can be transformed and operated on. Students with an elaborate concept of functions are supposed to be able to use the action, process or object conception depending on the mathematical situation (Dubinsky & Wilson, 2013).

## Aspects of functional thinking

The developmental stages of APOS are in line with key elements of a function concept, that are described as aspects of functional thinking (FT) by Vollrath (1989) as follows: the *correspondence* of an element of the definition set to exactly one element of the set of values; the *covariation* of the dependent variable when the independent variable is varied and the final aspect, in which the function is considered as an *object*. Although with the APOS perspective one might deduce a teaching sequence with an initial focus on correspondence, then covariation and finally object, current research advocates for a major role of covariation. Thompson and Carlson (2017) argue that the correspondence aspect alone does not evoke an intellectual need for the new concept function and difficulties with functional relationships are mainly rooted in lacking ability and opportunity to reason

covariationally. Johnson (2015) points out that correspondence induces a static view on a functional relationship, while a dynamic perspective is a prerequisite for covariation and a process concept. These arguments lead to the call for a qualitative approach to functional relationships in school.

### **Experimenting fosters functional thinking**

Learning environments with experimentation activities have proven to be beneficial for functional thinking (Lichti & Roth, 2018). One possible explanation could be the proximity of functional thinking to scientific experiments as illustrated by Doorman et al. (2012): with a given variable as starting point, a dependent variable is generated in an experiment. Relating the output to the input clearly addresses the correspondence aspect and the action concept. Following manipulations of the input and concurrent observation of the output make the covariation of both variables tangible and enables a process view. Another benefit of student experiment is the inherent constructivist learning approach that leads to higher learning gains in combination with digital technologies (Drijvers, 2019).

Lichti and Roth (2018) implement the scientific experimentation process – preparation (generate hypotheses), experimentation (test the hypotheses) and post-process (reflect results) – in a comparative intervention study to foster functional thinking of sixth graders with either hands-on material or simulations and report learning gains for both approaches (ibid.), but a closer look reveals disparities that can be explained with the instrumental genesis.

### **Hands-on experiments and simulations in the light of instrumental genesis**

The instrumental approach (Rabardel, 2002) and its distinction between artefact and instrument can be useful when interpreting these results: while the artefact is the object used as a tool, the instrument consists of the artefact and a corresponding utilization scheme that must be developed. This developmental process - the so-called instrumental genesis - depends on the subject, the artefact and the task in which the instrument is used. Hence, different artefacts lead to different schemes. Artefacts that are more suitable for the intended mathematical practice of a task appear to be more productive for the instrumental genesis and facilitate the learning process (Drijvers, 2019). When using simulations, schemes that develop are dynamic and concerned with variation as well as transition and hence support the covariation aspect (Lichti, 2019). Measurement procedures of the hands-on material induce static schemes for values and conditions, fostering the correspondence aspect (ibid.). While hands-on material stimulates basic modelling schemes, relating the situation to mathematical description, a simulation already contains a model of the situation. When used as multi-representational systems, the simulation illustrates connections between model and mathematical representations (e.g. graph and table) that evoke schemes for these representations and their transfer. The study presented here attempts to make use of both beneficial influences on the instrumental genesis through an appropriate combination of hands-on material and simulations in experimental activities to foster functional thinking.

### **Fostering the conceptual development**

The measurement procedure is laborious, giving it a dominant role, which sets a focus on correspondence and induces static view on the relationship. As stated above, it would be desirable to shift to a dynamic view, a process concept and covariation. Thus, we explicitly developed a non-

numerical approach for experimenting with immediate examination of covariation and compared this qualitative setting to a numerical one, following the implementation from Lichti and Roth (2018).

### The learning environments

Both settings use a story of two friends preparing to build a treehouse and contain identical overarching tasks. The contexts are implemented with the same hands-on material and simulations (see Figure 1 and 2), but different components of the simulations are visible in the settings. The student activities are structured in six contexts (see below for details), each one laid out like a scientific experimentation process with preparation, experimentation and post-processing phase. The students work in pairs (A and B), each working on three contexts (see Figure 1). The contexts are chosen to represent a linear and a quadratic relationship and one with varying change rate.

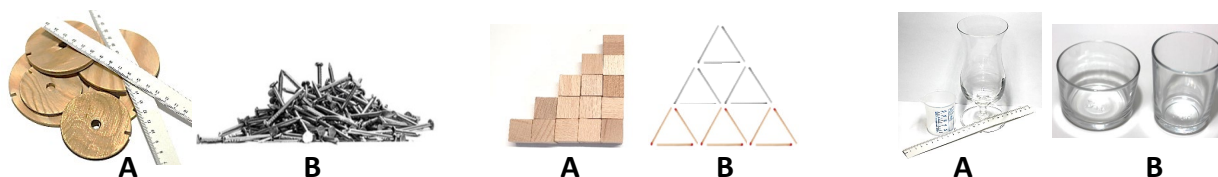


Figure 1: Hands-on material of the contexts for partner A and B

For partner A these are: the perimeter of a circular disc determined by its diameter, the number of cubes needed for a “staircase” determined by the number of steps and the fill height of a vessel determined by the volume of water filled into. Partner B examines the weight of a package of nails determined by the number of nails, the number of beams needed for a woodwork determined by the number of floors and the fill height of cylindric vessels with different diameters determined by the volume of water filled into. A bonus context for quick learning teams depicts the diameter of an unrolling tape determined by the length of tape that has been unrolled.

The **numerical setting** follows the scientific experimentation process: after initial hypotheses in the preparation phase, inspect hands-on material and estimate value pairs, students take a series of measurements and record their data in a table within a simulation (GeoGebra), which creates a graph from the data. The simulation also contains a model of the hands-on material, enabling systematic variation and parallel observation of the altering quantities in model and graph. In the post-processing phase the students verify their measurements, analyze the graph (interpreting and interpolating) and get back to the real material to check their estimations from preparation phase. The learners go through these phases for three contexts subsequently (see above), share their insight after each context with a partner and solve overarching tasks for each context as team.

In the **qualitative setting** the students also start off with hands-on material to activate modelling schemes and enable embodied experience. They are asked to make assumptions about a pattern and on that basis estimate subsequent values. With the aid of a simulation, where they can manipulate a model of the hands-on material, the students get a dynamic view of the relationship and are asked to identify the related quantities, which concludes the preparation phase. In the following experimentation phase students observe the variation and covariation of the quantities in the simulations and verbally describe the relationships discovered. Subsequently graphs are generated within the simulations to enable observing the covariation in multiple representations and in the post-

processing phase students are asked to analyze the form of the graphs and connect their insights with the relationship described in the previous phase. The students then team up with their partner, compare both contexts and identify similarities in the relations. In an additional experimentation phase, they take measurements in the context of their partner, represent the covariation in the measurement table and compare this to the results reported by their partner. As a final task the partners are asked to group the contexts by the kind of covariation, i.e. build pairs of similar contexts based on their findings. Both settings can be accessed in digital classrooms<sup>1</sup>.

## Study Design

A comparative intervention study (pre-post design) is implemented both as in-classroom and as home schooling with seventh and eighth graders at grammar and comprehensive schools. It contrasts the qualitative and numerical settings and includes an additional control group with the simulation only implementation of Lichti and Roth (see above). In a subsample the settings are laid out as individual learning paths, i.e. without team phases. The intervention is designed for six lessons (split into three sessions). It is preceded and followed by a short test on functional thinking (FT-short<sup>2</sup>), to compare the learning outcomes in both settings. Students work in teams of two pairs (except the individual work subsample). A pilot study (ibid.) verified the comparability of the two settings in terms of processing time and difficulty. With this layout we aim to answer the following research questions:

RQ 1: Which setting is more beneficial for FT?

RQ 2: Is the combination of hands-on material and simulations more effective regarding FT than the setting with simulations only?

RQ 3: Do the systematic constraints cooperation level (individual/team) and school form (grammar/comprehensive) have an impact on the learning gains in the compared settings?

## Method

Data analysis was conducted according to Item Response Theory. The dichotomous one-dimensional Rasch model and the virtual persons approach were used to estimate an item difficulty for every item of FT-short. The person ability was then estimated with fixed item difficulties. We applied mixed ANOVAs (between factors: setting, school form, teaching mode, cooperation level; within factor: time) after controlling data for normal distribution and homogeneity of variance. Pairwise t-tests were used to investigate differences of the settings. Due to the corona restrictions the distribution of the sample on the different constraints is somewhat imbalanced. For the mixed ANOVA of cooperation level, a subsample was selected out of the team sample and parallelized by pre-test (see values in brackets in Table 1 for team sample sizes).

A statistical power analysis (3 groups, 2 measurements, power .9,  $\alpha = .05$ ) for a medium effect ( $\eta_p^2 = .06$ ) in a mixed ANOVA gave a desired sample size of 204.

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<sup>1</sup> [www.geogebra.org/classroom](http://www.geogebra.org/classroom) numerical Setting: HQX7 UZRQ and covariational Setting: D3XM DDSB

<sup>2</sup> Rasch-scalable, 27 items, see Digel & Roth, 2020, online version of FT-short: [www.geogebra.org/m/undht8rb](http://www.geogebra.org/m/undht8rb)

## Results

Here we present quantitative results of the main study ( $N = 332$ , 121 female, 187 male, age  $M = 13.0$ ,  $SD = 4.8$ ). The distribution of the sample over the settings and constraints is shown in table 1.

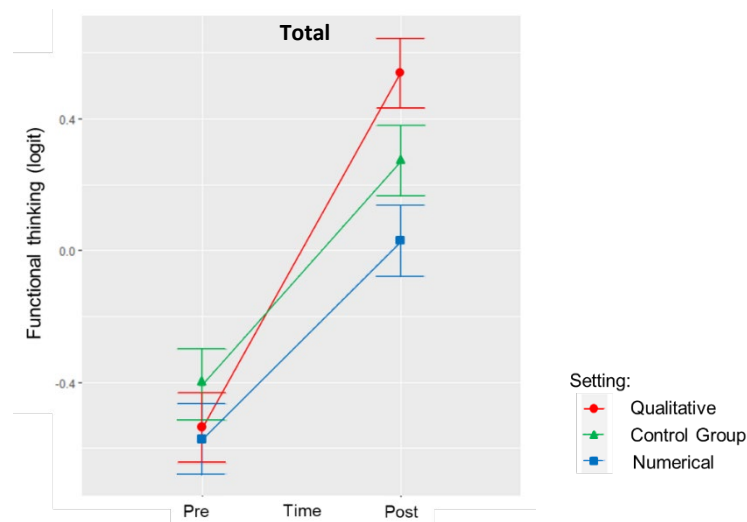
**Table 1: Data sample sizes of subgroups**

	Numerical Setting	Qualitative Setting	Control Group
Total	125	114	93
Comprehensive / Grammar	52 / 73	39 / 75	27 / 66
Individual work / Team	20 / 20 (105)	18 / 18 (96)	16 / 16 (77)

The estimation of the Rasch-model, used to determine the person abilities for the total sample, showed good reliabilities in the pre- and post-test:  $EAP-Rel_{pre} = .86$  and  $EAP-Rel_{post} = .80$  as well as  $WLE-Rel_{pre} = .85$  and  $WLE-Rel_{post} = .80$ .

### Comparison of the settings in total

The mixed ANOVA (see Figure 2) resulted in two significant and one minor significant effects: first, there was a significant main effect for time  $F(1, 329) = 188.17$ ,  $p < .001$ ,  $\eta_p^2 = .36$ . The results in FT-short for the total sample (numerical, qualitative and control setting together) increased significantly with a large effect from  $M = -.46$  logits ( $SD = 1.37$ ) up to  $M = .26$  logits ( $SD = 1.01$ ). Second, there was a minor significant main effect for setting  $F(1, 329) = 256.34$ ,  $p < .01$ ,  $\eta_p^2 = .04$ . The subsamples of both treatment groups (numeric/qualitative) did not differ before the intervention ( $t(198) = -.18$ ,  $p = .571$ ), but they did afterwards ( $t(198) = .26$ ,  $p < .001$ ,  $d = .32$ ) and both together did not differ from the control group before the intervention ( $t(134) = -.78$ ,  $p = .219$ ).



**Figure 2: Increase in FT pre to post by setting**

Results in all three settings increased significantly from pre- to post-test (see Table 2). The mixed ANOVA also showed a significant interaction between time and setting ( $F(2, 329) = 5.33$ ,  $p = .005$ ,  $\eta_p^2 = .03$ ) with a small effect. Due to limited space, the results of the

following ANOVAs are only reported briefly. If not stated otherwise, the remaining main and interaction effects were not significant.

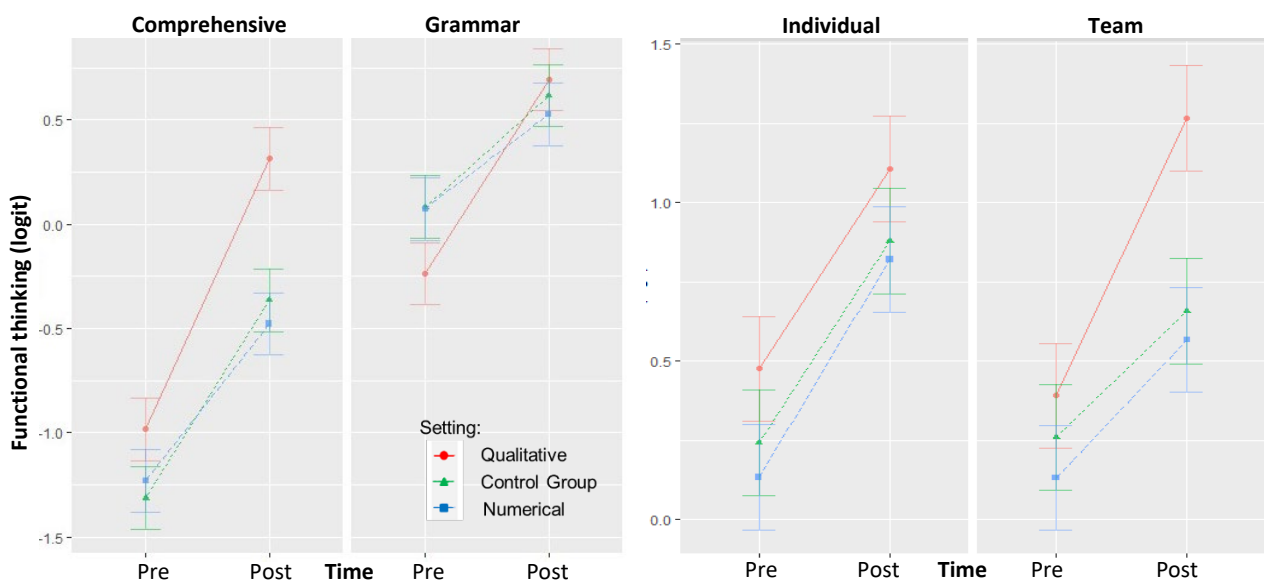
**Table 2: Learning Gains pre to post in subgroups per setting**

Reported are effect sizes (Cohens' d) with significance level \*\*\* ( $p < .001$ ) if not stated otherwise

	Numerical Setting	Qualitative Setting	Control Group
Total	.25	.51	.27
Grammar / Comprehensive	.27 / .32	.48 / .63	.28 / .34
Individual work / Team	.37** / .25*	.37** / .76	.26* / .23*

### Comparisons of the settings under constraints

Regarding the **school form** (see Figure 3 left) the mixed ANOVA showed a significant main effect for time ( $F(1, 326) = 197.34, p < .001, \eta_p^2 = .38$ ) and a significant effect of school form ( $F(1, 326) = 87.82, p < .001, \eta_p^2 = .21$ ). Above, there are two significant interaction effects: between time and setting ( $F(2, 326) = 5.92, p < .005, \eta_p^2 = .018$ ) and between time and school form ( $F(2, 326) = 9.57, p < .005, \eta_p^2 = .029$ ). The grammar school students outperformed the comprehensive school students in the pretest significantly ( $t(174) = 8.09, p < .001, d = .61$ ), but for both school forms students' ability increased significantly with a small to medium effect (grammar:  $t(425) = 7.08, p < .001, d = .34$ ; comprehensive:  $t(216) = 5.84, p < .001, d = .40$ ). In both school forms students in the qualitative settings showed the highest learning gains (see Table 2).



**Figure 3: Increase in FT pre/post by setting & school form (left) / by setting & cooperation level (right)**

The mixed ANOVA for **cooperation level** (see Figure 3 right) resulted in a significant main effect for time ( $F(1, 102) = 79.38, p < .001, \eta_p^2 = .44$ ) only and no significant interaction effects. This subsample is part of the grammar school sample (high abilities in the pretest with  $M = .36$  logits and  $SD = .87$ ). The effect sizes of the learning gains are reported in Table 2.

## Discussion

One of the major restrictions are the unbalanced subgroups, caused by requested flexibility towards the participating schools due to the pandemic restrictions. Above the results are not generalizable without reservation, since they depend on the concrete settings developed in the study. Nonetheless, the results show a significant increase of FT in the numerical (small effect  $d = .25$ ) and the qualitative settings (medium effect  $d = .51$ ), as well as the control group (small effect  $d = .27$ ), from pre- to post-test. Hence all three approaches are suitable to foster functional thinking of seventh and eighth graders. The learning effect for FT in the qualitative setting is significantly higher (small interaction effect of time and setting  $\eta_p^2 = .03$ ). Thus, we can conclude that the qualitative approach with a focus on covariation seems to be more beneficial for functional thinking than the other two (RQ1).

Since all three approaches in this study use identical (in case of the control group similar) simulations and contexts, it seems that the specific sequence and focusing of the tasks are decisive. Referring to our theoretical background, we consider two characteristics of the qualitative setting as influential aspects: first, the early focus on the dynamics of the observed variables in the qualitative approach provide opportunities to reason variationally and to develop a dynamic view on functions. Second, the shift of the measurement procedure to a very late step might also contribute to this view. We can assume that replacing early measurement with investigation and observation of the relationship initiates practice in covariational reasoning.

The learning effects in the numerical setting and the control group do not differ significantly as opposed to the qualitative setting. Regarding RQ2 we assert that the combination of hands-on material and simulations, as laid out in the qualitative and numerical setting, only lead to higher learning gains for FT (compared to the control group with simulations only), when the combination is embedded in a qualitative approach. From the perspective of instrumental genesis, we might conclude that the utilization schemes developed with hands-on material could have conflicting influences on FT. For instance, modelling schemes could be beneficial by facilitating the identification of independent and dependent variables, while schemes developed when investigating values and conditions of the hands-on material could hinder a dynamic view.

Regarding RQ3 the significantly different FT results of grammar and comprehensive school students in the pretest ( $d = 0.61$ ) are in line with PISA results reported by Reinhold et al. (2019). But the medium learning effect in the qualitative setting for comprehensive school students indicate that the covariational focus is also accessible to lower levels of FT and not restricted to high achievers. Since the sample size does not match the power analysis, especially the results regarding the cooperation level must be handled with caution and need to be verified. The contrast of comparable learning gains for all three settings in the subgroup “Individual” and higher learning gains in the subgroup “Team” for the qualitative setting might allude to the importance of the team discussion phases, only present in the “Team” subgroup. They might represent the opportunities for co-/variational reasoning, Thompson and Carlson (2017) call for.

To sum up, a qualitative approach to the concept of function with experiments (1) attains higher learning gains across competence levels, (2) makes the covariational aspect accessible for high and low achievers and (3) benefits from the combination of hands-on material and simulations, when (4)



opportunities to reason covariationally are included. In classroom practice, an approach to functions accommodating these aspects has the potential to enhance learning gains.

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