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# How do students describe and understand properties of special quadrilaterals with digital tools? – An epistemological perspective on mathematical interaction in inclusive settings

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*Both digital learning and inclusive learning become of increasing relevance in mathematics education research and practice. Still, the opportunities the first brings for the latter are underexplored, especially in secondary education. This paper presents a digital learning environment on quadrilaterals as considering design principles for inclusive learning, and its implementation. Through an epistemological analysis of a collaborative activity involving students with different learning abilities, the paper reconstructs how mathematical meaning evolves in signs, focusing in particular on the role of the digital tool in use.*

*Keywords: digital learning environment, geometry, inclusive settings, epistemological analysis*

## Introduction and theoretical foundations

Inclusive education and equity play an important role within mathematics education research and practice (e.g., Scherer et al., 2016). One approach addressing the heterogeneity in such settings is geared towards allowing students to work on the same basic mathematical idea in different ways and through different, student-led, approaches, following the idea of natural differentiation (Scherer & Krauthausen, 2010). However, little is known about how a mutual mathematical understanding of the idea actually develops in interaction among students with different learning abilities, especially in secondary mathematics classrooms. This paper presents a case study that adopts an epistemological perspective to better understand such a process of meaning-making between two students with and without special educational support. In particular, the case explores the role of a digital learning opportunity designed to facilitate the exploration and negotiation of concave quadrilaterals and their properties. Hence, it draws attention to the potential of digital learning tools in inclusive settings as they become suggested in this specific case.

We will first elaborate on the notion of inclusive education and present an epistemological perspective for reconstructing students' processes of constructing meaning as developing in interaction through and with signs, together with the epistemological triangle as its analytical tool (Steinbring, 2006). We will then describe the methodological background of the study, including the mathematical content in focus, the design of the digital learning opportunity, its empirical implementation, and methods of data collection and analysis. From the then following case analysis, we will draw tentative conclusions on the two research questions:

1. How can mathematical concepts evolve in interactions among students with different learning abilities through the interpretation and the use of signs?
2. Which role can the digital tool play within such an interaction process?

## Inclusive learning

Inclusive learning follows the idea of fostering all students with and without different learning conditions, taking into account the vast spectrum of abilities and needs (e.g., difficulties or special talents) encountered in the mathematics classroom. This understanding follows Article 24 of the Convention on the Rights of Persons with Disabilities, which claims equal opportunities for all people to participate in quality education and to develop their potential, regardless of special learning needs, gender, social and economic conditions (cf. United Nations, 2006). In inclusive learning, children should be supported individually and also learn collectively through collaborations and interactions (cf. Häsel-Weide & Nührenbörger, 2013; Scherer et al., 2016; Swidan & Daher, 2019). All students should get the same learning opportunities, but with individual learning aims. When implementing inclusive learning, enabling social participation and the explicit demand for social involvement should be taken into account. One challenge is to deal with heterogeneity and to develop learning environments accessible to all, while at the same time requiring social participation in collaborative learning phases. There is considerable need for research and development, particularly at secondary level. Didactic concepts suitable for inclusive teaching already exist for the primary level, such as *natural differentiation* through substantial learning environments (Scherer & Krauthausen, 2010). Although there are experiences of using digital tools in special education (Moyer-Packenham & Suh, 2012; Peltenburg et al., 2013), their use is often limited to learning and practice software for targeted training. In contrast, we understand digital tools more broadly as appropriate learning tools in the sense of enabling substantial mathematical experiences (Drijvers et al., 2016), acknowledging their potential for inclusive mathematics education. We therefore adapt the idea of ‘natural differentiation’ for the design of learning environments at the secondary level, taking into account the opportunities offered by digital tools.

## Epistemological perspective

Children's mathematical knowledge is situational and linked to certain learning contexts and experiences. These learning and experiential contexts can be extended through interactions by constantly negotiating and interpreting mathematical understandings (Steinbring, 2006). The epistemological triangle (see Figure 1) can be used for analyzing such processes of meaning-making in interaction as processes of learning mathematics. It is "used for modeling the nature of the (invisible) mathematical knowledge by means of representing relations and structures the learner constructs during the interaction" (p. 136). Central to this model is the interplay between *signs* that are interpreted, the *concept* about which the learner constructs knowledge as understood to be represented in some way in the sign, and the *reference context* corresponding to the meaning of the concept through the interpretation of the signs. Figure 1 shows a pictorial rectangle as a sign, which doesn't have a meaning of its own. The meaning has to be produced by the learner by means of establishing a mediation to suitable reference contexts.

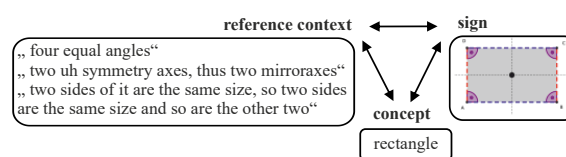


Figure 1: Exemplary epistemological triangle (adapted from Steinbring, 2006, p. 135)

In this case, the student describes the properties of the rectangle that represent the students' reference context, thereby establishing the meaning of the concept. This analytical tool allows a fine-grained reconstruction of evolving mathematical meaning in interaction processes. Therefore, the development of sign-use, the underlying reference contexts and the concepts involved can be analyzed in their synergy (cf. Steinbring, 2006).

## **Methods and methodology – from design to analysis**

### **Mathematical background and didactical approaches**

Quadrilaterals are characterized by their four sides and four angles and can be classified through hierarchy (De Villiers, 1994). Sub-concepts have the characteristics of the general concept and additional differentiating characteristics. A parallelogram has a center of rotational symmetry and opposite sides are parallel to each other, with sub-concepts (rectangle, rhombus, square) sharing these characteristics, thereby being considered special parallelograms. Conceptual understanding of quadrilaterals contains the understanding of a conceptual hierarchy in the sense of realizing ideas and sub-concepts *included* in the same concept (extensional definition of the concept), while also understanding the entirety of properties characterizing the concept and its sub-concepts (intensional definition of the concept) in order to *exclude* hierarchical relationships. Also, ideas like the operational principle consider actions crucial for conceptual development, with these being performed with real, visual and imagined objects, and described verbally. Designing a learning environment then means to offer all learners opportunities to openly approach and manipulate, such that properties can be explored on individually selected quadrilaterals.

### **The digital learning environment, the activities on it, and underlying design principles**

The purpose of the digital learning environment 'dynamic quadrilateral' is to provide opportunities to explore and classify the properties of quadrilaterals (Bebernik & Schacht, 2019; <https://www.geogebra.org/m/x7syhqny>). It mimics activities manually carried out on GeoBoards, realized digitally through dragging corners or sides with the finger on a touchscreen or with a mouse. In this specific design, properties of the quadrilaterals - such as right angles and axis of reflection - are highlighted by specific markers. It thereby allows students to explore and order quadrilaterals by means of their properties locally through operative manipulations and visual feedback. The tasks focus on convex quadrilaterals, which will appear grey in the digital learning environment. Functioning as an example generator, the digital tool allows for exploring quadrilateral properties, to make discoveries, and to order different types of quadrilaterals (e.g., square and rectangle) locally by identifying similarities and differences. To do this, it is first necessary to ascribe meaning to the markings, such as interpreting equally colored angles as angles of equal size. It is about understanding logical structures of special quadrilaterals rather than naming the quadrilateral. The digital learning environment is first worked on individually, along with prompts to facilitate systematization of what is found out. Later, students work in pairs on a reconstruction task (Wollring, 2012) in which each of them has a specific role. In the setting shown in Figure 2, one person (sender) manipulates a particular quadrilateral with the digital tool and then describes its properties. The partner (receiver) now has the task of reconstructing the quadrilateral with the digital tool. The pair changes roles several times.



- Assemble in pairs so that you can't see each other.
- Person 1: Create with the file 'dynamic quadrilateral' a particularly beautiful quadrilateral and describe its properties.
- Person 2: Create with the file 'dynamic quadrilateral' the described quadrilateral.
- Compare your quadrilaterals. Which one is it? Was the description good?
- Change the roles!

**Figure 2: Reconstruction task (Bebernik & Schacht, 2019; Wollring, 2012)**

The design of the digital learning environment and the accompanying tasks is informed by three main design principles specifically aligned to inclusive education. First, the already presented ***principle of natural differentiation***: In the digital learning environment, the conceptual understanding of individual quadrilaterals can be worked on at different levels – some children are able to interpret specific properties, other children can investigate similarities and differences between individual quadrilaterals. Free exploration and allowing different approaches to deal and engage with the content enable different learning objectives (from pre-formal explorations to formalizations). Second, the ***principle of representation networking***: In the sense of Duval (2006, p. 125), this involves operational variations in connection with different representations, considering *treatment* as transformation within one register of representation (e.g., within the geometric representation) or *conversion* from one register of representation to another one (e.g., the change from the geometric to the symbolic representation), whereby Duval considers the latter as being crucial for mathematical activity and learning (p. 125). With regard to the designed learning environment, the children should perform both treatment and conversion. For example, through the reconstruction tasks, the transition to symbolic representation or verbal description can be made by describing a quadrilateral. A third design principle concerns the relevance of ***integrating individual and cooperative learning phases***: Conceptual understanding is built through interactions with others, as interpretations of concepts are negotiated and constructed together. This is realized through the reconstruction task that stimulated exchange about quadrilateral properties, thereby enabling social participation, but also demanding for social involvement.

### **Data collection and analysis**

The data presented in this contribution has been drawn from a larger design research project on developing and researching substantial learning environments for inclusive classrooms. Following individual explorations with the digital learning environment, seven pairs of students (age 13-16) have been videographed while carrying out reconstruction tasks, stimulating the negotiation of mathematical understanding of the quadrilaterals and initiating social learning. The students had different learning conditions (with/without special educational needs) and attended different grades (5 to 8). In each pair, both students attended the same class. The students' interactions and the development of mathematical meaning are reconstructed using the epistemological triangle model (Steinbring, 2006). This paper provides an illustrative case study that allows to explore the research questions, providing rich conditions for reconstructing students' reference contexts through their use of signs while exploiting the affordances of the digital learning environment.

### **Results: Reconstruction of the learning processes**

The results presented in this section especially address the reconstruction of an interaction process emerging while using a digital tool. Here, the students **Sam** (sender) and **Ron** (receiver) (7th grade,

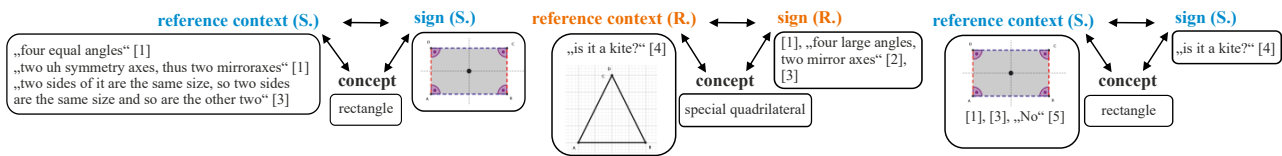
age 13-14) work on the reconstruction task as described above. Ron gets special educational support at school, he has been diagnosed with learning difficulties that encompass both math and written language. Writing and reading are difficult for him. In contrast, Sam has good mathematical and linguistic skills. This combination mirrors a prototypical inclusive learning setting. In the following transcript (translated from German), the two students use the dynamic geometry software GeoGebra to carry out the reconstruction task. Due to space limitations, the transcript only includes verbal interaction without a description of gestures and actions with the DGS.

- 1 S.: So, my, my shape has four equal angles. (...) And (...) two uh symmetry axes, thus two mirror axes.  
 2 R.: Wait, four large angles, two mirror axes.  
 3 S.: But only two sides of it are the same size, so two sides are the same size and so are the other two.  
 4 R.: Ah, I think, is it a kite?  
 5 S.: No.  
 6 R.: (...) Ok. It has two mirror axes and one rotation axis, right? Or doesn't it have a rotation axis?  
 7 S.: No, um, just two mirror axes and, well, two symmetry axes and a centre of symmetry.  
 8 R.: Ok, so yes. (.) That's what I mean by the rotation axis. Um, that should be the same (*manipulates the quadrilateral*). Trapezium?  
 9 S.: No.  
 10 R.: (...) So (...) hmm (14 sec.). Is that a, a (...), a, what's it called? (...) Uh a question, may I please have my two sheets back where I uh, or where the things are on (...). So where here, where you can read the names, of the angles, of the different quadrilateral patterns. (...) I can't remember the name.  
 11 I.: That's a rectangle. What you see there.  
 12 R.: Oh, ok. Is it a rectangle?  
 13 S.: Yes. (...) not like that, but it's also a rectangle.

The following analysis (1) first describes the specific structure of this interaction between the two students. In this context, we especially focus on the role of the digital tool. Secondly (2), an example of the development of reference contexts will be discussed. This will give insights into the conceptual development within a certain learning period based on the activities within the reconstruction tasks. A third analysis (3) will show the important distinction between sign-vehicle (*signifier*) and the underlying concept (*signified*) within a situation, in which a student uses the properties of a certain concept in order to determine the corresponding sign. Hence, the results presented in this section not only give insights into conceptual processes within the interaction, they also describe a process that starts with conceptual geometric activities with digital tools and leads to ways of signifying the underlying concept through verbal description.

### (1) *Evolution of meaning during interaction*

The following analysis shows how mathematical meaning can evolve during an interaction when working on reconstruction tasks. In a first step, Sam describes (to Ron) certain properties of the quadrilateral he has configured with his DGS (Figure 3 (left), [1] & [3]). For Ron these descriptions are signs to be interpreted first (Figure 3, in the middle is the ep. triangle of Ron being an interpreting agent). He partly repeats the described properties ([2]) and manipulates a quadrilateral with his DGS in order to find a quadrilateral that fits to the given signs by Sam. After a while, he has (supposedly) determined the corresponding configuration ( $\Delta$ , in process) and he asks: "is it a kite?" ([4]). Sam receives this question, which is now his sign, that he has to interpret (Figure 3 (right)). He compares the given sign (kite) with the properties of his initial quadrilateral. Because, for Sam, the underlying concept is the rectangle, he says "No" ([5]).

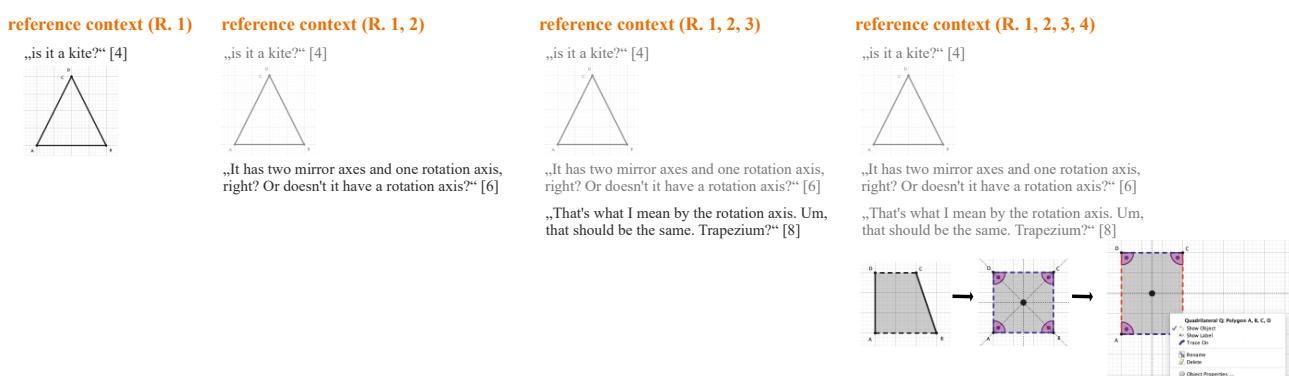


**Figure 3: Epistemological triangles - Evolvement of meaning during interaction**

The following interaction pattern can be identified here: First, a reference context (the description) is produced by Sam in the light of the sign (the geometric configuration) and the underlying concept. The description formulated by Sam is – in the next step – the sign to be interpreted by Ron. Ron produces a reference context and tries to find the matching quadrilateral with their own geometric configuration by using DGS. In the following reaction by Ron (in this case the question: “Is it a kite?”), Sam compares this sign to his initial situation (the geometric configuration and his properties) and formulates an answer (in this case: “No”). This pattern shows how the reference contexts produced by one of the two partners have to be interpreted by the second partner as a sign. It also shows that within this process of interpreting signs, producing reference context (descriptions) and integrating signs into one’s referential context, conceptual development can be observed in which properties are progressively made explicit by the students in the course of interaction. In this situation, Ron uses the possibilities of the tool in order to produce certain quadrilaterals which relate to the sign being given by Sam and his reference context.

### (2) Development of reference contexts

Figure 4 shows the development of Ron’s reference contexts during the interactive situation when working with the reconstruction task with Sam. Note that the first reference context in Figure 4 (left) equals Ron’s first reference context in Figure 3 (middle). The development of Ron’s reference contexts shows that he cumulatively integrates questions to it in order to interpret the underlying signs (the described properties by Sam) ([4], [6], [8]). He uses the information received to manipulate the geometric configuration with his DGS. Note also that “new” reactions in the next step of the interaction are in black font color whereas former reactions are in grey font color.



**Figure 4: Development of reference contexts from the receiver (Ron)**

Through the information Ron received in response to his questions, and the simultaneous tool use, the sign in demand from Sam (rectangle) becomes more and more meaningful. The conceptual development is manifested in this example by excluding properties and thus types of quadrilaterals. In this process the digital tool serves as an aid. Ron uses the digital tool as an example generator.



### (3) From conceptual content to the corresponding signifier (sign vehicle)

The last example in Figure 5 shows Ron's epistemological triangle at the end of this episode. Ron and Sam have exchanged a variety of properties, questions and assumptions. Ron has found a geometric configuration with the DGS that seems not to contradict the received information. Indeed, the picture in Figure 5 shows that he constructed a rectangle as Sam did in the first step (Figure 3 (left)). In this situation, Ron asks if he can get back a previously edited sheet with different quadrilaterals and their properties to determine the signifier ([10]). He doesn't know what the correct mathematical term is of this specific quadrilateral with the properties exchanged. The interviewer provides the conventional terminology ([11]). This example shows a process of the evolvement of mathematical meaning by working on a reconstruction task. Signs are interpreted through reference contexts and thus the meaning of the concept evolves. The students determine and use the properties of the concepts before and with the aim of determining the corresponding signifier (rectangle).

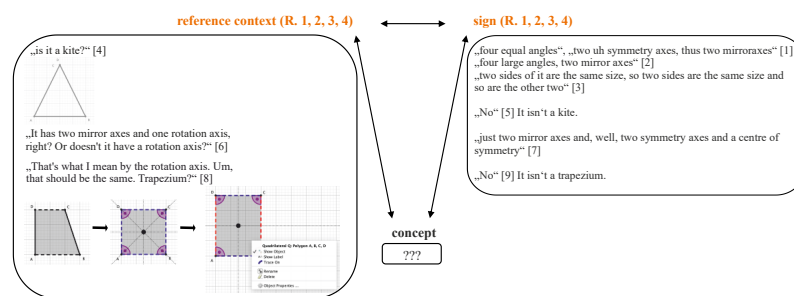


Figure 5: From conceptual content to the corresponding signifier

## Conclusion and outlook

The analyses provide first insights into how mathematical concepts – here specifically quadrilaterals – can evolve in interactions among students with different learning abilities through the interpretation and the use of signs. In this process, the evolvement of mathematical concepts is enhanced by the use of digital learning environments. Related to our example, a mutual mathematical idea develops through negotiation of quadrilateral properties during interaction. The analysis of the interaction processes of the students engaging with the reconstruction task shows how mathematical concepts develop by focusing on sign, reference context, and concept. Patterns of interaction emerge that reveal the interplay between signs and reference contexts. A crucial characteristic of the specific type of the reconstruction task is that given signs (information/description) have to be interpreted by producing reference contexts. Within this cascade-like process, a conceptual network is established by sending and receiving information based on the interpretation of given information. The digital tool plays important roles in this context: On the one hand, it highlights certain properties of the quadrilaterals, thereby inviting the students to interpret and communicate these properties represented in signs in the course of interaction and by doing so, extends their conceptual network. On the other hand, it serves as an example generator that has potential to constantly check and re-evaluate information and interpretations. Future work might also explore design-modifications against an embodiment framework to integrate how conceptual understanding can be fostered through the development of sensorimotor schemes (Abrahamson et al., 2020). Together with the epistemic analyses of the interaction processes, this might further enlighten the role of digital tools for conceptual development as encompassing the individual and the social, also in inclusive settings.



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