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2+2=4? Mathematics lost between the pitfalls of essentialism and alternative truths

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This essay problematises the epistemic status of mathematical knowledge. It is based on the observation that essentialist epistemologies provide no solid basis while relativist epistemologies have not yet convincingly succeeded to explain the objectivity of mathematical knowledge. I will start with three examples from popular media which illustrate that awareness for the problem discussed here has already reached the interested public. I will shortly address popular answers to the problem, only to refute them. I will end the essay by discussions which stay close to the example of $2+2=4$, ending with the presentation of possible directions for further understanding and research.

Keywords: Mathematical reasoning, epistemology, relativism, alternative truth, styles of reasoning.

Debates on $2+2=4$

Surprising as it may seem, the question whether $2+2=4$, which so prominently demarcated the border between ‘truth’ and ‘doublethink’ in George Orwell’s novel *Nineteen Eighty-Four*, has recently become the object of public attention. One example is the award-winning short film *Alternative Math* (Maddox, 2017), which is an alternative-facts parody, where the mathematics teacher’s rejection of a student’s claim that $2+2=22$ kicks off a chain of events in which the teacher has to face outraged parents, a public mob and undiscerning officials. It begins with the teacher showing the primary school student his test with $2+2=22$ marked wrong and with the teacher explaining $2+2=4$ with markers to the disavowing child. In the next scene, we see the teacher and the boy’s parents discussing the issue. When the parents ask the teacher who says that $2+2=22$ is ‘not the right answer’, the teacher replies ‘says maths’, whereupon the father asks ‘Who are you to say that your answer is right and his is wrong?’ Later, when a board of officials asks the teacher to proclaim that she is ‘open to the possibility there might be multiple correct answers’, the teacher replies ‘there is only one correct answer’. In the end, the teacher is fired but manages to demonstrate the consequences of allowing $2+2=22$, at least for the school’s finances. While the comedy of the short film builds on applying the logic of alternative facts to something as consensual as mathematical facts, the irony is that the teacher offers little more than authority and dogmatism to legitimise the epistemological claim made, which eventually does not dispel but invite further scepticism. Why does 2 plus 2 equal 4 and nothing else?

A different episode was brought to us by James Lindsay, a graduated mathematician, blogger and critic of post-structuralism. As a parody of the logic of post-structural narratives, Lindsay had posted the following on Twitter: ‘ $2+2=4$: A perspective in white, Western mathematics that marginalizes other possible values.’ What followed were intense debates on social networking services whether $2+2=4$ had to be accepted as a true statement and if $2+2$ could equal 5. Lindsay (2020) provided an (obviously factionary) overview of the discussions and rejected different proposals for $2+2$ not equalling 4. The proposals included non-trivial mathematical interpretations

of the expression $2+2$ (e.g., in a ternary numeral system, where we might note $2+2=11$, or in the residue system modulo 3, where we might note $2+2=1$), as well as peculiar contextualisations, for example, interpretations of addition as the bringing together of animal populations which might result in procreation or in the consumption of some specimen. Such proposals may be useful philosophical exercises that help us to explain what we mean by $2+2=4$, but in all cases the expression $2+2$ has been changed to mean something different than intended, thus losing its relation to the original problem.

A last example comes from the United States and even made it into the newspaper and television news in Germany and Austria. In early 2021, the Oregon Department of Education offered a course for in-service teachers based on the materials provided by the project 'A pathway to equitable math instruction' (Cintron et al., 2000). Most uproar was caused by the question whether mathematics was racist, but debates also kindled over the question whether or not mathematics provides unambiguous knowledge. Cintron et al. stated that 'the concept of mathematics being purely objective is unequivocally false, and teaching it is even much less so', that 'upholding the idea that there are always right and wrong answers perpetuate objectivity as well as fear of open conflict', that focussing 'on getting the "right" answer' is an instance of 'white supremacy', and that teachers should instead encourage students to 'come up with at least two answers that might solve this problem' (p. 65). Consequently, the material was interpreted by many by claiming that mathematics does not produce unambiguous knowledge, and that claims that it would were establishing mathematics as a racist endeavour. Meanwhile, the project leaders clarified on their homepage that this is not what they meant and changed their material, avoiding the problematic passages. The echo in the media with many rather emotional than epistemologically educated responses however indicates that there is a growing fear that the objectivity of mathematical knowledge may be called into question sooner or later.

Two popular explanations and their problems

Lindsay (2020) presented himself as a mathematical realist and argued that mathematics deduces propositions such as $2+2=4$ from fundamental premises (axioms) which are 'relatively simple and connect to the real world in a very obvious way'. Considering the axiom of induction from Peano's axiomatisation of natural numbers, such fundamental premises do not at all appear to be 'relatively simply', nor do they 'connect to the real world in a very obvious way'. But there is a more fundamental problem: Axioms for arithmetic were provided by Peano only at the end of the nineteenth century. Before that, it was impossible to prove the statement $2+2=4$ in the way described by Lindsay. So, does the validity of the statement $2+2=4$ really depend on mathematics as a deductive discipline? In fact, we could argue that we constructed mathematical theories in a way that provides for the truth of $2+2=4$ (and Lindsay implicitly does so as well when arguing 'if we chose to start with different fundamental assumptions, we'd have a different mathematics that doesn't seem remotely interested in reality at all').

A different explanation is that $2+2=4$ is abstracted from experience. (Remember the teacher explaining $2+2=4$ with markers in the above-mentioned short film.) We see that two and two apples make four apples, that two and two cows make four cows, that two and two fingers make four

fingers, and by induction we presume that $2+2=4$ holds true in general. But even if induction was the reason that we hold $2+2=4$ to be true, how about $74+26=100$? Surely, we would not claim that we saw 74 of something and 26 of that same thing make 100 so often that we hold the abstract $74+26=100$ to be true in general. Even worse, Greer (2005) made a strong point that ‘whether a situation is appropriately modeled, or not, by the equation $2+2=4$ [is] a matter of complex interpretation’ (p. 297). Already Frege (1884) had established that the use of arithmetic presupposes a specific interpretation of one’s perception, and Bishop (1988) informed us that the corresponding epistemology of what he calls ‘objectism’ has to be understood as a Western particularity. Consequently, it is not convincing either to establish the truth of basic arithmetic through induction.

The epistemological problem in more general terms

The discussion whether or why $2+2=4$ holds true is intimately (although often not explicitly) connected to an unresolved struggle in epistemological enquiry between essentialism and relativism, which can only be presented here in a brutally abridged version: Robertson & Atkins (2019) proposed that ‘essentialism is the doctrine that (at least some) objects *have independently of how they are referred to* (at least some) essential properties’. They add that such views have been fundamentally objected lately. In this sense, Lindsay (2020) argued that $2+2=4$ is a truth that exists independently of humans and can only be discovered. In contrast, relativism supposes that ‘things have the properties they have [...] only relative to a given framework of assessment’ (Baghramian & Carter, 2019).

In the philosophy of mathematics, the most prominent essentialist position is Platonism, where it is assumed that mathematical objects have a mind-independent and unalterable existence, that they provide shapes for the composition of our perceivable world, and that their properties can be discovered by the human mind. Linnebo (2018) reported that ‘platonism has been among the most hotly debated topics in the philosophy of mathematics over the past few decades’ and that ‘a variety of objections to mathematical platonism’ includes that ‘abstract mathematical objects are claimed to be epistemologically inaccessible and metaphysically problematic’ (p. 1).

In mathematics education research, the best-known objections against essentialism come from radical constructivism and post-structuralism. While the former stressed that humans lack the sensual apparatus to create an unbiased understanding of the world, and that knowledge is therefore necessarily relative to the limits of perception and interpretation (Glaserfeld, 1995), the latter was more interested in the refutability of truth claims and the historical deconstruction of their relations to power. For example, I discussed elsewhere how the laws of logic are far from natural or necessary but cultural products, which allied with specific social interests (Kolloche, 2014). So, is $2+2=4$ a ‘perspective in white, Western mathematics that marginalizes other possible values’ after all?

Interestingly, scepticism based on post-structuralism is not only guiding progressive research on equity in mathematics education and beyond, it also laid the epistemological ground for the phenomenon of alternative truth. As McIntyre (2018) documented in much detail, the intellectual pioneers of alternative truth explicitly adapted the post-structural claim that every discourse is necessarily constructed and socio-politically biased to a technique that then allows to refute any

truth claim. The very real effects of this development can be observed in the course of the denial of anthropogenic climate change or the COVID-19 pandemic by leading politicians. MacMullen (2020) argued that this epistemological scepticism is no longer only the academic background of the alternative truth phenomenon but an explicitly held position of some citizens. Elsewhere, I claim that this problem is not appropriately addressed in mathematics education research (Kollosche, 2021a).

Koertge (2017) regarded the phenomenon as a result of fundamental problems of relativism. She agreed that ‘all observations are laden with theory’ but questioned whether this means ‘that there is no objective/impartial perspective from which we can appraise and compare the truth-value of claims’ (p. 809). She warned that, ‘if it really were true that scientific assessments of truth and falsity could never be objective and could never be more than warring opinions, then we would be left with nothing but a clash of civilizations’ (p. 810). Indeed, it is one of the central problems of relativist epistemologies that they cannot sufficiently explain how we can avoid a situation where anybody can claim anything, how some sort of epistemic commitment can be guaranteed, and how specific ‘frameworks of assessment’ become accepted as scientific while others do not.

Skovsmose (2012) was early in warning the community of mathematics education researchers that post-structural deconstruction might be insightful but not productive in legitimising knowledge. In the light of the spread of alternative truths in public debate, Marcone et al. (2019) recently wondered if the fight of some scholars in our discipline against ‘the uncritical faith in mathematics’ and ‘the ideology of certainty’ has played a part in the post-factual ignorance of scientific facts (p. 186). They felt that ‘our arguments against universality and neutrality [of mathematics] have been trivialized and turned back against its original intention’ (p. 187). But have they? The fact that relativist arguments have been adopted by post-fact politics neither solves nor disvalues the original philosophical problem. It invites us to rearticulate it with greater urgency. Thus, we may ask: How can we explain the objectivity of mathematical knowledge without falling back to essentialist epistemologies?

Getting back to $2+2=4$

In a way, the above question is typical for Wittgenstein’s *Remarks on the Foundations of Mathematics* (1956), which I cannot address any further in the limits of this contribution. Recently, I have only seen the question further discussed by Azzouni (2006). I recommend to read his chapter, but this is not the place for me to discuss his sometimes fruitful and sometimes problematic ideas. Instead, I want to return to my initial example of $2+2=4$. Staying with such a basic example allows us to ground our complex philosophical thoughts in a matter that stays somewhat simple.

Let me shortly present my own understanding of why $2+2=4$ or $74+26=100$ holds true in a non-essentialist epistemology: We experience that there are several situations in which two of something and two of that same thing make four of that thing. There may also be situations in which this addition does not make sense. In the course of abstracting our experiences to the statement $2+2=4$, we also learn in which situation it makes sense to see $2+2$, and in which it does not. However, we do not arrive at the knowledge of $74+26=100$ this way. Instead, we arrive at the knowledge of $74+26=100$ through a line of algorithmic argumentation. From our empirically-based handling of

small numbers, we abstract a certain logic of calculation, which is suitable to inhabit our most simple and inductively true calculations and can be extended to quantities well beyond our experience. The study of the teaching and learning of arithmetic is a prominent field in mathematics education research and investigates such processes, in which the handling of large numbers is again and again made sense of through our handling of small numbers. We could say that calculation is a kind of reasoning that allows us to establish $74+26=100$ as a true claim, even though it is no longer directly connected to experience, nor are its truths established deductively.

A more sophisticated description of such an understanding has been provided by Piaget (1971) through the distinction of empirical and reflecting abstraction. Lensing (2018) discussed the importance of this distinction for mathematics education research and provides a good introduction. In short, empirical abstractions abstract from observables, while reflecting abstractions abstract from human action. In this sense, $2+2=4$ might be an abstraction from empirical cases of counting and adding, but the logic that lets us arrive at $74+26=100$ has been gained through an abstraction from our actions with numbers in easier cases. Piaget's distinction and its application to mathematics education shed light on the complex interplay of different epistemic processes that lead to the justification of knowledge. I will end this paper by a presentation of a more complex description of such processes and their interplay not from a psychological but from an epistemological perspective.

Styles of reasoning as a way out of the epistemological dead end

Here, we might pause and wonder whether there are fundamentally different ways to establish the result of an addition of natural numbers as a true claim. Such results might be abstractions from empirical observations, they may be won through the manipulation of material models such as markers, they might be products of an algorithmic technique, and they might be theorems proven in axiomatic theories. These ways to establish a truth claim such as $2+2=4$ reach the same judgement, but it is interesting that they arrive there by very different activities, which, each for itself, has the power to establish a claim as true. If we do not want to allow ourselves to fall back to essentialist explanations, this power and the consensus reached here are *curiosa* that demand an explanation.

Elsewhere, I have proposed to elaborate on Hacking's (1992) framework of *styles of reasoning* in order to arrive at a possible explanation (Kollosche, 2021b). A style of reasoning includes methods of argumentation for reaching truth claims, therewith also a set of statements whose truth-or-falsity can be decided scientifically, therewith also a set of objects that enter the scientific discourse in the first place. Hacking (1992) drew on the work of science historian Crombie (1994) who described the historical development of six distinct styles of reasoning: the postulation style, the experimental style, the modelling style, the taxonomic styles, the statistical style, and the genetic style.

It is already interesting that nearly all styles (maybe not so much the genetic style) are used in mathematics or use mathematics themselves: postulation in the axiomatic-deductive theories of mathematics, experimentation in experimental approaches to mathematics and in experiments on the suitability of mathematical models for specific applications, modelling for the understanding of our world through mathematics but also for gaining knowledge of mathematics objects through models such as diagrams, taxonomy as a principle for concept development in mathematics, and

statistics as a mathematical technique itself. Further, Hacking (1992) proposed to add ‘algorismic’ reasoning to Crombie’s list (p. 8), and was open to further additions or changes. We can recognise the postulation, the experimental, the modelling, and the algorismic styles of reasoning in our above-mentioned explanations of why $2+2=4$, which already shows the potential of this framework.

Here, I want to return to more general philosophical considerations one last time. Hacking’s (1992) framework of styles of reasoning clearly proposes a relativist epistemological position. However, it does not appear completely defenceless against the objections issued against epistemological relativism. Let us address Koertge’s (2017) concern that, in relativist epistemologies, ‘truth and falsity could never be objective and could never be more than warring opinions’ (p. 810). Note that opinions are individual views that do not require any justification (although opinions can be elaborated upon). I may hold the opinion that twelve is the most beautiful number, and may even present my thoughts associated with this opinion, but this does not establish ‘twelve is the most beautiful number’ as any kind of justified knowledge. When Koertge warns that relativism would degrade truth claims to opinions, she fears that we would be lost in opinions and that shared and accepted knowledge would no longer be possible. In the styles-of-reasoning framework, this is not the case. Here, we have a limited number of historically established frameworks of assessment, which are shared amongst people and used to justify knowledge on an interpersonal level. Especially, styles of reasoning do not allow to present just any opinion as knowledge, just as a mathematical theory does not allow to prove any statement whatsoever.

Admittedly, one might not be completely satisfied with the merely historical explanation of why Hacking’s (1992) styles of reasoning are accepted as truth-makers in science. Should there not be reasons specific to the respective styles which explain why they are acceptable for scientific inquiry and why other frameworks of assessment such as astrology are not? It is not easy to give an answer here. Hacking underlines that relativism implies that there is no objective ground from which the suitability of a framework of assessment for scientific enquiry can be decided. This means that answers can only be found in the styles and their interplay. He furthermore stresses that the acceptability of frameworks does underly historical changes, that all styles have appeared sooner or later in history, the statistical style being the most recent, that once-accepted styles such as the logic of resemblance as described by Foucault (1966) appear utterly ridiculous to us today, and that we cannot foreclose that new styles may emerge and gain acceptance. It may be productive to focus on the interplay of different styles of reasoning. Rupy (2011) introduced the concept of ‘ontological enrichment’ to capture that styles of reasoning do not only create new objects of scientific inquiry (in the way that Peano’s axiom system created numbers as axiomatic-deductive entities), but that these new objects mirror already-known objects (such as numbers as algorismic entities). Despite these ontic differences, we act ‘as if’ we were speaking about the same object in both styles, which allows us to gain understanding by studying this object in the various styles. Is it a fruitful hypothesis that new styles of reasoning gain acceptance through reproducing the knowledge gained by already-accepted styles while adding some new possibilities for understanding? For example, see how the postulation style mathematics of Ancient Greece as documented in Euclid’s *Elements* reproduced the already existing algorismic and diagrammatic mathematical knowledge of the time while adding new possibilities of expressing, justifying, ordering and interrelating knowledge – at

least until incommensurability problems provoked serious doubts concerning the suitability of this acting ‘as if’. I assume that the history of mathematics would be a good field to study the fruitfulness of this hypothesis, not only for the philosophy of mathematics itself but for epistemology in general.

Back to mathematics education

While the above considerations have been very philosophical and theoretical in nature, the mathematics classroom demands specific action. I cannot claim that I have thought-through recommendations for the mathematics classroom, but I have pressing questions, which mathematics education research might want to consider:

- While the styles-of-reasoning framework proposes that mathematics is an integral part of many styles of reasoning, the public image of mathematics and research in mathematics education focuses on one specific style of reasoning, namely on mathematical proof (Kolloche, 2021b). In how far does this bias hinder us to study mathematical reasoning more broadly in research and to address different forms of mathematical reasoning in the classroom? In how far is this bias socio-politically functional, for example for upholding an ‘ideology of certainty’ around mathematics?
- As the styles-of-reasoning framework allows to preserve both the ideas of relativism and of justified knowledge, may it be helpful, for example in the context of studies on what US-Americans call ‘race’ and mathematics, to articulate a critique of the socio-political bias of mathematics without questioning its objectivity altogether?
- In how far would a differentiated understanding of mathematical reasoning in the framework of styles of reasoning allow to deconstruct exactly which epistemic practices may interact with the preferred styles of knowing of possibly disadvantaged social groups without questioning the epistemic value of mathematical reasoning in general?
- Assuming that the growing public scepticism towards scientific knowledge leads to demands that schools should provide some sort of an epistemological education, that is education about the justification of knowledge, then would mathematics education not be a privileged place, given that mathematics is intimately connected to nearly all styles of reasoning? What would such an epistemological education in the mathematics classroom look like?

Eventually, answers to the above problems might also help to articulate in how far mathematics education helps adolescents to develop their reasoning skills – a claim that reoccurs in mathematics education research and educational policy in connection to mathematical argumentation, mathematical modelling, and problem solving in mathematics. Rethinking these fields from the perspective of a relativist epistemology can allow to articulate their theories more clearly, while the relativist framing would allow to more easily include a socio-political perspective in these fields of research.

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