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Investigating the processes of Mathematical Problem Solving with Technology of experienced mathematics teachers: The case of Sofia

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This paper aims to characterize the processes in which mathematics teachers engage while solving non-routine mathematical problems and express their reasoning with technology. The descriptive model Mathematical Problem Solving with Technology was used to analyse an experienced teacher's utterances and actions while solving a mathematical problem with a spreadsheet. Our findings reveal the complexity of expert problem solving with technology, through regulation processes and several micro-cycles involving the processes integrate and explore. The teachers' techno-mathematical fluency seems crucial to solving the problem and expressing the reasoning with technology.

Keywords: Mathematical problem-solving, techno-mathematical fluency, experienced mathematics teachers, expert problem solving.

Introduction

Digital technologies provide significant opportunities for enhancing mathematical thinking processes. Yet, the literature shows that most of the technology uses in mathematics learning consist of forms of replicating traditional classroom approaches with some improvements (Bray & Tangney, 2017).

Mathematical problem solving (PS) has been over decades a fertile field of research (Santos-Trigo, 2020). However, the role and impact of digital tools in the processes of solving mathematical problems remains an underexplored topic. Some studies address the strategies and ways of reasoning developed by means of digital tools (Santos-Trigo & Reyes-Martínez, 2019; Silva et al., 2021). Rott et al. (2021) propose a model of PS processes that can be used to characterize students' processes with dynamic geometry. Still, it is strongly influenced by Schoenfeld's (1985) model, which was based on paper-and-pencil work and did not account for the affordances of digital tools nor does it provide ways of explaining their role in the PS processes. In this paper we report on an exploratory case study developed to answer the following research question: what is the role of technology on mathematics teachers' processes of PS and how does it support their mathematical thinking?

Theoretical framework

Cognition in digital settings has been conceptualized as stemming from the interactions between individuals, technology, and the media; hence, *humans-with-media* entails the transformational and reorganizational power of the digital tools with which one thinks and acts (Borba & Villarreal, 2005). Technology plays a significant role in the development of mathematical thinking; it allows innovative ways of accessing information and affords new styles of thinking and knowing, producing a reorganization of cognitive activity, namely, in the PS processes.

Our research has been focusing on non-routine mathematical problems, i.e., challenging situations to

which the solver does not have a straightforward mathematical process that leads to the solution. Solving these problems by means of digital tools requires the engagement in a mathematisation activity that leads to a productive way of dealing with the challenging situation (Lesh & Zawojewski, 2007), entailing both mathematical and technological knowledge. The development of a conceptual model is consistent with a progressive mathematisation activity, where a *model of* a particular situation evolves into a *model for* explaining or justifying the solution (Gravemeijer, 2005). As the conceptual model portrays the mathematical reasoning developed, it becomes difficult to establish a clear boundary between the solving activity and the explanation of the reasoning. Often, they are so entangled that *solving-and-expressing* summarizes the synchronous processes of mathematisation and expression of mathematical thinking (Jacinto & Carreira, 2017).

Regarding teachers' use of technology in PS, Silva et al. (2021) discuss the ways of thinking-with-technology developed by collectives of teachers-with-media, concluding that the tools brought to the fore, GeoGebra and a Spreadsheet, influenced the exploration of the problem, not only visually, but also numerically and experimentally. Hernández et al. (2020) analysed pre-service teachers' mathematical understanding when dealing with problem solving in GeoGebra. They found that GeoGebra played a fundamental role in articulating different approaches and in the effective use of control strategies (such as, evaluating the solution or finding support for their conjectures). This leads to consider teachers' proficiency in using digital tools in their problem-solving-and-expressing activity.

The ability to articulate mathematical and technological skills, such as 'techno-mathematical literacies' (Hoyles et al., 2010), is seen as relevant to efficiently solve a problem from a mathematical perspective and to communicate its solution. The term 'fluency', adopted from Papert and Resnick (1995), seems appropriate to describe the ability of articulating a complex idea by means of a tool, being able to do or construct relevant things with it. Thus *techno-mathematical fluency* (TmF) refers to the ability to combine mathematical and technological knowledge for solving-and-expressing non-routine problems (Jacinto & Carreira, 2017). It entails bringing together digital tools and mathematics to create new understandings of the situations, develop techno-mathematical thinking and express it effectively. As with digital fluency, TmF involves to be able to select a useful technological tool from a pool of possibilities, the recognition of particular affordances in the tool, and knowing how it can be used to reach a mathematical outcome.

Research Method

This study investigated the processes of an experienced mathematics teacher, Sofia (pseudonym). A qualitative approach was used in collecting and analysing data. Data collection took place online, through Zoom. At first, in a semi-structured scoping interview the teacher was invited to talk about the role of PS and technology in mathematics teaching and learning. Then, she was asked to select one non-routine problem among four possibilities provided, and to solve it using the digital tools of her preference. She was asked to verbalize every thought and to make explicit every action, and to share her computer screen which enabled the video recording of actions and utterances while solving the problem. Data includes the video recording and the files produced by the teacher.

The teacher's processes were analysed, using NVivo, based on the transcript (utterances) and the video recording (actions), aiming to identify critical events (Powel et al., 2003) that would allow to

segment the activity. Deductive coding was performed on the segmented data, based on the ten processes of the descriptive model of Mathematical Problem Solving with Technology (MPST) developed earlier (Carreira & Jacinto, 2019; Jacinto & Carreira, 2017).

A keystone of the MPST model is the inseparability between the subject and the digital tool in solving the problems and expressing the solver's techno-mathematical thinking. The MPST model results from the combination of two theoretical lenses: Martin and Grudziecki's (2006) model for solving a technological problem, and Schoenfeld's (1985) mathematical problem-solving model. The MPST model includes ten processes: i) *Grasp* refers to the first encounter with the problem, either by reading or stating it, to an appropriation of the situation and early ideas involved; ii) *Notice* entails an initial attempt to understand what is at stake, the mathematics and the digital tools that may be useful; iii) *Interpret* is about placing affordances in the digital resources to ponder mathematical ways of approaching the solution; iv) *Integrate* refers to the combination of technological and mathematical resources within an exploratory approach; v) *Explore* entails the use of technological and mathematical resources to explore and analyse conceptual models that may enable the solution; vi) *Plan* involves the outlining of an approach to achieve the solution based on the analysis of the conjectures previously explored; vii) *Create* refers to carry out the outlined approach, recombining resources in new ways to enable the solution and synthesise new knowledge objects that will contribute to solve-and-express the problem; viii) *Verify* involves engaging in activities to explain and justify the solution based on the mathematical and technological resources available; ix) *Disseminate* refers to presenting the solution or outputs to relevant others and pondering on the success of the PS process; and x) *Communicate* comprises the interactions with relevant others while dealing with the problem.

The processes were used to code the segments and their analysis supported the writing of the case of the teacher Sofia solving-and-expressing a problem with the spreadsheet. In the next section, we describe the segments of her problem solving activity by summarizing, in the form of tables, the processes that she carried out at each stage.

Results and discussion

Sofia is a secondary mathematics teacher, with over 20 years of experience. She is highly enthusiastic about the use of technology, as she uses and promotes her students' use of digital tools (e.g. the calculator, GeoGebra, Kahoot). She thinks some tools are suited to particular kinds of problems and that the teachers' familiarity with such tools is fundamental to their successful integration. A detailed and clear explanation of the reasoning process is essential, and Sofia urges her students to do it.

Selecting the problem and grasping the conditions

Sofia started by reading the given problems but spent more time in some parts of the one she would choose (Figure 1), by reading out loud fragments of the statement.


<p>Leonor borrowed the video camera from her mother to film the general rehearsal of the play she is preparing with her colleagues at the Theatre Club. She knows that the camera's battery lasts 2 hours if it is in recording mode and lasts 3 hours in playback mode. Leonor wants to record the rehearsal and immediately watch that video with her colleagues, but cannot re-charge the battery. What is the maximum amount of time of the rehearsal that she can record, in minutes, to be able to view everything she recorded, right after?</p> <p>Don't forget to explain your problem-solving process!</p>	
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Figure 1: The problem chosen by Sofia: "How long does the battery last?"

It became clear that it was a new problem for her: “I’m not seeing a way to solve it. At first sight, I would say least common multiple... no, the greatest common divider... I don’t know... but it’s interesting. I might try this one out!” (*grasp*) (Table 1). Sofia notices that the spreadsheet is an appropriate tool to deal with the problem. Realizing that, if the recording lasts 2h, it will not be possible to play it (*interpret*) she begins to consider recording only 1h, and creates a table with Excel inserting titles and colouring cells A1 and B1, resizing the columns, and formatting boundaries (*integrate*). Then she tests the previous hypothesis, filling 1 in the recording column and 1 in the playing column, although she thinks this experiment will not lead to the solution (*interpret*).

Table 1: Utterances and actions of Sofia during the initial approach

What Sofia did or said	MPST processes
- reads the problems to appropriate the notions that may be involved, identifies a situation as familiar, chooses the problem to solve: “ <i>I like this one... it looks more challenging</i> ”	Grasp
- “ <i>I’m opening an Excel</i> ”, says, while overlapping that window with the page that contains the problems - reads the problem again: the battery lasts 2h when in recording and 3h when in playing mode	Notice
- hypothesis 1: “ <i>let’s see, a possibility... if she records 2h, she’s not able to see it. If she only records 1h...</i> ”	Interpret
- starts by organizing the information on a table: types “recording” in cell A1, and “playing” in cell B1 - formats the table: adjusts the cells dimensions, colours them in orange, formats table’s boundaries - inputs 1 in A2 and 1 in B2	Integrate
- “ <i>this is so basic... but I think this way won’t take me there</i> ”	Interpret

Testing with an erroneous approach

Sofia’s subsequent activity is characterized by a micro-cycle between the processes *integrate-interpret* (occasionally, *explore*) which entails introducing formulas in the spreadsheet using its syntax, testing concrete cases and analysing the results obtained in light of what she was expecting.

She assumes that 5h is the total battery duration and writes 2/5 and 3/5 in her spreadsheet model, meaning the ratio of the recording time and of the playing time, respectively. After some attempts, she decides to test a familiar case: if the recording takes 2h, she knows that the battery will be empty. But the result obtained with her model in Excel (0.8h of playing time) is not what she expected (0h). The tests, based on the initial erroneous assumption, disregard that the battery fully charged lasts 2h in recording mode and the same full charge lasts 3h in playing mode. These experiences support a perspective that will become crucial in the development of the solution: the *time left* after a certain recording. However, another difficulty seems to persist which is related to the perception of the existence of two variables of the same nature – the “*amount of time* spent using the camera” and the “*amount of time* that the battery lasts”, entailing an inverse proportion.

Testing and developing the conceptual model: “the percentage of the battery left”

The *exploratory* phase of Sofia’s activity (excerpts in Table 2) entails the test of a familiar case (1h in recording mode) and a new way of looking at the problem: the percentage of the battery left. When recording 1 hour, half the battery is spent and the remaining half allows watching a 1h30 video (*explore*). Realizing the potential of this approach, she adds a column - “LeftB” - where she considers the percentages of battery left after a certain recording time. That column is a new techno-mathematical object that reveals how she is conceiving the path towards the solution (*planning*).

Table 2: Utterances and actions of Sofia during the development of the conceptual model

What Sofia did or said	MPST processes
- "hum... 2h... I record 2h, its over, I can't, I can't watch it. (...) but if I have something previously recorded during some other time, I can play it for 3h. I can watch 3h of recording"	Interpret
- "If I record for instance 1h, let's see, I recorded 1h then I spent half of the battery. I recorded 1h, so 50% of the battery is left... 100% of the battery would allow to record 3h, 50% will allow for an hour and a half. There!" - "I'm in another line of thought now"	Explore
- Inputs 1.5 in cell B3 (testing) and explains "there will be 1h left of recording... the battery is left... if in here I record 1h, there will be 1h left"	Interpret
- Inserts 'LeftB' (what is left of the battery) in C1 and in C3 inputs '1' justifying "just to see if this allows me to generalize something faster"	Plan
- "there is 1h left, no, I have half of the battery left"	Interpret
- In the cell C3 she records 50% and in D3 she computes the percentage of the battery that is left to play the video, that is $D3 = C3 * B2$	Integrate
- "I have 50% left of the battery... so I recorded 1h, half is left, I kept half of the battery and that half battery will give, will be 50% of the 3h (...) I will have an hour and a half. An hour and a half to play the video"	Interpret
- Inputs 1.5 on A4, which corresponds to 1.5h in recording mode	Integrate
- "If I record 1h30 it is left... [sighs] I get a quarter of the battery left" - "I'm computing without generalizing and this should be generalized completely... so... I'm left with 25%..."	Interpret
...	...
- "is it enough to play? I don't know, I'll see... I don't want to make one by one" - Replaces the percentages of battery left after the recording with a formula, that is, $C3=50\%$ is replaced by $C3 = 1 - A3/2$ - "I have 50% left. This is what is left and here [C4], it will have to be 25%" - Drags the fill handle from C3 to C4 and confirms that the result is 25%.	Explore
- "I have 25% left, now what do I have to do? I have to calculate 25% of the 3h" [using the spreadsheet]	Interpret
- "let's see if it's enough, 1.7... 1.7", says, inputting 1.7 in cell A5 - inspects the formula, asking "What percentage is left?" and by dragging the fill handle from C4 to C5 concludes "15%"	Integrate
- "at the first sight there is still a little left over. Oh, but it's not enough to see, because then I need to play... I have to see what I recorded [laughs]. I was forgetting that detail of the problem, wasn't I? Then it's not working there [1.5 in recording mode]... right?" - Look, I think I've moved ahead but now I'm finding myself stuck somehow...	Interpret
- engages in reviewing the steps taken so far and her reasoning, explaining the meaning of lines 1 and 2 in the spreadsheet and going through the several experiments made with particular cases	Verify

Sofia's conceptual model (a *model of*) is being developed as she tests other values even if, at this point, they are worked out mentally and manually inserted on Excel. The processes integrate, interpret and explore follow each other in a cyclical way (Table 2), while Sofia keeps aiming to find a more robust approach that takes advantage of the spreadsheet affordances: "now how do I put it this here in a formula? (...) this should be completely generalized". The process interpret includes observations regarding the testing of particular cases, whilst the process explore is related to the use of those experiences in the refinement of the conceptual model.

The exploratory activity continues until she obtains a formula for the case corresponding to the recording of 1.5h, in which the battery is left with 25% of its capacity, and uses it to test 1.7h. She concludes that the solution must be somewhere between 1h and 1.5h but, as she fills stuck, she decides to review her reasoning and processes (*verify*).

Finding and expressing the solution

After concluding that the overall reasoning is correct and confirming that the formula used throughout column C (to compute what is left of the battery after a certain recording time, “LeftB”) is correct, Sofia realizes that she is looking for two equal values in different columns (A and D). She then colours these two columns in orange for her own “guidance” which is a critical action that sets the *creation* of the solution (Table 3), that is, she will continue her approach by carrying out the plan using the spreadsheet model to test between 1h and 1.5h. She finds the solution in the second attempt: the camera may record a video of 1.2h and its battery will allow to play the whole film.

Table 3: Utterances and actions of Sofia finding and expressing the solution

What Sofia did or said	MPST processes
<ul style="list-style-type: none"> - about to finish the review, she inserts “play” in D1 to become clearer what the values in the column refer to - “<i>now, hold on... let me do just one thing for guidance</i>”, says, formatting the colour of cells A2 to A8 to orange - “<i>to organize my ideas... here and here...</i>” and formats cells D3 to D7 with the same colour - tests 1.1 in A5 and obtains 1.35 in the column B (view) and claims “<i>they must both be equal</i>” referring to the orange shaded cells - tests 1.2 in A6 and drags the fill handle from C5 to C6 and from D5 to D6 while saying: “<i>hold on... technology could still be of more assistance to me... I could take further advantage if I worked with Excel</i>” - “<i>Look, 1.2... 1.2... I think that the maximum is 1.2... She could record 1h, 1h, and the 0.2 times 60... hum...</i>” - She starts to insert a formula in G4, but computes mentally first: “<i>12 minutes!</i>” <i>Oh, this is it! So... I think it is 1h and 12 minutes</i>” 	Create
<ul style="list-style-type: none"> - “<i>Now I have to recapitulate everything to see if this makes sense again</i>” 	Verify
...	...
<ul style="list-style-type: none"> - “<i>You have all the reasoning recorded, so it’s not necessary to explain it all</i>” - going back to statement, describes that she started by testing a concrete case because she already knew the solution, and then moves to explain how she obtained the formulas - Corrects the heading in the cell C1 changing it from “LeftB” to “%battery” to better adhere to the content - “<i>I think it’s fine now! I’m convinced!</i>” [laughs] - opens a text editor file and writes “Problem” as the title - uses the Snipping Tool to take a snapshot of the spreadsheet table and pastes it on the text file - starts typing and reading out loud a description of the processes followed, disregarding the initial erroneous path - presents the solution as 1.2 hours 	Verify
<ul style="list-style-type: none"> - “<i>now a confirmation is needed, oh, not a confirmation, the calculations confirm... a process, a mathematical formula to reach this value</i>” [the solution] - sends the spreadsheet and the text editor files to the researcher via e-mail 	Disseminate

As requested, Sofia engaged in explaining her problem solving processes by recapitulating, again, her thoughts and actions with Excel (*verify*). While doing so she revises the table, changing C1 heading from “LeftB” to “%battery” as she finds it to be more explicit. Even though she is continually seeking for a “mathematical formula”, she includes in the written explanation that she was “computing the percentage of the battery left after the recording (1- recording time/2)”, which worked as a *model for* explaining how the solution was achieved. Her reflections on the success of the activity and the files sent to the researcher containing the solution, characterize the process disseminate.

Conclusion

The Mathematical Problem Solving with Technology model allowed to analyse the role of the spreadsheet in the teacher’s processes. It also accounts for the complexity of expert successful problem-solving activity by revealing that metacognitive skills, namely control and regulation strategies are of paramount importance to progress (Schoenfeld, 1985; Hanin & Van Nieuwenhoven, 2020).

PS with technology takes place through micro-cycles of several processes, as others suggest (Carlson & Bloom, 2005; Jacinto & Carreira, 2021). Initially, within an erroneous approach, the processes *integrate-interpret* support experimentations that will disclose the basis of the conceptual model: the battery time that is left. Then, other cycles comprise the processes *integrate-explore-interpret*, as she perceives a different approach in using the spreadsheet to organize the testing of particular cases. The conceptual model evolves through cycles of *integrate-explore*, from testing with cases (model of) to a confirmation that the approach works and the spreadsheet supports a general solution (model for).

The teacher used the spreadsheet output to create the final answer, as a solving and an expressing tool. Even though the exploratory activity has induced a plan based on the percentage of battery left, the solution emerges from a retrospective analysis of her reasoning, that lead her to look for equal values of the recording time (column A) and the playing time (column D). This reinforces the idea that the ‘solving’ and the ‘expressing’ are simultaneous activities of mathematization.

Sofia’s techno-mathematical fluency is revealed by her familiarity with a diversity of digital tools useful in mathematics teaching and learning. In this case, the spreadsheet was chosen because she is familiar with its syntax, recognizes several of its affordances (tabular representations, formulas, automatic fill) in organizing and developing a numerical approach. Later on, a more robust conceptual model emerges as she is constantly seeking a generalization, a formula. The spreadsheet’s numerical feedback encouraged conjecture generation and exploration, by easily testing the effects of changing values or relations. By incorporating the formatted table on the text file, she created a techno-mathematical answer to the problem that represents her conceptual model of the solution. Her techno-mathematical fluency includes the recognition of affordances in the digital tools used with several purposes: to interpret the situation from a techno-mathematical point of view, to explore a conceptual model, and to produce the techno-mathematical solution.

Technology plays a paramount role throughout the mathematical problem solving and expressing activity, which suggests that techno-mathematical fluency is an essential skill for mathematics teachers to engage in successful problem solving with technology.

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