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# Two Primary School Teachers' Mathematical Knowledge of Content, Students, and Teaching Practices relevant to Lakatos-style Investigation of Proof Tasks

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*Despite recognition of the importance of Lakatos-style proving activity in the mathematics classroom, we know little about whether teachers' relevant mathematical knowledge is conducive to supporting it in their classrooms. We take a step towards addressing this research gap by reporting the results of an exploration of two primary teachers' mathematical knowledge of content, students, and teaching practices relevant to Lakatos-style investigation of proof tasks. Through vignettes-based, semi-structured interviews, we presented the participants with 19 illustrated classroom episodes covering a range of Lakatosian techniques and a range of student ways of engaging with supportive examples and counterexamples to formulate, validate, refute, and refine conjectures of different types. Participants' responses revealed both productive and counterproductive understandings highlighting that although Lakatos-style proof lies within teachers' reach, supporting their preparation is crucial.*

*Keywords: Primary school mathematics, Mathematical knowledge for teaching Lakatos-style proof, Conjectures, Supportive examples, Counterexamples.*

## Introduction

In his book “Proofs and Refutations”, Lakatos (1976) reconstructed prominent mathematicians' discussions on Euler's theorem to highlight the key role example use can play in proving and refuting conjectures. As a result of the increasing research interest about the incorporation of proof construction into mathematics classrooms (e.g., Stylianides, 2016), some studies (e.g., Balacheff, 1991; Komatsu et al., 2018) have investigated students' ability to comprehend and employ Lakatosian techniques to solve proof tasks and reported encouraging findings. However, few studies have centred upon primary school level (e.g., Komatsu, 2010; Reid, 2002) and none has had an explicit focus on teachers. Addressing these research gaps, our study sought to answer this research question: In what ways are two primary school teachers' knowledge of content, students, and teaching practices relevant to Lakatos-style investigation of proof tasks similar and different regarding the extent to which they may be able to support potentially their students' engagement with this style of proving?

## Theoretical Frameworks

### Phases of Lakatos-style Investigation of Proof Tasks

Drawing on Lakatos' (1976) original work and others' account of Lakatos-style proving activity in the mathematics classroom (e.g., Komatsu, 2010; Reid, 2002) we define Lakatos-style proving activity as an iterative and reflective process consisting of four interrelated phases (Deslis, 2020): in *Phase 1* students identify a conjecture which they wish to examine, based on a pattern, an educated guess, or simply following their teacher's suggestion. Then they start examining cases to investigate

its validity. The discovery of supportive examples in *Phase 2* may indicate that the conjecture is likely to be true. However, in *Phase 3* counterexamples may also emerge from the examination of cases suggesting that the conjecture is false. In *Phase 4* students reflect on the previously discovered examples to appropriately modify the false conjecture. The domains of the original conjecture can either be restricted or expanded, to exclude the discovered counterexamples, or to transform them into supportive examples, respectively. Once the conjecture has been altered, a new investigation cycle begins aiming to the further refinement of the conjecture. It is important to note that, although our framework is based on Lakatos' work, it focuses on certain aspects of Lakatos-style reasoning and does not purport to reflect the whole complexity of his philosophy. For example, our framework does not focus on the interplay between defining and proving, which is crucial in Lakatos' work.

### **Conjecture Types**

According to the classification of statements proposed by Tsamir et al. (2009), we identify three conjecture types: *Always-True* and *Never-True* conjectures are the two extremes, since only supportive examples and counterexamples emerge during their examination, respectively; in contrast, both example types emerge during the examination of *Sometimes-True* conjectures (see Figure 1).

### **Justification Schemes (JS) and Refutation Schemes (RS) for Student Understandings**

*Justification* and *Refutation* Schemes are two separate but interrelated three-level classifications describing a rather comprehensive range of student understandings about the interplay between examples and proving. According to Framework JS, which is based on Harel and Sowder's (1998) classification and its adaptation by Stylianides and Stylianides (2009), students at the least advanced level accept generalisations based on supportive evidence coming from a few easy-to-check cases. At the intermediate level they believe that only example-based evidence coming from the examination of representative cases can produce valid generalisations, whereas at the advanced level students are aware of the insufficiency of all types of example-based arguments. By analogy to JS, Framework RS, which we developed based on previous research on students' views around counterexamples (e.g., Balacheff, 1991; Lee, 2016), students at the advanced level consider the discovery a counterexample sufficient to refute a conjecture, while students at the intermediate level question the sufficiency of a single counterexample to refute a conjecture and demand the discovery of more counterexamples, preferably resulting from strategically selected cases. At the least advanced level students treat counterexamples as exceptions, resisting to the idea that the existence of counterexamples can affect the truth of a convincing conjecture. Students who hold the advanced schemes in relation to both example types can potentially also reach a meta-level that is key in the implementation of Phase 4 (Lakatos-style conjecture refinement). On top of the understanding that comes with the acquisition of the two advanced levels, students at this meta-level, which we call "*refinement scheme*", are able to reflect on the previously discovered supportive examples and counterexamples to get insights into how the refuted conjecture can be appropriately modified.

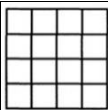
### **MaKTeLaP: Mathematical Knowledge for Teaching Lakatos-style Proof**

Building on previous research on the knowledge needed to teach mathematics (Ball et al., 2008), or specifically proof (Buchbinder & McCrone, 2020), we describe the mathematical knowledge relevant to bringing Lakatos-style activity into the classroom (Deslis et al., 2021). We identify three

knowledge components: (1) *CoLaP* which refers to the *Content* knowledge about what constitutes appropriate use of examples in *Lakatos-style Proof* tasks; (2) *StuLaP* which refers to the knowledge of *Students’* typical understandings around the example use in *Lakatos-style Proving*; and (3) *TeLaP* which refers to the knowledge of *Teaching* practices that can appropriately promote students’ efforts to productively engage with *Lakatos-style Proof*. Since all Lakatosian phases revolve around two example types, namely supportive examples (SEs) and counterexamples (CEs), we can identify two subcomponents within each component, each focusing on the knowledge around one example type. For example, CoLaP splits into CoLaP [SE] and CoLaP [CE].

### Methods

We collected our data through semi-structured interviews based on vignettes (Skilling & Stylianides, 2019), which are contextualised descriptions of classroom situations (Deslis et al., 2021). 10 in-service primary school teachers were presented with 19 classroom episodes with comic-style student characters discussing and exchanging arguments with their peers as they engaged in the phases of Lakatos-style investigation. Three student groups worked on the “Count the Squares” task (Zack, 1997) and each explored a conjecture of a different type (see Figure 1). The student dialogues we used were adapted versions of classroom episodes from fifth grade reported in Zack (1997) and Reid (2002) and covered the whole range of investigation phases and student understandings, as described by our theoretical frameworks. After each episode participants were asked to comment on the validity of arguments, evaluate students’ understandings, and discuss how they would respond to each student contribution. We analysed responses for themes and ranked the various teacher understandings relevant to the different MaKTeLaP components and example types according to their level of sophistication (Deslis et al., 2021). In this paper we focus on two teacher participants, identify similarities and differences in their responses, and discuss the degree to which their understandings put them in a good position to support the incorporation of Lakatos-style proving activity into their classrooms. The characteristics of the two teachers were reasonably similar. Alcyone and Nephele (nicknames) were both female, 29 and 27 years old, respectively. At the time of the study, they had 18 and 45 months of teaching experience, respectively, and both held a Bachelor’s and a Master’s degree in Education. Both participants taught middle-sized (21 and 23 students) fifth-grade classes (ages 10-11) in similar and neighbouring schools in Athens, Greece, with most of their students coming from middle-income households.

<div>  <div> How many squares are there in this 4-by-4 grid? Examine other similar grids of your choice. How many squares are there in each of them? Find a general rule that applies to grids of all sizes and prove your answer. </div> </div>		
Group 1- Always True Conjecture: “The number of squares in an n-by-n grid is $1^2+2^2+\dots+(n-1)^2+n^2$ .”	Group 2- Sometimes True Conjecture: “The number of squares in an n-by-n grid is a multiple of five.”	Group 3- Never True Conjecture: “If an n-by-n grid has N squares and an m-by-m grid has M squares, then the m×n-by-m×n grid has M×N squares.”
Note. The phrasing of the conjectures has been altered; the student characters in the vignettes presented and discussed these conjectures using language that reflects the mathematical knowledge of students of their age.		

**Figure 1: The proof task and the conjectures that were investigated by the three student groups**

## Findings and Discussion

### CoLaP: Knowledge of Content

Participants were asked to evaluate the arguments presented in the scenarios regarding their validity. The responses indicate that the two teachers' understandings about the role of examples in the investigation of conjectures had both similarities and differences (see Table 1). Commenting on whether it is appropriate to conclude that a conjecture is true based on the discovery of some supportive examples, Nephele preferred to judge on a case-by-case basis, while Alcyone's responses suggested a blanket rejection of this idea. For example, when the second group (Sometimes-True conjecture) found a few examples that supported their conjecture, the two teachers agreed that this does not allow us to conclude that the conjecture will work for any grid:

- Alcyone: The absence of counterexamples so far is not evidence that they do not exist. The examples so far are supportive, but a counterexample can emerge any time.  
Nephele: The generalisation is not permissible, because it is only based on two examples.

Alcyone maintained this opinion when commenting on a similar moment during the first group's investigation (Always-True conjecture). Yet, Nephele was convinced that it will work for all grids, although this generalisation would be based on the same number of examples as in the previous case:

- Nephele: All the examples so far have confirmed the conjecture, so we can safely conclude that it is correct and works for all grids.

Their comments on arguments against the idea that any example-based argument can lead to safe generalisations of conjectures also highlighted the divergence of views:

- Nephele: This is not necessarily true; sometimes even a few examples can provide undeniable evidence and thus enable us to conclude that the conjecture is true for all cases.  
Alcyone: The examples may show that there is a possibility for the conjecture to be true, but we cannot be sure about that if the only evidence we have comes from examples.

Overall, Alcyone showed concrete and stable appreciation of the usability and limitations of supportive examples. She valued their role in the investigation of conjectures while being aware that we cannot prove a general statement merely based on examples. In contrast, Nephele's responses varied from one episode to another, indicating weaker understanding which is highlighted by her erroneous belief that in some circumstances examples can be used to prove general statements.

In contrast to the previous example type, the two teachers were found to hold comparable views about the appropriate use of counterexamples in refuting conjectures. Specifically, both participants' responses showed awareness that a single counterexample can sufficiently refute a general statement:

- Nephele: Now that we have found a counterexample, we know that the conjecture is false.  
Alcyone: The number of supportive examples we have found is irrelevant; one counterexample is enough to show us that the conjecture does not hold.

**Table 1: Summary of Alcyone and Nephele's understandings relevant to CoLaP**

CoLaP [SE]	A: Supportive examples can be used to investigate general statements, but they cannot prove them.	N: Supportive examples can be used both to investigate and to prove general statements.
CoLaP [CE]	A & N: A single counterexample can sufficiently refute a conjecture.	

## **STuLaP: Knowledge of Student Understandings**

Turning to STuLaP, participants' judgements about the level of student understandings brought to the surface both similarities and differences in teachers' views (see Table 2). Alcyone not only recognised students' common belief that examples can be used to prove general statements is a misconception, but also was aware that the use of representative cases which have been strategically selected does not make the argument any more valid from a mathematical perspective.

Alcyone: Any student who is happy to accept generalisations that are merely based on examples as proofs (no matter how many examples there are or the process through which they have been identified) has a significantly less advanced level of understanding than students who reject all kinds of example-based proofs.

Unlike Alcyone, Nephele was occasionally favourable towards the use of examples as a means to prove, since she tended to accept arguments that reflected the intermediate JS level as valid:

Nephele: The student has not realised that the examples so far have covered the whole spectrum of possible grids and therefore proved that the rule is correct. His choice to reject example-based proofs in their entirety signifies weak understanding.

Unlike the case of supportive examples, the two participants had similar views about students' understandings around counterexamples. Specifically, both teachers spoke highly of students who believed that a single counterexample can sufficiently refute a conjecture:

Nephele: Students who discard a conjecture immediately after the discovery of the first counterexample and consider further checks as unnecessary have a more advanced level of understanding than those who demand a substantial number of counterexamples to be discovered before they refute the statement.

Furthermore, both participants consistently criticised students who continuously treated counterexamples as exceptions and maintained their initial opinion ignoring the evidence against it:

Alcyone: Students' refusal to reject a conjecture despite the existence of counterexamples and the treatment of counterexamples as exceptions indicates poor understanding.

Yet, Nephele also judged favourably students who were reluctant to reject conjectures after the discovery of one counterexample and instead demanded that a substantial number of counterexamples should be discovered, and characterised this practice as more productive than it actually is:

Nephele: Students' reluctance to reject a conjecture immediately after the discovery of one counterexample and their need to find additional counterexamples shows a scepticism that is desirable in the classroom of mathematics.

Although Alcyone occasionally criticised this practice, in other episodes she also expressed views that were similar to those of Nephele:

Alcyone: I like this student's critical attitude! It is always good to be reluctant and demand more evidence.

Overall, both teachers' responses showed a satisfactory degree of awareness about which student understandings about counterexamples signify an advanced level of understanding and which do not, mirroring, to an extent, their good content knowledge on counterexamples. Still, some of their reactions to students who resisted to conjecture refutation unless numerous counterexamples were discovered, were contradictory, thus showing knowledge fragility. The main differences lay in their

understandings about student conceptions around supportive examples: unlike Alcyone, Nephele did not consider students' tendency to overrely on empirical arguments as a misconception.

**Table 2: Summary of Alcyone and Nephele's understandings relevant to StuLaP**

StuLaP [SE]	A: Students' belief that conjectures can be proved through supportive examples, even if these are coming from the examination of strategically selected cases, is counterproductive.	N: Students' belief that conjectures can be proved through supportive examples is counterproductive unless these are coming from the examination of strategically selected cases.
StuLaP [CE]	A & N: Students' belief that a conjecture must be refuted after the discovery of one counterexample is productive, as is students' desire to discover more counterexamples before they refute a statement.	

### **TeLaP: Knowledge of Teaching Practices**

As for the third MaKTeLaP component (see Table 3), participants were asked how they would respond to students' contributions and how they would support students' efforts if they were their teachers. Teachers' responses showed that they promoted a slightly different usage of supportive examples in the exploration of proof tasks. To begin with the commonalities, both appreciated the importance of examining various examples in the beginning of the investigation:

Alcyone: The examination of several different cases is a reasonable thing to do after the formulation of a conjecture, since it can provide clues about how we can prove it.

Nephele's advice regarding which cases students should try first was even more specific:

Nephele: The examination of the easiest-to-check cases can be a convenient way to start.

However, Nephele also thought that it is appropriate to encourage students to terminate the investigation once much confirmatory example-based evidence has been found:

Nephele: The examples have shown that the rule works; now I'd advise students to stop, and I'd give them a new problem to solve.

Taking a different approach, Alcyone appreciated that despite the valuable contribution of supportive examples to the promotion of the investigation, it is inappropriate to conclude an investigation at the stage of the example examination even if several confirmatory cases have been discovered:

Alcyone: I'd tell the students that even if they have found several examples supporting their conjecture, it is prudent to remain cautious and continue the examination of cases in search of counterexamples, which can emerge anytime.

The two teachers' suggestions after the discovery of counterexamples were dissimilar, too. Nephele encouraged students to abandon the refuted conjecture and replace it with a new one:

Nephele: The conjecture clearly doesn't work. I'd encourage students to abandon this idea and try to formulate a new conjecture that is not related to the multiples of five.

Taking a step further, Alcyone not only advised students to replace the faulty conjectures, but also suggested that the new conjecture could be an improved version of the initial conjecture. She also added that reflecting on the characteristics of the previously discovered examples can provide clues for the appropriate modification of the refuted conjecture:

Alcyone: The students can review the examples they found to come up with a refined version of the conjecture. [...] There might be a subset of grids for which the conjecture works; for example, for the  $n$ -by- $n$  grids where  $n$  is a multiple of five.

All in all, both teachers' suggestions showed an appreciation of the important part example examination can play in the investigation of conjectures. However, Alcyone's suggestions in relation to both example types were clearly more sophisticated. Unlike Nephele, Alcyone was fully aware that supportive examples cannot prove general statements and judged the student arguments accordingly. Her suggestions to the students reflected an appreciation of the limitations of example use. A remarkable similarity between the two teachers' responses was that although both recognised that one counterexample can refute a conjecture, both also indulged in judging favourably students who attempted unnecessary checks to discover more. Yet, it was impressive that Alcyone encouraged the use of a technique that is surprisingly consistent with the spirit and the essence of Lakatos-style reasoning. Unlike Nephele who encouraged the replacement of a refuted conjecture with a new one, Alcyone said she would encourage students to use counterexamples not only to refute conjectures, but also to modify and refine them considering the characteristics of the previously examined examples. This practice lies at the heart of Lakatos-style activity and the fact that Alcyone reinvented it despite the lack of any prior relevant instruction in teacher education is encouraging.

**Table 3: Summary of Alcyone and Nephele's understandings relevant to TeLaP**

TeLaP [SE]	A: Use of supportive examples to initiate the conjecture exploration.	N: Use of supportive examples to initiate and terminate the conjecture exploration.
TeLaP [CE]	A: Use of counterexamples to refute the conjecture and reflection on their characteristics to refine it.	N: Use of counterexamples to refute the conjecture and then replace it with a new one.

## Conclusion

Research on classrooms of expert teachers (e.g., Zack, 1997) or teachers who worked closely with researchers (e.g., Komatsu et al., 2018) suggests that students can engage productively in Lakatos-style activity and benefit from it, even at primary school level. Still, we know little about how non-expert teachers understand various aspects of this activity and thus whether they would have the knowledge to be able to support it in their classrooms. The present study adds to the increasing literature on the incorporation of Lakatos-style reasoning into school mathematics by exploring ordinary teachers' relevant mathematical knowledge, an area that had previously been unexplored. We analysed and compared two primary school teachers' reactions to a set of 19 illustrated classroom episodes which enabled us, in an explorative way, to shed light on their understandings of content, students, and teaching practices relevant to Lakatos-style activity. Our study offers some encouraging findings while pointing in directions for future research. The case of Alcyone and her many productive intuitive understandings suggest that it is possible for primary school teachers to have necessary (though, arguably, not sufficient) knowledge to bring Lakatos-style proof into their classrooms. Yet, the case of Nephele suggests that it is also important to identify appropriate ways to support the refinement of teachers' understandings and highlights the crucial role teacher education has to play in preparing teachers to effectively engage their students with Lakatosian techniques.

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