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# On the dialectical relationship between truth and proof: Bolzano, Cauchy and the intermediate value theorem

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*Abstract - The question of the nature of the relationship between truth in an interpretation and proof in a mathematical theory is a complex epistemological issue. From a didactic perspective, we argue that it is worthwhile to address this issue in mathematics teaching and learning. In this paper, we illustrate this in the context of the Intermediate Value Theorem, focusing on the approaches of Bolzano and Cauchy, and we discuss possible didactic implications.*

*Keywords: Truth in an interpretation, proof in a theory, didactic and epistemology of mathematics, Intermediate Value Theorem.*

## Introduction

A recurrent issue raised by students concerns the need to prove mathematical statements that looks obvious. The prevalence of this issue has increased since the introduction of dynamic geometry software offering strong epistemic conviction through various means (Mariotti, 2006). Therefore, a crucial issue in mathematics education concerns the relationship between proof and truth. Indeed, it is well known in the literature that the motivation for proof and proving is difficult for students to endorse: most of them consider it as a teacher's requirement, not as an epistemic necessity. The need for truth is often considered to come from the possibility of raising doubt, for example because observations or results of actions are counter-intuitive, or surprising. Looking through the history of mathematics at the completeness of the set of real numbers, Bergé (2008, p. 220) points to another issue, namely the necessity, for a thorough study of Analysis, of the theoretical reconstruction of pre-constructed notions of Calculus such as the straight line or the continuity of functions. In this paper we intend to show that the relationship between *truth* and *proof and proving* is not one of subordination, as claimed by some authors considering that only what is proved is true, but rather a dialectic relationship. For this purpose, we will focus on the Intermediate Value Theorem, an existential statement that has long been taken for granted by mathematicians, who have provided proofs based on geometrical arguments.

One of the most important theorems about the continuum is intuitively obvious: if on a plane a continuous line has one of its extremities on one side of a right line and the other on the other side of the same right line, then the continuous line cuts through the right line (Longo, 2000, p.401).

Bolzano (1817) was the first to explicitly question the validity of proofs of the Intermediate Value Theorem based on geometrical arguments, thus paving the way for the recognition of the need for explicit constructions of the set of real numbers in the second half of the 19th century.

The choice of the intermediate value theorem is motivated 1/ by the epistemological relevance of the discussion on proof and truth; 2/ by the links with the question of  $\mathbb{Q}$ -incompleteness /  $\mathbb{R}$ -completeness; 3/ by its presence in many curricula at the transition from secondary to tertiary education, mainly because of its various applications.

In the first section of the paper, we present Bolzano's view that proofs of the Intermediate Value Theorem<sup>1</sup> based on geometrical arguments should not be admissible, while at the same time Cauchy provided such a proof in his course at the Ecole Polytechnique in Paris (Cauchy, 1821). In the second section, we discuss both approaches, with particular reference to a paper by Hourya Benis-Sinaceur. In the last section, we discuss possible didactic implications for proof and proving in mathematics education.

## ***Truth and proof and proving in the theory of equation***

### **Bolzano's discussion on geometrical arguments in proofs of the Intermediate Value Theorem**

In the preface of his paper "Rein analytischer Beweis des Lehrsatzes, dass zwischen je zwey Werthen, die ein entgegengesetztes Resultat gewähren, wenigstens eine reelle Wurzel der Gleichung liege", published in Prague in 1817, Bernard Bolzano wrote<sup>2</sup>:

There are two propositions in the theory of equations of which it could still be said, until recently, that a completely correct proof was unknown. One is the proposition: that between any two values of the unknown quantity which give results of opposite sign there must always lie at least one real root of the equation. (Russ, 1980, p.159)

It is noticeable that Bolzano does not raise doubt on the truth of this proposition, despite the lack in his view of a fully correct proof. He mentioned several names of mathematicians having provided a proof for this proposition, and claimed: "However, a more careful examination very soon shows that none of these proofs can be viewed as adequate" (Russ, 1980, p. 160). He then provided arguments for this claim.

The most common kind of proof depends on a truth borrowed from geometry, namely, that every continuous line of simple curvature of which the ordinates are first positive and then negative (or conversely) must necessarily intersect the x-axis somewhere at a point that lies in between those ordinates. There is certainly no question concerning the correctness, nor indeed the obviousness, of this geometrical proposition. But it is clear that it is an intolerable offense against correct method to derive truths of pure (or general) mathematics (i.e., arithmetic, algebra, analysis) from considerations which belong to a merely applied (or special) part, namely, geometry. (Russ, 1980, p.160).

This discussion by Bolzano brings to the fore the relationship between truth and proof and proving in Mathematics and prefigures Dedekind's concern in his essay "Stetigkeit und irrationale Zahlen" published in 1872 where he writes<sup>3</sup> that in teaching a course in Differential Calculus, he resorted to geometrical evidences "in proving the theorem that every magnitude which grows continuously, but not beyond all limits, must certainly approach a limiting value" (Dedekind, 1963, p. 1). He states that this is useful from a didactic point of view, but cannot be claimed as scientific, that leads him to

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<sup>1</sup> In fact, he deals with the case of two images of opposite signs.

<sup>2</sup> We use the English translation by S.B. Russ, published in Russ (1980, pp.159-181)

<sup>3</sup> We use the English translation by W.W. Beman in Dedekind (1963, pp. 1-27)

search for “a purely arithmetic and perfectly rigorous foundation for the principles of infinitesimal analysis.” (Dedekind, 1963, p.1)

Bolzano continues, claiming that proofs should not be simply confirmation (Gewissmachungen), but rather justification (Begründungen)<sup>4</sup>, and then, he moved back to the Theorem he is considering:

Consider now the objective reason why a line in the above-mentioned circumstances intersects the x-axis. Everyone will, no doubt, see very soon that this reason lies in nothing other than that general truth, as a result of which every continuous function of  $x$  which is positive for one value of  $x$ , and negative for another, must be zero for some intermediate value of  $x$ . And this is precisely the truth which is to be proved. It is therefore quite wrong to have allowed the latter to be derived from the former (...). Rather, conversely, the former must be derived from the latter if we wish to represent the truths in the science in the same way as they are connected to each other in their objective coherence. (Russ, 1980, pp. 160-161)

These criticisms concern the proofs mentioned by Bolzano at the beginning of the preface, which does not include the proof by Cauchy that was published four years later in 1821, but the criticism apply.

### Cauchy’s first proof of the Intermediate Value Theorem

In his famous “ Cours d’ Analyse De L’Ecole Royale Polytechnique”, Cauchy provides two proofs of the Intermediate Value Theorem. Nevertheless, as Longo writes, in the first proof of the intermediate value theorem in his Course of 1821, Cauchy “does not go further than the intuition of continuum that comes from strings and curves traced by a pencil and their crossing. (Longo, 2000. p.403)

Later, page 403, Longo provides an English translation of the theorem and of the proof by Cauchy:

Theorem of the mean value:<sup>5</sup> *If the function  $f(x)$  is continuous with respect to the variable  $x$  between  $a$  and  $b$ , and if we call  $c$  an intermediary value between  $f(a)$  and  $f(b)$ , then we can always satisfy the equation  $f(x) = c$  for at least one value of  $x$  between  $a$  and  $b$ .*

Proof (Cauchy, 1821): It is enough to see that the curve which has equation  $y = f(x)$  will meet one or more times the line  $y = c$  for at least one value of  $x$  between  $a$  and  $b$ .

**Figure 1: The theorem and his proof by Cauchy (1821) (English translation in Longo 2000, p. 403)**

The history of mathematics has long credited Cauchy’s role in developing a solid foundation for Analysis and Calculus, leaving Bolzano’s work in the shade. In the second half of the 20<sup>th</sup> century, comparisons between the work of Cauchy and Bolzano have been discussed in three papers (Grattan-Guinness, 1970; Freudenthal, 1971, Benis-Sinaceur, 1973). In the following section, we present the point of view developed by Hourya Benis-Sinaceur as an epistemological contribution to the reflection on the relationships between truth and proof and proving in a didactic perspective.

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<sup>4</sup> In the French translation by Sebsetik (1964), we have *certifications* (Gewissmachungen) and *fondements* (Begründungen).

<sup>5</sup> We recall that Longo refers to what we have named Intermediate Value Theorem

## **On a historical perspective between Bolzano and Cauchy**

In the abstract in English of her paper of 1973, H. Benis-Sinaceur writes:

*SUMMARY: The purpose of this article is to show the distance between Bolzano's analytic style with his profound logical tendencies, characteristic of the late called rigor of Weierstrass, and the Cauchy's way which remains, despite the important technical innovations, rooted in traditional geometrism. (Benis-Sinaceur, 1973, p. 97)*

She underlines the desire by Bolzano to found mathematics, and to privilege the proof of a result over the result itself and its application:

For him [Bolzano], what matters is to express explicitly and to underline the necessity of proving the results, even if they have been known and used for a long time, his own work is to try to make these proofs. (Benis-Sinaceur, 1973, p. 104)<sup>6</sup>

Concerning Cauchy, referring to the preface of the 1821 course, she writes:

[...] there are neither questions of founding, nor of refining the proofs, nor of rejecting geometric intuition (...). The aim is merely of doing mathematics, that is to say of solving problems and improving known solutions, by restriction, rather than by generalization, by specifying under which conditions the formulas are not a empty symbolism. (Benis-Sinaceur, 1973, p. 105)

In her paper, Benis-Sinaceur acknowledges the view of Freudenthal (1971), who against Grattan-Guinness (1970), rejected the idea that Cauchy might have plagiarised Bolzano. She emphasises the differences between the two authors' programmes, which leads to different practises: while Bolzano is primarily concerned with theoretical rigour and the search for purely analytical proofs, rejecting recourse to geometry as vicious circle, the qualities of Cauchy's course are those of synthesis rather than formal rigour (pp. 102-103). She concludes her paper by arguing that the work of Bolzano and Cauchy represent two distinct heterogeneous paths that would not meet until the works by Weierstrass, Cantor and so on, in contrast to the idea of a continuous development through the nineteenth century.

## **On Bolzano as a precursor of Dedekind**

It is a fact that, although it is hard to believe, neither Bolzano nor Cauchy defined the real numbers (Freudenthal 1971, p. 387). As mentioned above, in a certain sense, Bolzano's programme prefigured Dedekind's programme as expressed in the preface of his essay on continuity and irrational numbers. Dedekind uses the intuitive continuity of the line to define the notion of a cut in the set of rational numbers, to prove that some cuts are not operated by a rational number, and by creating for such cuts one and only one irrational number that operate this cut. Having proved that the system composed of all the rational and the irrational numbers is a complete ordered set (namely a continuous set), he was able to prove in a purely analytic way several theorems in Calculus<sup>7</sup>. In a letter to R. Lipschitz of July

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<sup>6</sup> Our translation from French; and the same for other quotations of the paper.

<sup>7</sup> For a presentation of Dedekind construction of the set of real numbers and didactic implication, see Durand-Guerrier (2016).

27<sup>th</sup>, 1876, Dedekind underlines that nothing is more dangerous than admitting existences without sufficient proof and ask how to recognise the licit and non-licit hypotheses of existences. Returning to Bolzano's geometric concerns, and to the intuitively obvious results mentioned by Longo and used in Cauchy's first proof, Dedekind's work reverses the burden of proof: if the domain of the continuous function at stake is a complete ordered set, then the equation has at least one solution, and thus the curve representing the function  $f$  will necessarily intersect the line of the abscissa (in the case of images of opposite signs).

## **Some didactic implications**

### **Contribution to the discussion on the relationships between *truth* and *proof and proving***

As already mentioned, Bolzano and Dedekind's concern was with the type of proof admissible in the mathematical field in question (Analysis, Differential Calculus). They questioned and rejected as unscientific the use of proofs based on geometrical arguments and considered the need to develop a theory in which it would be possible to prove these theorems within the theory and they did so. From a didactic point of view, this underlines the importance of considering that "what characterises a mathematical theorem is the system of statement, proof and theory" (Mariotti, 2006, p.185). In Durand-Guerrier & Tanguay (2018), we showed the impact of the definitional choices of the real numbers (through Dedekind's cuts; as limits of Cauchy sequences, as unlimited decimal expansions) in the proofs of the completeness of the set of real numbers, and conversely the contribution of working on these proofs to improve the understanding of the nature of the mathematical objects at stake and the topological relationships between rational and real numbers. The  $\mathbb{R}$ -completeness, compared with the  $\mathbb{Q}$ -incompleteness, provides strong proof of existential theorems such as the Intermediate Value Theorem, and some fixed-point theorems (for an example, see Durand-Guerrier, 2016 and below in the next section). As argued by Bolzano, for such theorems, the geometric evidence may be misleading. Indeed, in the graphical register, it is not possible to see the difference between a curve representing a function defined on a subset of  $\mathbb{R}$  that is dense-in-itself but not ordered-complete (e.g. the set of rational numbers), and a curve representing a function defined on an ordered-complete subset of the set of real numbers (e.g. an interval). Therefore, in the theory of equations in the set of rational numbers, the statement corresponding to the Intermediate Value Theorem is false, i.e. the existence of a solution to an equation satisfying the conditions of application is contingent. If we consider a formal axiomatic system of the theory of equations without including a completeness axiom, then there are models of this axiomatic system in which it is not possible to prove the Intermediate Value Theorem, and others in which it is possible to prove it. This was hidden in the proofs based on geometric arguments, such as the first proof of the intermediate value theorem by Cauchy (1821).

This accounts for Bolzano's point of view that what underlies the geometric truth used in the classical proofs that he criticises (the curve intersects the x-axis at a point between  $x_0$  and  $X$ ) derives from the truth of the Intermediate Values Theorem. Indeed, in a set of numbers that is dense-in-itself but not ordered-complete, the curve will always intersect in the geometric sense the x-axis, but it may not correspond to any point of the set of numbers under consideration. In our view, this supports our claim that the relationship between *truth* and *proof and proving* is not one of subordination, but rather

a dialectical relationship. In the case of Bolzano, and Dedekind, a statement can be accepted as true before it can be proven. The challenge is then to elaborate a theory in which it is possible to prove this statement true. As far as Cauchy is concerned, according to Benis-Sinaceur, the requirement for “geometric rigor is not a call to a strict formalism, but a control of conformity to an “intuition” of phenomena occurring in the familiar practice of Analysis” (Benis-Sinaceur, 1973, note 13, p. 107).

### Two situations aiming to discuss the relation between Analysis and graphical register

Due to the recurrent difficulties faced by students and teachers for what concerns the teaching and learning of proof and proving in mathematics, we consider that it would be valuable to put on the scene in didactical situations their dialectic relationships with truth. We briefly report two examples involving the context of discreteness-density-in-itself and continuity (completeness) and the associated graphical representations.

*A fixed-point problem.* In Durand-Guerrier (2016) we have presented and analysed a didactical situation involving a fixed-point problem for an increasing function, in four different cases: 1. the case of a finite segment of the set of natural numbers; 2. the cases of intervals  $[0,1]$  2.1. of the set of finite decimal numbers; 2.2. of the set of rational numbers; 2.3. of the set of real numbers. In case 1 (thanks to discreteness) and in case 2.3 (thanks to completeness), it is always possible to find a fixed-point; it is not the case for cases 2.2 and 2.3, due to incompleteness<sup>8</sup>. We have shown that this situation has the potentiality for questioning with students the relationships between discreteness, density-in-itself and continuity, and the interpretation in the graphical register. For example, after conjecturing that the statement was true in the set of interval  $[0,1]$  of decimal numbers and failing to adapt the proof they did in the first case, secondary students (grade 11) solving this problem in an experimental setting raised the following question: “Is it possible that the graph of an increasing function from the set  $D$  of finite decimal numbers to itself crosses the first bisector in a point with non-finite decimal coordinates? (Pontille et al., 1996, p. 25)<sup>9</sup>. Prospective secondary teachers faced difficulties very similar to the secondary students for what concerns the case of finite decimal numbers and rational numbers. Moving to the case of the interval  $[0,1]$  of the set of real numbers, they generally first considered that the continuity of the function is required, and then solved the problem with the Intermediate Value Theorem. The next step was to understand that the hypothesis of the continuity of the function is not needed, and to move to a proof without this hypothesis (e.g. relying on the existence of Supremum, or by construction of adjacent sequences).

*An algebraic definition of exponential.* In Durand-Guerrier, Montoya and Vivier (2019), we provided empirical evidence of students’ difficulties in distinguishing density-in-itself and continuity, and we hypothesised that the use of digital tools increases the need of conceptual tools to interpreting graphical representations and reconstructing the objects of Analysis. We then proposed a didactical situation with the aim with the aim of providing students with such tools.

The mathematical question consists in searching for the functions  $f: E \rightarrow R$  satisfying the functional equation  $\forall x \in E \forall y \in E f(x + y) = f(x) f(y)$ , where  $E$  is a usual set of numbers,

<sup>8</sup> This situation was first presented in Pontille, Feurly-Reynaud and Tisseron (1996)

<sup>9</sup> Our translation from French.

$\mathbb{Z}$ ,  $\mathbb{Q}$  and  $\mathbb{R}$ . We assume that  $f(1) = 2$  and it can be proved that  $f(0) = 1$  and  $f$  takes positive values. We ask to compute some values of  $f$  (respectively, for each set, value of  $f$  at: 3 and -3;  $1/3$ ,  $-1/3$ , 1.3; -1.3;  $\sqrt{3}$  and  $-\sqrt{3}$ ) and to draw graphical representations (on paper and with Geogebra software).

**Figure 2: The task posed to students (Durand-Guerrier, Montoya & Vivier, 2019, p. 89)**

While in the sets  $\mathbb{Z}$  and  $\mathbb{Q}$ , the function is completely determined by the algebraic relation and the calculation of the image of  $a$  yields to  $2^a$ , this is not the case for the set  $\mathbb{R}$  of real numbers. In the contexts of dynamic geometry and of paper-and-pencil geometry, the graphical representation of the function in the  $\mathbb{Q}$ -domain provides the same graph than the exponential function on the  $\mathbb{R}$ -domain. However, this graphical evidence is misleading: indeed, under the given hypothesis, there are a lot of functions defined on domain  $\mathbb{R}$  that satisfy the algebraic property. Therefore, the search for an analytic proof that the function coincides with the usual real exponential function is bound to fail. This should therefore lead students to question the graphical evidence. This situation was experimented in Chile and Peru with prospective teachers. The most advanced students try to prove algebraically that the image of  $\sqrt{3}$  is  $2^{\sqrt{3}}$ , which is not possible without additional assumption about the function (e.g. the function is increasing); others admitted without proof that the function was the exponential function  $2^x$  on the domain  $\mathbb{R}$ . After being asked to give a value (e.g.1) to the image of  $\sqrt{3}$ , students provided graphical representations that surprised them. In particular, in some cases there were two points on the same vertical, this making visible the distinction between the real line and the rational line, in other words, the distinction between a continuous (ordered-complete) domain and a dense-in-itself not continuous (not ordered-complete) domain. The first empirical results comfort our conjecture that this problem has the potential to let the need for proof emerge, beyond geometrical evidence, given that the organisation of the didactic situation allow a genuine engagement of students.

## Conclusion

In this paper, we have tried to highlight the dialectic relationship between *truth* and *proof and proving* from an epistemological point of view, in the case of the Intermediate Value Theorem. In the last section, we have given two examples of didactical situations that have the potential to stage this issue at the transition from secondary to tertiary level and for teacher training. Designing didactical situation with such potentialities is a challenging issue.

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