

Mathematical reasoning outside the classroom – A case study with primary school students solving math trail tasks

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Research findings underline the potentials of outdoor mathematics – leaving the classroom presents the students with every-day situations in which they can apply mathematics. By example, a math trail guides students along a path with several mathematical tasks to be solved on-site, e.g. by measuring or counting. Apart from long-term learning benefits and motivational aspects, it remains unclear how this form of experiential learning involves fundamental mathematics working methods and competences, i.e. mathematical reasoning. In this article, the method and results of a case study with 15 primary school students are presented concerning the frequency and qualitative use of mathematical reasoning during a math trail with eight different mathematical tasks from the areas of arithmetic, combinatorial and geometry. The results show that reasoning is involved in the students' solution processes, mainly in developing a solution plan and in geometry tasks, i.e. for estimations.

Keywords: Reasoning Skills, Outdoor Education, Mathematics Education, Experiential Learning

The Experiential Learning Theory and outdoor education

The learning theories by Lewin, Dewey and Piaget have – among other aspects – the importance of experiences with the environment in common (Kolb, 1984). Still, “learning was primarily a personal, internal process requiring only the limited environment of books, teacher, and classroom. Indeed, the wider ‘real world’ environment at times seems to be actively rejected by educational systems at all levels.” (Kolb, 1984, p. 34). In contrast, the Experiential Learning Theory (ELT) emphasizes the “central role that experience plays in the learning process” (Kolb, Boyatzis & Mainemelis, 2000, p. 1). According to the authors, activities such as concrete experiences, reflective observation, abstract conceptualization and active experimentation are essential for the learning process and, in particular, the acquisition of mathematics concepts and skills. Also, Hattie et al. (1997) highlight the importance of *out-of-class experiences* for education, i.e. in the sense of first-hand experiences and through the embodiment of “abstract mathematical concepts in concrete terms, using ideas and modes of reasoning grounded in the sensory-motor system” (Lakoff & Nuñez, 2009, p. 5).

In the context of mathematics education, the term *outdoor mathematics* describes the teaching and learning of mathematics outside the classroom in interaction with the environment. One approach is the *math trail* which describes a route with mathematics tasks to be discovered in and solved with the real-world environment (Zender & Ludwig, 2019). In the educational context, during a math trail, the students cooperate in groups of three and solve the tasks by means of mathematical activities. Furthermore, through first-hand, out-of-class experiences of mathematical concepts, math trails have the potential to foster the acquisition of mathematics skills and competencies, e.g. modelling, problem solving and reasoning (Buchholtz & Armbrust, 2018). The potential of the latter is focused on in the following.

Mathematical reasoning in the context of outdoor mathematics

Mathematical reasoning shows high relevance in the teaching and learning of mathematics at different ages, e.g. concerning the question of how task design can foster reasoning processes (Stylianides et al., 2019). In this paper, the focus is on reasoning in primary school. Therefore, a broad definition of reasoning is chosen, whereby both – arguing and proving – are regarded as subareas of reasoning. This definition is comparable to Stylianides' term *reasoning-and-proving* (Stylianides, 2008) and results in the description of reasoning through several activities, i.e. to identify patterns, make conjectures and provide arguments (Arnesen, Enge, Rø & Valenta, 2019). With the study being located in Germany, the following activities are highlighted in the curriculum for mathematics education in primary school by the Conference of Ministers of Education (KMK, 2004, p. 8): (a) question mathematical statements and check their correctness, (b) recognize mathematical relations and develop assumptions, and (c) search for and understand reasons.

Being listed as a possible potential of math trails, it is examined on a theoretical level how far outdoor mathematics and math trails, in particular, can foster mathematical reasoning in the sense of the three mathematical reasoning activities.

1. **Out-of-class experiences:** From ELT, it can be assumed that experiences play a major role in learning processes. In the setting of a math trail, the students have to interact with objects and situations in the real world – hereby, they collect experiences in this context. Still, it is not only the collection of real-world experiences. To solve a task of a math trail, the students have to reflect on their experiences by formulating mathematical statements and assumptions. In this reflection process, mathematical reasoning is mainly necessary in the sense of the reasoning activity (a) and (b).
2. **Group Interaction:** During a math trail, the students work in small groups. For the different activities, i.e. searching the task's object, planning the solving process, exchanging ideas, collecting data and validating the results, it can be assumed that the students interact with each other and reason for and against proposals and ideas. In this social process, mathematical reasoning is mainly necessary for reasoning activities (b) and (c).
3. **Transfer of Mathematical Knowledge:** Being outside the classroom, the students have to choose the data to be collected from all available data – whereby this number might be higher in the real world than in a school book. Hereby, the students have to transfer the mathematical knowledge that they acquired inside the classroom to a new context that was not primarily created for educational purposes. In contrast to calculation tasks where the mathematical content is often straightforward, the students have to decide (and reason) which mathematical characteristics and relations can be found in the real-world situation. In this reflection process, mathematical reasoning is necessary for the reasoning activities (a), (b) and (c).

State of the art and research question

From quantitative empirical studies, it results that math trails have the potential for positive (long-term) learning outcomes (Zender & Ludwig, 2019), motivational aspects (Gurjanow, Oliviera, Zender, Ludwig & Santos, 2019) and individual opportunities for the support of strengths and weaknesses (Buchholtz & Armbrust, 2018). With reference to ELT, math trails are a promising

approach for mathematics education. Still, there has been a lack of studies to examine the process-related mathematical competences when solving math trail tasks, i.e. mathematical reasoning. From the theoretical potentials, it can be assumed that math trails foster reasoning on different levels. Still, the considerations need empirical validation. In order to focus on the aspect of mathematical reasoning in more detail, the paper focuses on the research question:

To what extent and for what purpose do primary students use mathematical reasoning while solving math trail tasks outdoors?

In particular, the focus is on the question of what characterizes the solution processes of math trail tasks concerning mathematical reasoning. From this perspective, it can finally be concluded to what extent math trail tasks are suitable for learning mathematical content and fostering mathematical reasoning.

Methodology

In June 2021, a case study with primary students was conducted to answer the research question. In total, 14 students aged 9-11 years were divided into six groups of two, three or four students. In these groups, the students followed the route of a math trail located at Goethe University Frankfurt with eight different tasks, including combinatory, geometry and arithmetic tasks (see Figure 1). The tasks do not include an explicit question that initiates reasoning.

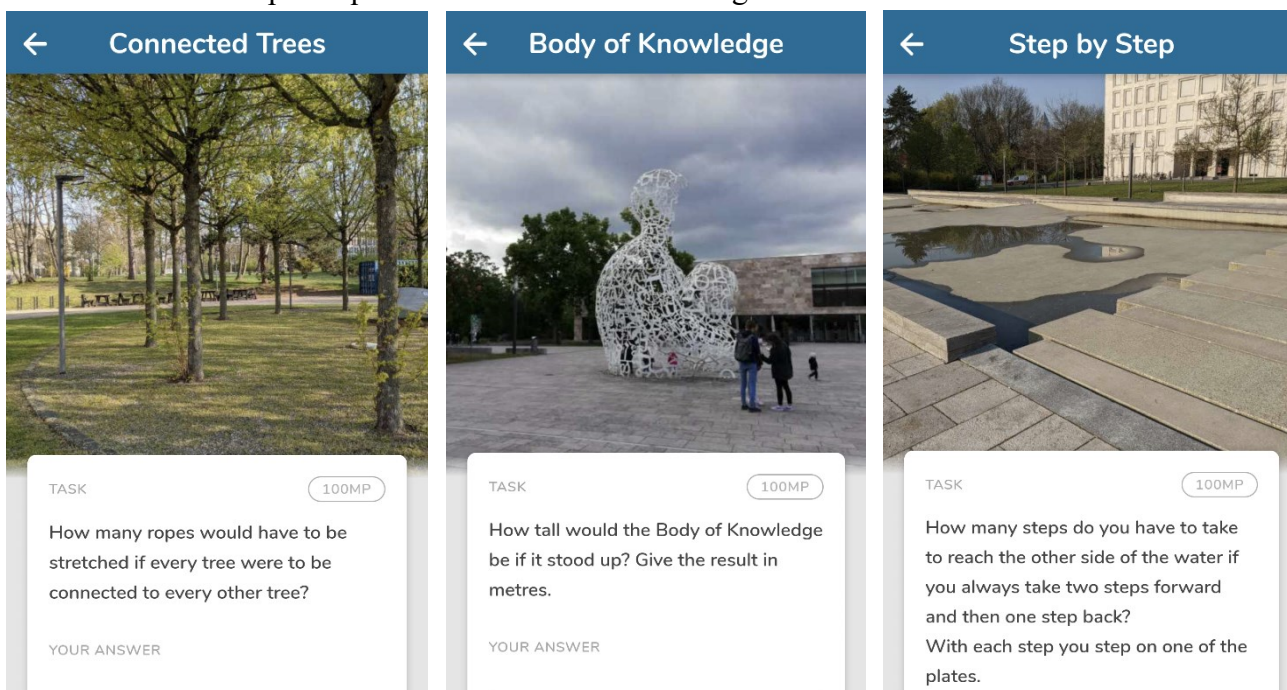


Figure 1: Three tasks from the study for combinatory (left), geometry (middle), arithmetic (right)

For the entire route, the groups needed between 45 and 70 minutes ($M = 62$ minutes). Hereby, the students were supported by the MathCityMap app through a map, hints and an answer validation (for more information see Zender & Ludwig, 2019). In addition, they were accompanied by a university student who filmed their solving processes and interacted with the students if clarification was necessary. This setting resembles the methodological adaptation of the *narrative walk-in-real-time interview* that Buchholtz, Orey & Rosa (2020) adapted to the context of math trails.

In total, the video material contains about 370 minutes. Especially the students' conversations during the actual solution processes are considered to analyze their way of reasoning. Therefore, the conversations are transcribed and analyzed in two different ways. First, on a quantitative level, the different mathematical activities are coded in accordance to Pólya's phases of problem solving: 'Understand Task', 'Develop Solution Plan', 'Solve Task' and 'Task Validation' (Pólya, 2004). In addition, the mathematical reasoning activities are coded and added to the respective activities in the solution processes. Through this, it is possible to specify the quantity of the students' reasoning in relation to the activities during the solution process. This is visualized by means of activity diagrams (Ärlebäck & Albarracín, 2019; see Figure 2). Afterwards, on a qualitative level, the students' identified reasoning activities are analyzed by means of a qualitative content analysis according to Mayring (2000). With this, different categories of mathematical reasoning can be identified inductively from the empirical material. The categories created in this process are to be considered disjoint. The results of the analyses on both levels are presented in the results section.

Results

Frequency of reasoning activities during the solution processes

Figure 2 shows the activity diagram of the solution processes during the math trail by one of the participating groups. The diagram includes the mathematical activities of the group according to Pólya and specifies, in addition, the activities in which mathematical reasoning is relevant through shading. The diagram gives an overview of the eight tasks of the trail, whereby the tasks from Figure 1 are Task 3 (Connected Trees), Task 6 (Body of Knowledge) and Task 7 (Step by Step). The time for navigation from one task to another is excluded in this presentation.

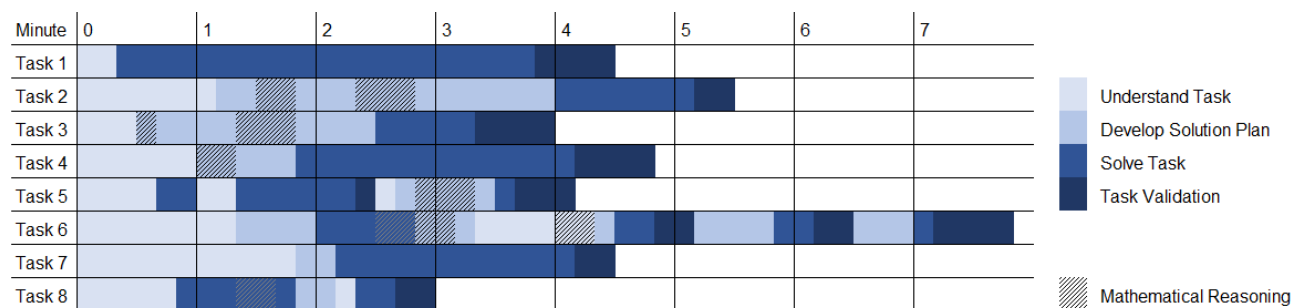


Figure 2: Activity diagram of solution processes and mathematical reasoning activities

In total, the amount of reasoning activities in this group is about 10 % of the time of the math trail task solution processes. Again, this number should be interpreted in the context of the tasks which do not include explicit claims for reasoning. During 25 % of the time, the students understand the tasks. This activity is relevant in all eight tasks. It happens mainly at the beginning of the solution processes through the actual task formulation and an analysis of the object and/or situation. In 4 % of the understanding activities, the students reasoned. The activity "Develop Solution Plan" is relevant in 26 % of the group's solution processes and – despite Task 1 – relevant in every task. In this activity, reasoning is coded most frequently, namely in 28 %. The students spend 35 % of the solution process on the actual task solving. About 5 % of this activity can be coded as reasoning activities. The group's task validation has a relative duration of 14 %. In this activity, they do not use any reasoning.

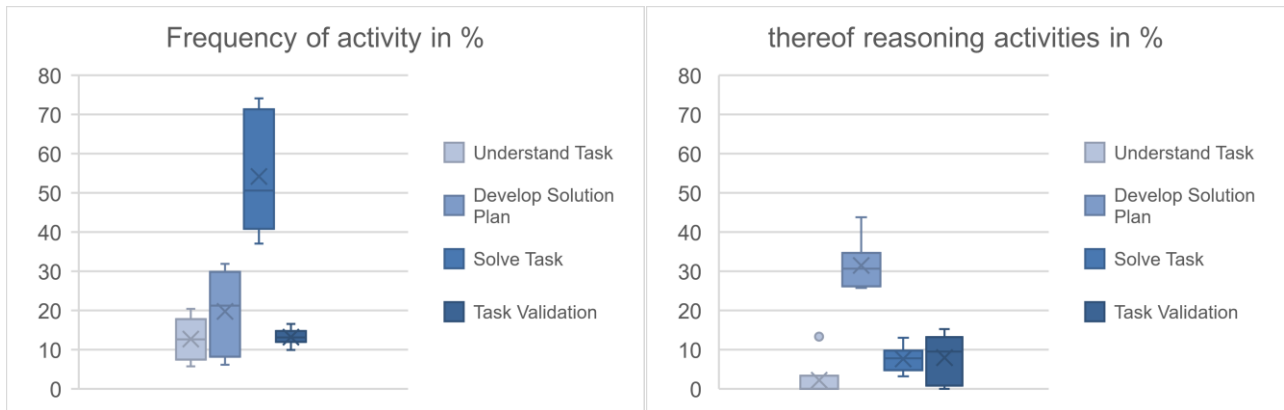


Figure 3: Quantitative overview of all solution processes and reasoning activities

Focusing on all six groups, in about 12 % of the math trail solution processes – excluding navigation – reasoning activities are coded. Figure 3 gives an overview of the frequency of the solution processes and the reasoning activities of all six groups. As a result of this, the beforehand described observations can be summarized for the sample as follows: The actual task solving activity is the most frequent activity in the solution process with more than 50 % on average. It is followed by the development of a solution plan whereby the high deviation shows differences between the groups for both activities. The activities of understanding and validating the task take both about 13 % on average. As presented in the example group, the students reason most frequently in developing a solution plan. From the 20 % of development activities, nearly one third involves reasoning activities. It is followed by the solving and validation of the task, in which 8 % of the activities involve reasoning activities. In the understanding of the task, reasoning plays a minor role.

With tasks from three different mathematical topics being involved in the study, it is possible to analyze the frequency of reasoning activities with regards to the different actions, i.e. counting in the arithmetic tasks, measuring and estimating in the geometry tasks and trying and sketching in the combinatory tasks. Hereby, it can be observed that the geometry tasks involve about two reasoning activities per task. Despite the development of a solution plan, the students reason frequently on their estimations in the activity ‘Solve Task’. In the combinatory tasks, about one and a half reasoning activities are coded on average, mainly in the development of a solution plan. In the arithmetic tasks, the amount is about one reasoning activity per task. This might be explained by the comparably low duration of the activities in which the students develop a solution plan. As the activity is mostly counting, the students tend to skip the development of a plan in which reasoning is most frequent.

Categorization of the reasoning activities

The qualitative content analysis results in different categories that describe the reasoning activities for the different steps in the solution process according to Pólya (2004) in more detail. Due to the low frequency of the activity ‘Understand Task’, it is excluded from the qualitative analysis.

With this analysis, it is possible to identify different ways of the reasoning activities the students used. Table 1 gives an overview of the categories with a representative example from the study (translated by the author from German to English).

Table 1: Qualitative categories describing the purpose of the reasoning activities in the different steps of the solution process

	Definition	Example
<i>Reasoning during Development of a Solution Plan</i>		
Choose a strategy	The students reason about a strategy they might use to solve the task. Aspects such as precision and efficiency are included where appropriate.	“We should count the number of stones in one row, because there are four [rows] [...] and it should be nearly the same [number of stones] in each row.”
Identify object/situation characteristics	The students use the local settings and refer to characteristics of the task object/the task situation to reason about a solution plan.	“We have to count with six people. Because it [the bench] has three parts where two persons can sit on each.”
<i>Reasoning during Solving of Task</i>		
Do measurements/ estimations/counts	The students reason about their approach to measuring, estimating or counting. In particular, considerations of the correct procedure are included.	<i>Student shows a way on a map (see Figure 4 left).</i> “No, this is not the shortest way. You have to go here and here and only measure up to this building.”
Perform a calculation	The students reason how to calculate, referring to known formulae and mathematical relations where appropriate.	“The first tree has 14 ropes. And then the second [tree] has only 13 [ropes], because it is already connected to the first one. [...] So we have 14 plus 13 plus 12 [...] and so on.” <i>(see Figure 4 middle)</i>
<i>Reasoning during Task Validation</i>		
Check result’s correctness	The students reason for or against a statement whether the achieved result might be correct.	<i>Student counts 16 regular elements on the task’s object (see Figure 4 right).</i> “I think it is possible with two [colors] because 16 is an even number.”
Make changes in the solution plan	The students reason why an incorrect result was obtained and how the solution plan should be changed.	“11 is too low, because we estimated 2 meters, but I think it is more. Can you stand next to the figure so that we can estimate with you?”



Figure 4: Impressions of the solving processes during the math trail

Discussion

With regard to the question to what extent and for what purpose primary students use mathematical reasoning while solving math trail tasks outdoors, the following conclusions are drawn. Math trails provoke reasoning activities without explicit questions for reasoning: The students reason especially during the activities in which they decide on a solution plan, i.e. reason about a strategy and by means of characteristics. From the theoretical considerations, in particular Kolb (1984), especially the *out-of-class experiences* and the *group interaction* seem to be relevant for the involvement of mathematical reasoning hereby. Still, the activities ‘Solve Task’ and ‘Task Validation’ involve reasoning. During the solving activities, they reason about measurements and calculations. From the theoretical considerations, especially the *transfer of mathematical knowledge* seems to be relevant, i.e. when choosing a suitable formula. During the task validation, again, the *out-of-class experience* seems to be an important factor, especially when it comes to the reflection of the solution process in the context of being outdoors.

The results are in-line with the theoretical considerations – mainly derived from Kolb (1984) and Lakoff & Nuñez (2009) – concerning the relevance of mathematical reasoning in the outdoor setting of a math trail. Still, the indicator that the tasks involve reasoning to a different amount, i.e. in relation to the mathematical topic, raises further questions. Also, the deviation between the groups indicates that the groups might follow different patterns of reasoning along the tasks. Finally, the question arises to what extent the reasoning outdoors differs from the reasoning inside the classroom in a comparable setting – on both, a qualitative and a quantitative level. Comparable data are not available at the current time. In a follow-up study, these questions will be answered to examine the potential of math trails to foster mathematical reasoning. This will be part of the MAP-Study “Modelling, Arguing and Problem Solving in Outdoor Mathematics” funded by Dr. Hans Messer Stiftung.

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