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Procedures in proof construction: five proving strategies of first-year university students

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As proof construction typically presents many difficulties to university students, there is an increasing interest in student-centred forms of in-process support. This is accompanied by a need for research on how students – effectively or non-effectively – proceed in proof construction. This study therefore examines the proof-construction behaviour of first-year university students in 24 cases. Following the principles of empirical type construction, the author identified five proving strategies, each of which is characterised by a specific combination of sub-processes. Comparing these strategies to each other provides new insights into both different approaches to proving and typical obstacles. However, a comparison between successful and non-successful students did not reveal a clear relationship between the strategy used and performance. Our results suggest that the crucial aspects of proof construction are related to implementation at the microscopic rather than macroscopic level.

Keywords: Reasoning and proof, proving strategies, tertiary education, type construction.

Introduction

Dealing with mathematical proof is a complex and demanding activity that causes great difficulties for many university students (Selden & Selden, 2008). One possible reason for these difficulties is related to a lack of transparency between proof as a product and its process of development (Hemmi, 2008). Therefore, there are many reasons to place more emphasis on underlying proving processes and to specifically train proving strategies in university teaching (Karunakaran, 2018; Schoenfeld, 1985). However, so far there has been little research on which such fostering programs could be based. This study, therefore, takes a process-oriented perspective on proving, and examines how university students behave in constructing proof. By comparing successful and non-successful students, the study, moreover, aims to identify procedures that are more effective than others.

A process-oriented perspective on proving

Although proof-writing processes run differently for each individual, previous research revealed some invariances occurring across different cases. On the one hand, there are efforts to describe proving processes as a combination of recurring sub-processes. Each sub-process focuses on different cognitive demands and pursues a specific sub-goal of proof construction. Another approach refers to general proving strategies that describe typical behaviour patterns within proof construction. The paper first gives a brief overview of both approaches before connecting them.

Sub-processes of proving

In the field of reasoning and proof, various models have been proposed to describe relevant sub-processes of proof construction (G. Stylianides et al., 2017). As the models differ in their target group (school students, university students and mathematicians), they consider different demands in terms of formalisation and rigor. While some models emphasise the creative part of proving and highlight

processes of exploration and discovery (Hsieh et al., 2012; Schwarz et al., 2010), others include precisising parts in the same way (Boero, 1999; Stein, 1986). However, from a synthesis of existing models four sub-processes emerge that are likely to be relevant in different contexts (Kirsten, 2018):

- Understanding the problem situation represented in the proving task,
- Identifying (informal) arguments that support the purported assertion,
- Structuring single proving steps in a logical chain,
- Formulating a proof according to the requirements of the community.

Taking the process of proving from the problem-solving perspective, it is reasonable to add a fifth sub-process, that is, checking the proof's validity (Carlson & Bloom, 2005; Schoenfeld, 1985). Although validation might be implicitly included in the sub-processes listed above, it seems useful for teaching and research to make validation visible in a separate sub-process. In fact, the assumption of a total of five sub-processes proved to be empirically valid in initial studies examining university students' proving processes (Kirsten, 2018). To that effect, there is empirical evidence that the five sub-processes display basic elements of proof construction and constitute an individual's proving process. However, there is limited research on how these sub-processes interact in proof construction and how, i.e., in what order and weighting, they can be fruitfully combined.

Proving strategies of university students

Proving strategies describe behavioural patterns that are characteristic for a proving process' overall structure. A proving strategy is thus characterised by features such as dealing with impasses, incorporating metacognitive processes, or planning an approach based on understanding (Zazkis et al., 2015). Examining the problem-solving behaviour of college students, Schoenfeld (1992) identified the *wild goose chase* strategy as a frequently used but highly inefficient approach to proving: "roughly 60% of the solution attempts are of the 'read, make a decision quickly, and pursue that direction come hell or high water' variety" (p. 356). Consistent with this, Karunakaran (2018) reports that first-year students often adhere to the linear structure of their proving process and therefore have difficulty with overcoming impasses. Both strategies reported here are likely to fail, because little time is spent on carefully analysing the problem situation or students do not adapt their process to the impasses occurred. Against this background, Zazkis et al. (2015) explicitly analysed the proving behaviours of six highly successful university students with reference to Pólya's four stages of problem-solving. Their analyses revealed two promising strategies, namely the *target strategy* and the *shotgun strategy*. The target strategy describes a highly systematic behaviour, where students chose their approach on the basis of a careful problem analysis, and continuously monitor their own progress. Students, on the other hand, who use the shotgun strategy, implement many different approaches in a short period of time. They spend little time on understanding the given statement or setting up a plan, but adjust their approach as soon as it proves to be of little use. In contrast to the wild goose chase approach, the shotgun strategy thus includes a higher level of process control and is, therefore, considered to be more effective (Zazkis et al., 2015).

Combining both frameworks, it can be assumed, that each proving strategy is reflected in a characteristic combination of sub-processes. For example, the target strategy proceeds in a straight line, while the shotgun strategy moves back and forth between identifying arguments and validating.

The current study

Summarising previous findings, process-oriented research provides initial evidence that proving processes consist of at least five different sub-processes, each of which makes its own contribution to proof construction. How students combine these sub-processes through proving strategies, however, has been studied scarcely, and predominantly in relation to episodes of problem solving (Schoenfeld 1985; D. Zazkis et al. 2015). To address this research gap, the current study aims to provide a holistic analysis of student proving processes that examines the interplay of individual sub-processes.

Research Questions

The current study pursues two main goals: first, the study aims to identify recurring proving strategies, each of which is characterised by a specific combination of sub-processes in terms of frequency, duration and sequence. Given the findings of Schoenfeld (1992) and Zazkis et al. (2015), it is reasonable that some proving strategies are more purposeful than others. Therefore, in a second step, we investigate the relationship between proving strategies and students' proof construction performance. Here, the aim is to identify effective and non-effective aspects of proving strategies in order to replicate and enhance previous findings. In particular, the research is guided by the following questions: (1) Which proving strategies do university students use in proof construction? (2) Is there a relationship between the proving strategy used and the students' performance? To address these questions, the study relies on an exploratory-descriptive design that allows fine-grained analyses.

Participants and data collection

To gain insights into students' proof-construction behaviour, the author conducted task-based interviews with first-year university students, including preservice teachers (Goldin, 1997). As part of a regular exercise, students were asked to complete the two proving tasks given in Figure 1. If students agreed, one of these tasks was solved under interview conditions. As is customary in the course, students were allowed to work in groups of two or three. In the interview, however, they received no support other than a standardised version of lecture notes (Schoenfeld, 1985). Following this procedure, the author and one of her colleagues conducted a total of 97 interviews, all of which were videotaped. From this total sample, we selected those cases for fine-grained analysis that were deemed representative of the entirety of cases. For this purpose, we first assessed the students' performance in proof construction by rating their final product on a four-point scale, with a score of 4 indicating a complete and valid proof and a score of 0 describing no substantial progress (Recio & Godino, 2001). Interrater reliability confirmed very good agreement for this rating ($\kappa = .82$). Based on the assessment of proving performance, the final sample was drawn by applying a qualitative sampling plan that follows the principles of maximum variation and homogenous sampling (Patton, 1990). Thus, the final sample includes 24 proving processes that are almost evenly distributed among the different performance scores and that cover both proving tasks equally.

Task 1 Let $f : [0, 1] \rightarrow [0, 1]$ be a continuous function. Show that f has a fixed point, that is, there exists a $x \in [0, 1]$ with $f(x) = x$.

Task 2 Consider a function $f : \mathbb{R} \rightarrow \mathbb{R}$ which is twice differentiable. Let $x_1 < x_2 < x_3$. Given $f(x_1) > f(x_2)$ and $f(x_3) > f(x_2)$, show that there exists a $y \in \mathbb{R}$ with $f''(y) \geq 0$.

Figure 1: Proving tasks used in the tasked-based interviews

Data analysis

In accordance with the research questions, data analysis was conducted in two stages. First, all proving processes were fully transcribed and coded leaning on Schoenfeld's (1985) method of protocol analysis. The proving protocols were partitioned into coherent episodes, each of which is assigned to one of the sub-processes listed above (Kirsten, 2018). The coding thus led to a macroscopic description of the students' processes, providing data on the frequency, duration and sequence of the sub-processes that occur ($\kappa = .92$). Based on this information, an empirical type construction was conducted in which processes with similar qualitative characteristics are grouped into types (Kluge, 2000). In particular, the transitions between sub-processes were considered in order to identify recurring sequences and empirical patterns that represent specific proving strategies. Each proving strategy thus describes a particular combination of sub-processes in terms of sequence and frequency. To identify particularly promising proving strategies, the second stage of analysis compared successful (score 3 or 4) and non-successful cases (score 0 or 1) in terms of strategy use.

Results

Our analyses of 24 proving processes revealed five different proving strategies that first-year students apply in the field of real analysis. The following section gives an overview about the proving strategies and its characteristics. Each strategy is illustrated by an exemplary process and its schematic representation.

Proving strategies and sub-processes used by first-year students

Proof constructions using the *step-by-step strategy* are characterised by a linear structure in which the individual sub-processes are run through in the natural sequence listed above. Doing so, all sub-processes are systematically built on one another, that is, each sub-process is worked through conclusively before the students move on to the next. In this strategy, the proving process is based on a careful analysis of the given statement and professes throughout. In particular, the choice of a proving approach is well-planned and highly target-oriented.

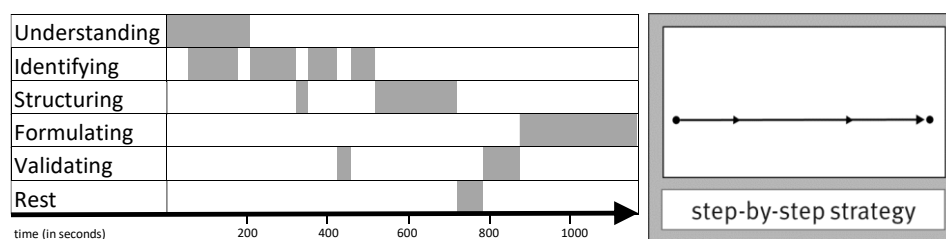


Figure 2: Exemplary sequence of sub-processes when using the step-by-step strategy (Alina & George)

The *scorekeeping strategy* is represented by proving processes in which a complete sequence of sub-processes is followed by an additional loop of brainstorming. This loop occurs rather accidentally and is initiated by a formulation process in which the students write down a response that is not satisfactory for them but serves to achieve partial points in the exercise. While formulating the response, a process of insight takes place and the students generate new ideas for proof construction. However, the students do not pursue the new approaches consistently and therefore end proof construction with a process of identifying arguments (see the case of Lisa, Pia & Laura in Figure 3).

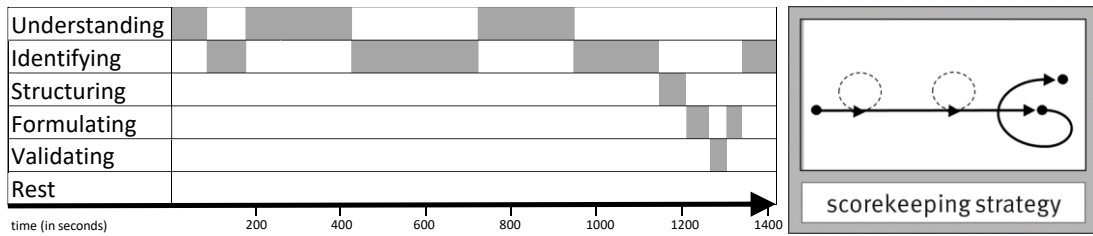


Figure 3: Exemplary sequence of sub-processes when using the scorekeeping strategy

Proving processes that are assigned to the *rebooting strategy* are characterised by a cyclical structure in which several approaches are implemented one after the other. Each cycle contains a linear, self-contained sequence of sub-processes such as understanding, identifying (informal) arguments and organising single proving steps. Thus, the rebooting strategy initially follows the step-by-step-strategy until the current approach proves to be of little use and a new cycle is initiated. The validation and rejection of an approach necessitates a restart which is characteristic for this strategy.

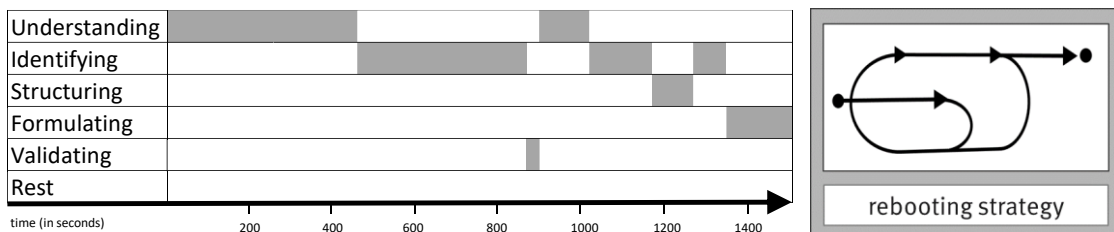


Figure 4: Exemplary sequence of sub-processes when using the rebooting strategy (Lukas & Tim)

When applying the *decomposing strategy*, a proving process shows a repeating pattern of development, elaboration and formulation, which is depicted in a staircase pattern in the schematic representation of the process. In contrast to the rebooting strategy, each cycle of development contributes to the final product, so there is a continuous progress. In our sample, students use the decomposing strategy in two different ways depending on the structure of the statement to be shown. Some proving tasks allow, for example, a distinction by cases which naturally divides the proving process into different parts. By considering sub-proofs, the complexity of proof construction is reduced and each part can be dealt with in a self-contained step-by-step approach (see the process of Andreas & Ibrahim in Figure 5). If no content or structural subdivision is created in the task, students perform a chronological division by writing down previous results from time to time. In some cases, formulating parts of the final response allows new insights and thus supports further progress.

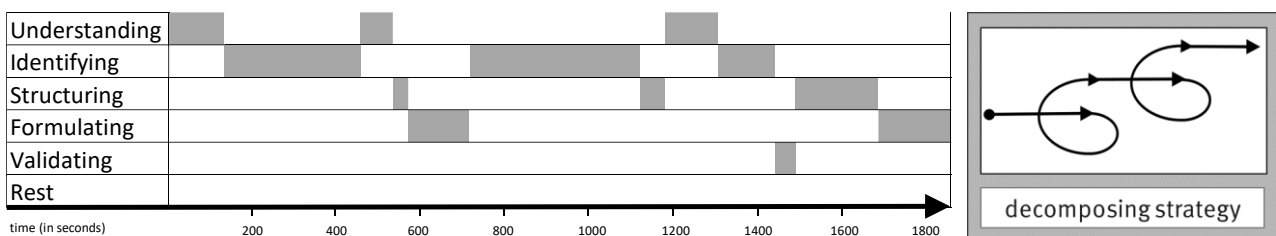


Figure 5: Exemplary sequence of sub-processes when using the decomposing strategy

The *looping strategy* describes a proof-writing behavior in which students frequently move back and forth between sub-processes. Due to an occurring uncertainty or impasse, students first return to the previous sub-process and then continue with their proving process based on the newly gained knowledge. Since the switches mainly occur between successive sub-processes such as understanding

and identifying or structuring and formulating, the looping strategy basically follows a linear approach but is interspersed with mini-cycles. In contrast to the step-by-step strategy, the looping strategy thus provides several deepening loops. Doing so, the individual sub-processes are worked out in interaction with each other and are thus more closely linked.

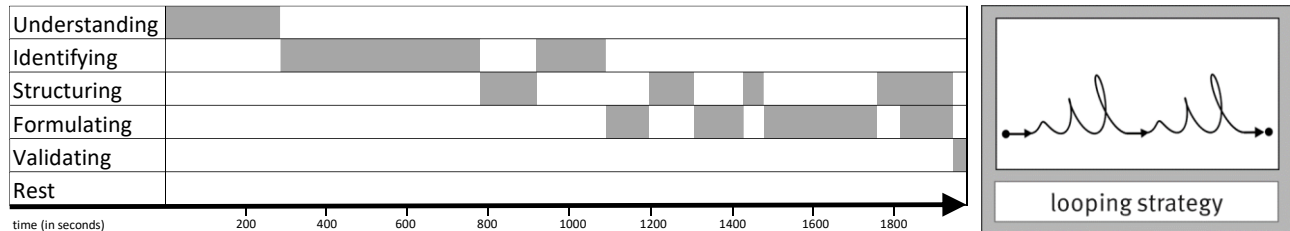


Figure 6: Exemplary sequence of sub-processes when using the looping strategy (Alina & Sascha)

Comparing the different descriptions, a basically linear structure emerges across all strategies. While the step-by-step strategy represents an idealized linear procedure, the decomposing and the rebooting strategies divide the process into linear subsequences and the looping strategy uses mini-cycles to deepen individual sub-processes within a mainly progressive procedure. Due to these modifications, each strategy places a different emphasis on the individual sub-processes.

Proving strategies and performance

To identify particularly effective strategies of proving, we compared successful and non-successful processes regarding the proving strategies used. Table 1 shows how the examined cases are distributed among the different strategies and performance categories. Each letter combination represents one proving process and refers to the initials of the students involved.

Performance	step-by-step	rebooting	decomposing	scorekeeping	looping
successful	ML, AG		FT		AS, TL
non-successful		LT, DP	AI, SDH	TH, LS, JH, OJ, LPL	MFY, JIA, LUM, LKH, DJ

Table 1: Distribution of the analysed processes among the different strategies and performances (successful = score 3 & 4, non-successful = score 0 & 1, score 2 excluded)

The distribution in Table 1 shows that with the step-by-step strategy, the rebooting and the scorekeeping strategy proof-writing behaviours exist, which are exclusively associated with a low or high performance. However, the observed pattern is only conditionally suitable for a hypothesis generation. While the low proving performance of scorekeeping is already anchored in its definition, the rebooting strategy describes an approach that, from a theoretical perspective, does have effective components because an unsuitable approach is rejected in favour of a new idea. A detailed review of the assigned cases suggests that the difficulties encountered here arise less from restarting, but rather are the result of a lack of conceptual, procedural or strategic knowledge. The decomposing and the looping strategy, on the other hand, occur across different scores. It is likely that these behavioural patterns are not effective per se, but that it is the specific implementation that matters.

Discussion and conclusion

In the presented study, we analysed the proof construction behaviour of first-year university students by looking at the frequency, duration and sequence of sub-processes typical for proving (Kirsten, 2018). Our research revealed five different proving strategies that replicate and complement previous findings. The step-by-step and the rebooting strategy are comparable to the target and the shotgun strategy reported by Zazkis et al. (2015). In addition, we identified two other strategies, namely the decomposing and the looping strategy, which can be considered as variants of the target strategy. Both strategies proceed target-oriented, but include several cycles of sub-processes in order to reduce complexity, handle impasses or create flexibility. The scorekeeping strategy is similar to wild goose chase processes in the way that students do not make progress over an extended period of time (Schoenfeld, 1985). Nevertheless, the students' efforts in this study go beyond a wild goose chase as there is a slight process control through the formulation of ideas. One possible reason for these differences is that proof construction in the university context is often associated with performance. Formulating partial solutions is a common strategy for coping with studies in the first year. Efforts in teaching should therefore aim at making this kind of strategy fruitful by valuing the late insights. In general, fostering programs such as heuristic worked examples, heuristic trainings, and learning videos should consider different types of proving strategies to provide authentic insights into proving.

Comparing successful and non-successful students, we could not establish a clear correlation between proving strategies and performance. Specifically, our results show that a proving strategy in which different approaches are implemented in the sense of the shotgun strategy is not necessarily associated with a high level of performance. As the study by Zazkis et al. (2015) only considers successful students, our results usefully complement the state of research and open the view for further aspects. On the one hand, it remains open whether proving strategies reflect an individual's disposition or whether they are task-, situation- or proof-specific. On the other hand, our results suggest that students might fail because they have difficulties with strategy implementation. Further research should therefore analyse students' proof-construction behaviour at a microscopic level. Especially those strategies that occur independently of performance should be analysed with particular care, as they can provide insights into effective and dysfunctional aspects of implementation.

References

- Boero, P. (1999). *Argumentation and mathematical proof: A complex, productive, unavoidable relationship in mathematics and mathematics education*. International Newsletter on the Teaching and Learning of Mathematical Proof.
<http://www.lettredelapreuve.org/OldPreuve/Newsletter/990708Theme/990708ThemeUK.html>
- Carlson, M. P., & Bloom, I. (2005). The cyclic nature of problem solving: An emergent multidimensional problem-solving framework. *Educational Studies in Mathematics*, 58(1), 45–75. <https://doi.org/10.1007/s10649-005-0808-x>
- Goldin, G. (1997). Observing mathematical problem solving through task-based interviews. In A. Teppo (Ed.), *Qualitative Research Methods in Mathematics Education, Journal for*

- Research in Mathematics Education Monograph No. 9* (pp. 40–62). National Council of Teachers of Mathematics. <https://doi.org/10.2307/749946>
- Hemmi, K. (2008). Students' encounter with proof: the condition of transparency. *ZDM*, *40*(3), 413–426. <https://doi.org/10.1007/s11858-008-0089-9>
- Hsieh, F.-J., Horng, W.-S., & Shy, H.-Y. (2012). From Exploration to Proof Production. In G. Hanna & M. de Villiers (Eds.), *New ICMI Study Series. Proof and Proving in Mathematics Education* (pp. 279–303). Springer Netherlands. <https://doi.org/10.1007/978-94-007-2129-6>
- Karunakaran, S. S. (2018). The Need for "Linearity" of Deductive Logic: An Examination of Expert and Novice Proving Process. In A. J. Stylianides & G. Harel (Eds.), *ICME-13 Monographs. Advances in Mathematics Education Research on Proof and Proving: An International Perspective* (pp. 171–183). Springer. <https://doi.org/10.1007/978-3-319-70996-3>
- Kirsten, K. (2018). Theoretical and Empirical Description of Phases in the Proving Processes of Undergraduates. In V. Durand-Guerrier, R. Hochmuth, S. Goodchild, & N. M. Hogstad (Eds.), *Proceedings of the second conference of the International Network for Didactic Research in University Mathematics (INDRUM)* (pp. 326–335). University of Agder and INDRUM.
- Kluge, S. (2000). Empirically Grounded Construction of Types and Typologies in Qualitative Social Research. *Forum Qualitative Sozialforschung*, *1*(1), Article 14. <https://www.qualitative-research.net/index.php/fqs/article/view/1124/2499>
- Patton, M. (1990). *Qualitative evaluation and research methods* (2nd ed.). Sage.
- Recio, A. M., & Godino, J. D. (2001). Institutional and personal meanings of mathematical proof. *Educational Studies in Mathematics*, *48*, 83–99. <https://doi.org/10.1023/A:1015553100103>
- Schoenfeld, A. H. (1985). *Mathematical problem solving*. Academic Press. <https://doi.org/10.1016/C2013-0-05012-8>
- Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition and sense making in mathematics. In D. Grouws (Ed.), *Handbook for Research on Mathematics Teaching and Learning* (pp. 334–370). Macmillan Publishing Co, Inc.
- Schwarz, B. B., Hershkowitz, R., & Prusak, N. (2010). Argumentation and mathematics. In K. Littleton & C. Howe (Eds.), *Educational Dialogues: Understanding and Promoting Productive Interaction* (pp. 115–141). Taylor & Francis. <https://doi.org/10.4324/9780203863510>
- Selden, A., & Selden, J. (2008). Overcoming students' difficulties in learning to understand and construct proofs. In M. P. Carlsson & C. Rasmussen (Eds.), *Making the Connection: Research and Practice in Undergraduate Mathematics* (pp. 95–110). Mathematical Association of America. <https://doi.org/10.5948/UPO9780883859759.009>
- Stein, M. (1986). *Beweisen. [Proving]*. Franzbecker. <https://doi.org/10.1007/BF03339266>
- Stylianides, G., Stylianides, A. J., & Weber, K. (2017). Research on the teaching and learning of proof. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 237–266). The National Council of Teachers of Mathematics, Inc.
- Zazkis, D., Weber, K., & Mejía-Ramos, J. P. (2015). Two proving strategies of highly successful mathematics majors. *The Journal of Mathematical Behavior*, *39*, 11–27. <https://doi.org/10.1016/j.jmathb.2015.04.003>