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# Student teachers' proofs and refutations on cyclic quadrilaterals

Magdalini Lada<sup>1</sup> and Tore Alexander Forbregd<sup>2</sup>

Norwegian University of Science and Technology (NTNU), Trondheim, Norway;

<sup>1</sup> [magdalini.lada@ntnu.no](mailto:magdalini.lada@ntnu.no), <sup>2</sup> [tore.a.forbregd@ntnu.no](mailto:tore.a.forbregd@ntnu.no)

*We follow and analyze three teacher students' behavior while they engage in a task with the goal of the reinvention of the characterization of cyclic quadrilaterals in terms of their opposite angles. We show how, during the exploration of the problem, the teacher students go through cycles of proofs and refutations, in the sense of Lakatos, and we discuss the role of the teacher educator in supporting and conceptualizing these processes. Our results may have the potential to be used in designing mathematical activities aiming at scaffolding teacher students' knowledge for teaching proof.*

*Keywords: Proof, counterexample, conjecture, task design, teacher knowledge.*

## Introduction.

Lakatos' *Proofs and refutations* (1976) marks the turn that Philosophy of Mathematics took towards fallibilism, the view that mathematical knowledge is corrigible and advances through a process of criticism and revision (Van Bendegem, 2018). Lakatos used the history of Euler's formula for polyhedra and its proofs as a basis for describing the road to mathematical discovery and at the same time presenting different views, over time, on what a satisfactory result and a valid proof consist of.

Besides its philosophical importance, Lakatos' work is also significant from a pedagogical point of view since it provides us with a window to original mathematicians' work presented in an educational setting. The idea of using Lakatosian notions in designing, implementing, and analyzing authentic mathematical activities is not new (Lampert, 1990). Several researchers have used Lakatos' ideas to develop frameworks that can be used for analyzing students' behavior in proving activities in various levels of mathematics education from primary (Komatsu, 2010), to secondary (Komatsu, 2012) and into undergraduate level (Larsen & Zandieh, 2008). Other studies have focused on designing activities that foster methods of proofs and refutations (Komatsu & Jones, 2017), or on teachers' mathematical knowledge for teaching Lakatosian proof (Deslis et al., 2021). Our study adds to this growing body of research-based literature first by providing some more examples of Lakatosian notions and processes, this time from teacher education, and second, by suggesting elements of instructional design aiming to build teachers students' mathematical knowledge for teaching Lakatosian proof. For the purpose of this article, we consider Lakatosian proof as it is described in (Deslis et al., 2021). However, discussed in the last section, authentic Lakatosian proving activities need to be further understood and defined. We seek to answer the following questions: 1) In which ways do teacher students' attempts to reinvent a characterization of cyclic quadrilaterals parallel the methods of Lakatos' proofs and refutations? And 2) How can teacher educators design and implement tasks that support teacher students' mathematical knowledge for teaching Lakatosian proof?

## Theoretical background.

With starting point the naive conjecture "All polyhedra satisfy the formula  $V - E + F = 2$ ", where  $V$  stands for vertices,  $E$  for edges and  $F$  for faces, and Cauchy's proof (Lakatos, 1976, p. 7), and after

counterexamples are discovered, Lakatos described techniques for improving the conjecture and its proof. In this process crucial role is played by the type of counterexamples. Lakatos referred to counterexamples refuting the initial naive conjecture as *global* and to those refuting at least one step of the proof as *local* (ibid., p.11). According to the type of the encountered counterexample, a method was suggested that would lead to an improved conjecture or/and proof. We only mention here those methods that are needed for analyzing our findings.

One of the first methods, *exception barring*, was proposed when the global and local counterexample of a picture-frame polyhedron was discovered and was initially described as restricting the domain of the conjecture to exclude the counterexample, without reference to the proof (ibid., p. 28). As opposed to this, *lemma incorporation* was suggested as investigating the proof to identify the step which is refuted by the counterexample and incorporating it as a condition in the domain of the conjecture (ibid., p. 35.) Later in the discussion the exception barring method is generalized to include “a whole continuum of exception barring attitudes” (ibid., p. 39), according to how much the proof is used to determine the domain of the conjecture, with lemma incorporation being the limiting case. Lemma incorporation can also be used in the opposite way to expand the domain of a conjecture that has been previously restricted by exception barring. According to Lakatos this is the approach of the *best exception-barrers* and it involves starting with a restricted *safe domain*, devising a proof, examining the proof for imposed conditions that were not used, and generalizing the initial modest theorem by incorporating only those conditions that proof relies on (ibid., p. 40).

Mathematicians’ views on *satisfactory* results are voiced via Lakatos’ fictional student Omega at a point where different proofs, corresponding to different domains had been proposed, like Gergonne’s (ibid., p. 63) and Legendre’s (ibid., p. 64) proofs, none of which was including the *local but not global* counterexample of the great stellated dodecahedron (ibid. p. 65). According to Omega, “a proof should explain the phenomenon of Eulerianness in its entire range” (ibid., p.67). Omega suggests that when encountering a local but not global counterexample, it might be needed to invent a “completely different, more embracing, *deeper* proof” (ibid., p. 62).

Two opposite paths towards mathematical discovery are discussed in (Lakatos, 1976). The first one, which Lakatos calls *analysis*, involves starting with a naive conjecture, to which one arrives by trial and error, and testing it by drawing consequences (ibid., p. 80). This path we learn later, can be followed even without a conjecture to start with. One can pretend that “the result is there and device an analysis” (ibid., p. 83). The other direction, which Lakatos called *synthesis* (ibid., p. 81), is what we normally call a deductive proof. For this direction too, starting with a conjecture is not necessary. One can immediately “device a synthesis [...] from a related proposition that is known to be true” (ibid., p. 83). In this way, synthesis corresponds to another term introduced by Lakatos, *deductive guessing*, as opposed to naive guessing, and to what Herbst (2004) refers to as building reasoned conjectures. As an aid to support students’ deductive guessing, Herbst proposes the *generative mode* of interaction between a student and a diagram which involves, among others, drawing new features on the diagram, transforming its shape and size, and changing location or orientation.

## **Method.**

### **The course and the participants.**

This study was conducted within the context of a course titled Historical and Philosophical Aspects of Mathematics aimed at student teachers in their 4<sup>th</sup> year of a five-year master program for grades 5-10 in a Norwegian university. Our informants are three of the courses participants and have gotten the fictional names Marie, Lars, and Siri. Our data consists of their solutions to a task included in one of the course's mandatory assignments supplemented by semi-structured follow-up interviews. The assignment was given after the last teaching session, which had focus on proof and proving. During the session, after an introduction to Lakatos' proofs and refutations were concepts like conjecture, proof analysis, global and local counterexample, and their role in proving were presented, the students discussed how Lakatos' ideas could be used in the mathematics classroom and how a teacher can support pupils in engaging in Lakatosian proof. Their discussions were based on the papers by Komatsu (2012) and Larsen & Zandieh (2008) that the students had read in advance as preparation.

### **Task selection.**

The task included in the students' assignments was meant to function as a starting point for involving them in genuine mathematical activity by reinventing a mathematical result. For this purpose, we considered the following characteristics that the target result should be: 1) new for teacher students but within reach using tools from school mathematics, 2) analogous to previously established result known to teacher students, and 3) valid under certain conditions – necessary or/and sufficient.

The characterization of quadrilaterals in terms of their opposite angles, which can be formulated as "A quadrilateral is cyclic if and only if its opposite angles are supplementary", has all the above characteristics. It is not normally included in the school curriculum in Norway, but its proof is based on the inscribed angle theorem which is. Also, our students are familiar with the circumcircle of a triangle which can serve as the motivation for investigating a similar question for quadrilaterals. Nevertheless, we decided to include this as the first part of the task, which was formulated as follows:

- a. Prove that the three perpendicular bisectors on the side of a triangle meet at one point and that this point is the center of a circle that goes through the triangle's vertices.
- b. Investigate if a similar result holds for quadrilaterals. State a conjecture and try to prove it.

Three students, all having delivered different solutions, were chosen for the follow-up interviews.

### **The interviews.**

The three students were interviewed after the end of the course and before the exam. The interviews varied from 40 to 50 minutes and took place in a digital environment. The interviewers were the authors of the paper where the first author was, in addition, the teacher of the course. With the students' different solutions as a starting point, we intended first to get an insight on how each one had arrived at their solution and then try to stimulate them to continue the investigation and reach a deeper result. Following Lakatos' heuristic methods, we aimed to offer, or lead the students to discover, counterexamples to their claims or arguments, anticipating that this would lead them to modify the proposed claim or argument. The initial goal was to lead the students to formulate and

prove the characterization of cyclic quadrilaterals as those that where opposite angles are supplementary. This goal soon showed to be too optimistic, so we ended all interviews once a necessary condition was conjectured by the students, namely “If a quadrilateral is cyclic, then its opposite angles are supplementary”. As a tool to assist in exploring the problem, we used the dynamic geometry software Geogebra on a shared screen. In Marie’s interview, the researcher controls the software and shares his screen, while in the other two interviews, the students do so.

### Analysis.

In what follows, the first author is referred to as the teacher, the second as the researcher. The analysis is based on and structured over Marie’s interview, but supplementary comments based on data coming from the other two students are also included.

### Surrendering.

Marie presents in her solution a random quadrilateral with two circles drawn, each one going through two of the vertices. She concludes that “it will therefore not work to make a circle that goes through all vertices at the same time, but one can make a set of circles so that all vertices lie on one of them”.

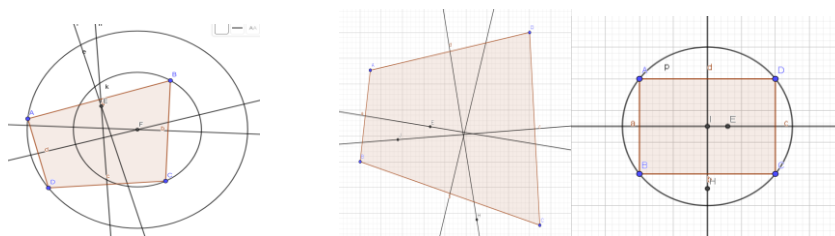


Figure 1: Marie’s multiple circles and Lars’ two examples

When asked if she had an initial feeling that led to her strategy the student replied that she thought it should not work (to have a circumscribed circle for a quadrilateral) since she had never heard of it. The drawing she presented was her first move towards dealing with the task and since it confirmed her suspicion, she decided to end the investigation following the method of surrender (Lakatos, p. 14). She seemed to think that this was enough for what the problem was asking: “I got a result and I thought, ok, I found it out”.

Another student, Lars, that had managed to come a bit further in his investigation and had concluded that some quadrilaterals are cyclic while others are not, explained in the interview that if he was to go any further and try to find out what all cyclic quadrilaterals have in common, this should have been explicitly asked for as the next part of the problem. He also felt that his conclusion was enough.

### The two ways of using lemma incorporation.

In the interview, Marie admitted that she could have worked more with the task, which was the starting point for us to trigger her to investigate a bit further. The teacher asked the student if she could think of any quadrilateral that could be inscribed in a circle.

- Marie: If I had drawn a quadrilateral with right angles, and maybe equal sides, a square, then I would have gotten a center if I am right [...] I just see that when the sides are equal you will get this diagonal’s cross and you can have the center there.
- Teacher: What is it that makes it work for squares? Can we find more classes?

Marie: All rectangles then? Because then you will get that diagonal's cross [...] which is equally far [from the vertices]

Marie retreats to quadrilaterals with right angles and equal sides, which includes only squares, as the safe domain of her theorem. She argues that, in this case, the diagonals' intersection point can be used as a center for a circle that goes through all vertices. This argument can, of course, work for any quadrilateral having right angles, that is, any rectangle. It can thus be incorporated as a condition expanding the domain of the theorem. Marie manages to do so as soon as the teacher turns the focus on the argument. This process parallels what Lakatos (1976) described as a combination of exception barring and lemma incorporation.

Siri, another of our students, used lemma incorporation to restrict the domain of her claim. She had overgeneralized from rectangles the property that the perpendicular bisectors of opposite sides coincide and stated that "all parallelograms are cyclic" because of this reason. When encountering a local and global counterexample offered by the researcher - a random non-rectangular parallelogram - she restricted the domain to include this property as a condition ending up with only rectangles as the domain of the theorem.

### **More local counterexamples and the need of a deeper proof.**

Recognizing that the argument concerning the intersection point of the quadrilateral cannot be used to include other classes than rectangles the educators suggest that we look at the problem in another way, namely starting with a quadrilateral that lies on a circle. All drawings and their manipulations are performed and shared by the researcher as the discussion goes on.

Researcher: If you start at the opposite end can you draw a quadrilateral that lies on a circle?  
Marie: ... (seems confused)  
Teacher: so if you start by drawing a circle  
Marie: then I can set two random..., if you want to have a rectangle then, parallel lines anywhere on the circle so that you get four intersection points

Marie aims at constructing an inscribed rectangle, but her instructions lead to a non-rectangular isosceles trapezoid. This is a local counterexample since it refutes her argument but still supports the conclusion of her theorem. At first, Marie gets surprised with the outcome and resists in expanding the domain of the theorem.

Researcher: Was this a rectangle now?  
Marie: No, it wasn't ... (laughs)  
Researcher: What does this tell us?  
Marie: That it is possible to draw a quadrilateral in a circle. But it doesn't say anything about the center, the intersection point of the perpendicular bisectors.

The researcher draws the perpendicular bisectors to confirm that the intersection point coincides with the circle's center. This seems to confuse her even more.

Marie: Now I am so confused that I don't know anymore  
Researcher: What did you get confused about?  
Marie: About what I have done in the problem

The researcher explains the steps that led us from starting with a circle to drawing an inscribed isosceles trapezoid and eventually Marie accepts that the domain should be expanded.

Marie: So, while it works for trapezoids and rectangles and squares, it is just for my odd quadrilateral that it doesn't work...that it doesn't have any parallel sides.

Here, there is a missed opportunity of offering or leading the student to discover another counterexample. Maria has not noticed that the trapezoids need to be isosceles. This could have led to focusing on investigating earlier and more naturally the properties of the angles of cyclic quadrilaterals. Instead, the researcher chooses to focus on the parallel sides' condition imposed by the student and draws a random inscribed quadrilateral with no pair of parallel sides to offer as a counterexample. The student gets surprised once more.

Marie: It works here too

Researcher: You seem surprised

Marie: I am confused

Researcher: Why are you so surprised? If you think what we have done until now...

Marie: Here it makes sense because we have kept us in a circle, but when I tried to do it the opposite way it didn't work, or at least I couldn't make it work

Assuming that the conclusion is true for a random quadrilateral – that is, the quadrilateral is cyclic – and drawing consequences was particularly challenging for all three students that we interviewed. This heuristic method, which parallels what Lakatos called analysis (Lakatos, 1976, p. 80), can, in our case lead to discovering the necessary and sufficient conditions so that a quadrilateral is cyclic, and synthesizing the proof arriving in a final theorem. However, the student here does not seem to be aware of this possibility. Similar reactions were observed with the other two students we interviewed. The teacher educator, at this point, paused and explained how taking this direction can help us explore the properties that all cyclic quadrilaterals have in common.

### Arriving to a conjecture by generative interaction with diagrams.

The researcher now sets it as a goal to describe all quadrilaterals inscribed in the same circle and asks the student how she would go on towards this goal.

Marie: it was easy to start with the squares because all vertices are equally far from a center...but in such rare quadrilaterals it is hard to find a center.

Because of symmetry, it is easy to visualize the center for the case of squares and rectangles. The student knows that the diagonals' intersection point is equally far from the vertices of a square and reasoned from this to get her initial conjecture/theorem. Even with the diagram showing a "rare" quadrilateral inscribed in a circle the student hesitates to take further action.

At this point the researcher interacts with the diagrams to lead the student to arrive at a new conjecture. In what follows, we describe some of the actions the researcher took.

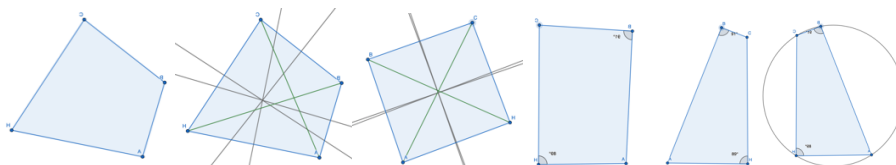


Figure 2: Generative mode of interaction

The researcher removes both the circle and circle's center from the drawing to move the focus away from the circle's center. To show to the student that the argument must be about something different from the diagonals the researcher draws the diagonals and the perpendicular bisectors and shows that

it is only in the case of the rectangles that the intersection points of those coincide. To turn the focus to the angles, he marks and measures a pair of opposite angles, moves around the vertices, and lets the student observe what happens to the angles and make a hypothesis. After discussing what changes and what remains the same the student manages to arrive at the conjecture: “if a quadrilateral is cyclic their opposite angles are supplementary”, which ended our investigation.

Although the mode of interaction with the diagram is generative, the way the student reaches the conclusion is rather empirical. The conjecture lays on observations, while the reason it works is still hidden and, to get closer to what Lakatos would call deductive guessing, one needs to make it visible.

### **Discussion and conclusions.**

Several methods Lakatos (1976) described were observed during our students’ explorations. A difference between our task and the problem the fictional students of Lakatos were investigating is that while for the Euler formula the number of supportive examples, together with what was then thought to be a polyhedron, made it natural to consider all polyhedra as the domain of the naive conjecture, in the case of quadrilaterals one quickly realizes that most of them are not cyclic. This resulted in our students restricting the domains of their theorems too much. Then, counterexamples like the isosceles trapezoid that Marie came upon or a random quadrilateral inscribed in a given circle, although local in the sense that they refuted her main argument, cannot be considered as non-global since they were not included in the domain of the conjecture/theorem. For the same reason, the method of analysis as exemplified in our investigation which lacks a broad naive conjecture, would be starting by assuming that a quadrilateral is cyclic and drawing consequences. Another deviation is that in Lakatos’ classroom the counterexamples stem from the definition of a polyhedron and how this changed over time while theorizing. In our activity, this interplay between defining and proving, is absent. Although the criteria under which our task was designed were adequate for capturing some Lakatosian processes, authentic Lakatosian activities should also invite for exploring the role of definitions in proving and, in the opposite way, the role of proving in concept formation.

Our students’ solutions showed that they stopped their investigation before reaching a mathematically satisfactory result. From the interviews, it became clear that presenting a partial solution like Marie’s and Lars’, as shown in figure 2, was experienced by the students as satisfactory enough for the purpose of the problem. Besides the students’ reluctance to further investigate the problem, it was also revealed that they might lack the techniques for doing so. This became apparent when we suggested working with the problem in the opposite way, starting with a circle and drawing an inscribed quadrilateral to determine necessary conditions for a random quadrilateral to be cyclic. The students seemed confused and could not anticipate what this would lead to. These findings, call for negotiating the didactical contract and offering more tasks that demand taking on responsibility in mathematical discovery in general and in exploring necessary conditions in particular. The same conclusion regarding instructional norms is also supported by analyzing how students interacted with the diagrams. The dynamic software environment that we used during the interviews, and which the students also used spontaneously when working independently with the problem, proved to be a useful tool for finding examples, counterexamples, and possible arguments. But just offering this as a tool was not enough to build reasoned conjectures. As Herbst (2004) also concluded, norms have



to be breached in engaging students in generative interaction. A reformulation of the task may also be needed to assist students in deductive guessing. Last, conducting the study in a physical classroom environment may enable the students to interact more independently with the diagrams.

Our analysis gave insight into our students' mathematical knowledge for teaching Lakatosian proof, particularly the component Deslis et al. (2021) called knowledge of content. It revealed that the students could not always use examples in ways that would lead them to a satisfactory mathematical result, without further guidance. This, in turn, might translate into students' low competency in guiding pupils in using examples in productive ways, which is associated with the students' knowledge of teaching (Deslis et al., 2021). These observations suggest that besides exposing students to tasks that would naturally engage them in Lakatosian proof, instruction that aims to build their mathematical knowledge for teaching should also include discussions centered on specific Lakatosian techniques that can be followed after a supportive/counter example has been encountered.

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