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The use of a deductive reasoning when solving a word problem

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The response of problem solvers to a request like ‘Explain your answer’ does not necessarily provide the assurance that the mathematical representation describing a word problem situation is correct. We suggest to compliment a traditional algebraic word problem with a request to establish a mathematical representation generalizing the problem situation and then to justify its derivation of this mathematical representation. Next, we suggest the construction of a valid deductive argument and evaluate the justification. An example executing the suggestions is given.

Keywords: Word problems, deductive argument, proof validation, magnitudes.

Introduction

Recognition of the importance of proof in school mathematics has increased during recent decades (Balacheff, 1988; Hanna, 2000; Harel & Sowder, 2007; A.J. Stylianides, 2007; G.J. Stylianides, 2008). There are several books presenting new forms of problems that are designed to engage teachers in the practice of reasoning and proving (Arbaugh et al., 2019; A.J. Stylianides, 2016). However, despite these trends in the mathematical research area, the choice of ways to learn to prove in a classroom seem to be quite poor when compared with the available resources for traditional tasks.

Usually word problems have no relationship to reasoning or proving. Their aim is to learn to apply mathematics in contexts outside of mathematics. Word problems are designed just to find an answer to the formulated question. In itself, solving word problems is a great challenge for students as well as for teachers (Verschaffel et al., 2000). By modifying word problems from a plain search for an answer to a tool for reasoning, we hope to help students to make better progress in this type of task. We therefore suggest practising mathematical reasoning in the process of solving word problems. Uncovering logical relationships in the search for mathematical representations of word problems is a way to understand solutions and to make sense of them.

Our research question is how can deductive reasoning be taught and learnt when solving word problems?

The logical connections between statements constitute an important knowledge which students should learn about a deductive reasoning in school mathematics. For example, the rules for drawing the Modus Ponens inferences appear well-developed by early adolescence according to various studies from psychological research (Stylianides & Stylianides, 2008). Therefore the modified word problems considered below are appropriate for students of lower secondary education. This immensely increase opportunities to engage in reasoning-and-proving activities (also see Chapter 5 in Arbaugh et al., 2019, for other arguments).

The discourse of a classroom should be central to the teaching and learning when using modified word problems. This facilitates the exchange of ideas between students and a teacher. Combining different levels of discourse could benefit students' deductive reasoning (p. 22 in Blanke, 2018).

Algebraic word problems

To explain the idea we comment on the supposed phases and components of solving a word problem. The solution is obtained by the following six parts of a *problem solving process* (Verschaffel et al., 2020, pp. 909–910):

- (a) the construction of an internal model of the problem situation, reflecting an understanding of the elements and relations in the problem situation; (b) the transformation of this situation model into a mathematical model of the elements and relations that are essential for the solution; (c) working through the mathematical model to derive mathematical result(s); (d) interpreting the outcome of the computational work; (e) evaluating whether the interpreted mathematical outcome is computationally correct and reasonable; and (f) communicating the obtained solution.

The question is what makes us sure that the solution that is obtained is correct with respect to the original problem. We argue that the evaluation suggested by part (e) does not provide this assurance. Part (e) refers to the answer obtained from part (c) and the interpretation made in part (d). However, the question that remains is whether the mathematical model constructed in part (b) is correct. In other words, if the solver's understanding of the problem or the transformation to the mathematical model are incorrect then the actions suggested by part (e) do not help. We therefore suggest using reasoning-and-proving activity to make sure that the mathematical model in part (b) is correct.

From now on we replace the phrase 'mathematical model' in the description of the problem solving process by the phrase 'symbolic representation', for two reasons. First, we consider traditional word problems as a tool for learning the abstract concepts of school mathematics rather than as a tool for relating mathematics to the real world. In this paper a *symbolic representation* is a generalization of the relationships in a problem expressed in terms of mathematical symbols, such as an equation, a system of equations, inequalities, and so on.

The second reason for the change in terminology is the discovery that the symbolic representations of the problem situations examined in this article are based on a suitable mathematical structure. We hypothesize that such a situation is typical for traditional algebraic word problems. The mathematical structure comes from the hidden properties of the magnitudes describing the word problem. These properties surface when we search for the premises of deductive reasoning. Thus, when considering word problems we pay special attention to magnitudes and their properties.

Now we can specify the above mentioned discovery that a mathematical representation of an algebraic word problem is based on a suitable mathematical structure. Let the three magnitudes A, B and U be given. Let $[A : U]$ and $[B : U]$ be numerical values of A and B with respect to the unit magnitude U. Then the numerical value $[A + B : U]$ of the (non-arithmetical) sum $A + B$ of magnitudes is the arithmetical sum of the numerical values $[A : U]$ and $[B : U]$, or in symbols we have the equality:

$$[A + B : U] = [A : U] + [B : U]. \quad (1)$$

For example, if A is a length and B is a length then one can take two line segments with lengths A and B , respectively. Suppose that $[A : U]$ and $[B : U]$ are the corresponding numerical values measured with respect to a unit length U . Then the numerical value of the length $A + B$ of the two end-to-end concatenated segments is obtained by the equality (1).

Magnitudes usually appear in a contextual representation of the word problem contained in part (a) of the problem solving process. Some of them are unknown and have to be found. By part (b), the contextual representation is transformed into a symbolic representation by means of the relationships between the numerical values of the magnitudes. For example, all symbolic representations of the word problems considered in this paper are equations relating numerical values of magnitudes to each other. The unit names are added back to the numerical values when the result of the computational work is interpreted, as listed in part (d) of the problem solving process.

We are now ready to name the class of word problems studied in this paper. An *algebraic word problem* (AWP) is a verbal description of a problem in which one or more questions are raised, the answers to which can be obtained by establishing and solving an equation with respect to a numerical value of unknown magnitude. This is a modification of the description of an arithmetic word problem given by Verschaffel et al. (2000).

The primary goal of our suggestion is to teach and learn mathematical reasoning in school mathematics. With this aim, in addition to the usual question in an algebraic word problem we have two tasks. The first of these is to justify a symbolic representation that answers the question, and the second is to prove the stated representation by constructing a valid deductive argument. The second task is required since in this paper a ‘justification’ is a broader term than a ‘proof’: a *justification* means a set of arguments used to give reasons why a conjecture is true. We use the symbol AWP+2T to denote an AWP together with the two added tasks. The proof, in the form of deductive reasoning, assures us that the overall solution is correct with respect to the original problem. In sum, a problem solving process of an AWP+2T has a new part, (b’), consisting of proving the symbolic representation, while parts (a) and (b) now provide a response to the first task.

Framework for reasoning-and-proving

In this paper, reasoning-and-proving activities are considered when solving algebraic word problems. Here we recall the analytic framework describing the meaning of reasoning-and-proving given by G.J. Stylianides (2008) (see also Arbaugh et al., 2019).

In mathematics, a proof of a new piece of knowledge is the final step of a work researching a mathematical phenomenon. It is preceded by asking questions, searching for patterns, making conjectures, and going back and forth. We use the mathematical component of the *analytic framework* that integrates three activities: identifying patterns, making conjectures, and providing arguments which may or may not qualify as proofs.

How does solving word problems with reasoned judgement fit into the analytic framework? Briefly, the three activities of the mathematical component correspond to parts (a), (b) and (b’) of the problem solving process of AWP+2T. In other words, identification of patterns corresponds to searching for

mathematical relationships in the problem, making conjectures corresponds to formulating a symbolic representation, and giving arguments should prove the correctness of the symbolic representation. Analysis of the symbolic representation of the word problem and the computational work are therefore not part of the analytic framework, and nor are they a focus of the present paper.

An AWP problem is usually described in terms of magnitudes. We interpret an attempt to understand the mathematical relationships between the magnitudes as a search for patterns in the context of the analytic framework. On what basis does a problem solver choose one possible mathematical relationship over other possible relationships? This question corresponds in the framework to the separation between plausible patterns and definite patterns. The patterning activity is called ‘definite’ if it is mathematically possible for a problem solver to provide conclusive evidence for the selection of one specific pattern over other patterns that also fit the data. In some tasks, the necessary information for the definiteness of a pattern is given explicitly (G.J. Stylianides, 2008; Arbaugh et al., 2019). How can we make definite choices between possible patterns when solving AWP+2T? The basis for this choice is given by the mathematical structure describing the properties of magnitudes. The structure we use below is relation (1) as explained in the preceding section.

The third and final framework activity is the construction of arguments which may or may not qualify as proofs. The term ‘proof’ is used with the meaning suggested by A.J. Stylianides (2007) – a valid argument based on accepted truths for or against a mathematical claim. The term ‘valid’ indicates that the assertions making up the argument are connected by means of accepted canons of correct inference such as modus ponens and modus tollens. The term ‘accepted truth’ refers to a class of statements like axioms, theorems, definitions, modes of reasoning and representational tools that a classroom community may take as shared at a given time. We call such statements mathematical. An argument that qualifies as a proof makes explicit reference to the accepted truths that it uses. Next, we elaborate on the meaning of accepted truth, since it is too narrow when solving word problems with reasoned judgement.

In the process of solving an AWP+2T problem, one needs to construct arguments proving a justification of the symbolic representation. The proof is based on a deductive argument, which is a series of statements consisting of premises and a conclusion. Clearly, the conclusion must be the symbolic representation of the problem. To specify a class of possible premises, we note that mathematical statements may not be sufficient. Statements about a word problem may refer to a contextual representation of facts about an imaginary world and may not be a part of reality. The truth of such statements is therefore of a different kind. The new term ‘accepted truth in context’ will refer to a class of mathematical statements as well as statements about a word problem that a classroom community may take as shared at a given time.

Definition 1. *Proof by context* is a valid deductive argument with the premises being accepted truth in context and with the conclusion being a symbolic representation of the problem.

A deductive argument is said to be valid if and only if it takes a form that makes it impossible for the premises to be true and the conclusion nevertheless to be false. Otherwise, a deductive argument is said to be invalid.

Having a precise meaning of the phrase ‘proof by context’, we can work on a definition of the term ‘justification’ by adopting the ‘proof schemes’ approach of Harel and Sowder (1998). Similarly, we suggest using three levels of justification: by external sources, as a collection of facts, and proof by context. The changes reflect the specific features of word problems. Justifications by external sources are those in which problem solvers give reasons based on (a) the ritual or the form of the appearance of the argument, (b) the word of an authority, such as a teacher, and (c) some symbolic manipulation without reference to meaning. For example, the reason is ritual if there is no mention of constant speed in situations discussing the movement of objects. Justification as a collection of facts appears when there is no mention of the logical relations between statements leading to a hypothetical conclusion. Logical connections ensure the validity of a deductive argument, leading to the third level of justification.

An example of AWP+2T

This section answers the first research question of how deductive reasoning can be taught and learnt when solving word problems. We take *any* algebraic word problem (AWP) and complement it with two tasks. The first task is to justify a symbolic representation that answers the question, and the second is to prove the justification of the symbolic representation. The symbol AWP+2T denotes the resulting modified AWP.

A solution of AWP+2T is obtained by the following modified problem solving process:

- I. Justifying a symbolic representation of the problem situation.
- II. Answering the word problem question.
- III. Constructing and validating a deductive argument proving the stated representation.

Relating this to the traditional problem solving process described at the beginning of Section 2, Part I corresponds to (a) and (b), part II corresponds to (c), (d), (e), (f), and part III is the new (b’).

Recall the concept of a *constant speed* (Wu, 2011).

Definition 2. An object moves at a constant speed along a straight line if there is a real number v such that for each real number $t > 0$, the distance $s(t)$ (measured in kilometres, metres, ...) covered by the moving object during the time period from 0 to t (measured in hours, seconds, ...) is equal to the product $v \cdot t$. This number v is called the speed of the motion, and it is the derived magnitude (measured in kilometres per hour, metres per second, ..., respectively) related to the fundamental magnitudes of distance and duration by the equality of numerical values

$$v = \frac{s(t)}{t}, \quad \text{for each } t > 0.$$

This example of an AWP is taken from a Lithuanian book for teachers. The original formulation has no hypothesis of the constancy of speed. We use a justification taken from this book.

Word Problem. *Tourists walking at a constant speed planned to cover a distance between a river and a tourist camp in 6 hours. However, after 2 hours’ walking they slowed down their initial speed by 0.5 km/h and were 30 minutes late arriving at the camp.*

1. *What was the initial speed of the tourists?*

2. *Justify a symbolic representation of the problem situation.*
3. *Create and validate a deductive argument proving the justification.*

Solution I. Justification. (1) Suppose that the initial speed of the tourists is x km/h. (2) Since distance = speed \times time and since the tourists intended to walk from the start to the end in 6 hours, the distance between the river and the camp is equal to $6x$ km. (3) In fact the tourists walked at different speeds: for the first 2 hours at the initial speed of x km/h and for 4.5 hours (since they were 30 minutes late) at a speed of $(x-0.5)$ km/h. (4) The distances they travelled are therefore equal to $2x$ km and $4.5(x-0.5)$ km, respectively. (5) Combining the resulting distances we obtain the desired equation:

$$2x + 4.5(x - 0.5) = 6x. \quad (2)$$

II. Finding an answer. Solving this equation one obtains $x = 4.5$. Thus the initial speed is 4.5 km/h, which is the answer to the problem.

III. Deductive argument and validation.



The figure depicts a mathematical model of the trip described by the word problem. The point A denotes the location near the river where the tourists began their walk. The point C denotes the location of the tourist camp, the final destination of the trip. In between, the point B is the place where the tourists slowed down from their initial speed. The trip itself can be imagined as a moving point along the line segment AC at the speed described by the word problem.

Next we construct the deductive argument

$$P_1, P_2, P_3, P_4, P_5 \vdash \text{equation (2)}, \quad (3)$$

with the following premises.

P_1 If the length of the line segment AB is S_1 km and the length of the line segment BC is S_2 km then the length of the line segment AC is $S_1 + S_2$ km.

P_2 If an object moves at a constant speed v km/h during the time duration t h then the distance covered $s = v \cdot t$ km.

P_3 Fact I: the distance between points A and B is travelled at the speed of x km/h in a time of 2 hours.

P_4 Fact II: the distance between points B and C is travelled at the speed of $x - 0.5$ km/h in a time of 4.5 hours.

P_5 Fact III: the distance between points A and C is travelled at the speed of x km/h in a time of 6 hours.

Now we show that the deductive argument (3) is valid (that is, that the truth of all the premises entails the truth of the conclusion).

P_6 According to premises P_2 and P_3 and the inference rule modus ponens, $S_1 = 2x$.

P_7 According to premises P_2 and P_4 and the inference rule modus ponens, $S_2 = 4.5(x - 0.5)$.

P_8 According to premises P_2 and P_5 and the inference rule modus ponens, the distance between points A and C is equal to $6x$ km.

P_9 According to premise P_1 , statements P_6 , P_7 and P_8 and the inference rule modus ponens, equation (2) holds true.

Evaluation of the level of justification of (2). This is done by interpreting each sentence of the justification.

(1) The unknown initial speed of tourists is called the unknown variable and is denoted by the symbol x . This is used to represent the distances travelled by the tourists. (2) Fact III is obtained using the hypothesis of constancy of speed. (3) The value of the reduced speed is obtained using the relations described by the problem statement. (4) Facts I and II are obtained using the hypothesis of constancy of speed. (5) While it is not stated explicitly, the stated equation (2) is obtained by the property (1) of measurements.

In conclusion, all the facts of the deductive argument are mentioned in the justification with no mention of logical links between them. Therefore the justification is at the second level.

Conclusions

The meaning of a symbolic representation of an algebraic word problem lies in the mathematical system of the properties of continuous magnitudes. This conclusion is confirmed by examples of solutions of modified algebraic word problems using reasoning-and-proving. The deductive argument appears to be a necessary instrument for this discovery. We do not need additional instruments to see a link between an arithmetical operation and a relationship between discrete magnitudes in an arithmetical word problem.

A stepping stone to success in solving a modified word problem is the question of how to construct a deductive argument having built a non-proof argument. For example, having built a justification at the second level, to obtain a valid deductive argument one needs to find proper logical links between statements presumably implying a symbolic representation. This is the kind of task that is familiar in propositional logic. In conclusion, solving word problems becomes an exercise in logic.

The most promising conclusion is that solving word problems with reasoned judgement in the sense of the present essay provides an unlimited source of reasoning-and-proving activities.

More examples of AWP+2T and advantages of using reasoning to solve word problems are discussed in Kilienė & Norvaiša (2021).

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