



# On the implementation of proving in primary school

Melanie Platz

## ► To cite this version:

Melanie Platz. On the implementation of proving in primary school. Twelfth Congress of the European Society for Research in Mathematics Education (CERME12), Feb 2022, Bozen-Bolzano, Italy. hal-03746834v2

**HAL Id: hal-03746834**

**<https://hal.science/hal-03746834v2>**

Submitted on 17 Oct 2022

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# On the implementation of proving in primary school

Melanie Platz

Saarland University, Germany; [melanie.platz@uni-saarland.de](mailto:melanie.platz@uni-saarland.de)

*The project Prim-E-Proof aims to generate substantial learning environments to support proving skills at the primary level. The learning environment “The Pythagoreans” on the theorem “the sum of two odd numbers is always even” is constantly further developed through design science research. The evaluation showed that the majority of the primary school pupils use empirical justifications and that the solutions of younger primary school children do not seem to differ much from those of older ones. A first concept for the reflection of justifications is supposed to support learners in developing valid justifications and teachers in implementing proof in primary school.*

*Keywords: Primary education, “inhaltlich-anschaulich” proving, reasoning, instruction support.*

## Introduction

“Reasoning” can be understood as a generic term for forms of argumentation and proof (Brunner, 2014, p. 31). However:

A formal-deductive understanding of proof is not suitable to describe the aspect of reasoning in primary school [in Germany]. The same is valid for the “inhaltlich-anschaulich” [a loose translation could be “content-visual”] proving, although its use in primary education is undoubtedly possible. However, a content-visual proof also aims at proving the validity or non-validity of general mathematical facts such as rules, conspicuousness, relationships, or the like. The detachment from the individual case, the generalization, represents an indispensable aspect of content-visual proofs. In contrast, mathematics lessons often require reasoning to lend validity to a very concrete example or a particular assertion. (Peterßen, 2012, p. 20)

Stylianides (2016) formulates two crucial goals from a pedagogical standpoint that can be reached through a more central place of proving in the mathematical experiences of children:

First, elementary students will have more opportunities to learn deeply in mathematics and engage with it as a sense-making activity. Second, when students get to secondary school they will not only be better prepared to engage with proving, which they will see as a natural extension of their prior mathematical experiences, but also better able to reason mathematically in disciplined ways [...]. (Stylianides, 2016, p. 10)

In the context of primary school, especially the research perspective of Blum and Kirsch (among others 1991), together with Wittmann and Müller (among others 1990), which Reid and Knipping (2010, p. 43) call a “preformalist” research perspective, seems suitable. In content-visual proofs, preformalists loosen the requirement that proofs must be formal. Still, they expect deductive reasoning to be used and that proofs based on intuitive or obvious foundations are as convincing (or even more convincing) as formal proofs (ibid.). A *content-visual proof* is based on a concrete, visually perceptible object, on which something general is proven and which is usually presented in an iconic way. It can be understood as an object of a more general kind by trained observation. To produce a content-visual proof, the more general in the special of an example must be mentally seen

(Krumsdorf, 2015). But in particular, to disengage from examples, i.e., to generalize the effects of operations, confronts primary school children with a problem that they cannot solve spontaneously and without support (Sturm, 2018). Another challenge for primary school learners is that the *content-visual proof* must be verbalized. What is initially subjectively found to be universally valid can be socially shared and recognized by others (Wittmann & Ziegenbalg, 2007). The pupils face the requirement of generalizing without knowledge of the algebraic language. They interpret a general structure into the given mathematical signs and use the signs they know from other contexts to describe the general patterns and structures (Akinwunmi, 2012). Besides the challenges for learners, proving in primary school brings challenges to teachers that make an implementation into mathematics teaching difficult. These are:

[...] the weak knowledge that many elementary teachers have about proof (factor 1) and their presumed beliefs that proving is an advanced mathematical topic beyond the reach of elementary students (factor 2); the high pedagogical demands placed on elementary teachers who strive to engage their students in proving (factor 3); and the inadequate instructional support offered or available to elementary teachers about how to achieve that goal in their classrooms (factor 4). (Stylianides, 2016, p. 21)

Melhuish et al. (2020) evaluated the efficacy of a 3-year teacher professional development program with participants from 25 elementary schools. They concluded: “Even when teachers are engaged in a PD designed to promote high-level reasoning forms (in our case justifying and generalization) and are provided with relevant definitions, teachers may continue to revert to nonstandard views of those reasoning forms.” (p. 63). This emphasizes the complexity of designing support measures for teachers.

The *Prim-E-Proof* project’s objective is to develop *substantial learning environments* for primary school mathematics lessons, among others inspired by the tasks stimulating *operative proving* in the primary school book *Zahlenbuch* (based on Wittmann & Müller, 1990), to support proving skills with a particular focus on the factors identified by Stylianides (2016). Wittmann (1998) characterizes substantial learning environments based on four criteria. They must (1) represent central goals, content, and principles of mathematics education; (2) provide rich opportunities for students’ mathematical activities; (3) be flexible and easily adaptable to the particular circumstances of a given class; (4) integrate mathematical, psychological, and pedagogical aspects of teaching and learning holistically and therefore offer a broad potential for empirical research. The starting point is the theorem: *the sum of two odd numbers is always even*.

### **The learning environment “The Pythagoreans”**

The tasks within the learning environment are as follows:

- (1) *What was again an even number, and what was an odd number?* With this question, the childrens’ previous knowledge is linked up.
- (2) *In the past, calculation stones were used for calculating instead of reversible tiles. The Pythagoreans alienated the calculating stones to perform mathematical proofs – among other things about even and odd numbers. These calculation stones of the Pythagoreans were further developed*

in this applet [besides the applet also analog material is provided]. *Let me show you what the applet can do* [the functions of the applet ([www.melanie-platz.com/Steinchen-Applet/Steinchen.html](http://www.melanie-platz.com/Steinchen-Applet/Steinchen.html), see Fig. 1) are presented]. *How might the Pythagoreans have represented even and odd numbers with stones? Use the applet and represent an even and an odd number.* To avoid a defensive reaction (Platz, 2019) and to be able to arouse a need for proof and thus provide rich opportunities for mathematical activities for pupils, a historical digression (Krauthausen, 2018) is initiated.

(3) *Always add two odd numbers. What do you notice?* Bezold (2009, p. 37) describes four activities or steps in an argumentation chain. The first two activities are stimulated by this task: the discovery of mathematical features as *prerequisites* for argumentative activities. *Step 1* represents the description of discoveries (ibid.) or a reference to common prior knowledge (Neumann et al., 2014).

(4) *Why is this so? Give reasons!* With this task, *step 2* of the argumentation chain, according to Bezold (2009), the questioning of discoveries, and *step 3*, the finding of justifications or ideas for justification of mathematical regularities and connections, are stimulated.

## Research Questions

In this paper, the following research questions are targeted: *RQ(1) Which level of proving do learners working on the learning environment “The Pythagoreans” reach when providing support measures to support and supplement the learners’ activities?; RQ(2) Which consequences can be drawn for the further development of the learning environment?*

## Methods and procedures for collecting and analyzing data & obtained data

A multiple case study (Yin, 2018) using qualitative data collection and analysis methods was conducted. A clinical interview with only key questions defined and the requirement to follow children’s thinking is analogous in principle to classroom management in implementing a substantial learning environment (Wittmann, 1998). The collected data can provide information “[...] about teaching/learning processes, thinking processes and learning progress of students [...]. On the other hand, they help to evaluate and revise the learning environments to design teaching/learning processes even more effectively.” (ibid., p. 339).

A course concept for student teachers, who did not have experiences with proving in primary school before, was developed: the students elaborated various support options for the learners that were applied (if necessary) depending on the proof phase, e.g., variation of suitable materials or representations (Krauthausen, 2001), variation of appropriate example-based proof processes (approaches) (Krumsdorf, 2015), use of Big Numbers (Martin & Harel, 1989) or merely presented examples (Krumsdorf, 2015) or use of pre-exercises for proof (Brunner, 2014). Before the students performed the clinical interviews, the learning instruction was focused which should help to enable them to support and supplement the children’s activities with suitable impulses that do not go too far and to be a rich source of reliable factual information for the children (Wollring, 2008).

The learning environment was tested by student teachers at the University of Münster in the winter semester 2020/21 and the University College of Teacher Education Tyrol in the summer semester 2021 in 40-minute clinical interviews with primary school children (grades 1-4). The interviews were videotaped and transcribed. The solutions of 17 pupils were analyzed regarding the handling of

examples (abstraction) and generalization, the level of justification, the need for support during proving and classified in proving levels. Almeida (2001, p. 55) adds ‘*proof*’ by *authority* (level 1) and ‘*proof*’ by *intuition* (level 2) implied by Cobb (1986) to Balacheff’s (1991) widely accepted descriptions of the levels of proving: (3) ‘*proof*’ by *naive empiricism*, (4) ‘*proof*’ by *crucial experiment*, (5) *proof* by *generic example* and (6) *proof* by *thought experiment*. Proofs on levels 5 and 6 can be regarded as valid.

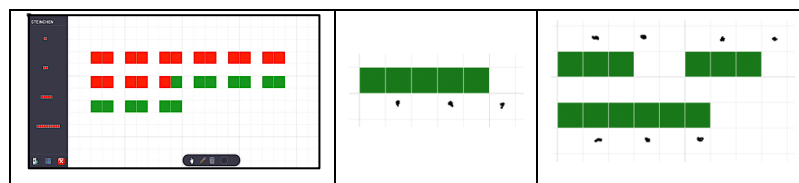
## Results & Discussion

### RQ(1)

Analyzing the proving level of the pupils from grades 3 and 4 (8-12 years), seven pupils used empirical justifications, three *generic examples*, and one a *proof by thought experiment*. Two could not be convinced that the assertion is true. This draws a similar picture to the results of a study of the proof perceptions and practices of “nineteen year ten pupils” (p. 55): Almeida (2001) observed that the majority of pupils preferred empirical justifications. In our case, four younger children (grade 1 & 2, 6-7 years) were interviewed, and the approach to proving did not seem to differ that much from that of the older children. Neumann et al. (2014), who assessed students’ mathematical reasoning skills from two 3rd grades, five 4th grades, and five 6th grades, observed something similar. The comparison of the groups showed no significant differences between grades 3, 4, and 6 ( $n=243$ ). Focusing on the proving skills of the youngest pupils, one interesting case is the following: Emma (grade 1, 7 years) was asked by the interviewer, what pattern all the odd numbers have in common and how she can tell straight away that she laid an odd number with the tiles. Emma explains: “[...] because you can’t always count two together with them. For example, you can’t count to four now, when in steps of two, two, four, then one tile is missing.” (see Fig. 1, middle). Then this scene follows:

- 1 Interviewer: And if you then put two such numbers together? (...) What happens then (...) with the steps of two?
- 2 Emma: (...) Then it works.
- 3 Interviewer: Then it works.
- 4 Emma: Yes
- 5 Interviewer: Um (hesitating). Why is this always the case?
- 6 Emma: (...) Hm (thoughtful) (...) Because (...) the odd numbers are odd numbers?

Stimulated by the interviewer, Emma uses the self-chosen example  $3+3=6$  to show that the sum of 6 is even by counting in steps of two (see Fig. 1, right).



**Figure 1: Screenshot of the “Steinchen-Applet” and visualization of Emma’s solution**

Emma describes odd numbers through the basic idea of dividing by grouping (pairing) the tiles (counting in steps of two). If no tile is missing to form a pair, it is an even number. If there is a tile missing, it is an odd number. Emma does not express the proof idea (that the two individual tiles can be joined together and that an even number can be produced when forming the sum by pairing). She just states “then it works” (*turn 2*) and responds to the *why-question* of the interviewer (*turn 5*)

“Because the odd numbers are odd numbers?” (*turn 6*). Afterward, Emma verifies with an example that her procedure works. The reaction of Emma to the *why-question* could be compared to the “I see it this way” (cf. *proof by intuition*) described as an answer of children to the question “Why? How do you know?” by Freudenthal (1978, p. 262) focusing on a geometric context. This can be transferred to the geometric-visualized pattern structuring in our case: “One thing I know for sure today: the answer is not an excuse, it is true. It is not a symptom of guessing, nor is it a confession of powerlessness in linguistic expression. On the contrary: that the child sees it prevents the linguistic effort.” (ibid., p. 263). Consequently, Emma seems to start with a *proof by thought experiment*, because she verifies the statement by appealing to the structural properties of mathematics independent of examples. In *turn 6*, she seems to switch to a *proof by intuition* and is based on stimulation by the interviewer directed to a *proof by a generic example* (which is not completed).

## RQ(2)

The mathematical reasoning competencies of the students mostly do not correspond to the curricular requirements of the educational standards, which can also be attributed to a lack of meta-knowledge of the students regarding mathematics-specific reasoning (Vogt, 2020). But a targeted further development of mathematical reasoning skills in primary school mathematics lessons seems to be possible in principle (Bezold, 2009; Peterßen, 2012; Vogt, 2020). Peterßen (2012) reports on a study where most grade 3 pupils ( $n=20$ ) used valid justifications in the categories *generic example* and *pure thought experiment* to validate a claim; empirical justifications accounted for only about one-quarter of the students’ justifications. “One possible explanation could be the culture of justification already embedded in the class. The students were used to being asked *why* over and over again. Furthermore, the quality of pupils’ explanations was also the subject of the lesson.” (ibid., p. 63). Understanding proof in the following way makes the negotiation for a common argumentation basis apparent: “A set of propositions that are accepted as true, together with inferences that are accepted as admissible, shall be called a basis of argumentation. Reasoning based on a given argumentation basis be called a proof concerning that argumentation basis.” (Fischer & Malle, 1985, p. 180). In terms of a common argumentation basis, Peterßen (2012, p. 348) emphasizes the *reflection of justifications* with pupils. Pupils need to understand what is required of a justification, i.e., which justification is accepted or rejected, when, and why. Justifications in themselves and their respective persuasiveness should be made the subject of teaching. Such a reflection of justifications was not stimulated in the learning environment *the Pythagoreans*: The tasks enable to pass through the four activities or steps in an argumentation chain described by Bezold (2009) as mentioned above. Comparing this argumentation chain with the *process model of the proof process of experts* (the seven phases do not necessarily have to be passed through in the order given here; rather, a frequent alternation between these phases is to be expected, especially for experienced mathematicians), the first three phases seem to coincide partly: (1) *Finding a conjecture from the mathematical problem field*. (2) *Formulate the conjecture according to common standards*. (3) *Exploring the conjecture with the limits of its validity; making connections to mathematical framework theory; identifying suitable arguments to support the conjecture* (Reiss & Ufer, 2009, p. 162). The phases (4)-(6) are not stimulated with the tasks: (4) *Selecting arguments that can be organized in a deductive chain to form a proof*. (5) *Fixing the chain of arguments according to current mathematical standards*. (6) *Approximation to a formal proof*

(Phase (6) is only realized in a few cases. Usually, the proof found in phase (5) is reviewed and, if necessary, published by the mathematical community at conferences and in the context of peer review.) (ibid.). (7) *Acceptance by the mathematical community* (ibid.), whereby in the school context, this role of the mathematical community is usually performed by the teacher. As the model is based on an expert's (optimal) proof process, it cannot directly be transferred to purposes of teaching (ibid., p. 163), but it can be used as a starting point for the development of a teaching concept. With this intention, the reflection of justification is focused for helping students understand what is required of a justification (Peterßen, 2012) and to find a common argumentation base (Fischer & Malle, 1985) to be able to select arguments that can be organized in a deductive chain to form a proof (phase (4)). In the next step of the project, the reflection of justifications is focused on, which is supposed to be done via analyzing authentic pupils' solutions. These solutions can include errors or not. What should be emphasized is that most of the pupils' answers are not wrong; the pupils just had another understanding of the argumentation base. The *reflection of errors or justifications that are to be rejected* can lead to negative knowledge (Oser et al., 1999). This includes two main components: the differentiation knowledge (To what extent does something not belong to a specific item, concept, or procedure?) and the error knowledge (What must not be done in a particular situation?). The use of solutions of other (anonymous) pupils can be justified because not each error has to be done by oneself to learn from it, and pupils usually prefer to work on mistakes from others (Prediger & Wittmann, 2009). The aim of *reflecting "good" solutions or justifications that are to be accepted* is not for every child to learn and use as many ways of proving as possible. It is essential that they find ways to prove according to their individual preferences. This way, the lack of familiarity with adequate tools (Krauthausen, 2001) and the interpretative openness of figural representations (Krumsdorf, 2015) in tile-proofs can be counteracted. Training in the flexible perception of different states (one representation, different perspectives) can be supportive. It can be helpful if the same situation can be experienced and explicitly compared using different representations (ibid.). For analyzing the students' solutions, the teacher can guide the pupils through stages with increasing metacognitive demand similar to those proposed by Prediger & Wittmann (2009, p. 8) for analyzing errors: (1) *Find calculation errors or errors in the procedure of the student* (What is wrong here? Justify why.; What rule was broken?). (2) *Compare two solutions* (Here are two different solutions. Which one is better?). (3) *Explain the procedure of a student* (How did the student think here?). (4) *Reconstruction of the reason why a justification was given that must be rejected/accepted* (What idea is behind the solution?; Explain why the justification is rejected/accepted.). (5) *Development of optimization measures* (How would you help the student?; What would you do same wise/different?). The following scene could be used as starting point for a discussion (especially items (3)–(5); in comparison with Emma's solution also (2)): After checking on several examples, Ben (grade 3, 8 years) was asked why his procedure (pairing) always works:

Ben: If you can partition the little tiles exactly. Then it must be an even number. But. (5) Two odd numbers make an even one. (...) Because it remains. If you divide them up, there is always one remaining. And if you combine the one leftover with the other one/ one is left over. If you then put them together, you have two again. And (..) if you had three odd numbers now, they would have remained odd.

## Conclusion & further Development

Peterßen (2012) identifies the following aspects that positively affect the development of reasoning competence and a subjective need for reasoning: discovery learning as a teaching principle, good tasks, regularity, pupil-focused communication, and subject-specific competent teachers. By developing substantial learning environments that should be implementable regularly, discovery learning is realized, and good tasks are designed. Focusing on the reflection of justifications, informative scenes from the clinical interviews will be extracted and prepared for use in the primary school classroom. Melhuish et al. (2020) suspect that

[...] the ability to notice complex mathematical-reasoning forms (a necessary prerequisite for fostering them in classrooms) may rely not only on providing teachers with mathematical definitions and rich examples but also on promoting explicit connections between definitions and the examples themselves. (p. 64)

With the focus of promoting explicit connections between definitions and the example scenes, teacher training courses will be developed, building on the experiences of Melhuish et al. (2020) and the concept *professional vision* (Sherin, 2007) to support teachers subject-specifically and in pupil-focused communication.

## Acknowledgment

I thank all the students and participating children for their excellent work and patience!

## References

- Akinwunmi, K. (2012). *Zur Entwicklung von Variablenkonzepten beim Verallgemeinern mathematischer Muster*. Springer. <https://doi.org/10.1007/978-3-8348-2545-2>
- Almeida, D. (2001). Pupils' proof potential. *International Journal of Mathematical Education in Science and Technology*, 32(1), 53–60. <https://doi.org/10.1080/00207390119535>
- Balacheff, N. (1991). The benefits and limits of social interaction: The case of mathematical proof. In A. J. Bishop, S. Mellin-Olsen, J. van Dormolen (Eds.), *Mathematical knowledge: Its growth through teaching* (pp. 173–192). Kluwer Academic Publishers. [https://doi.org/10.1007/978-94-017-2195-0\\_9](https://doi.org/10.1007/978-94-017-2195-0_9)
- Bezold, A. (2009). *Förderung von Argumentationskompetenzen durch selbstdifferenzierende Lernangebote*. Dr. Kovač. <https://doi.org/10.1007/BF03339083>
- Blum, W., & Kirsch, A. (1991). Preformal proving: Examples and reflections. *Educational Studies in Mathematics*, 22(2), 183–203. <https://doi.org/10.1007/BF00555722>
- Brunner, E. (2014). *Mathematisches Argumentieren, Begründen und Beweisen*. Berlin, Springer. <https://doi.org/10.1007/978-3-642-41864-8>
- Cobb, P. (1986). Contexts, goals, beliefs, and learning mathematics. *For the Learning of Mathematics*, 6(2), 2–9.
- Fischer, H., & Malle, G. (1985). *Mensch und Mathematik*. Profil.
- Freudenthal, H. (1978). *Vorrede zu einer Wissenschaft vom Mathematikunterricht*. Oldenbourg.



- Krauthausen, G. (2001). »Wann fängt das Beweisen an? Jedenfalls, ehe es einen Namen hat.« In W. Weiser, & B. Wollring (Eds.), *Beiträge zur Didaktik der Mathematik in der Primarstufe* (pp. 99–113). Dr. Kovač.
- Krauthausen, G. (2018). *Einführung in die Mathematikdidaktik – Grundschule*. Springer. <https://doi.org/10.1007/978-3-662-54692-5>
- Krumsdorf, J. (2015). *Beispielgebundenes Beweisen*. (PhD thesis, Univ. Münster).
- Martin, W., & Harel, G. (1989). Proof Frames of Preservice Elementary Teachers. *Journal for Research in Mathematics Education*, 20, 41–51. <https://doi.org/10.2307/749097>
- Melhuish, K., Thanheiser, E., & Guyot, L. (2020). Elementary school teachers' noticing of essential mathematical reasoning forms: justification and generalization. *Journal of Mathematics Teacher Education*, 23(1), 35–67. <https://doi.org/10.1007/s10857-018-9408-4>
- Neumann, A., Beier, F., & Ruwisch, S. (2014). Schriftliches Begründen im Mathematikunterricht. *Zeitschrift für Grundschulforschung*, 7(1), 113–125.
- Oser, F., Hascher, T., & Spychiger, M. (1999). Lernen aus Fehlern. In W. Althof (Ed.), *Fehlerwelten* (pp. 11–41). VS Verlag für Sozialwissenschaften. [https://doi.org/10.1007/978-3-663-07878-4\\_1](https://doi.org/10.1007/978-3-663-07878-4_1)
- Platz, M. (2019). Learning environments applying digital learning tools to support argumentation skills in primary school: first insights into the project. In U.T. Jankvist, M. van den Heuvel-Panhuizen, & M. Veldhuis (Eds.), *Proceedings of CERME11* (pp. 2908–2915). Freudenthal Group, in collaboration with the Freudenthal Institute, Utrecht University & ERME.
- Peterßen, K. (2012). *Begründungskultur im Mathematikunterricht der Grundschule*. Franzbecker.
- Prediger, S., & Wittmann, G. (2009). Aus Fehlern lernen–(wie) ist das möglich? *Praxis der Mathematik in der Schule*, 51(3), 1–8.
- Reid, D. A., & Knipping, C. (2010). *Proof in mathematics education. Research, learning and teaching*. Sense Publisher. <https://doi.org/10.1163/9789460912467>
- Sherin, M. (2007). The development of teachers' professional vision in video clubs. In R. Goldman, R. Pea, B. Barron, & S.J. Derry (Eds.), *Video Research in the Learning Sciences* (pp. 397–410). Routledge.
- Sturm, N. (2018). *Problemhaltige Textaufgaben lösen: Einfluss eines Repräsentationstrainings auf den Lösungsprozess von Drittklässlern*. Springer. <https://doi.org/10.1007/978-3-658-21398-5>
- Stylianides, A. J. (2016). *Proving in the elementary mathematics classroom*. Oxford Univ. Press. <https://doi.org/10.1093/acprof:oso/9780198723066.001.0001>
- Vogt, M. (2020). *Schriftliche Begründungskompetenzen in stochastischen Aufgabenkontexten*. Springer. <https://doi.org/10.1007/978-3-658-33053-8>
- Wittmann, E. C. (1998). Design und Erforschung von Lernumgebungen als Kern der Mathematikdidaktik. *Beiträge zur Lehrerbildung*, 16(3), 329–342.
- Wittmann, E. C., & Müller, G. (1990). *Handbuch produktiver Rechenübungen, Band I*. Klett.
- Wittmann, E. C., & Ziegenbalg, J. (2007). Sich Zahl um Zahl hochhangeln. In G. Müller, H. Steinbring & E. C. Wittmann (Eds.), *Arithmetik als Prozess* (pp.35–53). Kallmeyer.
- Wollring, B. (2008). Zur Kennzeichnung von Lernumgebungen für den Mathematikunterricht in der Grundschule. In Kasseler Forschergruppe (Eds.), *Lernumgebungen auf dem Prüfstand* (pp. 9–26). kassel university press GmbH.
- Yin, R. K. (2018). *Case study research and Applications: Design and methods*. Sage.