



# Asymptotic-Möbius maps

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# Asymptotic-Möbius maps.

Georg Grützner

## Abstract

Roughly speaking, a map between metric spaces is an asymptotic-Möbius map if it almost preserves cross-ratios of points being a large distance apart from each other. We show that under such maps, some large-scale notions of dimension increase: asymptotic dimension for finitely generated nilpotent groups, telescopic dimension for  $CAT(0)$  spaces.

## 1 Introduction

In 1999, Oded Schramm investigated the scaling limits of loop-erased random walks (LERW) [18]. Schramm observed that the conformal invariance of the limit process implies its encoding in corresponding conformal mappings  $f_t$  satisfying Loewner's differential equation

$$\frac{\partial f_t}{\partial t} = z \frac{\zeta(t) + z}{\zeta(t) - z} \frac{\partial f_t}{\partial z}.$$

This observation revolutionized two-dimensional statistical physics (see, e.g., a review of SLE by G. F. Lawler [11]).

There are generalizations for dimensions higher than 2 [20]. However, the strength of conformal techniques in higher dimensions is severely limited. This is particularly clear in Liouville's 1850 theorem.<sup>1</sup> While all simply connected domains in  $\mathbb{R}^2$  are conformally equivalent by the Riemann mapping theorem, any smooth conformal mapping on a domain of  $\mathbb{R}^n$ , with  $n > 2$ , is a composition of translations, similarities, orthogonal transformations and inversions.

Suitable classes of functions on metric spaces that capture conformal behavior are the classes of quasisymmetric (QS) maps, quasi-Möbius (QM) maps, and quasiconformal (QC) maps. Quasisymmetric maps transform balls into "quasiballs." The definition is based on the fact that conformal maps transform infinitesimal balls into infinitesimal ellipsoids. Quasi-Möbius maps preserve almost the cross ratio of points, and quasiconformal maps are an infinitesimal version of quasisymmetric maps. We always have  $QS \Rightarrow QM \Rightarrow QC$  [19].

We are therefore interested in classes of functions that converge under scaling limits to the classes QS, QM or QC. In the eyes of a geometric

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<sup>1</sup>Monge 1850, pp. 609 - 616, a note contributed by Liouville as editor (Note VI: Extension au cas des trois dimensions de la question du tracé géographique.)

group theorist, we are looking for classes of maps between finitely generated groups  $\Gamma$ , with the word metric, which remain invariant under small changes in topology and which become conformal in the above sense when viewed from a distance. Specifically, we require that the class of maps is invariant under composition by quasi-isometries.

This chapter presents and examines a class of maps that mimics quasi-Möbius maps in the context of geometric group theory.

In [16] Pierre Pansu introduces a notion of large-scale conformal maps that mimics the infinitesimal behavior of conformal maps. In short, large-scale conformal maps map families of disjoint balls onto families of weakly overlapping quasiballs. It is a very flexible notion that includes, for example, coarse embeddings. However, this flexibility makes the asymptotic behavior of such maps less predictable.

Instead, we introduce a more restrictive and semantically simpler notion that mimics quasi-Möbius behavior. Roughly speaking, a map between metric spaces is an asymptotic-Möbius map if it nearly preserves the cross ratio of points that are a large distance apart. As a result, the results presented in this paper are much stronger than those in [16].

We postpone the precise definition of asymptotic-Möbius ( $AM$ ) maps until later. Sources of examples of  $AM$ -maps are:

1. Quasi-isometric embeddings,
2. Sublinear-bi-Lipschitz equivalences (i.e., maps inducing Lipschitz equivalences on asymptotic cones [5]),
3. Snowflaking (i.e., replacing a metric by a power of it),
4. Assouad maps from doubling metric spaces to  $\mathbb{R}^N$ .

For example, any nilpotent Lie group or finitely generated group can be  $AM$ -embedded in a Euclidean space of sufficiently high dimension [1]. Moreover, we will encounter infinite dimensional examples.

## 1.1 Results

Our main result is that under  $AM$ -mappings a large-scale term of dimension increases. The relevant term depends on the class of the groups considered.

**Theorem 3.** *Let  $G$  and  $G'$  be finitely generated nilpotent groups and  $f : G \rightarrow G'$  an  $AM$ -map, then  $\text{asdim}(G) \leq \text{asdim}(G')$ . Furthermore, if  $\text{asdim}(G) = \text{asdim}(G')$ , then the asymptotic cones of  $G$  and  $G'$  are isomorphic. Conversely, given nilpotent groups with isomorphic asymptotic cones, there exists an asymptotic-Möbius map between them.*

In the world of  $\text{CAT}(0)$ -spaces, the analogous theorem takes the following form.

**Theorem 4.** *Let  $X$  and  $Y$  be  $\text{CAT}(0)$  spaces. If there exists an  $AM$ -map  $f : X \rightarrow Y$ , then the telescopic dimension increases e.g.  $\text{tele-dim}(X) \leq \text{tele-dim}(Y)$ .*

## 2 Asymptotic-Möbius maps

### 2.1 Quasi-Möbius maps

Quasi-Möbius maps were introduced by Jussi Väisälä, among others, as a means of studying quasimetric maps and quasiconformal maps. Unlike quasimetric maps, quasi-Möbius maps do not have a point fixed at infinity.

Let  $X$  be a metric space, and  $x, y, z, w \in X$  a quadruple of distinct points. Their *cross-ratio* is

$$[x, y, z, w] = \frac{d(x, z)d(y, w)}{d(x, w)d(y, z)}.$$

If  $X$  is unbounded, then the cross ratio extends to the one-point compactification of  $X$  [19]. A map  $f : X \rightarrow Y$  is *quasi-Möbius* if there is a homeomorphism  $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that for all quadruples of distinct points  $x, y, z, w \in X$ ,

$$[f(x), f(y), f(z), f(w)] \leq \eta([x, y, z, w]).$$

If  $f$  is a homeomorphism, then  $f$  is quasi-Möbius iff its inverse is quasi-Möbius.

Examples of quasi-Möbius maps are

1. the stereographic projection  $\mathbb{R}^n \rightarrow S^n$ ,
2. the Cayley transformations (the complex, quaternionic and octonionic analogues of the stereographic projection),
3. the inversions  $x \mapsto \frac{x}{|x|^2}$  in Banach spaces.

### 2.2 Asymptotic-Möbius maps

We need a criterion to tell when two points in a space are far apart. One way to do this is to separate them by sublinear growing functions. This idea was used by Y. Cornuier to define sublinear-bi-Lipschitz equivalences [5].

Given  $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ , we say that the points  $x$  and  $y$  in  $X$  are separated by  $u$  from a fixed origin  $o$  if  $d(x, y) > u(d(x, o) + d(o, y))$ . For simplicity, we often write  $x, y > u$  and  $|x| = d(o, x)$  for the distance from the origin. We can call  $u$  a *gauge*, i.e., a function that sets a scale of "sight" as a function of "location". To fix a gauge means to decide what is "near" and "far away".

**Definition 1.** A function  $u : \mathbb{R}_+ \rightarrow \mathbb{R}$  is an *admissible gauge* if

1. it is non-decreasing, and
2.  $u$  grows sublinearly, i.e.  $\limsup_{r \rightarrow \infty} \frac{u(r)}{r} = 0$ .

Morally, asymptotic-Möbius maps at widely spaced points are quasi-Möbius.

**Definition 2.** A map  $f : X \rightarrow Y$  is an *asymptotic-Möbius map* or *AM-map* if there exists an admissible gauge  $u$  and a homeomorphism  $\eta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  such that for every  $x, y, z, w$  in  $X$  such that all 6 distances between  $x, y, z, w > u$ , then

$$[f(x), f(y), f(z), f(w)] < \eta([x, y, z, w]).$$

At some point we will need a coarse version of path connectivity.

**Definition 3.** A metric space  $X$  is *asymptotically chained*, if for every origin  $o \in X$  there exists an admissible gauge  $v$  such that for all  $x, y \in X$  there exists a chain  $x_1 = x, \dots, x_{k+1} = y$  satisfying

$$\max_{i \in 1 \dots k} \{d(x_i, x_{i+1})\} < v(|x| + |y|).$$

Our most important technical step will be the following theorem, the proof of which will occupy section 2.4 after some introductory definitions.

**Theorem 1.** Let  $f$  be an AM-map from an asymptotically chained space  $X$  to a space  $Y$  that preserves diverging and bounded sequences, then  $f$  induces a continuous, injective, quasisymmetric map  $g$  between some asymptotic cones of  $X$  and  $Y$ .

## 2.3 Asymptotic-cones

The *asymptotic cone* of a metric space  $(X, d)$ , captures the geometry on the large scale of  $X$ . Roughly speaking, it formalizes the idea of snapshots of the space  $X$  taken by an observer moving farther and farther away from  $X$ . This sequence of snapshots can stabilize and the observer has the impression of seeing a single object. We call this object the asymptotic cone of  $X$ .

If  $R$ -balls  $B(o, R)$  with metric  $R^{-1}d$  are uniformly pre-compact, then the asymptotic cones can be constructed concretely by pointed Gromov-Hausdorff convergence [6], [8] In general, however, we have to resort to constructions via ultrafilters.

**Definition 4.** A (nonprincipal) ultrafilter  $\omega$  over  $\mathbb{N}$  is a set of subsets of  $\mathbb{N}$  satisfying the following conditions:

1. If  $A, B \in \omega$  then  $A \cap B \in \omega$ .
2. If  $A \in \omega, A \subset B \subset \mathbb{N}$ , then  $B \in \omega$ .
3. For every  $A \subset \mathbb{N}$ , either  $A \in \omega$  or  $\mathbb{N} \setminus A \in \omega$ .
4. No finite subset of  $\mathbb{N}$  is in  $\omega$ .

Equivalently,  $\omega$  is a finitely additive measure on  $\mathbb{N}$  such that every subset has measure either 0 or 1 and every finite subset has measure 0.

If a statement  $A(n)$  holds for all  $n \in B$ , where  $B$  belongs to an ultrafilter  $\omega$ , then  $A(n)$  is said to hold  $\omega$ -almost surely.

**Definition 5.** Let  $\omega$  be a (nonprincipal) ultrafilter over  $\mathbb{N}$ . An  $\omega$ -limit of a sequence of points  $\{x_n\}$  in a topological space  $X$  is a point  $x$  in  $X$  such that for any neighborhood  $U$  of  $x$  the relation  $x_n \in U$  holds  $\omega$ -almost surely.

If  $X$  is a Hausdorff space, then the  $\omega$ -limit of a sequence is unique. We denote this point by  $\lim_{\omega} x_n$ .

**Definition 6.** *The ultrapower of a set  $X$  with respect to an ultrafilter  $\omega$ , denoted by  $X^{\omega}$ , consists of equivalence classes of sequences  $\{x_n\}$ ,  $x_n \in X$ , where two sequences  $\{x_n\}$  and  $\{y_n\}$  are identical if and only if  $x_n = y_n$   $\omega$ -almost surely.*

We adopt the notation  $\{x_n\}_{\omega}$  for the equivalence class of sequences.

Let  $(X, d)$  be a metric space,  $\omega$  an ultrafilter over  $\mathbb{N}$  and  $o$  an origin in  $X$ . Furthermore, let a sequence of numbers  $\{d_n\}$  be given, with  $\lim_{\omega} d_n = \infty$ . Then there is a pseudo-metric on the ultrapower  $X^{\omega}$  by setting the distance between  $x = \{x_n\}_{\omega}$  and  $y = \{y_n\}_{\omega}$  to

$$d_{\omega}(x, y) = \lim_{\omega} \frac{d(x_n, y_n)}{d_n}.$$

Let  $X_o^{\omega}$  be the set of equivalence classes of elements  $x = \{x_n\}_{\omega}$  in  $X^{\omega}$  satisfying  $\lim_{\omega} \frac{d(x_n, o)}{d_n} < \infty$  and two elements  $x = \{x_n\}_{\omega}$  and  $y = \{y_n\}_{\omega}$  of  $X^{\omega}$  are identical if and only if  $d_{\omega}(x, y) = 0$ . The metric space  $(X_o^{\omega}, d_{\omega})$  is called an *asymptotic cone* of  $(X, d)$ .

For points in the asymptotic cone  $(X_o^{\omega}, d_{\omega})$  of  $(X, d)$  we use the notation  $x_{\omega}$ .

**Lemma 7.** *If  $x_{\omega} \neq y_{\omega}$ , then for every admissible gauge  $u$  and all sequences  $\{x_n\}$ ,  $\{y_n\}$  representing  $x_{\omega}$  and  $y_{\omega}$ , respectively,*

$$d(x_n, y_n) > u(|x_n| + |y_n|), \quad \omega - a.s.$$

**Proof:** If there is an admissible function  $u$  and sequences  $\{x_n\}$ ,  $\{y_n\}$  such that

$$d(x_n, y_n) \leq u(|x_n| + |y_n|), \quad \omega - a.s.$$

then  $\omega - a.s.$ ,

$$\frac{d(x_n, y_n)}{d_n} \leq \frac{u(|x_n| + |y_n|)}{d_n}.$$

So

$$\lim_{\omega} \frac{d(x_n, y_n)}{d_n} = 0.$$

■

**Lemma 8.** *Let  $X$  be an asymptotically chained metric space. Let  $\{x_n\}$  and  $\{y_n\}$  be two sequences in  $X$  representing the same point  $x_{\omega}$  different from the origin  $o_{\omega}$  such that  $x_n \neq y_n$   $\omega - a.s.$  Then, for any admissible gauge  $u$ , either*

$$d(x_n, y_n) > u, \quad \omega - a.s.$$

*or there exists a sequence  $\{w_n\}$  representing the same point s.t*

$$d(x_n, w_n) > u \quad \omega - a.s.$$

*and*

$$d(y_n, w_n) > u \quad \omega - a.s.$$

**Proof:** Suppose  $d(x_n, y_n) \leq u \omega - a.s.$  and assuming  $|x_n|, |y_n| \rightarrow +\infty$ .

Since  $X$  is asymptotically chained, there is an admissible gauge  $v$  and a chain  $x_1^n = o, \dots, x_{k+1}^n = y_n$  such that

$$\max_{i \in 1, \dots, k} \{d(x_i^n, x_{i+1}^n)\} < v(|y_n|).$$

Assuming  $n$  is large enough so that  $2u(4|y_n|) < |y_n|$ , then at least one point  $x_{i_n}^n$  in the chain satisfies

$$2u(4|y_n|) < d(y_n, x_{i_n}^n) \leq 2u(4|y_n|) + v(|y_n|).$$

Define  $w_n = x_{i_n}^n$ .

For  $n$  large enough,

$$\begin{aligned} 2d(x_n, y_n) &\leq 2u(|x_n| + |y_n|) \\ &\leq |x_n| + |y_n| \\ &\leq 2|y_n| + d(x_n, y_n). \end{aligned}$$

It follows,

$$d(x_n, y_n) \leq u(4|y_n|).$$

and therefore

$$u(4|y_n|) < d(x_n, w_n) \leq 3u(4|y_n|) + v(|y_n|).$$

Choose  $N$  s.t.  $\forall n \geq N$

$$3u(4|y_n|) + v(|y_n|) \leq 2|y_n|.$$

Then

$$\begin{aligned} u(|x_n| + |w_n|) &\leq u(|x_n| + |y_n| + d(y_n, w_n)) \\ &\leq u(2|y_n| + d(x_n, y_n) + 2u(4|y_n|) + v(|y_n|)) \\ &\leq u(2|y_n| + 3u(4|y_n|) + v(|y_n|)) \\ &\leq u(4|y_n|) \\ &< d(x_n, w_n). \end{aligned}$$

This means  $x_n, w_n > u$  and

$$\frac{d(x_n, w_n)}{d_n} \leq \frac{3u(4|y_n|) + v(|y_n|)}{d_n} \rightarrow 0.$$

Related,

$$\begin{aligned} u(|y_n| + |w_n|) &\leq u(2|y_n| + d(y_n, w_n)) \\ &\leq u(2|y_n| + 2u(4|y_n|) + v(|y_n|)) \\ &\leq u(4|y_n|) \\ &< d(y_n, w_n). \end{aligned}$$

■

## 2.4 Proof of Theorem 1.

**Proof:** Let  $u$  be an admissible gauge as given by the definition of  $f$ . Fix an origin  $o \in X$ . Let  $B_n$  be the ball of radius  $n$  around  $o$ . For each  $n$ , choose a point  $z_n$  in  $B_n$  such that

$$d(f(o), f(z_n))$$

is 'maximal'. More precisely, let  $\epsilon > 0$  and choose  $z_n$  such that

$$d(f(o), f(z_n)) > \sup_{x \in B_n} d(f(o), f(x)) - \epsilon.$$

The supremum on the right hand side exists. Otherwise, one could construct a bounded sequence in  $B_n$  mapped by  $f$  to an unbounded sequence. This contradicts the assumptions.

Moreover, the four-point condition

$$[f(x), f(w), f(y), f(z)] \leq \eta([x, w, y, z]),$$

reduces to the three-point condition

$$\frac{d(f(x), f(y))}{d(f(x), f(z))} \leq \eta\left(\frac{d(x, y)}{d(x, z)}\right),$$

by bringing  $w$  to a point at  $\infty$ .

In what follows, we often abbreviate the notation  $f(x)$  to  $x'$ .

Define

$$d_n = d(o, z_n)$$

and

$$d'_n = d(o', z'_n).$$

By construction and assumption over  $f$ , the sequence  $d'_n$  is divergent. Consequently, the sequence  $d_n$  must also be divergent by the assumption over  $f$ .

(1) We show that if  $\frac{d(o, x_n)}{d_n}$  is bounded, then  $\frac{d(o', f(x_n))}{d'_n}$  is also bounded.

If  $x_\omega = o_\omega$ , then  $d(o, x_n) \leq d_n$  for  $n$  large, therefore  $d(o', x'_n) \leq d'_n$ .

If  $x_\omega \neq z_\omega$ ,  $x_\omega \neq o_\omega$ , then by lemma 7  $\{x_n\}, \{z_n\}, \{o\}$  are separated by  $u$   $\omega$ -a.s., and the Möbius- condition yields

$$\frac{d(o', x'_n)}{d'_n} = \frac{d(o', x'_n)}{d(o', z'_n)} < \eta\left(\frac{d(o, x_n)}{d(o, z_n)}\right) = \eta\left(\frac{d(o, x_n)}{d_n}\right) \quad \omega - a.s.$$

Thus  $\frac{d(o', x'_n)}{d'_n}$  is bounded.

If  $x_\omega = z_\omega$ , then by lemma 8 there are  $\{w_n\}$  representing the same point, s.t.  $\{w_n\}, \{x_n\}$  are  $u$  separated and  $\{w_n\}, \{z_n\}$  are  $u$  separated. In particular,  $\{w_n\}, \{x_n\}, \{o\}$  and  $\{w_n\}, \{z_n\}, \{o\}$  are  $u$  separated. Again



using the Möbius condition,

$$\begin{aligned}
\frac{d(o', x'_n)}{d'_n} &= \frac{d(o', x'_n)}{d(o', z'_n)} \\
&= \frac{d(o', x'_n)}{d(o', w'_n)} \frac{d(o', w'_n)}{d(o', z'_n)} \\
&< \eta\left(\frac{d(o, x_n)}{d(o, w_n)}\right) \eta\left(\frac{d(o, w_n)}{d(o, z_n)}\right) \\
&= \eta\left(\frac{d(o, x_n)}{d_n} \frac{d_n}{d(o, w_n)}\right) \eta\left(\frac{d(o, w_n)}{d_n}\right) \\
&\rightarrow \eta(1)^2
\end{aligned}$$

Thus  $\frac{d(o', x'_n)}{d'_n}$  is bounded.

(2) The mapping  $\{x_n\} \rightarrow \{f(x_n)\}$  is well-defined from  $X_\omega(d_n)$  to  $Y_\omega(d'_n)$ .

If  $\{x_n\}$  and  $\{y_n\}$  are two sequences representing the same point different from the origin  $o_\omega$ , then by Lemma 8 and Lemma 7 either  $\{x_n\}, \{y_n\}, \{o\}$  are separated by  $u$ , or there exists a sequence  $\{w_n\}$  representing the same point, s.t.  $\{x_n\}, \{w_n\}, \{o\}$  and  $\{y_n\}, \{w_n\}, \{o\}$  are separated by  $u$ .

In the first case,

$$\begin{aligned}
\frac{d(x'_n, y'_n)}{d'_n} &= \frac{d(o', x'_n)}{d'_n} \frac{d(x'_n, y'_n)}{d(x'_n, o')} \\
&\leq \frac{d(o', x'_n)}{d'_n} \eta\left(\frac{d(x_n, y_n)}{d(x_n, o)}\right) \\
&= \frac{d(o', x'_n)}{d'_n} \eta\left(\frac{d(x_n, y_n)}{d_n} \frac{d_n}{d(x_n, o)}\right) \\
&\rightarrow 0
\end{aligned}$$

In the second case,

$$\begin{aligned}
\frac{d(x'_n, y'_n)}{d'_n} &\leq \frac{d(x'_n, w'_n)}{d'_n} + \frac{d(w'_n, y'_n)}{d'_n} \\
&\leq \frac{d(o', x'_n)}{d'_n} \frac{d(x'_n, w'_n)}{d(x'_n, o')} + \frac{d(o', y'_n)}{d'_n} \frac{d(y'_n, w'_n)}{d(y'_n, o')} \\
&\leq \frac{d(o', x'_n)}{d'_n} \eta\left(\frac{d(x_n, w_n)}{d(x_n, o)}\right) + \frac{d(o', y'_n)}{d'_n} \eta\left(\frac{d(y_n, w_n)}{d(y_n, o)}\right) \\
&\leq \frac{d(o', x'_n)}{d'_n} \eta\left(\frac{d(x_n, w_n)}{d_n} \frac{d_n}{d(x_n, o)}\right) + \frac{d(o', y'_n)}{d'_n} \eta\left(\frac{d(y_n, w_n)}{d_n} \frac{d_n}{d(y_n, o)}\right) \\
&\rightarrow 0
\end{aligned}$$

Therefore, in both cases  $x'_\omega = y'_\omega$ .

Let  $\{x_n\}$  be a sequence representing the origin  $o_\omega$ . If  $\{x_n\}$  is a bounded sequence, then under the assumption over  $f$   $\{x'_n\}$  is also bounded and thus  $x'_\omega = o'_\omega$ .

If  $d(o, x_n)$  diverges, then  $\{x_n\}$  and  $\{o\}$  are  $u$ -separated. We can choose

$y_\omega \neq o_\omega$ , then by lemma 7,  $\{x_n\}, \{y_n\}, \{o\}$  are  $u$ -separated. Thus,

$$\begin{aligned} \frac{d(o', x'_n)}{d'_n} &= \frac{d(o', y'_n)}{d'_n} \frac{d(o', x'_n)}{d(o', y'_n)} \\ &\leq \frac{d(o', y'_n)}{d'_n} \eta\left(\frac{d(o, x_n)}{d(o, y_n)}\right) \\ &= \frac{d(o', y'_n)}{d'_n} \eta\left(\frac{d(o, x_n)}{d_n} \frac{d_n}{d(o, y_n)}\right) \\ &\rightarrow 0. \end{aligned}$$

Therefore,  $o'_\omega = x'_\omega$ .

(3) The map  $g : \{x_n\} \rightarrow \{f(x_n)\}$  is quasisymmetric and in particular continuous and injective.

Let  $\{w_n\}, \{x_n\}, \{y_n\}$  be sequences representing three distinct points  $w_\omega, x_\omega$  and  $y_\omega$  in  $X_\omega(d_n)$ . By lemma 7,  $\{w_n\}, \{x_n\}, \{y_n\}$  are pairwise  $u$  separated. So

$$\frac{d(x'_n, y'_n)}{d(x'_n, w'_n)} < \eta\left(\frac{d(x_n, y_n)}{d(x_n, w_n)}\right),$$

equivalently

$$\frac{d(x'_n, y'_n)}{d'_n} \frac{d'_n}{d(x'_n, w'_n)} < \eta\left(\frac{d(x_n, y_n)}{d_n} \frac{d_n}{d(x_n, w_n)}\right).$$

Taking the  $\omega$  limit, we get

$$\frac{d_\omega(x'_\omega, y'_\omega)}{d_\omega(x'_\omega, w'_\omega)} \leq \eta\left(\frac{d_\omega(x_\omega, y_\omega)}{d_\omega(x_\omega, w_\omega)}\right).$$

So, if  $g : \{x_n\} \mapsto \{f(x_n)\}$  is not constant, then it is continuous and injective. Note that  $\{o'\}$  and  $\{f(z_n)\}$  are not the same points, so  $g$  is indeed not constant. ■

## 3 Examples

### 3.1 Sublinear-Lipschitz equivalences

In [5], Y. Cornulier introduces *sublinear-Lipschitz maps*. A map  $f : X \rightarrow Y$  between metric spaces is a sublinear-Lipschitz map if there is an admissible gauge  $u : \mathbb{R}_+ \rightarrow \mathbb{R}$  such that

$$d(f(x), f(y)) \leq C d(x, y) + C' u(|x| + |y|), \quad \forall x, y \in X,$$

for some constants  $C, C' > 0$ .

Two sublinear-Lipschitz maps  $f, f'$  are equivalent if there is an admissible gauge  $v$  and a constant  $C'' > 0$  such that

$$d(f(x), f'(x)) \leq C'' v(|x|)$$

for all  $x \in X$ .

Sublinear-Lipschitz maps between metric spaces form a category. Taking asymptotic cones, we obtain a functor from the sublinear-Lipschitz category to the Lipschitz category. The sublinear-Lipschitz category is in a sense the maximal category with such a property.

The isomorphisms in the sublinear-Lipschitz category are called sublinear-Lipschitz equivalences or SBE maps.

**Proposition 9.** *Every SBE-mapping  $f : X \rightarrow Y$  is an AM-mapping with linear  $\eta$ .*

**Proof:** If  $f$  is SBE, then  $f$  is bi-Lipschitz except at scales below an admissible gauge  $u$ . Indeed, let  $x, y \in X$  s.t.

$$d(x, y) \geq u(|x| + |y|).$$

Then

$$\begin{aligned} d(f(x), f(y)) &\leq cd(x, y) + Cu(|x| + |y|) \\ &\leq (c + C)d(x, y), \end{aligned}$$

and

$$\begin{aligned} d(f(x), f(y)) &\geq c'd(x, y) - C'u(|x| + |y|) \\ &\geq (c' - C')d(x, y) \end{aligned}$$

So there exists  $D > 0$ , s.t. for all  $x, y > u$ ,

$$\frac{1}{D}d(x, y) \leq d(f(x), f(y)) \leq Dd(x, y).$$

In particular,  $f$  is an AM-mapping with linear  $\eta$ . ■

### 3.2 Assouad-type maps

How can one recognize when a metric space is bi-Lipschitz equivalent to an Euclidean space? (S. Semmes, On the nonexistence of bilipschitz parametrizations, 1996)

Simple as it sounds, this question is not obvious. If a metric space admits a bi-Lipschitz embedding in  $\mathbb{R}^n$ , then it is clearly doubling. The converse is not true, however; the 3-dimensional Heisenberg group with the Carnot-Carathéodory metric is doubling, but does not admit a bi-Lipschitz embedding in  $\mathbb{R}^n$  for any  $n$ .

Assouad's embedding theorem [1] [13] states that any snowflake  $X^\alpha = (X, d^\alpha)$ ,  $0 < \alpha < 1$ , of a doubling metric space admits a bi-Lipschitz embedding in a Euclidean space.

It is clear that the Assouad embedding of a doubling metric space into Euclidean space is an asymptotic-Möbius map. In what follows, we construct an example of an Assouad mapping from an infinite dimensional Heisenberg group into a Hilbert space. This construction follows straightforwardly from a construction of Lee and Naor for the finite dimensional case [12].

Let  $H$  be an infinite dimensional complex Hilbert space.  $H$  carries the symplectic form  $\Omega(a, b) = \text{Im}(\langle a, b \rangle)$ .

The *infinite-dimensional Heisenberg group*  $\mathcal{H}_\Omega$ , is the set of tuples  $(a, t)$  with  $a \in H$ ,  $t \in \mathbb{R}$  and the group law

$$(a, t)(a', t') = (a + a', t + t' + 2\Omega(a, a')).$$

Let  $G$  be a group with identity element  $e$ . A group seminorm on  $G$  is a function  $G \rightarrow [0, \infty)$  satisfying  $N(g^{-1}) = N(g)$  for all  $g \in G$ ,  $N(gh) \leq N(g) + N(h)$  for all  $g, h \in G$  and  $N(e) = 0$ . Moreover, if  $N(g) = 0$  if and only if  $g = e$ , then  $N$  is a group norm on  $G$ .

The function

$$N(a, t) = \sqrt{\sqrt{\|a\|^4 + t^2} + \|a\|^2},$$

is a group norm on  $\mathcal{H}_\Omega$ . Namely, if  $N_1$  and  $N_2$  are group seminorms, then  $\sqrt{N_1^2 + N_2^2}$  is a group seminorm. In this case,  $N_1 = (\|a\|^4 + t^2)^{\frac{1}{4}}$  is the Koranyi norm. Given  $N$ ,

$$d_N((a, t), (a', t')) = N((a, t)(a', t')^{-1})$$

is a right invariant metric on  $\mathcal{H}_\Omega$ .

Let  $G$  be a group, then a Hermitian kernel on  $G$  is a complex-valued function  $K : G \times G \rightarrow \mathbb{C}$  satisfying  $K(g, h) = \overline{K(h, g)}$  for all  $g, h \in G$ .

A Hermitian kernel on  $G$  is positive definite if

$$\sum_{i,j}^n K(g_i, g_j) c_i \overline{c_j} \geq 0$$

for all  $g_1, \dots, g_n \in G$  and for all complex numbers  $c_1, \dots, c_n \in \mathbb{C}$ , with equality if and only if the  $c_i$  vanish. A Hermitian kernel on  $G$  is conditionally negative definite if

$$\sum_{i,j}^n K(g_i, g_j) c_i \overline{c_j} \leq 0$$

for all  $g_1, \dots, g_n \in G$  and for all complex numbers  $c_1, \dots, c_n \in \mathbb{C}$  satisfying  $\sum_i^n c_i = 0$ .

**Theorem 2** (Schönberg [17]). *Let  $G$  be a group,  $K : G \times G \rightarrow \mathbb{R}$  a real-valued kernel on  $G$  satisfying  $K(g, g) = 0 \ \forall g \in G$ . Then  $K$  is negative definite if and only if there exists a Hilbert space  $H$  and a function  $T : G \rightarrow H$  such that  $\forall g, h \in G : K(g, h) = \|T(g) - T(h)\|^2$ .*

Thus, if  $d_N$  is negative definite, then there is a Hilbert space  $H$  and an isometry  $T : (\mathcal{H}_\Omega, \sqrt{d_N}) \rightarrow H$ . In particular, the mapping  $T : (\mathcal{H}_\Omega, d_N) \rightarrow H$  is an AM mapping.

A complex-valued function  $F$  on a group  $G$  satisfying  $F(g^{-1}) = \overline{F(g)}$  for all  $g \in G$  induces a Hermitian kernel on  $G$  by defining  $K_F(g, h) := F(gh^{-1})$ . The function  $F$  is called positive definite if  $K_F$  is a positive definite Hermitian kernel on  $G$ .

**Lemma 10.** For any  $\lambda \in \mathbb{R}$ , the function

$$\Phi_\lambda(a, t) = e^{-|\lambda|\|a\|^2 + i\lambda t}$$

is positive definite on  $\mathcal{H}_\Omega$ .

**Proof:** Indeed, the function  $\Phi_\lambda$  satisfies the condition  $\Phi_\lambda((a, t)^{-1}) = \overline{\Phi_\lambda((a, t))}$  and induces the kernel

$$K((a, s), (b, t)) := \exp(-|\lambda|\|a - b\|^2 + i\lambda(s - t - 2\Omega(a, b))).$$

The kernel can be rewritten as a product of three exponentials

$$\exp(-|\lambda|(\|a\|^2 + \|b\|^2)) \exp(i\lambda(s - t)) \exp(2|\lambda|(\operatorname{Re}(\langle a, b \rangle) - i \operatorname{sign}(\lambda) \Omega(a, b))).$$

The product of positive definite kernels is positive definite. The first two factors are positive definite because for all  $c_1, \dots, c_n \in \mathbb{C}$  the matrix  $(c_i \overline{c_j})_{i,j}$  is positive semidefinite. In particular, also the matrix  $(\exp(i\lambda(s_i - t_j)) c_i \overline{c_j})_{i,j}$ . The exponential of any positive definite kernel is again a positive definite kernel ([3] Proposition 8.2.). Thus the kernel  $K$  is positive definite if

$$\operatorname{Re}(\langle a, b \rangle) - i \operatorname{sign}(\lambda) \Omega(a, b)$$

is positive definite. This is  $\langle a, b \rangle$  if  $\lambda < 0$  and  $\overline{\langle a, b \rangle}$  if  $\lambda \geq 0$  and thus clearly positive definite.  $\blacksquare$

**Proposition 11.**  $d_N : \mathcal{H}_\Omega \times \mathcal{H}_\Omega \rightarrow H$  is conditionally negative definite.

**Proof:** The existence of  $\frac{1}{2}$ -stable distributions implies that for all  $\epsilon > 0$  there exists a non-negative integrable function  $\varphi_\epsilon : \mathbb{R} \rightarrow [0, \infty)$  s.t. its Fourier transform  $\hat{\varphi}_\epsilon(t) = e^{-\epsilon\sqrt{|t|}}$ . Note that  $\frac{1}{2\pi} \hat{\varphi}_\epsilon(x) = \varphi_\epsilon(x)$ .

By the above lemma,

$$F_\epsilon(a, t) = \int_{\mathbb{R}} e^{-|\lambda|\|a\|^2 + i\lambda t} \varphi_\epsilon(\lambda) d\lambda$$

is positive definite on  $\mathcal{H}_\Omega$ . Let

$$h_k(x) = \frac{k}{\pi} \frac{1}{k^2 + x^2}$$

denote the Cauchy distribution with scale parameter  $k > 0$ . We write around

$$\begin{aligned} F_\epsilon(a, t) &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\lambda t} \hat{h}_{\|a\|^2}(\lambda) \hat{\varphi}_\epsilon(\lambda) d\lambda \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\lambda t} \widehat{h_{\|a\|^2} * \hat{\varphi}_\epsilon}(\lambda) d\lambda \\ &= (h_{\|a\|^2} * \hat{\varphi}_\epsilon)(t). \end{aligned}$$

Since  $F_\epsilon$  is positive definite,  $\frac{1-F_\epsilon}{\epsilon}$  is conditionally negative definite and

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \frac{1 - F_\epsilon(a, t)}{\epsilon} &= \lim_{\epsilon \rightarrow 0} \left[ h_{\|a\|^2} * \frac{1 - \hat{\varphi}_\epsilon}{\epsilon} \right] \\ &= \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}} \frac{1 - e^{-\epsilon\sqrt{|x|}}}{\epsilon} h_{\|a\|^2}(t - x) dx \\ &= \frac{\|a\|^2}{\pi} \int_{\mathbb{R}} \frac{\sqrt{|x|}}{\|a\|^4 + (t - x)^2} dx. \end{aligned}$$

We now show that for all  $r, t \in \mathbb{R}$

$$r^2 \int_{\mathbb{R}} \frac{\sqrt{|x|}}{r^4 + (t-x)^2} dx = \pi \sqrt{\sqrt{r^4 + t^2} + r^2}.$$

By changing variables  $x = r^2 y$  and  $s = t/r^2$ , the left hand side can be written as

$$\int_0^\infty \left( \frac{1}{1+(s-y)^2} + \frac{1}{1+(s+y)^2} \right) \sqrt{y} dy$$

This integral is equal to

$$\lim_{r \rightarrow 0} \lim_{R \rightarrow \infty} \frac{1}{2} \int_{C_{r,R}} \left( \frac{1}{1+(s-z)^2} + \frac{1}{1+(s+z)^2} \right) \sqrt{z} dz,$$

where  $C_{r,R}$  is the keyhole contour with a branch cut along the positive real axis. The integrand has simple poles at  $i \pm s$  and  $-i \pm s$ . By  $\text{Res}(\frac{\sqrt{z}}{1+(s-z)^2}, i+s) = \frac{\sqrt{i+s}}{2i}$ ,  $\text{Res}(\frac{\sqrt{z}}{1+(s-z)^2}, -i+s) = \frac{\sqrt{-i+s}}{-2i}$ ,  $\text{Res}(\frac{\sqrt{z}}{1+(s+z)^2}, i-s) = \frac{\sqrt{i-s}}{2i}$  and  $\text{Res}(\frac{\sqrt{z}}{1+(s+z)^2}, -i-s) = \frac{\sqrt{-i-s}}{-2i}$ . Now adding all the residuals and multiplying by  $2\pi i$ , we obtain the above equation as

$$\frac{\pi}{2} (\sqrt{i+s} - \sqrt{-i+s} + \sqrt{i-s} - \sqrt{-i-s}).$$

This further simplifies to

$$\begin{aligned} \pi \text{Re}(\sqrt{i+s} + \sqrt{i-s}) &= \pi (\text{Re}(\sqrt{i+s}) + \text{Im}(\sqrt{i+s})) \\ &= \pi \left( \sqrt{\frac{\sqrt{1+s^2}+s}{2}} + \sqrt{\frac{\sqrt{1+s^2}-s}{2}} \right) \\ &= \pi \sqrt{\sqrt{1+s^2}+1} \end{aligned}$$

So we have seen that the  $N$ -norm is a limit of conditionally negative definite functions on  $\mathcal{H}_\Omega$ , so it is conditionally negative definite.  $\blacksquare$

## 4 Applications to dimension theory

### 4.1 Dimension theory of finitely-generated groups

In 1993 M. Gromov introduced the notion of asymptotic dimension as a large scale analogue of Lebesgue's covering dimension [7]. The asymptotic dimension of a finitely generated group is a quasi-isometric invariant. Its most prominent application goes back to Guoliang Yu, who showed that any finitely generated group with finite homotopy type and finite asymptotic dimension satisfies the Novikov conjecture [21].

**Definition 12.** *Let  $X$  be a metric space. We say that the asymptotic dimension of  $X$  does not exceed  $n$  if for every uniformly bounded open cover  $\mathcal{V}$  of  $X$  there exists a uniformly bounded open cover  $\mathcal{U}$  of  $X$  of multiplicity  $\leq n+1$  such that  $\mathcal{V}$  refines  $\mathcal{U}$ . We write  $\text{asdim} X = n$  if it is true that  $\text{asdim} X \leq n$  and  $\text{asdim} X \not\leq n-1$*

The asymptotic dimension of a finitely generated group  $\Gamma$  has several interesting implications. For example,  $\text{asdim } \Gamma = 0$  if and only if  $\Gamma$  is finite (Proposition 65. in [2]) and  $\text{asdim } \Gamma = 1$  if and only if  $\Gamma$  is virtually free (Theorem 66. in [2]).

If  $G$  is a finitely generated nilpotent group, then its asymptotic cone  $G_\omega$  is a Carnot group [14] and the asymptotic dimension of  $G$  coincides with the topological dimension of its asymptotic cone [2].

**Theorem 3.** *Let  $G$  and  $G'$  be a finitely generated nilpotent groups and  $f : G \rightarrow G'$  an AM-map, then  $\text{asdim}(G) \leq \text{asdim}(G')$ . If  $G$  and  $G'$  are nilpotent Lie groups, then  $\dim(G) \leq \dim(G')$ . Furthermore, if  $\text{asdim}(G) = \text{asdim}(G')$ , then the asymptotic cones of  $G$  and  $G'$  are isomorphic graded Lie groups. Conversely, given nilpotent groups with isomorphic asymptotic cones, there exists an asymptotic-Möbius map between them.*

**Proof:** The mapping  $f$  induces a quasimetric mapping  $g$  between the asymptotic cones  $G_\omega$  and  $G'_\omega$  of  $G$  and  $G'$ , respectively.

In particular,  $g$  is an injective mapping from  $G_\omega$  to  $G'_\omega$  and hence the topological dimensions increase. Since  $G$  and  $G'$  are nilpotent,  $\text{asdim}(G) = \text{top-dim}(G_\omega) \leq \text{top-dim}(G'_\omega) = \text{asdim}(G')$ .

The case for nilpotent Lie groups follows from the fact that for nilpotent Lie groups  $\text{top-dim}(G) = \text{top-dim}(G_\omega)$ .

If  $\text{asdim}(G) = \text{asdim}(G')$ , the invariance of domain theorem implies that  $g$  is a homeomorphism. Since both  $g$  and  $g^{-1}$  are quasimetric, they are differentiable by Pansu's theorem a.e., and the differential of  $g$  is a graded isomorphism of groups at almost every point [15]. ■

## 4.2 Dimension theory of CAT(0)-spaces

Heinrich Jung established in the early 1900s that the diameter of a set in Euclidean space  $\mathbb{R}^n$  is related to the radius of the smallest enclosing ball by an inequality whose constant depends only on the dimension  $n$  [9]. More precisely, for any set  $A \subset \mathbb{R}^n$

$$r \leq d(A) \sqrt{\frac{n}{2(n+1)}}, \quad (1)$$

where  $r$  is the radius of the smallest enclosing ball of  $A$  and  $d(A)$  is the diameter of  $A$ .

The above inequality can be used to define a geometric dimension of CAT(0)-spaces. A CAT(0)-space  $X$  has *geometric dimension* less than  $n$  if and only if the Jung's inequality (1) is true in  $X$  [4].

**Remark 13.** *It turns out that the geometric dimension of a CAT(0)-space  $X$  is equal to the maximal topological dimension of all compact subsets in  $X$  [10].*

The geometric dimension can be transformed into a large-scale term in the following way. A CAT(0)-space  $X$  has telescopic dimension less than  $n$  if and only if for every  $\delta > 0$  there is a  $D > 0$  such that for every set  $A$

in  $X$  with diameter greater than  $D$ ,

$$r \leq \left( \delta + \sqrt{\frac{n}{2(n+1)}} \right) d(A). \quad (2)$$

Somewhat more general,

**Definition 14.** *A space  $X$  whose asymptotic cones are  $\text{CAT}(0)$  has telescopic dimension  $\leq n$  if every asymptotic cone has geometric dimension  $\leq n$ .*

We adopt the notation  $\text{tele-dim}(X)$  for the telescopic dimension of  $X$ .

**Theorem 4.** *Let  $X$  and  $Y$  be spaces, whose asymptotic cones are  $\text{CAT}(0)$ . If there exists an  $AM$ -map  $f : X \rightarrow Y$ , then the telescopic dimension increases e.g.  $\text{tele-dim}(X) \leq \text{tele-dim}(Y)$ .*

The theorem follows straight from theorem 1, remark 13 and the fact that the topological dimension increases under injective continuous maps.

It follows that there can be no  $AM$ -mapping from Euclidean buildings of finite rank  $r$  and hence telescopic dimension  $r$  to a Euclidean building of lower rank.

If there is a  $AM$ -mapping from a  $\text{CAT}(0)$ -space to a hyperbolic metric space, then  $X$  is also a hyperbolic metric space.



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